Image Compression by Partial Differential Equations

(Séminaire Doctorants IRIMAS)

Thomas Jacumin

sous la direction de Zakaria Belhachmi

Goal : reconstruct missing parts of an image $f : D \to [0, 1]$ from $K \subset D$.

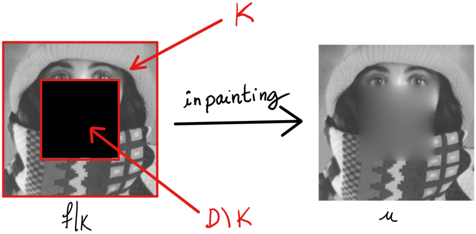


Figure – Inpainting

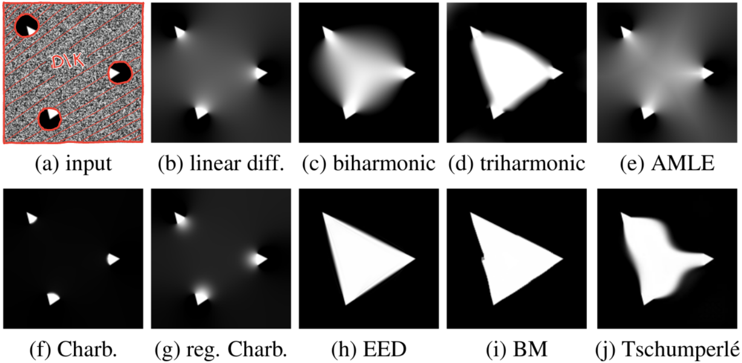


Figure – Examples [2]

JPEG

Each block 8×8 of an image is described as a function

$$F : \{0, \dots, 7\}^2 \to \mathbb{R}.$$

JPEG

Each block 8×8 of an image is described as a function

$$F : \{0, \dots, 7\}^2 \to \mathbb{R}.$$

For $0 \leq u, v \leq 7$,

$$F_{u,v} : (x, y) \mapsto C \cos\frac{(2x+1)u\pi}{16} \cos\frac{(2y+1)v\pi}{16}.$$

JPEG

Each block 8×8 of an image is described as a function

$$F : \{0, \ldots, 7\}^2 \to \mathbb{R}.$$

For $0 \leq u, v \leq 7$,

$$F_{u,v} : (x, y) \mapsto C \;\cos \frac{(2x+1)u\pi}{16} \;\cos \frac{(2y+1)v\pi}{16}.$$

Then, we can decompose F as :

$$F = \sum_{0 \leq u,v \leq 7} c_{u,v}\, F_{u,v}.$$

JPEG

Each block 8×8 of an image is described as a function

$$F : \{0, \ldots, 7\}^2 \to \mathbb{R}.$$

For $0 \leq u, v \leq 7$,

$$F_{u,v} : (x, y) \mapsto C \ \cos \frac{(2x+1)u\pi}{16} \ \cos \frac{(2y+1)v\pi}{16}.$$

Then, we can decompose F as :

$$F = \sum_{0 \leq u,v \leq 7} c_{u,v} \, F_{u,v}.$$

For compression purpose, we can neglect small $c_{u,v}$.

We do not do this : we remove parts of the image and reconstruct them by inpainting :

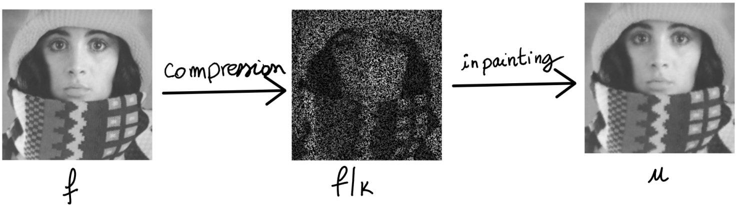


Figure – Compression by Inpainting.

Question 1

How to choose pixels to keep ?

Question 1

How to choose pixels to keep ?

Answer 1

We should keep relevant pixels to obtain a good reconstruction.

Question 1

How to choose pixels to keep ?

Answer 1

We should keep relevant pixels to obtain a good reconstruction.

Questions 2-3

What is a good reconstruction ? How to find these pixels ?

Let $D \subset \mathbb{R}^2$ the support of an image $f : D \longrightarrow [0,1]$. Some errors examples

Example 1

$$\|u - f\|_{L^2(D)} := \left(\int_D (u - f)^2 \, dx \right)^{1/2}.$$

Example 2

$$|u - f|_{H^1(D)} := \left(\int_D |\nabla u - \nabla f|^2 \, dx \right)^{1/2}.$$

Shape Optimisation Problem

$$\min_{K \subseteq D} \{\mathcal{E}(u_K) \mid u_K \text{ solution of an inpainting problem}\},$$

Shape Optimisation Problem

$$\min_{K \subseteq D} \{\mathcal{E}(u_K) \mid u_K \text{ solution of an inpainting problem}\},$$

A trivial solution is $K = D$. We need to add constraint on the "size" of K.

Shape Optimisation Problem

$$\min_{K\subseteq D}\{\mathcal{E}(u_K)\mid u_K \text{ solution of an inpainting problem}\},$$

A trivial solution is $K = D$. We need to add constraint on the "size" of K.

Compression problem

$$\min_{K\subseteq D, m(K)\leq c}\{\mathcal{E}(u_K)\mid u_K \text{ solution of an inpainting problem}\},$$

Shape Optimisation Problem

$$\min_{K \subseteq D} \{\mathcal{E}(u_K) \mid u_K \text{ solution of an inpainting problem}\},$$

A trivial solution is $K = D$. We need to add constraint on the "size" of K.

Compression problem

$$\min_{K \subseteq D, m(K) \leq c} \{\mathcal{E}(u_K) \mid u_K \text{ solution of an inpainting problem}\},$$

The existence of a solution depends on the inpainting problem, on the error $\mathcal{E}$ and on m.

Problem (Homogeneous Diffusion Inpainting [1])

Find u in $H^1(D)$ such that

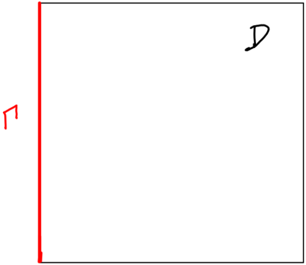
$$\begin{cases} -\Delta u = 0, & \text{in } D \setminus K, \\ u = f, & \text{in } K, \\ \frac{\partial u}{\partial \mathrm{n}} = 0, & \text{on } \partial D. \end{cases} \tag{1}$$

Problem (Heat equation)

Find u such that

$$\left\{\begin{array}{rl}\partial_t u - \Delta u = 0, & \text{in } [0,+\infty[\times D,\\ u = 100, & \text{on } [0,+\infty[\times\Gamma,\\ \partial_n u = 0, & \text{on } [0,+\infty[\times\partial D\setminus\Gamma.\end{array}\right. \tag{2}$$

and $u(0,\cdot) = 0$ *in* D.

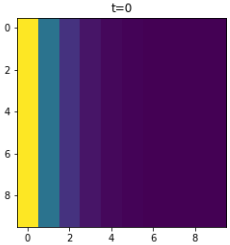


Problem (Heat equation)

Find u such that

$$\left\{ \begin{aligned} \partial_t u - \Delta u &= 0, \quad \textit{in } [0,+\infty[\times D, \\ u &= 100, \quad \textit{on } [0,+\infty[\times \Gamma, \\ \partial_n u &= 0, \quad \textit{on } [0,+\infty[\times \partial D \setminus \Gamma. \end{aligned} \right. \tag{3}$$

and $u(0,\cdot) = 0$ *in* D.

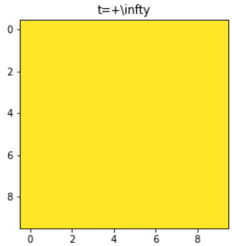


Problem (Heat equation ($t \to +\infty$))

Find u such that

$$\begin{cases} \cancel{\partial_t u} - \Delta u = 0, & \textit{in } [0, +\infty[\times D, \\ u = 100, & \textit{on } [0, +\infty[\times \Gamma, \\ \partial_n u = 0, & \textit{on } [0, +\infty[\times \partial D \setminus \Gamma. \end{cases} \tag{4}$$

and $u(0, \cdot) = 0$ *in* D.



Theorem

If the inpainting is the homogeneous diffusion inpaiting

$$\mathcal{E}(u) = |u - f|_{H^1(D)} := \left(\int_D |\nabla u - \nabla f|^2 \, dx \right)^{1/2},$$

and if

$$m(K) = cap(K) := \inf \left\{ \int_D |\nabla u|^2 \, dx + \int_D u^2 \, dx \;\middle|\; u \in H_0^1(D),\ u \geq 1 \text{ } p.p. \text{ } dans \text{ } K \right\},$$

Then, the compression problem admits at least one solution.

Question

In practice, how to construct K ?

Topological Gradient

Goal : Determine the influence of make a hole in K on the error.
Let $K_\varepsilon := K \setminus B(x_0, \varepsilon)$ such that $B(x_0, \varepsilon) \subset K$.

Topological Gradient

Goal : Determine the influence of make a hole in K on the error.
Let $K_\varepsilon := K \setminus B(x_0, \varepsilon)$ such that $B(x_0, \varepsilon) \subset K$.

Topological Gradient

Goal : Determine the influence of make a hole in K on the error.
Let $K_\varepsilon := K \setminus B(x_0, \varepsilon)$ such that $B(x_0, \varepsilon) \subset K$.

$$j : A \subset D \mapsto \min_{v \in H^1(D),\ v=f \text{ in } A} \int_D |\nabla v|^2 \, dx.$$

Proposition

The compression problem is equivalent to $\displaystyle\max_{K \subset D,\ m(K) \leq c} j(K)$.

Topological Gradient

$$\max_{K\subset D,\ m(K)\leq c} j(K) = \max_{K\subset D,\ m(K)\leq c}\ \min_{v\in H^1(D),\ v=f \text{ in } K} \int_D |\nabla v|^2\ dx$$

$$= \max_{K\subset D,\ m(K)\leq c} \left(\underbrace{\left(\min_{v\in H^1(D),\ v=f \text{ in } K} \int_{D\setminus K} |\nabla v|^2\ dx \right)}_{\geq 0} + \int_K |\nabla f|^2\ dx \right).$$

Then,

$$\max_{K\subset D,\ m(K)\leq c} \int_K |\nabla f|^2\ dx \leq \max_{K\subset D,\ m(K)\leq c} j(K).$$

Topological Gradient

Goal : Determine the influence of make a hole in K on the error.
Let $K_\varepsilon := K \setminus B(x_0, \varepsilon)$ such that $B(x_0, \varepsilon) \subset K$.

$$j : A \subset D \mapsto \min_{v \in H^1(D),\ v = f \text{ in } A} \int_D |\nabla v|^2 \, dx.$$

Proposition

The compression problem is equivalent to $\displaystyle \max_{K \subset D,\ m(K) \leq c} j(K)$.

Proposition

When ε *tends to* 0*,*

$$j(K_\varepsilon) - j(K) = -|\Delta f(x_0)|^2 \frac{\pi}{2} \varepsilon^4 + o(\varepsilon^4).$$

Topological Gradient

Figure – f et $|\Delta f|$.

Topological Gradient

Question

How to build K ?

Topological Gradient

Question

How to build K ?

Answer

Keep points x_0 where the quantity $|\Delta f(x_0)|$ is maximal.

Topological Gradient

Question

How to build K ?

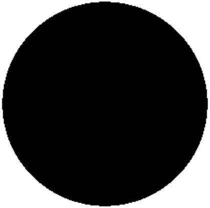
Answer

Keep points x_0 where the quantity $|\Delta f(x_0)|$ is maximal.

Fat Pixels

For $m > 0$ and $n \in \mathbb{N}^*$, we set

$$\mathcal{A}_{m,n} := \left\{ \bigcup_{i=1}^{n} \overline{B(x_i, r)} \;\middle|\; x_i \in D,\ r = mn^{-1/2} \right\}.$$



Fat Pixels

Question

How to build K ?

Fat Pixels

Question

How to build K ?

Answer

Built K such that the pixels density increase with the quantity $|\Delta f|$.

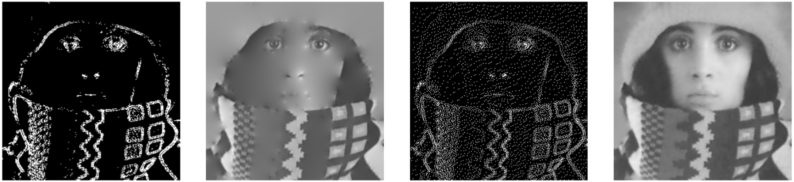


Figure – Topological Gradient vs Fat Pixels.

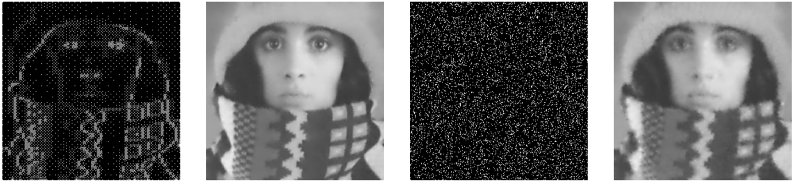


Figure – B-Tree vs Random.

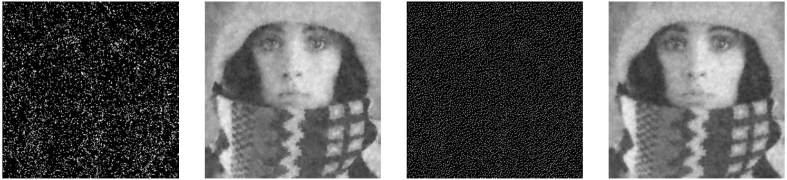


Figure – Image with Gaussian Noise.

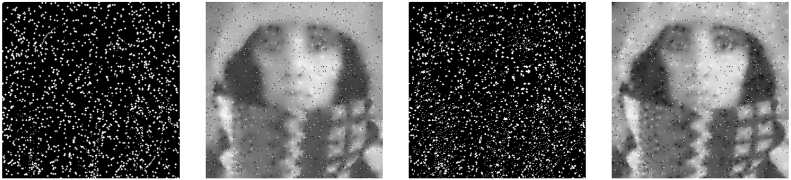


Figure – Image with Salt and Pepper Noise.

Figure – Image with Gaussian Noise.

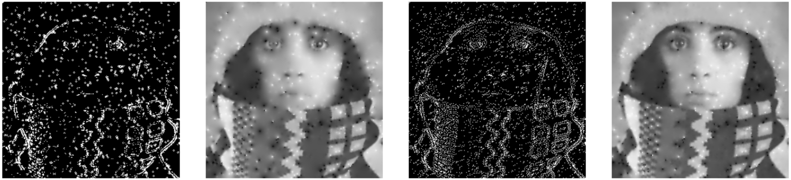


Figure – Image with Salt and Pepper Noise.

Thanks for your attention !

- Z. Belhachmi, D. Bucur, B. Burgeth, and J. Weickert, *How to choose interpolation data in images*, 70, pp. 333–352.
- C. Schmaltz, P. Peter, M. Mainberger, F. Huth, J. Weickert, and A. Bruhn, *Understanding, optimising, and extending data compression with anisotropic diffusion*, 108.