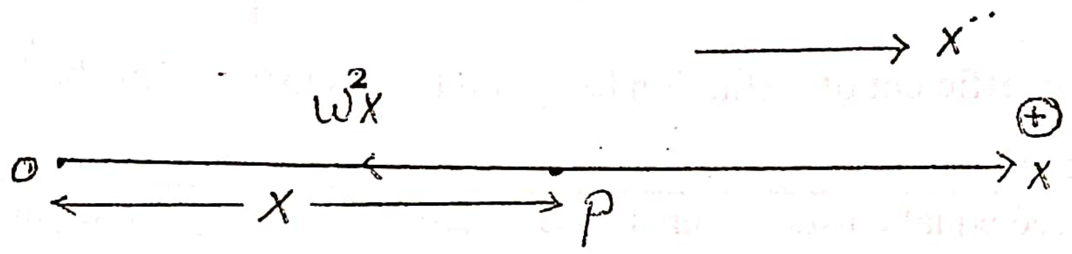
Simple Harmonic Motion

1. Definition:

If a particle moves in a straight line such that its acceleration is always directed towards a fixed point on that line and of magnitude which is proportional to the distance between the particle and that point, the motion of the particle is known as simple harmonic motion (S. H. M.). This means that the particle is acted upon by a central attractive force directed always to the fixed point (centre of attraction) and having magnitude which is proportional to the distance between the particle and that point.

Let 0 be the fixed point and P the position of the particle at time t where $OP = x$.



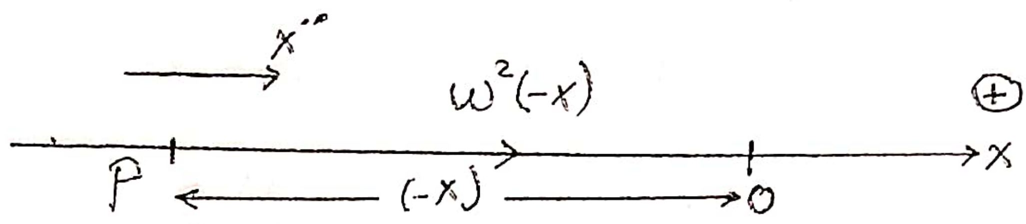
Since the magnitude of the acceleration of the particle at P is proportional to the distance OP, therefore this magnitude is equal to a positive constant w^2 multiplied by OP,

i.e. $\mathbf{w^2\,(OP)}$.

If the point P is to the right of O, the direction of the acceleration is the direction of x decreasing but $\ddot{x}$ is the acceleration in the direction of x increasing,

$$\therefore \quad \ddot{x} = -w^2 x \quad . \qquad (1)$$

But if P is to the left of O, x will be negative, i.e. the distance between the particle and the point O in this case is equal to (-x) and the magnitude of the acceleration will be equal to $w^2(-x)$ and its direction is again towards O, i.e. in the direction of x increasing.



The equation of motion in this case takes the form

$$x^{\bullet\bullet} = w^2(-x) = -w^2 x$$

which is the same as equation (1).

Equation (1) is the standard equation of motion which represents a simple harmonic motion in a straight line. In general, any motion which is represented by equation (1) where x is the displacement from a certain position is considered as simple harmonic motion.

For example, the variable x may be an angle θ, in this case equation (1) takes the form

$$\theta^{\bullet\bullet} = -w^2\theta ,$$

or the variable x may be the arc length s on a given curve, in this case equation (1) becomes

$$s^{\bullet\bullet} = -w^2 s .$$

2. Simple Harmonic Motion in a straight line :

$$\therefore \quad x^{\bullet\bullet} = -w^2 x \qquad (1)$$

$$\therefore \quad v\frac{dv}{dx} = -w^2 x ,$$

separating the variables

$$\therefore \quad v\,dv = -w^2 x\ dx .$$

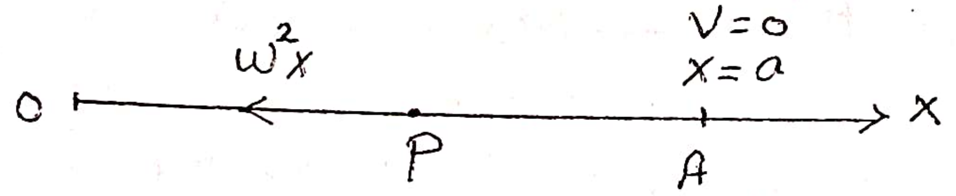
Integrating both sides

$$\therefore \quad \frac{1}{2}v^2 = -\frac{1}{2}w^2 x^2 + C.$$

Assume the particle was at rest at the point A to the right of O where OA = a, i.e.

$$v = 0 \ \ at \ \ x=a \ \ , \ \therefore \ C = \frac{1}{2} w^2 a^2 ,$$

$$v^2 = w^2 (a^2 - x^2). \qquad (2)$$



From equation (2), we have

$$v = \frac{dx}{dt} = -w \sqrt{a^2 - x^2} . \qquad (3)$$

The negative sign is taken since x decreases with the increase of time t. Separating the variables we get

$$\frac{-dx}{\sqrt{a^2 - x^2}} = w dt .$$

Integrating both sides

$$\therefore \qquad \cos^{-1} \frac{x}{a} = wt + \epsilon ,$$

where ϵ is a constant of integration which could be determined from the initial conditions.

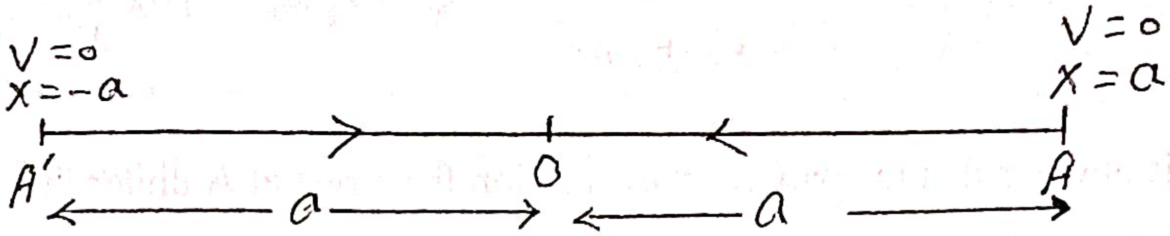
$$\therefore \qquad x = a \cos(wt + \epsilon) . \qquad (4)$$

Notice that $-a \le x \le a$, i.e. the particle will oscillate between the two points A(x = a), A'(x = -a) which are equidistant from the origin O known as the centre of oscillation (centre of S.H.M.). The range, OA or OA', of the moving point on either side of the centre O is called the amplitude of the motion.

$$\therefore \qquad \text{Amplitude of the S.H.M.} = a .$$

From equation (2), we see that the velocity of the particle vanishes at the two points A, A'.

These two points represent the positions of instantaneous rest and are called the ends of the motion.



To find v as a function of time t we differentiate (4) with respect to t, hence.

$$v = -wa \sin(wt + \in) . \qquad (5)$$

From (2), we find that the velocity of the particle is maximum when it passes by the point O,

i.e. the maximum velocity is given by

$$v_{\max} = w\,a . \qquad (6)$$

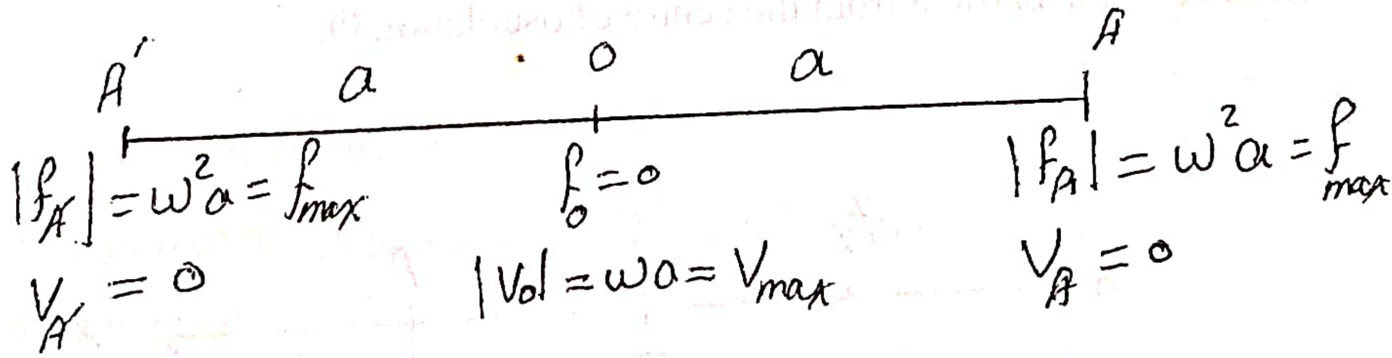
To find the acceleration f as a function of time t we differentiate (5) with respect to t, hence

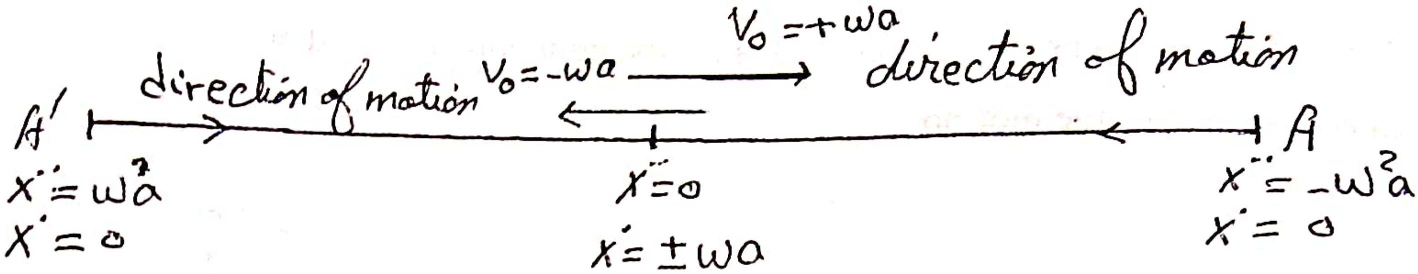
$$f = -w^2\, a \cos(wt + \in). \qquad (7)$$

(Equation (7) could be directly obtained from (1), (4))

From (1), it is clear that the maximum value of the acceleration of the particle through its motion is at the two ends A, A' and the value of this maximum acceleration is given by

$$f_{\max} = w^2\; a . \qquad (8)$$





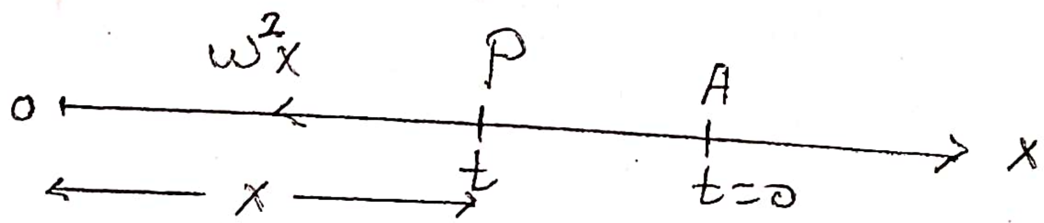
Now, it is obvious that the particle starts motion from rest at A under the action of the maximum value of the central force which attracts it towards O, and is equal to $m\, w^2 a$ then the value of the acceleration, which is proportional to the distance from O always, will decrease till it vanishes at O where the velocity of the particle is maximum and is equal to w a. The particle will continue to move to the left of O with a retardation till its velocity vanishes at A' then it returns from A' to A and so on.

3. <u>The Phase Angle</u>:

The angle $(wt+\epsilon)$ is known as the phase angle, Initially when t = o, the value of this angle is equal to ϵ, therefore ϵ is the initial phase angle. If the time t is measured from A, i.e. $x = a$ at $t = o$, we find from (4) that $\epsilon = 0$, and equations (4), (5)& (7) take the following forms:

$$x = a \cos wt\ ,$$
$$v = -wa \sin wt,$$
$$f = -w^2 a \cos wt = -w^2 x\ .$$

On using these relations, we must notice that if P represents the position of the particle at time t then t is the time taken from A to P while x is the distance of the particle from the centre of oscillation O.



4. Periodic Time:

If the particle moves from A to A' and returns back from A' to A we say that it has done a complete oscillation. The time taken in this complete oscillation is called the periodic time of the motion,

By substituting x = 0 in the relation

$$x = a \cos wt$$

we get $\cos wt = 0 \quad i.e.\ wt = \frac{\pi}{2}.$

∴ The time taken by the particle in moving from A to the centre O is equal to $\frac{\pi}{2w}$.

Multiplying by 4, we obtain the periodic time (P.T.)

$$\tau = \frac{2\pi}{w}. \qquad (9)$$

In general, the periodic time represents the time that elapses from any instant till the instant in which the moving particle is again moving through the same position with the same velocity and direction.

It will be noted that the periodic time, $\frac{2\pi}{w}$, is independent of the amplitude of the motion, but depends only on the value of w.

5. Frequency:

The number of complete oscillations made by the particle per second is equal to

$$\gamma = \frac{1}{\tau} = \frac{w}{2\pi}, \qquad (10)$$

γ is known as the frequency of the S.H.M.

6. Another form of the general solution of the equation of motion:

We have found that

$$x = a \cos (wt + \epsilon)$$

is the general solution of the differential equation

$$x^{\bullet\bullet} = -w^2 x ,$$

where a is the amplitude and $\in$ is the initial phase angle.

This solution could always be writ then in the form

$$x = a(\cos wt \cos \in - \sin wt \sin \in)$$

i.e.

$$x = A \cos wt + \mathbf{B} \sin wt ,$$

where A , B are constants such that

$A = a \cos \in \quad , \quad \mathbf{B} = -a \sin \in ,$

$\therefore \quad A^2 + B^2 = a^2 \quad , \quad \dfrac{-B}{A} = \tan \in$

i.e. the amplitude is given by $a = \sqrt{A^2 + B^2}$, and the initial phase angle is $\in = \tan^{-1}(\dfrac{-B}{A})$.

When using the solution in the form:

$$x = A \cos wt + B \sin wt ,$$

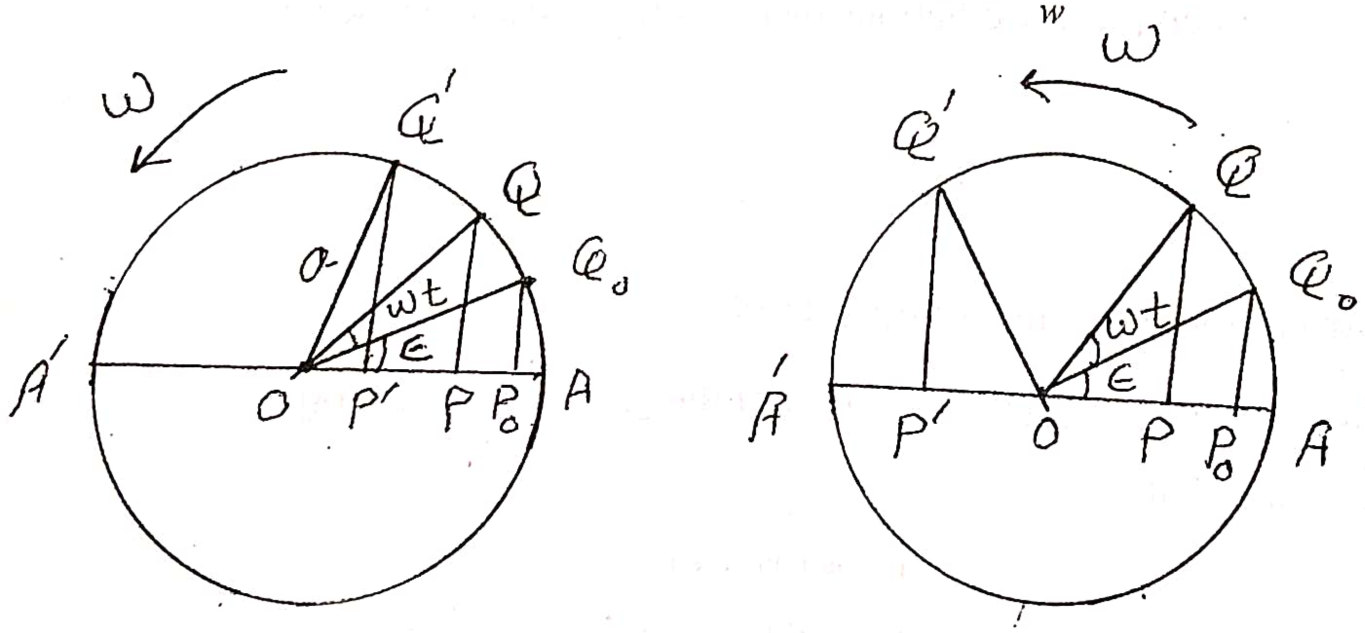
we have

$$x^{\bullet} = -Aw \sin wt + Bw \cos wt ,$$

and so from the initial conditions we obtain two linear equations in A and B from them we can find the values of these constants.

7. <u>Geometric representation of the simple harmonic motion</u>:

If a point Q describes a circle of radius a with uniform angular velocity w, and if P be always the foot of the perpendicular from Q upon a fixed diameter AOA' of the circle, we will prove that P moves with simple harmonic motion of amplitude a and periodic time $\dfrac{2\pi}{w}$.



Let AOA' be a fixed diameter of the circle whose centre is at O and radius a and let the point Q describe the circle with uniform angular velocity w about O.

Let Q_o be the position of the point at t = 0 where $A\hat{O}Q_o = \in$, Q be the position of the point at time t and P_o, P are the feet of the perpendiculars from Q_o, Q upon the diameter AOA'.

$\because$ the angular velocity is uniform and is equal to w rad./sec.

$$\therefore\ Q_o\hat{O}Q = w\,t\,,$$

$$A\hat{O}Q = w\,t + \in .$$

If $OP_o = x_o$, $OP = x$,

$$\therefore\quad x = OQ\ \cos\ A\hat{O}Q$$

$$= a\ \cos\ (wt + \in),$$

where $\in = \cos^{-1}\dfrac{x_o}{a}$.

This is the same relation which represents a simple harmonic motion, i.e. as Q describes the circle, its foot P on the diameter AOA' will move with S.H.M. of amplitude a and frequency $\dfrac{w}{2\pi}$.

This geometric representation helps in finding the time taken between any two positions P , P' on the diameter AOA' since this time is equal to the time taken between the corresponding positions Q , Q' in the circular motion with uniform angular velocity w. This time is given by:

$$t_{P \to P'} = t_{Q \to Q'} = \frac{Q\hat{O}Q'}{\omega},$$

where $Q\hat{O}Q'$ is in radians.

If P, P' are in the same side from 0 (fig. (i)), this time will be equal to

$$t_{P \to P'} = \frac{1}{w}\left(\cos^{-1}\frac{OP'}{a} - \cos^{-1}\frac{OP}{a}\right),$$

but if P , P' are on opposite sides from O (fig. (ii)), this time is given by

$$t_{P \to P'} = \frac{1}{w}\left(\pi - \cos^{-1}\frac{OP'}{a} - \cos^{-1}\frac{OP}{a}\right).$$

The velocity of the point P is given by

$$\begin{aligned} v &= -w\,a \sin(w+\in) \\ &= -w(OQ)\sin A\hat{O}Q \\ &= -w\,(PQ), \end{aligned}$$

i.e. the magnitude of the velocity at P is proportional to the ordinate of the corresponding point Q .

When P makes a complete oscillation on the diameter AOA', the point Q makes a complete revolution on the circle, i.e. Q revolves an angle equal to 2π radians with uniform angular velocity w. Hence the periodic time τ is equal to $\frac{2\pi}{w}$ as before.

8. <u>Compounding of two simple harmonic motions of the same period and in the same straight line:</u>

The most general displacements of this kind are given by

$$a\cos(wt+\in) \text{ and } b\cos(wt+\in')$$

so that

$$\begin{aligned} x &= a\cos(wt+\in) + b\cos(wt+\in') \\ &= a\,(\cos wt\cos\in - \sin wt\sin\in) \\ &\quad + b\,(\cos wt\cos\in' - \sin wt\sin\in') \\ &= \cos wt\,(a\cos\in + b\cos\in') \\ &\quad - \sin wt\,(a\sin\in + b\sin\in'). \end{aligned}$$

Let $a\cos\in + b\cos\in' = A\cos E$,

and $a\sin\in + b\sin\in' = A\sin E$

so that

$$A = \sqrt{a^2+b^2+2ab\cos(\in-\in')}\ , \text{ and}$$

$$\tan E = \frac{a\sin\in + b\sin\in'}{a\cos\in + b\cos\in'}.$$

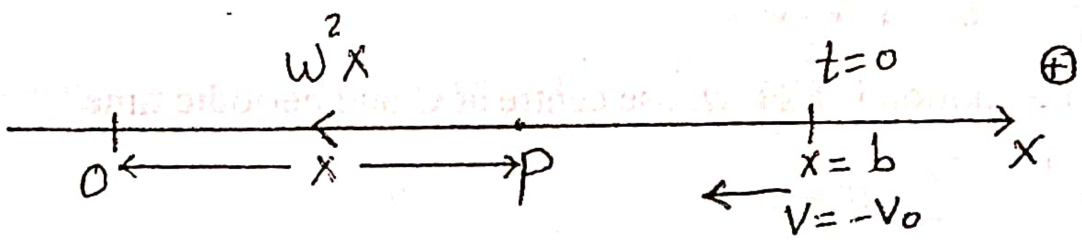
Then $x = A\cos(wt+E)$,

So that the composition of the two given motions gives a similar motion of the same period whose amplitude and initial phase angle are known. So with more than two such motions of the same period.

9. Examples:

1- A particle moves in a straight line with acceleration which is always directed towards fixed point O, and is equal to wx^2 where x is the distance from O. If the particle starts to move with velocity V_0 towards O and at a distance b from it, prove that its motion is simple harmonic motion with amplitude $\sqrt{b^2+\frac{V_0^2}{w^2}}$ and

$x = b\cos wt - \frac{V_0}{w}\sin wt$, where t is measured from the starting point.



Since $\quad x^{\bullet\bullet} = -w^2\, x$, then the motion is S.H. with center at O.

Then the general solution is given by $x = A\cos wt + B\sin wt$

Initially t = 0, x = b and $v = V_0$

i.e A = b $\quad$ and $\quad B = -\frac{V_0}{w} \Rightarrow x = b\cos wt - \frac{V_0}{w}\sin wt$, and the amplitude

is equal to $\sqrt{A^2+B^2} = \sqrt{b^2+\frac{V_0^2}{w^2}}$.

2- A particle moves in a straight line such that

$$x = 3\cos\frac{\pi t}{5} + 4\sin\frac{\pi t}{5}$$

where t is in seconds and x in cms.

Prove that the motion is S.H. and find :

(i) Periodic time and amplitude.

(ii) Maximum velocity and maximum acceleration.

(iii) The initial phase angle.

(iv) The velocity and acceleration of the particle at a distance 3 cms. from the centre of oscillation.

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i) since $x = 3\cos\frac{\pi t}{5} + 4\sin\frac{\pi t}{5}$, then $\dot{x} = \frac{-3\pi}{5}\sin\frac{\pi t}{5} + \frac{4\pi}{5}\cos\frac{\pi t}{5}$

and $\ddot{x} = -\left(\frac{\pi}{5}\right)^2\left(3\cos\frac{\pi t}{5} + 4\sin\frac{\pi t}{5}\right) = -\left(\frac{\pi}{5}\right)^2 x$

i.e $\ddot{x} = -w^2 x$

i.e the motion is S.H. whose centre at O and periodic time

$$\tau = \frac{2\pi}{w} = \frac{2\pi}{\frac{\pi}{5}} = 10\,\text{sec.}$$

since A = 3 and B = 4, then the amplitude $a = \sqrt{A^2 + B^2} = 5$.

ii) $V_{\max} = wa = \pi\ cm./\sec.$ and $f_{\max} = w^2 a = \frac{\pi^2}{5} cm./\sec^2$.

iii) The initial phase angle ε is given by $\varepsilon = \tan^{-1}\left(\frac{-B}{A}\right) = \tan^{-1}\left(\frac{-4}{3}\right)$

iv) To evaluate V and f at x = 3, we get

$V^2 = w^2\left(a^2 - x^2\right) = \frac{\pi^2}{25}(25-9) \Rightarrow V = \frac{4\pi}{5} cm./\sec.$ The magnitude of acceleration at x = 3 is given by $f = w^2 x = \frac{3\pi^2}{25} cm./\sec^2$.

3- A particle moves in a straight line with S.H.M. P and Q are two points on that line distant 6 cms. And 8 cms. respectively from the centre of motion.

If the velocities of the particle when passing through these two points are 8 m./sec. and 6 m./sec. respectively. Find the frequency and amplitude of the motion. Find also the time taken from P to Q if :

(i) P , Q are in the same side from O .

(ii) P , Q are on opposite sides from O .

Since $V^2 = w^2(a^2 - x^2)$, then at P x = 6 and V = 8

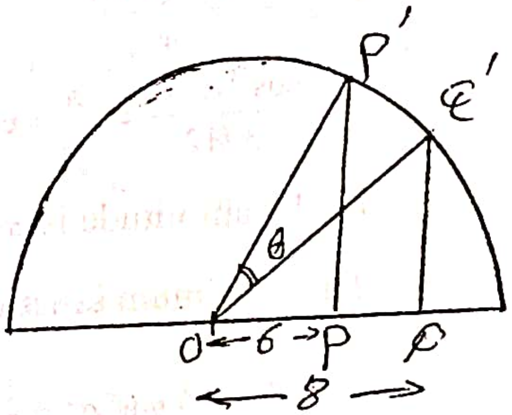
$$64 = w^2(a^2 - 36) \qquad (1)$$

Also at Q x = 8 and V = 6 .

$$36 = w^2(a^2 - 64) \qquad (2)$$

From (1) and (2), we get $w = 1$ *and* $a = 10$.

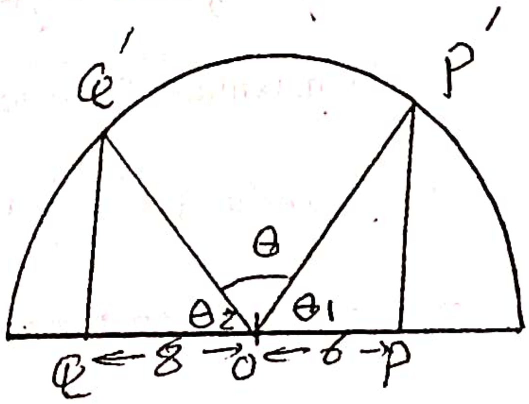
The frequency $\upsilon = \frac{1}{\tau} = \frac{w}{2\pi} = \frac{1}{2\pi}$ *cycle/sec.*



Case (i)

$$t_{P\to Q} = \frac{\theta}{w} = \cos^{-1}\left(\frac{6}{10}\right) - \cos^{-1}\left(\frac{8}{10}\right)$$

$$= \cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{4}{5}\right) \text{ sec.}$$



Case (ii)

$$t_{P\to Q} = \frac{\theta}{w} = \frac{\pi - (\theta_1 + \theta_2)}{w}$$

$$= \pi - \left[\cos^{-1}\left(\frac{6}{10}\right) \underset{+}{-} \cos^{-1}\left(\frac{8}{10}\right)\right]$$

$$= \pi - \left[\cos^{-1}\left(\frac{3}{5}\right) \underset{+}{-} \cos^{-1}\left(\frac{4}{5}\right)\right] \text{ sec.}$$

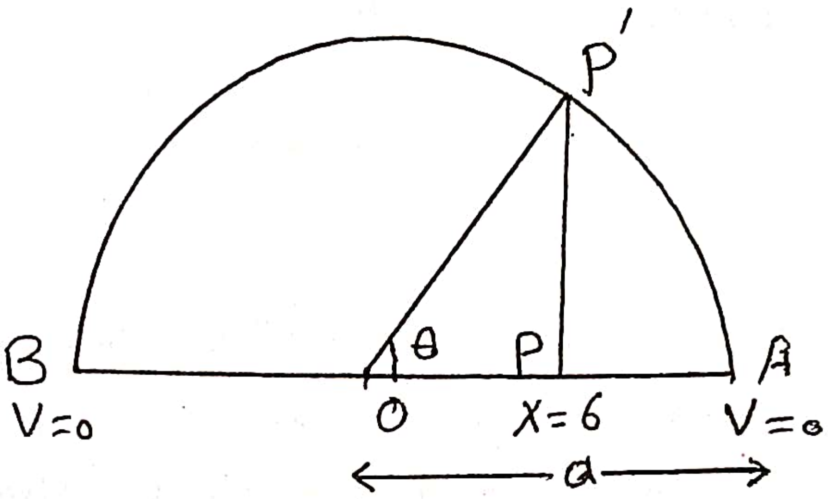
(4-) A particle of mass 24 lbs. moves in a straight line with S.H.M. The ends of this motion are at A , B and P in a point on the path of the particle. If the time from A to P is 2 secs, and from P to B is 10 secs when P is at a distance of 6 ft. from the centre of oscillation, find the maximum kinetic energy of the particle and the maximum force acting on it.

The periodic time is

$$\tau = 2(t_{A\to P} + t_{P\to B}) =$$

$$= 2(2+10) = 24 = \frac{2\pi}{w}$$

$$\therefore \quad w = \frac{\pi}{12}$$



$$t_{A\to P} = \frac{\cos^{-1}(6/a)}{w}$$

$$2 = \frac{\cos^{-1}(6/a)}{\pi/12} \Rightarrow \frac{\pi}{6} = \cos^{-1}(6/a)$$

i.e the amplitude is $a = 4\sqrt{3}$

The maximum kinetic energy is

$$\frac{1}{2}mV_{\max.}^2 = \frac{1}{2}mw^2a^2 = \frac{1}{2}(24)\left(\frac{\pi}{12}\right)^2\left(4\sqrt{3}\right)^2 = 4\pi^2$$

The maximum force acting on it is

$$m f_{\max.} = mw^2a = (24)\left(\frac{\pi}{12}\right)^2\left(4\sqrt{3}\right) = \frac{2\pi^2\sqrt{3}}{3}$$

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10. <u>Elastic strings and spiral springs (Hooke's law):</u>

The strain in elastic strings is defined as the ratio between the increase in length (extension) and the natural length, i.e. the increase in unit length of the string.

If a is the natural length of the string, and x is the extension, the strain is given by $\text{strain} = \frac{x}{a}$.

The stress is defined as the force per unit area.

If T is the tension in the string, S its cross- sectional area, the stress is given by

$\text{stress} = \frac{T}{S}$.

Hooke's law states that stress is proportional to strain, i.e.

$$\frac{T}{S} = E\left(\frac{x}{a}\right)$$

$$\therefore \quad T = ES\left(\frac{x}{a}\right)$$

where E is a constant depending on the material of the string and is known as Younge's modulus.

This law could be written in either of the two forms:

$$T = kx$$

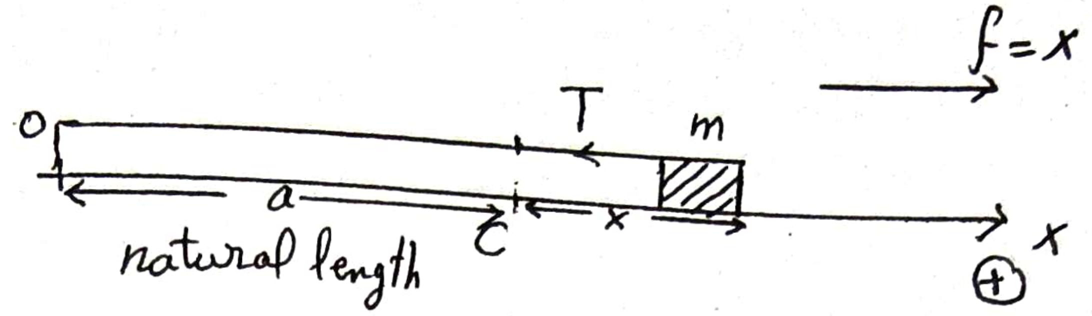
$$or \quad T = \frac{\lambda}{a}x$$

where k is a constant known as the stiffness constant, and λ is another constant known as the modulus of elasticity.

Hooke's law could also be used to determine the tension in spiral springs where x , a are measured on the axis of the spring. Also it could be used to determine the pressure in springs considered as a negative tension where x is the contraction.

Finally Hooke's law is an experimental law and it holds when x is less than the elastic limit for the string or the spring.

11- **Motion of a particle attached to a spiral spring on a smooth horizontal table :**



Let a particle of mass m be attached to one end of a spiral spring on a smooth horizontal table while the other end is fastened to a fixed point O on the table. Let the spring be of unstretched length a and stiffness constant k.

Consider the particle is moving on the straight line Ox, where C is the position of equilibrium (OC = a).

If x (> 0) is the distance between the particle and the point C at the instant t and T is the tension in the spring at this position.

The equation of motion of the particle is

$$m\ddot{x} = -T = -kx$$

i.e.

$$\ddot{x} = -\frac{k}{m}x \qquad (1)$$

$$or, \quad \ddot{x} = -\frac{\lambda}{ma}x$$

where λ is the modulus of elasticity of the spring.

The same equation is obtained if $x < 0$,

i.e. the spring is under contraction.

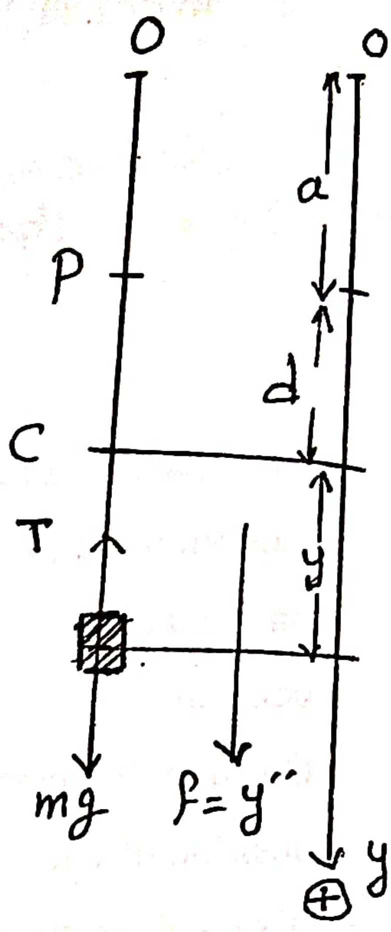
Therefore the motion of the particle in simple harmonic with centre at the equilibrium position C and periodic time given by

$$\tau = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ma}{\lambda}} ,$$

while the amplitude depends on the initial conditions of the motion.

12. <u>Motion of a particle suspended by a spiral spring:</u>

Consider a particle of mass m suspended from a fixed point O by a spiral spring of natural length a and stiffness constant k .



The forces acting on the particle are its weight mg and the tension or compression in the spring T, these two forces are equal when the particle is at the position of equilibrium.

Let the axis Oy be taken vertically downwards and assume P is the end of the natural length of the spring, C is the equilibrium position of the particle. If d is the extension of the spring at C, we have

$$m\,g = k\,d\,. \qquad (1)$$

Let the particle be below P at a distance y from C and T is the tension in the spring.

The equation of motion of the particle is

$$m\,\ddot{y} = m\,g - T$$

$$= m\,g - k\,(d+y)$$

and from the condition (1). Therefore

$$m\,\ddot{y} = -\,k\,y \;, \qquad i.e.$$

$$\ddot{y} = -\frac{k}{m}\,y \qquad (2)$$

$$or, \qquad \ddot{y} = -\frac{\lambda}{m\,a}\,y\,.$$

The same equation is obtained for the motion of the particle above P, i.e. when the spring is compressed.

Therefore the motion of the particle is simple harmonic about the equilibrium position C with periodic time given by

$$\tau = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m\,a}{\lambda}}\ ,$$

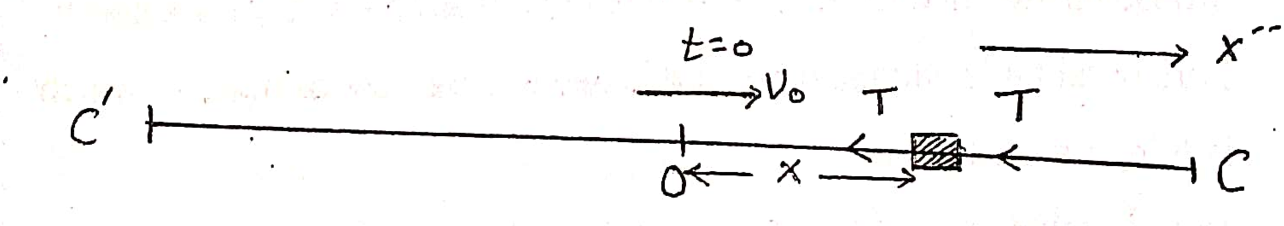
while the amplitude depends on the initial conditions of the motion.

13. <u>Remark</u>:

It is known that strings in general may not be compressed but may only be under tension. So if the spring in the previous two articles be replaced by an elastic string, the results obtained will represent the motion so long as the distance between the particle and the fixed point O is greater than the natural length of the string. But if this distance is less than the natural length, the particle will move with uniform velocity if the string is horizontal or under gravity (i.e. with uniform acceleration = $\pm g$) if the string in vertical.

14. <u>Examples</u>:

1) A particle of mass 1 Kgm. Is attached to two equal springs each of natural length 28 cms. And stiffness constant 100 gm. wt./cm. and the system rests on a smooth horizontal table with the other ends of the two springs fastened to two points distant 56 cms. in between. The particle receives a blow of magnitude 100 gms. wt. sec. in the direction of the springs. Prove that the motion of the particle is simple harmonic, and find its position after time t.



Let C , C' be the two fixed points, where CC' = 56 cms., and let O be the middle point. Therefore O is the equilibrium position of the particle. After the blow the particle will move from O with initial velocity

$$V_o = \frac{100 \times 980}{1000} = 98 \ cm./sec.$$

After time t let the position of the particle be at a distance x from 0.

Since OC and OC' are each equal to the natural length of either spring, one will be under tension where the extension is x while the other is compressed where the contraction is also equal to x.

$\therefore$ The tension in the left spring is kx

i.e. 100 x 980x and this value also represents the compression in the right spring.

The equation of motion of the particle is therefore given by :

$$\begin{aligned} 1000\, \ddot{x} &= -T - T \\ &= -2T \\ &= -2 \times 100 \times 980 x \end{aligned}$$

$$i.e. \qquad \ddot{x} = -196\, x$$

and this represents a S.H.M. about the equilibrium position O. The general solution of this equation is therefore

$$x = A \cos 14t + B \sin 14t \qquad (1)$$

where A , B are constants to be determined from the initial conditions.

Now, $\qquad \dot{x} = -14\, A \sin 14t + 14 B \cos 14\, t \qquad (2)$

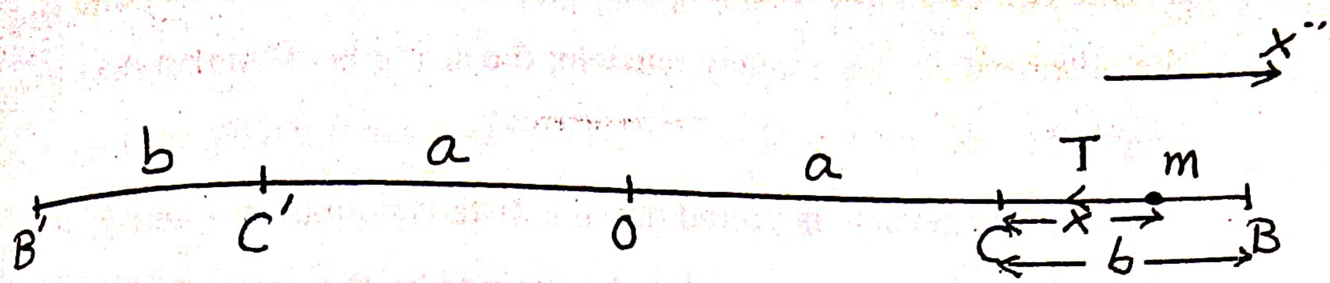
When t = 0 , x = 0 and $\dot{x} = V_o = 98$, substituting in equations (1) and (2), we get : A = 0 , B = 7 .

$\therefore$ The position of the particle after time t is given by: $x = 7 \sin 14t$.

---

(2) One end of an elastic string, whose modulus of elasticity is λ and whose unstretched length is a, is fixed to a fixed point O on a smooth horizontal table and the other end is tied to a particle of mass m which is lying on the table.

The particle is pulled to a distance where the extension of the string is b and then let go; show that the time of a complete oscillation is $4\sqrt{\frac{am}{\lambda}}\left(\frac{\pi}{2}+\frac{a}{b}\right)$.



Let C be the position of the particle when the string is in its natural length a , B be its position at the beginning of its motion where the length of the string is $a+b$, i.e. $CB = b$.

Through the motion from B to C , the force acting on the particle is the tension T.
If x is the distance from C at time t, the equation of motion from B to C is given by :

$$m\ddot{x} = -T = -\frac{\lambda}{a}x,$$

i.e.

$$\ddot{x} = -\frac{\lambda}{ma}x$$

i.e. this motion is simple harmonic whose centre is at C, amplitude b, and the time taken in this distance is quarter the periodic time, i.e. is equal to $\frac{\tau}{4} = \frac{\pi}{2}\sqrt{\frac{ma}{\lambda}}$.

When the particle reaches the point C, its velocity is maximum and is given by $V = b\sqrt{\frac{\lambda}{ma}}$.

After that point, the particle m will move with uniform velocity equal to $b\sqrt{\frac{\lambda}{ma}}$ .till it reaches C' where OC' = OC = a and then the string become stretched again and so on.

The time taken from C to O is given by $\frac{a}{V} = \frac{a}{b}\sqrt{\frac{ma}{\lambda}}$.

∴ The time taken from B to O is equal to

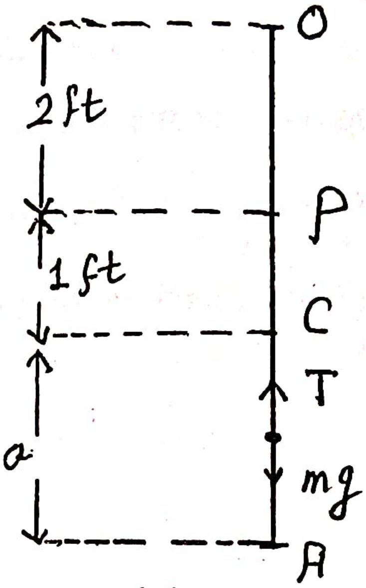
$$\frac{\pi}{2}\sqrt{\frac{ma}{\lambda}} + \frac{a}{b}\sqrt{\frac{ma}{\lambda}} = \sqrt{\frac{ma}{\lambda}}\left(\frac{\pi}{2} + \frac{a}{b}\right),$$

and the time of a complete oscillation is $4\sqrt{\frac{ma}{\lambda}}\left(\frac{\pi}{2} + \frac{a}{b}\right)$.

3) One end of a light extensible string is fastened to a fixed point and the other end carries a heavy particle; the string is of unstretched length 2 ft. and its modulus of elasticity is twice the weight of the particle. The particle is pulled down a distance equal to a and then released. Find the maximum height attained by the particle and the time of a complete oscillation when:

(i) $a = \frac{2}{3}$ ft. , (ii) $a = 2$ ft.

Let OP = 2 ft. be the natural length of the string, C be the position of equilibrium of the particle of mass m.



At C, $mg = \frac{\lambda}{2}(PC)$

where $\lambda = 2mg$,

$\therefore\ PC = 1$ ft.

If y is the distance of the particle from C measured downwards after time t from the initial position at A, the equation of motion of the particle below P is given by :

$$m\,\ddot{y} = mg - T$$

$$= mg - \frac{2mg}{2}(y+1),$$

i.e. $\ddot{y} = -gy = -32y$. (1)

Therefore the motion of the particle below P is simple harmonic with centre at C and periodic time $\frac{2\pi}{\sqrt{g}} = \frac{2\pi}{\sqrt{32}} = \frac{\pi\sqrt{2}}{4}$.

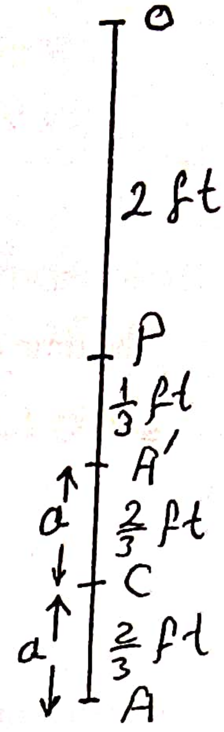
The amplitude of this motion is AC = a .

(i) If $a = \frac{2}{3}$ ft.

In this case, the amplitude of the S.H.M.

a < CP, therefore the motion of the

particle is always below P and is given by equation (1).



$\therefore v_{A'}=0$ where $CA'=a=\frac{2}{3}$ ft.

and the maximum height is

$AA'=2a=\frac{4}{3}$ ft.

The time of a complete oscillation is equal to the periodic time of the S.H.M., i.e. is equal to $\frac{\pi\sqrt{2}}{4}$ sec.

(ii) If a = 2 ft.

In this case, the amplitude of the S.H.M. a > CP, therefore the motion of the particle is S.H. only from A to P and is given by (1).

Using the law of velocity

$v^2=w^2(a^2-y^2)=32(4-y^2)$.

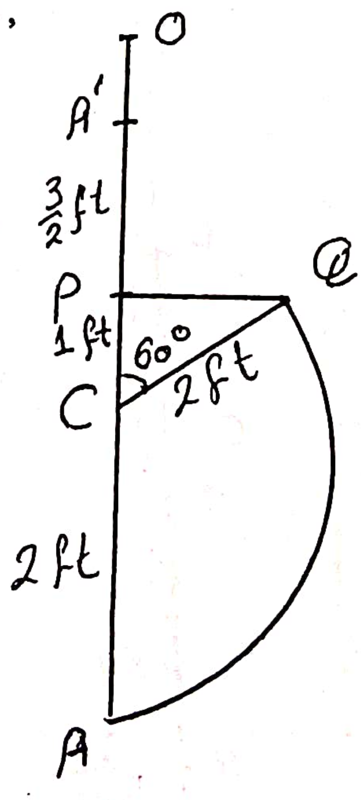
At P, y = -1, $\therefore |v_p|=4\sqrt{6}$ ft./sec.

After P, the motion of the particle is under gravity, i.e. with acceleration (-g).

Let the velocity vanishes at a point A' above P where PA' = h say.

$$\therefore \quad O=(4\sqrt{6})^2-2gh,$$

$$\text{i.e.} \quad h=\frac{96}{2\times 32}=\frac{3}{2} \text{ ft.}$$



$\therefore$ The maximum height is

$AA'=AP+h=3+\frac{3}{2}=4.5$ ft.

To find the time of the motion from A to P we draw a circle of centre at C and radius 2 ft.

$\therefore t_1=t_{A\to P}=t_{A\to Q}$ for the motion on this circle with uniform angular velocity

$$\omega = \sqrt{32} = 4\sqrt{2}\ rad./sec.$$

$$\therefore \quad t_1 = \frac{A\hat{C}Q}{\omega} = \frac{1}{4\sqrt{2}}(\pi - \cos^{-1}\frac{1}{2})$$

$$= \frac{\pi\sqrt{2}}{12}\ \text{sec.}$$

The time of the motion under

gravity from P to A' is given by $0 = 4\sqrt{6} - g t_2$,

i.e. $$t_2 = \frac{4\sqrt{6}}{32} = \frac{\sqrt{6}}{8}\ \text{sec.}$$

$$\therefore t_{A \to A'} = t_1 + t_2 = (\frac{\pi\sqrt{2}}{12} + \frac{\sqrt{6}}{8})\ \text{sec.}$$

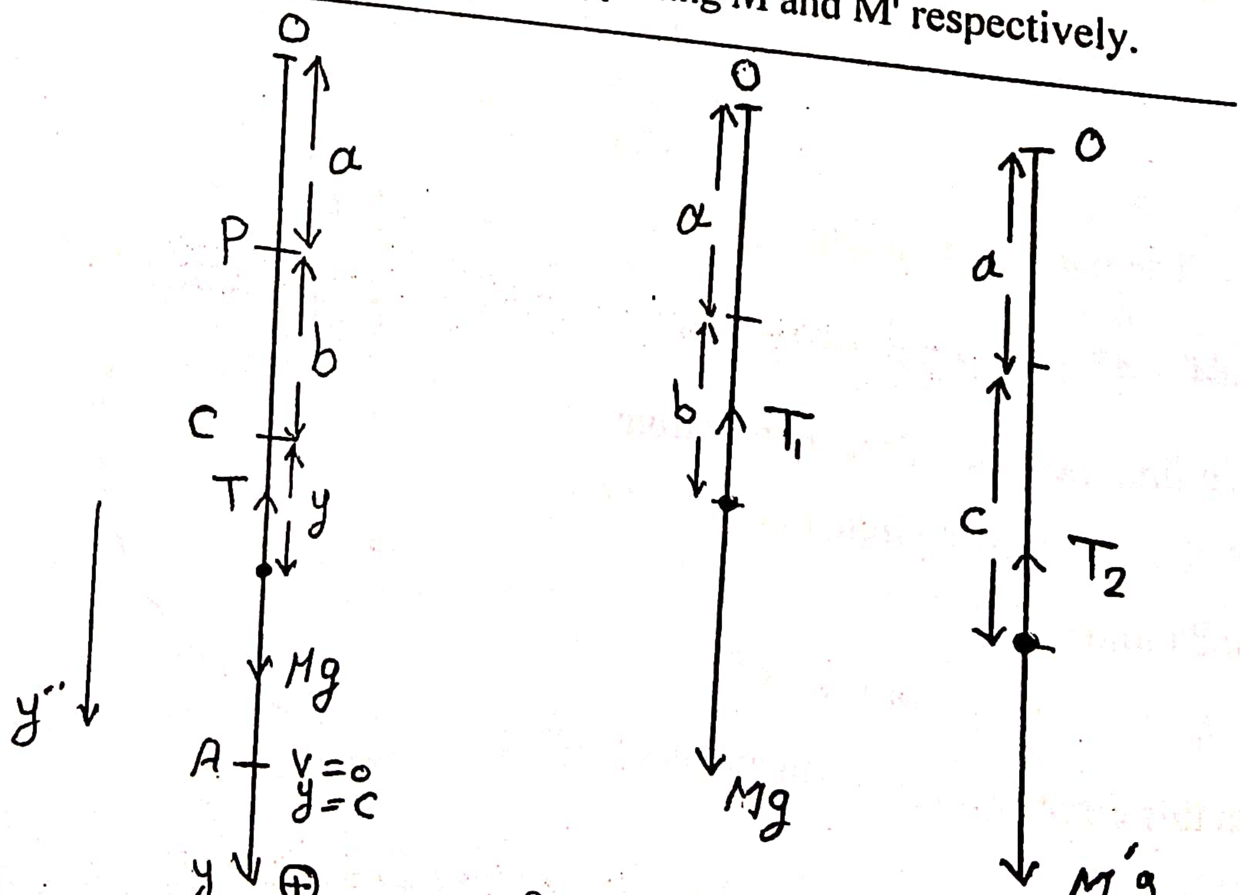
and the time of a complete oscillation is

$$2(t_1 + t_2) = (\frac{\pi\sqrt{2}}{6} + \frac{\sqrt{6}}{4})\ \text{sec.}$$

4- Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off suddenly; show that the distance of M from the upper end of the string at time t is

$$a + b + c\ \cos(\sqrt{\frac{g}{b}}\ t),$$

where a is the natural length of the string, and b and c the distances by which it would be stretched when supporting M and M' respectively.



Let OP = a be the natural length of the string, C be the equilibrium position of the mass M, and A be the equilibrium position of the two masses M , M' together, i.e. of the mass (M + M').

From the conditions of equilibrium of the masses M , M' separately we have:

$$M\ g = T_1 = k b\ , \qquad (1)$$

$$M'\ g = T_2 = k c\ . \qquad (2)$$

Adding, $\therefore (M+M')g = k(b+c)$

i.e. $PA = b + c$

$$\therefore \qquad CA = c\ .$$

If y is the distance of the mass M from C, measured downwards, after time t from the instant when M' falls off, the equation of motion, so long as the string is tight, is given by :

$$M\ y^{\bullet\bullet} = M\ g - T$$

$$= M\ g - k(b+y).$$

Using the condition (1), we get:

$$M\ y^{\bullet\bullet} = -k\ y$$

i.e. $$y^{\bullet\bullet} = -\frac{k}{M}y = -\frac{g}{b}y\ .$$

The motion is simple harmonic whose general solution is :

$$y = A\cos\sqrt{\frac{g}{b}}t + B\sin\sqrt{\frac{g}{b}}t\ .$$

$$\therefore \quad y^{\bullet} = -A\sqrt{\frac{g}{b}}\sin\sqrt{\frac{g}{b}}t + B\sqrt{\frac{g}{b}}\cos\sqrt{\frac{g}{b}}t\ .$$

At A , i.e. when $t = 0\ ,\ y = c\ ,\ y^{\bullet} = 0$

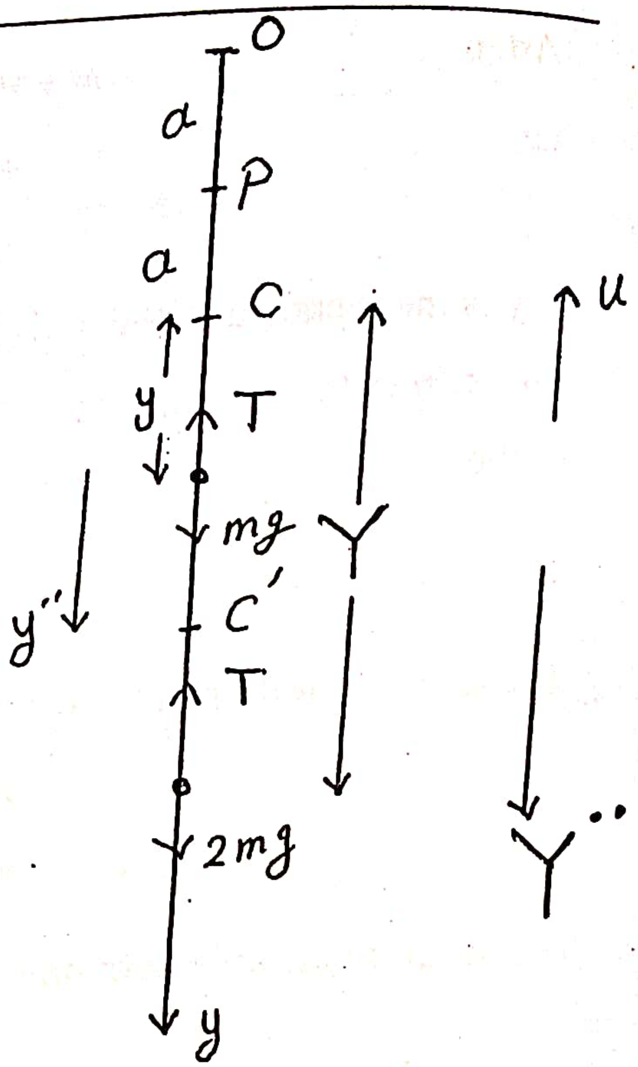
$$\therefore \quad A = c\ ,\ B = 0\ ,$$

hence $$y = c\ \cos(\sqrt{\frac{g}{b}}\ t)\ .$$

$\therefore$ The required distance $= a + b + y = a + b + c\ \cos(\sqrt{\frac{g}{b}}\ t).$

5. A particle is suspended by an elastic string of natural length a and modulus of elasticity equal to the weight of the particle. The particle was moving in a simple harmonic motion of amplitude b, and when it passes through its equilibrium position moving upwards, it catches another particle of the same mass. Prove that the new amplitude is $\sqrt{a^2+\frac{1}{2}b^2}$.

Let OP = a be the natural length of the string, C the position of equilibrium of the mass m.



At C , $mg = T = \frac{\lambda}{a}(PC)$,

where $\lambda = mg$,

$\therefore\ PC = a$.

If y is as shown, the equation of motion of the particle m is given by:

$$m\ddot{y} = mg - T$$
$$= mg - \frac{mg}{a}(y+a),$$

i.e. $\quad \ddot{y} = -\frac{g}{a}y$.

This motion is simple harmonic having amplitude equal to b.

$$\therefore\ v^2 = w^2(b^2-y^2) = \frac{g}{a}(b^2-y^2).$$

At C , $y=0$, $\therefore\ v = \pm\, wb = \pm b\sqrt{\frac{g}{a}}$.

When the mass is doubled at C, let the new velocity of the mass 2m be u.

Applying the law of conservation of momentum for the two masses we get:

$$mb\sqrt{\frac{g}{a}} + 0 = 2mu,$$

$$\therefore \quad u = \frac{1}{2} b \sqrt{\frac{g}{a}}$$, in the upward direction.

Motion of the mass 2m :

Let C' be the equilibrium position of this mass, therefore

$$2\,mg = T = \frac{mg}{a}(a + CC'),$$

i.e. $\quad CC' = a$.

Equation of motion of 2m is

$$2mY^{\bullet\bullet} = 2mg - T$$

$$= 2mg - \frac{mg}{a}(Y + a)$$

$$\therefore \quad Y^{\bullet\bullet} = -\frac{g}{2a}(Y - a).$$

This motion is simple harmonic with centre at Y = a , i.e. at C' (equilibrium position of 2m).

In this case, the law of velocity becomes:

$$v^2 = \frac{g}{2a}\left[b'^2 - (Y - a)^2\right], \qquad (1)$$

where b' is the amplitude of the new motion.

Now, at C, we have:

$$y = Y = 0,$$

$$v = Y^{\bullet} = -u = -\frac{1}{2} b \sqrt{\frac{g}{a}}.$$

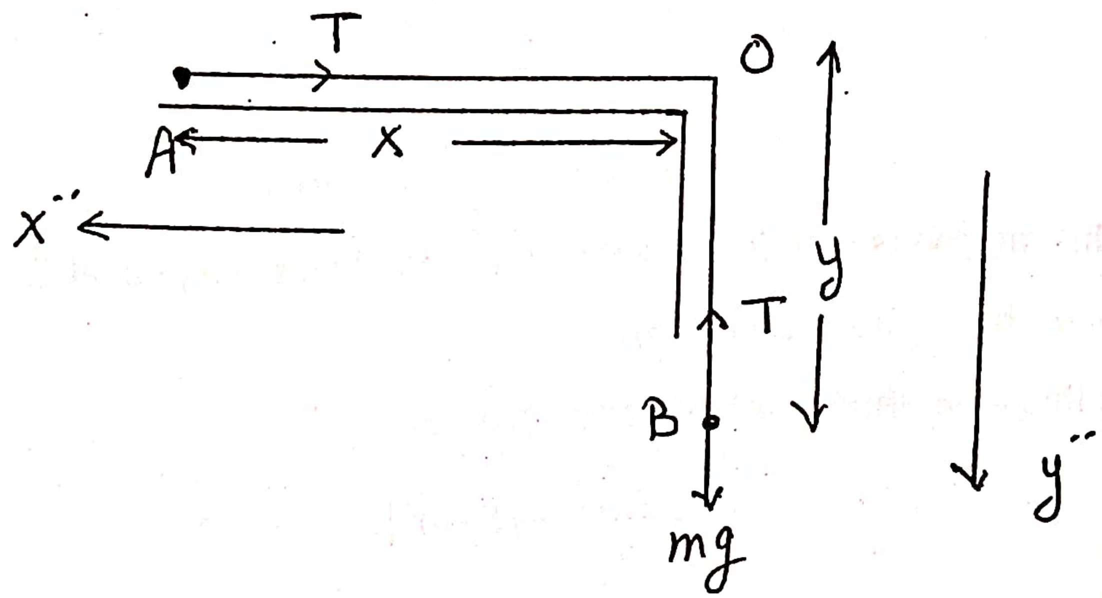
Substituting in equation (1) we get :

$$\frac{1}{4}\frac{gb^2}{a} = \frac{g}{2a}(b'^2 - a^2),$$

$$\therefore \quad b' = \sqrt{a^2 + \frac{1}{2}b^2} .$$

6- A weightless elastic string, of natural length ℓ and modulus of elasticity λ, has two equal particles of mass m at its ends and lies on a smooth horizontal table perpendicular to an edge with one particle just hanging over. Show that the other particle will pass over at the end of time t given by the equation

$$2\,\ell+\frac{mg\ell}{\lambda}\sin^2\sqrt{\frac{\lambda}{2m\ell}}\,t=\frac{1}{2}gt^2 .$$



Let the particle on the table be at the point A distant x from the edge O at time t and let the other particle be at the point B distant y from O.

∴ The length of the string at any instant is equal to x + y .

The equations of motion of A , B are given by :

$$m\ddot{x} = -T \quad , \qquad (1)$$

$$m\ddot{y} = mg - T \qquad (2)$$

where $T=\frac{\lambda}{\ell}(x+y-\ell)$.

Adding (1) and (2) we get :

$$m(\ddot{x}+\ddot{y}) = mg - 2T ,$$

$$= mg - \frac{2\lambda}{\ell}(x+y-\ell)$$

$$\therefore \quad \ddot{x}+\ddot{y} = g - \frac{2\lambda}{m\ell}(x+y-\ell) . \qquad (3)$$

Let $x + y = z$, $\quad \therefore \ \ddot{x} + \ddot{y} = \ddot{z}$

and (3) takes the form :

$$\ddot{z} = g - \frac{2\lambda}{m\ell}(z-\ell)$$

$$= -\frac{2\lambda}{m\ell}\left(z-\ell-\frac{mg\ell}{2\lambda}\right).$$

This represents a S.H.M. whose general solution is given by :

$$z - \ell - \frac{mg\ell}{2\lambda} = A\cos\sqrt{\frac{2\lambda}{m\ell}}\,t + B\sin\sqrt{\frac{2\lambda}{m\ell}}\,t .$$

$$\therefore \dot{z} = -A\sqrt{\frac{2\lambda}{m\ell}}\sin\sqrt{\frac{2\lambda}{m\ell}}\,t + B\sqrt{\frac{2\lambda}{m\ell}}\cos\sqrt{\frac{2\lambda}{m\ell}}\,t .$$

At $t = 0$, $x = \ell$, $y = 0$, $\dot{x} = \dot{y} = 0$

$\therefore \quad z = \ell$, $\dot{z} = \dot{x} + \dot{y} = 0$.

Hence $A = -\dfrac{mg\ell}{2\lambda}$, $\quad B = 0$, *i.e.*

$$z = \ell + \frac{mg\ell}{2\lambda}\left(1-\cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right)$$

or, $$x + y = \ell + \frac{mg\ell}{2\lambda}\left(1-\cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right)$$

The tension in the string at any instant is now given by

$$T = \frac{mg}{2}\left(1-\cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right).$$

Substituting this value in equation (1) we get :

$$\ddot{x} = -\frac{g}{2}\left(1-\cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right).$$

Integrating with respect to t,

$$\therefore x^{\bullet} = -\frac{g}{2}\left(t - \sqrt{\frac{m\ell}{2\lambda}}\sin\sqrt{\frac{2\lambda}{m\ell}}\,t\right) + C_1.$$

when $t = 0$, $x^{\bullet} = 0$, $\therefore$ $C_1 = 0$, *i.e.*

$$x^{\bullet} = -\frac{g}{2}\left(t - \sqrt{\frac{m\ell}{2\lambda}}\sin\sqrt{\frac{2\lambda}{m\ell}}\,t\right).$$

Integrating again,

$$\therefore x = -\frac{g}{2}\left(\frac{t^2}{2} + \frac{m\ell}{2\lambda}\cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right) + C_2.$$

when $t = 0$, $x = \ell$, $\therefore$ $C_2 = \ell + \frac{mg\ell}{4\lambda}$.

$$\therefore\ x = \ell + \frac{mg\ell}{4\lambda} - \frac{g}{2}\left(\frac{t^2}{2} + \frac{m\ell}{2\lambda}\cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right)$$

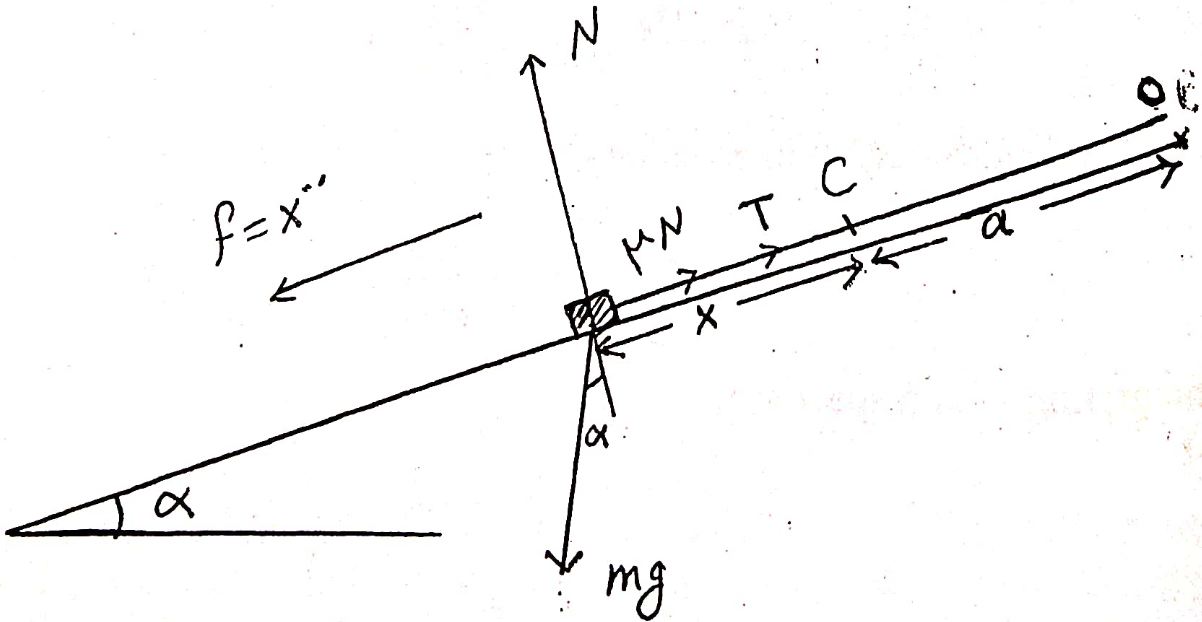
$$= \ell + \frac{mg\ell}{4\lambda}\left(1 - \cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right) - \frac{1}{4}gt^2.$$

When A reaches the edge O, x = O,

$$\therefore\ \ell + \frac{mg\ell}{4\lambda}\left(1 - \cos\sqrt{\frac{2\lambda}{m\ell}}\,t\right) = \frac{1}{4}gt^2,$$

$$\text{i.e. } 2\ell + \frac{mg\ell}{\lambda}\sin^2\sqrt{\frac{\lambda}{2m\ell}}\,t = \frac{1}{2}gt^2.$$

(7) A particle is attached by means of an elastic string to a point in a rough plane inclined at an angle α to the horizontal, originally the string was unstretched and lay along a line of greatest slope; show that the particle will oscillate only if the coefficient of friction is $< \frac{1}{3}\tan\alpha$.



Let the string be fixed at the point O, and C be the end of the natural length where OC = a. If x is the distance between the particle and the point C at any instant, the equation of motion down the plane is given by :

$$m\ddot{x} = mg\sin\alpha - \mu N - T$$

where μ is the coefficient of friction and $N = mg\cos\alpha$ is the normal reaction.

The tension in the elastic string is given by

$$T = kx$$

where k is the stiffness constant.

$$\therefore \quad \ddot{x} = g(\sin\alpha - \mu\cos\alpha) - \frac{k}{m}x\,;$$

For the particle to start motion at C,
i.e. at $x = 0$, we must have
i.e. $\sin\alpha - \mu\cos\alpha > 0$,

$$\therefore \quad \mu < \tan\alpha\,.$$

writing the equation of motion in the form

$$v\frac{dv}{dx} = g(\sin\alpha - \mu\cos\alpha) - \frac{k}{m}x\,.$$

Separating the variables and integrating we get :

$$\frac{v^2}{2} = g(\sin\alpha - \mu\cos\alpha)x - \frac{kx^2}{2m} + C\,.$$

At $x = 0$, $v = 0$, $\therefore\ C = O$ and so

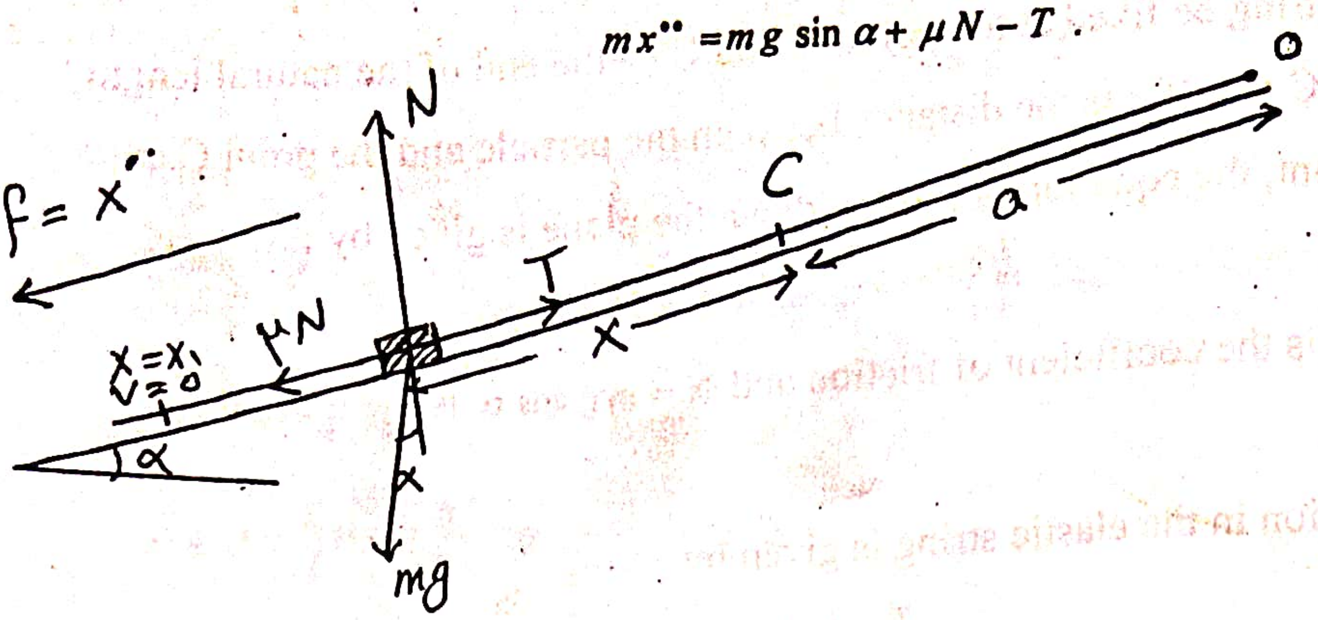
$$v^2 = 2g(\sin\alpha - \mu\cos\alpha)x - \frac{k}{m}x^2\,.$$

Substituting v = 0 we find that the particle will come to rest again at $x = x_1$, say, where

$$x_1 = \frac{2mg}{k}(\sin\alpha - \mu\cos\alpha)\,. \qquad (1)$$

If we assume that the particle will move from this position upwards, its equation of motion will take the form:

$$m x'' = m g \sin\alpha + \mu N - T \,.$$



As before, $N = mg\cos\alpha \,, \quad T = kx \,,$

therefore

$$m x'' = m g \sin\alpha + \mu m g \cos\alpha - k x \,.$$

i.e. $$x'' = g(\sin\alpha + \mu\cos\alpha) - \frac{k}{m} x \,.$$

The particle will move upwards, if

$$x'' < 0 \quad at \quad x = x_1 \,.$$

(This means that it has an initial upward acceleration.)

$\therefore \quad g(\sin\alpha + \mu\cos\alpha) - \frac{k}{m} x_1 < 0 \,.$

using (1), we get :

$$g(\sin\alpha + \mu\cos\alpha) < 2g(\sin\alpha - \mu\cos\alpha)$$

$$\therefore \quad 3\mu\cos\alpha < \sin\alpha$$

$$i.e. \quad \mu < \frac{1}{3}\tan\alpha$$

and the particle will oscillate only under this condition, if it starts motion from rest at C.

<u>Exercises</u>

1- A particle moves in a straight line with S.H.M. of periodic time 4 secs. If the particle started to move from rest at a distance 4 ft. from the centre of the motion find the time taken in moving a distance of 1 ft. and it velocity then.

2- A particle moves in a straight line with S.H.M, of amplitude a and periodic time $\frac{2\pi}{w}$. At the centre of motion, the particle receives a blow in the direction of motion which increases the magnitude of its velocity by the amount 3wa. Find the new amplitude.

3- A particle moves in a straight line with an acceleration which is always directed towards a fixed point 0 and is equal to $w^2 x$ where x is the distance from 0.

If the particle starts to move with velocity V towards O at a distance a from it, prove that its motion is S.H. with amplitude $\sqrt{a^2+\frac{V^2}{w^2}}$ and $x = a\cos wt - \frac{V}{w}\sin wt$ where t is measured from the starting point.

4- A particle moves in a straight line with S.H.M. such that its velocity is equal to 3 ft./sec. and 4 ft./sec. at the distances 4 ft. and 3 ft. respectively from the centre of motion. Find the periodic time and the amplitude.

5- A particle moves about a fixed point O, in a straight line AOA', a simple harmonic motion of periodic time $\frac{2\pi}{w}$ and amplitude OA = a.

After time $\frac{\pi}{6w}$ from reaching the end A, the particle receives a blow in the direction of motion which increases the magnitude of its velocity by the amount w a. Show that the new amplitude is $a\sqrt{3}$.

6-A rod AB moves in a plane such that the end B describes a circle of centre C and radius a with uniform angular velocity w and the end A moves in a straight line passing through C. Prove that if $AB = a$, the motion of A is simple harmonic. Find the amplitude and the periodic time of this motion.

7- The velocity v for a particle moving in the axis Ox in terms of its distance x from O is given by the relation

$$v^2 = n^2 (8ax - x^2 - 12a^2)$$

where n , a are constants. Prove that the motion is simple harmonic of amplitude 2a, and that the time of the motion from $x = 4a$ to $x = 6a$ is equal to $\frac{\pi}{2n}$. Also find the time of motion from $x = 3a$ to $x = 5a$.

8- Assuming that the earth attracts points inside it with a force which varies as the distance from its centre , show that, if a straight frictionless airless tunnel be made from one point of the earth's surface to any other point, a train would traverse the tunnel in slightly less than three – quarters of an hour.

9- 7- A particle moves in a straight line with an acceleration which is always directed towards a fixed point O and is equal to $w^2 x$ where x is the distance from O. If the particle starts to move with velocity V away from O and at a distance b from it, prove that

$$x = b \cos wt + \frac{V}{w} \sin wt .$$

Find the amplitude and the initial phase angle.

10-A , B are the ends of the path of a particle of mass 4.9 gms. moving with S.H.M. in a straight line. If the velocity of the particle at a point P is equal to 44 cms./sec. and the time from A to P is 1.5 secs. and from P to B is 4.5 secs. find the two distances PA and PB . Find also the maximum momentum of the particle and the maximum force acting on it.

11-A particle moves in a straight line with S,H.M. It passes by two points A , B on its path at a distance 22 inches appart with the same velocity. If the time taken for the motion from A to B is 2 secs. and the time taken to return to B again is also 2 secs., find the amplitude and the frequency of the motion.

12-A particle of mass 3 lbs. moves with S.H.M. in a straight line whose ends are A , B. P is a point on the path distant 16 ft. from A and 36 ft. from B.

If the velocity of the particle at P in equal to 144 ft./sec. find the two times of motion from A to P and from P to B .

Find the maximum force acting on the particle and the maximum kinetic energy.

13- The length of a spring increases by $\frac{1}{2}$ inch if the suspended weight increases by 1 lb. The spring is suspended vertically carrying a mass 4 lbs. in equilibrium. The mass is pulled down a distance of 3 inches and then released. Find the periodic time. Also find the velocity and acceleration of the particle when its distance from the equilibrium position is 1 inch.

14- One end of an elastic string of natural length 2b and modulus of elasticity mg is fixed to a point O on a smooth horizontal table and the other end is tied to a particle of mass m which rests on the table at O. If the particle is projected horizontally with velocity $2\sqrt{bg}$ find the time of a complete oscillation.

15- A particle of mass m is tied at the middle point of an elastic string PQ of natural length 2b and modulus of elasticity mg. The two ends P , Q are fixed at two points on a smooth horizontal table at a distance 2b apart. The particle is initially on the table at a distance $\frac{3}{2}b$ from P and $\frac{1}{2}b$ from Q and then released. Prove that its motion is S.H. Find the amplitude and the periodic time.

16- A particle of mass 10 lbs. is suspended by a spring. At the position of equilibrium the extension of the spring is 10 inches. The particle is pulled down a distance of $\frac{1}{2}$ **inch** and then released. Find the time of a complete oscillation, and the velocity of the particle after rising 1 inch, and the force in the spring at the maximum height. Is it tension or compression ?

17- A bead is tied at the middle point of an elastic string joining two points P,Q on a smooth horizontal table. If the bead is given a small displacement in a direction perpendicular to PQ and then let go. Prove that the resulting motion is approximately simple harmonic. (consider the tension constant throughout the motion).

If $PQ = 9$ ft. and the tension in the string is twice the weight of the bead and the given displacement is $\frac{1}{2}$ **inch** find the periodic time and the maximum velocity of the bead.

18- The lower end of a spring of natural length 4 ft. and modulus of elasticity 4 mg pdls is fixed such that the axis of the spring is vertical, A particle of mass m rests on the other end. Another particle of the same mass falls from a height of 1 ft. on the first particle. Find the distance described by the common mass till it first comes to rest, and the time taken to reach this position.

19- A particle of mass m is tied to one endof an elastic string of natural length a and modulus of elasticity $\frac{1}{2} mg\tan^2\theta$ where θ is fixed at the point C. If the particle is let to fall from rest at C, prove that the distance described downwards is equal to

$a\cot^2\frac{\theta}{2}$, in time given by $\sqrt{\frac{2a}{g}}\left[1+(\pi-\theta)\cot\theta\right]$.

20- A small ring slides on a smooth straight horizontal wire. The ring is tied to one end of an elastic string of natural length a and modulus of elasticity λ. The other end is fixed at a point in the same horizontal plane passing through the wire and at a distance $b > a$ from it. If the ring is given a small displacement on the wire from the equilibrium position and then released, prove that the motion is simple harmonic and find its periodic time.

21. A light spring is kept compressed by the action of a given force; the force is suddenly reversed; prove that the greatest subsequent extension of the string is three times its initial contraction.

22. A particle is suspended by an elastic string of natural length a from a fixed point C. If the particle is let to fall from rest when the string is vertical of length a and the particle is below C, and when the particle reaches its lowest point half its mass falls. Prove that the other half will rise a distance 2d above the initial position where d is the extension of the string in the original equilibrium position.

If the particle when at its lowest position receives a downward blow to move with velocity u, show that it will rise a distance $\frac{u^2}{2g}$ above the initial position. Find τ in each case.

23- Two particles, of masses m and m', are connected by an elastic string whose coefficient of elasticity is λ ; they are placed on a smooth table, the distance between them being a, the natural length of the string.

The particle m is projected with velocity V along the direction of the string produced; find the motion of each particle, and show that in the subsequent motion the greatest length of the string is $a+Vp$, and that the string is next at its natural length after time πp, where $p^2 = \frac{mm'}{m+m'}\frac{a}{\lambda}$.