Chapter (3)

The Derivatives

1. Tangents and the Derivative at a Point

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1).

If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

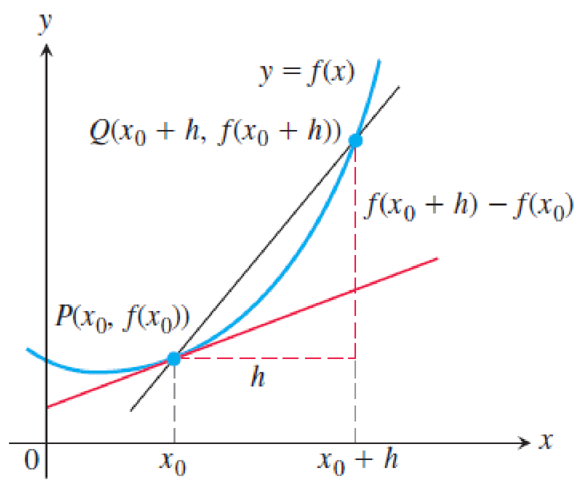


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

Is called the difference quotient of the function at x_0 with increment h.

If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Summary

We have been discussing slopes of curves, lines tangent to a curve, the rate of change of a function, and the derivative of a function at a point.

All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

2. The Derivative as a Function

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

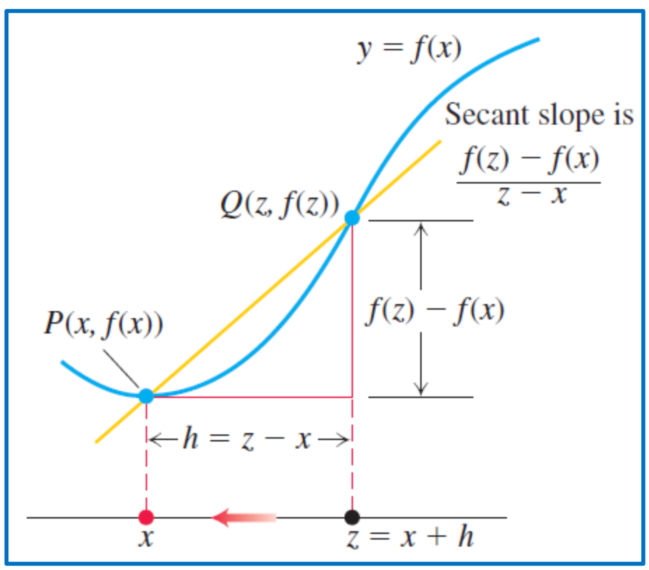
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

If we write $z = x + h$, then $h = z - x$ and $h \to 0$ if and only if $z \to x$. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function, and focuses on the point z that approaches x.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$



Derivative of $f(x)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Example 1: Differentiate $f(x) = \frac{x}{x-1}$.

Solution: We use the definition of derivative, which requires us to calculate $f(x + h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} && \text{Definition} \\
&= \lim_{h \to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} && \frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd} \\
&= \lim_{h \to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} && \text{Simplify.} \\
&= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. && \text{Cancel } h \neq 0.
\end{aligned}$$

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Example 2:

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution:

(a) We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right).\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \ .$$

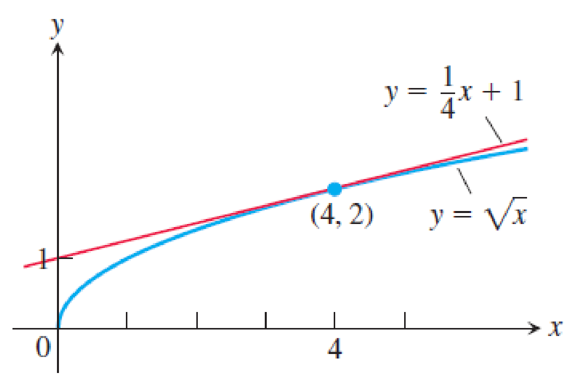


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

(b) the slope of the curve at $x = 4$ is

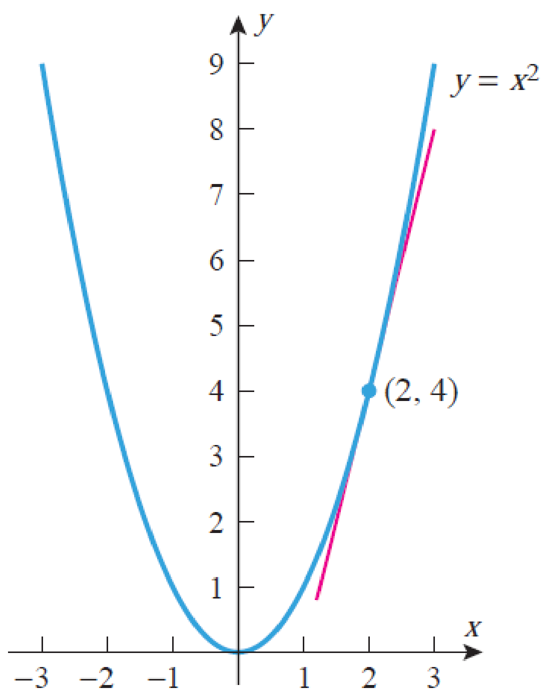
$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point $(4, 2)$ with slope $\frac{1}{4}$ is

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1\,.$$

Example 3: Find the derivative with respect to x of $f(x) = x^2$, and use it to find the equation of the tangent line to the curve $y = x^2$ at $x = 2$.



▲ **Figure 2.2.1**

Solution. It follows from (2) that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

Thus, the slope of the tangent line to $y = x^2$ at $x = 2$ is $f'(2) = 4$. Since $y = 4$ if $x = 2$, the point-slope form of the tangent line is

$$y - 4 = 4(x - 2)$$

which we can rewrite in slope-intercept form as $y = 4x - 4$ (Figure 2.2.1). ◀

Notations:

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = D(f)(x) = D_x f(x).$$

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx} f(x)\right|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Remark:

We shall say that a function $f(x)$ is differentiable at $x = c$, if $f'(c)$ exists and finite.

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exists at the endpoints.

Theorem (Differentiability Implies Continuity)

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof

Given that $f'(c)$ exists, we must show that $\lim\limits_{x \to c} f(x) = f(c)$, or equivalently, that $\lim\limits_{h \to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$f(h+c) = f(c) + (f(c+h) - f(c))$$

$$= f(c) + \frac{f(c+h) - f(c)}{h} \, .h$$

Now take limits as $h \to 0$.

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \, . \lim_{h \to 0} h$$

$$= f(c) + f'(c).0 \;\; = f(c).$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$, then f is continuous from that side at $x = c$.

The previous theorem shows that differentiability at a point $x = c$ implies continuity at that point. But the converse is false.

Example 4: Given $f(x) = \begin{cases} \frac{1}{x} & , \; if\ 0 < x < k \\ 1 - \frac{x}{4}, & if\ x \geq k \end{cases}$.

(a) Determine the value of k such that f is continuous at $x = k$.

(b) Is $f(x)$ differentiable function at the value of k found in part (a).

Solution:

(a) The function $f(x)$ will be continuous at $x = k$ if

$$\lim_{x \to k^-} f(x) = \lim_{x \to k^+} f(x) = f(k) \ .$$

$$\lim_{x \to k^-} \frac{1}{x} = \frac{1}{k} \ , \qquad \lim_{x \to k^+} 1 - \frac{x}{4} = 1 - \frac{k}{4} \ .$$

Therefore $f(x)$ will be continuous at k if

$$\frac{1}{k} = 1 - \frac{k}{4}$$

$$4 = 4k - k^2 \qquad \to \qquad k = 2$$

Thus $f(x) = \begin{cases} \frac{1}{x} & , \; if\ 0 < x < 2 \\ 1 - \frac{x}{4}, & if\ x \geq 2 \end{cases}$

and $f(x)$ is continuous at $x = 2$.

(b) To determine if f is differentiable at x=2, we compute $f'_-(2)$ and $f'_+(2)$ by using the alternative formulas.

$$f'_-(2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \lim_{x \to 2^-} \frac{2 - x}{2x(x - 2)} = -\frac{1}{4}\ .$$

And $f'_+(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} \quad = \lim_{x \to 2^+} \frac{\left(1 - \frac{x}{4}\right) - \frac{1}{2}}{x - 2}$

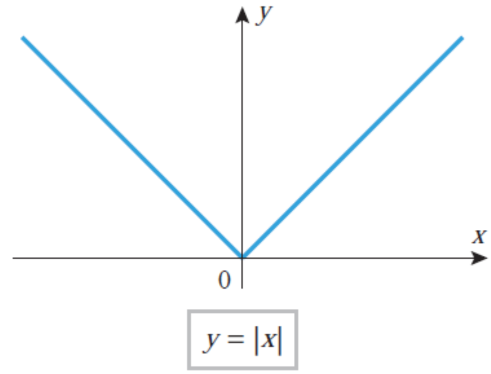
$$= \lim_{x \to 2^+} \frac{\left(\frac{1}{2} - \frac{x}{4}\right)}{x - 2} = \lim_{x \to 2^+} \frac{2 - x}{4(x - 2)} = -\frac{1}{4}\ .$$

Since $f'_-(2) = f'_+(2) = -\frac{1}{4}$, it follows that $f'(2)$ exists, hence $f(x)$ is differentiable at $x = 2$.

Example 5: If the function $f(x) = |x|$ is continuous for all x .

(a) Show that $f(x) = |x|$ is not differentiable at $x = 0$.

(b) Find a formula for $f'(x)$.



▲ **Figure 2.2.10**

Solution:

(a) From the definition of differentiability at $x_0 = 0$, that is the value of $f'(0)$ is given by

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{|h|-|0|}{h} = \lim_{h\to 0}\frac{|h|}{h}\ .$$

But

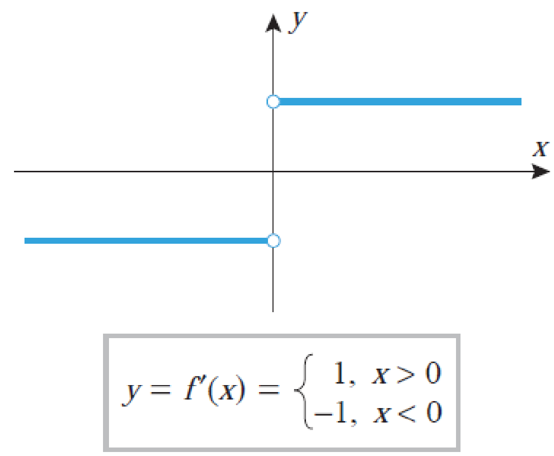
$$\frac{|h|}{h} = \begin{cases} 1, & h>0 \\ -1, & h<0 \end{cases}$$

so that

$$\lim_{h\to 0^-}\frac{|h|}{h} = -1 \quad \text{and} \quad \lim_{h\to 0^+}\frac{|h|}{h} = 1$$

Since these one-sided limits are not equal, the two-sided limit does not exist, and hence $f(x) = |x|$ is not differentiable at x = 0.

(b) A formula for the derivative of $f(x) = |x|$ can be obtained by writing $|x|$ in piecewise form and treating the cases $x > 0$ and $x < 0$ separately.
If $x > 0$, then $f(x) = |x| = x$ and $f'(x) = 1$; if $x < 0$, then $f(x) = |x| = -x$ and $f'(x) = -1$.



▲ **Figure 2.2.11**

Thus

$$f'(x) = \begin{cases} 1\,, & x > 0 \\ -1\,, & x < 0 \end{cases}$$

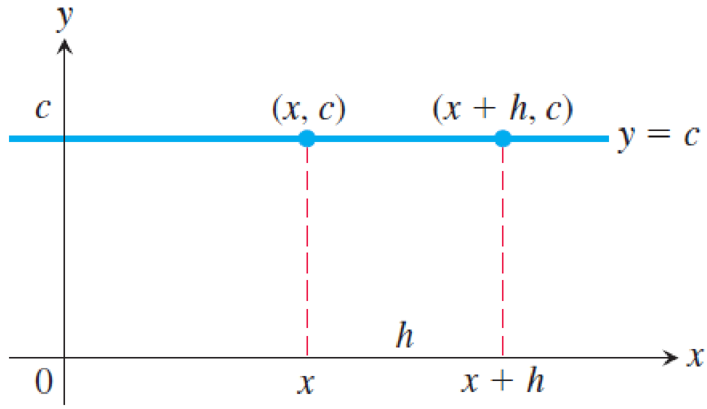
The graph of $f(x) = |x|$ is shown in Figure 2.2.11. Observe that $f'(x)$ is not continuous at $x = 0$, so this example shows that a function that is continuous everywhere may have a derivative that fails to be continuous everywhere.

Differentiation Rules

1) Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$



Proof We apply the definition of the derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.9). At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0. \qquad \blacksquare$$

2) Derivative of Power Rule

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

for all x where the powers x^n and x^{n-1} are defined.

Proof of the Positive Integer Power Rule The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \qquad n \text{ terms} \\ &= nx^{n-1}. \end{aligned}$$

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3) Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

4) Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

5) Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

6) Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 1: Differentiate the following functions of x.

1) $y = x^7 - 2x^4 + 6x - x^{\sqrt{2}} + \frac{1}{x^4} - x^{-4/3} - 5$

2) $y = (x^2 + 1)(x^3 + 3)$

3) $y = \frac{x^2 - 1}{x^3 + 1}$

4) $y = \frac{(x-1)(x^2 - 2x)}{x^4}$

5) $y = 3\sqrt{(x)^{2+\pi}}$

Solution:

1) $\frac{dy}{dx} = 7x^6 - 8x^3 + 6 - \sqrt{2}\, x^{\sqrt{2}-1} - 4x^{-5} + \frac{4}{3}x^{-\frac{7}{3}}$.

2) From the product rule, we find

$$\frac{dy}{dx} = 3x^2(x^2 + 1) + 2x(x^3 + 3)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$
$$= 5x^4 + 3x^2 + 6x \, .$$

3) Apply the Quotient rule, we find

$$\frac{dy}{dx} = \frac{2x(x^3 + 1) - 3x^2(x^2 - 1)}{(x^3 + 1)^2}$$

$$= \frac{2x^4 + 2x - 3x^4 + 3x^2}{(x^3 + 1)^2}$$

$$= \frac{-x^4 + 3x^2 + 2x}{(x^3 + 1)^2}$$

4) We write the function $y = \frac{(x-1)(x^2-2x)}{x^4}$ as a product of 3 functions

$$y = (x-1)(x^2-2x)x^{-4}$$

Then apply the product rule, we get

$$\frac{dy}{dx} = (x^2-2x)x^{-4} + (2x-2)(x-1)x^{-4} - 4x^{-5}(x-1)(x^2-2x)$$

$$= (x^2-2x)x^{-4} + 2(x-1)^2x^{-4} - 4x^{-5}(x-1)(x^2-2x)\,.$$

5) $\frac{dy}{dx} = 3\,\frac{(2+\pi)\,x^{1+\pi}}{2\sqrt{x^{2+\pi}}}$.

Remark: y' or $f'(x)$ denoting the derivative of the function $y = f(x)$. We shall denote by y'' or $f''(x)$ the derivative of $f'(x)$, y''' or $f'''(x)$ is the derivative of $f''(x)$, $\cdots$ and so on. We call y' the first derivative of $f(x)$, y'' the second derivative of $f(x), \cdots$

The following notation will also be adopted,

$$y'' = \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(f) = f''(x)\,, y''' = \frac{d^3y}{dx^3} = \frac{d^3}{dx^3}(f) = f'''(x), \cdots$$

Example 2: Find the equation of the tangent line to the curve $y = x + \frac{2}{x}$ at the point (1,3).

Solution: The slope of the tangent to the curve is

$$\frac{dy}{dx} = 1 - \frac{2}{x^2}\,.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x\,=\,1} = \left.\left[1 - \frac{2}{x^2}\right]\right|_{x\,=\,1} = 1 - 2 = -1\ .$$

The line through the point $(1,3)$ with slope $m = -1$ is

$$y - 3 = -1(x - 1) \qquad \rightarrow \qquad y + x = 4$$

Example 3: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

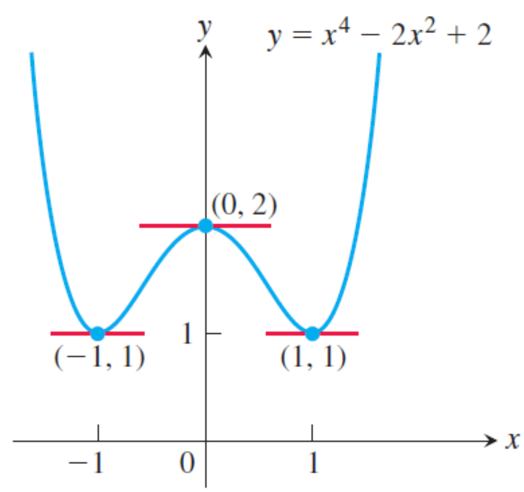
Solution: The horizontal tangents, if any, occur where the slope $\frac{dy}{dx} = 0$.

We have

$$\frac{dy}{dx} = 4x^3 - 4x\ .$$

Now solve the equation $\frac{dy}{dx} = 0$ for x :

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$



The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1,$ and -1. The corresponding points on the curve are (0, 2), (1, 1), and (-1, 1). As shown in the figure.

Derivatives of Trigonometric Functions

1. Derivative of the Sine Function

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

Proof

From the angle sum identity for the sine function

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = sin\ x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

2. Derivative of the Cosine Function

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Proof

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x.
\end{aligned}$$

Derivatives of the Other Basic Trigonometric Functions

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 4: Find the derivatives of the following functions:

a) $y = x^2 - \sin x$

b) $y = \frac{\sin x}{x} + \sin x \cos x$

c) $y = \frac{\cos x}{1 - \sin x}$

d) $y = x^3 \tan x$

Solution:

a) $\frac{dy}{dx} = 2x - cos\ x$

b) $\frac{dy}{dx} = \frac{x\ cos\ x - sin\ x}{x^2} + sin\ x\ (-sin\ x) + cos\ x\ (cos\ x)$

$$= \frac{x\ cos\ x - sin\ x}{x^2} + cos^2 x - sin^2 x$$

$$= \frac{x\ cos\ x - sin\ x}{x^2} + cos\ (2x).$$

c) $\frac{dy}{dx} = \frac{-(1 - \sin x) \sin x - \cos x\ (-\cos x)}{(1 - \sin x)^2}$

$$= \frac{sin^2 x - \sin x + cos^2 x}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}.$$

d) $\frac{dy}{dx} = 3x^2 \tan x + x^3 \sec^2 x$.

The Chain Rule (Derivative of a Composite Function)

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example 5:

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution:

$$g'(t) = \frac{d}{dt}(\tan(5 - \sin 2t))$$

$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t)$$

Derivative of $\tan u$ with $u = 5 - \sin 2t$

$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right)$$

Derivative of $5 - \sin u$ with $u = 2t$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t). \qquad \blacksquare$$

Example 6:

The function

$$y = (3x^2 + 1)^2$$

is the composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$.

Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du}\cdot\frac{du}{dx} &= 2u\cdot 6x\\ &= 2(3x^2+1)\cdot 6x \qquad \text{Substitute for } u\\ &= 36x^3+12x.\end{aligned}$$

The following table contains a list of generalized derivative formulas:

$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$	$\frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$
$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx}$	$\frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$
$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx}$	$\frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}[\sec u] = \sec u\tan u\frac{du}{dx}$	$\frac{d}{dx}[\csc u] = -\csc u\cot u\frac{du}{dx}$

Note that If $f(g(x))$ is a composition of functions in which the inside function g and the outside function f are differentiable, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x)).g'(x).$$

Example 7: Find the derivatives of the following functions:

(a) $$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$$
$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$$
$$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

(b) $$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$
$$= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2)$$
$$= -1(3x - 2)^{-2}(3)$$
$$= -\frac{3}{(3x - 2)^2}$$

(c) $$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$
$$= 5\sin^4 x \cos x$$

Example 8: Find the derivatives of the following functions:

a) $y = \sqrt{x^3 + csc\ x}$
b) $y = (1 + x^5\ cot\ x)^{-8}$
c) $y = cos^4(\pi x)$
d) $y = sin\ \sqrt{1 + cos\ x}$

Solution:

a) Taking $u = x^3 + \csc x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sqrt{x^3 + csc\ x}\right] = \frac{d}{dx}\left[\sqrt{u}\right] = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^3 + \csc x}} \left[3x^2 - \csc x \cot x\right] .$$

b) Taking $u = 1 + x^5\, cot\ x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[(1 + x^5\, cot\ x)^{-8}\right] = \frac{d}{dx}\left[(u)^{-8}\right] = -8(u)^{-9}\,\frac{du}{dx}$$

$$\frac{dy}{dx} = -8(1 + x^5\, cot\ x)^{-9}\,(5x^4 \cot x - x^5 \csc^2 x) .$$

c) $y = [\cos(\pi x)]^4$, taking $u = \cos(\pi x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\cos(\pi x)\right]^4 = \frac{d}{dx}\left[(u)^4\right] = 4(u)^3\,\frac{du}{dx}$$

$$\frac{dy}{dx} = 4(\cos(\pi x))^3.(-\sin \pi x) = -4cos^3(\pi x).\sin(\pi x) .$$

d) Taking $u = \ 1 + cos\ x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin\left(\sqrt{1 + cos\ x}\right)\right] = \frac{d}{dx}\left[\sin\left(\sqrt{u}\right)\right] = \cos u\,\frac{1}{2\sqrt{u}}\,u'$$

$$\frac{dy}{dx} = \cos\left(\sqrt{1 + cos\ x}\right)\frac{1}{2\sqrt{1 + cos\ x}}\,.(-\sin x)$$

$$\therefore\ \frac{dy}{dx} = \frac{-\sin x \cos\left(\sqrt{1 + cos\ x}\right)}{2\sqrt{1 + cos\ x}} .$$

Inverse Trigonometric Functions

Definitions:

For each x in the interval $[-1,1]$, we define the number $y = \sin^{-1} x$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = x$; that is, for the stated restrictions on x and y the equations $\sin y = x$, $y = \sin^{-1} x$ are equivalent .

In a similar way the other inverse trigonometric functions can be investigsted as follows:

1) $y = \sin^{-1} x$, $\quad -\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

2) $y = \cos^{-1} x$, $\quad 0 \le \cos^{-1} x \le \pi$

3) $y = \tan^{-1} x$, $\quad -\frac{\pi}{2} \le \tan^{-1} x \le \frac{\pi}{2}$

4) $y = \cot^{-1} x$, $\quad 0 \le \cot^{-1} x \le \pi$

5) $y = \sec^{-1} x$, $\quad 0 \le \sec^{-1} x \le \pi \ , y \ne \frac{\pi}{2}$

6) $y = \csc^{-1} x$, $\quad -\frac{\pi}{2} \le \csc^{-1} x \le \frac{\pi}{2} \ , y \ne 0$

The cancellation formulas for the trigonometric functions are the following:

$\sin(\sin^{-1} x) = x$, $\quad if\ (-1 < x < 1)$ $\sin^{-1}(\sin x) = x$, $\quad if\ (-\frac{\pi}{2} < x < \frac{\pi}{2})$	$\cos(cos^{-1} x) = x$, $\quad if\ (-1 < x < 1)$ $\cos^{-1}(\cos x) = x$, $\quad if\ (0 < x < \pi)$
$\tan(tan^{-1} x) = x$, $\quad if\ (-\infty < x < \infty)$ $tan^{-1}(\tan x) = x$, $\quad if\ (-\frac{\pi}{2} < x < \frac{\pi}{2})$	$\cot(cot^{-1} x) = x$, $\quad if\ (-\infty < x < \infty)$ $cot^{-1}(\cot x) = x$, $\quad if\ (0 < x < \pi)$

$\sec(sec^{-1} x) = x\,, \quad if\ (\lvert x \rvert \geq 1)$ $sec^{-1}(\sec x) = x\,,$ $if\ (0 < x < \pi),\ x \neq \frac{\pi}{2}$	$\csc(csc^{-1} x) = x\,, \quad if\ (\lvert x \rvert \geq 1)$ $csc^{-1}(\csc x) = x\,,$ $if\ \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right),\ x \neq 0$

Derivatives of Inverse Trigonometric Functions

1) $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$, $(\lvert x \rvert < 1)$	**2)** $\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$, $(\lvert x \rvert < 1)$
3) $\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$	**4)** $\frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}$
5) $\frac{d}{dx}[\sec^{-1} x] = \frac{1}{\lvert x \rvert\sqrt{x^2-1}}$, $(\lvert x \rvert > 1)$	**6)** $\frac{d}{dx}[\csc^{-1} x] = \frac{-1}{\lvert x \rvert\sqrt{x^2-1}}$, $(\lvert x \rvert > 1)$

To prove (1), we first prove $\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$.

If $y = \sin^{-1} x$, then $x = \sin y$, $\frac{dx}{dy} = \cos y$

$$\therefore \frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)} = \frac{1}{y} = \frac{1}{\sqrt{1-sin^2 y}} = \frac{1}{\sqrt{1-x^2}}\,.$$

$$\therefore \frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}} \quad , \qquad |x| < 1\,.$$

If $u(x)$ is a differentiable function of x ,

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad , \qquad |u| < 1\,.$$

Theorem: If $u(x)$ is a differentiable function of x , then

1) $\frac{d}{dx}[\sin^{-1}u] = \frac{u'}{\sqrt{1-u^2}}$, $(\lvert u\rvert < 1)$	2) $\frac{d}{dx}[\cos^{-1}x] = \frac{-u'}{\sqrt{1-x^2}}$, $(\lvert u\rvert < 1)$
3) $\frac{d}{dx}[\tan^{-1}u] = \frac{u'}{1+u^2}$	4) $\frac{d}{dx}[\cot^{-1}u] = \frac{-u'}{1+u^2}$
5) $\frac{d}{dx}[\sec^{-1}u] = \frac{u'}{\lvert u\rvert\sqrt{u^2-1}}$, $(\lvert u\rvert > 1)$	6) $\frac{d}{dx}[\csc^{-1}u] = \frac{-u'}{\lvert u\rvert\sqrt{u^2-1}}$, $(\lvert u\rvert > 1)$

Example 9: Differentiate $y = \sec^{-1}(5x^4)$

Solution:

Using the Chain Rule and derivative of the arcsecant function, we find

$$\frac{d}{dx}\sec^{-1}(5x^4) = \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}\frac{d}{dx}(5x^4)$$

$$= \frac{1}{5x^4\sqrt{25x^8-1}}(20x^3) \qquad 5x^4 > 1 > 0$$

$$= \frac{4}{x\sqrt{25x^8-1}}.$$

Example 10: Differentiate the following functions

a) $y = \sin^{-1}\sqrt{x} + 4x\sec^{-1}x^2$

b) $y = \frac{\tan^{-1}x}{x}$

c) $y = \cot^{-1}(\tan x)$

d) $y = \cos^{-1}(2x^3 - 1)$.

Example 11: If $y = \ln(\sec x + \tan x)$, prove that $y'' = \sec x \tan x$.

Proof: Differentiate both sides with respect to x, we get

$$y' = \frac{\sec x \tan x + \sec^2 x}{(sec\ x + tan\ x)}$$

$$\therefore y' = \frac{sec\ x \,.\, \cancel{(tan\ x + sec\ x)}}{\cancel{(sec\ x + tan\ x)}} = \sec x$$

Thus $y'' = \frac{d}{dx}(\sec x) = \sec x \tan x$.

Example 12: If $y = \tan(\cos^{-1} x)$, prove that $y' = \frac{-(y^2+1)}{\sqrt{1-x^2}}$.

Proof: Differentiate both sides with respect to x, we get

$$y' = sec^2(\cos^{-1} x).\ \frac{-1}{\sqrt{1-x^2}}$$

Since $\boxed{\boldsymbol{sec^2(cos^{-1} x) = tan^2(cos^{-1} x) + 1}}$

$$\therefore y' = \underbrace{\{tan^2(\cos^{-1} x) + 1\}}_{sec^2(\cos^{-1} x)} .\ \frac{-1}{\sqrt{1-x^2}}$$

$$\therefore y' = \{y^2 + 1\}.\ \frac{-1}{\sqrt{1-x^2}} = \frac{-(y^2+1)}{\sqrt{1-x^2}} .$$

Derivatives of Exponential Functions

$\frac{d}{dx}\ [a^x] = a^x \ln a$	$\frac{d}{dx}\ [a^{u(x)}] = a^{u(x)} \ln a\, u'$
$\frac{d}{dx}\ [e^x] = e^x$	$\frac{d}{dx}\ [e^{u(x)}] = e^{u(x)}\ \ u'$

For example,

$$\frac{d}{dx}(e^{kx}) = e^{kx} \cdot \frac{d}{dx}(kx) = ke^{kx}, \qquad \text{for any constant } k$$

and

$$\frac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot \frac{d}{dx}(x^2) = 2xe^{x^2}.$$

Example 1: the derivative of $y = e^{-x}$ is $\frac{dy}{dx} = -e^{-x}$.

Example 2: the derivative of $y = e^{3x}$ is $\frac{dy}{dx} = 3e^{3x}$.

EXAMPLE 3 Differentiate $\sin(x^2 + e^x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin(\underbrace{x^2 + e^x}_{\text{inside}}) = \cos(\underbrace{x^2 + e^x}_{\text{inside left alone}}) \cdot (\underbrace{2x + e^x}_{\text{derivative of the inside}}).$$

■

EXAMPLE 4 Differentiate $y = e^{\cos x}$.

Solution Here the inside function is $u = g(x) = \cos x$ and the outside function is the exponential function $f(x) = e^x$. Applying the Chain Rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\cos x}) = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = -e^{\cos x}\sin x.$$

■

Example 5: Find $\frac{dy}{dx}$ for the following functions

a) $y = e^{x \tan x}$

b) $y = ln\,(cos\ e^x)$

c) $y = \sqrt{1 + e^{sec\,x}}$

d) $y = e^{\sqrt{1+5x^3}}$

e) $y = \csc^{-1}(3)^{x+1}$

Solution:

a) $\frac{dy}{dx} = (\tan x + xsec^2 x)e^{x\,tan\,x}$.

b) $\frac{dy}{dx} = \frac{-e^x \sin e^x}{cos\ e^x} = -e^x \tan e^x$.

c) $\frac{dy}{dx} = \frac{\sec x\ \tan x\ e^{\sec x}}{2\sqrt{1+e^{sec\,x}}}$.

d) $\frac{dy}{dx} = \frac{15x^2}{2\sqrt{1+5x^3}}\ e^{\sqrt{1+5x^3}}$

e) $\frac{dy}{dx} = -\frac{3^{(x+1)}\ln 3}{(3)^{x+1}\sqrt{(3)^{2(x+1)}-1}} = -\frac{\ln 3}{\sqrt{(3)^{2x+2}-1}}$

Derivatives of Logarithmic Functions

$\frac{d}{dx}[\log_a x] = \frac{1}{x}.\frac{1}{\ln a}$, $a > 0, \quad a \neq 1$	$\frac{d}{dx}[\log_a u(x)] = \frac{u'}{u(x)}.\frac{1}{\ln a}$, $a > 0, \quad a \neq 1$
$\frac{d}{dx}[\ln x] = \frac{1}{x}$, $x > 0$	$\frac{d}{dx}[\ln u(x)] = \frac{u'}{u(x)}$

To prove the derivative of $\log_a u(x)$ for an arbitrary base ($a > 0,\ a \neq 1$), we start with the change-of-base formula for logarithms and express $\log_a u(x)$ in terms of natural logarithms,

$$\log_a x = \frac{\ln x}{\ln a}.$$

Taking derivatives, we have

$$\begin{aligned}\frac{d}{dx}\log_a x &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)\\ &= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x \qquad \text{ln } a \text{ is a constant.}\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}.\end{aligned}$$

If u is a differentiable function of x and $u > 0$, the Chain Rule gives a more general formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}.$$

Example 6: Differentiate

1) $y = \ln(x^2+1)$.
2) $y = 5^{\tan x}\ln(\sin x)$.
3) $y = \log_3\left(\frac{x+1}{x-1}\right)^{\ln 3}$
4) $y = \log_3\sqrt{1+\sec x}$

Solution:

1) $\frac{dy}{dx} = \frac{2x}{(x^2+1)}$.
2) $\frac{dy}{dx} = 5^{\tan x}.\frac{\cos x}{\sin x} + 5^{\tan x}\, sec^2\, x\; ln\,(sin\; x).$
3) First, we write

$$y = \ln 3 \, . \log_3 \frac{x+1}{x-1} = \ln 3 \left[\log_3(x+1) - \log_3(x-1) \right] .$$

Then differentiate both sides, we get

$$y' = \ln 3 \left[\frac{1}{(x+1)} - \frac{1}{(x-1)} \right] \frac{1}{\ln 3} = \frac{-2}{(x^2-1)} \; .$$

4) From the properties of logarithm function

$$y = \frac{1}{2} \log_3(1 + \sec x)$$

$$y' = \frac{\sec x \tan x}{2(1 + \sec x)} \cdot \frac{1}{\ln 3} .$$

Derivatives of Hyperbolic Functions:

Theorem: If $u(x)$ is a differentiable function of x, then

1) $If \;\, y = \sinh(u) \quad \Rightarrow \quad \dfrac{dy}{dx} = \cosh(u) \, . \, u'$

2) $If \;\, y = \cosh(u) \quad \Rightarrow \quad \dfrac{dy}{dx} = \sinh(u) \, . \, u'$

3) $If \;\, y = \tanh(u) \quad \Rightarrow \quad \dfrac{dy}{dx} = \operatorname{sech}^2(u) \, . \, u'$

4) $If \;\, y = \coth(u) \quad \Rightarrow \quad \dfrac{dy}{dx} = -\operatorname{csch}^2(u) \, . \, u'$

5) $If \;\, y = \operatorname{sech}(u) \quad \Rightarrow \quad \dfrac{dy}{dx} = -\, sech(u) \tanh(u) \, . \, u'$

6) $If \;\, y = \operatorname{csch}(u) \quad \Rightarrow \quad \dfrac{dy}{dx} = -\, csch(u) \coth(u) \, . \, u'$

Example 7: Find y' for the following functions

1) $y = x^2 coth^3\sqrt{x}$ 2) $y = \ln\sqrt{\tanh 3x}$

3) $y = \sinh x^3$ 4) $y = \ln\left[\frac{(x+1)^2 e^{cosech\, x}}{\sqrt{x^3-1}}\right]$

Solution:

1) $y' = x^2 \,.\, 3\left(\coth\sqrt{x}\right)^2 .\ -\operatorname{csch}^2\sqrt{x} \,.\, \frac{1}{2\sqrt{x}} + 2x \,.\left(\coth\sqrt{x}\right)^3$

$$\therefore y' = -\frac{3x^2}{2\sqrt{x}}\left(\coth\sqrt{x}\right)^2 \operatorname{csch}^2\left(\sqrt{x}\right) + 2x \,.\left(\coth\sqrt{x}\right)^3.$$

2) We can write $y = ln(tanh\, 3x)^{1/2} = \frac{1}{2}\, ln(tanh\, 3x)$

$$y' = \frac{1}{2}\, \frac{\operatorname{sech}^2(3x)\,.\,3}{\tanh 3x}$$

$$\therefore y' = \frac{3}{2}\, \frac{\operatorname{sech}^2(3x)}{\tanh 3x}$$

3) $y' = \cosh(\boldsymbol{x^3})\,.\ 3x^2$.

4) Using the properties of ln function, we get

$$y = ln(x+1)^2 + \ln \boldsymbol{e}^{\boldsymbol{cosech\, x}} - \ln(\boldsymbol{x^3} - 1)^{1/2}$$

$$y = 2\, ln(x+1) + cosech\, x . \ln \boldsymbol{e} - \frac{1}{2}\ln(\boldsymbol{x^3} - 1)$$

$$\therefore y' = 2\frac{1}{(x+1)} - cosech\, x \coth x - \frac{1}{2}\, \frac{3x^2}{(\boldsymbol{x^3} - 1)}.$$

Derivatives of Inverse Hyperbolic Functions:

Theorem: If $u(x)$ is a differentiable function of x, then

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1 - u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1 + u^2}}\frac{du}{dx}, \quad u \neq 0$$

Example 8: Find the derivative of the following functions

1) $y = \sinh^{-1}(\tan x)$
2) $y = \ln(\cosh^{-1} x)$
3) $y = e^x \operatorname{sech}^{-1}\sqrt{x}$

Topics in Differentiation

(1) Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the laws of logarithms to simplify the formulas before differentiating. The process, called logarithmic differentiation, is illustrated in the next example.

Example 1:

Find dy/dx if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution: We take the natural logarithm of both sides and simplify the result with the algebraic properties of logarithms

$$\begin{aligned}
\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\
&= \ln ((x^2 + 1)(x + 3)^{1/2}) - \ln (x - 1) \\
&= \ln (x^2 + 1) + \ln (x + 3)^{1/2} - \ln (x - 1) \\
&= \ln (x^2 + 1) + \frac{1}{2} \ln (x + 3) - \ln (x - 1).
\end{aligned}$$

We then take derivatives of both sides with respect to x :

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Example 2: Find the derivative of

$$y = \frac{x^2\sqrt[3]{7x - 14}}{(1 + x^2)^4}$$

Solution:

We can solve this example by taking ln of both sides and use the properties of the natural logarithm

$$\ln y = 2\ln x + \frac{1}{3}\ln(7x - 14) - 4\ln(1 + x^2)\,.$$

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7}{3(7x - 14)} - \frac{8x}{(1 + x^2)}$$

Thus, solve for $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{x^2\sqrt[3]{7x - 14}}{(1 + x^2)^4}\left\{\frac{2}{x} + \frac{7}{3(7x - 14)} - \frac{8x}{(1 + x^2)}\right\}.$$

Example 3: Find

$$\frac{d}{dx}\left[\frac{(x^2+1)^{10}\sin^3(\sqrt{x})}{\sqrt{1+\csc x}}\right]$$

Solution: We let $y = \frac{(x^2+1)^{10} sin^3(\sqrt{x})}{\sqrt{1+csc\,x}}$

By taking ln of both sides, we get

$$\ln y = 10\ln(x^2+1) + 3\ln\sin(\sqrt{x}) - \frac{1}{2}\ln(1+\csc x)$$

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{20x}{(x^2+1)} + \frac{3\ \cos(\sqrt{x})}{\sin(\sqrt{x})} + \frac{\csc x\cot x}{2(1+\csc x)}\ .$$

$$\therefore\ \frac{dy}{dx} = y\left\{\frac{20x}{(x^2+1)} + 3\cot\sqrt{x} + \frac{\csc x\cot x}{2(1+\csc x)}\right\}.$$

Thus

$$\frac{dy}{dx} = \frac{(x^2+1)^{10} sin^3(\sqrt{x})}{\sqrt{1+csc\ x}}\ .\left\{\frac{20x}{(x^2+1)} + 3\cot\sqrt{x} + \frac{\csc x\cot x}{2(1+\csc x)}\right\}\ .$$

Example 4: Differentiate $f(x) = x^x\ , x > 0$.

Solution:

We note that $f(x) = x^x = e^{x\ln x}$, so differentiation gives

$$f'(x) = \frac{d}{dx}(e^{x\ln x})$$

$$= e^{x \ln x} \frac{d}{dx}(x \ln x) \qquad \frac{d}{dx} e^u,\ u = x \ln x$$

$$= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right)$$

$$= x^x (\ln x + 1). \qquad x > 0$$

We can also find the derivative of $y = x^x$ using logarithmic differentiation, assuming y' exists.

Example 5:

Differentiate $y = (\tan x)^{\cos x}$.

Solution:

We can solve this example by taking ln both sides,

$$\ln y = \ln(\tan x)^{\cos x}$$

$$\ln y = \cos x \,.\, \ln(\tan x)$$

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{\sec^2 x}{\tan x} - \sin x \,.\, \ln(\tan x)$$

$$\frac{dy}{dx} = y\left\{\frac{\sec x}{\tan x} - \sin x \,.\, \ln(\tan x)\right\}.$$

Thus, the derivative is

$$\frac{dy}{dx} = (\tan x)^{\cos x}\{\csc x - \sin x \,.\, \ln(\tan x)\}\,.$$

Example 6:

If $y = (x^3 + sinh\,x)^x$. Find y'.

Solution:

Taking ln function to both sides

$$\ln y = x\, ln(x^3 + sinh\,x)$$

$$\frac{y'}{y} = x\,.\frac{(3x^2 + \cosh x)}{(x^3 + sinh\,x)} + ln(x^3 + sinh\,x)$$

$$\therefore y' = y.\left\{x\,.\frac{(3x^2 + cosh\,x)}{(x^3 + sinh\,x)} + ln(x^3 + sinh\,x)\right\}$$

Example 7:

Differentiate $y = (1 + \sin x)^{\sec x}$.

Solution:

We can solve this example by taking ln both sides,

$$\ln y = \ln(1 + \sin x)^{\sec x}$$

$$\ln y = \sec x\,.\ln(1 + \sin x)$$

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \sec x\,\frac{\cos x}{(1 + \sin x)} + \sec x \tan x\,.\ln(1 + \sin x)$$

$$\frac{dy}{dx} = y\left\{\frac{1}{(1 + \sin x)} + \sec x \tan x\,.\ln(1 + \sin x)\right\}.$$

Thus, the derivative is

$$\frac{dy}{dx} = (1 + \sin x)^{\sec x}\left\{\frac{1}{(1 + \sin x)} + \sec x \tan x\,.\ln(1 + \sin x)\right\}.$$

(2) Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0, \qquad y^2 - x = 0, \qquad \text{or} \qquad x^2 + y^2 - 25 = 0.$$

These equations define an implicit relation between the variables x and y. In some cases, we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find $\frac{dy}{dx}$ by implicit differentiation. This section describes the technique.

EXAMPLE 1 Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.29). We know how to calculate the derivative of each of these for $x > 0$:

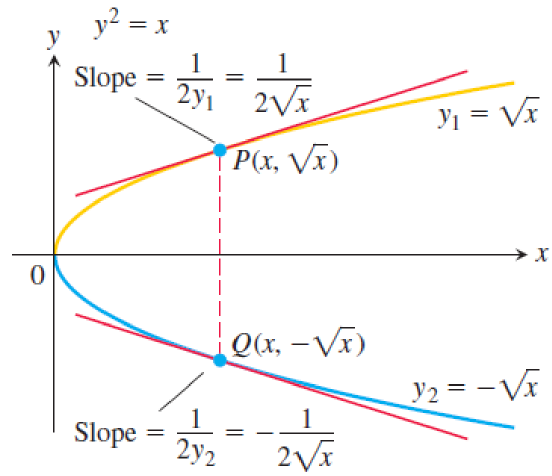
$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$
\begin{aligned}
y^2 &= x \\
2y\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{2y}.
\end{aligned}
$$

The Chain Rule gives $\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.



This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

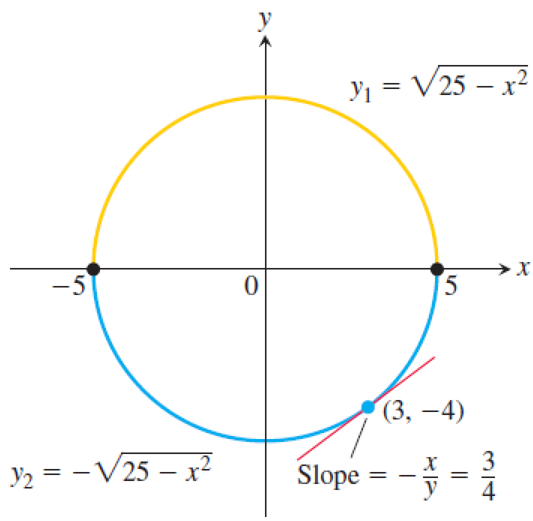
$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Rather, it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.30). The point $(3, -4)$ lies on the graph of y_2, so we can find the slope by calculating the derivative directly, using the Power Chain Rule:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

$$\frac{d}{dx}\left(-(25 - x^2)^{1/2}\right) = -\frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$



We can solve this problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0 \qquad \text{See Example 1.}$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $\left.-\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right)$$ Differentiate both sides with respect to x . . .

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$ . . . treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx.

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx.

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution: To start, we differentiate both sides of the equation with respect to x in order to find $y' = \frac{dy}{dx}$

$$\frac{d}{dx}\left(2x^3 - 3y^2\right) = \frac{d}{dx}(8)$$
$$6x^2 - 6yy' = 0$$
$$y' = \frac{x^2}{y}, \qquad \text{when } y \neq 0$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \qquad \text{when } y \neq 0$$

Example 5:

Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

Solution:

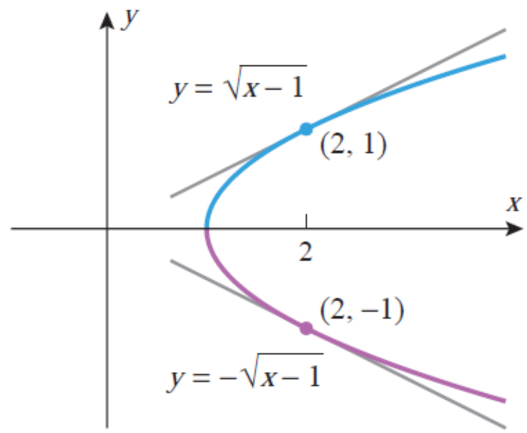
$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$
$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$
$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$
$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Solving for dy/dx we obtain
$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

Example 6: Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

Solution: Differentiating implicitly yields

$$\frac{d}{dx}[y^2 - x + 1] = \frac{d}{dx}[0]$$
$$\frac{d}{dx}[y^2] - \frac{d}{dx}[x] + \frac{d}{dx}[1] = \frac{d}{dx}[0]$$
$$2y\frac{dy}{dx} - 1 = 0$$
$$\frac{dy}{dx} = \frac{1}{2y}$$



At $(2, -1)$ we have $y = -1$, and at $(2, 1)$ we have $y = 1$, so the slopes of the two tangent lines to the curve at those points are

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-1}} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{\substack{x=2\\y=1}} = \frac{1}{2} \blacktriangleleft$$

(3) Parametric Differentiation

We may apply the next theorem to find $\frac{dy}{dx}$ when x and y are given functions of a parameter t.

Theorem:

Let $x = x(t),\ y = y(t)$ are parametric functions of t. Then, if we assume that $x = x(t)$ has an inverse we may consider y as a function of x, and we shall have

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \left(\frac{dy}{dt}\right) \cdot \left(\frac{dt}{dx}\right)$$

The second derivative for the parametric functions is

$$\frac{d^2y}{dx^2} = \left(\frac{dy'}{dt}\right) \cdot \left(\frac{dt}{dx}\right),$$

where $\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}$.

Example 1: If $y = \tan t$, $x = \sin^{-1} t$

We get $\frac{dy}{dt} = sec^2 t$, $\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}}$.

Then, $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{sec^2 t}{\frac{1}{\sqrt{1-t^2}}} = \sqrt{1-t^2}\,.\,sec^2 t$.

Example 2: If $x = \sin^{-1} t\,,\; y = \sqrt{1-t^2},\;\; find\; y''$.

Solution:

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-t^2}} \quad , \qquad \frac{dy}{dt} = \frac{-\cancel{2}t}{\cancel{2}\sqrt{1-t^2}}$$

$$\therefore y' = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{-t}{\sqrt{1-t^2}}}{\frac{1}{\sqrt{1-t^2}}} = -t$$

$$y'' = \frac{d}{dt}(y').\frac{dt}{dx} = \frac{d}{dt}(-t).\frac{dt}{dx}$$

$$= (-1).\sqrt{1-t^2} = -\sqrt{1-t^2}\,.$$

Example 3: If $x = e^t \cosh t\,,\; y = e^t \sinh t\,.$ Prove that $y'' = 0\,.$

Solution:

$$\frac{dx}{dt} = e^t \sinh t + e^t \cosh t\;,$$

$$\frac{dy}{dt} = e^t \cosh t + e^t \sinh t$$

$$\therefore y' = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{e^t \cosh t + e^t \sinh t}{e^t \sinh t + e^t \cosh t} = \frac{\cancel{e^t}\cancel{(\cosh t + \sinh t)}}{\cancel{e^t}\cancel{(\sinh t + \cosh t)}} = 1$$

And

$$y'' = \frac{d}{dt}(y').\frac{dt}{dx} = (0).\frac{1}{e^t \sinh t + e^t \cosh t} = 0\,.$$

Example 4: If $x = \sin t$, $y = \sin nt$, prove that

$$(1 + x^2)y'' - xy' + n^2y = 0$$

Solution:

We get $\frac{dy}{dt} = n\cos nt$, $\frac{dx}{dt} = \cos t$.

Then, $y' = \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{n\cos nt}{\cos t}$.

and $y'' = \frac{dy'}{dt} . \frac{dt}{dx}$

$$y'' = \frac{\cos t\ (-n^2\sin nt) - n\cos nt\ (-\sin t)}{\cos^2 t} . \frac{1}{\cos t}$$

$$= \frac{\cos t\ (-n^2 y) - n\cos nt\ (-x)}{\cos^2 t} . \frac{1}{\cos t}$$

$$= \left[-\frac{n^2 y}{\cos t} + \frac{y'}{\cos t}\ x\right] . \frac{1}{\cos t} = \left[\frac{xy' - n^2 y}{\cos^2 t}\right]$$

$$\left[\frac{xy' - n^2 y}{1 - \sin^2 t}\right] = \left[\frac{xy' - n^2 y}{1 - x^2}\right].$$

$$\therefore (1 + x^2)y'' = xy' - n^2 y$$

Thus $(1 + x^2)y'' - xy' + n^2y = 0$

Exercise

[1] Find $\frac{dy}{dx}$ for the following functions:

1) $y = \dfrac{5x+1}{2\sqrt{x}}$

2) $y = (x^2+1)^2\left(x+5+\dfrac{1}{x}\right)^3$

3) $y = \dfrac{1}{(x^2-1)(x^2+x+1)}$

4) $y = \sqrt[7]{x^2} - x^3 + 8$

5) $y = \dfrac{1+x-4\sqrt{x}}{x}$

6) $y = \sqrt{\tan^2 x + x\cos^3 x}$

7) $y = \dfrac{3}{\sqrt[3]{(5x^2-2x+1)^2}}$

8) $y = \left(\dfrac{1+x^2}{1-x^2}\right)^{17}$

9) $y = \sqrt{2x^2+1}\,(x^4-2)^6$

10) $y = \dfrac{\sqrt[4]{3-2x}}{\sqrt{5+\sin x}}$

11) $y = \left(\sqrt[4]{x^3-1} - 5x^2\right)^{\frac{3}{2}}$

12) $y = \left(\dfrac{\cos x}{1+\sin x}\right)^2$

13) $y = \sqrt{x}\,(1-\sqrt{x})^6$

14) $y = (x^2+2)^6\,(1-2x)^7$

15) $y = \dfrac{x\sin(2x)}{1-cos^2(3x)}$

16) $y = x^3\cos(x^2) - \cot(x^{-3})$

17) $y = tan(x^2 + sin\,x) + \sec\left(\dfrac{x}{x-1}\right)$

19) $y = sec^2\left(\sqrt{cos^2 x + 1}\right)$

20) $y = \frac{\tan(3x)}{(x+7)^4} + \sin\left(\frac{x}{\sqrt{x+1}}\right)$

21) $y=\sqrt{x^2+1}\tan^{-1}(x^3)+e^{x^2}\sec^{-1}(\sin x)$

22) $y=csc^4(cos^3x+1)$

23) $y=\left(\dfrac{2-\sin(3x)}{2+4\cos(2x)}\right)^8$

24) $y=\left(\dfrac{1+\tan^2(3x)}{\sec^2(3x)}\right)^6$

25) $y=x^3\sin^{-1}\sqrt{x}\;-2\csc^{-1}(x^2)$

26) $y=x\cot^{-1}\sqrt{1-x^2}$

27) $y=\left(x^5+\sin\sqrt{x}\right)^3.\tan^4 x$

[2] Find the first derivative for the following functions

1) $y=(\ln(\sin 3x))^{\cos x}$

2) $x^{\sin x}.y=(\sin x)^x$

3) $x^y=\sin(x^3+e^{7x^2})$

4) $y=\log_7\left(\frac{\sin x\cos x}{e^{2x}\,2^x}\right)^5$

5) $y=x[\log_5(x^2-2x)]^3$

6) $y=x^{\sin x}.\cot^4(\sqrt{1-x^2})$

7) $y=\frac{x^x(2-\cot^{-1}x)^{x^2}}{x^{\cos x}\,(1-2\ln x)^5}.$

[3] Find $\frac{dy}{dx}$ by implicit differentiation

1) $x\cos(2x+3y) = y\sin x$

2) $x^3 + \tan(xy) = 4^{\sin x}$

3) $\cot^2(xy^2 + y) = \sin(xy)$

4) $\ln(x+y) + x^2 + 3y^3 = 15$

[4] If $y = \cos(\ln x^2) + \sin(\ln x^2)$. Prove that $x^2y'' + xy' + 4y = 0$

[5] If $y = \cot(3\sin^{-1} x)$. Prove that

i. $y' = \frac{-3(1+y^2)}{\sqrt{1-x^2}}$.

ii. $(1-x^2)y'' - xy' - 18y\,(1+y^2) = 0$

[6] If $x = \tan\left(\frac{\sqrt{t}}{t^2+1}\right)$, $y = \sec\left(\frac{\sqrt{t}}{t^2+1}\right)$. Show that $y'' = y^{-3}$.

[7] If $x = 2\,(1-\cos\theta)$, $y = 2\sin^2\theta$. Show that $y'' = 1$.
