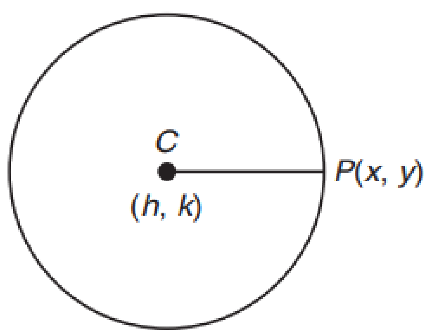
Circle

Definition 4.1.1: A circle is the locus of a point in a plane such that its distance from a fixed point in the plane is a constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.

4.2 EQUATION OF A CIRCLE WHOSE CENTRE IS (h, k) AND RADIUS r

Let C (h, k) be the centre of the circle and P (x, y) be any point on the circle. $CP = r$ is the radius of the circle. $CP^2 = r^2$ (i.e.) $(x-h)^2 + (y-k)^2 = r^2$. This is the equation of the required circle.



Note 4.2.1: If the centre of the circle is at the origin, then the equation of the circle is $x^2 + y^2 = r^2$.

4.3 CENTRE AND RADIUS OF A CIRCLE REPRESENTED BY THE EQUATION $x^2 + y^2 + 2gx + 2fy + c = 0$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$x^2 + y^2 + 2gx + 2fy = -c$$

Adding $g^2 + f^2$ to both sides, we get

$$\begin{aligned} & x^2 + y^2 + 2gx + 2fy + g^2 + f^2 = g^2 + f^2 - c \\ & \Rightarrow (x+g)^2 + (y+f)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2 \end{aligned} \tag{4.1}$$

This equation is of the form $(x-h)^2 + (y-k)^2 = r^2$, which is a circle with centre (h, k) and radius r. Thus, equation (4.1) represents a circle whose centre is $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$.

Note 4.3.1: A second degree equation in x and y will represent a circle if the coefficients of x^2 and y^2 are equal and the xy term is absent.

Note 4.3.2:

1. If $g^2 + f^2 - c$ is positive, then the equation represents a real circle.
2. If $g^2 + f^2 - c$ is zero, then the equation represents a point.
3. If $g^2 + f^2 - c$ is negative, then the equation represents an imaginary circle.

Example 4.1

Find the equation of the circle whose centre is (3, –2) and radius 3 units.

Solution

The equation of the circle is $(x-h)^2+(y-k)^2=r^2$

(i.e.) $(x-3)^2+(y+2)^2=3^2$

(i.e.) $x^2-6x+9+y^2+4y+4=9 \Rightarrow x^2+y^2-6x+4y+4=0$

Example 4.2

Find the equation of the circle whose centre is $(a, -a)$ and radius 'a'.

Solution

The centre of the circle is $(a, -a)$. The radius of the circle is a. The equation of the circle is $(x-a)^2+(y+a)^2=a^2$ (i.e.) $x^2-2ax+a^2+y^2+2ay+a^2=a^2$ (i.e.) $x^2+y^2-2ax+2ay+a^2=0$.

Example 4.3

Find the centre and radius of the following circles:

(i) $x^2 + y^2 - 14x + 6y + 9 = 0$

(ii) $5x^2 + 5y^2 + 4x - 8y - 16 = 0$

Solution

(i) $x^2 + y^2 - 14x + 6y + 9 = 0$

Centre is $(7, -3)$

Radius $= \sqrt{g^2 + f^2 = c} = \sqrt{49 + 9 - 9} = \sqrt{49} = 7$ units

(ii) $5x^2 + 5y^2 + 4x - 8y - 16 = 0$

Dividing this by 5, we get

$$x^2 + y^2 + \frac{4}{5}x - \frac{8}{5}y - \frac{16}{5} = 0$$

Centre is $\left(\frac{-2}{5}, \frac{4}{5}\right)$

Radius $= \sqrt{\frac{4}{25} + \frac{16}{25} + \frac{16}{5}} = \sqrt{\frac{100}{25}} = 2$ units

Example

Find the equation of the circle that has center $C(-4,-2)$ and contains the point (-1,2)

$$(x-h)^2+(\mathrm{y}-k)^2=r^2$$
$$(x+4)^2+(y+2)^2=r^2$$

The circle contains the point $(-1,2)$, the point satisfies the equation of the circle

$$\therefore (-1+4)^2+(2+2)^2=r^2$$
$$r^2=9+6=25$$
$$(x+4)^2+(2+\sqrt{2})^2=25$$

Example

Find the equation of the circle whose center in the point $(5,-3)$ and its radius equals to the radius of the circle

$$x^2+y^2-8x+4y-29=0$$

Solution

From the equation of the given circle

Center (4,-2), $\quad r=\sqrt{4^2+(-2)^2-(-29)}$

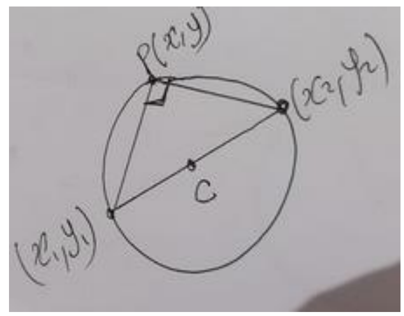
$$\therefore r=7$$

The radius of the required circle is 7and the center is $(5,-3)$ then its equation takes the form

$$(x-5)^2+(y+3)^2-7^2$$
$$x^2-10x+25+y^2+6y+9-49-0$$
$$x^2+y^2-10x+6y-15-0$$

The equation of the circle with (x_1, y_1), (x_2, y_2) are the ends of a diameter takes the form:

$$\boldsymbol{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$$



The two lines are perpendicular, then

$$m_1.m_2=-1$$

$$\Rightarrow\left(\frac{y-y_1}{x-x_1}\right)\left(\frac{y-y_2}{x-x_2}\right)=-1$$

$$(y-y_1)(y-y_1)=-(x-x_1)(x-x_2)$$
$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

Example

Find the equation of the Circle that has the points $(1,8)$, $(5,-6)$ as the endpoints of a diameter

Solution

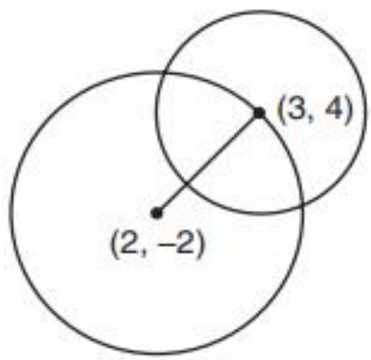
$$(x-1)(x-5)+(y-8)(y+6)=0$$
$$x^2-6x+5+y^2-2y-48=0$$
$$x^2+y^2-6x-2y-43=0$$

Example 4.4

Find the equation of the circle whose centre is $(2, -2)$ and which passes through the centre of the circle $x^2 + y^2 - 6x - 8y - 5 = 0$

Solution

The centre of the required circle is $(2, -2)$. The centre of the circle $x^2 + y^2 - 6x - 8y - 5 = 0$ is $(3, 4)$. The radius of the required circle is given by $r^2 = (2-3)^2 + (-2-4)^2 = 1 + 36 = 37$.



Therefore, the equation of the required circle is $(x-2)^2 + (y+2)^2 = 37$
(i.e.) $x^2 + y^2 - 4x + 4y - 29 = 0$

Tangent of a circle

suppose that we have the equation of the Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0,$$

and the line $\qquad ax + by + d = 0.$

The tangency Condition of this line to the given Circle is $\mathbf{P} = \mathbf{r}$, where P is the length of the perpendicular distance from the Center of the Circle to the to the tangent line and r is the radius of the circle

i.e. $\quad \frac{|-ag-bf+d|}{\sqrt{a^2+b^2}} = \sqrt{g^2 + f^2 - C}$

Example Prove that the line $2x + y = 4$ is a tangent to the circle

$$x^2 + y^2 + 6x - 10y + 29 = 0$$

Solution

From the equation of the circle

$$C(-3,5)$$
$$r = \sqrt{(-3)^2 + (5)^2 - 29} = \sqrt{9 + 25 - 29}$$
$$\therefore \quad r = \sqrt{5}$$

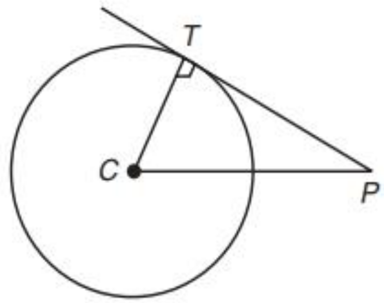
Then the distend from the center to the line is

$$P = \frac{|2(-3) + 5 - 4|}{\sqrt{(2)^2 + (1)^2}} = \frac{|-6 + 5 - 4|}{\sqrt{5}} = \frac{|-5|}{\sqrt{5}}$$
$$P = \frac{5}{\sqrt{5}} = \sqrt{5}$$
$$\text{Then} \quad P = r$$

and the line is the tangent to the circle.

4.4 LENGTH OF TANGENT FROM POINT $P(x_1, y_1)$ TO THE CIRCLE $x^2 + y^2 + 2gx + 2fy + c = 0$

The centre of the circle is $C(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$. Let PT be the tangent from P to the circle.



$$\begin{aligned} \text{Then } PT^2 = PC^2 - r^2 &= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c) \\ &= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \\ PT &= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} \end{aligned}$$

Note 4.4.1:

(1) If $PT^2 > 0$ then point $P(x_1, y_1)$ lies outside the circle.

(2) If $PT^2 = 0$ then the point $P(x_1, y_1)$ lies on the circle.

(3) If $PT^2 < 0$ then point $P(x_1, y_1)$ lies inside the circle.

Example 4.15

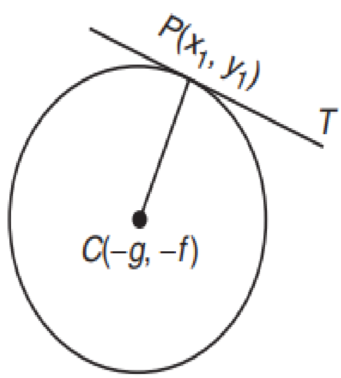
Find the length of the tangent from the point (2, 3) to the circle $x^2 + y^2 + 8x + 4y + 8 = 0$.

Solution

The length of the tangent from $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $PT^2 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$. Here, the length of the tangent from $P(2, 3)$ to the given circle is $PT = \sqrt{4+9+16+12+8} = \sqrt{49} = 7$ units.

4.5 EQUATION OF TANGENT AT (x_1, y_1) TO THE CIRCLE $x^2 + y^2 + 2gx + 2fy + c = 0$

The centre of the circle is $(-g, -f)$. The slope of the radius $CP = \dfrac{y_1 + f}{x_1 + g}$.



Hence, the equation of tangent at (x_1, y_1) is $(y - y_1) = m(x - x_1)$

(i.e.) $\quad y - y_1 = \dfrac{-(x_1 + g)}{(y_1 + f)}(x - x_1) \Rightarrow (y - y_1)(y_1 + f) = -(x_1 + g)(x - x_1)$

(i.e. $\quad yy_1 - y_1^2 + fy - fy_1 = -x_1 x - gx + x_1^2 + gx_1$

(i.e.) $\quad xx_1 + yy_1 + gx + fy = x_1^2 + y_1^2 + gx_1 + fy_1$

Adding $gx_1 + fy_1 + c$ to both sides,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

since the point (x_1, y_1) lies on the circle.

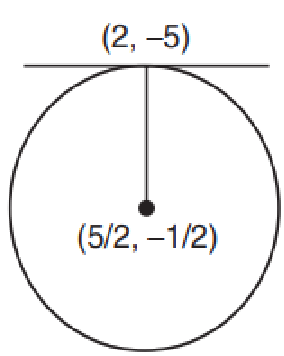
Hence, the equation of the tangent at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

Example 4.17

Find the equation of the tangent at the point (2, −5) on the circle $x^2 + y^2 - 5x + y - 14 = 0$.

Solution

Given $x^2 + y^2 - 5x + y - 14 = 0$



The equation of the tangent is given by

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Here we have: $x_1 = 2,\ y_1 = -5,\ g = \frac{-5}{2},\ f = \frac{1}{2}$, and $c = -14$.

$$2x - 5y - \frac{5}{2}(x + 2) + \frac{1}{2}(y - 5) - 14 = 0$$
$$x\left(2 - \frac{5}{2}\right) + y\left(-5 + \frac{1}{2}\right) + \left(-5 - \frac{5}{2} - 14\right) = 0$$
$$-\frac{1}{2}x - \frac{9}{2}y - \frac{43}{2} = 0$$
$$x + 9y + 43 = 0$$

Then the equation of the tangent is

$$x + 9y + 43 = 0.$$

Example 4.30

Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5).

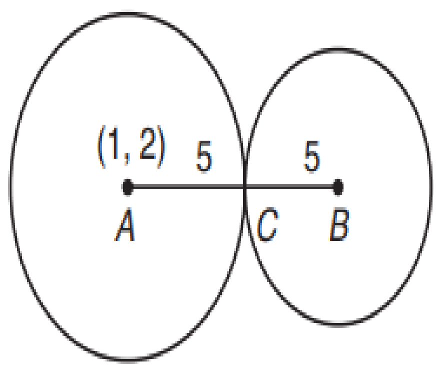
Solution

Given $x^2 + y^2 - 2x - 4y - 20 = 0$.

Centre is (1, 2) and radius $= r = \sqrt{1+4+20} = 5$

Let the centre of the required circle be (x_1, y_1). The point of contact is the midpoint of AB.

(i.e.) $\dfrac{(x+1)}{2} = 5,\ \dfrac{(y+2)}{2} = 5$



$\therefore\ x = 9$ and $y = 8$

Thus, B is (9, 8). Hence, the equation of the required circle is

$$(x-9)^2 + (y-8)^2 = 25$$

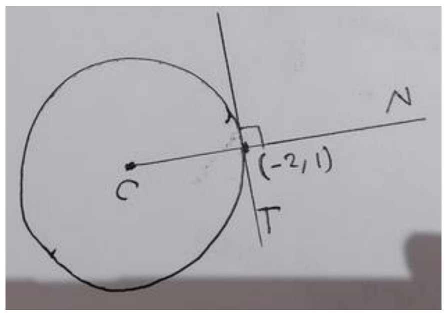
$$\text{(i.e.)}\ x^2 + y^2 - 18x - 16y + 120 = 0$$

Example

Find the equation of the tangent and the normal to the circle

$$x^2 + y^2 + 8x - 10y + 21 = 0 \text{ at the point } (-2,1)$$

Solution



From the equation of the circle

$$g = 4, \qquad f = -5, \qquad c = 21$$

then the equation of the tangent is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
$$-2x + y + 4(x - 2) - 5(y + 1) + 21 = 0$$
$$-2x + y + 4x - 8 - 5y - 5 + 21 = 0$$
$$2x - 4y + 8 = 0 \rightarrow x - 2y + 4 = 0$$

Slope of the tangent $= \dfrac{-1}{-2} = \dfrac{1}{2}$

Slope of the normal $= -2$

The equation of the normal

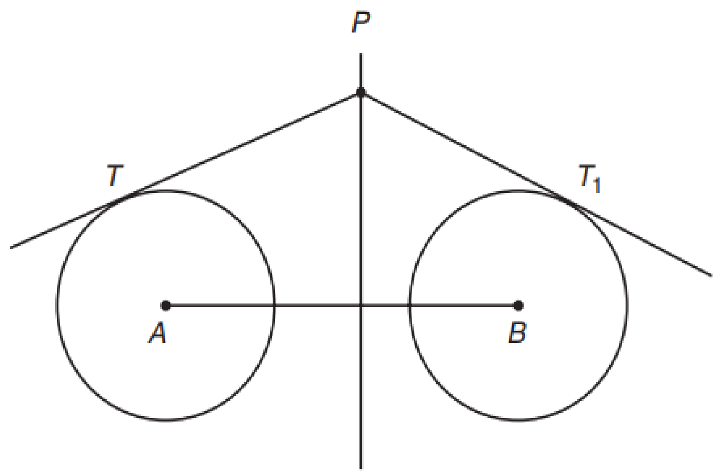
$$\frac{y - y_1}{x - x_1} = -2 \rightarrow \frac{y - 1}{x + 2} = -2 \rightarrow 2x + y + 3 = 0.$$

System of Circles

5.1 RADICAL AXIS OF TWO CIRCLES

Definition 5.1.1: The radical axis of two circles is defined as the locus of a point such that the lengths of tangents from it to the two circles are equal.

Obtain the equation of the radical axis of the two circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ and $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$.



Let $P(x_1, y_1)$ be a point such that the lengths of tangents to the two circles are equal.

$$\text{(i.e.)} \quad PT = PT_1$$
$$\Rightarrow PT^2 = PT_1^2$$
$$\therefore x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$
$$2(g - g_1)x_1 + 2(f - f_1)y_1 + c - c_1 = 0$$

The locus of (x_1, y_1) is $2(g - g_1)x + 2(f - f_1)y + (c - c_1) = 0$ which is a straight line.

Therefore, the radical axis of two given circle is a straight line.

Note 5.1.1: If $S = 0$ and $S_1 = 0$ are the equations of two circles with unit coefficients for x^2 and y^2 terms then the equation of the radical axis is $S - S_1 = 0$.

Note 5.1.2: Radical axis of two circles is a straight line perpendicular to the line of centres.

The centres of the two circles are $A(-g, -f)$ and $B(-g_1, -f_1)$.

The slope of the line of centres is $m_1 = \dfrac{-f + f_1}{-g + g_1} = \dfrac{f - f_1}{g - g_1}$

The slope of the radical axis is $m_2 = \dfrac{-(g - g_1)}{f_1 - f_1}$

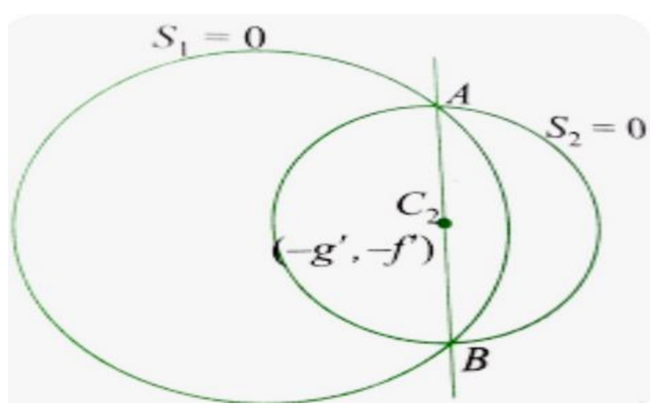
$$\therefore\ m_1 m_2 = -1$$

Therefore, the radical axis is perpendicular to the line of centres.

Note 5.1.3: If the two circles $S = 0$ and $S_1 = 0$ intersect then the radical axis is the common chord of the two circles.

Note 5.1.4: If the two circles touch each other, then the radical axis is the common tangent to the circles.

Note 5.1.5: If a circle bisects the circumference of another circle then the radical axis passes through the centre of the second circle.



Example 5.5.1

Find the radical axis of the two circles $x^2 + y^2 + 2x + 4y - 7 = 0$ and $x^2 + y^2 - 6x + 2y - 5 = 0$ and show that it is at right angles to the line of centres of the two circles.

Solution

$$x^2 + y^2 + 2x + 4y - 7 = 0$$
$$x^2 + y^2 - 6x + 2y - 5 = 0$$

The radical axis of the circles is $S - S_1 = 0$.

$$\text{(i.e.)} \quad 8x + 2y - 2 = 0$$
$$\text{(i.e.)} \quad 4x + y - 1 = 0$$

The slope of the radical axis is $m_1 = -4$.

The centres of the two circles are $(-1, -2)$ and $(3, -1)$.

The slope of the line of centres is $m_2 = \dfrac{-2+1}{-1-3} = \dfrac{1}{4}$.

$$m_1 m_2 = (-4)\left(\frac{1}{4}\right) = -1$$

Therefore, the radical axis is perpendicular to the line of centres.

Example 5.5.3

Show that the circles $x^2 + y^2 - 4x + 6y + 8 = 0$ and $x^2 + y^2 - 10x - 6y + 14 = 0$ touch each other and find the coordinates of the point of contact.

Solution

$$x^2 + y^2 - 4x + 6y + 8 = 0$$
$$x^2 + y^2 - 10x - 6y + 14 = 0$$

The radical axis of these two circles is $6x + 12y - 6 = 0$.

$$\text{(i.e.)} \quad x + 2y - 1 = 0$$

The centres of the circles are A (2, –3) and B (5, 3).

The radii of the circles are $r_1 = \sqrt{4+9-8} = \sqrt{5}$ and $r_2 = \sqrt{25+9-14} = 2\sqrt{5}$.

The perpendicular distance from $A(2, -3)$ on the radical axis $x + 2y - 1 = 0$ is $\frac{12-6-11}{\sqrt{1+4}} = \frac{5}{\sqrt{5}} = \sqrt{5}$ = radius of the first circle.

Therefore, radical axis touches the first circle and hence the two circles touch each other.

The equation of the lines of centres is $\frac{y+3}{x-2} = \frac{-3-3}{2-5} = \frac{-6}{-3} = 2$

or

$$2x - 4 = y + 3 \Rightarrow 2x - y - 7 = 0$$

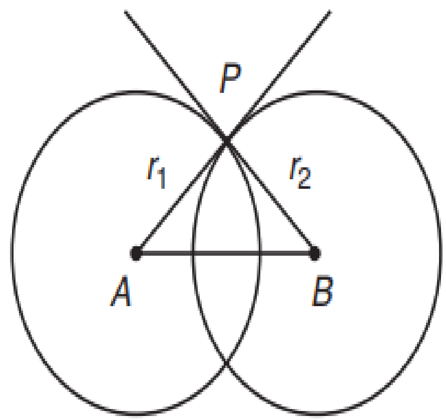
Solving (5.18) and (5.19), we get the point of contact.

Therefore, the point of contact is (3, –1).

5.2 ORTHOGONAL CIRCLES

Definition 5.2.1: Two circles are defined to be orthogonal if the tangents at their point of intersection are at right angles.

Find the condition for the circles $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ to be orthogonal.



Let P be a point of intersection of the two circles $S = 0$ and $S_1 = 0$. The centres are $A(-g, -f)$, $B(-g_1, -f_1)$.

The radii are $r_1 = \sqrt{g^2 + f^2 - c}$ and $r_2 = \sqrt{g_2^2 + f_2^2 - c_1}$.

Since the two circles are orthogonal, PA is perpendicular to PB.

(i.e.) APB is a right triangle.

$$\therefore AB^2 = PA^2 + PB^2$$

$$\Rightarrow (-g + g_1)^2 + (-f + f_1)^2 = (g^2 + f^2 - c) + (g_1^2 + f_1^2 - c_1)$$

$$\Rightarrow g^2 + g_1^2 - 2gg_1 + f^2 + f_1^2 - 2ff_1 = g^2 + g_1^2 + f^2 + f_1^2 - c - c_1$$

$$\Rightarrow 2gg_1 + 2ff_1 = c + c_1$$

Example 5.5.11

Find the equation to the circle which cuts orthogonally the three circles $x^2 + y^2 + 2x + 17y + 4 = 0$, $x^2 + y^2 + 7x + 6y + 11 = 0$ and $x^2 + y^2 - x + 22f + 33 = 0$.

Solution

Let the equation of the circle which cuts orthogonally the three given circles be $x^2 + y^2 + 2gx + 2fy + c = 0$.

Then the conditions for orthogonality are

$$2g + 17f = c + 4 \tag{5.41}$$

$$7g + 6f = c + 11 \tag{5.42}$$

$$-g + 22f = c + 3 \tag{5.43}$$

(5.41) – (5.42) gives $$-5g + 11f = -7 \tag{5.44}$$

(5.42) – (5.43) gives $$3g - 5f = 1 \tag{5.45}$$

(5.44) × 3 gives $$-15g + 33f = -21$$

(5.45) × 5 gives $$\underline{15g - 25f = 5}$$

$$8f = 16$$

$$\therefore f = -2$$

From (5.45), we get $3g + 10 = 1$

$$g = \frac{-9}{3}$$

$$= -3$$

From (5.41), we get $-6 - 34 = c + 4$ or $c = -44$

Therefore, the equation of the circle which cuts orthogonally the three given circles is $x^2 + y^2 - 6x - 4y - 44 = 0$.

Example

Find the equation of the circle which passes through the origin and cuts orthogonally each of the circles

$$S_1: x^2 + y^2 - 6x + 8 = 0$$
$$S_2: x^2 + y^2 - 2x - 2y - 7 = 0$$

Solution

Let the equation of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Since the circle passes through the origin, then $C = 0$.
From $S_1: g_1 = -3, f_1 = 0,\ c_1 = 8$
From the Condition of or thoyonality, we have

$$-6g + 0 = 8 + 0 \rightarrow g = -\frac{4}{3}$$

From $S_2: g_2 = -1, f_2 = -1, C_2 = -7$

$$\therefore -2g - 2f = -7 + 0$$
$$\frac{8}{3} - 2f = -7 \rightarrow f = \frac{29}{6}$$

$\therefore$ The equation is $x^2 + y^2 - \frac{8}{3}x + \frac{29}{3}y = 0$