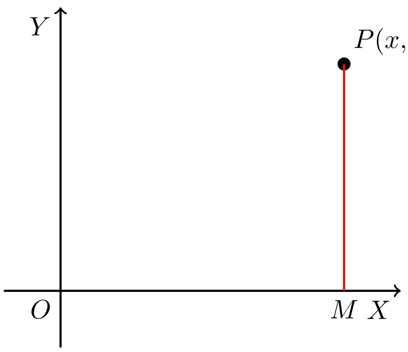
Straight Lines

Cartesian Coordinates

Let OX and OY be two perpendicular straight lines, and they are called the axes of the coordinates. Were, OX is called the x-axis and OY is called the y-axis. The point O is called the origin and the plane containing the axes is the XY-plane.

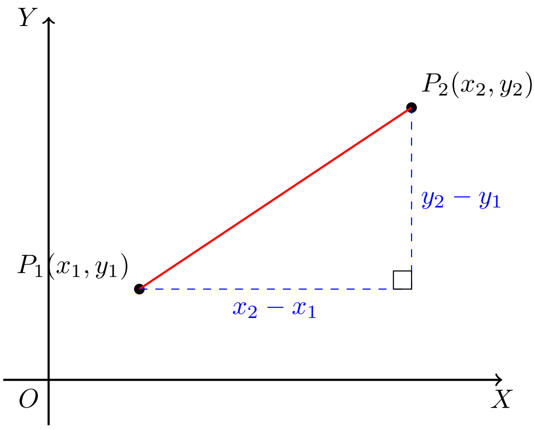
- The coordinates of a point $P(x, y)$ are determined by the point's position relative to the x-axis and y-axis.



The distance between two points

The distance between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

$$\overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example

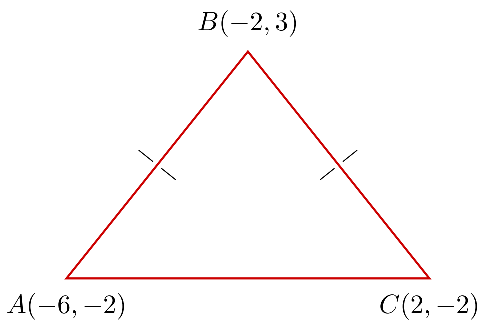
Find the length from the point $(2, 3)$ to the point $(5, 7)$.

– Soln –

$$\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
d &= \sqrt{(5 - 2)^2 + (7 - 3)^2} \\
d &= \sqrt{(3)^2 + (4)^2} \\
d &= \sqrt{9 + 16} \\
d &= \sqrt{25} = 5
\end{aligned}$$

Example

Show that the points $A(-6,-2)$, $B(-2,3)$, and $C(2,-2)$ form an isosceles triangle.



– Proof –

$$\begin{aligned} AB &= \sqrt{(-2-(-6))^2+(3-(-2))^2} \\ &= \sqrt{(4)^2+(5)^2} = \sqrt{16+25} = \sqrt{41} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(2-(-2))^2+(-2-3)^2} \\ &= \sqrt{(4)^2+(-5)^2} = \sqrt{16+25} = \sqrt{41} \end{aligned}$$

$$\begin{aligned} AC &= \sqrt{(2-(-6))^2+(-2-(-2))^2} \\ &= \sqrt{(8)^2+(0)^2} = \sqrt{64} = 8 \end{aligned}$$

Since $AB = BC = \sqrt{41}$, then $\triangle ABC$ is an isosceles triangle.

Example

Show that the points $A(5,1)$, $B(-3,7)$, and $C(8,5)$ are the vertices of a right triangle.

By finding the length of each side, we get:

- $AB = 10$
- $BC = 5\sqrt{5}$
- $CA = 5$

To show it is a right triangle, we check if the sides satisfy the Pythagorean theorem ($a^2+b^2=c^2$). The longest side is BC, so it must be the hypotenuse.

$$\begin{aligned} (AB)^2+(CA)^2 &= (10)^2+(5)^2 \\ &= 100+25 \\ &= 125 \end{aligned}$$

$$\begin{aligned} (BC)^2 &= (5\sqrt{5})^2 \\ &= 25\times 5 \\ &= 125 \end{aligned}$$

Since $(AB)^2+(CA)^2=(BC)^2$, then $\triangle ABC$ is a right triangle with the right angle at vertex A.

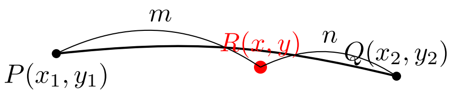
Division of a Line

The point dividing a segment in a given ratio:

Interior

$$x = \frac{mx_2 + nx_1}{m+n}$$

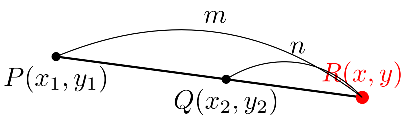
$$y = \frac{my_2 + ny_1}{m+n}$$



Exterior

$$x = \frac{mx_2 - nx_1}{m-n}$$

$$y = \frac{my_2 - ny_1}{m-n}$$

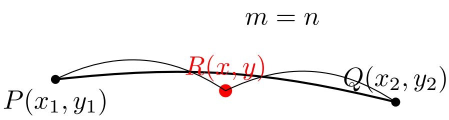


Special Case (Midpoint)

The midpoint is the case where $m = n$. The formulas simplify to:

$$x = \frac{x_1 + x_2}{2}$$

$$y = \frac{y_1 + y_2}{2}$$



Example

Find the point which divides the line joining the points $(-4, 5)$ and $(11, -4)$ in the ratio $1 : 2$ internally and externally.

– Soln –

Let the points be $P_1(x_1, y_1) = (-4, 5)$ and $P_2(x_2, y_2) = (11, -4)$. The ratio is $m : n = 1 : 2$.

Internally

For the x-coordinate:

$$\begin{aligned} x &= \frac{mx_2 + nx_1}{m+n} \\ &= \frac{(1)(11) + (2)(-4)}{1+2} \\ &= \frac{11-8}{3} = \frac{3}{3} = 1 \end{aligned}$$

For the y-coordinate:

$$\begin{aligned} y &= \frac{my_2 + ny_1}{m+n} \\ &= \frac{(1)(-4) + (2)(5)}{1+2} \\ &= \frac{-4+10}{3} = \frac{6}{3} = 2 \end{aligned}$$

The point of internal division is $\mathbf{P(x, y) = (1, 2)}$.

Externally

For the x-coordinate:

$$\begin{aligned} x &= \frac{mx_2 - nx_1}{m-n} \\ &= \frac{(1)(11)-(2)(-4)}{1-2} \\ &= \frac{11+8}{-1} = -19 \end{aligned}$$

For the y-coordinate:

$$\begin{aligned} y &= \frac{my_2 - ny_1}{m-n} \\ &= \frac{(1)(-4)-(2)(5)}{1-2} \\ &= \frac{-4-10}{-1} = 14 \end{aligned}$$

The point of external division is $\mathbf{P(x,y)} = (\mathbf{-19}, \mathbf{14})$.

Example

In what ratio does the point $(-1,-1)$ divide the segment joining the points $(-5,-3)$ and $(5,2)$? Determine if the division is internal or external.

– Soln –

Let the dividing point be $P(x,y) = (-1,-1)$. Let the segment points be $P_1(x_1,y_1) = (-5,-3)$ and $P_2(x_2,y_2) = (5,2)$. Let the ratio be $m_1 : m_2$.

We can assume the division is internal and use the section formula. If the resulting ratio is positive, our assumption is correct. If it's negative, the division is external.

Using the formula for the x-coordinate:

$$\begin{aligned} x &= \frac{m_1x_2 + m_2x_1}{m_1+m_2} \\ -1 &= \frac{m_1(5) + m_2(-5)}{m_1+m_2} \\ -1(m_1+m_2) &= 5m_1 - 5m_2 \\ -m_1 - m_2 &= 5m_1 - 5m_2 \\ 5m_2 - m_2 &= 5m_1 + m_1 \\ 4m_2 &= 6m_1 \\ \frac{m_1}{m_2} &= \frac{4}{6} = \frac{2}{3} \end{aligned}$$

So, the ratio $m_1 : m_2$ is $2:3$.

Since the ratio is positive, the point divides the segment **internally**.

Verification using the y-coordinate: We can check our result with the y-values and the ratio $2:3$.

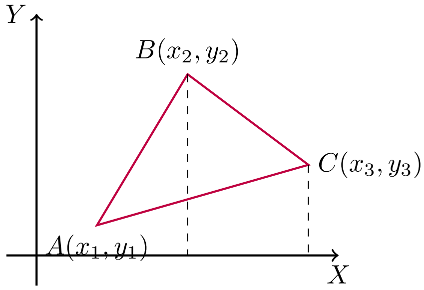
$$\begin{aligned} y &= \frac{m_1y_2 + m_2y_1}{m_1+m_2} \\ &= \frac{2(2)+3(-3)}{2+3} \\ &= \frac{4-9}{5} = \frac{-5}{5} = -1 \end{aligned}$$

This matches the y-coordinate of the given point, so our calculation is correct.

Triangle Area

The area of the triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is given by the determinant formula. Since area must be a positive quantity, we take the absolute value of the result.

$$\text{Area} = \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$



Example

Find the area of the triangle whose vertices are $(-2, 3)$, $(4, 3)$, and $(1, 1)$.

– Soln –

We set up the determinant with the given coordinates:

$$\begin{aligned}
\text{Area} &= \frac{1}{2} \left| \begin{vmatrix} -2 & 3 & 1 \\ 4 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} \right| \\
&= \frac{1}{2} \left| (-2) \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} - (3) \begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix} + (1) \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} \right| \\
&= \frac{1}{2} \left| -2(3 \cdot 1 - 1 \cdot 1) - 3(4 \cdot 1 - 1 \cdot 1) + 1(4 \cdot 1 - 1 \cdot 3) \right| \\
&= \frac{1}{2} \left| -2(2) - 3(3) + 1(1) \right| \\
&= \frac{1}{2} \left| -4 - 9 + 1 \right| \\
&= \frac{1}{2} \left| -12 \right| \\
&= 6 \text{ square units}
\end{aligned}$$

Collinear Points

The points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are **collinear** (meaning they lie on the same straight line) if the area of the triangle formed by them is zero.

This gives the condition:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example

Show that the points $(-4, 1)$, $(2, 3)$, and $(-1, 2)$ are collinear.

– Soln –

We evaluate the determinant, which we'll call Δ. If $\Delta = 0$, the points are collinear.

$$\begin{aligned}
\Delta &= \begin{vmatrix} -4 & 1 & 1 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{vmatrix} \\
&= (-4)\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} - (1)\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} + (1)\begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \\
&= -4(3 \cdot 1 - 2 \cdot 1) - 1(2 \cdot 1 - (-1) \cdot 1) + 1(2 \cdot 2 - (-1) \cdot 3) \\
&= -4(3 - 2) - 1(2 + 1) + 1(4 + 3) \\
&= -4(1) - 1(3) + 1(7) \\
&= -4 - 3 + 7 \\
&= 0
\end{aligned}$$

Since the determinant is **zero**, the points are collinear.

Example

Show that the three points $(a, b+c)$, $(b, a+c)$, and $(c, a+b)$ are collinear.

– Soln –

We set up the determinant Δ. The points are collinear if $\Delta = 0$.

$$\Delta = \begin{vmatrix} a & b+c & 1 \\ b & a+c & 1 \\ c & a+b & 1 \end{vmatrix}$$

Apply the column operation $C_1 \to C_1 + C_2$:

$$\Delta = \begin{vmatrix} a+(b+c) & b+c & 1 \\ b+(a+c) & a+c & 1 \\ c+(a+b) & a+b & 1 \end{vmatrix} = \begin{vmatrix} a+b+c & b+c & 1 \\ a+b+c & a+c & 1 \\ a+b+c & a+b & 1 \end{vmatrix}$$

Factor out the common term $(a+b+c)$ from the first column:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b+c & 1 \\ 1 & a+c & 1 \\ 1 & a+b & 1 \end{vmatrix}$$

The determinant is zero because the first and third columns are identical.

$$\Delta = (a+b+c) \cdot 0 = 0$$

Since $\Delta = 0$, the points are **collinear**.

Example

Find the value of λ which makes the points $(0, \lambda)$, $(-2, -1)$, and $(-3, -2)$ collinear.

– Soln –

Set the determinant of the coordinates to zero:

$$\begin{aligned}
\begin{vmatrix} 0 & \lambda & 1 \\ -2 & -1 & 1 \\ -3 & -2 & 1 \end{vmatrix} &= 0 \\
0\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} - \lambda\begin{vmatrix} -2 & 1 \\ -3 & 1 \end{vmatrix} + 1\begin{vmatrix} -2 & -1 \\ -3 & -2 \end{vmatrix} &= 0 \\
0 - \lambda((-2)(1) - (-3)(1)) + 1((-2)(-2) - (-3)(-1)) &= 0 \\
-\lambda(-2+3) + 1(4-3) &= 0 \\
-\lambda(1) + 1(1) &= 0 \\
-\lambda + 1 &= 0 \\
\lambda &= 1
\end{aligned}$$

The value is $\lambda = \mathbf{1}$.

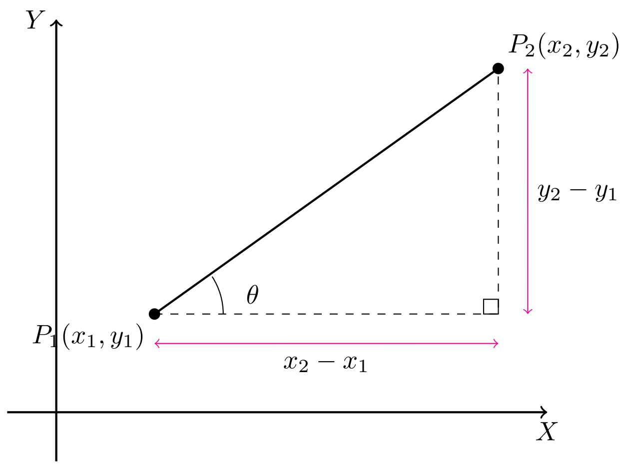
Straight Line

The Slope of a Line

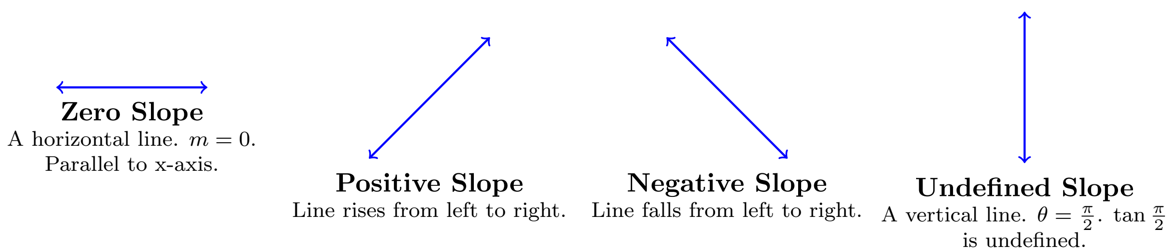
The slope, denoted by m, of a non-vertical line passing through the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by:

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan\theta$$

where θ is the angle of inclination that the line makes with the positive x-axis.



Notes on Slope



Example

Find the slopes of the lines containing the given pairs of points:

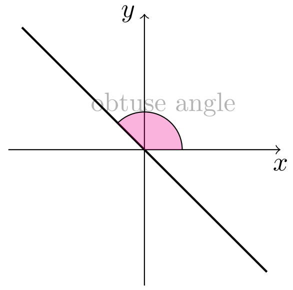
1. $(2,3)$ and $(-1,4)$
2. $(1,-3)$ and $(4,0)$

– Soln –

1. Points $(2,3)$ and $(-1,4)$

Let $(x_1, y_1) = (2,3)$ and $(x_2, y_2) = (-1,4)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-3}{-1-2} \\ &= \frac{1}{-3} = -\frac{1}{3} \end{aligned}$$

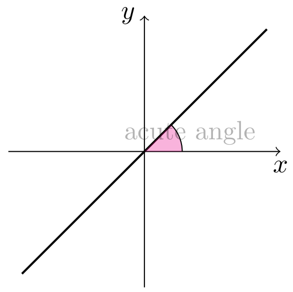


The slope is negative, so the line forms an obtuse angle with the positive x-axis.

2. Points $(1,-3)$ and $(4,0)$

Let $(x_1, y_1) = (1,-3)$ and $(x_2, y_2) = (4,0)$.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0-(-3)}{4-1} \\ &= \frac{3}{3} = 1 \end{aligned}$$



The slope is positive, so the line forms an acute angle with the positive x-axis.

The Equation of a Straight Line

The **General Form** of the equation of a straight line is given by:

$$ax + by + c = 0$$

Where a, b, and c are constants.

We can determine the equation of a straight line if we have one of the following sets of data:

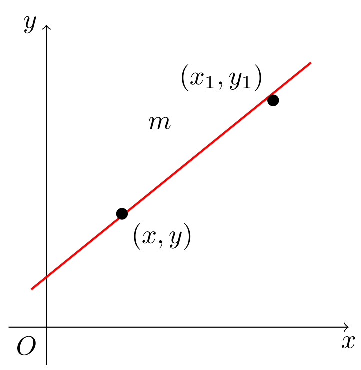
1. A Point and the Slope

The equation of a line passing through a given point (x_1, y_1) with a known slope m is derived from the slope formula. For any other point (x, y) on the line, we have:

$$m = \frac{y - y_1}{x - x_1}$$

This is more commonly written in the **point-slope form**:

$$y - y_1 = m(x - x_1)$$

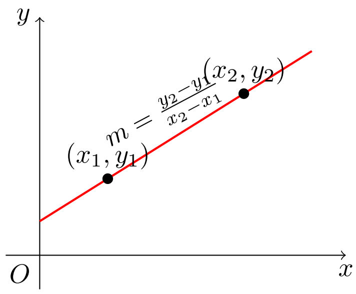


2. Two Points

The equation of a straight line passing through two given points, (x_1, y_1) and (x_2, y_2), is given by the **two-point form**:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

This equates the slope between any point (x, y) and the first point to the slope between the two given points.

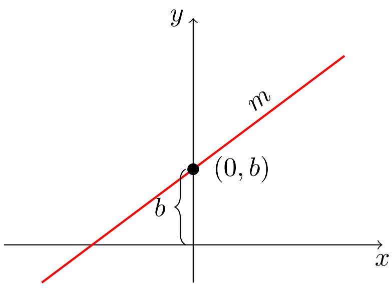


3. Slope and y-intercept

The equation of a line with a known slope m and y-intercept b is given by the **slope-intercept form**:

$$y = mx + b$$

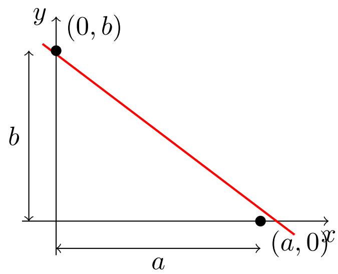
If a line passes through the origin, its y-intercept is $b = 0$, so its equation simplifies to $y = mx$.



4. x-intercept and y-intercept (Intercept Form)

The equation of a line can be expressed in terms of its x-intercept, a, and y-intercept, b. This is known as the **intercept form**:

$$\frac{x}{a} + \frac{y}{b} = 1$$



Example

Find the equation of the straight line passing through the points $(4, -5)$ and $(-3, 7)$.

– Soln –

We start with the two-point form and rearrange it into the general form $ax + by + c = 0$. Let

$(x_1, y_1) = (4, -5)$ and $(x_2, y_2) = (-3, 7)$.

$$\begin{aligned} \frac{y - y_1}{x - x_1} &= \frac{y_2 - y_1}{x_2 - x_1} \\ \frac{y - (-5)}{x - 4} &= \frac{7 - (-5)}{-3 - 4} \\ \frac{y + 5}{x - 4} &= \frac{12}{-7} \\ 7(y + 5) &= -12(x - 4) \\ 7y + 35 &= -12x + 48 \\ 12x + 7y + 35 - 48 &= 0 \\ 12x + 7y - 13 &= 0 \end{aligned}$$

The equation of the line is $\mathbf{12x + 7y - 13 = 0}$.

Example

Find the equation of the line that passes through the point $(1, 4)$ and makes an angle of $45°$ with the positive x-axis.

– Soln –

First, find the slope m from the given angle $\theta = 45°$.

$$m = \tan\theta = \tan 45° = 1$$

Now, use the point-slope form with $m = 1$ and the point $(x_1, y_1) = (1, 4)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= 1(x - 1) \\ y - 4 &= x - 1 \\ 0 &= x - y - 1 + 4 \\ x - y + 3 &= 0 \end{aligned}$$

The equation of the line is $\mathbf{x - y + 3 = 0}$.

Example

Find the equation of the line with slope 5 and y-intercept 7.

– Soln –

Using the slope-intercept form, $y = mx + b$, with $m = 5$ and $b = 7$:

$$y = 5x + 7$$

To write this in the general form $ax + by + c = 0$, we rearrange the terms:

$$5x - y + 7 = 0$$

The equation of the line is $\mathbf{5x - y + 7 = 0}$.

Example

Find the equation of the line passing through the point $(-3, 2)$ that has a slope of 2.

– Soln –

We are given the slope $m = 2$. Using the slope-intercept form $y = mx + c$:

$$y = 2x + c$$

Now, substitute the point $(x, y) = (-3, 2)$ into the equation to find the y-intercept, c.

$$\begin{aligned} 2 &= 2(-3) + c \\ 2 &= -6 + c \\ c &= 8 \end{aligned}$$

The equation of the line is $\mathbf{y = 2x + 8}$, or in the general form, $\mathbf{2x - y + 8 = 0}$.

Example

Find the equation of the line whose x-intercept is -5 and y-intercept is 3.

– Soln –

Using the intercept form, $\frac{x}{a} + \frac{y}{b} = 1$, with $a = -5$ and $b = 3$:

$$\frac{x}{-5} + \frac{y}{3} = 1$$

To clear the denominators, we can multiply the entire equation by the least common multiple, which is 15:

$$\begin{aligned} 15 \cdot \left(\frac{x}{-5}\right) + 15 \cdot \left(\frac{y}{3}\right) &= 15 \cdot 1 \\ -3x + 5y &= 15 \end{aligned}$$

Rearranging to the general form $ax + by + c = 0$:

$$-3x + 5y - 15 = 0$$

Or, multiplying by -1 to make the x-term positive:

$$3x - 5y + 15 = 0$$

The equation of the line is $\mathbf{3x - 5y + 15 = 0}$.

Properties of Lines

1. **Parallel Lines:** If two lines are parallel, the angle between them is $0°$. Their slopes are equal.

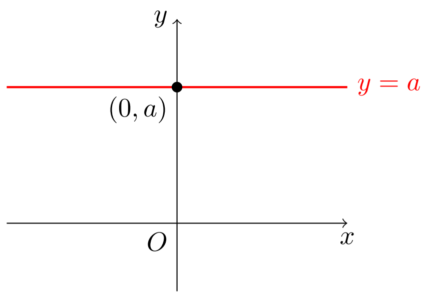
$$m_1 - m_2 = 0 \implies \boxed{m_1 = m_2}$$

2. **Perpendicular Lines:** If two lines are perpendicular, the angle between them is $90°$ or $\frac{\pi}{2}$. The product of their slopes is -1.

$$1 + m_1 m_2 = 0 \implies \boxed{m_1 = -\frac{1}{m_2}}$$

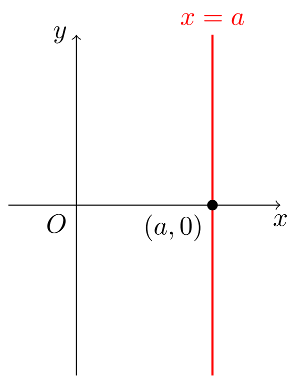
3. Horizontal Line

- A line parallel to the x-axis has a slope of zero.
- Its equation is given by $\mathbf{y} = \mathbf{a}$, where a is the y-intercept.
- The equation of the **x-axis** itself is $\mathbf{y} = \mathbf{0}$.



4. Vertical Line

- A line parallel to the y-axis has an undefined slope.
- Its equation is given by $\mathbf{x} = \mathbf{a}$, where a is the x-intercept.
- The equation of the **y-axis** itself is $\mathbf{x} = \mathbf{0}$.



Note: Slope from the General Form

The slope of a line given in the general form $Ax + By + C = 0$ can be found directly with the formula:

$$\text{Slope} = m = -\frac{A}{B}$$

Example

Find the equation of the lines that are:

1. Through the point $(1, 2)$ and **parallel** to the line $y = 3x + 4$.
2. Through the point $(5, -3)$ and **perpendicular** to the line $2x - 3y = 5$.

– Soln –

1. Parallel Line Example

The given line is $y = 3x + 4$. We can see from the slope-intercept form that its slope, let's call it m_1, is 3.

$$m_1 = 3$$

Since the new line is **parallel**, its slope, m_2, must be the same.

$$\boxed{m_2 = m_1 = 3}$$

Now we find the equation of the line with slope $m = 3$ passing through the point $(x_1, y_1) = (1, 2)$ using the point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 3(x - 1) \\ y - 2 &= 3x - 3 \\ 0 &= 3x - y - 1 \end{aligned}$$

The equation of the line is $\mathbf{3x - y - 1 = 0}$.

2. Perpendicular Line Example

The given line is $2x - 3y = 5$, or $2x - 3y - 5 = 0$. Its slope, m_1, is:

$$m_1 = -\frac{A}{B} = -\frac{2}{-3} = \frac{2}{3}$$

Since the new line is **perpendicular**, its slope, m_2, is the negative reciprocal of m_1.

$$\boxed{m_2 = -\frac{1}{m_1} = -\frac{3}{2}}$$

Now we find the equation of the line with slope $m = -3/2$ passing through the point $(x_1, y_1) = (5, -3)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-3) &= -\frac{3}{2}(x - 5) \\ y + 3 &= -\frac{3}{2}(x - 5) \\ 2(y + 3) &= -3(x - 5) \\ 2y + 6 &= -3x + 15 \\ 3x + 2y + 6 - 15 &= 0 \\ 3x + 2y - 9 &= 0 \end{aligned}$$

The equation of the line is $\mathbf{3x + 2y - 9 = 0}$.

Example

Find the equation of the line passing through the point $(3, -2)$ that has equal intercepts.

– Soln –

Let the x-intercept be a and the y-intercept be b. Since the intercepts are **equal**, we can say that $a = b = k$ for some constant k.

We start with the intercept form of the line:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Substitute $a = k$ and $b = k$:

$$\begin{aligned} \frac{x}{k} + \frac{y}{k} &= 1 \\ x + y &= k \end{aligned}$$

Since the line passes through the point $(3, -2)$, we can substitute $x = 3$ and $y = -2$ into the equation to find the value of k.

$$\begin{aligned} 3 + (-2) &= k \\ 1 &= k \end{aligned}$$

Now, substitute $k = 1$ back into the simplified equation $x + y = k$.

$$x + y = 1$$

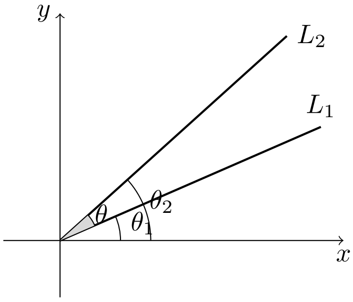
The equation of the line in the general form is $\mathbf{x + y - 1 = 0}$.

Angle Between Two Lines

Let the equations of two lines, L_1 and L_2, be:

$$\begin{aligned} y &= m_1 x + c_1 \\ y &= m_2 x + c_2 \end{aligned}$$

Let θ_1 and θ_2 be the angles of inclination for L_1 and L_2 respectively, so that $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.
If θ is the angle between the lines, then from the geometry of the lines we have $\theta = \theta_2 - \theta_1$.



Taking the tangent of both sides:

$$\begin{aligned} \tan\theta &= \tan(\theta_2 - \theta_1) \\ \tan\theta &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} \end{aligned}$$

Substituting the slopes gives the final formula. The absolute value is often used to find the acute angle.

$$\tan\theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$$

Example Find the angle between the lines $L_1 : x - 7y + 1 = 0$ and $L_2 : 2x + 3y - 5 = 0$.

– Soln – First, find the slope of each line using $m = -A/B$.

- For $L_1 : x - 7y + 1 = 0$, the slope is $m_1 = -\frac{1}{-7} = \frac{1}{7}$.
- For $L_2 : 2x + 3y - 5 = 0$, the slope is $m_2 = -\frac{2}{3}$.

Now, substitute these slopes into the formula for the angle:

$$\begin{aligned} \tan\theta &= \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \\ &= \left| \frac{-\frac{2}{3} - \frac{1}{7}}{1 + (-\frac{2}{3})(\frac{1}{7})} \right| \\ &= \left| \frac{\frac{-14-3}{21}}{1 - \frac{2}{21}} \right| \\ &= \left| \frac{\frac{-17}{21}}{\frac{19}{21}} \right| = \left| -\frac{17}{19} \right| = \frac{17}{19} \end{aligned}$$

Therefore, the acute angle is $\theta = \arctan\left(\mathbf{\frac{17}{19}}\right)$.

Line Through the Intersection of Two Lines

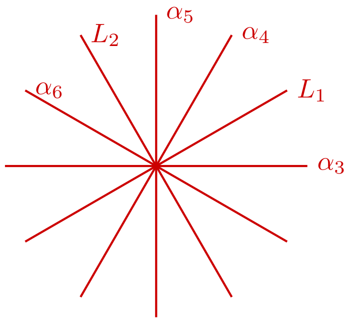
Suppose two lines, L_1 and L_2, are given in the general form:

$$L_1 : a_1x + b_1y + c_1 = 0$$
$$L_2 : a_2x + b_2y + c_2 = 0$$

Then, the equation of the **family of lines** passing through the intersection point of L_1 and L_2 is given by the form:

$$(a_1x + b_1y + c_1) + \alpha(a_2x + b_2y + c_2) = 0$$

where α is a parameter. Each value of α defines a unique line passing through the common intersection point.



Example

Find the equation of the straight line L that passes through the intersection point of the two lines $x - 7y + 29 = 0$ and $4x + 5y - 16 = 0$, and also passes through the point $(2, 3)$.

– Soln –

The equation of the line L passing through the intersection of $L_1 : x - 7y + 29 = 0$ and $L_2 : 4x + 5y - 16 = 0$ is given by:

$$(x - 7y + 29) + \alpha(4x + 5y - 16) = 0$$

Since the line also passes through the point $(2, 3)$, we can substitute $x = 2$ and $y = 3$ into the equation to solve for the parameter α.

$$\begin{aligned} (2 - 7(3) + 29) + \alpha(4(2) + 5(3) - 16) &= 0 \\ (2 - 21 + 29) + \alpha(8 + 15 - 16) &= 0 \\ (10) + \alpha(7) &= 0 \\ 7\alpha &= -10 \\ \alpha &= -\frac{10}{7} \end{aligned}$$

Now, substitute this value of α back into the family of lines equation:

$$(x - 7y + 29) - \frac{10}{7}(4x + 5y - 16) = 0$$

Multiply the entire equation by 7 to eliminate the fraction:

$$\begin{aligned} 7(x - 7y + 29) - 10(4x + 5y - 16) &= 0 \\ (7x - 49y + 203) - (40x + 50y - 160) &= 0 \\ 7x - 49y + 203 - 40x - 50y + 160 &= 0 \\ -33x - 99y + 363 &= 0 \end{aligned}$$

Finally, divide the entire equation by -33 for the simplest form:

$$x + 3y - 11 = 0$$

The equation of the line is $\mathbf{x + 3y - 11 = 0}$.

Example

Find the equation of the line that passes through the intersection of the lines $2x - y + 4 = 0$ and $x + y - 3 = 0$ and has a slope of 3.

– Soln –

Let the two lines be $L_1 : 2x - y + 4 = 0$ and $L_2 : x + y - 3 = 0$. The equation for the family of lines passing through their intersection is:

$$(2x - y + 4) + \lambda(x + y - 3) = 0$$

To find the slope, we first rearrange this equation into the general form $Ax + By + C = 0$ by grouping the terms:

$$\begin{aligned}(2x + \lambda x) + (-y + \lambda y) + (4 - 3\lambda) &= 0\\ (2 + \lambda)x + (-1 + \lambda)y + (4 - 3\lambda) &= 0\end{aligned}$$

The slope m of a line in this form is $m = -A/B$. We are given that the slope is 3.

$$\begin{aligned}m = -\frac{A}{B} &= 3\\ -\frac{2+\lambda}{-1+\lambda} &= 3\\ -(2+\lambda) &= 3(-1+\lambda)\\ -2-\lambda &= -3+3\lambda\\ 1 &= 4\lambda\\ \lambda &= \frac{1}{4}\end{aligned}$$

Now, substitute $\lambda = 1/4$ back into the grouped general form:

$$\begin{aligned}\left(2 + \frac{1}{4}\right)x + \left(-1 + \frac{1}{4}\right)y + \left(4 - \frac{3}{4}\right) &= 0\\ \left(\frac{9}{4}\right)x + \left(-\frac{3}{4}\right)y + \left(\frac{13}{4}\right) &= 0\end{aligned}$$

Multiply the entire equation by 4 to clear the denominators:

$$9x - 3y + 13 = 0$$

The equation of the line is $\mathbf{9x - 3y + 13 = 0}$.

Example

Find the equation of the line that passes through the intersection of the lines $2x - y + 4 = 0$ and $x + y - 3 = 0$ and has a slope of 3.

– Soln –

Let the two lines be $L_1 : 2x - y + 4 = 0$ and $L_2 : x + y - 3 = 0$. The equation for the family of lines passing through their intersection is:

$$(2x - y + 4) + \lambda(x + y - 3) = 0$$

To find the slope, we first rearrange this equation into the general form $Ax + By + C = 0$ by grouping the terms:

$$\begin{aligned}(2x + \lambda x) + (-y + \lambda y) + (4 - 3\lambda) &= 0\\ (2 + \lambda)x + (-1 + \lambda)y + (4 - 3\lambda) &= 0\end{aligned}$$

The slope m of a line in this form is $m = -A/B$. We are given that the slope is 3.

$$\begin{aligned} m = -\frac{A}{B} &= 3 \\ -\frac{2+\lambda}{-1+\lambda} &= 3 \\ -(2+\lambda) &= 3(-1+\lambda) \\ -2-\lambda &= -3+3\lambda \\ 1 &= 4\lambda \\ \lambda &= \frac{1}{4} \end{aligned}$$

Now, substitute $\lambda = 1/4$ back into the grouped general form:

$$\begin{aligned} \left(2+\frac{1}{4}\right)x + \left(-1+\frac{1}{4}\right)y + \left(4-\frac{3}{4}\right) &= 0 \\ \left(\frac{9}{4}\right)x + \left(-\frac{3}{4}\right)y + \left(\frac{13}{4}\right) &= 0 \end{aligned}$$

Multiply the entire equation by 4 to clear the denominators:

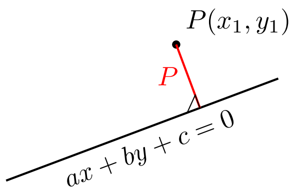
$$9x - 3y + 13 = 0$$

The equation of the line is $\mathbf{9x - 3y + 13 = 0}$.

Normal Distance (from a Point to a Line)

The normal (or shortest) distance from a point $P(x_1, y_1)$ to a line with the equation $ax + by + c = 0$ is given by the formula:

$$P = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



Example

Find the normal distance from the point $(3, -2)$ to the line $4x - 3y - 4 = 0$.

– Soln – Here, $(x_1, y_1) = (3, -2)$ and the line gives us $a = 4$, $b = -3$, and $c = -4$.

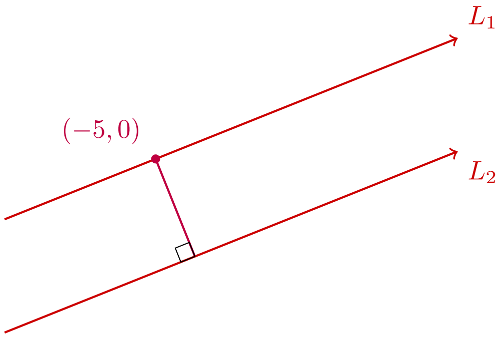
$$\begin{aligned} P &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|4(3) + (-3)(-2) + (-4)|}{\sqrt{(4)^2 + (-3)^2}} \\ &= \frac{|12 + 6 - 4|}{\sqrt{16 + 9}} \\ &= \frac{|14|}{\sqrt{25}} = \frac{14}{5} \end{aligned}$$

Example Find the shortest distance between the two parallel lines:

$$\begin{aligned} L_1 &: x + 3y + 5 = 0 \\ L_2 &: x + 3y - 1 = 0 \end{aligned}$$

– Soln –

To find the shortest distance between two parallel lines, we can pick any point on one line and calculate its normal distance to the other line.
The first line is $L_1 : x + 3y + 5 = 0$. Let's find a point on it. If we set $y = 0$, then $x + 5 = 0$, which gives $x = -5$. So, the point $(-5, 0)$ is on L_1.
Now, we find the distance from $(x_1, y_1) = (-5, 0)$ to the second line, $L_2 : x + 3y - 1 = 0$.



$$\begin{aligned} d &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\ &= \frac{|1(-5) + 3(0) - 1|}{\sqrt{(1)^2 + (3)^2}} \\ &= \frac{|-5-1|}{\sqrt{1+9}} = \frac{|-6|}{\sqrt{10}} = \frac{6}{\sqrt{10}} \end{aligned}$$

Rationalizing the denominator, the distance is $\boldsymbol{\frac{3\sqrt{10}}{5}}$.

Example

Find the equation of the line passing through the intersection point of the lines $x - 2y - 1 = 0$ and $2x + y + 3 = 0$, and has a distance of 2 from the point $(1, -3)$.

– Soln –

Let the two lines be $L_1 : x - 2y - 1 = 0$ and $L_2 : 2x + y + 3 = 0$. The family of lines passing through their intersection is:

$$(x - 2y - 1) + \lambda(2x + y + 3) = 0$$

First, we group the terms to get the general form $Ax + By + C = 0$:

$$\begin{aligned} (x + 2\lambda x) + (-2y + \lambda y) + (-1 + 3\lambda) &= 0 \\ (1 + 2\lambda)x + (-2 + \lambda)y + (-1 + 3\lambda) &= 0 \end{aligned}$$

The distance from the point $(x_1, y_1) = (1, -3)$ to this line must be 2. Using the distance formula $d = \frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$:

$$2 = \frac{|(1 + 2\lambda)(1) + (-2 + \lambda)(-3) + (-1 + 3\lambda)|}{\sqrt{(1 + 2\lambda)^2 + (-2 + \lambda)^2}}$$

Let's simplify the numerator and the denominator separately. **Numerator**:

$$|1 + 2\lambda + 6 - 3\lambda - 1 + 3\lambda| = |2\lambda + 6|$$

Denominator:

$$\sqrt{(1 + 4\lambda + 4\lambda^2) + (4 - 4\lambda + \lambda^2)} = \sqrt{5\lambda^2 + 5}$$

Now, substitute these back into the equation and solve for λ:

$$
\begin{aligned}
2 &= \frac{|2\lambda + 6|}{\sqrt{5\lambda^2 + 5}} \\
4 &= \frac{(2\lambda + 6)^2}{5\lambda^2 + 5} \\
4(5\lambda^2 + 5) &= (2\lambda + 6)^2 \\
20\lambda^2 + 20 &= 4\lambda^2 + 24\lambda + 36 \\
16\lambda^2 - 24\lambda - 16 &= 0 \\
2\lambda^2 - 3\lambda - 2 &= 0
\end{aligned}
$$

Factoring the quadratic equation gives $(2\lambda + 1)(\lambda - 2) = 0$. The two possible values for the parameter are $\lambda = 2$ and $\lambda = -1/2$. This means there are two lines that satisfy the condition.

Case 1: $\lambda = 2$ Substitute $\lambda = 2$ into the grouped general form:

$$
\begin{aligned}
(1 + 2(2))x + (-2 + 2)y + (-1 + 3(2)) &= 0 \\
5x + 0y + 5 &= 0 \\
5x = -5 &\implies \mathbf{x + 1 = 0}
\end{aligned}
$$

Case 2: $\lambda = -1/2$ Substitute $\lambda = -1/2$ into the grouped general form:

$$
\begin{aligned}
\left(1 + 2\left(-\frac{1}{2}\right)\right)x + \left(-2 + \left(-\frac{1}{2}\right)\right)y + \left(-1 + 3\left(-\frac{1}{2}\right)\right) &= 0 \\
(1 - 1)x + \left(-\frac{5}{2}\right)y + \left(-\frac{5}{2}\right) &= 0 \\
0x - \frac{5}{2}y - \frac{5}{2} &= 0 \\
-\frac{5}{2}y = \frac{5}{2} &\implies \mathbf{y + 1 = 0}
\end{aligned}
$$

The two possible equations are $\mathbf{x + 1 = 0}$ and $\mathbf{y + 1 = 0}$.

Example

If the normal distance from the point $(-1, -3)$ to the line $2x + 3y + C = 0$ is $\frac{17}{\sqrt{13}}$, evaluate the constant C.

– Soln –

We are given the distance $d = \frac{17}{\sqrt{13}}$, the point $(x_1, y_1) = (-1, -3)$, and the line parameters $a = 2$ and $b = 3$. We set up the distance formula and solve for C.

$$
\begin{aligned}
d &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}} \\
\frac{17}{\sqrt{13}} &= \frac{|2(-1) + 3(-3) + C|}{\sqrt{(2)^2 + (3)^2}} \\
\frac{17}{\sqrt{13}} &= \frac{|-2 - 9 + C|}{\sqrt{4 + 9}} \\
\frac{17}{\sqrt{13}} &= \frac{|C - 11|}{\sqrt{13}}
\end{aligned}
$$

Multiplying both sides by $\sqrt{13}$ gives:

$$|C - 11| = 17$$

This absolute value equation yields two possible solutions for C:

- **Case 1:** $C - 11 = 17 \implies C = 17 + 11 = 28$
- **Case 2:** $-(C - 11) = 17 \implies -C + 11 = 17 \implies -C = 6 \implies C = -6$

The two possible values for the constant are $\mathbf{C = 28}$ and $\mathbf{C = -6}$.

Pair of Lines

I. Homogeneous Equation

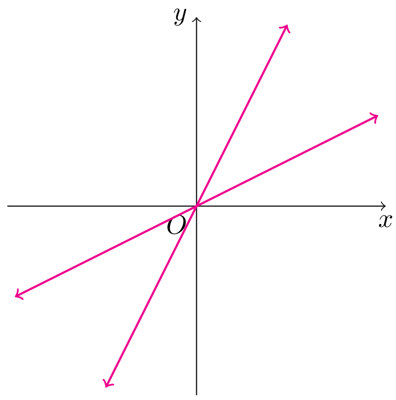
Consider two straight lines passing through the origin, given by the equations:

$$y - 2x = 0 \tag{1}$$
$$2y - x = 0 \tag{2}$$

Multiplying the two equations (1) and (2), we get a single equation that represents both lines:

$$\begin{aligned}(y - 2x)(2y - x) &= 0 \\ 2y^2 - xy - 4xy + 2x^2 &= 0 \\ 2x^2 - 5xy + 2y^2 &= 0 \qquad (3)\end{aligned}$$

Equation (3) is the combined equation, or the **equation of the pair of lines**, for lines (1) and (2).



The general form of the homogeneous equation of the second degree is:

$$\boxed{ax^2 + 2hxy + by^2 = 0}$$

This equation represents a pair of straight lines passing through the origin, in the form $y = m_1x$ and $y = m_2x$.

To find the equations of the individual lines from the general form, we can treat it as a quadratic equation in terms of y:

$$by^2 + (2hx)y + (ax^2) = 0$$

Now, we apply the quadratic formula, $y = \frac{-B\pm\sqrt{B^2-4AC}}{2A}$, where $A = b$, $B = 2hx$, and $C = ax^2$.

$$\begin{aligned}
y &= \frac{-(2hx) \pm \sqrt{(2hx)^2 - 4(b)(ax^2)}}{2b} \\
&= \frac{-2hx \pm \sqrt{4h^2x^2 - 4abx^2}}{2b} \\
&= \frac{-2hx \pm \sqrt{4x^2(h^2 - ab)}}{2b} \\
&= \frac{-2hx \pm 2x\sqrt{h^2 - ab}}{2b} \\
&= \frac{2x(-h \pm \sqrt{h^2 - ab})}{2b} \\
&= \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right) x
\end{aligned}$$

This single equation represents the two separate lines passing through the origin:

$$\begin{aligned}
y &= \left(\frac{-h + \sqrt{h^2 - ab}}{b}\right) x \\
y &= \left(\frac{-h - \sqrt{h^2 - ab}}{b}\right) x
\end{aligned}$$

Nature of the Lines from $ax^2 + 2hxy + by^2 = 0$

Based on the discriminant, $h^2 - ab$, from the slope formula, we have three cases for the nature of the two lines:

1. If $\mathbf{h^2 > ab}$: The slopes are real and distinct. The equation represents two unique lines intersecting at the origin.
2. If $\mathbf{h^2 < ab}$: The slopes are imaginary. The equation represents two imaginary lines that intersect only at the real point $(0, 0)$.
3. If $\mathbf{h^2 = ab}$: The slopes are real and identical. The equation represents two coincident lines (a single line).

Therefore, the equation represents a pair of real lines if $\mathbf{h^2 \geq ab}$.

Angle Between the Pair of Lines

To find the angle θ between the two lines, we can use the formula involving their slopes, m_1 and m_2.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

From the previous page, we know the two slopes are $m_1 = \frac{-h+\sqrt{h^2-ab}}{b}$ and $m_2 = \frac{-h-\sqrt{h^2-ab}}{b}$. **1.**

Difference of Slopes ($m_1 - m_2$):

$$\begin{aligned}
m_1 - m_2 &= \frac{(-h + \sqrt{h^2 - ab}) - (-h - \sqrt{h^2 - ab})}{b} \\
&= \frac{2\sqrt{h^2 - ab}}{b}
\end{aligned}$$

2. Product of Slopes ($m_1 m_2$):

$$m_1 m_2 = \left(\frac{-h+\sqrt{h^2-ab}}{b}\right)\left(\frac{-h-\sqrt{h^2-ab}}{b}\right)$$
$$= \frac{(-h)^2-(\sqrt{h^2-ab})^2}{b^2} = \frac{h^2-(h^2-ab)}{b^2} = \frac{a}{b}$$

3. Substitute into the tangent formula:

$$\tan\theta = \left|\frac{\frac{2\sqrt{h^2-ab}}{b}}{1+\frac{a}{b}}\right| = \left|\frac{\frac{2\sqrt{h^2-ab}}{b}}{\frac{b+a}{b}}\right|$$

This gives the formula for the angle between the lines represented by $ax^2+2hxy+by^2=0$:

$$\boxed{\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}}$$

(Note: The $\pm$ in the original notes accounts for the acute and obtuse angles, which is handled by the absolute value in the standard formula.)

Special Cases for the Angle

From the angle formula $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$, we can deduce two important conditions:

1. If $\mathbf{h^2 - ab = 0}$, the numerator is zero, so $\tan\theta = 0$. This means $\theta = 0°$, and the two lines are **coincident** (the same line).
2. If $\mathbf{a + b = 0}$, the denominator is zero, so $\tan\theta$ is undefined. This means $\theta = 90°$ or $\pi/2$, and the two lines are **perpendicular**.

Example

Show that the equation $10x^2 - 17xy + 3y^2 = 0$ represents a pair of real lines. Then, find the equation of each line and the angle between them.

– Soln –

We compare the given equation to the general form $ax^2+2hxy+by^2=0$.

$$a = 10, \quad b = 3, \quad 2h = -17 \implies h = -\frac{17}{2}$$

1. Check if the lines are real: The lines are real if $h^2 \geq ab$.

$$h^2 = \left(-\frac{17}{2}\right)^2 = \frac{289}{4} = 72.25$$
$$ab = (10)(3) = 30$$

Since $72.25 > 30$, the condition $h^2 > ab$ is met, so the equation represents two **real and distinct lines**.

2. Find the equation of each line: We find the individual lines by factoring the homogeneous equation.

$$\begin{aligned} 10x^2 - 17xy + 3y^2 &= 0 \\ 10x^2 - 15xy - 2xy + 3y^2 &= 0 \\ 5x(2x-3y) - y(2x-3y) &= 0 \\ (5x-y)(2x-3y) &= 0 \end{aligned}$$

This gives the two separate line equations:

$$\boxed{5x - y = 0} \quad \text{and} \quad \boxed{2x - 3y = 0}$$

3. Find the angle between them: Using the formula $\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}$:

$$\begin{aligned}\tan\theta &= \frac{2\sqrt{72.25 - 30}}{10+3} \\ &= \frac{2\sqrt{42.25}}{13} \\ &= \frac{2(6.5)}{13} = \frac{13}{13} = 1\end{aligned}$$

Since $\tan\theta = 1$, the angle is $\theta = \arctan(\mathbf{1}) = \mathbf{45}°$ or $\frac{\pi}{\mathbf{4}}$.

II. Non-Homogeneous Equation

The general form of the equation of a pair of lines of the second degree is given by:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The necessary condition that this general equation represents a pair of straight lines is that the following determinant is zero:

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

If this condition is met, the second-degree terms $(ax^2 + 2hxy + by^2)$ still determine the slopes of the lines. Therefore, the formulas for the angle between them, and the conditions for being parallel or perpendicular, are the same as in the homogeneous case.

- The **angle** θ between the lines is given by:

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

- The two lines are **parallel** (or coincident) if:

$$h^2 - ab = 0$$

- The two lines are **perpendicular** if:

$$a + b = 0$$

Example

Prove that the equation $x^2 + 3xy - 4y^2 + 5x + 5y + 6 = 0$ represents a pair of lines.

– Soln –

We compare the equation to the general form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to find the coefficients:

$$a = 1, \quad h = \frac{3}{2}, \quad b = -4, \quad g = \frac{5}{2}, \quad f = \frac{5}{2}, \quad c = 6$$

The equation represents a pair of lines if the determinant $\Delta = 0$.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & 3/2 & 5/2 \\ 3/2 & -4 & 5/2 \\ 5/2 & 5/2 & 6 \end{vmatrix}$$

$$= 1\left((-4)(6) - \left(\frac{5}{2}\right)^2\right) - \frac{3}{2}\left(\left(\frac{3}{2}\right)(6) - \left(\frac{5}{2}\right)^2\right) + \frac{5}{2}\left(\left(\frac{3}{2}\right)\left(\frac{5}{2}\right) - (-4)\left(\frac{5}{2}\right)\right)$$

$$= 1\left(-24 - \frac{25}{4}\right) - \frac{3}{2}\left(9 - \frac{25}{4}\right) + \frac{5}{2}\left(\frac{15}{4} + 10\right)$$

$$= \left(-\frac{96+25}{4}\right) - \frac{3}{2}\left(\frac{36-25}{4}\right) + \frac{5}{2}\left(\frac{15+40}{4}\right)$$

$$= -\frac{121}{4} - \frac{3}{2}\left(\frac{11}{4}\right) + \frac{5}{2}\left(\frac{55}{4}\right)$$

$$= -\frac{121}{4} - \frac{33}{8} + \frac{275}{8}$$

$$= \frac{-242 - 33 + 275}{8} = \frac{-275 + 275}{8} = 0$$

Since $\Delta = 0$, the equation **does represent a pair of lines**.

Example

Find the value of k that makes the equation $4x^2 - 3xy + 2y^2 + 3x - 4y + k = 0$ represent a pair of lines.

– Soln – First, we find the coefficients:

$$a = 4, \quad h = -\frac{3}{2}, \quad b = 2, \quad g = \frac{3}{2}, \quad f = -2, \quad c = k$$

Now, we set the determinant $\Delta = 0$ and solve for k.

$$\begin{vmatrix} 4 & -3/2 & 3/2 \\ -3/2 & 2 & -2 \\ 3/2 & -2 & k \end{vmatrix} = 0$$

$$4(2k-4) - \left(-\frac{3}{2}\right)\left(-\frac{3}{2}k + 3\right) + \frac{3}{2}(3-3) = 0$$

$$8k - 16 + \frac{3}{2}\left(-\frac{3}{2}k + 3\right) + 0 = 0$$

$$8k - 16 - \frac{9}{4}k + \frac{9}{2} = 0$$

$$\left(8 - \frac{9}{4}\right)k - \left(16 - \frac{9}{2}\right) = 0$$

$$\left(\frac{32-9}{4}\right)k - \left(\frac{32-9}{2}\right) = 0$$

$$\frac{23}{4}k - \frac{23}{2} = 0$$

$$\frac{23}{4}k = \frac{23}{2}$$

$$k = \frac{23}{2} \cdot \frac{4}{23} = 2$$

The value that makes the equation represent a pair of lines is $\mathbf{k = 2}$.

Example

Show that the equation $2x^2 - 3xy + y^2 - 3x + y - 2 = 0$ represents a pair of lines. Then, find the equation of each line and the angle between them.

– Soln –

We are given the equation $2x^2 - 3xy + y^2 - 3x + y - 2 = 0$.

1. Condition for a Pair of Lines

First, we identify the coefficients by comparing to the general form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$:

$$a = 2, \quad h = -\frac{3}{2}, \quad b = 1, \quad g = -\frac{3}{2}, \quad f = \frac{1}{2}, \quad c = -2$$

The equation represents a pair of lines if the determinant $\Delta = 0$.

$$\begin{aligned}
\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} &= \begin{vmatrix} 2 & -3/2 & -3/2 \\ -3/2 & 1 & 1/2 \\ -3/2 & 1/2 & -2 \end{vmatrix} \\
&= 2\left(-2 - \frac{1}{4}\right) - \left(-\frac{3}{2}\right)\left(3 + \frac{3}{4}\right) + \left(-\frac{3}{2}\right)\left(-\frac{3}{4} + \frac{3}{2}\right) \\
&= 2\left(-\frac{9}{4}\right) + \frac{3}{2}\left(\frac{15}{4}\right) - \frac{3}{2}\left(\frac{3}{4}\right) \\
&= -\frac{18}{4} + \frac{45}{8} - \frac{9}{8} = -\frac{36}{8} + \frac{45}{8} - \frac{9}{8} = \frac{0}{8} = 0
\end{aligned}$$

Since $\Delta = 0$, the equation **represents a pair of lines**.

2. Angle Between the Lines

We use the formula $\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$:

$$\begin{aligned}
h^2 - ab &= \left(-\frac{3}{2}\right)^2 - (2)(1) = \frac{9}{4} - 2 = \frac{1}{4} \\
a + b &= 2 + 1 = 3 \\
\tan\theta &= \frac{2\sqrt{1/4}}{3} = \frac{2(1/2)}{3} = \frac{1}{3}
\end{aligned}$$

The angle is $\theta = \arctan\left(\frac{1}{3}\right)$.

3. Equations of the Individual Lines

We find the individual lines by factoring. First, factor the second-degree terms:

$$2x^2 - 3xy + y^2 = (2x - y)(x - y)$$

This means the full equation can be factored in the form $(2x - y + \alpha)(x - y + \beta) = 0$. Expanding this gives:

$$2x^2 - 3xy + y^2 + (2\beta + \alpha)x + (-\beta - \alpha)y + \alpha\beta = 0$$

By comparing the coefficients of the x and y terms with our original equation $(\ldots -3x + y - 2 = 0)$, we get a system of equations:

$$\begin{aligned}
\text{coeff of x:} \quad & 2\beta + \alpha = -3 \\
\text{coeff of y:} \quad & -\beta - \alpha = 1
\end{aligned}$$

Adding the two equations gives: $\beta = -2$. Substituting $\beta = -2$ into the second equation: $-(-2) - \alpha = 1 \implies 2 - \alpha = 1 \implies \alpha = 1$. (We can check the constant term: $\alpha\beta = (1)(-2) = -2$, which matches the original equation.)

Thus, the two lines are:

$$\mathbf{2x - y + 1 = 0} \quad \text{and} \quad \mathbf{x - y - 2 = 0}$$