# Assignment 1

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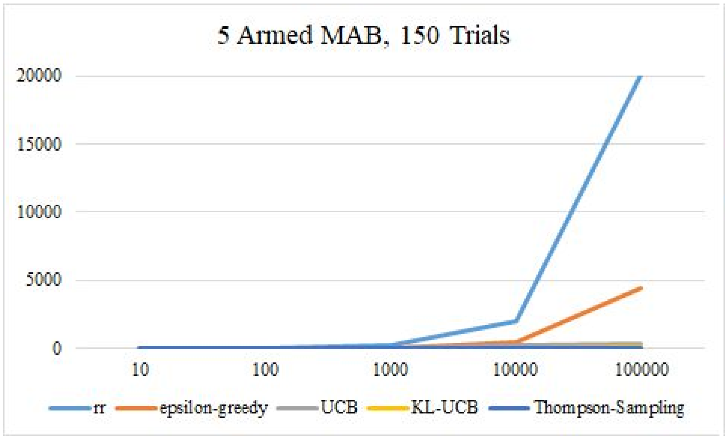
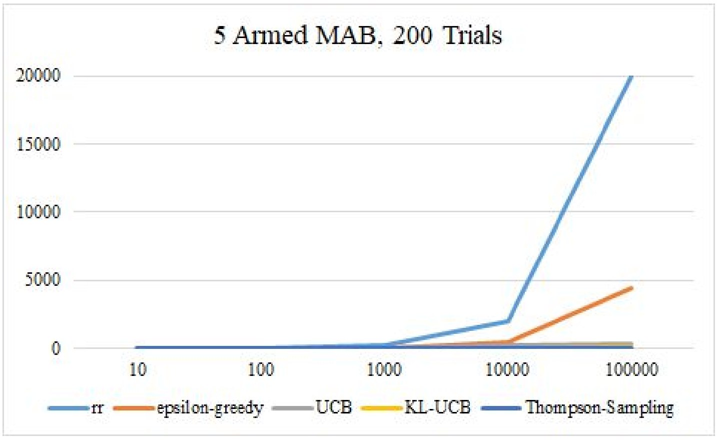
## Plots

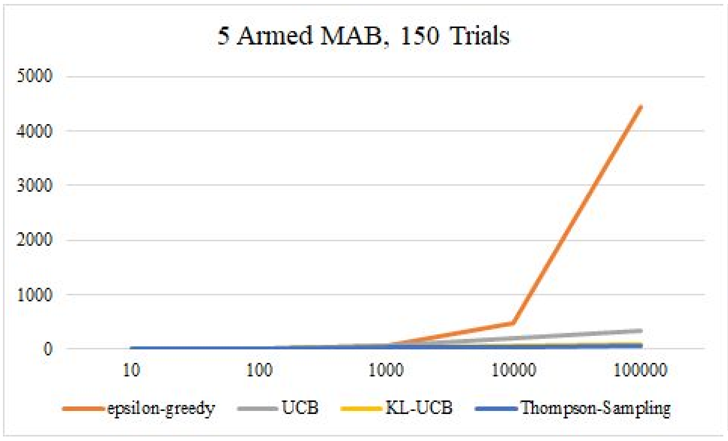
### 5 Arm Instance

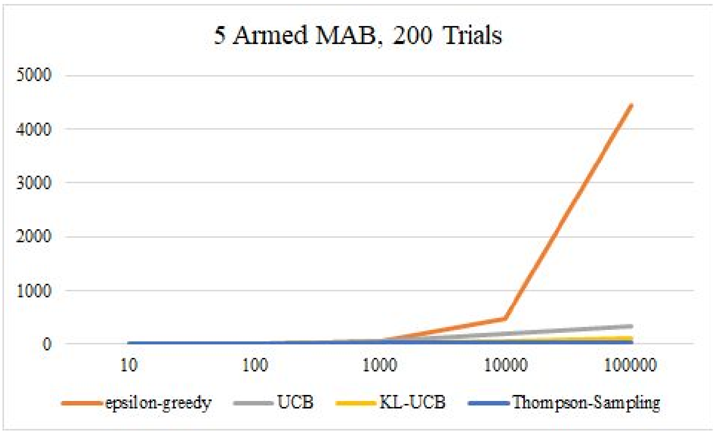
Following are the graphs of expected regret (Y axis) vs Number of Pulls (X axis) for 150 and 200 trials respectively for the 5 armed bandit.



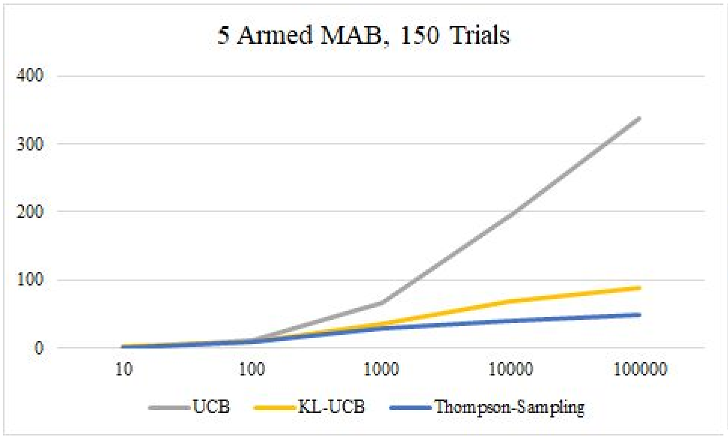


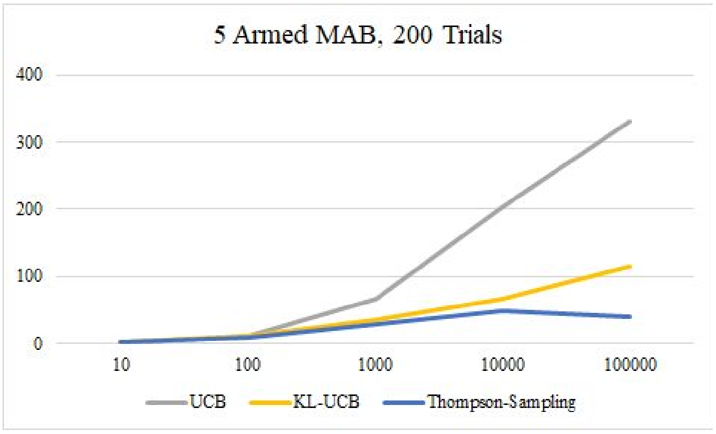
From these graphs, it is evident that all the algorithms implemented viz. Epsilon-Greedy, UCB, KL-UCB and Thompson-Sampling outperform round robin sampling approach. But, comparison between remaining algorithms is not evident from these graphs. So, by removing the line corresponding to round robin and rescaling, following graphs were obtained.





Here, it can be observed that Epsilon-Greedy has cumulative regret of 4500 (less than quarter of that with round robin) in both the 150 trials and 200 trials cases. Also, the regret is roughly exponentially increasing with experiments with increasing number of pulls. UCB algorithm has 10 times less cumulative regret than Epsilon-Greedy.

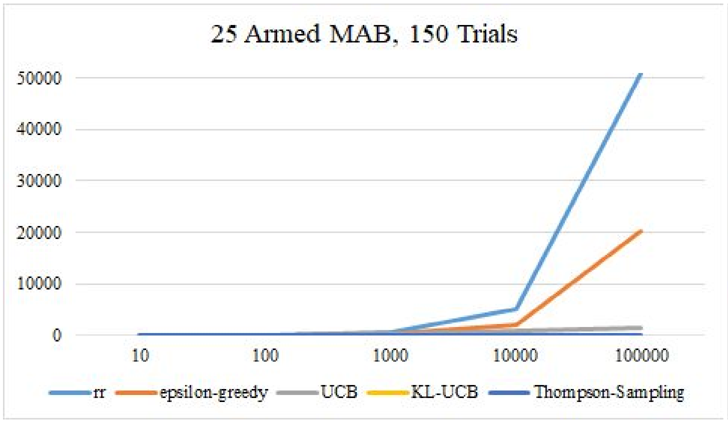


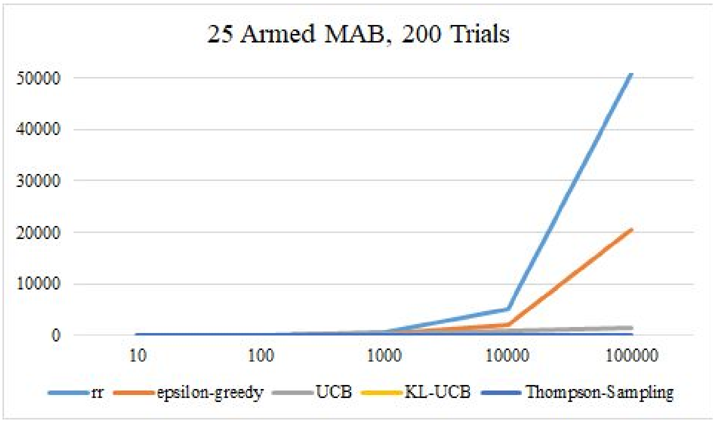


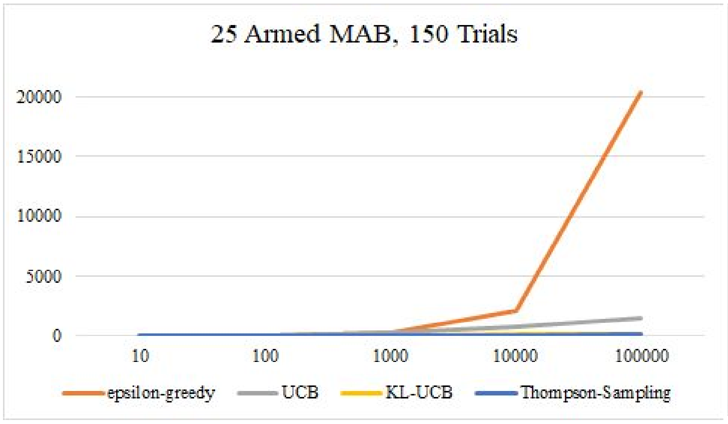
Thompson-Sampling is the best performing algorithm and KL-UCB performed close to it. It can be seen that regret goes on increasing more and more in the order Thompson-Sampling with lowest regret, then KL-UCB, UCB and finally Epsilon-Greedy. Also, the regret does not increase exponentially, graph bends more towards X axis between 10000 and 100000 pulls. These are more observable in 25 armed case.

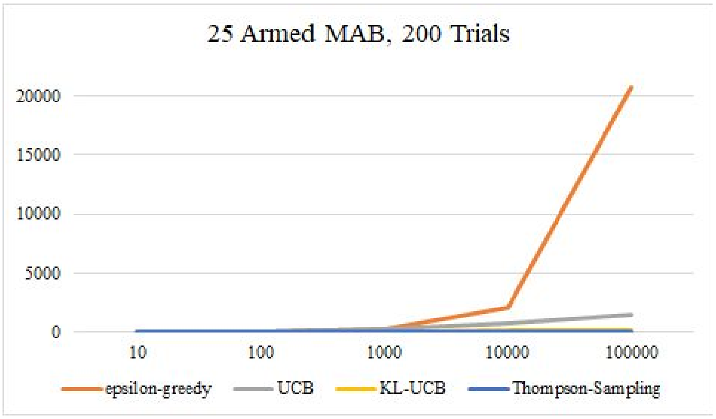
## 25 Arm Instance

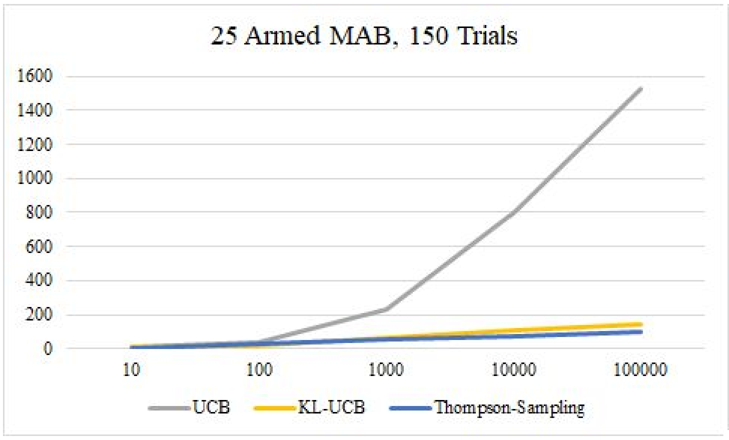
Trends similar to 5 armed bandit are observable in 25 armed counterpart. But, as the number of arms is more the regret is higher in all the algorithms.

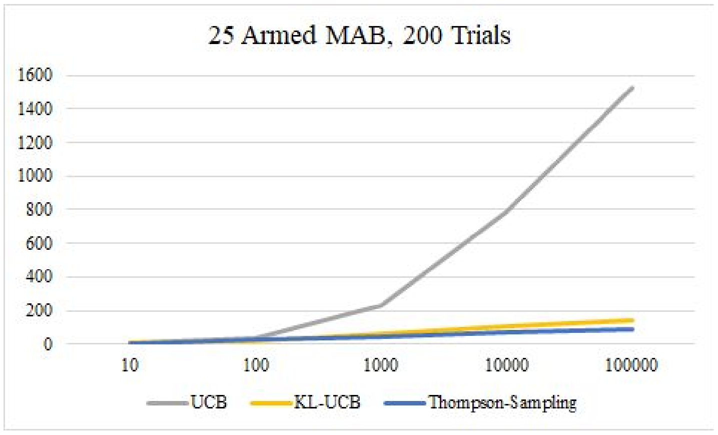


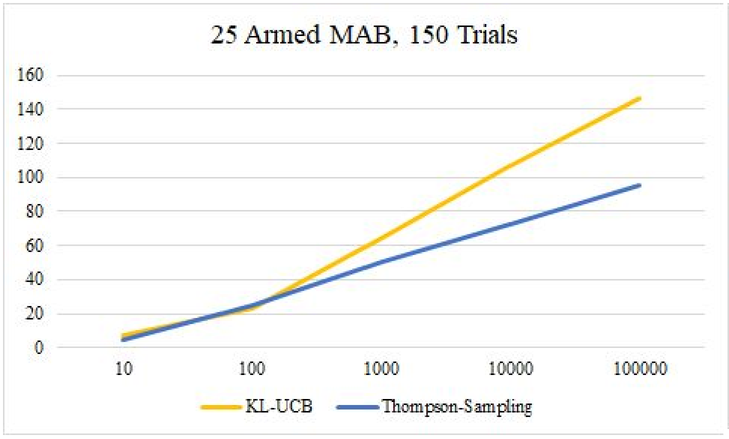


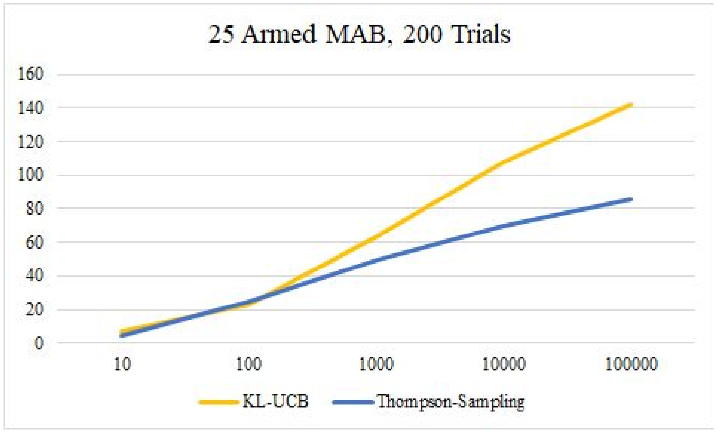












There is a slight dip in KL-UCB than Thompson-Sampling for the experiment with 100 pulls which can also be confirmed from numerical results. However, the bending toward X axis is only observable in 200 trials case.

## Results

### 5 Armed Bandit 150 trials

| Pulls | rr | epsilon-greedy | UCB | KL-UCB | Thompson-Sampling |
|---|---|---|---|---|---|
| 10 | 1.9466 | 2.9066 | 1.2733 | 1.86 | 1.5466 |
| 100 | 19.9333 | 15.32 | 11.8533 | 10.0733 | 10.0266 |
| 1000 | 200.14 | 61.2333 | 66.3333 | 34.7466 | 28.7266 |
| 10000 | 2005.2133 | 462.8533 | 194.7733 | 67.92 | 40.2466 |
| 100000 | 20017.073 | 4433.84 | 338.4733 | 88 | 49.4733 |

### 5 Armed Bandit 200 trials

| Pulls | rr | epsilon-greedy | UCB | KL-UCB | Thompson-Sampling |
|---|---|---|---|---|---|
| 10 | 2.015 | 2.435 | 1.43 | 1.965 | 1.715 |
| 100 | 19.835 | 13.875 | 11.68 | 10.665 | 10.26 |
| 1000 | 199.515 | 63.635 | 65.785 | 35.36 | 29.91 |
| 10000 | 1996.23 | 461.885 | 204.265 | 66.545 | 48.105 |
| 100000 | 20011.565 | 4429.835 | 331.055 | 115.825 | 40.03 |

### 25 Armed Bandit 150 Trials

| Pulls | rr | epsilon-greedy | UCB | KL-UCB | Thompson-Sampling |
|---|---|---|---|---|---|
| 10 | 7.4866 | 5.7 | 7.4333 | 7.5133 | 4.6 |
| 100 | 51.2 | 41.9533 | 37.3866 | 22.8 | 24.9133 |
| 1000 | 506.6866 | 246.4866 | 235.22 | 64.22 | 50.88 |
| 10000 | 5082.1666 | 2088.6733 | 794.7 | 106.5133 | 72.14 |
| 100000 | 50842.8333 | 20429.6466 | 1529.6066 | 146.7066 | 95.4933 |

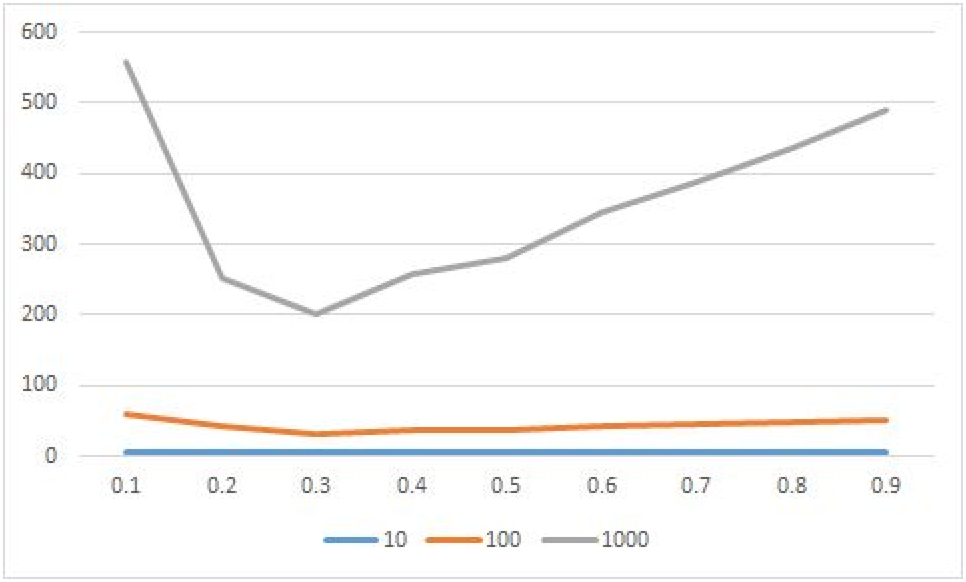
### 25 Armed Bandit 200 Trials

| Pulls | rr | epsilon-greedy | UCB | KL-UCB | Thompson-Sampling |
|---|---|---|---|---|---|
| 10 | 7.47 | 5.66 | 7.41 | 7.41 | 4.525 |
| 100 | 50.445 | 40.91 | 37.5 | 22.72 | 24.66 |
| 1000 | 508.08 | 248.15 | 234.93 | 63.33 | 49.77 |
| 10000 | 5084.735 | 2089.485 | 788.545 | 107.54 | 69.43 |
| 100000 | 50835.245 | 20690.895 | 1524.815 | 141.695 | 86.03 |

## Observations

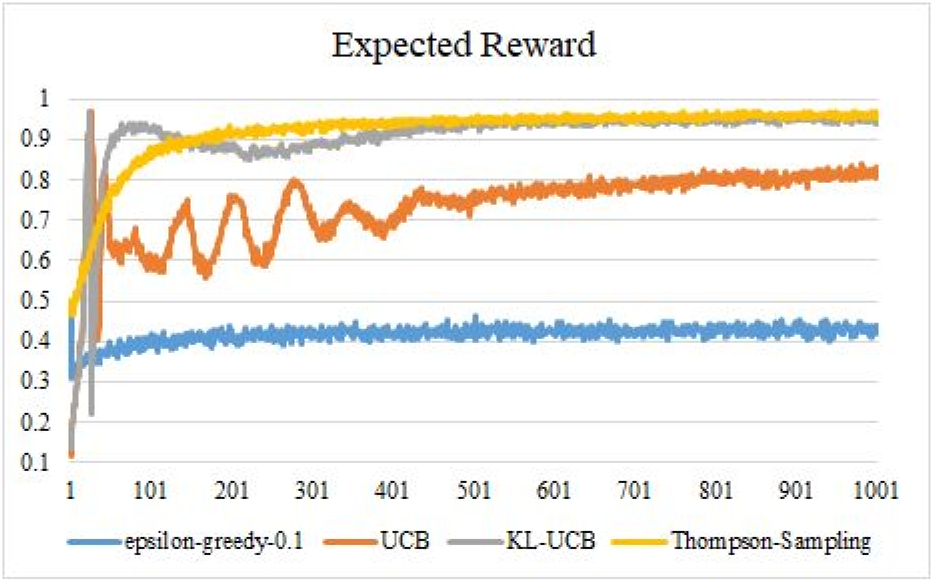
In 5 armed bandit, there is no dip of KL-UCB below Thompson-Sampling at 100 pulls, but regret values are comparable. In 25 armed counterpart, KL-UCB has slightly less regret (about 2). This is in agreement with Fig 2 of reference 1, Kaufmann et. al.

Following graph shows variation of expected regret (Y-axis) vs epsilon (X-axis) for Epsilon-Greedy Algorithm for different number of pulls over 150 instances.



The minimum regret for all the cases is around 0.3. Hence, this is the optimal value of epsilon.

Following graphs shows performance of algorithms in terms of expected reward at each pull over 2000 trials. Large number of trials are required to get smooth curves.



Here, the variations in UCB reward curve are unexpected. While the other algorithms perform exactly as expected. The dip of KL UCB near 100 pulls is due to better rewards than Thompson-Sampling during this period.

## Implementation Details

- Arm pulled in previous iteration was passed on to SampleArm function
- Index of largest element from array was found out using function getIndexOfLargestElement from reference 3

1. **Epsilon-Greedy Algorithm**
    a. A random number was generated between 0 and 99. For the first pull, arrays were initialized with all zero values for keeping track of total reward, pulls done and empirical mean reward for each of the arm and an arm was selected randomly by using modulo operator according to number of arms on the previously generated random number.
    b. For all next pulls, empirical mean was updated and if the random number generated was greater than epsilon times 100, the arm with maximum empirical mean was pulled. If this condition was not met, an arm was selected randomly as above.
2. **UCB Algorithm**
    a. For the first pull, arrays were initialized with all zero values for keeping track of total reward, pulls done, empirical mean and a separate array to update empirical mean with sqrt(2* Total Pulls/Pulls of Arm i) for each of the arm.
    b. These arrays were updated in all the subsequent pulls. If total pulls were less than number of arms then the arm was selected with index equal to total pulls, thus ensuring that each arm is pulled once.
    c. Otherwise, arm with highest UCB term was pulled.
3. **KL-UCB Algorithm**
    a. For the first pull, arrays were initialized with all zero values for keeping track of total reward, pulls done, empirical mean and qmax for each of the arm.
    b. These arrays were updated in all the subsequent pulls. If total pulls were less than number of arms then the arm was selected with index equal to total pulls, thus ensuring that each arm is pulled once.
    c. Otherwise, arm with highest qmax term was pulled.
    d. qmax was found out by Newton Raphson method according to reference 2 (explanation on next page).

$$a = \underset{1 \leq a \leq K}{\operatorname{argmax}} \ \max_q \left\{ q \in \overbrace{\left( \frac{\$ R_a}{N_a}, 1 \right]}^{\text{Ref } 4} : KL\left(\frac{R_a}{N_a}, q\right) \leq \frac{\log T}{N_a} \right\}$$

where $KL(p,q) = p \log \frac{p}{q} + (1-p) \log \frac{(1-p)}{(1-q)}$ and

$$p = \frac{R_a}{N_a} \begin{array}{l} \leftarrow \text{Rewards of arm } a \\ \leftarrow \text{Number of pulls of arm } a \end{array}$$

This can be formulated as equality

$$\left[ \frac{\log T}{N_a} - KL(p,q) \right]^2 = 0 \qquad \because KL(p,q) \text{ is monotonically increasing for } q \in [p, 1]$$

Let $c = \frac{\log T}{N_a}$ and $f = \frac{\log T}{N_a} - KL(p,q)$

$$\therefore f = c - KL(p,q)$$

$$\therefore f' = -\frac{\partial}{\partial q}\left( p \log \frac{p}{q} - (1-p) \log \frac{(1-p)}{(1-q)} \right)$$

$$= -\frac{q-p}{q(1-q)}$$

$\therefore$ By Newton Rhapson method,

$$q^{new} = q^{old} - \frac{f}{f'}$$

Let $\delta$ be a ~~very small~~ an arbitarily small number greater than 0

$$\because q \in (p, 1]$$

Initial value for $q$ can be taken as $p+\delta$

If $q^{old} - \frac{f}{f'} < p + \delta$ then $q^{new} = p+\delta$, thus

$$q^{new} = \max\left(q^{old} - \frac{f}{f'}, p+\delta\right)$$

4. **Thompson Sampling**
    a. For the first pull, arrays were initialized with all zero values for keeping track of total reward, pulls done, empirical mean and random sample from beta distribution whose parameters change for each pull for each of the arm.
    b. These arrays were updated in all the subsequent pulls. An arm with largest randomly drawn sample from beta distribution was pulled.
    c. Beta distribution was generated using randist of GSL library.

## References

1. Kaufmann, Emilie et al. "Thompson Sampling: An Asymptotically Optimal Finite-Time Analysis." ALT (2012).
2. Junpei Komiyama's Github Library
https://github.com/jkomiyama/banditlib/blob/master/policy/policy_klucb.hpp
3. Obeahon Okaiwele's blog post "C++: Get Index of Largest Element in Array"
https://obtalk.wordpress.com/2013/02/27/c-get-index-of-largest-element-in-array/
4. Olivier Cappe, Aurelien Garivier "The KL-UCB Algorithm for Bounded Stochastic Bandits and Beyond"
https://perso.limsi.fr/cappe/Research/Talks/11-gatsby.pdf