Chapter 1

Functions

1.1 Introduction to Functions

A **function** is a rule that assigns to each element in a set called the **domain** to exactly one element in a set called the **range** or codomain. It can be thought of as a machine that takes an input and produces a unique output. We generally denote a function as $f(x)$, where f is the name of the function, x is the input value from the domain, and $f(x)$ is the output value in the range.

FIGURE 1.1 A diagram showing a function as a kind of machine.

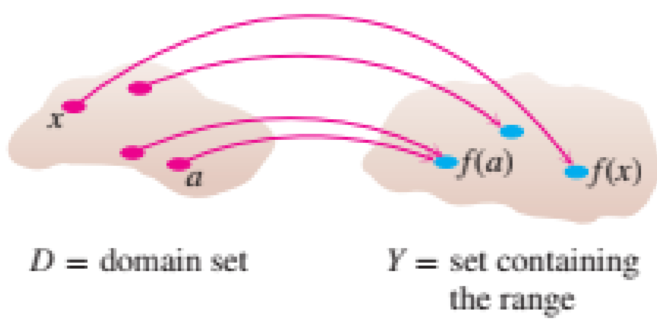


FIGURE 1.2 A function from a set D to a set Y assigns a unique element of Y to each element in D.

1.1.1 Domain and Range

Domain: The set of all possible input values (x) for which the function is defined. **Range**: The set of all possible output values ($f(x)$) produced by the function.

EXAMPLE 1 Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of x for which the formula makes sense.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution The formula $y = x^2$ gives a real y-value for any real number x, so the domain is $(-\infty, \infty)$. The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and every nonnegative number y is the square of its own square root, $y = (\sqrt{y})^2$ for $y \geq 0$.

The formula $y = 1/x$ gives a real y-value for every x except $x = 0$. For consistency in the rules of arithmetic, *we cannot divide any number by zero*. The range of $y = 1/x$, the set of reciprocals of all nonzero real numbers, is the set of all nonzero real numbers, since $y = 1/(1/y)$. That is, for $y \neq 0$ the number $x = 1/y$ is the input assigned to the output value y.

The formula $y = \sqrt{x}$ gives a real y-value only if $x \geq 0$. The range of $y = \sqrt{x}$ is $[0, \infty)$ because every nonnegative number is some number's square root (namely, it is the square root of its own square).

In $y = \sqrt{4 - x}$, the quantity $4 - x$ cannot be negative. That is, $4 - x \geq 0$, or $x \leq 4$. The formula gives real y-values for all $x \leq 4$. The range of $\sqrt{4 - x}$ is $[0, \infty)$, the set of all nonnegative numbers.

Examples

1. $f(x) = \sqrt{x}$

- **Domain**: All non-negative real numbers, since the square root of a negative number is not defined in the real numbers. So, $x \geq 0$.
- **Range**: All non-negative real numbers, as the output of a square root is always non-negative. So, $f(x) \geq 0$.

2. $g(x) = \frac{1}{x-3}$

   - **Domain**: All real numbers except $x = 3$, because division by zero is undefined.
   - **Range**: All real numbers except 0.

3. $h(x) = 5 - \sqrt{x-1}$

   - **Domain**: We need $x - 1 \geq 0$, which means $x \geq 1$. The domain is $[1, \infty)$.
   - **Range**: The term $\sqrt{x-1}$ is always non-negative. So, $5 - \sqrt{x-1}$ will be less than or equal to 5. The range is $(-\infty, 5]$.

1.2 Inverse Functions

Definition: If a function f maps each element x in its domain to a unique element y in its range, then the **inverse function** f^{-1} is the function that maps y back to x.

Condition: For a function f to have an inverse, it must be **one-to-one**, meaning that every output (y) corresponds to exactly one input (x). This can be tested using the **Horizontal Line Test** on the function's graph.

1.2.1 How to Find an Inverse Function

1. Start with the function in the form $y = f(x)$.
2. Swap x and y in the equation.
3. Solve the new equation for y.
4. Replace y with $f^{-1}(x)$.

Example: Find the inverse of the function $f(x) = 2x + 1$.

1. Write the function as $y = 2x + 1$.
2. Swap variables: $x = 2y + 1$.
3. Solve for y:
$$x - 1 = 2y$$
$$y = \frac{x-1}{2}$$
4. Thus, the inverse function is $f^{-1}(x) = \frac{x-1}{2}$.

1.3 Types of Functions

There are many types of functions in mathematics. A few common ones are:

1.4 Polynomial Functions

A **polynomial function** is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where n is a non-negative integer called the degree, and the coefficients $a_0, a_1, \ldots, a_n$ are real numbers.

1.4.1 Linear Functions

A **linear function** is a polynomial of degree 1.

Definition: A function of the form $f(x) = mx + b$, where m and b are constants.

Example: $f(x) = 2x+3$. The graph is a straight line with a slope of 2 and a y-intercept of 3.

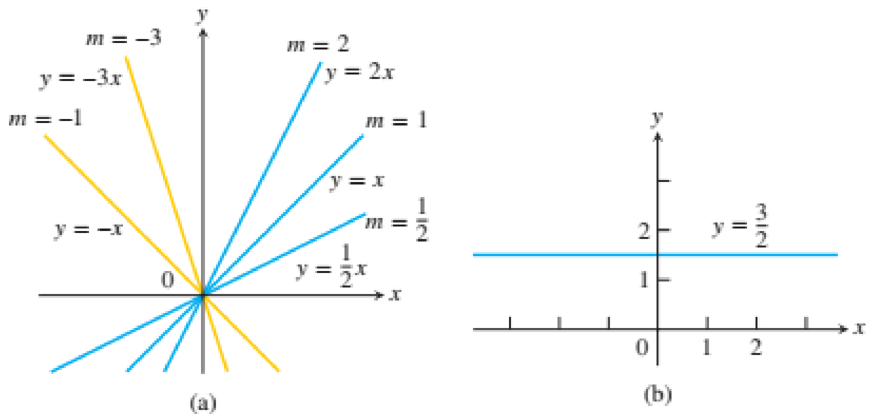


Figure 1.1: Linear Function $f(x) = mx + b$.

1.4.2 Quadratic Functions

A **quadratic function** is a polynomial of degree 2.

Definition: A function of the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$.

Example: $f(x) = x^2 - 4x + 3$. The graph is a parabola that opens upwards.

1.4.3 Cubic Functions

A **cubic function** is a polynomial of degree 3.

Definition: A function of the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants and $a \neq 0$.

Example: $f(x) = x^3 - 2x$.

1.5 Rational Functions

Definition: A **rational function** is a function that can be expressed as the ratio of two polynomial functions, $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ is not the zero polynomial. The domain is all real numbers where $Q(x) \neq 0$.

Example: $f(x) = \frac{x+1}{x-2}$. The domain is $\{x|x \neq 2\}$. The graph has a vertical asymptote at $x = 2$.

1.6 Radical Functions

Definition: A **radical function** is a function that involves a radical expression, such as a square root or cube root, containing the independent variable.

Example: $f(x) = \sqrt{x+4}$. The domain is $\{x|x + 4 \geq 0\}$, which simplifies to $[-4, \infty)$.

1.7 Exponential and Logarithmic Functions

1.7.1 Exponential Functions

Definition: An **exponential function** is a function of the form $f(x) = a^x$, where a is a positive constant called the base.

Example: $f(x) = 10^x$.

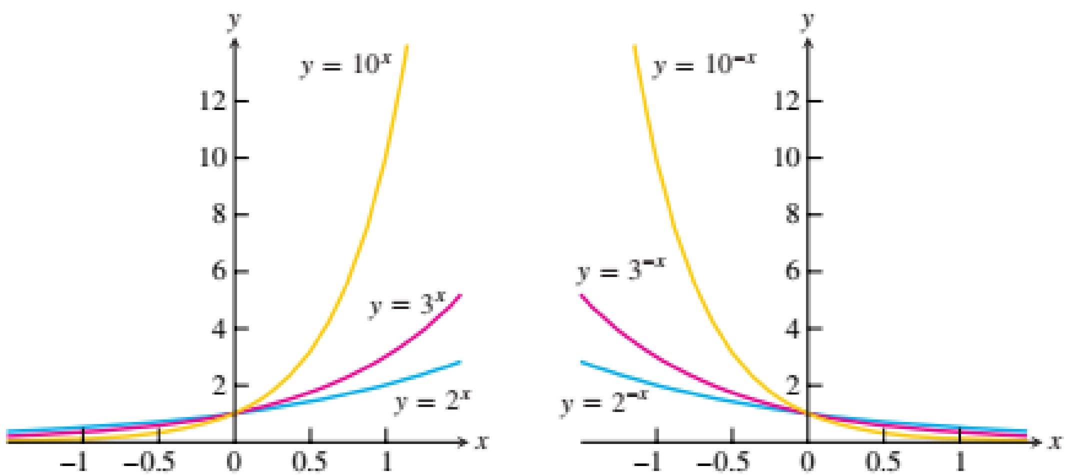


Figure 1.2: Exponential Functions $f(x) = a^x$

1.8 Algebraic Rules for Exponential Functions

These rules govern how exponents behave in various operations.

- **Product Rule:**
$$a^x a^y = a^{x+y}$$

- **Quotient Rule:**
$$\frac{a^x}{a^y} = a^{x-y}$$

- **Power Rule:**
$$(a^x)^y = a^{xy}$$

- **Power of a Product Rule:**
$$(ab)^x = a^x b^x$$

- **Power of a Quotient Rule:**
$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

- **Zero Exponent Rule:**
$$a^0 = 1 \quad \text{for } a \neq 0$$

- **Negative Exponent Rule:**
$$a^{-x} = \frac{1}{a^x}$$

A Special Note on the Natural Base, e

When the base of an exponential function is the irrational number **e** (approximately 2.71828), it is called the **natural exponential function**, written as $f(x) = e^x$. This function holds immense importance in mathematics, science, and finance because it naturally models continuous growth and decay processes.

Its properties are consistent with all exponential functions where the base is greater than 1: it is always increasing, passes through the point $(0, 1)$, and has the x-axis as a horizontal asymptote.

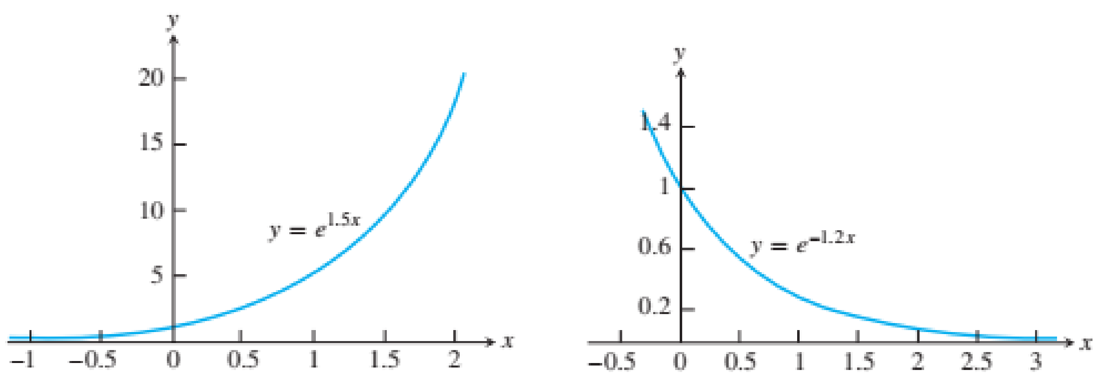


Figure 1.3: exponential function $f(x) = e^{ax}$.

1.8.1 Logarithmic Functions

Definition: A **logarithmic function** is the inverse of an exponential function. It is defined by the relationship:
$y = \log_a(x)$ if and only if $x = a^y$.

Example: $f(x) = \log_2(x)$.

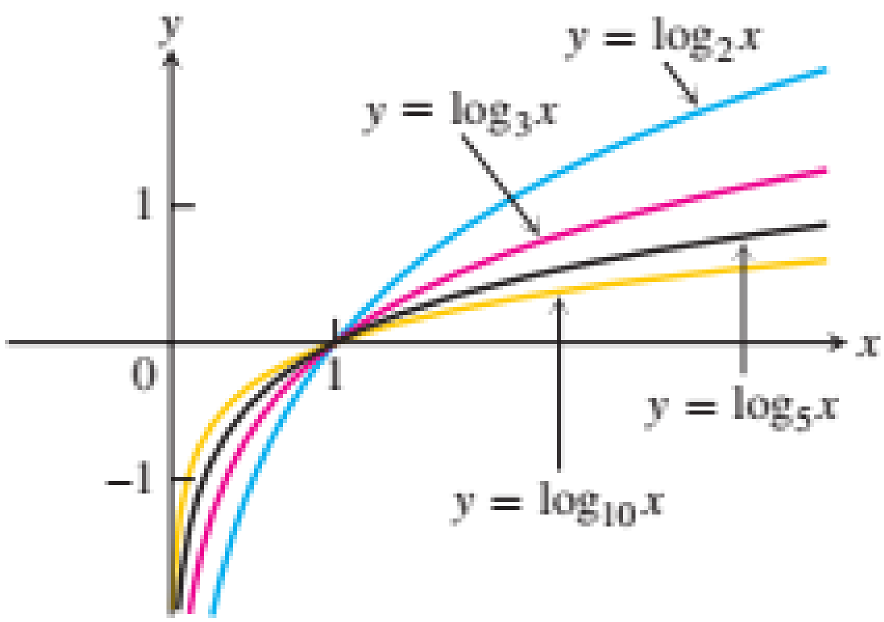


Figure 1.4: logarithmic Function$y = \log_a(x)$

1.9 Algebraic Rules for Logarithmic Functions

These rules are the counterparts to the exponential rules, designed to simplify logarithmic expressions.

- **Product Rule:**

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

- **Quotient Rule:**

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

- **Power Rule:**

$$\log_b(x^n) = n \cdot \log_b(x)$$

Special Rules:

- **Change of Base Formula:**

$$\log_b(x) = \frac{\ln(x)}{\ln(b)}$$

- **Logarithm of One:**

$$\log_b(1) = 0$$

- **Logarithm of the Base:**

$$\log_b(b) = 1$$

The Natural Logarithm

The inverse of the natural exponential function is the **natural logarithm**. It is written as $\ln(x)$ or $\log_e(x)$.

- For example: $\ln(e) = 1$ because $e^1 = e$.

The natural logarithm is the most common and widely used type of logarithm because it simplifies many calculations in calculus and higher-level mathematics. Its properties are the inverse of the natural exponential function's properties:

THEOREM 1—Algebraic Properties of the Natural Logarithm For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$
2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $b = 1$
4. *Power Rule*: $\ln x^r = r \ln x$

1.10 Trigonometric Functions

Trigonometric functions are a group of functions that relate the angles of right-angled triangles to the ratios of their side lengths. These functions are fundamental in mathematics, physics, and engineering, and are used to describe periodic phenomena such as waves and sounds.

For any acute angle θ in a right-angled triangle, the basic functions are defined as follows:

- **Sine Function** ($\sin(\theta)$)**:**

Definition: The ratio of the length of the side opposite the angle to the length of the hypotenuse.

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

- **Cosine Function ($\cos(\theta)$):**

Definition: The ratio of the length of the side adjacent to the angle to the length of the hypotenuse.

$$\cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

- **Tangent Function ($\tan(\theta)$):**

Definition: The ratio of the length of the side opposite the angle to the length of the side adjacent. It can also be defined as the ratio of sine to cosine.

$$\tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{\sin(\theta)}{\cos(\theta)}$$

1.10.1 Reciprocal Functions

There are three other functions that are the reciprocals of the basic trigonometric functions:

- **Cosecant ($\csc(\theta)$):** The reciprocal of the sine function.

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

- **Secant ($\sec(\theta)$):** The reciprocal of the cosine function.

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

- **Cotangent ($\cot(\theta)$):** The reciprocal of the tangent function.

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

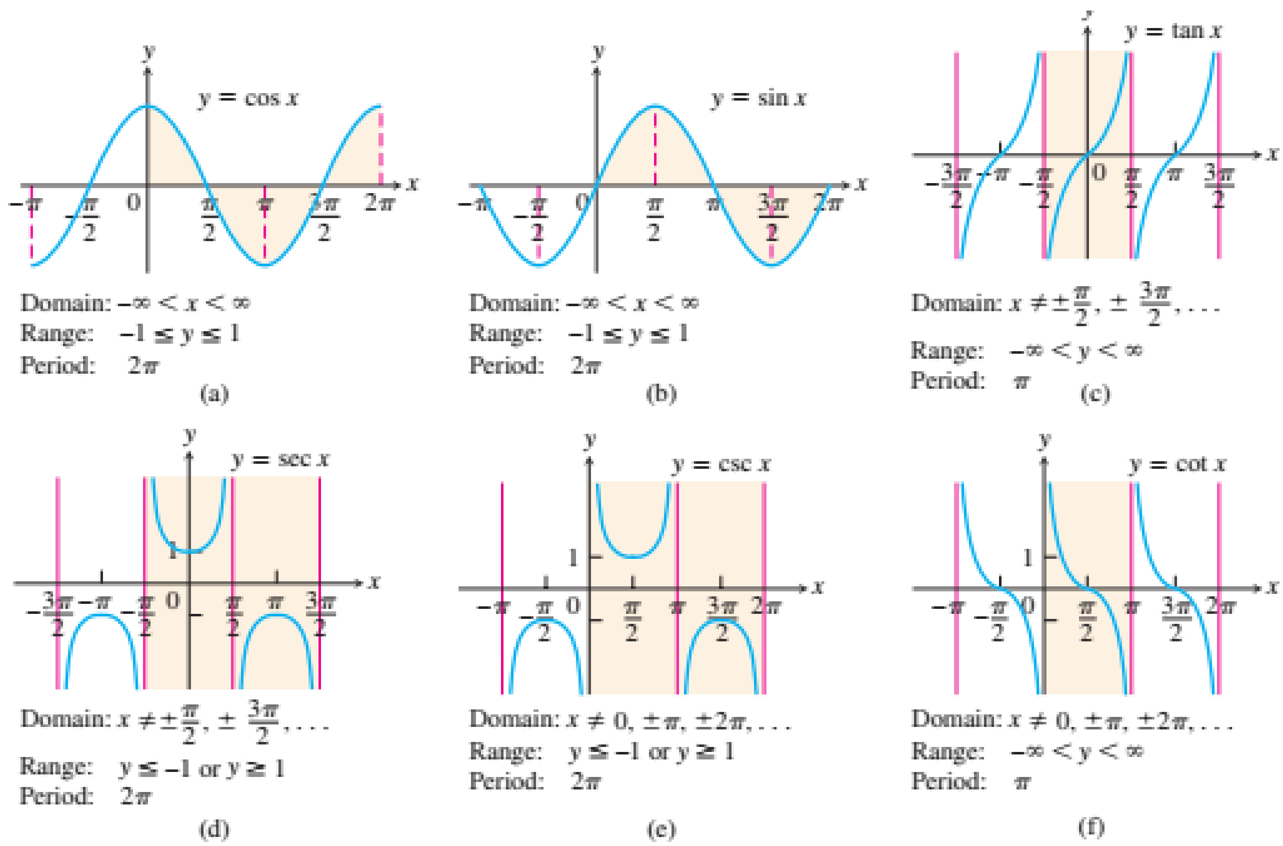


Figure 1.5: Trigonometric Functions .

1.10.2 Fundamental Trigonometric Identities

These identities are equations that are true for all defined values of the variables.

- **Pythagorean Identity:**

$$\sin^2(x) + \cos^2(x) = 1$$

- **Other Derived Identities:**

$$1 + \tan^2(x) = \sec^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

Angle addition and subtraction

$$\begin{aligned}
\sin(a \pm b) &= \sin a \cos b \pm \cos a \sin b, \\
\cos(a \pm b) &= \cos a \cos b \mp \sin a \sin b, \\
\tan(a \pm b) &= \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \quad \text{(where denominators } \neq 0).
\end{aligned}$$

Double-angle and half-angle

$$\begin{aligned}
\sin 2a &= 2 \sin a \cos a, \\
\cos 2a &= \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a, \\
\tan 2a &= \frac{2 \tan a}{1 - \tan^2 a}.
\end{aligned}$$

Half-angle formulas (sign depends on quadrant):

$$\sin \frac{a}{2} = \pm \sqrt{\frac{1 - \cos a}{2}}, \qquad \cos \frac{a}{2} = \pm \sqrt{\frac{1 + \cos a}{2}}.$$

Product-to-sum and sum-to-product

$$\begin{aligned}
\cos A \cos B &= \tfrac{1}{2}[\cos(A - B) + \cos(A + B)], \\
\sin A \sin B &= \tfrac{1}{2}[\cos(A - B) - \cos(A + B)], \\
\sin A \cos B &= \tfrac{1}{2}[\sin(A + B) + \sin(A - B)].
\end{aligned}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2},$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

Double-angle rules

$$\sin(2\theta) = 2\sin\theta\cos\theta,$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta,$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \quad (\tan\theta \neq \pm 1).$$

Half-angle formulas

$$\sin\frac{a}{2} = \pm\sqrt{\frac{1-\cos a}{2}}, \qquad \cos\frac{a}{2} = \pm\sqrt{\frac{1+\cos a}{2}}.$$

1.11 Inverse Trigonometric Functions:

DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

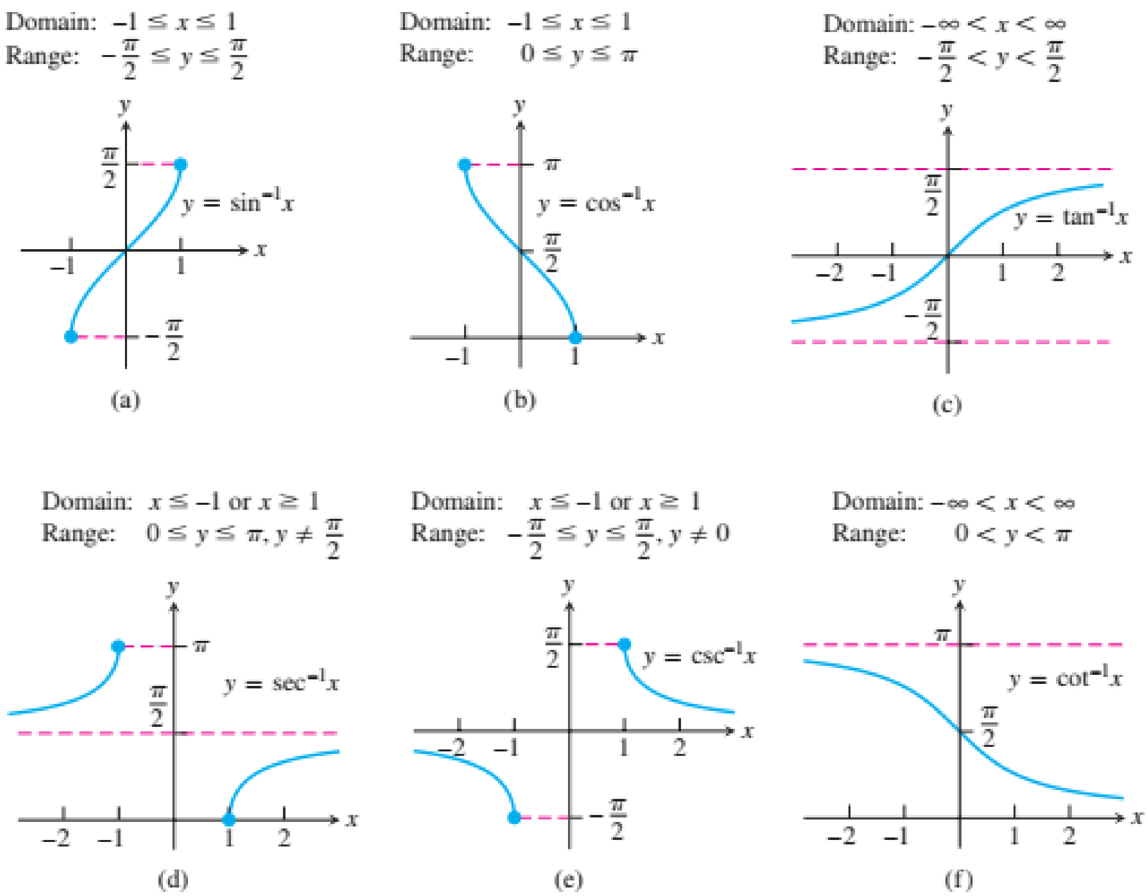
Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{aligned}
y &= \sin^{-1} x \quad &\text{or} \quad y &= \arcsin x \\
y &= \cos^{-1} x \quad &\text{or} \quad y &= \arccos x \\
y &= \tan^{-1} x \quad &\text{or} \quad y &= \arctan x \\
y &= \cot^{-1} x \quad &\text{or} \quad y &= \operatorname{arccot} x \\
y &= \sec^{-1} x \quad &\text{or} \quad y &= \operatorname{arcsec} x \\
y &= \csc^{-1} x \quad &\text{or} \quad y &= \operatorname{arccsc} x
\end{aligned}
$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on.

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

The graphs of the six inverse trigonometric functions are shown in Figure 1.66. We can obtain these graphs by reflecting the graphs of the restricted trigonometric functions through the line $y = x$. We now take a closer look at two of these functions.



Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \sin^{-1}x$
(a)
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$
$y = \cos^{-1}x$
(b)
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \tan^{-1}x$
(c)
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$
$y = \sec^{-1}x$
(d)
Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
$y = \csc^{-1}x$
(e)
Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$
$y = \cot^{-1}x$
(f)

1.12 Hyperbolic functions

1.12.1 Definitions

The six basic hyperbolic functions are defined by the exponential functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \;\; \cosh x = \frac{e^x + e^{-x}}{2}, \;\; \tanh x = \frac{\sinh x}{\cosh x},$$
$$\coth x = \frac{\cosh x}{\sinh x}, \qquad \operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{csch} x = \frac{1}{\sinh x}.$$

1.12.2 Basic identity

$$\cosh^2 x - \sinh^2 x = 1,$$

and hence

$$1 - \tanh^2 x = \operatorname{sech}^2 x, \qquad \coth^2 x - \operatorname{csch}^2 x = 1.$$

1.12.3 Addition formulas

$$\begin{aligned}
\sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y,\\
\cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y,\\
\tanh(x \pm y) &= \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \; \tanh y}.
\end{aligned}$$

1.12.4 Double-angle formulas

$$\sinh(2x) = 2\sinh x \cosh x, \qquad \cosh(2x) = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x, \qquad \tanh($$

1.12.5 Inverse hyperbolic functions (principal forms)

$$\operatorname{arsinh} x = \ln\left(x + \sqrt{x^2+1}\right), \quad \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right) \ (x \geq 1),$$

$$\operatorname{artanh} x = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (|x| < 1).$$

1.13 Absolute Value Functions

Definition: An **absolute value function** is a function that contains an absolute value expression, such as $f(x) = |x|$. It returns the non-negative value of its input.

Example: $f(x) = |x - 3|$. The graph is a "V" shape with its vertex at $(3, 0)$.

1.14 Piecewise Functions

Definition: A **piecewise function** is a function defined by multiple sub-functions, with each sub-function applying to a different interval in the domain.

Example: $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$

1.15 Constant and Identity Functions

1.15.1 Constant Functions

Definition: A **constant function** is a function of the form $f(x) = c$, where c is a constant. The output is the same regardless of the input.

Example: $f(x) = 5$. The graph is a horizontal line.

1.15.2 Identity Functions

Definition: An **identity function** is a function that maps every element to itself, defined as $f(x) = x$.

Example: $f(x) = x$. The graph is a straight line through the origin with a slope of 1.

1.16 Function Properties

1.16.1 Even and Odd Functions

- An **even function** is a function $f(x)$ that satisfies $f(-x) = f(x)$. For example, $f(x) = x^4$ is an even function.

- An **odd function** is a function $f(x)$ that satisfies $f(-x) = -f(x)$. For example, $g(x) = x^3$ is an odd function.

1.16.2 Monotonic Functions

- An **increasing function** is one where $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.

- A **decreasing function** is one where $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

- Both increasing and decreasing functions are called **monotonic**.

1.17 Practice Problems

1. Find the domain and range for the following functions:

    - $f(x) = 4x$

- $f(x) = \sqrt{x+1}$
- $f(x) = \frac{1}{x-2}$

Solutions for Domain and Range:

- $f(x) = 4x$: Domain is $(-\infty, \infty)$, Range is $(-\infty, \infty)$.
- $f(x) = \sqrt{x+1}$: Domain is $[-1, \infty)$, Range is $[0, \infty)$.
- $f(x) = \frac{1}{x-2}$: Domain is all real numbers except 2, Range is all real numbers except 0.

2. Determine whether the following functions are even, odd, or neither:

- $f(x) = 3x^2 + 6x^4$
- $f(x) = x^3 - 2x$
- $f(x) = \sin(x)$
- $f(x) = \cos(x) + \sin(x)$

Solutions for Even/Odd:

- $f(-x) = 3(-x)^2 + 6(-x)^4 = 3x^2 + 6x^4 = f(x)$, so it is **even**.
- $f(-x) = (-x)^3 - 2(-x) = -x^3 + 2x = -(x^3 - 2x) = -f(x)$, so it is **odd**.
- $f(-x) = \sin(-x) = -\sin(x) = -f(x)$, so it is **odd**.

- $f(-x) = \cos(-x) + \sin(-x) = \cos(x) - \sin(x)$, which is not equal to $f(x)$ or $-f(x)$, so it is **neither**.

Exercises:
(A) Find the domain and range of each function.
1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(x) = \sqrt{5x + 10}$

4. $g(x) = \sqrt{x^2 - 3x}$

5. $f(t) = \dfrac{4}{3 - t}$

6. $G(t) = \dfrac{2}{t^2 - 16}$

(B)Find the domain of the following functions:

1. $f(x) = 5 - 2x$

2 $f(x) = 1 - 2x - x^2$

3. $g(x) = \sqrt{x + 1}$

4. $g(x) = \sqrt{-x}$

5. $F(t) = t/|t|$

6. $G(t) = 1/|t|$

7. Find the domain of $y = \dfrac{x+3}{4-\sqrt{x^2-9}}$

8. Find the range of $y = 2 + \dfrac{x^2}{x^2+4}$

Chapter 2

Limits and continuity

2.1 Introduction

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior near a particular point , but not at . This might be the case, for instance, if is an irrational number, like or , whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in y by considering the quotient function for h closer and closer to zero. Here's a specific example where we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

2.2 Definition of a Limit

Definition 2.2.1 *Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself.*
*We say that the **limit** of $f(x)$ as x approaches x_0 is the number L, and write*

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon.$$

Example 2.2.1 *Show that*

$$\lim_{x \to 1}(5x - 3) = 2.$$

Solution: *Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, whenever*

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ε of $L = 2$, so

$$|f(x) - 2| < \varepsilon.$$

We find δ by working backward from the ε-inequality:

$$|(5x - 3) - 2| = |5x - 5| < \varepsilon$$

$$5|x-1| < \varepsilon$$
$$|x-1| < \frac{\varepsilon}{5}.$$

Thus, we can take $\delta = \frac{\varepsilon}{5}$. *If* $0 < |x-1| < \delta = \frac{\varepsilon}{5}$, *then*

$$|(5x-3)-2| = |5x-5| = 5|x-1| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon,$$

which proves that

$$\lim_{x \to 1}(5x-3) = 2.$$

The value $\delta = \frac{\varepsilon}{5}$ *is not the only value that will make* $0 < |x-1| < \delta$ *imply* $|5x-5| < \varepsilon$. *Any smaller positive* δ *will do as well. The definition does not ask for a "best" positive* δ, *just one that will work.*

Theorem 2.2.1 (Limit Laws) *If* $L, M, c,$ *and* k *are real numbers and*

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M,$$

then:

1. ***Sum Rule:***

$$\lim_{x \to c} \big(f(x) + g(x)\big) = L + M$$

2. ***Difference Rule:***

$$\lim_{x \to c} \big(f(x) - g(x)\big) = L - M$$

3. ***Constant Multiple Rule:***

$$\lim_{x\to c}\big(k\cdot f(x)\big)=k\cdot L$$

4. ***Product Rule:***

$$\lim_{x\to c}\big(f(x)\cdot g(x)\big)=L\cdot M$$

5. ***Quotient Rule:***

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M},\quad M\neq 0$$

6. ***Power Rule:***

$$\lim_{x\to c}\big(f(x)\big)^n=L^n,\quad n \textit{ a positive integer}$$

7. ***Root Rule:***

$$\lim_{x\to c}\sqrt[n]{f(x)}=\sqrt[n]{L}=L^{1/n},\quad n \textit{ a positive integer}$$

(If n is even, we assume that $\lim_{x\to c} f(x)=L>0$.)

EX.1 Suppose $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = -5$. Evaluate

$$\lim_{x\to c} \frac{2f(x) - g(x)}{(f(x) + 7))^{2/3}}$$

Solution:

$$\lim_{x\to c} \frac{2f(x) - g(x)}{(f(x) + 7))^{2/3}} = \frac{\lim_{x\to c}(2f(x) - g(x))}{(\lim_{x\to c}(f(x) + 7))^{2/3}}$$

$$\frac{2\lim_{x\to c} f(x) - \lim_{x\to c} g(x)}{(\lim_{x\to c} f(x) + \lim_{x\to c} 7)^{2/3}} = \frac{2(1) - (-5)}{(1 + 7)^{2/3}} = \frac{7}{4}$$

EX.2 Let $\lim_{x\to c} h(x) = 5$, $\lim_{x\to c} p(x) = 1$, and $\lim_{x\to c} r(x) = 2$. Evaluate

$$\lim_{x\to c} \frac{\sqrt{5h(x)}}{p(x)(4 - r(x))}$$

Solution:

$$\frac{\lim_{x\to c} \sqrt{5h(x)}}{\lim_{x\to c}[p(x)(4 - r(x))]}$$

$$= \frac{\sqrt{\lim_{x\to c} 5h(x)}}{(\lim_{x\to c} p(x))(\lim_{x\to c} 4 - \lim_{x\to c} r(x))}$$

$$= \frac{\sqrt{5(5)}}{(1)(4 - 2)} = \frac{\sqrt{25}}{(1)(2)} = \frac{5}{2}$$

Exercise Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find:

1. $\lim_{x\to c} f(x)g(x)$
2. $\lim_{x\to c} 2f(x)g(x)$

EX.3 If

$$\lim_{x\to 4} \frac{f(x)-5}{x-2} = 1,$$

find $\lim_{x\to 4} f(x)$.

Solution:
As $x \to 4$, the denominator tends to 2. So

$$\lim_{x\to 4} \frac{f(x)-5}{x-2} = \frac{\lim_{x\to 4}(f(x)-5)}{2} = 1.$$

Thus,

$$\lim_{x\to 4}(f(x)-5) = 2 \quad \Longrightarrow \quad \lim_{x\to 4} f(x) = 7.$$

EX.4 If

$$\lim_{x\to 2} \frac{f(x)}{x^2} = 1,$$

find:

1. $\lim_{x\to 2} f(x)$
2. $\lim_{x\to 2} \frac{f(x)}{x}$

Solution: (a) Since $\lim_{x\to 2} \frac{f(x)}{x^2} = 1$, then

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x^2 \cdot \tfrac{f(x)}{x^2}) = (2^2) \cdot 1 = 4.$$

(b) We write:

$$\lim_{x\to 2} \frac{f(x)}{x} = \lim_{x\to 2} \left(\frac{f(x)}{x^2} \cdot x \right) = (1)(2) = 2.$$

EX.5 Evaluate

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x}$$

Solution: We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x-1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x-1)(x+2)}{x(x-1)} = \frac{x+2}{x} \quad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution.

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{x+2}{x} = \frac{1+2}{1} = 3.$$

EX.6 Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$$

Solution: We can create a common factor by multiplyingboth numerato

$$\begin{aligned}
\frac{\sqrt{x^2+100}-10}{x^2} &= \frac{\sqrt{x^2+100}-10}{x^2} \cdot \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \\
&= \frac{x^2+100-100}{x^2(\sqrt{x^2+100}+10)} \\
&= \frac{x^2}{x^2(\sqrt{x^2+100}+10)} \\
&= \frac{1}{\sqrt{x^2+100}+10}
\end{aligned}$$

Therefore,

$$\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10}$$

$$= \frac{1}{\sqrt{0^2 + 100} + 10}$$

$$= \frac{1}{\sqrt{100} + 10}$$

$$= \frac{1}{10 + 10} = \frac{1}{20} = 0.05$$

EX.7 Evaluate

$$\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$$

Solution.
Factor the denominator: $x^2 - 25 = (x-5)(x+5)$. For $x \neq 5$ we simplify

$$\lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \lim_{x \to 5} \frac{x - 5}{(x-5)(x+5)} = \lim_{x \to 5} \frac{1}{x + 5}.$$

Now substitute $x = 5$:

$$\lim_{x \to 5} \frac{x - 5}{x^2 - 25} = \frac{1}{5 + 5} = \frac{1}{10}.$$

EX.8 Evaluate

$$\lim_{x \to -3} \frac{x + 3}{x^2 + 4x + 3}$$

Solution.
Factor denominator: $x^2 + 4x + 3 = (x + 1)(x + 3)$. For $x \neq -3$,

$$\lim_{x\to-3} \frac{x+3}{x^2+4x+3} = \lim_{x\to-3} \frac{x+3}{(x+3)(x+1)} = \lim_{x\to-3} \frac{1}{x+1}.$$

Substitute $x = -3$:

$$\lim_{x\to-3} \frac{x+3}{x^2+4x+3} = \frac{1}{-3+1} = \frac{1}{-2} = -\tfrac{1}{2}.$$

EX.9 Evaluate

$$\lim_{x\to 2} \frac{4x - x^2}{2 - \sqrt{x}}$$

Solution. Rewrite numerator: $4x - x^2 = x(4 - x) = -x(x - 4)$. It is convenient to factor so that a factor $x - 4$ pairs with the conjugate. Instead multiply numerator and denominator by the conjugate of the denominator $2 + \sqrt{x}$:

$$\lim_{x\to 2} \frac{4x - x^2}{2 - \sqrt{x}} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} = \lim_{x\to 2} \frac{(4x - x^2)(2 + \sqrt{x})}{4 - x}.$$

Note $4 - x = -(x - 4)$, and factor numerator $4x - x^2 = x(4 - x) = -x(x - 4)$. Thus

$$\lim_{x\to 2} \frac{(4x - x^2)(2 + \sqrt{x})}{4 - x} = \lim_{x\to 2} \frac{-x(x - 4)(2 + \sqrt{x})}{-(x - 4)} = \lim_{x\to 2} x(2+\sqrt{x}).$$

Now substitute $x = 2$:

$$\lim_{x\to 2} \frac{4x - x^2}{2 - \sqrt{x}} = 2(2 + \sqrt{2}) = 4 + 2\sqrt{2}.$$

EX.10 Evaluate

$$\lim_{x\to -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3}$$

Solution.
Multiply numerator and denominator by the conjugate $2+\sqrt{x^2 - 5}$:

$$\lim_{x\to -3} \frac{2 - \sqrt{x^2 - 5}}{x + 3} \cdot \frac{2 + \sqrt{x^2 - 5}}{2 + \sqrt{x^2 - 5}}$$

$$= \lim_{x\to -3} \frac{4 - (x^2 - 5)}{(x + 3)(2 + \sqrt{x^2 - 5})}$$

$$= \lim_{x\to -3} \frac{9 - x^2}{(x + 3)(2 + \sqrt{x^2 - 5})}.$$

Factor $9 - x^2 = (3 - x)(3 + x) = -(x - 3)(x + 3)$. So

$$\lim_{x\to -3} \frac{9 - x^2}{(x + 3)(2 + \sqrt{x^2 - 5})} = \lim_{x\to -3} \frac{-(x - 3)(x + 3)}{(x + 3)(2 + \sqrt{x^2 - 5})}$$

$$= -\lim_{x\to -3} \frac{x - 3}{2 + \sqrt{x^2 - 5}}.$$

Now substitute $x = -3$: numerator $= -3 - 3 = -6$, denominator $= 2 + \sqrt{9-5} = 2 + 2 = 4$, so value $= -(-6)/4$? Careful: plugging gives

$$-\frac{(-3)-3}{2+\sqrt{9-5}} = -\frac{-6}{2+2} = -\frac{-6}{4} = \frac{6}{4} = \tfrac{3}{2}.$$

Thus the limit is $\frac{3}{2}$.

2.3 Limits Involving Trigonometric Functions

In calculus, certain limits involving trigonometric functions occur frequently. The following are the most important standard limits:

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\displaystyle\lim_{x\to 0} \frac{\tan x}{x} = 1$
3. $\displaystyle\lim_{x\to 0} \frac{\sin(ax)}{x} = a$, where a is a constant.
4. $\displaystyle\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)} = \frac{a}{b}$, where $a, b \neq 0$.

Note: All trigonometric functions in these limits are in *radians*.

Example 1

Evaluate:

$$\lim_{x \to 0} \frac{\sin(3x)}{x}$$

Solution: We use the standard limit $\lim_{x \to 0} \frac{\sin x}{x} = 1$:

$$\lim_{x \to 0} \frac{\sin(3x)}{x} = \lim_{3x \to 0} \left(\frac{\sin(3x)}{3x} \cdot 3 \right) = (1) \cdot 3 = 3$$

Example 2

Evaluate:

$$\lim_{x \to 0} \frac{1 - \cos(2x)}{x^2}$$

Solution: We recall the identity:

$$1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$$

Here, $\theta = 2x$, so:

$$1 - \cos(2x) = 2\sin^2(x)$$

The limit becomes:

$$\lim_{x \to 0} \frac{2\sin^2(x)}{x^2} = 2 \cdot \lim_{x \to 0} \left(\frac{\sin(x)}{x} \right)^2 = 2 \cdot (1)^2 = 2$$

Solved Exercises:

(1) $\lim\limits_{x\to 2} \dfrac{x^4-16}{x-2}$

Solution:

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)(x^2+4)}{x-2}$$

So,

$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2}(x+2)(x^2+4) = 32$$

(2) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x}$

Solution: Multiply numerator and denominator by $\sqrt{1+x}+1$:

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \cdot \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$= \lim_{x\to 0} \frac{x}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1}.$$

So,

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}.$$

(3) $\lim\limits_{x\to -3} \dfrac{x^2-9}{\sqrt{x^2+7}-4}$

Solution: At $x = -3$: numerator $= 9 - 9 = 0$, denominator $= \sqrt{9+7} - 4 = 4 - 4 = 0$. Use conjugate:

$$\lim_{x \to -3} \frac{x^2 - 9}{\sqrt{x^2+7} - 4} \cdot \frac{\sqrt{x^2+7} + 4}{\sqrt{x^2+7} + 4}$$

$$= \lim_{x \to -3} \frac{(x^2-9)(\sqrt{x^2+7}+4)}{x^2+7-16}.$$

$$= \lim_{x \to -3} \frac{(x-3)(x+3)(\sqrt{x^2+7}+4)}{x^2-9}.$$

Carefully simplifying gives

$$\lim_{x \to -3} \frac{x^2 - 9}{\sqrt{x^2+7} - 4}$$

$$= \lim_{x \to -3} (\sqrt{x^2+7}+4) \cdot \frac{(x-3)(x+3)}{(x-3)(x+3)}$$

$$= \lim_{x \to -3} \sqrt{x^2+7} + 4 = \sqrt{9+7} + 4 = 4 + 4 = 8..$$

(4) $\displaystyle \lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$

Solution: Use identity $1 - \cos x = 2\sin^2(x/2)$:

$$\lim_{x \to 0} \frac{1 - \cos x}{x \sin x} = \lim_{x \to 0} \frac{2\sin^2(x/2)}{x \sin x}.$$

$$= \lim_{x \to 0} \frac{2\sin^2(x/2)}{x^2} \cdot \lim_{x \to 0} \frac{x}{\sin x}.$$

$$= 2(\frac{1}{4})(1) = \frac{1}{2}$$

(5) $\lim\limits_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

Solution: Use $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$:

$$\lim_{x\to 0} \frac{2x^2}{3 - 3\cos x} = \lim_{x\to 0} \frac{2x^2}{6\sin^2\left(\frac{x}{2}\right)}.$$

So,

$$= \frac{1}{3}\lim_{x\to 0}(\frac{x}{\sin\frac{x}{2}})^2 = \frac{4}{3}.$$

Exercises:

1. $\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}$

2. $\lim_{x\to a} \frac{x^2-a^2}{x^4-a^4}$

3. $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$

4. $\lim_{x\to 0} \frac{(x+h)^2-x^2}{h}$

5. $\lim_{x\to 0} \frac{\frac{1}{2+x}-\frac{1}{2}}{x}$

6. $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$

7. $\lim_{x\to 1} \frac{x^{1/3}-1}{\sqrt{x}-1}$

8. $\lim_{x\to 64} \frac{x^{2/3}-16}{\sqrt{x}-8}$

9. $\lim_{x\to 0} \frac{\tan(2x)}{\tan(\pi x)}$

10. $\lim_{x\to \pi} \csc x$

11. $\lim_{x\to \pi} \sin\left(\frac{x}{2} + \sin x\right)$

12. $\lim_{x\to \pi} \cos^2(x - \tan x)$

13. $\lim_{x\to 0} \frac{8x}{3\sin x - x}$

14. $\lim_{x\to 0} \frac{\cos 2x - 1}{\sin x}$

Theorem 2.3.1 *Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that*

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then

$$\lim_{x\to c} f(x) = L$$

.

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EX.12 Given that

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x\to 0} u(x)$.

Solution Since

$$\lim_{x\to 0}(1 - x^2/4) = 1 \quad \text{and} \quad \lim_{x\to 0}(1 + x^2/2) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$.
EX.13 For any function f, $\lim_{x\to c} |f(x)| = 0$, prove that $\lim_{x\to c} f(x) = 0$.

Solution
Since $-|f(x)| \le f(x) \le |f(x)|$ and $\lim_{x\to c} |f(x)|$ and $\lim_{x\to c} |f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x\to c} f(x) = 0$.

Limits at Infinity

Find the limits in Exercises .

1.

$$\lim_{x\to\infty} \frac{2x+3}{5x+7}$$

Solution: Divide numerator and denominator by x:

$$\lim_{x\to\infty} \frac{2+\frac{3}{x}}{5+\frac{7}{x}} = \frac{2}{5}.$$

2.

$$\lim_{x\to\infty} \frac{2x^2+3}{5x^2+7}$$

Solution: Divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{2+\frac{3}{x^2}}{5+\frac{7}{x^2}} = \frac{2}{5}.$$

3.

$$\lim_{x\to\infty} \frac{x^2-4x+8}{3x^3}$$

Solution: Divide numerator and denominator by x^3:

$$\lim_{x\to\infty} \frac{\frac{1}{x}-\frac{4}{x^2}+\frac{8}{x^3}}{3} = 0.$$

4.

$$\lim_{x\to\infty} \frac{1}{x^2-7x+1}$$

Solution: For large x, denominator grows like x^2:

$$\lim_{x\to\infty} \frac{1}{x^2-7x+1} = 0.$$

5.

$$\lim_{x\to\infty} \frac{x^2-7x}{x+1}$$

Solution: Divide numerator and denominator by x:

$$\lim_{x\to\infty} \frac{x-7}{1+\frac{1}{x}} = \infty.$$

6.

$$\lim_{x\to\infty} \frac{x^4+x^3}{12x^3+128}$$

Solution: Divide numerator and denominator by x^3:

$$\lim_{x\to\infty} \frac{x+1}{12+\frac{128}{x^3}} = \infty.$$

7.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$$

Solution: Divide numerator and denominator by x:

$$\lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \frac{1}{1} = 1.$$

8.

$$\lim_{x\to\infty} \frac{x-3}{\sqrt{4x^2+25}}$$

Solution: Divide numerator and denominator by x:

$$\lim_{x\to\infty} \frac{1-\frac{3}{x}}{\sqrt{4+\frac{25}{x^2}}} = \frac{1}{2}.$$

Continuity

Theorem 2.3.2 *A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:*

$$\lim_{x\to c} f(x) = L \quad \Longleftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \textit{and} \quad \lim_{x\to c^+} f(x) = L.$$

<u>**Interior point:**</u> A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \quad \text{respectively.}$$

<u>**Continuity test**</u>

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists $\qquad$ (c lies in the domain of f).

2. $\lim_{x\to c} f(x)$ exists $\qquad$ (f has a limit as $x \to c$).

3. $\lim_{x\to c} f(x) = f(c)$ $\qquad$ (the limit equals the function value).

Ex.1: discuss the continuity at $x = 3$ for the following functions:

$$f(x) = \frac{x^2 - 9}{x - 3}, \quad g(x) = \begin{cases} \frac{x^2-9}{x-3} & 4, x \neq 3 \\ 4 & x = 3 \end{cases}, \quad h(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$$

Solution

1. The function $f(x)$ is undefined at $x = 3$, and hence is not continuous at that point.

2. $lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x\to 3}(x + 3) = 6$, $f(3) = 4$

   The function value and the limit aren't the same and so the function is discontinuous at $x = 3$.

3. $\lim_{x\to 3} f(x) = 6$ $\quad$ and $\quad$ $f(3) = 6$
   The function is continuous at $x = 3$ since the function and limit have the same value.

Ex.2: Find values of a, b where the function

$$f(x) = \begin{cases} ax^2 + 1, & x > 2 \\ -11, & x = 2 \\ x^3 + b, & x < 2 \end{cases}$$

is continuous.

Solution:

f is continuous everywhere. In particular, f is continuous at $x = 2$. This implies that:

1. $\lim_{x \to 2} f(x)$ exist. This implies that:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$$

$$\Longrightarrow \lim_{x \to 2^+} (ax^2 + 1) = \lim_{x \to 2^-} (x^3 + b)$$

$$\Longrightarrow 4a + 1 = 8 + b \tag{1}$$

2. $\lim_{x \to 2} f(x) = f(2)$. this implies that

$$\lim_{x \to 2} f(x) = -11$$

$$\Longrightarrow \lim_{x\to 2}(4a+1) = -11 \qquad (\text{or } 8+b=-11)$$

$$\Longrightarrow 4a = -12$$

$$\Longrightarrow a = -3$$

Substitute $a = -3$ in (1), getting

$$b = -19$$

Ex.4: Find all values of the constant k that make

$$f(x) = \begin{cases} \frac{2x}{\tan kx}, & x < 0 \\ 3x + k, & x \geq 0 \end{cases}$$

is continuous at $x = 0$.

Solution: To make f continuous at $x = 0$, we must have $\lim\limits_{x\to 0} f(x)$ exist.

i.e.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\lim_{x \to 0^+} (3x + k) = \lim_{x \to 0^-} \frac{2x}{\tan kx} \implies 3(0) + k$$

$$= \lim_{x \to 0^-} \frac{2}{\frac{\tan kx}{x}} \implies k = \frac{2}{k}(1), k \neq 0$$

$$k^2 = 2 + k \implies k^2 - k - 2 = 0$$

$$\implies (k-2)(k+1) = 0 \implies k = 2 \text{ and } k = -1$$

Ex.5: Find value(s) for the constant k so that

$$f(x) = \begin{cases} \frac{\sin kx}{x}, & x < 0 \\ 3x + 2k^2, & x \geq 0 \end{cases}$$

will be continuous at $x = 0$.

Solution: To make f continuous at $x = 0$, we must have $\lim_{x \to 0} f(x)$ exist.

i.e.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\lim_{x\to 0^+}(3x+2k^2) = \lim_{x\to 0^-}\frac{\sin kx}{x} \implies 3(0)+2k^2$$

$$= k \implies 2k^2 - k = 0 \implies k(2k-1) = 0 \implies k = 0, k = \frac{1}{2}$$

EX.6 Discuss the continuity of $g(x)$ at $x = 3$

$$g(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \neq 3, \\ 5, & x = 3. \end{cases}$$

Solution. For $x \neq 3$, factor the numerator:

$$x^2 - x - 6 = (x-3)(x+2).$$

Thus

$$g(x) = x + 2 \qquad (x \neq 3).$$

Now evaluate the limit:

$$\lim_{x\to 3} g(x) = \lim_{x\to 3}(x+2) = 3 + 2 = 5.$$

Since $g(3) = 5$, we have

$$\lim_{x\to 3} g(x) = g(3) = 5.$$

Therefore, g is continuous at $x = 3$.

EX.7 Discuss the continuity of $f(x)$ at $x = 2$ and $x = -2$

$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \neq 2,\ x \neq -2, \\ 3, & x = 2, \\ 4, & x = -2. \end{cases}$$

Solution. First, **Limit at** $x = 2$**:**

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4} = \frac{3}{2}(2)^{3-2} = 3.$$

Given that $f(2) = 3$, we have

$$\lim_{x \to 2} f(x) = f(2) = 3,$$

so f is continuous at $x = 2$.

Limit at $x = -2$**:**

$$\lim_{x \to -2} \frac{x^3 - 8}{x^2 - 4} = \frac{(x-2)(x^2 + 2x + 4)}{(x-2)(x+2)} = \frac{x^2 + 2x + 4}{x + 2}.$$

At $x = -2$, the numerator is $4 - 4 + 4 = 4$ (nonzero), while the denominator tends to 0. Thus the expression diverges to $+\infty$ from one side and $-\infty$ from the other. Therefore the limit does not exist.

Since $f(-2) = 4$ but the limit at -2 does not exist, the function is *discontinuous* at $x = -2$.