Kinetics of motion, Projectiles

1. Introduction:

We have discussed in chapter III the kinematics of motion of a particle, i.e. all about the velocity and acceleration in terms of distance and time. In this chapter, the kinetics of motion means the study of the forces and their effect on the motion of particles.

2. The momentum:

If a particle P of mass m moves and its velocity at the moment t is $\vec{v}$, the momentum of that particle at this moment is defined as $m\,\vec{v}$, i.e. the momentum is a vector of magnitude m v in the direction of the velocity. Therefore the momentum is in the direction of the tangent to the path of the mass m and since the velocity of the particle is equal to $\frac{d\vec{r}}{dt}$ where $\vec{r}$ is the position vector of P relative to an origin O at time t, therefore

The momentum of the particle $= m\frac{d\vec{r}}{dt}$.

In the case of linear motion, if the particle moves in the positive direction of the x-axis Ox, its momentum is equal to $m\frac{dx}{dt} = m\dot{x}$,

In the positive direction of that axis.

3- Newton's Laws of Motion:

Law I.

Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it be compelled by external impressed force to change that state.

Law II.

The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the straight line in which the force acts.

Law III.

To every action there is an equal and opposite reaction.

If $\vec{F}$ is the force acting on a particle of mass m which is moving with velocity $\vec{v}$

$$\therefore \quad \vec{F} = \frac{d}{dt}(m\vec{v}) \qquad (1)$$

$$\text{i.e.} \quad \vec{F} = m\frac{d\vec{v}}{dt} + \frac{dm}{dt}\vec{v}. \qquad (2)$$

If m is constant, $\frac{dm}{dt}=0$ and equation (2) gives

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\text{i.e.} \quad \vec{F} = m\,\vec{f}\,. \qquad (3)$$

where $\vec{f}$ is the acceleration of the moving particle.

Equation (3) is the equation of motion of the particle and could be written in the form $\quad m\vec{f} = \vec{F}$.

In the case of linear motion, the equation of motion of the particle becomes

$$mf = F.$$

Where F is the force (or the resultant of acting forces) acting in the positive direction of the x-axis, and as before

$$f = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{dv^2}{dx}.$$

4- The weight of the particle:

The weight of the particle at a certain place on the earth is defined as the force of attraction of the earth acting on the particle at that place. If a body is let to fall towards the earth in vacuo, it will move with an acceleration which is always the same at the same place on the earth, but which varies slightly for different places.

The value of this acceleration, which is called the acceleration due to gravity, is always denoted by the letter g.

By Newton's second law, the weight W of a particle of mass m is given by

$$W = m g .$$

Hence the weight of the unit of mass = g units of force.

Therefore the weight of one gramme is equal to g units of force where in the C.G.S. system of unit $g = 980\ cm./sec^2$ approximately,

i.e. $$1\,gm.wt. = 1 \times 980\ gm.cm./sec^2 = 980\ dynes.$$

Hence a dyne is equal to the weight of about $\frac{1}{980}$ of a gramme.

Similarly the weight of one pound (1 lb.) is equal to g units of force where in the F.P.S. system of units $g = 32\ ft./sec^2$ approximately,

i.e. $$1\,lb.wt. = 1 \times 32\,lb.\ ft./sec^2. = 32\ poundals.$$

Hence a poundal is approximately equal to the weight of $\frac{1}{32}$ of a pound.

$$\because 1\ kilogramme = 1000\ grammes,$$

$$\therefore 1\,kgm.wt. = 10^3\ gm.wt. = 10^3 \times 980\ dynes,$$

and in F.P.S. system (English system),

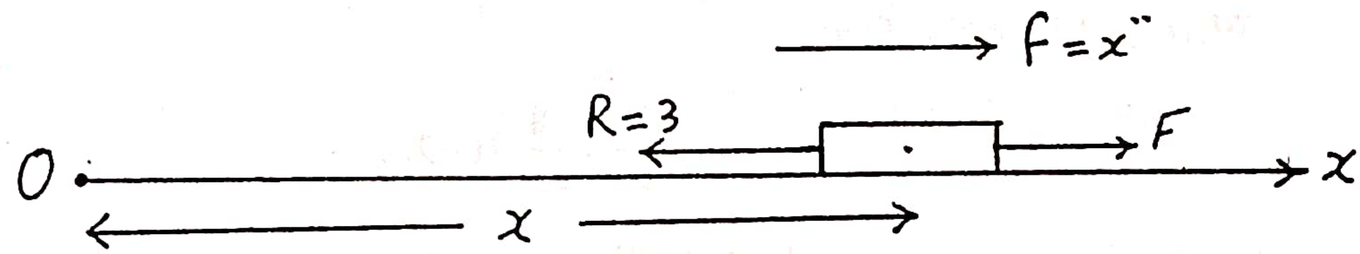
1 ton = 2240 lbs.

$$\therefore 1\ ton\,wt. = 2240\ lb.wt. = 2240 \times 32\ poundals = 71680\ pdls.$$

5. Examples:

1- A train, of mass 300 tons, is originally at rest upon a level track. It is acted on by a horizontal force which increases uniformly with the time in such a way that $F = 0$ when $t = 0$, and $F = 5$ tons wt. when $t = 15$ secs.

When in motion the train is assumed to be acted upon by a constant frictional force equal to 3 tons wt. Find the instant of starting, and show that, when $t = 15$ secs., the speed of the train is 0.64 foot per second.



From Newton's second law

$$m\bar{f} = \bar{F}.$$

$\therefore$ the equation of motion of the train is

$$mf = F - R \qquad (1)$$

where F increases uniformly with the time

$\therefore \qquad F = at + b$

where a , b are constants. .

$\because$ $F = 0$ when $t = 0$,

$F = 5$ tons wt. when $t = 15$ secs.

$$\therefore \quad b = 0 \quad , \quad a = \frac{1}{3}$$

$$\therefore \quad F = \frac{1}{3}t. \qquad (2)$$

From (1) & (2) :

$$\therefore \quad mf = \frac{1}{3}t - 3$$

$$= \frac{1}{3}(t-9) \text{ tons wt.}$$

The units in this equation must be changed to absolute units, i.e. poundals, as follows:

m = 300 tons = 300 x 2240 lbs.,

f is considered in ft./ sec^2.,

$$\therefore \ 300 \times 2240 \ f = \frac{1}{3}(t-9) \times 2240 \times 32$$

i.e. $$f = \frac{8}{225}(t-9). \qquad (3)$$

From this equation we see that $f = 0$ when $t = 9$ i.e. the train starts motion after 9 secs .

Writing (3) in the form

$$\frac{dv}{dt} = \frac{8}{225}(t-9).$$

Separating variables and integrating we get :

$$v = \frac{8}{225} \cdot \frac{(t-9)^2}{2} + C$$

$$= \frac{4}{225}(t-9)^2 + C$$

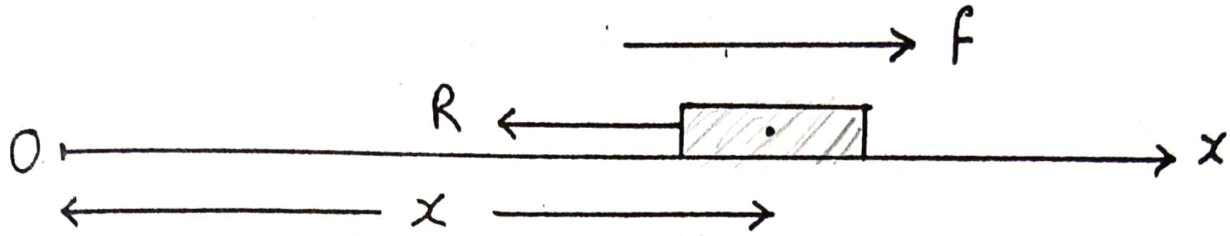
where $v = 0$ at $t=9$, $\therefore C=0,$

$$v = \frac{4}{225}(t-9)^2 .$$

when $t = 15$ secs., we obtain

$$v = \frac{4}{225} \times 36 = \frac{16}{25} = 0.64 \text{ ft./sec.}$$

2- When the brakes are applied in a motor car it was found that the resistance due to friction is equal to $(1000 + 0.08\,v^2)$ lb.wt. per ton of its weight where v is the velocity of the car at any moment in miles per hour. Prove that if the brakes were first applied when the car was moving with velocity 50 m.p.h. it will stop after describing a distance of 57 yards approximately.



The resistance R is equal to

$$\left[1000+0.08\left(v\times\frac{15}{22}\right)2\right]\times m\times 32 \text{ pdls.}$$

where m is the mass of the car in tons, v its velocity in ft./sec.

The equation of motion of the car is

$$m\,f=-R$$

i.e. $$m\times 2240 f=-\left[1000+\frac{8}{100}\times\left(\frac{15}{22}\right)^2 v^2\right]\times m\times 32$$

$$\therefore\quad f=-\frac{1}{70}\left[1000+\frac{9}{242}v^2\right]$$

$$=-\frac{9}{16940}\left(\frac{242000}{9}+v^2\right).$$

writing $v\frac{dv}{dx}$ instead of f and separating the variables we get :

$$v\frac{dv}{dx}=-\frac{9}{16940}\left(\frac{242000}{9}+v^2\right)$$

i.e. $$dx=-\frac{16940}{9}\,\frac{v\,dv}{\frac{242000}{9}+v^2}.$$

Integrating both sides,

$$\therefore x=-\frac{8470}{9}\ln\left(\frac{242000}{9}+v^2\right)+C$$

$$\therefore v=\frac{50\times 22}{15}\text{ ft/sec.}\quad \text{at } x=0$$

$$\therefore C = -\frac{8470}{9} \ln \left(\frac{242000}{9} + \frac{48400}{9}\right)$$

$$= -\frac{8470}{9} \ln \frac{290400}{9},$$

$$\therefore \quad x = \frac{8470}{9} \ln \left(\frac{290400}{242000 + 9v^2}\right).$$

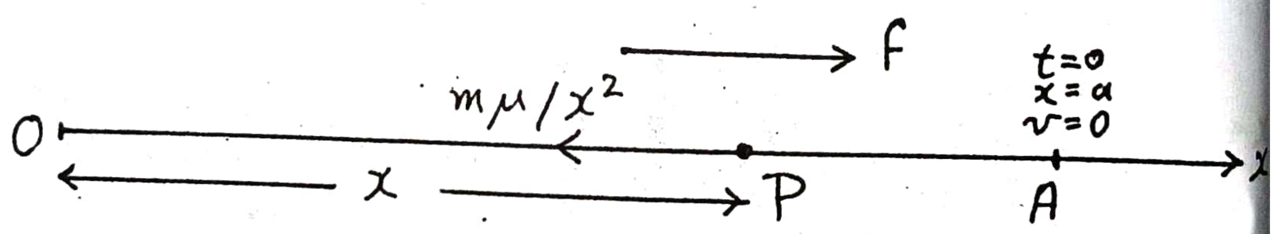
When the car is stopped, put $v = 0$, the required distance will be

$$x = \frac{8470}{9} \ln \ 1.2 \cong 171.58 \text{ ft.}$$

$$= 57.19 \text{ yards}$$

i.e. 57 yards approximately.

3- A particle moves in a straight line Ox under a central attractive force $m\mu/x^2$ towards the origin O where m is the mass of the particle and μ is a constant. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi a^{3/2}/\sqrt{8\mu}$.



From Newton's second law

$$mf = -\frac{m\mu}{x^2}$$

$$\therefore \quad f = -\frac{\mu}{x^2}$$

i.e. $\quad v\frac{dv}{dx} = -\frac{\mu}{x^2}.$

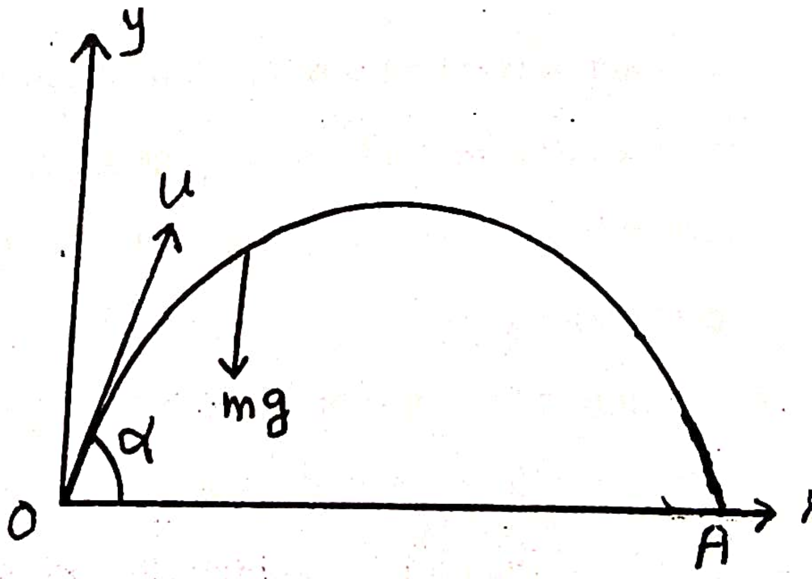
Separating the variables and integrating we get :

$$\frac{1}{2}v^2 = \frac{\mu}{x} + C_1$$

From the initial conditions, $v = 0$ at $x = a$

Projectiles:

If a particle is projected from a point O into the air with any direction and velocity and we neglect the resistance of the air, it will move under the action of its weight mg only.



Let the velocity of projection be u in a direction making an angle α with the horizontal plane through the point of projection. α is called the angle of projection. The path which the particle describes is called its trajectory;

The distance between the point of projection and the point where the path meets any plane drawn through the point of projection is its range on the plane; and the time that elapses before it again meets the horizontal plane through the point of projection is called the time of flight.

Consider two axes at the point of projection O, one of them Ox is horizontal and the other Oy vertically upwards.

The equations of motion of the particle relating to these axes are

$$m\ddot{x}=0 \quad , \quad m\ddot{y}=-mg$$

i.e. $$\ddot{x}=0 \quad , \quad \ddot{y}=-g . \qquad (1)$$

$$\therefore \quad \dot{x}=C_1 \quad , \quad \dot{y}=-gt+C_2 .$$

Since $\dot{x}=u\cos\alpha$, $\dot{y}=u\sin\alpha$ at $t=0$,

$$\therefore C_1=u\cos\alpha \quad , \quad C_2=u\sin\alpha$$

i.e. $$\dot{x}=u\cos\alpha \; , \; \dot{y}=u\sin\alpha-gt . \qquad (2)$$

Integrating these two equations we get :

$$x=(u\cos\alpha)t+C_3 \; , \; y=(u\sin\alpha)t-\frac{1}{2}gt^2+C_4 .$$

But $x=y=0$ when $t=0$, $\therefore C_3=C_4=0$

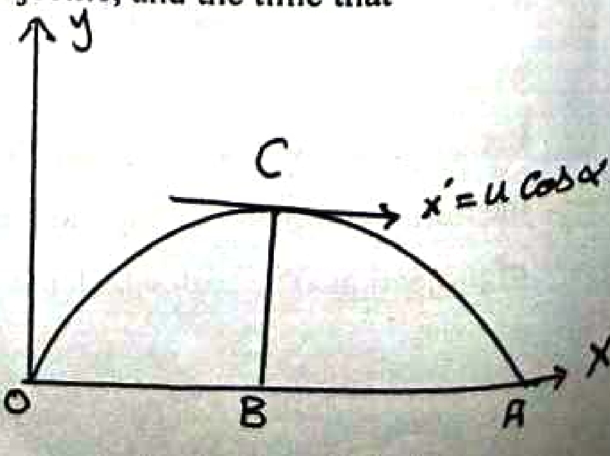
i.e. $$x=(u\cos\alpha)t \; , \; y=(u\sin\alpha)t-\frac{1}{2}gt^2 . \qquad (3)$$

Equations (1), (2) and (3) show that the motion of the projectile is the resultant of two motions, one horizontal with uniform velocity $u\cos\alpha$, and the other vertical which is the same as that of a particle projected vertically upwards with initial velocity $u\sin\alpha$, and moving with acceleration (-g) .

7- To find the greatest height attained by a projectile, and the time that elapses before it is at its greatest height.

Let C, be the highest point of the path.

The projectile must at C be



Moving horizontally, and hence the vertical velocity at C must be zero.

From (2), the time from O to C is equal to $\frac{u\sin\alpha}{g}$. Substituting this val in the second of equations (3) we get the greatest height attained as

$$H = \frac{u^2\sin^2\alpha}{2g}. \qquad (4)$$

8- To find the range on the horizontal plane and the time of flight. At A , y = 0 and from (3) we have

$$0 = (u\sin\alpha)t - \frac{1}{2}gt^2.$$

If T be the time of flight,

$$\therefore \quad T = \frac{2u\sin\alpha}{g}. \qquad (5)$$

Notice that T = twice the time to the highest point.

During this time T the horizontal velocity remains constant and equal to $u\cos\alpha$. If R be the horizontal range,

$$\therefore \quad R = (u\cos\alpha)T = \frac{2u^2\sin\alpha\,\cos\alpha}{g},$$

i.e.

$$\therefore \quad R = \frac{u^2\sin 2\alpha}{g}. \qquad (6)$$

Hence the horizontal range is equal to twice the product of the initial vertical and horizontal velocities divided by g.

When the angle of projection is $\frac{\pi}{2}-\alpha$, the range

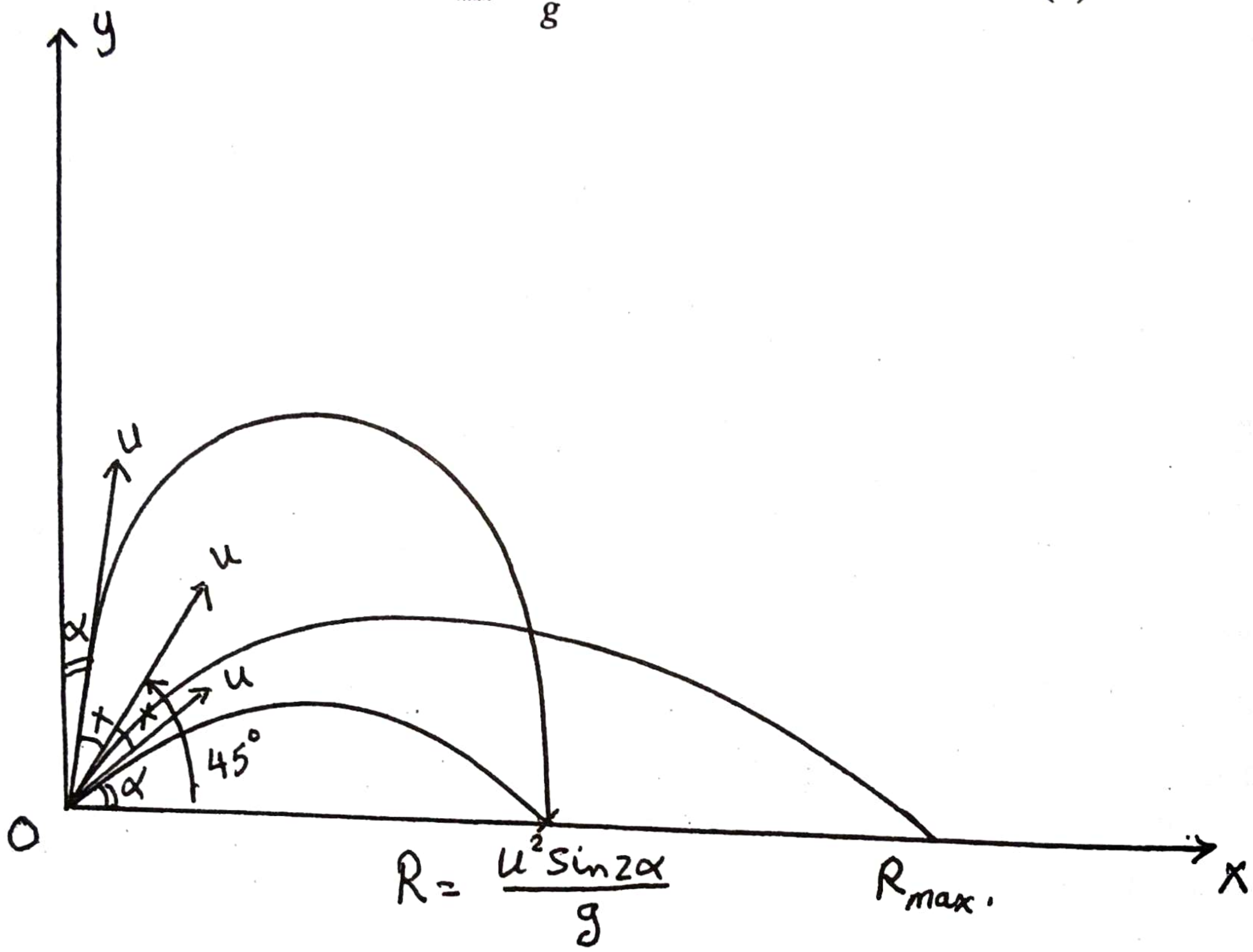
$$= \frac{u^2}{g}\sin 2\left(\frac{\pi}{2}-\alpha\right) = \frac{u^2}{g}\sin(\pi-2\alpha) = \frac{u^2\sin 2\alpha}{g}$$

Hence we have the same horizontal range for the angles of projection α and $\frac{\pi}{2}-\alpha$.

These directions are equally inclined to the horizontal and the vertical respectively.

$\because$ **sin 2α** is greatest when $2\alpha = \frac{\pi}{2}$, i.e. when $\alpha = \frac{\pi}{4}$, the range on a horizontal plane is greatest and is given by

$$R_{max} = \frac{u^2}{g}. \tag{7}$$



9- <u>The trajectory</u>:

Eliminating t between the two equations (3) we obtain the equation of the path. of the projectile (the trajectory.) in the form:

$$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha}, \tag{8}$$

which is the equation of a parabola whose axis is vertically downwards. Equation (8) could be written in the form :

$$x^2 - \frac{x u^2 \sin 2\alpha}{g} = -\frac{2u^2 \sin^2 \alpha}{g} y$$

or,

$$(x-\frac{u^2\sin 2\alpha}{2g})^2=-\frac{2u^2\cos^2\alpha}{g}(y-\frac{u^2\sin 2\alpha}{2g})$$

showing that the vertex of the parabola is at the point

$$(\frac{u^2\sin 2\alpha}{2g},\frac{u^2\sin^2\alpha}{2g})=(\frac{R}{2},H).$$

From equations (2), (3) we have

$$\begin{aligned} v^2 &= \dot{x}^2+\dot{y}^2 \\ &= u^2\cos^2\alpha+(u\sin\alpha-gt)^2 \\ &= u^2-2(u\sin\alpha)gt+g^2t^2 \\ &= u^2-2g\left[(u\sin\alpha)t-\frac{1}{2}gt^2\right]. \end{aligned}$$

i.e.

$$v^2=u^2-2gy. \tag{9}$$

This equation means that the velocity of the particle has the same magnitude at any two points on the path on the same horizontal line. Equation (9) could also be obtained by applying the principle of work, namely

$$\frac{1}{2}mv^2-\frac{1}{2}mu^2=-mgy.$$

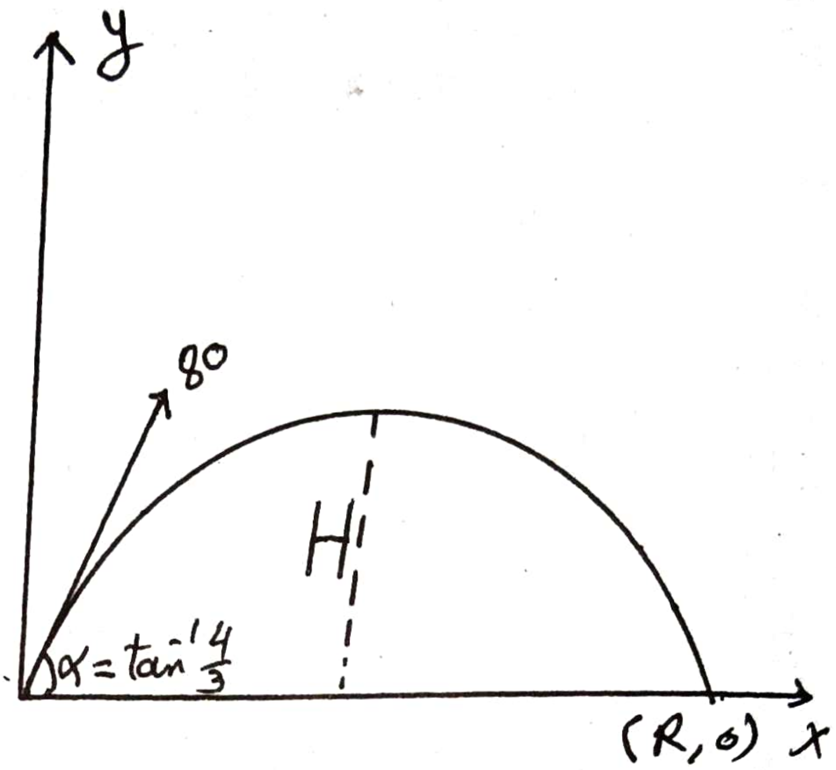
10- Examples:

1) A particle is projected with a velocity of 80 ft./sec. at an angle $\tan^{-1}\frac{4}{3}$ with the horizontal. Find the greatest height attained, the time of flight, the horizontal range, the equation of the path and determine the direction of motion of the particle when it is at a height of 48 ft.

$u=80\,ft./\sec.$, $\alpha=\tan^{-1}\frac{4}{3}$

The greatest height is given by

$$H = \frac{u^2 \sin^2 \alpha}{2g} = \frac{(u \sin \alpha)^2}{2g} = 64 \ ft.$$



The time of flight is given by

$$T = \frac{2u \sin \alpha}{g} = 4 \text{ sec.}$$

The horizontal range is given by

$$R = \frac{u^2 \sin 2\alpha}{g} = \frac{2(u \sin \alpha)(u \cos \alpha)}{g}$$

$$= \frac{2(48)(64)}{32} = 192 \ ft.$$

And the equation of the path is

$$y = x \tan \alpha - \frac{g x^2}{2u^2 \cos^2 \alpha} = \frac{4}{3}x - \frac{x^2}{144}$$

$\because \dot{x} = u \cos \alpha = 48$, $\dot{y} = u \sin \alpha - gt$

$\qquad = 64 - 32t$

& $y = (u \sin \alpha)t - \frac{1}{2} g t^2 = 64t - 16t^2$,

$\therefore$ when y = 48 ft.

$$64t - 16t^2 = 48.$$

i.e. $t^2 - 4t + 3 = 0$

$\therefore$ $(t-1)(t-3) = 0$

$\therefore$ $t_1 = 1 \text{sec.}$, $t_2 = 3 \text{sec.}$

$\because$ $\tan \psi = \frac{\dot{y}}{\dot{x}}$

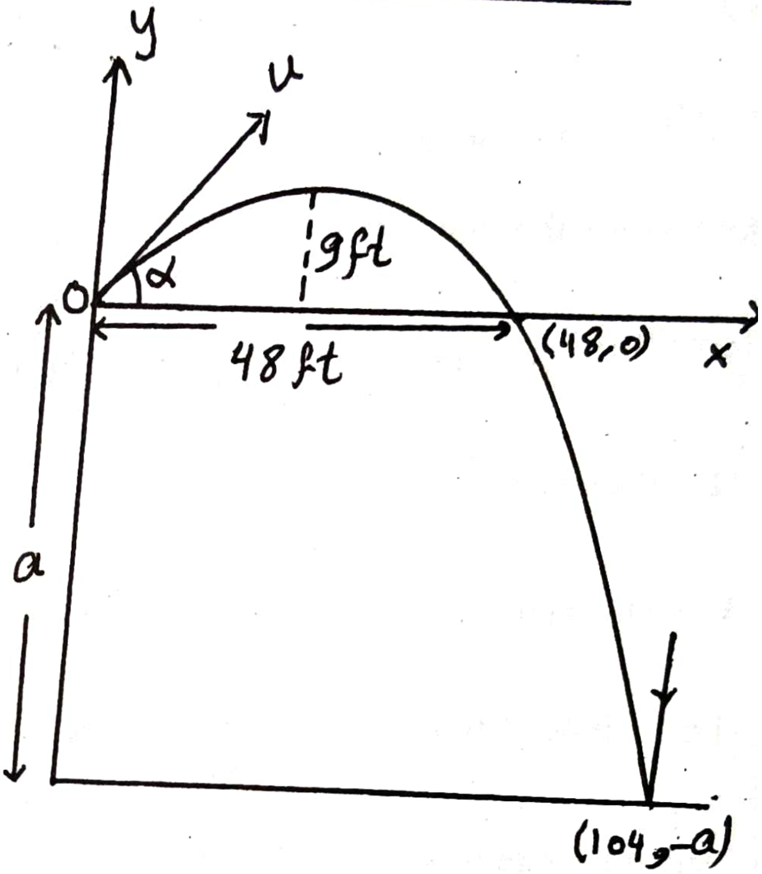
$$= \frac{64 - 32t}{48}$$

$\therefore$ *at* $t = 1$ sec , $\tan \psi = \frac{2}{3}$,

and at $t = 3$ sec , $\tan \psi = -\frac{2}{3}$.

2) A projectile is fired from the top of a tower with a velocity of u. in a direction making an angle α with the horizontal and it hits a target at

a distance 104 ft. from the foot of the tower. If the horizontal range is 48 ft. and the maximum height above the plane passes through the point of projection is 9 ft., find u & α and prove that the height of the tower is 91 ft.



Take axes at the point Of projection 0 as shown.

Since the greatest height and the range on the horizontal plane passing thought O are

$H = \dfrac{u^2 \sin^2 \alpha}{2g}$ and $R = \dfrac{u^2 \sin 2\alpha}{g}$, then

$$9 = \frac{u^2 \sin^2 \alpha}{2g} \qquad (1)$$

$$48 = \frac{u^2 \sin 2\alpha}{g} \qquad (2)$$

From (i) and (2), we get $\tan\alpha = \dfrac{3}{4}$ and $u = 40\ ft./\text{sec}$.

Relative to these axes, the equation of the path of the projectile is given by:

$$y = x\tan\alpha - \frac{g x^2}{2u^2 \cos^2 \alpha}$$

i.e.

$$y = \frac{3}{4}x - \frac{32x^2}{2(32)^2}$$

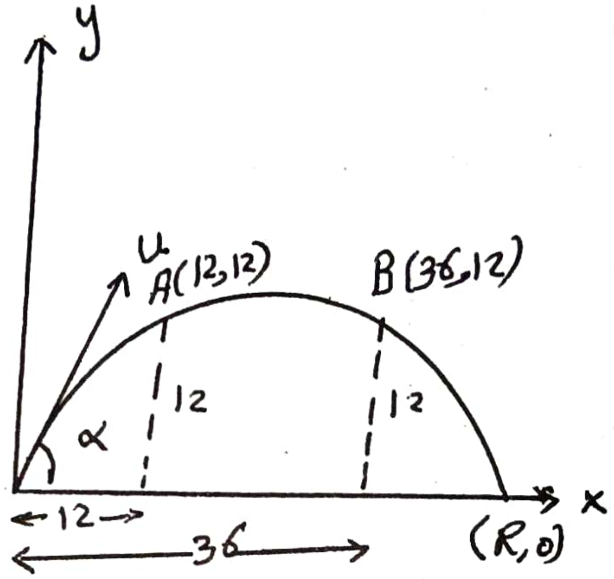
$$\therefore \quad y = \frac{3}{4}x - \frac{x^2}{64}.$$

Put $y = -a$ *and* $x = 104$, therefore

$$\therefore \quad -a = \frac{3}{4}(104) - \frac{(104)^2}{64} \Rightarrow a = 91\ ft.$$

3) A particle is projected from the origin O and passes through the two points A(12,12), B(36,12). Find the velocity and the angle of projection, also find the time of flight and the horizontal range.

Relative to these axes, the equation of the path of the projectile is given by:



$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$$

And passes through A and B, then

$$12 = 12\tan\alpha - \frac{32(12)^2}{2u^2\cos^2\alpha} \qquad (1)$$

$$12 = 36\tan\alpha - \frac{32(36)^2}{2u^2\cos^2\alpha} \qquad (2)$$

From (1) and (2), we get $\tan\alpha = \frac{4}{3}$ and $u = 40\ ft./\sec.$

i,e the time of flight is $T = \frac{2u\sin\alpha}{g} = 2$ sec.

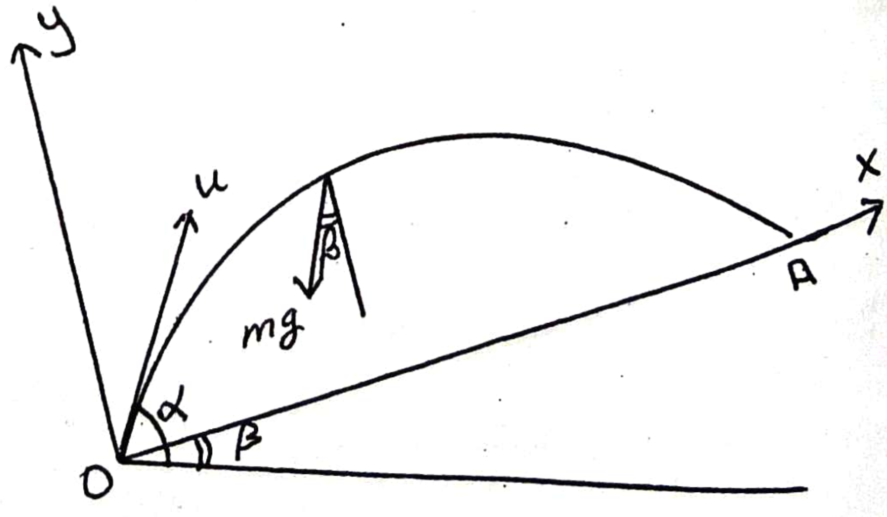
and the range is $R = \frac{u^2\sin 2\alpha}{g} = 48\ ft.$

11. Range on an inclined plane:

Let a particle be projected from a point 0 on a plane of inclination β, in the vertical plane through OA, the line of greatest slope of the inclined plane as shown in figure.

Let the velocity of projection be u at an elevation α to the horizontal.

Take axes at O, one Ox Along OA, and the other Oy perpendicular to it. If (x , y) are the coordinates of the projectile at any time t, the equations of motion are :



$$\ddot{x} = -g \sin\beta \quad , \quad \ddot{y} = -g \cos\beta$$

$$\therefore \dot{x} = u\cos(\alpha-\beta) - (g\sin\beta)t,$$

$$\dot{y} = u\sin(\alpha-\beta) - (g\cos\beta)t,$$

$$x = ut\cos(\alpha-\beta) - \frac{1}{2}gt^2\sin\beta, \qquad (1)$$

$$y = ut\sin(\alpha-\beta) - \frac{1}{2}gt^2\cos\beta.$$

At A, $y = 0$ and so the time of flight T from 0 to A is given by

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \qquad (2)$$

Substituting in (1), the time t by the value T given by (2) we get the range $R = OA$ on the inclined plane in the form

$$R = \frac{2u^2\sin(\alpha-\beta)\cos(\alpha-\beta)}{g\cos\beta} - \frac{2u^2\sin^2(\alpha-\beta)\sin\beta}{g\cos^2\beta}$$

$$= \frac{2u^2\sin(\alpha-\beta)}{g\cos\beta}\left[\cos(\alpha-\beta)\cos\beta - \sin(\alpha-\beta)\sin\beta\right]$$

$$= \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta},$$

i.e. $$R = \frac{u^2\left[\sin(2\alpha-\beta) - \sin\beta\right]}{g\cos^2\beta}. \qquad (3)$$

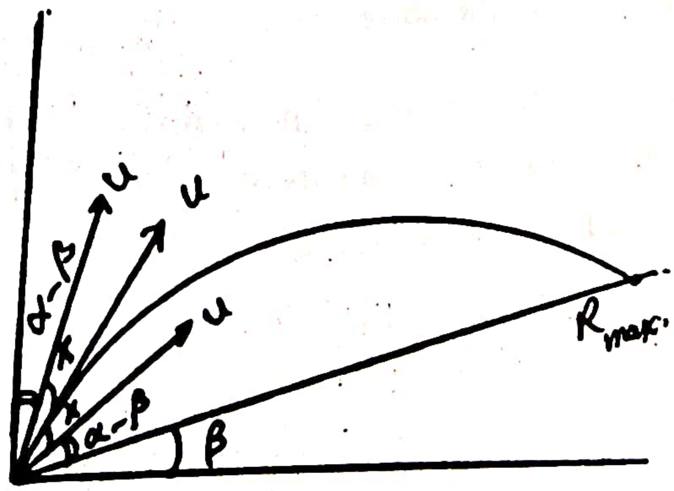
$$\because \sin(2\alpha-\beta) = \sin(\pi - 2\alpha + \beta)$$

$$= \sin\left[2\left(\frac{\pi}{2} - \alpha + \beta\right) - \beta\right]$$

$$= \sin\left[2\left\{\frac{\pi}{2} - (\alpha-\beta)\right\} - \beta\right].$$

it is clear from (3) that we obtain the same range on the inclined plane for the angles of projection α and $\frac{\pi}{2} - (\alpha-\beta)$.

These two directions are equally Inclined to the vertical and the Inclined plane by the angle $(\alpha-\beta)$.



Also for a given velocity of projection u, the range OA is maximum when

$$2\alpha-\beta=\frac{\pi}{2}, \text{ i.e. } \quad \alpha=\frac{\pi}{4}+\frac{\beta}{2}.$$

Hence for maximum range, the direction of projection must bisect the angle between the vertical and the inclined plane.

Notice that, for a given range, the two directions of projection are equally inclined to the direction for maximum range.

The value of the maximum range is given by

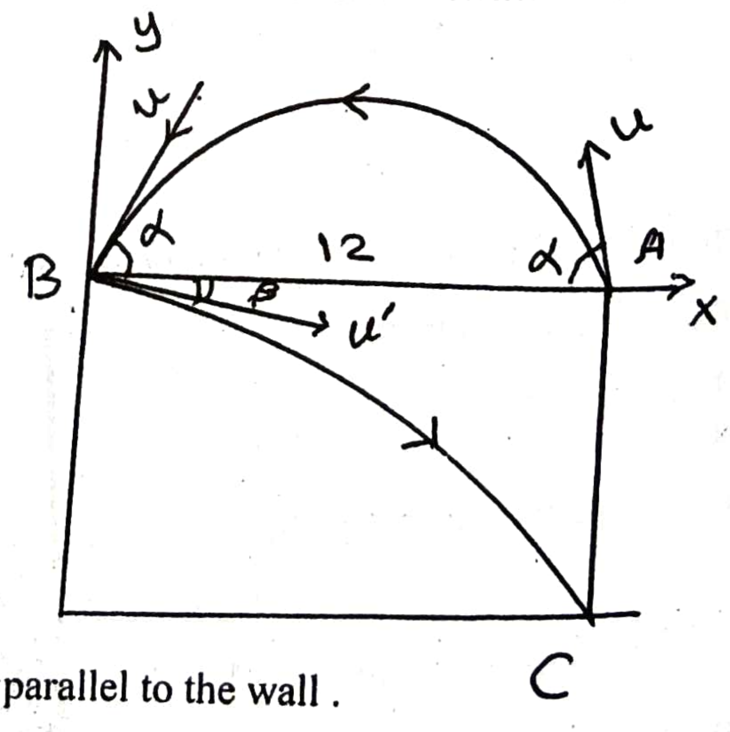
$$R_{\max}=\frac{u^2(1-\sin\beta)}{g\cos^2\beta}$$

$$=\frac{u^2}{g(1+\sin\beta)}. \qquad (4)$$

12- <u>Examples</u>:

1) A ball is projected from a point A in a direction making an angle $\tan^{-1}\frac{3}{4}$ with the horizontal to hit a smooth vertical wall distant 12 ft. from A at a point B such that AB is horizontal. If the ball rebounds and passes through a point C which lies vertically below A at a distance of $\frac{400}{9}$ ft. and the coefficient of restitution is equal to $\frac{9}{16}$, find the velocity of projection and the time of motion from A to C .

$\because$ **A,B** lie on the same horizontal line, the velocity of the ball at B before impact will be equal to the velocity of projection at A and make an angle $\tan^{-1}\frac{3}{4}$ with the horizontal.



Let the initial velocity of projection be u, therefore the velocity of the ball at B before impact has components $u\cos\alpha=\frac{4}{5}u$ along AB and $u\sin\alpha=\frac{3}{5}u$ parallel to the wall .

Since the wall is smooth, the vertical component is unaltered by impact while the horizontal component after impact will be

$$e\,u\cos\alpha = \frac{9}{16}\left(\frac{4}{5}u\right) = \frac{9}{20}u.$$

The equation of the path of the ball from B to C relative to the axes at B shown is

$$y = -x\tan\beta - \frac{g x^2}{2u'^2\cos^2\beta}$$

where $\tan\beta = \dfrac{\frac{3}{5}u}{\frac{9}{20}u} = \dfrac{4}{3}$, $u'\cos\beta = \dfrac{9}{20}u$

$$\therefore\ y = -\frac{4}{3}x - \frac{32x^2}{2\left(\frac{9}{20}u\right)^2}$$

$$= -\frac{4}{3}x - \left(\frac{32x}{9u}\right)^2$$

The point $C(12, -\frac{400}{9})$ lies on this path,

$$\therefore\ -\frac{400}{9} = -16 - \left(\frac{320}{3u}\right)^2,$$

$$\therefore\ \left(\frac{320}{3u}\right)^2 = \frac{256}{9} = \left(\frac{16}{3}\right)^2,$$

i.e. $u = 20$ ft./sec.

[This value of u could directly be obtained by using $R = \dfrac{u^2\sin 2\alpha}{g} = 12$,

$$\therefore u^2 = \frac{12g}{2\sin\alpha\cos\alpha} = \frac{6\times 32}{\frac{3}{5}\times\frac{4}{5}} = 400]$$

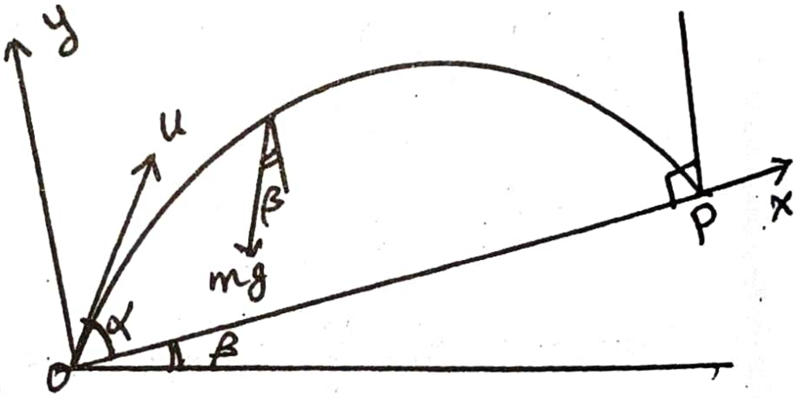
Now, $t_{A\to B} = \dfrac{12}{u\cos\alpha} = \dfrac{12}{\frac{4}{5}u} = \dfrac{12}{16} = \dfrac{3}{4}$ sec.,

$$t_{B\to C} = \frac{12}{e u\cos\alpha} = \frac{12}{\frac{9}{20}u} = \frac{12}{9} = \frac{3}{4}\text{ sec.}$$

$$\therefore \quad t_{A\to C} = \frac{3}{4} + \frac{4}{3} = \frac{25}{12} = 2\frac{1}{12} \text{ sec.}$$

2) A particle is projected from a point on a plane of inclination β to the horizontal, in the vertical plane through the line of greatest slope in a direction making an angle α with the plane given by $2\tan\alpha = \cot\beta$. Prove that the particle meets the plane in a direction perpendicular to it, and that the range on the inclined plane is given by $\dfrac{2u^2\sin\beta}{g(1+3\sin^2\beta)}$ where u is the velocity of projection.

Take axes at the point of projection 0, 0x in the direction of the line of greatest slope and 0y perpendicular to it. Equations of motion of the particle are :



$$\ddot{x} = -g\sin\beta \quad , \quad \ddot{y} = -g\cos\beta$$

$$\therefore \quad \dot{x} = u\cos\alpha - gt\sin\beta, \qquad (1)$$

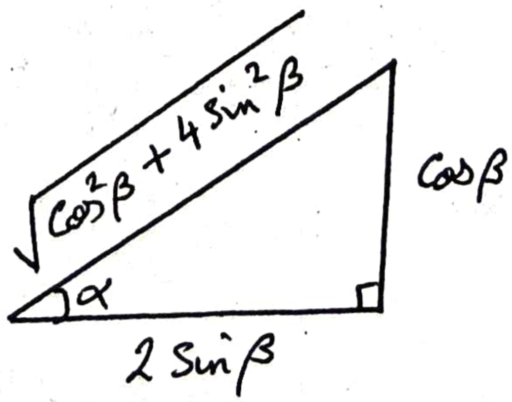
$$\dot{y} = u\sin\alpha - gt\cos\beta, \qquad (2)$$

$$x = ut\cos\alpha - \frac{1}{2}gt^2\sin\beta, \qquad (3)$$

$$y = ut\sin\alpha - \frac{1}{2}gt^2\cos\beta. \qquad (4)$$

Let the particle meets the plane at P where y = 0, therefore (4) gives the time from 0 to P as

$$t = \frac{2u\sin\alpha}{g\cos\beta} \qquad (5)$$



Substituting in (1) we get

$$\dot{x}_p = u\cos\alpha - 2u\sin\alpha\tan\beta$$

$$= u\cos\alpha\,(1 - 2\tan\alpha\tan\beta).$$

Using the relation $2\tan\alpha = \cot\beta$,

$\therefore \quad x_p^{\bullet} = 0$, i.e. the particle meets the plane at P normal to it.

Substituting from (5) in (3), we get

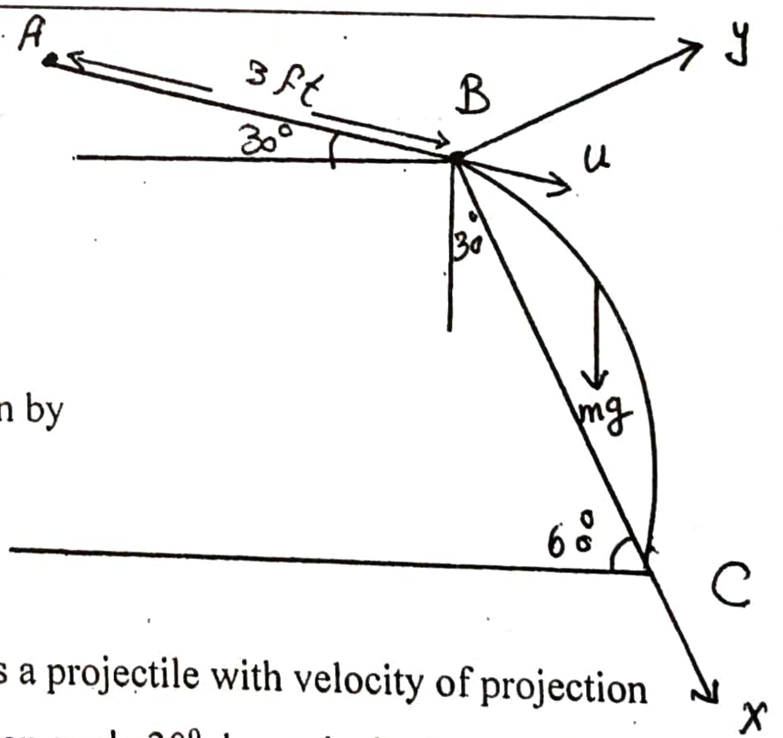
$$R = \frac{2u^2 \sin\alpha \cos\alpha}{g \cos\beta} - \frac{2u^2 \sin^2\alpha \sin\beta}{g \cos^2\beta}$$

$$= \frac{2u^2 \sin^2\alpha}{g \cos\beta} (\cot\alpha - \tan\beta)$$

$$= \frac{2u^2 \cos\beta (2\tan\beta - \tan\beta)}{g (\cos^2\beta + 4\sin^2\beta)}$$

$$= \frac{2u^2 \sin\beta}{g (1 + 3\sin^2\beta)}.$$

3) A particle starts to slide with velocity 2 ft./sec. at the point A down a smooth plane AB of length 3ft. which is inclined at an angle 30° to the horizontal . If the particle continues its motion after B and meet another plane BC , which is inclined at an angle 60° to the horizontal, at C. Find the length BC and the time of motion from A to C.



Velocity of the particle at B is given by

$$u^2 = 2^2 + 2g(3\sin 30)$$
$$= 4 + 96 = 100$$
$$\therefore \quad u = 10 \ ft./\sec.$$

and the time from A to B is given by

$$u = 2 + (g\sin 30)t_1$$

$$\therefore \quad t_1 = \frac{10-2}{16} = \frac{1}{2} \ \sec.$$

The particle will move after B as a projectile with velocity of projection 10 ft./sec. in a direction making an angle 30° down the horizontal. Take axes Bx, By as shown, the equations of motion of the particle between B, C are given by :

$$x^{\bullet\bullet} = g\cos 30 = 16\sqrt{3} \qquad , \qquad y^{\bullet\bullet} = -g\sin 30 = -16$$

$$\therefore \quad \dot{x} = u\cos 30 + 16\sqrt{3}t \quad , \quad \dot{y} = u\sin 30 - 16t$$
$$= 5\sqrt{3} + 16\sqrt{3}t \qquad\qquad = 5 - 16t$$

$$\& \quad x = 5\sqrt{3}t + 8\sqrt{3}t^2 \quad , \quad y = 5t - 8t^2$$

At C, y = 0, and the time from B to C is given by

$$t_2 = \frac{5}{8} \text{ sec.}$$

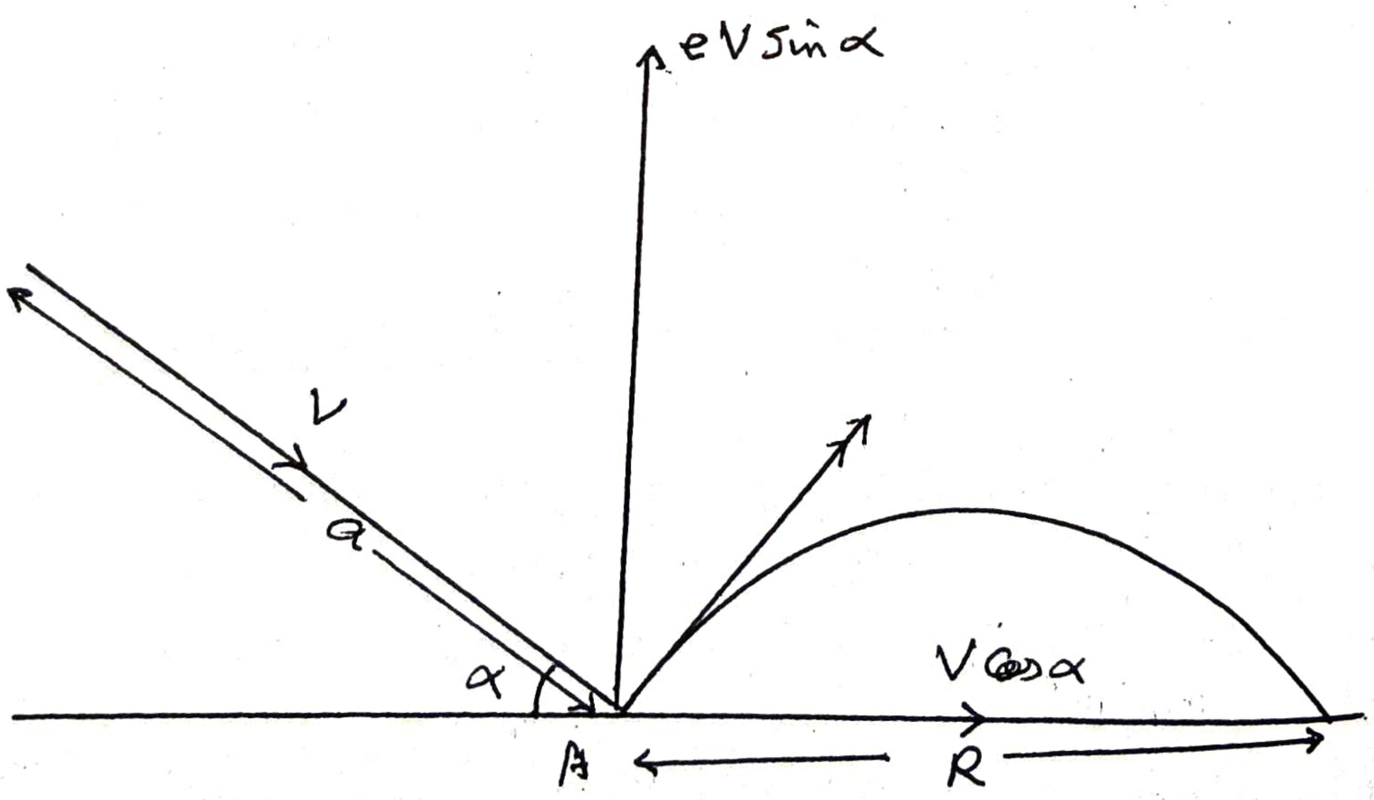
Substituting this value in x we obtain

$$BC = 5\sqrt{3}\left(\frac{5}{8}\right) + 8\sqrt{3}\left(\frac{5}{8}\right)^2 = \frac{25\sqrt{3}}{4} \text{ ft. and} \qquad t_{A\to C} = t_1 + t_2 = \frac{1}{2} + \frac{5}{8} = \frac{9}{8} \text{ sec.}$$

---

4) A particle slides from rest on a smooth plane which is inclined at an angle α to the horizontal. After moving a distance a on the plane, the particle hits another smooth horizontal plane. If e is the coefficient of restitution, prove that the particle will return again to the horizontal plane after a horizontal distance equal to

$$2ae\sin\alpha\ \sin 2\alpha\,.$$



The particle slides down the inclined plane with acceleration $g\sin\alpha$ through the distance a, therefore its velocity V at A before impact with the horizontal plane is given by

$$V^2 = 2ga\sin\alpha. \qquad (1)$$

After impact, the horizontal component $V\cos\alpha$ will remain unaltered while the vertical component will be $eV\sin\alpha$. The particle will move after A as a projectile and will return again to the horizontal plane after a horizontal distance R equal to the horizontal range,

i.e. $$R = \frac{2(V\cos\alpha)(eV\sin\alpha)}{g}$$

$$= \frac{eV^2\sin 2\alpha}{g}.$$

using (1). We get:

$$R = 2a\,e\sin\alpha\sin 2\alpha.$$

Exercises

1. A train of mass 225 tons was moving with velocity 48m.p.h. when it starts to ascend an inclined plane of slope $\frac{1}{75}$. If the engine force is 2.5 tons wt. and the resistance due to friction is constant and is equal to 3360 lbs. wt. find the distance described by the train before it comes to rest. (consider the plane makes an angle α to the horizontal, i.e. $\tan\alpha = \frac{1}{75}$, and since α is small $\therefore \sin\alpha \cong \frac{1}{75}$).
2. An engine, of mass 105 tons, pulls after it a train of mass 30 tons by a chain between them. If the resistance acting on the engine is $\frac{1}{100}$ of its weight, while that acting on the train is $\frac{1}{150}$ of its weight, find the tension in the chain if the engine force is equal to 6000 lbs, wt.
3. A train of mass 210 tons whose engine force is 4 tons wt. and the resistance to its motion is 20 lbs. wt. per ton while the force of its brakes is 400 lbs. wt. per ton. If the train starts motion from rest up an inclined plane of slope $\frac{1}{224}$ and when its velocity becomes 45m.p.h. the steam is shut off and the brakes are used. Find the time that elapses before the train comes to rest and the distance described in this time.
4. A train, whose mass is 112 tons, is traveling at the uniform rate of 25m.p.h. on a horizontal track, and the resistance due to air, friction, etc. is 16 lbs. wt. per ton. Part of the train, of mass 12 tons, becomes detached. Assuming that the force exerted by the engine is the same throughout, find the velocity of the train and the distance between it and the detached part when this part comes to rest.

5. A particle of mass 200 lbs. is attached to one end of a light string which passes over a fixed smooth pulley. The other end of the string is pulled and the particle starts to move upwards. If the tension is initially equal to 250 lbs. wt. and decreases uniformly with the rate of 1 lb. wt. per ft. through which the particle rises, find the velocity of the particle after rising a distance of 30 ft.

6. To determine the coefficient of friction between two materials experimentally, a particle of mass 10 lbs. made from one material is projected with velocity 30 ft./sec. over a horizontal surface made from the second material. Find the coefficient of friction if the particle comes to rest after moving a distance of 45 ft.

7. A driver uses the brakes of its car suddenly. The car slides a distance of 32 ft. in 2 secs. before stopping. If the retardation was uniform in this interval prove that the coefficient of friction between the car and the ground is $\frac{1}{2}$.

8. A particle of mass 2 Kgms. is projected with velocity 10m./sec. up an inclined plane whose slope is $\frac{3}{4}$. If the particle comes to rest after moving 8.3m. find the time taken and the coefficient of friction.

9. A car ascends an inclined plane of slope $\frac{1}{20}$. The air resistance is proportional to the square of the velocity. If the motor is shut off when the velocity of the car was 90 km.p.h. and the air resistance was 5% the weight of the car at this velocity; prove that the car has moved a distance of 435 metres before it comes to rest. (assume $g = 10$ m. /sec^2)

10-A particle is projected vertically upwards with velocity u from a point A, and when it reaches a point B, another particle is projected upwards with the same velocity u from A. If the two particles meet at B prove that the distance AB is equal to $\frac{8}{9}$ the maximum height attained by the first particle.

11-A heavy particle is projected vertically upwards with velocity u in a medium, the resistance of which is $\frac{mg}{u^2}\tan^2\alpha$ times the square of the velocity, where m is the mass of the particle, α is a constant.

Show that the particle will return to the point of projection with velocity $u\cos\alpha$, after a time

$$\frac{u}{g}\cot\alpha\,(\alpha+\ln\frac{\cos\alpha}{1-\sin\alpha}).$$

12-In starting a train the pull of the engine on the rails is at first constant and equal to P; and after the speed attains a certain value V the pull of the engine becomes equal to $\frac{PV}{v}$, where v is the velocity of the train.

Prove that the time t and distance x from the start are given by

$$t=\frac{M}{2pv}(v^2+V^2),$$

$$x=\frac{M}{3pv}(v^3+\frac{1}{2}V^3),$$

Where M is the combined mass of the engine and train.

13- A particle is projected with velocity 20 ft./sec. at an angle $\sin^{-1}\frac{3}{5}$ with the horizontal. Find the greatest height, the time of flight, the horizontal range and the equation of the path. Find also when and where the direction of motion makes an angle. $\tan^{-1}\frac{3}{4}$ with the horizontal and find the velocity there.

14- Find the velocity and direction of projection of a shot which passes in a horizontal direction just over the top of a wall of height 75 ft. at a distance of 150 ft. from the point of projection.

15- A shot is fired from a tower whose height is 3z with a velocity of $\sqrt{2gz}$. Find the greatest horizontal distance from the foot of the tower and the angle of projection in this case. If the shot is fired at the end of that distance to hit the top of the tower, find the least velocity of projection.

16- A shot is fired horizontally from the top of a tower. If it hits a target at a horizontal distance a and a vertical distance b downwards, find the velocity of projection and prove that there is another direction with which this velocity could hit the same target. Find the ratio between the times of flight in the two cases.

17- If the maximum range for a particle which is projected up an inelined plane at an angle of 30° with the horizontal is $\frac{64}{3}$ ft., find the velocity of projection. If the particle is projected with the same velocity down the plane, find the maximum range and the time of flight. Find the angle of projection in each case.

18- A ball is projected from a point at a height z above the ground and at a distance a from a smooth vertical wall.

If the ball is projected with a horizontal velocity V_0 towards the wall and $a < \sqrt{\frac{2z}{g}}\,V_0$, prove that it will hit the ground at a distance

$e\left(\sqrt{\frac{2z}{g}}\,V_0 - a\right)$ from the wall where e is the coefficient of restitution between the ball and the wall.

19-A particle is projected with a velocity of 26 ft./sec. at an angle $\tan^{-1}\frac{5}{12}$ with the horizontal. Find the greatest height attained the time of flight, the horizontal range and determine the direction of motion of the particle when it is at a height of 1 ft.

20-A projectile is fired from the top of a tower whose height is 64 ft. with a velocity of 80 ft./sec. in a direction making an angle $\tan^{-1}\frac{3}{4}$ with the horizontal. Prove that it hits a target at a distance of 256 ft. from the foot of the tower. Find the maximum height above the ground and the direction of motion when hitting the target.

21-A particle moves in a straight line Ox under a central attractive force $m\mu/x^2$ towards the origin O where m is the mass of the particle and μ is a constant. If it starts from rest at a distance a, show that it will arrive at the origin in time $\pi a^{3/2}/\sqrt{8\mu}$.

22-A particle is projected vertically upwards with initial velocity V_1 in a medium whose resistance is mkv^2 where k is a constant.

If the particle returns to the point of projection with velocity V_2 prove that $\frac{1}{V_2^2}=\frac{1}{V_1^2}+\frac{k}{g}$

where g is the acceleration of gravity.

23-A particle of mass m is projected vertically upwards with initial velocity V_0 in a medium whose resistance is mkv^2 where k is a constant and v is the velocity of the particle at any moment. Prove that the velocity v is given, in terms of the time t, measured from the initial position, by the equation

$$v = u\tan(\alpha - kut)$$

where u is the limiting velocity of the particle and $\alpha = \tan^{-1}(v_0/u)$. Find the relation between distance and time and prove that the maximum height is equal to $\frac{1}{k}\ln(\sec\alpha)$.