

Calculus I and Analytical Geometry

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Chapter 4

INTEGRATION

Methods of integration

1. Integration by parts

Let u, v are functions of x

$$\therefore \frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$
$$d(uv) = u \, dv + v \, du$$

$$\int d(uv) = \int u \, dv + \int v \, du$$

$$uv = \int u \, dv + \int v \, du$$

$$\therefore \int u \, dv = uv - \int v \, du$$

This method is useful to solve problems like:

$$\int x^n \sin x \, dx, \int x^n \cos x \, dx, \int x^n \ln x \, dx, \int \sin^{-1} x \, dx, \dots \dots$$

Example 1 : Evaluate the following integrals

$$\int x \sin x \, dx$$

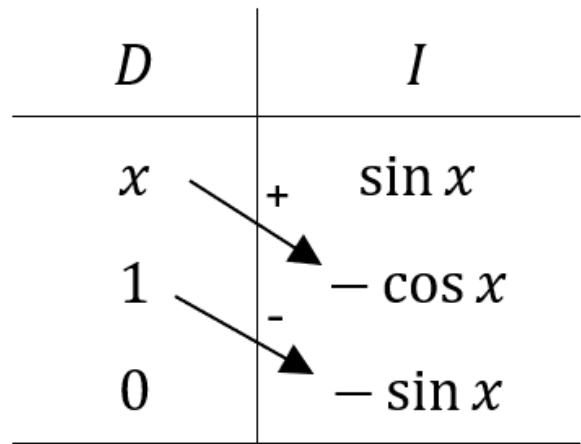
Solution

$$\begin{array}{ccc} u = x & & dv = \sin x \, dx \\ du = dx & \xrightarrow{\quad} & v = -\cos x \end{array}$$

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$$

Tabular method

$$\int x \sin x \, dx = -x \cos x + \sin x + c$$



Example 2:

$$\int x^2 e^{-2x} dx$$

Solution

$$\begin{array}{ccc} u = x^2 & & dv = e^{-2x} dx \\ & \searrow & \\ du = 2x dx & & v = \frac{-e^{-2x}}{2} \end{array}$$

$$\int x^2 e^{-2x} dx = -\frac{1}{2} x^2 e^{-2x} - \int \frac{-e^{-2x}}{2} \cdot 2x dx = -\frac{1}{2} x^2 e^{-2x} + \int x e^{-2x} dx$$

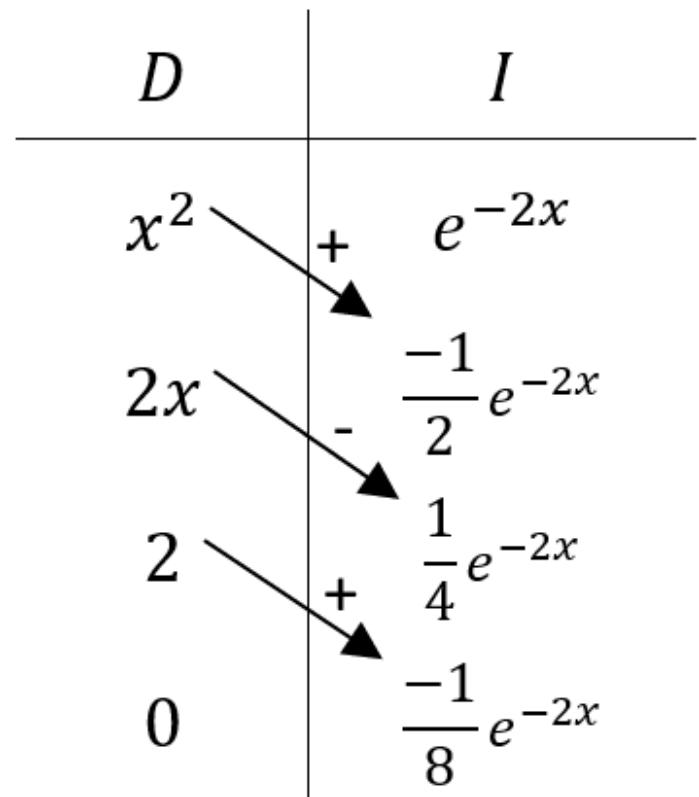
Now, we want to solve $\int x e^{-2x} dx$ by parts

$$\begin{array}{ccc} u = x & & dv = e^{-2x} dx \\ & \searrow & \\ du = dx & & v = \frac{-e^{-2x}}{2} \end{array}$$

$$\therefore \int x^2 e^{-2x} dx = \frac{-1}{2} x^2 e^{-2x} + x \cdot \frac{-e^{-2x}}{2} - \int \frac{-e^{-2x}}{2} dx = -\frac{1}{2} x^2 e^{-2x} - \frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C$$

Tabular method

$$\int x^2 e^{-2x} dx = -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

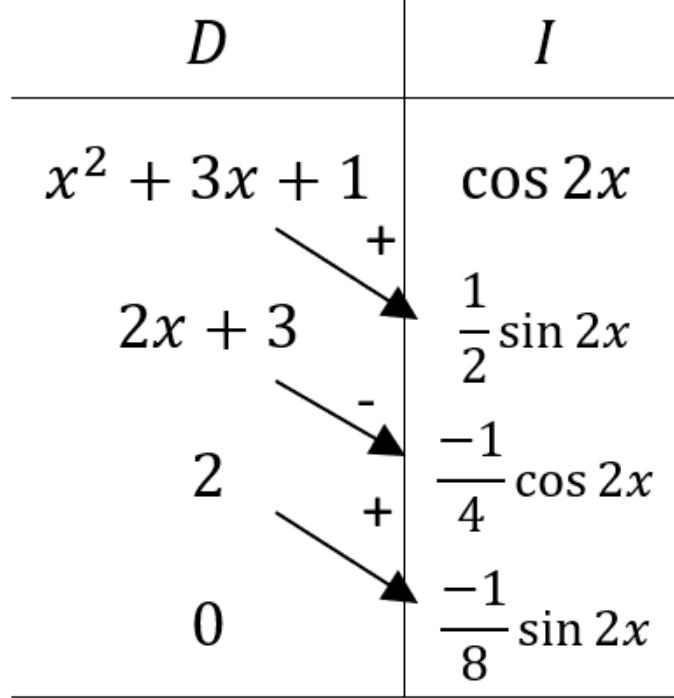


Example 3 :

$$\int (x^2 + 3x + 1) \cos 2x \, dx$$

Solution

$$I = \int (x^2 + 3x + 1) \cos 2x \, dx = (x^2 + 3x + 1) \frac{\sin 2x}{2} + (2x + 3) \frac{\cos 2x}{4} - 2 \cdot \frac{\sin 2x}{8} + C$$

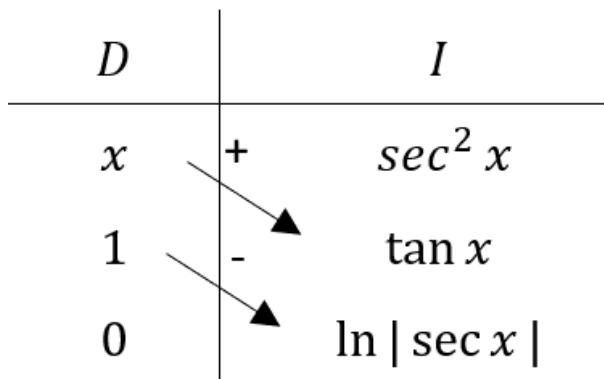


Example 4:

$$\int x \sec^2 x \, dx$$

Solution

$$I = x \tan x - \ln|\sec x| + C$$



Example 5 :

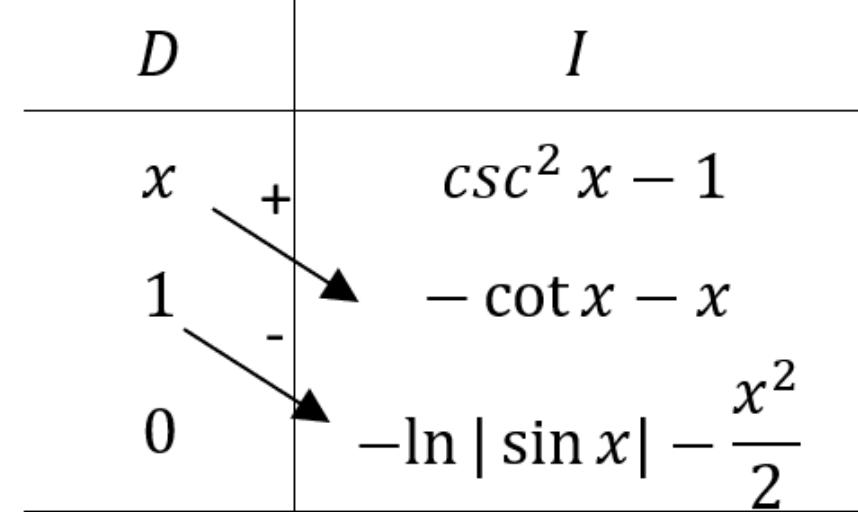
$$\int x \cot^2 x \, dx$$

Solution

$$\int x \cot^2 x \, dx = \int x (\csc^2 x - 1) \, dx$$

$$= x(-\cot x - x) - \left(-\ln |\sin x| - \frac{x^2}{2} \right) + C$$

$$= -x \cot x + \ln |\sin x| - \frac{x^2}{2} + C$$



Example 6:

$$\int x^2 \ln x \, dx$$

Solution

$$u = \ln x \qquad \qquad dv = x^2 dx$$

$$du = \frac{1}{x} dx \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad v = \frac{x^3}{3}$$

$$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} * \frac{x^3}{3} + C$$

$$= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C$$

Example 7:

$$\int \ln(x^2 + 1) \, dx$$

Solution

$$u = \ln(x^2 + 1) \quad dv = dx$$

$$du = \frac{2x}{x^2 + 1} dx \quad v = x$$

$$\int \ln(x^2 + 1) \, dx = x \ln(x^2 + 1) - 2 \int \frac{x^2}{x^2 + 1} \, dx$$

$$= x \ln(x^2 + 1) - 2 \int \frac{x^2 + 1 - 1}{x^2 + 1} \, dx$$

$$= x \ln(x^2 + 1) - 2 \int \left[1 - \frac{1}{x^2 + 1} \right] \, dx$$

$$= x \ln(x^2 + 1) - 2[x - \tan^{-1} x] + C$$

Example 8:

$$\int \sinh^{-1} x \ dx$$

Solution

$$u = \sinh^{-1} x \quad dv = dx$$

$$du = \frac{1}{\sqrt{1+x^2}} dx \quad v = x$$

$$\begin{aligned} I &= x \sinh^{-1} x - \int \frac{x}{\sqrt{1+x^2}} dx = x \sinh^{-1} x - \frac{1}{2} \int \frac{2x}{\sqrt{1+x^2}} dx \\ &= x \sinh^{-1} x - \frac{1}{2} \cdot 2 \sqrt{1+x^2} + C \end{aligned}$$

Example 9:

$$\int x \tan^{-1} x \, dx$$

Solution

$$u = \tan^{-1} x \quad dv = x \, dx$$

$$du = \frac{1}{1+x^2} \, dx \quad v = \frac{x^2}{2}$$

$$\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx$$

$$\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{x^2 + 1} \, dx$$

$$\int x \tan^{-1} x \, dx = \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C$$

Example 10:

$$\int e^{2x} \cos x \, dx$$

Solution

Let $I = \int e^{2x} \cos x \, dx$

$$\begin{aligned} u &= e^{2x} & dv &= \cos x \, dx \\ du &= 2e^{2x} \, dx & v &= \sin x \end{aligned}$$

$$I = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx$$

Now, we want to evaluate the integral $\int e^{2x} \sin x \, dx$ by parts

$$\begin{aligned} u &= e^{2x} & dv &= \sin x \, dx \\ du &= 2e^{2x} \, dx & v &= -\cos x \end{aligned}$$

$$I = e^{2x} \sin x - 2 \left[-e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \right]$$

$$I = e^{2x} \sin x + 2e^{2x} \cos x - 4I$$

$$I + 4I = e^{2x} \sin x + 2e^{2x} \cos x$$

$$5I = e^{2x} \sin x + 2e^{2x} \cos x$$

$$I = \frac{1}{5} [e^{2x} \sin x + 2e^{2x} \cos x] + c = \frac{e^{2x}}{5} [\sin x + 2 \cos x] + c$$

(Homework)

$$\int e^x \sin x \, dx$$

Example 11:

$$\int \sin(\ln x) dx$$

Solution

Let $I = \int \sin(\ln x) dx$

$$u = \sin(\ln x) \quad dv = dx$$

$$du = \cos(\ln x) \cdot \frac{1}{x} dx \quad v = x$$

$$I = x \sin(\ln x) - \int x \cdot \cos(\ln x) \cdot \frac{1}{x} dx = x \sin(\ln x) - \int \cos(\ln x) dx$$

$$u = \cos(\ln x) \quad dv = dx$$

$$du = -\sin(\ln x) \cdot \frac{1}{x} dx \quad v = x$$

Now,

$$I = x \sin(\ln x) - \left[x \cos(\ln x) + \int x \cdot \sin(\ln x) \cdot \frac{1}{x} dx \right]$$

$$I = x \sin(\ln x) - x \cos(\ln x) - I$$

$$2I = x \sin(\ln x) - x \cos(\ln x)$$

$$\therefore I = \frac{1}{2} [x \sin(\ln x) - x \cos(\ln x)] + C$$

(Homework)

$$\int \cos(\ln x) dx$$

Example 12:

$$\int \sec^3 x \, dx$$

Solution

Let $I = \int \sec^3 x \, dx$

$$I = \int \sec^3 x \, dx = \int \sec^2 x \sec x \, dx$$

$$\begin{aligned} u &= \sec x & dv &= \sec^2 x \, dx \\ du &= \sec x \tan x \, dx & v &= \tan x \end{aligned}$$

$$I = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$I = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$$

$$I = \sec x \tan x - \int [\sec^3 x - \sec x] dx$$

$$I = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$I = \sec x \tan x - I + \ln |\sec x + \tan x|$$

$$2I = \sec x \tan x + \ln |\sec x + \tan x|$$

$$I = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + C$$

(Homework)

$$\int \csc^3 x dx$$

2. Integration by Completing the Square

Use the method of completing the square to integrate integrals of the form

$$\int \frac{Ax + B}{ax^2 + bx + c} dx \quad \text{and} \quad \int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx$$

I) Integrals of the form

$$\int \frac{Ax + B}{ax^2 + bx + c} dx$$

Completing the square and use the rules:

$$\int \frac{u'}{u^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + c$$

$$\int \frac{u'}{a^2 - u^2} dx = \frac{1}{a} \tanh^{-1} \left(\frac{u}{a} \right) + c$$

Example 1:

$$\int \frac{dx}{x^2 + 2x + 5} dx$$

Solution

$$\int \frac{dx}{x^2 + 2x + 5} dx = \int \frac{dx}{(x+1)^2 + 4} = \frac{1}{2} \tan^{-1} \left(\frac{x+1}{2} \right) + c$$

Example 2:

$$\int \frac{x}{x^2 + 6x + 15} dx$$

Solution

$$\begin{aligned} \int \frac{x}{x^2 + 6x + 15} dx &= \int \frac{\frac{1}{2}(2x+6)-3}{x^2 + 6x + 15} dx = \frac{1}{2} \int \frac{2x+6}{x^2 + 6x + 15} dx - 3 \int \frac{dx}{x^2 + 6x + 15} \\ &= \frac{1}{2} \ln|x^2 + 6x + 15| - 3 \int \frac{dx}{(x+3)^2 + 6} = \frac{1}{2} \ln|x^2 + 6x + 15| - \frac{3}{\sqrt{6}} \tan^{-1} \left(\frac{x+3}{\sqrt{6}} \right) + c \end{aligned}$$

$$\begin{aligned} x &= \frac{1}{2}(2x+6) - 3 \quad \& \\ x^2 + 6x + 15 &= \\ (x+3)^2 - 9 + 15 &= \\ (x+3)^2 + 6 & \end{aligned}$$

Example 3:

$$\int \frac{dx}{2x^2 - 4x - 4}$$

Solution

$$2x^2 - 4x - 4 = 2[(x - 1)^2 - 3]$$

$$\int \frac{dx}{2x^2 - 4x - 4} = \int \frac{dx}{2[(x - 1)^2 - 3]} = \frac{1}{2} \int \frac{dx}{(x - 1)^2 - 3}$$

$$= \frac{-1}{2} \int \frac{dx}{3 - (x - 1)^2} = \frac{-1}{2} * \frac{1}{\sqrt{3}} \tanh^{-1} \left(\frac{x - 1}{\sqrt{3}} \right) + c$$

Example 4 :

$$\int \frac{8x - 1}{4x^2 - 4x - 3} dx$$

Solution

$$\begin{aligned} I &= \int \frac{8x - 1}{4x^2 - 4x - 3} dx = \int \frac{(8x - 4) + 3}{4x^2 - 4x - 3} dx \\ I &= \int \frac{8x - 4}{4x^2 - 4x - 3} dx + \int \frac{3}{4x^2 - 4x - 3} dx \\ I &= \int \frac{8x - 4}{4x^2 - 4x - 3} dx + \frac{3}{2} \int \frac{2}{(2x - 1)^2 - (2)^2} dx \\ &\quad \int \frac{8x - 4}{4x^2 - 4x - 3} dx - \frac{3}{2} \int \frac{2}{(2)^2 - (2x - 1)^2} dx \\ &= \ln(4x^2 - 4x + 3) - \frac{3}{2} \cdot \frac{1}{2} \tanh^{-1} \left(\frac{2x - 1}{2} \right) + C \end{aligned}$$

$$\begin{aligned} 4x^2 - 4x - 3 &= (4x^2 - 4x) - 3 \\ &= 4(x^2 - x) - 3 \\ &= 4\left(x^2 - x + \frac{1}{4}\right) - 3 - 1 \\ &= 4\left(x - \frac{1}{2}\right)^2 - 4 \\ &= (2x - 1)^2 - 2^2 \end{aligned}$$

II) Integrals of the form:

$$\int \frac{Ax + B}{\sqrt{ax^2 + bx + c}} dx$$

Completing the square and use one of the rules:

$$\int \frac{u'}{\sqrt{a^2 - u^2}} dx = \sin^{-1} \left(\frac{u}{a} \right) + c$$

$$\int \frac{u'}{\sqrt{a^2 + u^2}} dx = \sinh^{-1} \left(\frac{u}{a} \right) + c$$

$$\int \frac{u'}{\sqrt{u^2 - a^2}} dx = \cosh^{-1} \left(\frac{u}{a} \right) + c$$

Example 5:

$$\int \frac{2+x}{\sqrt{x^2 + 2x + 3}} dx$$

Solution

$$\begin{aligned} \int \frac{2+x}{\sqrt{x^2 + 2x + 3}} dx &= \int \frac{\frac{1}{2}(2x+2) + 1}{\sqrt{x^2 + 2x + 3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2 + 2x + 3}} dx + \int \frac{dx}{\sqrt{x^2 + 2x + 3}} \\ &= \frac{1}{2} \cdot 2\sqrt{x^2 + 2x + 3} + \int \frac{dx}{\sqrt{(x+1)^2 - 1 + 3}} \\ &= \sqrt{x^2 + 2x + 3} + \int \frac{dx}{\sqrt{(x+1)^2 + 2}} = \sqrt{x^2 + 2x + 3} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + c \end{aligned}$$

Example 6:

$$\int \frac{dx}{\sqrt{x^2 + 4x + 3}}$$

Solution

$$\int \frac{dx}{\sqrt{x^2 + 4x + 3}} = \int \frac{dx}{\sqrt{(x+2)^2 - 1}} = \cosh^{-1}\left(\frac{x+2}{1}\right) + c$$

$$\begin{aligned}x^2 + 4x + 3 \\= (x+2)^2 - 4 + 3 \\= (x+2)^2 - 1\end{aligned}$$

Example 7:

$$\int \frac{dx}{\sqrt{7 + 6x - x^2}}$$

Solution

$$\begin{aligned}\int \frac{dx}{\sqrt{7 + 6x - x^2}} &= \int \frac{dx}{\sqrt{-[x^2 - 6x - 7]}} = \int \frac{dx}{\sqrt{-[(x-3)^2 - 16]}} \\&= \int \frac{dx}{\sqrt{16 - (x-3)^2}} = \sin^{-1}\left(\frac{x-3}{4}\right) + c\end{aligned}$$

$$\begin{aligned}x^2 - 6x - 7 \\= (x-3)^2 - 9 - 7 \\= (x-3)^2 - 16\end{aligned}$$

Example 8:

$$\int \frac{2x + 3}{\sqrt{5 + 4x - x^2}} dx$$

Solution

$$\begin{aligned}\int \frac{2x + 3}{\sqrt{5 + 4x - x^2}} dx &= - \int \frac{-2x - 3}{\sqrt{5 + 4x - x^2}} dx = - \int \frac{(-2x + 4) - 7}{\sqrt{5 + 4x - x^2}} dx \\&= - \int \frac{(-2x + 4)}{\sqrt{5 + 4x - x^2}} dx + \int \frac{7}{\sqrt{5 + 4x - x^2}} dx \\&= - \int \frac{(-2x + 4)}{\sqrt{5 + 4x - x^2}} dx + 7 \int \frac{1}{\sqrt{3^2 - (x - 2)^2}} dx \\&= -2\sqrt{5 + 4x - x^2} + 7 \sin^{-1}\left(\frac{x - 2}{3}\right) + C\end{aligned}$$

$$\begin{aligned}5 - x^2 + 4x \\&= -(x^2 - 4x) + 5 \\&= -(x^2 - 4x + 4) + 5 + 4 \\&= 9 - (x - 2)^2\end{aligned}$$

Example 8:

$$\int \frac{\cos 2x \, dx}{\sqrt{\sin^2 2x + 2 \sin 2x + 5}}$$

Solution

$$\begin{aligned}\int \frac{\cos 2x \, dx}{\sqrt{\sin^2 2x + 2 \sin 2x + 5}} &= \int \frac{\cos 2x \, dx}{\sqrt{(\sin 2x + 1)^2 - 1 + 5}} \\ &= \int \frac{\cos 2x \, dx}{\sqrt{(\sin 2x + 1)^2 + 4}} = \frac{1}{2} \sinh^{-1} \left(\frac{\sin 2x + 1}{2} \right) + c\end{aligned}$$

3) Integrals of the form

$$1) \int \cos(mx) \cos(nx) dx$$

$$2) \int \sin(mx) \sin(nx) dx$$

$$3) \int \sin(mx) \cos(nx) dx$$

Rules

$$1) \cos x \cdot \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)]$$

$$2) \sin x \cdot \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)]$$

$$3) \sin x \cdot \cos y = \frac{1}{2} [\sin(x - y) + \sin(x + y)]$$

Example 1:

$$\int \sin 3x \cos 5x \, dx$$

Solution

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(3x - 5x) + \sin(3x + 5x)] \, dx \\ &= \frac{1}{2} \int [\sin(-2x) + \sin(8x)] \, dx = \frac{1}{2} \int [-\sin(2x) + \sin(8x)] \, dx = \frac{1}{2} \left[\frac{\cos 2x}{2} - \frac{\cos 8x}{8} \right] + C\end{aligned}$$

Example 2:

$$\int \cos 3x \cos 7x \, dx$$

Solution

$$\int \cos 3x \cos 7x \, dx = \frac{1}{2} \int [\cos 4x + \cos 10x] \, dx = \frac{1}{2} \left[\frac{\sin 4x}{4} + \frac{\sin 10x}{10} \right] + C$$

Example 3:

$$\int \sin 4x \sin 2x \, dx$$

Solution

$$\int \sin 4x \sin 2x \, dx = \frac{1}{2} \int [\cos 2x - \cos 6x] \, dx$$

$$= \frac{1}{2} \left[\frac{\sin 2x}{2} - \frac{\sin 6x}{6} \right] + C$$