# Kinematics of motion

## 1. Introduction:

When we discuss the motion of particles, the principal functions like velocity and acceleration are vector functions of a scalar variable which is the time.

Let $\vec{v}$ be defined as a vector function of the scalar variable t, i.e.

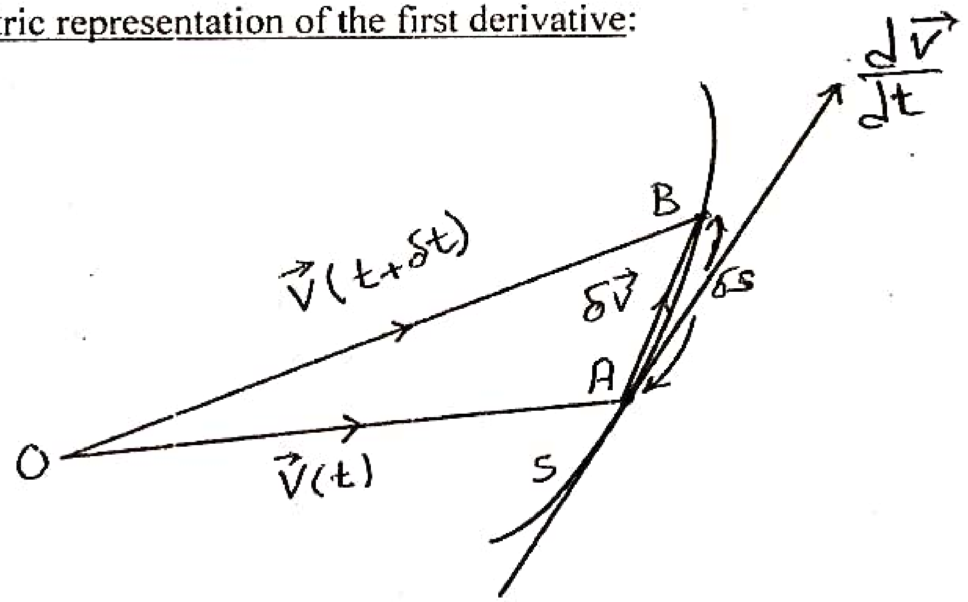
$$\vec{v} = \vec{v}(t)$$

The first derivative of this function with respect to the variable t is defined as the limit

$$\lim_{\delta t \to o} \frac{\vec{v}(t+\delta t) - \vec{v}(t)}{\delta t} = \lim_{\delta t \to o} \frac{\delta \vec{v}}{\delta t}$$

and is denoted by $\overrightarrow{\frac{dv}{dt}}$ and we say that $\vec{v}$ is differentiable if this limit exists.

## 2. Geometric representation of the first derivative:




Let $\overline{OA}$ represent the function $\vec{v}$ at any value t and let $\overline{OB}$ represent this vector function at the value $t+\delta t$.

From figure $\quad \delta\vec{v} = \overrightarrow{AB}$

i.e. $\quad \delta\vec{v} = \delta s \ \hat{A}B$

where s represents the are length on the curve drawn by $A.\hat{AB}$ is a unit vector along $\overline{AB}$

$$\therefore \frac{d\vec{v}}{dt} = \lim_{\delta t \to 0} \frac{\delta \vec{v}}{\delta t} = \lim_{\delta t \to 0} \frac{\delta s}{\delta t} \hat{A}B = \frac{ds}{dt} \vec{T}$$

where $\vec{T}$ is a unit vector along the tangent to the curve at A.

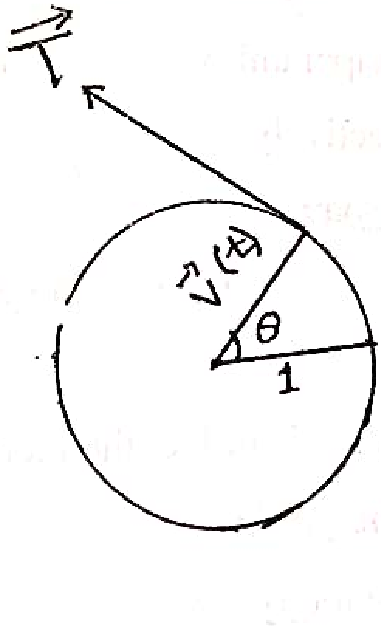
$\therefore \frac{d\vec{v}}{dt}$ is a vector whose magnitude is given by $\frac{ds}{dt} = \dot{s}$ and its direction is along the tangent to the curve.

3. First derivative of the unit vector:

When $|\vec{v}| = 1$ for all values of t, the curve drawn by A lies on the surface of a sphere whose centre is at O and its radius is unity (unit sphere). In the cases where $\vec{v}$ lies in a fixed plane for all values of t, this curve becomes a circle in that plane whose centre is at O and of unit radius (unit circle).

In this last case we get

$$\frac{d\vec{v}}{dt} = \frac{d\theta}{dt} \vec{T}$$

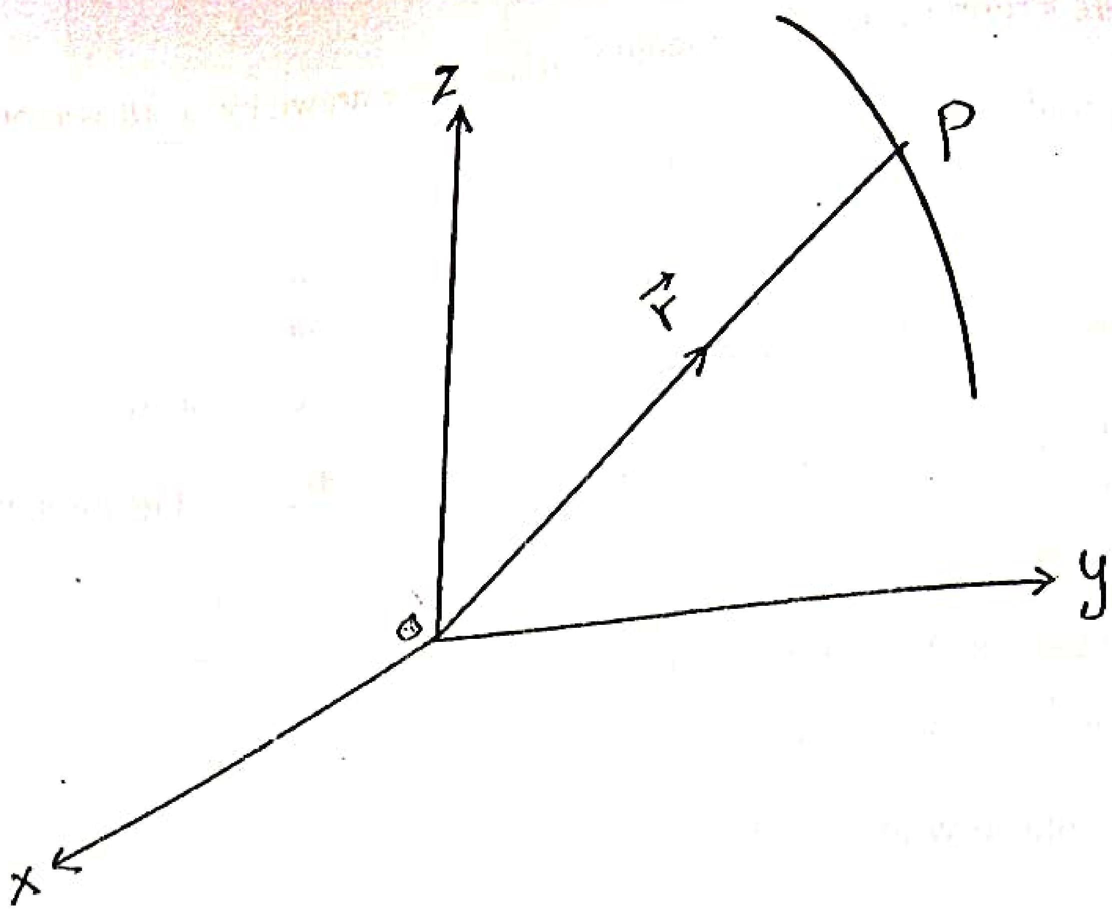



where $\theta$ is the angle between $\vec{v}$ and a fixed direction in the plane, and $\vec{T}$ in this case is the unit vector perpendicular to $\vec{v}$ in the direction of $\theta$ increasing.

4. The position Vector:

If O is the origin, P is a variable point, the vector $\overline{OP}$ is called the position vector of P relative to O and is denoted by $\vec{r}$.

Now, $\vec{r} = \vec{i}\, x + \vec{j}\, y + \vec{k}\, z$



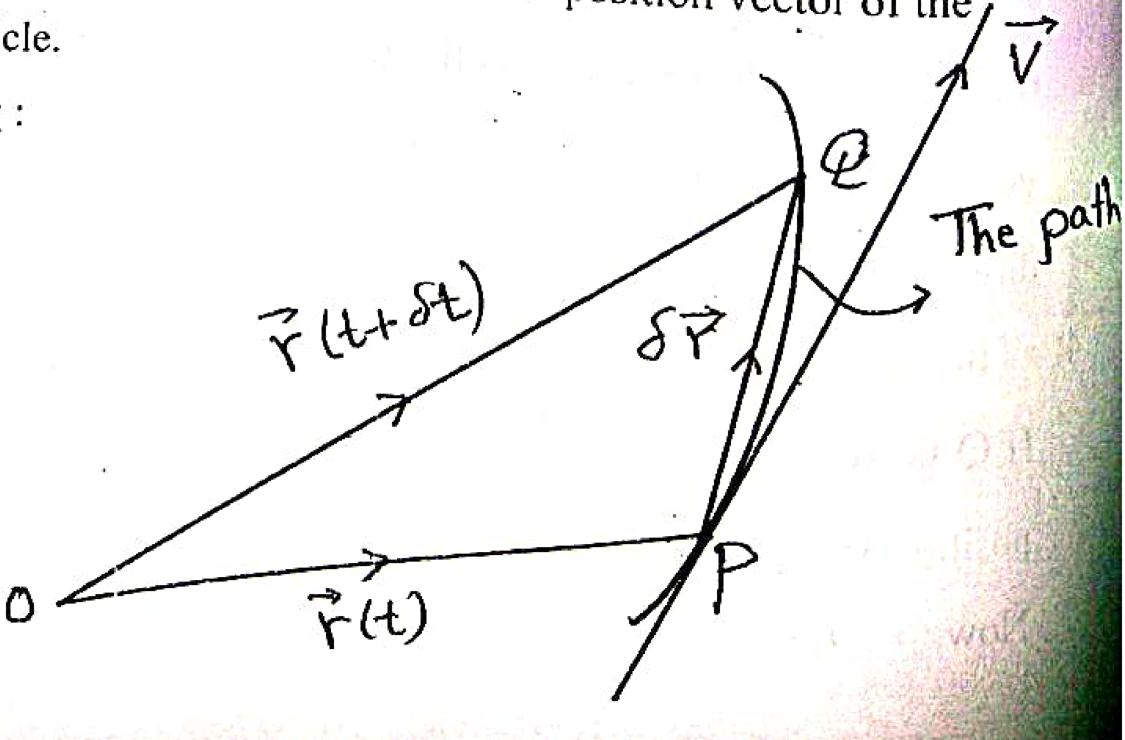

where x, y, z are the cartisian coordinates of the point P; $\bar{i}$ , $\bar{j}$ , $\bar{k}$ are the principal unit vectors along the perpendicular axes Ox, Oy, Oz respectively.

5. The path:

If a particle moves in space, it will draw a curve called the path of the particle.

This path is in fact the locus of the end of the position vector of the moving particle.

6. The velocity :



The velocity of a moving particle is defined as the rate of change of its position vector with respect to the time.

i.e. $$\vec{v} = \lim_{\delta t \to 0} = \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t}$$

$$\therefore \quad \vec{v} = \frac{d\vec{r}}{dt}$$

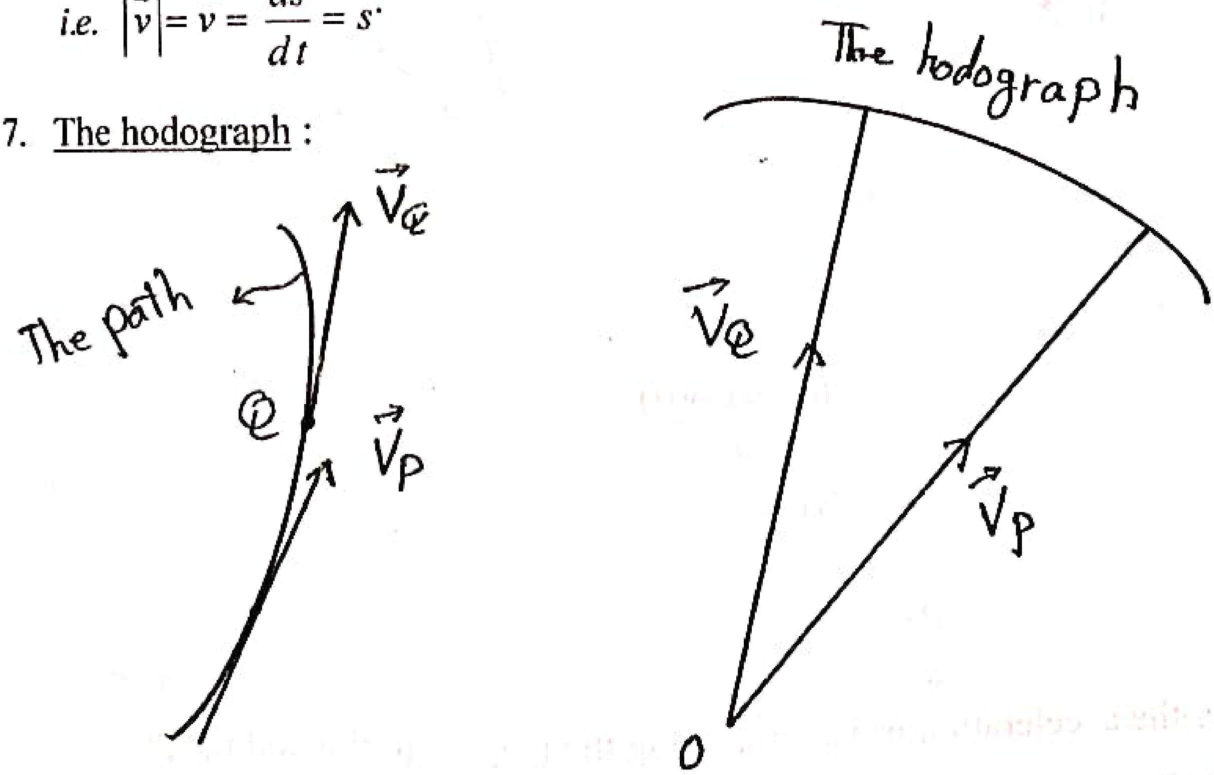
and the velocity vector $\vec{v}$ at any point P is along the tangent to the curve drawn by the vector $\vec{r}$, i.e. $\vec{v}$ is along the tangent to the path at P.

From § 2 we find that

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{ds}{dt}\vec{T}$$

i.e. $$|\vec{v}| = v = \frac{ds}{dt} = s^{\cdot}$$

7. <u>The hodograph</u> :

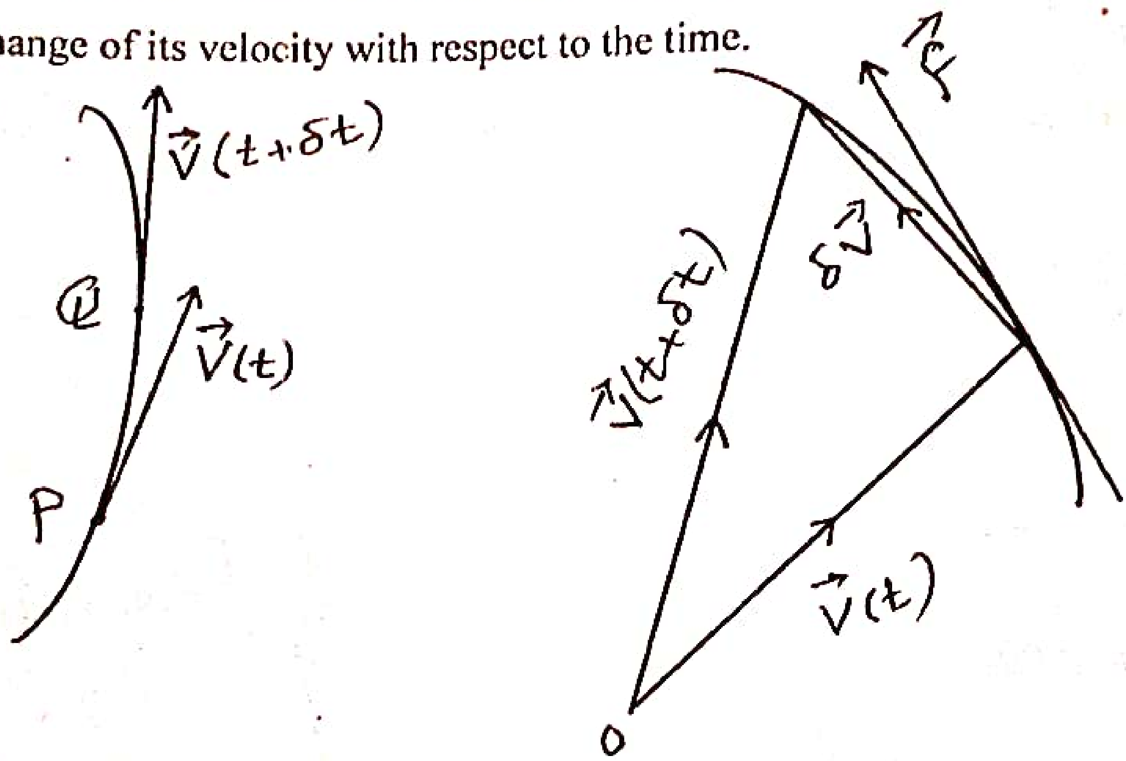


If we draw from a point O, vectors which represent the velocities of a moving particle at different moments, the ends of these vectors lie on a curve called the hodograph.

If the motion is in a plane, i.e. the path is a plane curve, the hodograph will be also a plane curve.

8. <u>The acceleration</u>:

The acceleration of a moving particle is defined as the rate of change of its velocity with respect to the time.




i.e. $\bar{f} = \lim\limits_{\delta t \to 0} \dfrac{\bar{v}(t+\delta t) - \bar{v}(t)}{\delta t}$

$= \lim\limits_{\delta t \to 0} \dfrac{\delta \bar{v}}{\delta t}$

$\therefore \quad \bar{f} = \dfrac{d\bar{v}}{dt} = \dfrac{d^2 \bar{r}}{dt^2}$

$\therefore$ the acceleration is a vector along the tangent to the hodograph.

9. <u>Components of velocity and acceleration in the different kinds of motion</u>:

Let the position vector of a moving particle be

$$\vec{r} = \vec{i}\,x + \vec{j}\,y + \vec{k}\,z$$

The velocity vector becomes

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{i}\,\dot{x} + \vec{j}\,\dot{y} + \vec{k}\,\dot{z} \;,$$

and the acceleration vector becomes

$$\vec{f} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \vec{i}\ddot{x} + \vec{j}\ddot{y} + \vec{k}\ddot{z}$$

i.e. the components of velocity are

$$v_x = x^{\bullet} \quad , \quad v_y = y^{\bullet} \quad , \quad v_z = z^{\bullet}$$

and the components of acceleration are

$$f_x = x^{\bullet\bullet} \quad , \quad f_y = y^{\bullet\bullet} \quad , \quad f_z = z^{\bullet\bullet}$$

This case represents the motion in space i.e. in three dimensions.

In the case of the motion in a plane i.e. in two dimensions we have

$\vec{v} = \vec{i}\,x^{\bullet} + \vec{j}\,y^{\bullet}$

i.e. the velocity components are $x^{\bullet}, y^{\bullet}$ and $\vec{f} = \vec{i}\,x^{\bullet\bullet} + \vec{j}\,y^{\bullet\bullet}$

i.e. the acceleration components are $x^{\bullet\bullet}, y^{\bullet\bullet}$.

In the case of the motion in a straight line i.e. in one dimension we have

$$\vec{v} = \vec{i}\,x^{\bullet}$$

$$\therefore\; v = x^{\bullet} = \frac{dx}{dt}$$

also

$$\vec{f} = \vec{i}\,x^{\bullet\bullet}$$

$$\therefore\; f = x^{\bullet\bullet} = \frac{d^2x}{dt^2}\,.$$

We can find f by using

$$f = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}\,.$$

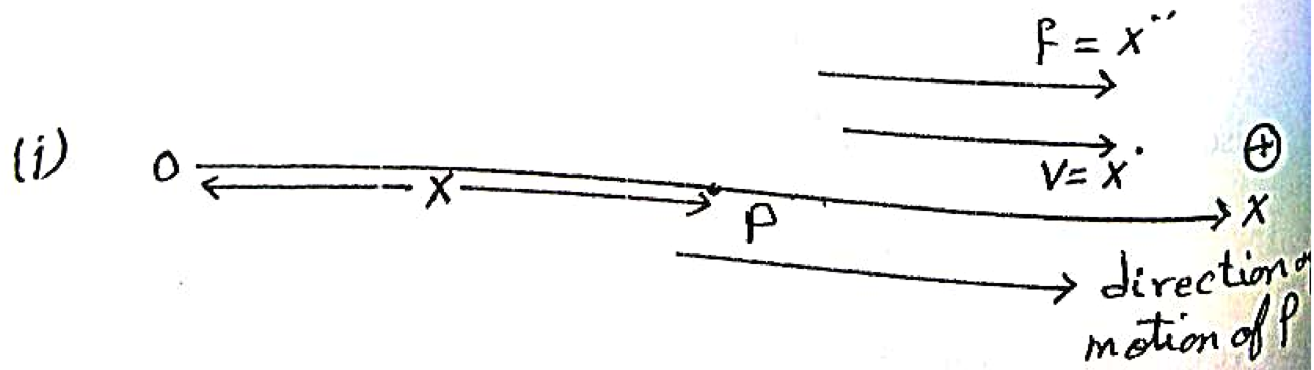
The form $\frac{dv}{dt}$ is used to find the acceleration f if v is known as a function of time t while the form $v\frac{dv}{dx}$ or $\frac{1}{2}\frac{dv^2}{dx}$ is used to find f if v is known as a function of x.

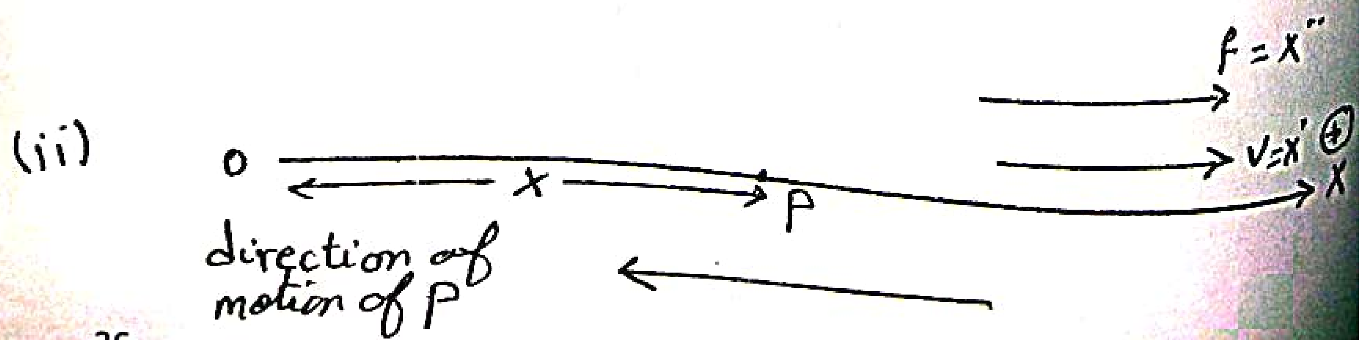
10. The sign of both velocity and acceleration:

If x is an increasing function of $t$, $\frac{dx}{dt}$ is positive but if x is a decreasing function of $t$, $\frac{dx}{dt}$ is negative and so if a particle moves in the positive direction of the x-axis, i.e. in the direction of x increasing, its velocity is positive and if the particle moves in the negative direction of the x-axis, its velocity is negative.

In both cases $x^{\bullet}$ represents the velocity of the particle in the positive direction of the x-axis, i.e. in the direction of x increasing. Similarly, $x^{\bullet\bullet}$ represents always the acceleration of the particle in the direction of x increasing. This is shown in the following figure if the motion of the particle P is far from O or towards O.

In both cases, $x^{\bullet}$, $x^{\bullet\bullet}$ are the velocity and acceleration of the particle in the direction of x increasing.



## 11. Motion in a straight line with a given variable acceleration:

If a particle moves in a straight line with a known acceleration f (f is known if the force acting on the particle is known) we must calculate the velocity of the particle v and the distance x by using integration. To determine the arbitrary constants which appear in the process of integration we have to use the initial conditions related to each case. If the acceleration of the moving particle is given, it will be known as a function of time t or a function of velocity v or as a function of the distance x.

First case:

If the acceleration is known as a function of time.

Let $f = \phi(t)$.

writing $\frac{dv}{dt}$ instead of f we obtain

$$\frac{dv}{dt} = \phi(t).$$

Separating the variables

$$\therefore \quad dv = \phi(t)\, dt.$$

Integrating both sides we get

$$v = \int \phi(t)dt = \psi(t) + c_1$$

where $c_1$ is an arbitrary constant which could be determined if the velocity of the particle is known at a certain moment.

To determine the distance described in terms of the time t we write $\frac{dx}{dt}$ instead of v and so we have

$$\frac{dx}{dt} = \psi(t) + c_1$$

$$\therefore x = \int \psi(t)\, dt + c_1 t + c_2$$

where $c_2$ is a constant of integration which could be determined if x is known at a certain moment.

Second case:

If the acceleration is known as a function of velocity.

Assume $$f = \phi(v). \tag{1}$$

In this case we can write $\frac{dv}{dt}$ or $v\frac{dv}{dx}$ instead of f and it is clear that we use the first form when it is required to find the relation between v and t. The second form is used when it is required to find the relation between v and x.

$$\therefore \quad \frac{dv}{dt} = \phi(v).$$

separating the variables we get

$$dt = \frac{dv}{\phi(v)},$$

$$\therefore \quad t = \int \frac{dv}{\phi(v)} + c.$$

Also, by writing (1) in the form

$$v\frac{dv}{dx} = \phi(v),$$

again by separating the variables, we obtain

$$dx = \frac{v\, dv}{\phi(v)},$$

$$\therefore \quad x = \int \frac{v\, dv}{\phi(v)} + c'$$

The constants of integration c, c' could be determined from the initial conditions.

Third case:

If the acceleration is known as a function of the distance x.

Assume $f = \phi(x)$

In this case we must replace the acceleration f by $v\dfrac{dv}{dx}$ or equivalently $\dfrac{1}{2}\dfrac{dv^2}{dx}$ and after separating the variables we get

$$v\,dv = \phi(x)\,dx\,.$$

Integrating both sides we obtain

$$\frac{1}{2}v^2 = \int \phi(x)\,dx + c_1$$

where $c_1$ is an arbitrary constant which could be determined if the velocity is known at a certain position, and we can write this equation in the form

$$v^2 = \psi(x)$$

where $\psi(x) = 2\int \phi(x)\,dx + 2c_1\,.$

$$\therefore\ v = \frac{dx}{dt} = \pm\sqrt{\psi(x)}\,.$$

The positive sign is taken if the particle is moving such that x increases with time t, while the negative sign is considered when x is a decreasing function of time t.

Separating the variables x , t and integrating we get

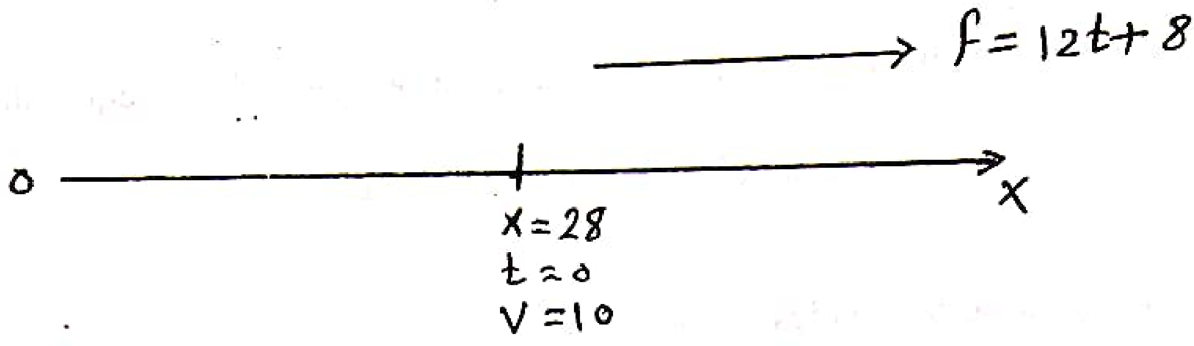
$$t = \pm\int \frac{dx}{\sqrt{\psi(x)}} + c_2\,,$$

where $c_2$ is a constant which could be evaluated if x is known at a certain moment.

12. Examples:

1. A particle moves in the positive direction of the x-axis with an acceleration of magnitude $12t+8$ $ft./\sec^2$ .If the particle starts motion with velocity 10 $ft./\sec$. from a point at a distance 28 $ft.$ from the origin

O, find the velocity of the particle after 5 sec. When and where the velocity of the particle becomes 50 $ft./\sec.$ and then find its acceleration.




$$f = 12t + 8 \qquad (1)$$

writing $\frac{dv}{dt}$ instead of f and separating the variables we get

$$dv = (12t + 8)\, dt.$$

Integrating both sides

$$\therefore v = 6t^2 + 8t + c_1$$

where the constant of integration $c_1$ could be determined from the initial condition $v = 50\ ft./\sec.$ when $t = 0$.

$$\therefore c_1 = 10$$

$$\therefore v = 6t^2 + 8t + 10 \qquad (2)$$

writing $\frac{dx}{dt}$ instead of v and separating the variables we get

$$\therefore dx = (6t^2 + 8t + 10)dt$$

Integrating both sides

$$\therefore x = 2t^3 + 4t^2 + 10t + c_2$$

$c_2$ could be determined from the condition x = 28

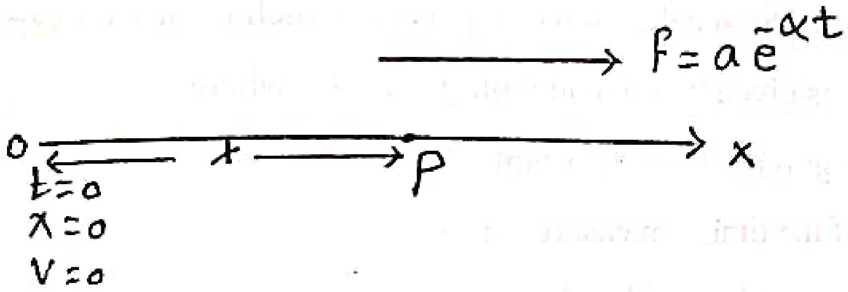
when $t = 0$. $\therefore c_2 = 28$,

i.e. $\therefore x = 2t^3 + 4t^2 + 10t + 28 \qquad (3)$

Put t = 5 sec. in ( 2 ) and ( 3 ), we get $v = 200\ ft./\sec.$ and $x = 428\ ft.$

Put $v=50\ ft./sec.$ in ( 2 ), we get $\therefore 0=6t^2+8t-40 \Rightarrow t=2\sec.$ and also put $t=2\sec.$ in ( 3 ) and ( 1 ), we get $x=80\ ft.\ and\ f=32\ ft./\sec^2$.

2. A particle starts to move from rest in a straight line Ox from the origin O. If the acceleration at time t measured from O is equal to $ae^{-\alpha t}\ ft./\sec^2$ where $a\ and\ \alpha$ are constants. Find its velocity and displacement at any time.




$$f = ae^{-\alpha t} \quad ft./\sec^2 \qquad (1)$$

writing $\frac{dv}{dt}$ instead of f and separating the variables we get

$$dv = ae^{-\alpha t}dt$$

Integrating both sides

$$v = \frac{a}{-\alpha}e^{-\alpha t} + c_1 \qquad (1)$$

where the constant of integration $c_1$ could be determined from the initial condition $v=0$ when $t=0$.

$$\therefore\ c_1 = \frac{a}{\alpha}$$

$$\therefore\ v = \frac{a}{\alpha}\left(1-e^{-\alpha t}\right) \qquad (2)$$

writing $\frac{dx}{dt}$ instead of v and separating the variables we get

$$\therefore dx = \frac{a}{\alpha}\left(1-e^{-\alpha t}\right)dt$$

Integrating both sides

$$\therefore\ x=\frac{a}{\alpha}\left(t+\frac{1}{\alpha}e^{-\alpha t}\right)+c_2$$

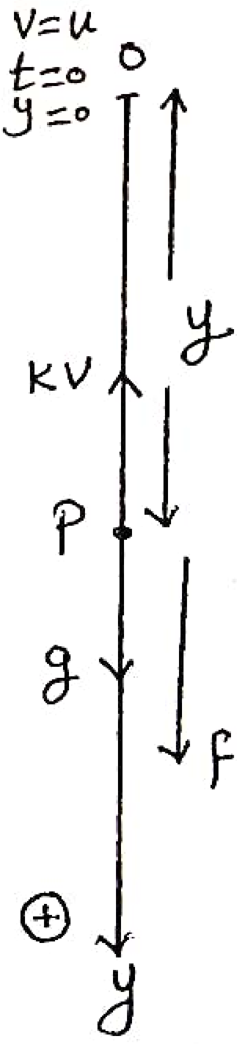
$c_2$ could be determined from the condition x = 0

when $t=0$. $\quad \therefore\ c_2=\frac{-a}{\alpha^2}$,

i.e. $$\therefore\ x=\frac{a}{\alpha}\left(t+\frac{1}{\alpha}e^{-\alpha t}-\frac{1}{\alpha}\right) \qquad (3)$$

3.The acceleration of a particle which is moving vertically downwards is given by the equation $f=g-kv$ where g is the acceleration of gravity, k is a constant.

If the time is measured from the moment at which the particle was descending with velocity u and the distance y is measured from the same position. Find v, y as functions of the time t. Find also the relation between v and y.




$$\therefore\ f=g-kv$$

$$\therefore\ \frac{dv}{dt}=g-kv$$

By separating the variables

$$\therefore\ dt=\frac{dv}{g-kv}$$

Integrating both sides

$$\therefore\ t=-\frac{1}{k}\ln(g-kv)+C_1$$

From initial conditions,

v = u when t = 0

$$\therefore\ C_1=\frac{1}{k}\ln(g-ku)$$

$$\therefore\ t=\frac{1}{k}\ln\frac{g-ku}{g-kv} \qquad (1)$$

i.e. $$\frac{g-ku}{g-kv}=e^{kt} \qquad (2)$$

$$\therefore\ g-kv=(g-ku)\,e^{-kt}$$

$$\therefore\ v=\frac{g}{k}-(\frac{g}{k}-u)\,e^{-kt}$$

writing $\frac{dy}{dt}$ instead of v and separating the variables we get

$$dy=\left[\frac{g}{k}-(\frac{g}{k}-u)e^{-kt}\right]dt.$$

Integrating both sides

$$\therefore\quad y=\frac{g}{k}t+(\frac{g}{k^2}-\frac{u}{k})\,e^{-kt}+C_2.$$

From the condition y = 0 when t = 0 .

$$C_2=-(\frac{g}{k^2}-\frac{u}{k}),$$

$$\therefore\ y=\frac{g}{k}t+(\frac{g}{k^2}-\frac{u}{k})(e^{-kt}-1).$$

To find the relation between v and y we can find by using the relations (1),(2)

$$y=\frac{g}{k^2}\ln\frac{g-ku}{g-kv}+(\frac{g}{k^2}-\frac{u}{k})(\frac{g-kv}{g-ku}-1).$$

$$=\frac{g}{k^2}\ln\frac{g-ku}{g-kv}+\frac{1}{k}(u-v). \qquad (3)$$

This last relation could also be obtained as follows:

Writing $v\frac{dv}{dy}$ instead of f

$$\therefore\quad v\frac{dv}{dy}=g-kv.$$

Separating the variables we get

$$dy=\frac{v\,dv}{g-kv}$$

$$=-k^{-1}(\frac{g-kv-g}{g-kv})\,dv$$

$$=-k^{-1}(1-\frac{g}{g-kv})\,dv$$

$$\therefore\quad y=-\frac{1}{k}(v+\frac{g}{k}\ln(g-kv))+C_3.$$

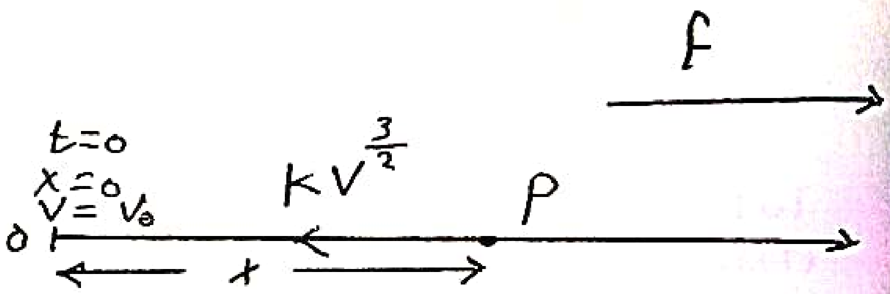
From the condition v = u at y = 0 ,

$$\therefore \; C_3 = \frac{1}{k}\left(u + \frac{g}{k}\ln(g - ku)\right)$$

$$\therefore \quad y = \frac{1}{k}(u - v) + \frac{g}{k^2}\ln\frac{g - ku}{g - kv} \;,$$

which is the same result (3).

4. A particle is projected from the origin in the positive direction of the x-axis with initial velocity $v_0$ and moves with a retardation of magnitude $kv^{3/2}$ at any moment, where $v$ is the velocity of the particle at this moment and $k$ is a constant. Find its velocity and displacement at any time.




$$\therefore \; f = -kv^{3/2}$$

$$\therefore \; \frac{dv}{dt} = -kv^{3/2}$$

By separating the variables

$$\therefore \; v^{-3/2}\,dv = -k\,dt$$

Integrating both sides

$$\therefore \; -2v^{-1/2} = -kt + C_1$$

From initial conditions,

$v = v_0$ when $t = 0$

$$\therefore \; C_1 = \frac{-2}{\sqrt{v_0}}$$

$$\therefore \; \frac{2}{\sqrt{v}} = kt + \frac{2}{\sqrt{v_0}} = \frac{2 + k\sqrt{v_0}\,t}{\sqrt{v_0}}$$

i.e. $\therefore \; \sqrt{v} = \dfrac{2\sqrt{v_0}}{2 + k\sqrt{v_0}\,t}$

$$\therefore \; v = \frac{4v_0}{\left(2 + k\sqrt{v_0}\,t\right)^2}$$

writing $\frac{dx}{dt}$ instead of v and separating the variables we get

$$\therefore \; dx = \frac{4v_0}{\left(2+k\sqrt{v_0}\,t\right)^2}dt = 4v_0\left(2+k\sqrt{v_0}\,t\right)^{-2}dt$$

Integrating both sides

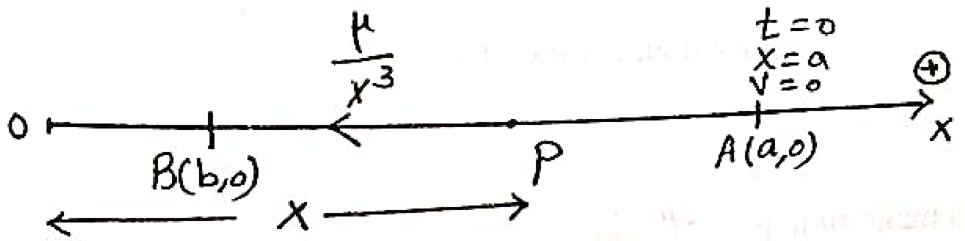
$$\therefore \; x = \frac{-4v_0}{k\sqrt{v_0}}\left(2+k\sqrt{v_0}\,t\right)^{-1} + c_2$$

From the condition x= 0 when t = 0 .

$$C_2 = \frac{2}{k}\sqrt{v_0},$$

$$\therefore \; x = \frac{2\sqrt{v_0}\,t}{\left(2+k\sqrt{v_0}\,t\right)}$$

5. A point moves in a straight line towards a centre of force 0 starting from rest at a distance a from 0. If the acceleration which is directed towards 0 varies inversely as the cube of the distance from 0, find the time of reaching a point distant b from the centre of force and its velocity then.




Let the central acceleration be equal to $\frac{\mu}{x^3}$ where μ is a constant.

$$\therefore \; f = -\frac{\mu}{x^3}$$

$$\therefore \; v\frac{dv}{dx} = -\frac{\mu}{x^3}$$

$$\text{i.e.} \quad v\,dv = -\frac{\mu}{x^3}dx$$

Integrating both sides

$$\therefore \; \frac{v^2}{2} = +\frac{\mu}{2x^2} + C_1$$

$$\because \; at \; x = a \;,\; v = 0 \;,\quad \therefore \; C_1 = -\frac{\mu}{2a^2}$$

$$\therefore \; v^2 = \frac{\mu}{x^2} - \frac{\mu}{a^2} = \frac{\mu(a^2 - x^2)}{a^2 x^2} \qquad (1)$$

$$\therefore \; v = \frac{dx}{dt} = \pm \frac{\sqrt{\mu(a^2 - x^2)}}{a x}.$$

Since the point is moving towards 0, we have to take the negative sign.

$$\therefore \; \frac{dx}{dt} = -\frac{\sqrt{\mu}}{a} \frac{\sqrt{a^2 - x^2}}{x}.$$

Separating the variables

$$\therefore \; dt = -\frac{a}{\sqrt{\mu}} \frac{x \, dx}{\sqrt{a^2 - x^2}}.$$

Integrating both sides

$$\therefore \; t = -\frac{a}{\sqrt{\mu}} \int \frac{x \, dx}{\sqrt{a^2 - x^2}} + C_2$$

$$= \frac{a}{\sqrt{\mu}} \sqrt{a^2 - x^2} + C_2 \, .$$

$\because \; at \; t = 0 \;, x = a \;, \quad \therefore \; C_2 = 0$

i.e $\; t = \frac{a}{\sqrt{\mu}} \sqrt{a^2 - x^2}$ .

At x = b , the required time is now given by $t = \frac{a}{\sqrt{\mu}} \sqrt{a^2 - b^2}$ .

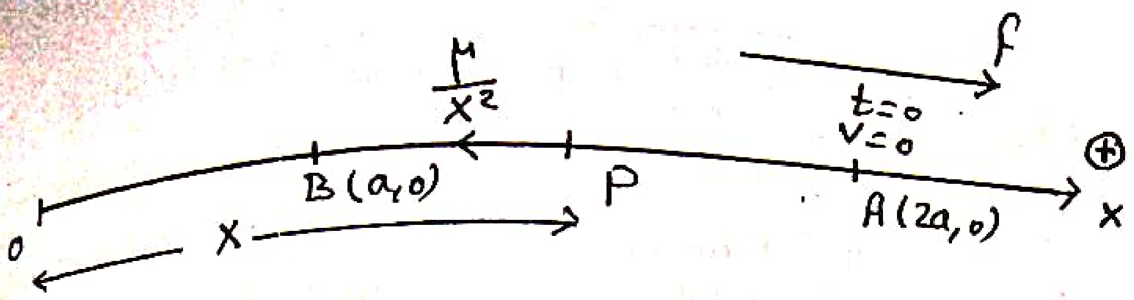
From (1), the velocity then is given by

$v^2 = \frac{\mu(a^2 - b^2)}{a^2 b^2}$

i.e. its magnitude is $\frac{\sqrt{\mu}}{ab} \sqrt{a^2 - b^2}$ .

6. A particle moves in a straight line 0x, with an acceleration which is always directed towards 0 and varies inversely as the square of its distance from 0. If initially the particle were at rest at the point A(2a , 0)

prove that the ratio between the times of motion from A to the point B (a, 0) and from B to 0 is equal to $\frac{\pi + 2}{\pi - 2}$.




Let the central acceleration be equal to $\frac{\mu}{x^2}$.

$$\therefore \quad f = -\frac{\mu}{x^2}$$

writing $v\frac{dv}{dx}$ instead of f and separating the variables we get

$$v\,dv = -\frac{\mu}{x^2}\,dx\ .$$

Integrating both sides we have

$$\frac{1}{2}v^2 = \frac{\mu}{x} + C_1 .$$

$$\because \quad v = 0 \quad at \quad x = 2a\ , \qquad \therefore C_1 = -\frac{\mu}{2a}$$

$$\therefore \quad v^2 = 2\mu\left(\frac{1}{x} - \frac{1}{2a}\right)$$

$$= 2\mu\left(\frac{2a-x}{2ax}\right) = \frac{\mu(2a-x)}{ax}\ .$$

$$\therefore \quad v = \frac{dx}{dt} = -\sqrt{\frac{\mu(2a-x)}{ax}}\ .$$

The negative sign is taken because the motion of P is towards 0, i.e. in the direction of x decreasing.

Separating the variables and integrating

$$\therefore \quad t = -\sqrt{\frac{a}{\mu}}\int\sqrt{\frac{x}{2a-x}}\,dx + C_2\ .$$

To integrate the right-hand side, we use the substitution $x = 2a\sin^2\theta$

$$\therefore \quad dx = 4a\sin\theta\cos\theta\,d\theta$$

$$\therefore \int \sqrt{\frac{x}{2a-x}}\,dx = \int \sqrt{\frac{2a\sin^2\theta}{2a\cos^2\theta}}\,.\,4a\sin\theta\,\cos\theta\,d\theta$$

$$= 4a\int \sin^2\theta\,d\theta$$

$$= 2a\int (1-\cos 2\theta)\,d\theta$$

$$= 2a(\theta - \frac{1}{2}\sin 2\theta)$$

$$= 2a(\theta - \sin\theta\cos\theta)$$

$$= 2a\left[\sin^{-1}\sqrt{\frac{x}{2a}} - \sqrt{\frac{x}{2a}}\sqrt{\frac{2a-x}{2a}}\right]$$

$$= 2a\sin^{-1}\sqrt{\frac{x}{2a}} - \sqrt{x(2a-x)}$$

$$\therefore t = C_2 - \sqrt{\frac{a}{\mu}}\left[2a\sin^{-1}\sqrt{\frac{x}{2a}} - \sqrt{x(2a-x)}\right]$$

If the time is measured from the point A, we have

$t=0$ at $x=2a$, $\therefore C_2 = \pi a\sqrt{\frac{a}{\mu}}$,

i.e. $t = \sqrt{\frac{a}{\mu}}\left[\pi a - 2a\sin^{-1}\sqrt{\frac{x}{2a}} + \sqrt{x(2a-x)}\right]$

At B, x = a

$$\therefore\ t_1 = t_{A\to B} = \sqrt{\frac{a}{\mu}}\left[\pi a - \frac{\pi a}{2} + a\right]$$

$$= \frac{a}{2}\sqrt{\frac{a}{\mu}}(\pi+2).$$

At O , x = 0

$$\therefore\ t_2 = t_{A\to O} = \pi a\sqrt{\frac{a}{\mu}},$$

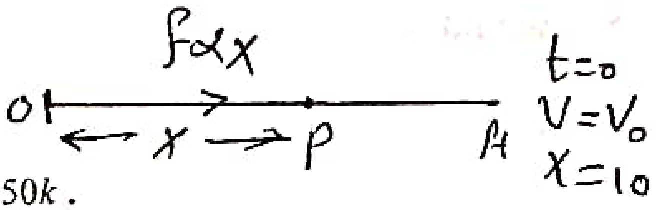
i.e. $$t_{B\to O} = t_2 - t_1 = \frac{a}{2}\sqrt{\frac{a}{\mu}}(\pi - 2)$$

$$\therefore\ \frac{t_{A\to B}}{t_{B\to O}} = \frac{\pi+2}{\pi-2}.$$

7. A particle moves in a straight line under the action of a central repulsive force from a fixed point O on that line and of magnitude proportional to the distance from it. The particle is projected towards O from the point A at the distance 10 ft. from O with velocity $v_0$ just enough for the particle to reach O. Prove that if the particle is projected again from A towards O with velocity $\frac{1}{2}v_0$. It will come to rest instantaneously at a point distance $5\sqrt{3}$ ft. from O.

Since $f \propto x$, then $f = kx$, where $k$ is unknown constant

$$v\frac{dv}{dx} = kx \Rightarrow \frac{v^2}{2} = k\frac{x^2}{2} + c_1$$




Since $v = v_0$ at $x = 10$, we get $c_1 = \frac{v_0^2}{2} - 50k$.

So, we have $\frac{v^2}{2} = k\frac{x^2}{2} + \frac{v_0^2}{2} - 50k$. Now since $v = 0$ at $x = 0$, then $k = \frac{v_0^2}{100}$

Then $f = \frac{v_0^2}{100}x$.

Second case:


$$v\frac{dv}{dx} = \frac{v_0^2}{100}x \Rightarrow \frac{v^2}{2} = \frac{v_0^2}{100}\frac{x^2}{2} + c_2$$

Since $v = \frac{1}{2}v_0$ at $x = 10$, then $\frac{1}{8}v_0^2 = \frac{v_0^2}{2} + c_2 \Rightarrow c_2 = -\frac{3v_0^2}{8}$.

$\frac{v^2}{2} = \frac{v_0^2}{100}\frac{x^2}{2} + \frac{v_0^2}{2} - \frac{3}{8}v_0^2$, we have to find $x$ when $v = 0$

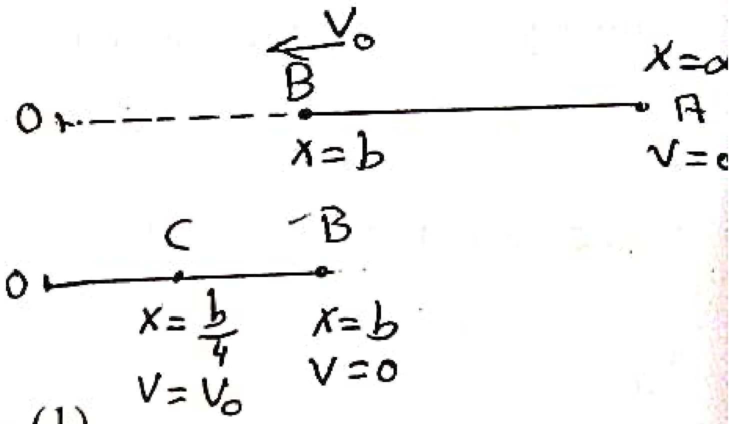
$x^2 = 75 \Rightarrow x = 5\sqrt{3}$.

8. A particle is attracted by a force to a fixed point varying inversely as the $n^{th}$ power of the distance $(n \neq 1)$.
If the velocity acquired by it in falling from rest from an infinite distance to a distance $b$ from the center is equal to the velocity that would be acquired by it in falling from rest at a distance $b$ to a distance $\frac{b}{4}$, show that $n = \frac{3}{2}$.

Motion from A to B

Since $f \propto \frac{1}{x^n}$, then $f = -\frac{k}{x^n}$, where $k$ is unknown constant

$$v\frac{dv}{dx} = -\frac{k}{x^n} \Rightarrow \frac{v^2}{2} = -k\frac{x^{-n+1}}{-n+1} + c_1$$




Since $v = 0$ at $x = \infty$, we get $c_1 = 0$.

So, we have $v^2 = 2k\frac{x^{1-n}}{-1+n}$

at B $v = v_0$ at $x = b$, then $\frac{2k}{n-1}b^{1-n} = v_0^2$ --- (1)

Motion from B to C:

$$v\frac{dv}{dx} = -\frac{k}{x^n} \Rightarrow \frac{v^2}{2} = -k\frac{x^{-n+1}}{-n+1} + c_2$$

Since $v = v_0$ at $x = b$, then $c_2 = \frac{kb^{1-n}}{1-n}$.

$v^2 = -2k\frac{x^{1-n}}{1-n} + 2k\frac{b^{1-n}}{1-n}$, so at C: $v = v_0$, $x = \frac{b}{4}$, then

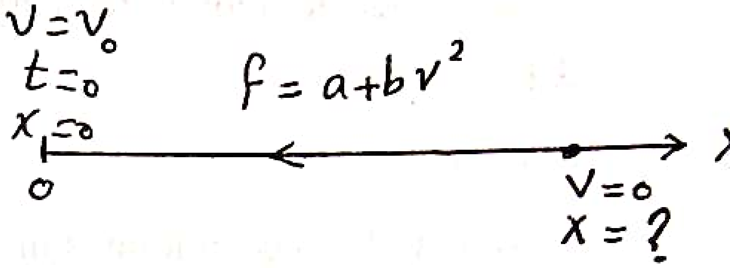
$$\frac{2k}{1-n}b^{1-n} - 2k\frac{b^{1-n}}{4^{1-n}(1-n)} = v_0^2 \quad \text{------------------- (2)}$$

From (1) and (2), we have $\frac{2k}{1-n}b^{1-n} - 2k\frac{b^{1-n}}{4^{1-n}(1-n)} = \frac{-2kb^{1-n}}{1-n}$

Then $b^{1-n} = \frac{b^{1-n}}{4^{1-n}} - b^{-n} \Rightarrow 2b^{1-n} = b^{1-n}2^{2n-2} \Rightarrow 2 = 2^{2n-2}$

$$\therefore \ 1 = 2n - 2 \Rightarrow n = \frac{3}{2}$$

9. The friction force acting upon a train which is moving in a straight line with velocity $v$ ft./sec. causes a retardation of magnitude $a+bv^2$ where $a \,\&\, b$ are constants. If the engine is stopped when the train was moving with velocity $v_0$ ft./sec., prove that it moves a distance $\frac{1}{2b}\ln\left(1+\frac{bv_0^{\,2}}{a}\right)$ before it come to rest.




$\because\ f=-(a+bv^2)$, then

$$v\frac{dv}{dx}=-(a+bv^2)$$

$$\int\frac{vdv}{a+bv^2}=-\int dx$$

$$\frac{1}{2b}\ln(a+bv^2)=-x+c_1$$

At $x=0,\ v=v_0 \Rightarrow c_1=\frac{1}{2b}\ln(a+bv_0^{\,2})$

So $\frac{1}{2b}\ln(a+bv^2)=-x+\frac{1}{2b}\ln(a+bv_0^{\,2})$ --------------- (1)

To find $x$ when $v=0$, from (1)

$$x=\frac{1}{2b}\ln\left(\frac{a+bv^2}{a}\right)=\frac{1}{2b}\ln\left(1+\frac{bv_0^{\,2}}{a}\right)$$

## Exercises

1- A particle moves on a straight line such that its distance x ft. measured from a fixed point 0 on the line is given in terms of the time t secs by the equation $x = t^3 - 4t^2 + 5t - 2$.

a- Where is the particle when its velocity vanishes.

b- Find the position of the particle when its acceleration vanishes.

c- Find the acceleration of the particle when its velocity is equal to 8 ft./ sec.

2- If $x^2 = t^2 + 6t + 5$

prove that the acceleration is inversely proportional to the cube of the distance .

3- If the relation between distance x and time t for a particle which moves in a straight line is given by

$$x = 3\cos 2t + 4\sin 2t$$

prove that

$$v^2 = 4(25 - x^2)\ ; \quad f = -4x.$$

4- A particle moves in the positive direction of the x-axis with acceleration $(4t+3)\,cm/sec^2$. The particle starts motion from a point distant 9 cms. From the origin 0 and after 3 secs its velocity becomes 22 cms/sec. Find its velocity and distances from O after time t secs. When and where the velocity of the particle vanishes.

5- A particle starts motion from rest at the origin in the positive direction of the x-axis. If the acceleration of the particle after time t is given by

$$f = k\,e^{-nt}$$

where k , n are constants. Prove that the acceleration f and the velocity v after the particle has moved a distance x satisfy the relation $nx = (nt-1)v + ft.$

6- The distance x of a moving particle on a straight line from a fixed point O on that line is related to the time t by the equation

$$x = Ae^{nt} + Be^{-nt}$$

where A , B , n are constants. Prove that

$$v^2 = n^2(x^2 - 4AB) \quad , \quad f = n^2 x.$$

7- For the motion of a particle in a straight line, if $v^2$ is quadratic in x, prove that the acceleration of the particle at any position varies as the distance between this position and a fixed point on the line.

8- A particle starts motion from rest in a straight line such that its acceleration after time t secs is equal to $\left[3\sin t + (t+1)^{-2}\right] ft./sec^2$. Find the distance described after time t secs from the beginning.

9- A particle moves in a straight line such that its distance x ft. from a fixed point O on the line is given by the equation

$$x = 2t^3 - 3t^2 - 12t + 18$$

where t is the time in secs. Find the position of the particle and its acceleration when its velocity vanishes.

10- A ladder of length 30 ft. slides in a vertical plane such that its upper end touches a vertical wall and its lower end touches a horizontal ground. Find the velocity of the upper end when the lower one is at a distance of 18 ft. from the wall and is moving with velocity 8 ft./sec.

11- A particle starts to move from rest in a straight line with an acceleration which increases with constant time rate from $1 ft./sec^2$. to $4 ft./sec^2$. in one second. Prove that the particle will move a distance of 1 ft. in this second.

12- A particle moves in a straight line Ox with an acceleration of magnitude $w^2 x$ directed always far from the origin O. If the particle is initially projected towards O with velocity w a from a point distant a from O, prove that its distance from O at time t is given by $x = ae^{-wt}$.

13- A particle moves in a straight line with a retardation of magnitude $\mu v^2$ where v is the velocity, $\mu$ is a constant. If the particle starts motion from the origin with velocity V prove that the velocity v and the time t depend on x by the following relations:

$$v = V e^{-\mu x} \qquad , \qquad t = \frac{1}{\mu V}(e^{\mu x} - 1).$$

14- The relation between the velocity and the distance for a particle moving in a straight line is given by $v = V_0 e^{-kx}$ where $V_0, k$ are constants. Find x, v, f as functions of time t.

15-. The frictional forces acting upon a train which is moving in a straight line with velocity v ft./sec. causes a retardation of magnitude $a + bv^2$ where a,b are constants. If the engine is stopped when the train was moving with velocity $v_0$ ft./sec., prove that it moves a distance $\frac{1}{2b}\ln(1+\frac{bv_0^2}{a})$ before it comes to rest.

16- A particle moves in a straight line under the action of a central repulsive force from a fixed point O on that line and of magnitude proportional to the distance from it. The particle is projected towards O from the point p distant 10 ft. from O with velocity $v_0$ just enough for the particle to reach O. Prove that if the particle is projected from P towards O with velocity $\frac{1}{2}v_0$, it will come to rest instantaneously at a point distant $5\sqrt{3}$ ft. from O.

17- A particle is attracted by a force to a fixed point varying inversely as the $n^{th}$ power of the distance; if the velocity acquired by it in falling from an infinite distance to a distance a from the centre is equal to the velocity that would be acquired by it in falling from rest at a distance a to a distance $\frac{a}{4}$, show that $n = 3/2$.

18- A particle is moving in a straight line under the action of an acceleration of magnitude $k^2(x+\frac{a^4}{x^3})$ towards the origin. If it starts from rest at a distance a, show that it will arrive at the origin in time $\frac{\pi}{4k}$. Find also the time taken by the particle from the initial position to the point $x=\frac{a}{\sqrt{2}}$.

19. A particle is projected from the origin in the positive direction of the x-axis with initial velocity $v_o$ and moves with a retardation of magnitude $kv^2$ at any moment, where v is the velocity of the particle at this moment and k is a constant. Prove that

$$v=\frac{v_o}{1+kv_o t}=v_o e^{-kx}, \quad x=\frac{1}{k}\ln(1+kv_o t).$$

20. A particle moves in a straight line with acceleration equal to $\mu \div$ the $n^{th}$ power of the distance from a fixed point O in the straight line. If it be projected towards 0, from a point at a distance a, with the velocity it would have acquired in falling from infinity, show that it will reach 0 in time

$$\frac{2}{n+1}\sqrt{\frac{n-1}{2\mu}}\, a^{\frac{n+1}{2}}$$