Calculus I and Analytical Geometry

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Chapter 4

INTEGRATION

Methods of integration

1. Integration by parts

Let u, v are functions of x

$$\therefore \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$d(uv) = u\,dv + v\,du$$

$$\int d(uv) = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$\therefore \int u\,dv = uv - \int v\,du$$

This method is useful to solve problems like:

$$\int x^n \sin x\,dx, \int x^n \cos x\,dx, \int x^n \ln x\ dx, \int \sin^{-1} x\ dx, \dots\dots$$

Example 1 : Evaluate the following integrals

$$\int x \sin x \, dx$$

Solution

$$u = x \qquad\qquad dv = \sin x \, dx$$

$$du = dx \qquad\qquad v = -\cos x$$

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

Tabular method

$$\int x \sin x \, dx = -x \cos x + \sin x + c$$

D		I
x	+	$\sin x$
1	-	$-\cos x$
0		$-\sin x$

Example 2:

$$\int x^2\, e^{-2x} dx$$

Solution

$$u = x^2 \qquad dv = e^{-2x} dx$$

$$du = 2x\, dx \qquad v = \frac{-e^{-2x}}{2}$$

$$\int x^2 e^{-2x} dx = -\frac{1}{2} x^2 e^{-2x} - \int \frac{-e^{-2x}}{2} .\, 2x\, dx = -\frac{1}{2} x^2 e^{-2x} + \int x\, e^{-2x}\, dx$$

Now, we want to solve $\int x e^{-2x}\, dx$ by parts

$$u = x \qquad dv = e^{-2x} dx$$

$$du = dx \qquad v = \frac{-e^{-2x}}{2}$$

$$\therefore \int x^2 e^{-2x} dx = \frac{-1}{2} x^2 e^{-2x} + x . \frac{-e^{-2x}}{2} - \int \frac{-e^{-2x}}{2}\, dx = -\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C$$

Tabular method

$$\int x^2 e^{-2x} dx = -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C$$

D		I
x^2	+	e^{-2x}
$2x$	-	$\frac{-1}{2}e^{-2x}$
2	+	$\frac{1}{4}e^{-2x}$
0		$\frac{-1}{8}e^{-2x}$

Example 3 :

$$\int (x^2 + 3x + 1)\cos 2x\, dx$$

D		I
$x^2 + 3x + 1$		$\cos 2x$
	+	
$2x + 3$		$\frac{1}{2}\sin 2x$
	-	
2		$\frac{-1}{4}\cos 2x$
	+	
0		$\frac{-1}{8}\sin 2x$

Solution

$$I = \int (x^2 + 3x + 1)\cos 2x\, dx = (x^2 + 3x + 1)\frac{\sin 2x}{2} + (2x + 3)\frac{\cos 2x}{4} - 2.\frac{\sin 2x}{8} + C$$

Example 4:

$$\int x\sec^2 x\, dx$$

Solution

D		I
x	+	$sec^2 x$
1	-	$\tan x$
0		$\ln\lvert\sec x\rvert$

$$I = x\tan x - \ln|\sec x| + C$$

Example 5 :

$$\int x \cot^2 x \, dx$$

Solution

$$\int x \cot^2 x \, dx = \int x \left(csc^2 x - 1 \right) dx$$

$$= x(-\cot x - x) - \left(-\ln |sin\, x| - \frac{x^2}{2}\right) + C$$

$$= -x \cot x + \ln|\sin x| - \frac{x^2}{2} + C$$

D		I
x	$+$	$csc^2 x - 1$
1	$-$	$-\cot x - x$
0		$-\ln \vert \sin x \vert - \dfrac{x^2}{2}$

Example 6:

$$\int x^2 \, ln \, x \, dx$$

Solution

$$u = ln \, x \qquad\qquad dv = x^2 dx$$

$$du = \frac{1}{x} dx \qquad\qquad v = \frac{x^3}{3}$$

$$\int x^2 \, ln \, x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} . \frac{1}{x} \, dx = \frac{x^3}{3} \, ln \, x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \, ln \, x - \frac{1}{3} * \frac{x^3}{3} + C$$

$$= \frac{x^3}{3} \, ln \, x - \frac{1}{9} x^3 + C$$

Example 7:

$$\int \ln(x^2+1)\ dx$$

Solution

$$u = \ln(x^2+1) \qquad\qquad dv = dx$$

$$du = \frac{2x}{x^2+1}dx \qquad\qquad v = x$$

$$\int \ln(x^2+1)\ dx = x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1}dx$$

$$= x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}dx$$

$$= x\ln(x^2+1) - 2\int \left[1 - \frac{1}{x^2+1}\right]dx$$

$$= x\ln(x^2+1) - 2[x - \tan^{-1}x] + C$$

Example 8:

$$\int \sinh^{-1}x \ dx$$

Solution

$$u = \sinh^{-1} x \qquad\qquad dv = dx$$

$$du = \frac{1}{\sqrt{1+x^2}}dx \qquad\qquad v = x$$

$$I = x\sinh^{-1} x - \int \frac{x}{\sqrt{1+x^2}}\ dx \quad = x\sinh^{-1} x - \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}\ dx$$

$$= x\sinh^{-1} x - \frac{1}{2}.2\sqrt{1+x^2} + C$$

Example 9:

$$\int x\ \tan^{-1} x\, dx$$

Solution

$$u = \tan^{-1} x \qquad dv = x\, dx$$

$$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^2}{2}$$

$$\int x\ \tan^{-1} x\, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2}\, dx$$

$$\int x\ \tan^{-1} x\, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1}\, dx$$

$$\int x\ \tan^{-1} x\, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C$$

Example 10:

$$\int e^{2x} \cos x \, dx$$

Solution

Let $I = \int e^{2x} \cos x \, dx$

$$u = e^{2x} \qquad dv = \cos x \; dx$$

$$du = 2e^{2x} dx \qquad v = \sin x$$

$$I = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx$$

Now, we want to evaluate the integral $\int e^{2x} \sin x \, dx$ by parts

$$u = e^{2x} \qquad dv = \sin x \; dx$$

$$du = 2e^{2x} dx \qquad v = -\cos x$$

$$I = e^{2x} \sin x - 2 \left[-e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \right]$$

$$I = e^{2x} \sin x + 2e^{2x} \cos x - 4I$$

$$\mathrm{I} + 4\mathrm{I} = e^{2x}\sin x + 2e^{2x}\cos x$$

$$5\mathrm{I} = e^{2x}\sin x + 2e^{2x}\cos x$$

$$\mathrm{I} = \frac{1}{5}[e^{2x}\sin x + 2e^{2x}\cos x] + c = \frac{e^{2x}}{5}[\sin x + 2\cos x] + c$$

(Homework)

$$\int e^{x}\sin x\, dx$$

Example 11:

$$\int \sin(\ln x)\, dx$$

Solution

Let $I = \int \sin(\ln x)\, dx$

$$u = \sin(\ln x) \qquad\qquad dv = dx$$

$$du = \cos(\ln x).\frac{1}{x}dx \qquad\qquad v = x$$

$$\mathrm{I} = x\sin(\ln x) - \int x.\cos(\ln x).\frac{1}{x}dx = x\sin(\ln x) - \int \cos(\ln x)dx$$

$$u = \cos(\ln x) \qquad\qquad dv = dx$$

$$du = -\sin(\ln x).\frac{1}{x}dx \qquad\qquad v = x$$

Now,

$$I = x\, sin(\ln x) - \left[x\cos(\ln x) + \int x.\sin(\ln x).\frac{1}{x}\, dx \right]$$

$$I = x\sin(\ln x) - x\cos(\ln x) - I$$

$$2I = x\sin(\ln x) - x\cos(\ln x)$$

$$\therefore \mathrm{I} = \frac{1}{2}[x\sin(\ln x) - x\cos(\ln x)] + C$$

(Homework)

$$\int \cos(\ln x)\, dx$$

Example 12:

$$\int \sec^3 x \, dx$$

Solution

Let $I = \int \sec^3 x \, dx$

$$I = \int \sec^3 x \, dx = \int sec^2 x \sec x \, dx$$

$$u = \sec x \qquad\qquad dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx \qquad\qquad v = \tan x$$

$$I = \sec x \tan x - \int \sec x \ \tan^2 x \, dx$$

$$I = \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$$

$$I = \sec x \tan x - \int [\sec^3 x - \sec x]\, dx$$

$$I = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$I = \sec x \tan x - I + \ln|\sec x + \tan x|$$

$$2I = \sec x \tan x + \ln|\sec x + \tan x|$$

$$I = \frac{1}{2}[\sec x \tan x + ln\,|\sec x + \tan x|] + C$$

(Homework)

$$\int \csc^3 x\, dx$$

2. Integration by Completing the Square

Use the method of completing the square to integrate integrals of the form

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx \quad \text{and} \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$$

I) Integrals of the form

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx$$

Completing the square and use the rules:

$$\int \frac{u'}{u^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)+c$$

$$\int \frac{u'}{a^2-u^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right)+c$$

Example 1:

$$\int \frac{dx}{x^2+2x+5}\,dx$$

Solution

$$\int \frac{dx}{x^2+2x+5}\,dx = \int \frac{dx}{(x+1)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$$

Example 2:

$$\int \frac{x}{x^2+6x+15}\,dx$$

Solution

$$\int \frac{x}{x^2+6x+15}\,dx = \int \frac{\frac{1}{2}(2x+6)-3}{x^2+6x+15}\,dx = \frac{1}{2}\int \frac{2x+6}{x^2+6x+15}\,dx - 3\int \frac{dx}{x^2+6x+15}$$

$$= \frac{1}{2}\ln|x^2+6x+15| - 3\int \frac{dx}{(x+3)^2+6} = \frac{1}{2}\ln|x^2+6x+15| - \frac{3}{\sqrt{6}}\tan^{-1}\left(\frac{x+3}{\sqrt{6}}\right) + c$$

$\boldsymbol{x = \frac{1}{2}(2x+6) - 3}$ **&**

$x^2+6x+15 =$

$\boldsymbol{(x+3)^2 - 9 + 15}$

$\boldsymbol{= (x+3)^2 + 6}$

Example 3:

$$\int \frac{dx}{2x^2 - 4x - 4}$$

Solution

$$2x^2 - 4x - 4 = 2[(x-1)^2 - 3]$$

$$\int \frac{dx}{2x^2 - 4x - 4} = \int \frac{dx}{2[(x-1)^2 - 3]} = \frac{1}{2}\int \frac{dx}{(x-1)^2 - 3}$$

$$= \frac{-1}{2}\int \frac{dx}{3 - (x-1)^2} = \frac{-1}{2} * \frac{1}{\sqrt{3}} \tanh^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$$

Example 4 :

$$\int \frac{8x-1}{4x^2-4x-3}\,dx$$

Solution

$$I = \int \frac{8x-1}{4x^2-4x-3}\,dx \quad = \int \frac{(8x-4)+3}{4x^2-4x-3}\,dx$$

$$I = \int \frac{8x-4}{4x^2-4x-3}\,dx + \int \frac{3}{4x^2-4x-3}\,dx$$

$$I = \int \frac{8x-4}{4x^2-4x-3}\,dx + \frac{3}{2}\int \frac{2}{(2x-1)^2-(2)^2}\,dx$$

$$\int \frac{8x-4}{4x^2-4x-3}\,dx - \frac{3}{2}\int \frac{2}{(2)^2-(2x-1)^2}\,dx$$

$$= \ln(4x^2-4x+3) - \frac{3}{2}.\frac{1}{2}\tanh^{-1}\left(\frac{2x-1}{2}\right) + C$$

$$\boxed{\begin{aligned} 4x^2-4x-3 &= (4x^2-4x)-3 \\ &= 4(x^2-x)-3 \\ &= 4\left(x^2-x+\frac{1}{4}\right)-3-1 \\ &= 4\left(x-\frac{1}{2}\right)^2-4 \\ &= (2x-1)^2-2^2 \end{aligned}}$$

II) Integrals of the form:

$$\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}}\, dx$$

Completing the square and use one of the rules:

$$\int \frac{u'}{\sqrt{a^2 - u^2}}\, dx = \sin^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \frac{u'}{\sqrt{a^2 + u^2}}\, dx = \sinh^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \frac{u'}{\sqrt{u^2 - a^2}}\, dx = \cosh^{-1}\left(\frac{u}{a}\right) + c$$

Example 5:

$$\int \frac{2+x}{\sqrt{x^2+2x+3}}\,dx$$

Solution

$$\int \frac{2+x}{\sqrt{x^2+2x+3}}\,dx = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$= \frac{1}{2}\,.\,2\sqrt{x^2+2x+3} + \int \frac{dx}{\sqrt{(x+1)^2-1+3}}$$

$$= \sqrt{x^2+2x+3} + \int \frac{dx}{\sqrt{(x+1)^2+2}} = \sqrt{x^2+2x+3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c$$

Example 6:

$$\int \frac{dx}{\sqrt{x^2 + 4x + 3}}$$

Solution

$$\int \frac{dx}{\sqrt{x^2 + 4x + 3}} = \int \frac{dx}{\sqrt{(x+2)^2 - 1}} = \cosh^{-1}\left(\frac{x+2}{1}\right) + c$$

$$\boxed{\begin{aligned} &\boldsymbol{x^2 + 4x + 3} \\ &= \boldsymbol{(x+2)^2 - 4 + 3} \\ &= \boldsymbol{(x+2)^2 - 1} \end{aligned}}$$

Example 7:

$$\int \frac{dx}{\sqrt{7 + 6x - x^2}}$$

Solution

$$\int \frac{dx}{\sqrt{7 + 6x - x^2}} = \int \frac{dx}{\sqrt{-[x^2 - 6x - 7]}} = \int \frac{dx}{\sqrt{-[(\boldsymbol{x-3})^{\boldsymbol{2}} - \boldsymbol{16}]}}$$

$$= \int \frac{dx}{\sqrt{16 - (x-3)^2}} = \sin^{-1}\left(\frac{x-3}{4}\right) + c$$

$$\boxed{\begin{aligned} &\boldsymbol{x^2 - 6x - 7} \\ &= \boldsymbol{(x-3)^2 - 9 - 7} \\ &= \boldsymbol{(x-3)^2 - 16} \end{aligned}}$$

Example 8:

$$\int \frac{\mathbf{2x+3}}{\sqrt{\mathbf{5+4x-x^2}}}\,\mathbf{dx}$$

Solution

$$\int \frac{2x+3}{\sqrt{5+4x-x^2}}dx = -\int \frac{-2x-3}{\sqrt{5+4x-x^2}}dx \;= -\int \frac{(-2x+4)-7}{\sqrt{5+4x-x^2}}dx$$

$$= -\int \frac{(-2x+4)}{\sqrt{5+4x-x^2}}dx + \int \frac{7}{\sqrt{5+4x-x^2}}dx$$

$$= -\int \frac{(-2x+4)}{\sqrt{5+4x-x^2}}dx + 7\int \frac{1}{\sqrt{3^2-(x-2)^2}}dx$$

$$= -2\sqrt{5+4x-x^2} + 7\,sin^{-1}\left(\frac{x-2}{3}\right) + C$$

$5 - x^2 + 4x$

$= -(x^2 - 4x) + 5$

$= -(x^2 - 4x + 4) + 5 + 4$

$= 9 - (x-2)^2$

Example 8:

$$\int \frac{\cos 2x \, dx}{\sqrt{\sin^2 2x + 2\sin 2x + 5}}$$

Solution

$$\int \frac{\cos 2x \, dx}{\sqrt{\sin^2 2x + 2\sin 2x + 5}} = \int \frac{\cos 2x \, dx}{\sqrt{(\sin 2x + 1)^2 - 1 + 5}}$$

$$= \int \frac{\cos 2x \, dx}{\sqrt{(\sin 2x + 1)^2 + 4}} = \frac{1}{2} \sinh^{-1}\left(\frac{\sin 2x + 1}{2}\right) + c$$

3) Integrals of the form

$$1)\int \boldsymbol{cos(mx)\cos(nx)dx}$$

$$2)\int \boldsymbol{sin(mx)\,sin(nx)dx}$$

$$3)\int \boldsymbol{sin(mx)\cos(nx)dx}$$

Rules

$$1)\ \cos x \,.\cos y = \frac{1}{2}[\cos(x-y)+\cos(x+y)]$$

$$2)\ \sin x \,.\sin y = \frac{1}{2}[\cos(x-y)-\cos(x+y)]$$

$$3)\ \sin x \,.\cos y = \frac{1}{2}[\sin(x-y)+\sin(x+y)]$$

Example 1:

$$\int \sin 3x \ \cos 5x \, dx$$

Solution

$$\int \sin 3x \ \cos 5x \, dx = \frac{1}{2}\int [\sin(3x - 5x) + \sin(3x + 5x)] \, dx$$

$$= \frac{1}{2}\int [\sin(-2x) + \sin(8x)] \, dx = \frac{1}{2}\int [-\sin(2x) + \sin(8x)] \, dx = \frac{1}{2}\left[\frac{\cos 2x}{2} - \frac{\cos 8x}{8}\right] + C$$

Example 2:

$$\int \cos 3x \ \cos 7x \, dx$$

Solution

$$\int \cos 3x \ \cos 7x \, dx = \frac{1}{2}\int [\cos 4x + \cos 10x] \, dx = \frac{1}{2}\left[\frac{\sin 4x}{4} + \frac{\sin 10x}{10}\right] + C$$

Example 3:

$$\int \sin 4x \, \sin 2x \, dx$$

Solution

$$\int \sin 4x \, \sin 2x \, dx = \frac{1}{2} \int [\cos 2x - \cos 6x] \, dx$$

$$= \frac{1}{2} \left[\frac{\sin 2x}{2} - \frac{\sin 6x}{6} \right] + C$$