Reference Answers to Assignment 2 of Numerical Algorithms with Case Studies

Here are TA's reference answers, not standard answers. If you have any different ideas, please feel free to let us know. Anyone who first finds out errors in this doc will receive additional rewards(up to 5) for this assigment.

Contents

Question 1

(10 pts) Conduct Gaussian elimination on

$$G = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 1 & -3 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 3 & -2 & 0 \end{bmatrix}$$

and find its LU decomposition.

Answer:

$$\begin{aligned}
G = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 1 & -3 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 3 & -2 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 1 & -3 & -1 \\ 2 & 3 & -1 & 1 \\ -2 & 3 & -2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & & 1 & \\ -2 & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ & 1 & -3 & -6 \\ & 3 & -1 & -9 \\ & 3 & -2 & -10 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 3 & 1 & \\ -2 & 3 & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ & 1 & -3 & -6 \\ & & 8 & -9 \\ & & 7 & 28 \end{bmatrix} \\
&= \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 3 & 1 & \\ -2 & 3 & \frac{7}{8} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 5 \\ & 1 & -3 & -6 \\ & & 8 & -9 \\ & & & \frac{161}{8} \end{bmatrix}
\end{aligned}$$

Question 2

(10 pts) Write your own codes for forward substitution or back substitution (choose the one you like). Conduct random tests to show that its computational complexity is proportional to n^2(n is the size of the matrix) by making a plot. (Hint:for each tested n, conduct for example 10 random tests and compute the average time.)

Answer:
The following function *tri_system_solver* offers two algorithms to solve a triangular system: row oriented method and column oriented method given the triangular T and vector b as the first two parameters of the function. To use different algorithm, change its third parameter *row_oriented* to be true or false.

If the input matrix is an upper triangular matix, the function will apply back substitution method. If the input matrix is a lower triangular matrix, the function will apply forward substitution method. Be sure to input a triangular matrix, or the result may be wrong.

The two algorithms perform different in efficiency. If you noticed the different performance of matrix multiplication in assignment 1, you will know why column oriented method is faster immediantly.

```
function [ x ] = tri_system_solver(T, b, row_oriented)
if nargin<3
      row_oriented = true;
end
n = size(T,1);
x = zeros(n,1);
if row_oriented
    if T(1,n) == 0 %Forward Substitution for Lower Triangular System
        for i = 1:n
            x(i) = (b(i) - T(i,1:i-1)*x(1:i-1))/T(i,i);
        end
    else %Back Substitution for Upper Triangular System
        for i = n:-1:1
            x(i) = (b(i) - T(i,i +1:n)*x(i+1:n))/T(i,i);
        end
    end
else
    if T(1,n) == 0 %%Forward Substitution for lower triangular system
        index = 1:n;
    else %Back Substitution for upper triangular system
        index = n:-1:1;
    end
    for i = index
```

```
        x(i) = b(i)/T(i ,i);
        b = b - T(:, i) * x(i);
    end
end
```

Function *rand_tri* in the codes above is for generating random triangular matrix and listed with its details in the end of this doc. Now check the correctness of this solver.

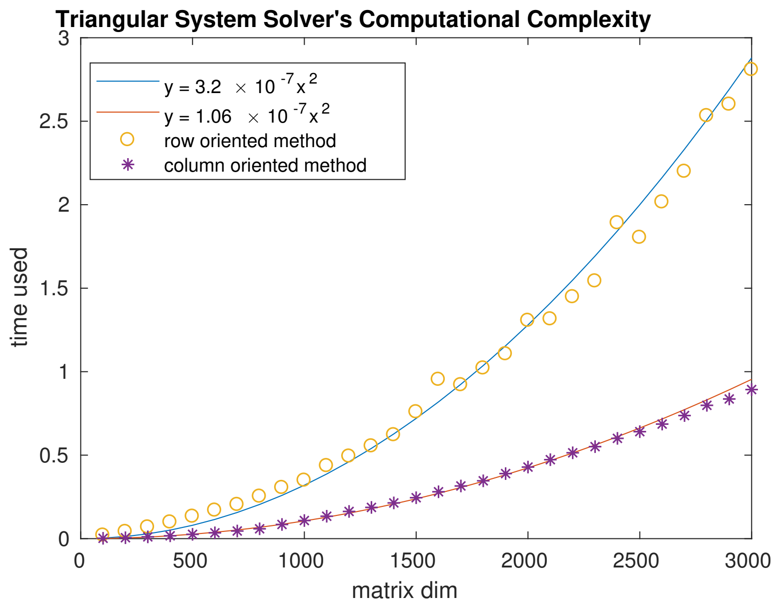
```
error = 0;
n = 200;
for i=1:1000
    T = rand_tri(n,rand>0.5); %generate lower or upper triangular system randomly
    b = rand(n,1);
    re = tri_system_solver(T, b, rand>0.5);%use the two method randomly
    loss = sum((T*re-b).^2)/sum(b.^2);
    error = max(loss,error);
end
fprintf('The relative error is %g.\n',error);
```

```
The relative error is 7.25271e-28.
```

We can confirm the correctness of the solver as the relative error is smaller than machine precision.

Next I will conduct random tests to show its computational complexity. The function *time_test_tri* in the codes below is for computing the time used with matrix dim n and given algorithm for 100 computations and is listed with its details in the end.

```
ns = (1:30)*100;
times = zeros(2,30);
for i = 1:30
    times(1,i) = time_test_tri(ns(i), 1);
    times(2,i) = time_test_tri(ns(i), 0);
end
plot(ns,3.2*1e-7*ns.^2,ns,1.06*1e-7*ns.^2,ns,times(1,:),'o',ns,times(2,:),'*');
xlabel('matrix dim');
ylabel('time used');
legend('y = 3.2\times 10^{-7}x^{2}','y = 1.06\times 10^{-7}x^{2}',...
    'row oriented method','column oriented method','Location','best');
title("Triangular System Solver's Computational Complexity");
```



The figure shows that the complexity is $o(n^2)$ for both algorithms but the coefficients are different.

Question 3

(20 pts) Write your own codes for the LU decomposition of a matrix A and test the correctness of your codes by computing the error between A and LU, where L and U are the lower triangular and upper triangular matrices obtained from your codes. Conduct random tests to show that its computational complexity is proportional to n^3(n is the size of the matrix) by making a plot.

Answer:
The following two functions offer two algorithms to solve a LU decomposition: row oriented and column oriented. The row oriented algorithm uses the pseudocodes discussed in class.

The column oriented algorithm will return a unit upper triangular matrix, which is different from the method discussed in class. But they all return a LU decompostion. It's easy for column oriented algorithm to change itself to return a unit lower triangular matrix. If you have no idea how to make it, feel free to contact me.

You will find the two algorithms perform different in efficiency.

Function LU_s applies the method discussed in class.

```
function [L,U] = LU_s(A)
```

```
[m,n] = size(A);
L = eye(n);
U = A;
if m~=n
    fprintf('Dim error\n');
    return
end
for k= 1:n
    for j= (k+1):n
        L(j,k) = U(j,k)/(U(k,k)+eps);
        for i = k:n
            U(j,i) = U(j,i) - L(j,k)*U(k,i);
        end
    end
end
end
```

Function LU_f will be faster.

```
function [L,U] = LU_f(A)
[m,n] = size(A);
U = eye(n);  % initialize
L = A;
if m~=n
    fprintf('Dim error\n');
    return
end
for k= 1:n
    for j= (k+1):n
        U(k,j) = L(k,j)/(L(k,k)+eps);
        L(:,j) = L(:,j) - U(k,j)*L(:,k);
    end
end
end
```

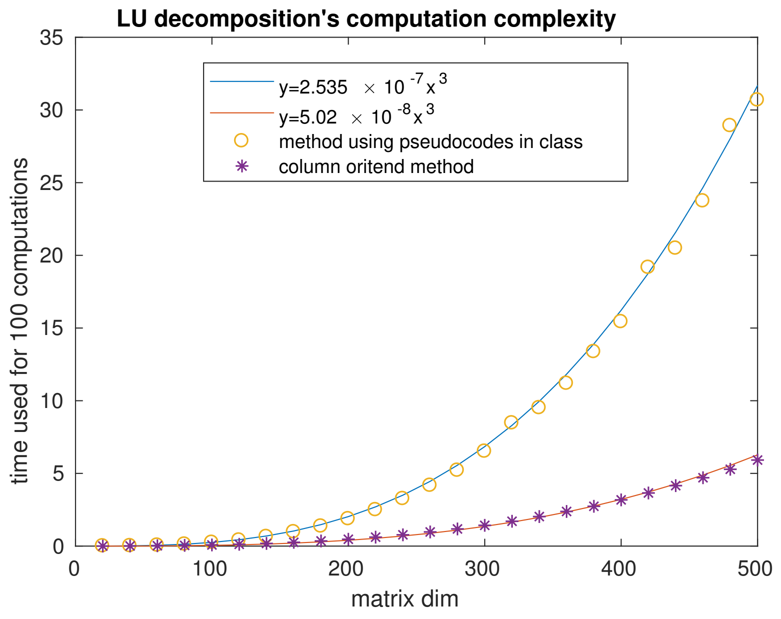
Now check the correctness.

```
error = 0;
loss = 0;
for i = 1:100
    A = rand(100) + eye(100);
    [L1,U1] = LU_f(A);
    [L2,U2] = LU_s(A);
    error = max(error, norm(L1-tril(L1),'fro'));
    error = max(error, norm(L2-tril(L2),'fro'));
    error = max(error, norm(U1-triu(U1),'fro'));
    error = max(error, norm(U2-triu(U2),'fro')); % check the triangular matrix
    loss = max(loss, norm(L1*U1-A,'fro'));
    loss = max(loss, norm(L2*U2-A,'fro')); % check the decomposition
end
fprintf('The max difference between the results and triangular matrix is %g.\n',error);
fprintf('The max decomposition loss is %g.\n',loss);
```

```
The max difference between the results and triangular matrix is 9.17958e-10.
The max decomposition loss is 1.42533e-09.
```

Now I will conduct random tests to show its computational complexity. Function *time_test_lu* is for computing the time used for 100 computations and is listed with its details in the end.

```
ns = (1:25)*20;
times = zeros(25,2);
for i = 1:25
    times(i,:) = time_test_lu(ns(i));
end
plot(ns,2.535*1e-7*ns.^3,ns,5.02*1e-8*ns.^3,ns,times(:,1)','o',ns,times(:,2)','*');
xlabel('matrix dim');
ylabel('time used for 100 computations');
legend('y=2.535\times 10^{-7}x^3','y=5.02\times 10^{-8}x^3',...
    'method using pseudocodes in class','column oritend method','Location','best');
title("LU decomposition's computation complexity");
```



Functions metioned in the doc.

Function *rand_tri* is for generating random triangular matrix in Question 2.

```
function [ T ] = rand_tri( n , islower )
T = ones(n);
for i=1:n
    T(i+1:n,i) = 0;
end
if islower
    T = T';
end
T = T.*rand(n)+eye(n);% make sure T is well-conditioned.
end
```

Funtion *time_test_tri* is for computing the time used with 100 computations for triangular system solver in Question 2.

```
function [ t ] = time_test_tri(n, row_oriented)
T = rand_tri(n,0);
b = rand(n,1);
tic;
for i=1:500
    tri_system_solver(T,b,row_oriented);
end
t = toc;
end
```

Function *time_test_lu* is for computing the time used with 100 computations for LU decomposition in Question 3.

```
function [ t ] = time_test_lu(n)
A = rand(n);
A = A*A';
t = [0 0];
tic;
for i=1:200
    LU_s(A);
end
t(1) = toc;
tic;
for i=1:200
    LU_f(A);
end
t(2) = toc;
end
```