Chapter I
VECTORS

VECTORS IN THE SPACE:

Geometric Vectors:

In science, mathematics, and engineering we distinguish two important quantities: scalars and vectors. A scalar is simply a real number or a quantity that has magnitude. For example, length, temperature, area, and volume. A vector, on the other hand, is usually described as a quantity that has both magnitude and direction.

A vector is written either as a boldface symbol **v** or as $\vec{v}$ or $\overline{AB}$. Examples of vector quantities are weight W, velocity V, and the retarding force of friction F.

A vector whose initial point is A and whose terminal point is B is written $\overline{AB}$. The magnitude of a vector is written $|\overline{AB}|$. Two vectors that have the same magnitude and same direction are said two be equal. Thus in figure 1, we have $\overline{AB} = \overline{CD}$. Vectors are said to be free, which means that a vector can be moved from one position to another provided its magnitude and direction are not changed. The negative of a vector $\overline{AB}$, written $-\overline{AB}$, is a vector that has the same magnitude as $\overline{AB}$ but is opposite in direction. If $k \neq 0$, is a scalar, the scalar multiple of a vector, $k\overline{AB}$ is a vector that is $|k|$ times as long as $\overline{AB}$. If $k \succ 0$, then $k\overline{AB}$ has the same direction as the vector $\overline{AB}$; if $k \prec 0$, then $k\overline{AB}$ has the direction opposite that of $\overline{AB}$. When $k = 0$, we say $0\,\overline{AB} = 0$ is the zero vector. Two vectors are parallel if and only if they are nonzero scalar multiples of each other. See Figure 2.

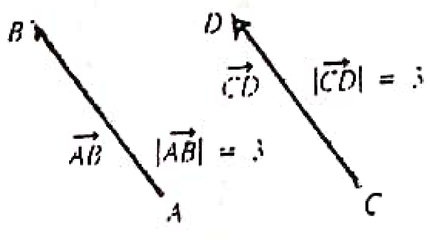

Figure 1

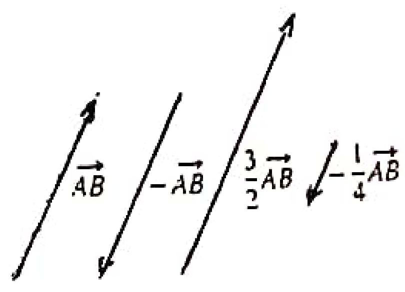

Figure 2

Addition and Subtraction:

Two vectors can be considered as having common initial point, such as A in figure 3(a). Thus, if nonparallel vectors $\overline{AB}$ and $\overline{AC}$ are the sides of a parallelogram as in Figure 3(b), we say the vector that is the main diagonal, or $\overline{AD}$, is the sum of $\overline{AB}$ and $\overline{AC}$. We write $\overline{AD} = \overline{AB} + \overline{AC}$.

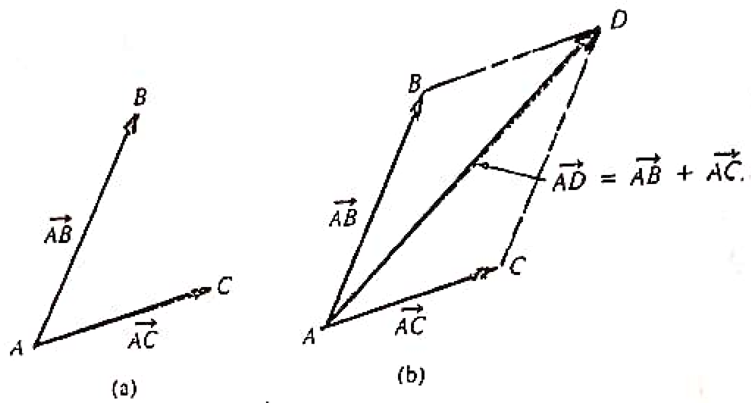

Figure 3

The difference of two vectors $\overline{AB}$ and $\overline{AC}$ is defined by

$$\overline{AB} - \overline{AC} = \overline{AB} + \left(-\overline{AC}\right)$$

As seen in Figure 4(a), the difference $\overline{AB} - \overline{AC}$ can be interpreted as the main diagonal of the parallelogram with sides $\overline{AB}$ and $\left(-\overline{AC}\right)$. However, as shown in Figure 4(b), we can also interpret the same vector difference as the third side of a triangle with sides $\overline{AB}$ and $\overline{AC}$. In this second interpretation, observe that the vector difference $\overline{AB} - \overline{AC} = \overline{CB}$ points toward the terminal point of the vector from which we are subtracting the second vector. If $\overline{AB} = \overline{AC}$, then $\overline{AB} - \overline{AC} = 0$.

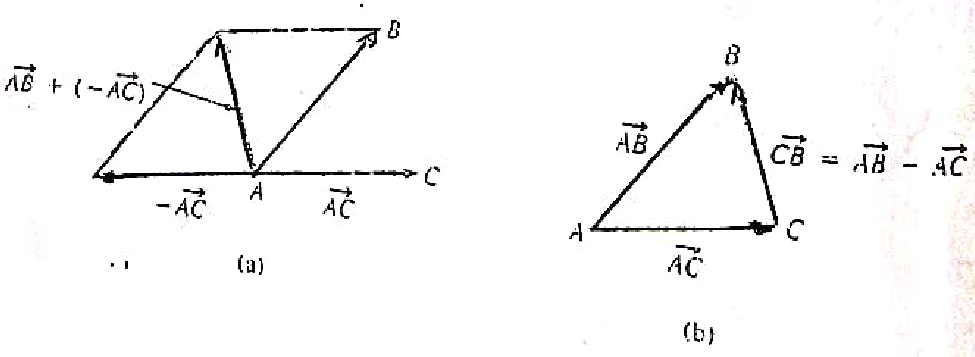

Figure 4

Vectors in a coordinate plane:

To describe a vector analytically, let us suppose the remainder of this section that the vectors we are considering lie in a two – dimensional coordinate plane. The vector shown in Figure 5, with initial point the origin O and terminal point $P(x_1, y_1)$, is called the position vector of the point P and is written $\overrightarrow{OP} = \langle x_1, y_1 \rangle$.

In general, a vector $\bar{a}$ in the plane is any ordered pair of real numbers, $\bar{a} = \langle a_1, a_2 \rangle$.

The numbers a_1 and a_2 are said to be the components of the vector $\bar{a}$.

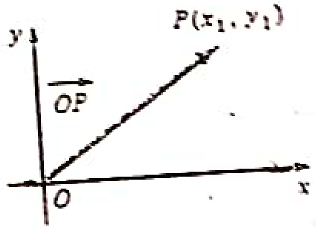

Figure 5

Addition and subtraction of vectors, multiplication of vectors by scalars, and so on, are defined in terms of their components.

Theorem1:

Let $\bar{a} = \langle a_1, a_2 \rangle$ and $\bar{b} = \langle b_1, b_2 \rangle$ be vectors in the plane, then

(1) $\bar{a} + \bar{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$

(2) $k\bar{a} = \langle ka_1, ka_2 \rangle$

(3) $\bar{a} = \bar{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$

Using (2), we define the negative of a vector $\bar{b}$ by

$-\bar{b} = (-1)\bar{b} = \langle -b_1, -b_2 \rangle$

We can define the subtraction of two vectors as

$\bar{a} - \bar{b} = \bar{a} + \left(-\bar{b}\right) = \langle a_1 - b_1, a_2 - b_2 \rangle$

In Figure 6(a), we have illustrated the sum of two vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$.

In Figure 6(b), the vector $\overrightarrow{P_1P_2}$ with initial point P_1 and terminal point P_2, is the difference of position vectors $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \langle x_2 - x_1, y_2 - y_2 \rangle$.

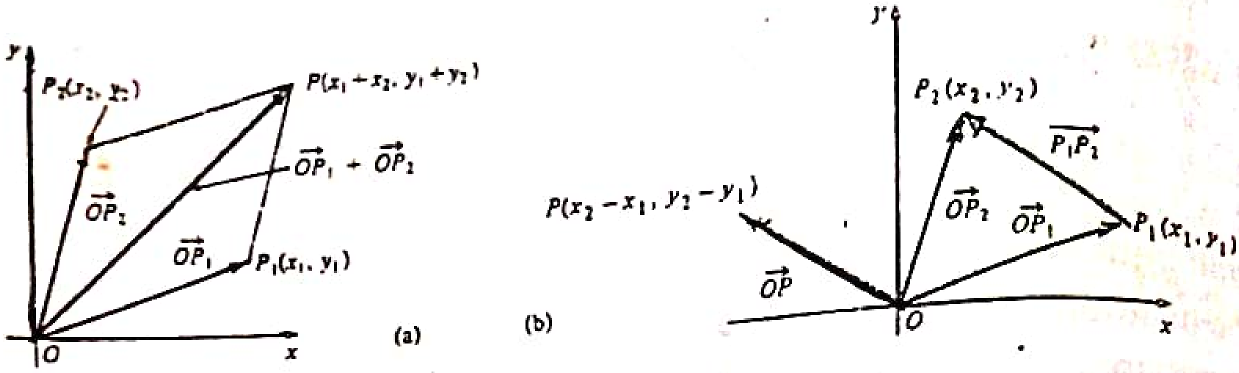

Figure 6

Example (1) If $\bar{a}=\langle 1,4\rangle$ and $\bar{b}=\langle -6,3\rangle$, find $\bar{a}+\bar{b}$, $\bar{a}-\bar{b}$, and $2\bar{a}+3\bar{b}$. Solution: $\bar{a}+\bar{b}=\langle 1+(-6),4+3\rangle=\langle -5,7\rangle$

$$\bar{a}-\bar{b}=\langle 1-(-6),4-3\rangle=\langle 7,1\rangle$$

$$2\bar{a}+3\bar{b}=\langle 2,8\rangle+\langle -18,9\rangle=\langle -16,17\rangle$$

The magnitude or length of a vector $\bar{a}=\langle a_1,a_2\rangle$ is denoted by $|\bar{a}|=\sqrt{a^2{}_1+a^2{}_2}$.

Unit vectors: a vector that has magnitude one is called a unit vector. We can obtain a unit vector $\bar{u}$ in the same direction as a nonzero vector $\bar{a}$ and $\bar{u}=\dfrac{\bar{a}}{|\bar{a}|}$. For example, the unit vector of $\bar{a}=\langle 2,-1\rangle$ is given by $\bar{u}=\dfrac{\langle 2,-1\rangle}{\sqrt{4+1}}=\langle\dfrac{2}{\sqrt{5}},\dfrac{-1}{\sqrt{5}}\rangle$.

Basic unit vectors: any vector $\bar{a}=\langle a_1,a_2\rangle$ can be written as $\bar{a}=\langle a_1,a_2\rangle=\langle a_1,0\rangle+\langle 0,a_2\rangle=a_1\langle 1,0\rangle+a_2\langle 0,1\rangle=a_1\bar{i}+a_2\bar{j}$, where $\bar{i}=\langle 1,0\rangle$ and $\bar{j}=\langle 0,1\rangle$ are said to form a basis for the system of two – dimensional vectors. The scalar a_1 is called the horizontal component of $\bar{a}$, and the scalar a_2 is called the vertical component of $\bar{a}$. See Figure 7.

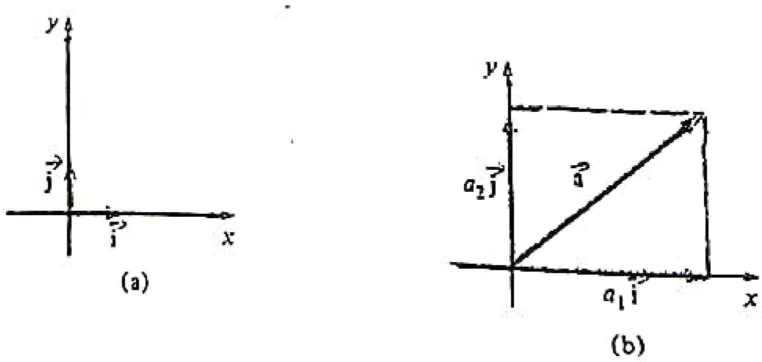

Figure 7

Example (2) (a) $\langle 4,7\rangle = 4\,\vec{i} + 7\,\vec{j}$

(b) $\left(2\,\vec{i} - 5\,\vec{j}\right) + \left(8\,\vec{i} + 13\,\vec{j}\right) = 10\,\vec{i} + 8\,\vec{j}$

(c) $\left|\vec{i} + \vec{j}\right| = \sqrt{2}$

(d) $10\left(3\,\vec{i} - \vec{j}\right) = 30\,\vec{i} - 10\,\vec{j}$

(e) $\bar{a} = 6\,\vec{i} + 4\,\vec{j}$ and $\bar{b} = 9\,\vec{i} + 6\,\vec{j}$ are parallel, since $\bar{b}$ is a scalar multiple of $\bar{a}$. We see that $\bar{b} = \frac{3}{2}\bar{a}$.

Vectors in the space:

A vector in three-space is any ordered triple of real numbers $\bar{a} = \langle a_1, a_2, a_3\rangle$ where a_1, a_2 and a_3 are the components of the vector. The component definitions of addition, subtraction, scalar multiplication, and so on are natural generalization of those given for vectors in three-space are given by the following definition:

Theorem 2:

Let $\bar{a} = \langle a_1, a_2, a_3\rangle$ and $\bar{b} = \langle b_1, b_2, b_3\rangle$ be vectors in the space, then

(1) $\bar{a} + \bar{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$

(2) $k\bar{a} = \langle ka_1, ka_2, ka_3\rangle$

(3) $\bar{a} = \bar{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$

(4) $\bar{a} - \bar{b} = \bar{a} + \left(-\bar{b}\right) = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$

(5) $\left|\bar{a}\right| = \sqrt{a^2{}_1 + a^2{}_2 + a^2{}_3}$

If $\overline{OP_1}$ and $\overline{OP_2}$ are the position vectors of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, then the vector $\overline{P_1P_2}$ is given by $\overline{P_1P_2} = \overline{OP_2} - \overline{OP_1} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$. See Figure 8

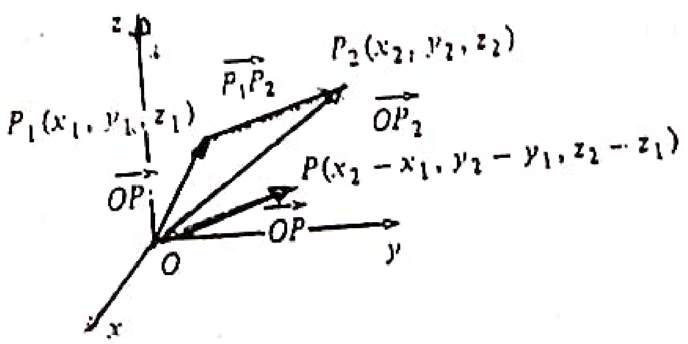

Figure 8

Example (3): Find the vector $\overline{P_1P_2}$ if the points P_1 and P_2 are given by $P_1(4,6,-2)$ and $P_2(1,8,3)$.

Solution: If the position vectors of the points are $\overline{OP_1} = \langle 4,6,-2 \rangle$ and $\overline{OP_2} = \langle 1,8,3 \rangle$, then we have

$\overline{P_1P_2} = \overline{OP_2} - \overline{OP_1} = \langle 1-4, 8-6, 3-(-2) \rangle = \langle -3,2,5 \rangle$.

Basic unit vectors in the space:

Any vector in the three-space can be written in the form $\bar{a} = \langle a_1, a_2, a_3 \rangle = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, where $\bar{i} = \langle 1,0,0 \rangle$, $\bar{j} = \langle 0,1,0 \rangle$ and $\bar{k} = \langle 0,0,1 \rangle$ are the unit vectors in the three-space, which lie along the coordinate axes and have the origin as a common initial point. See Figure 9.

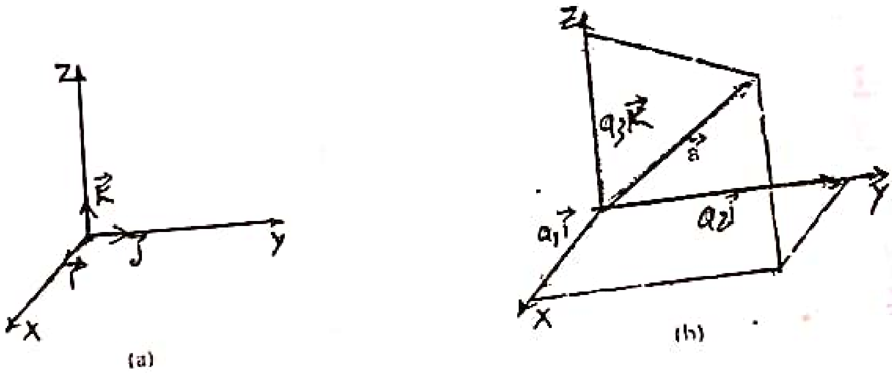

Figure 9

Example (4): If $\bar{a} = 3\bar{i} - 4\bar{j} + 3\bar{k}$ and $\bar{b} = \bar{i} - 4\bar{k}$, find $5\bar{a} - 2\bar{b}$

Solution: $5\bar{a} - 2\bar{b} = (15\bar{i} - 20\bar{j} + \underset{15}{40}\bar{k}) - (2\bar{i} - 0\bar{j} - 8\bar{k}) = 13\bar{i} - 20\bar{j} + 48\bar{k}$

The Dot (Scalar) Product:

Theorem 3:

The dot product of two vectors $\bar{a}$ and $\bar{b}$ is the scalar

$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}|\cos\theta$

Where θ is the angle between the two vectors such that $0 \le \theta \le \pi$. If the vectors $\bar{a}$ and $\bar{b}$ are not parallel, then θ is the smaller of the two possible angles between them.

The dot product possesses the following properties:

1) $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$

2) $\bar{a} \cdot \bar{b} = 0$ If $\bar{a} = 0$ or $\bar{b} = 0$ or $\bar{a}$ and $\bar{b}$ are orthogonal.

From 2), we get $\bar{i}\cdot\bar{j}=0$, $\bar{j}\cdot\bar{k}=0$, $\bar{k}\cdot\bar{i}=0$

3) $\bar{a}\cdot\bar{a}=|\bar{a}|^2$, and from 3), we get $\bar{i}\cdot\bar{i}=1$, $\bar{j}\cdot\bar{j}=1$, $\bar{k}\cdot\bar{k}=1$

4) If $\bar{a}=\langle a_1,a_2,a_3\rangle=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ and $\bar{b}=\langle b_1,b_2,b_3\rangle=b_1\bar{i}+b_2\bar{j}+b_3\bar{k}$, then

$$\bar{a}\cdot\bar{b}=a_1b_1+a_2b_2+a_3b_3$$

5) The angle between two nonzero vectors $\bar{a}$ and $\bar{b}$ is given by:

$$\cos\theta=\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a^2{}_1+a^2{}_2+a^2{}_3}\sqrt{b^2{}_1+b^2{}_2+b^2{}_3}}.$$

Also, we get

sin | all
tan | cos

i) $\bar{a}\cdot\bar{b}>0$, if and only if θ is acute. $\theta<90$ & $\theta>270$

ii) $\bar{a}\cdot\bar{b}<0$, if and only if θ is obtuse, and $90<\theta<270$
$\frac{\pi}{2}$ $\frac{3\pi}{2}$

iii) $\bar{a}\cdot\bar{b}=0$, if and only if $\cos\theta=0$.

Example (5): Find the angle between $\bar{a}=2\bar{i}+3\bar{j}+\bar{k}$ and $\bar{b}=-\bar{i}+5\bar{j}+\bar{k}$.

Solution: $$\cos\theta=\frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a^2{}_1+a^2{}_2+a^2{}_3}\sqrt{b^2{}_1+b^2{}_2+b^2{}_3}}=\frac{14}{\sqrt{14}\sqrt{27}}=\frac{\sqrt{42}}{9}$$

and so $\theta=\cos^{-1}\left(\frac{\sqrt{42}}{9}\right)\approx 0.77$ radian or $\theta=44.9^\circ$.

<u>Applications of the dot product</u>:

(1) <u>Component of $\bar{a}$ on $\bar{b}$</u>: if $\bar{a}=\langle a_1,a_2,a_3\rangle=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$, then

$$\boxed{a_1=\bar{a}\cdot\bar{i}=comp_{\bar{i}}\,\bar{a},\ a_2=\bar{a}\cdot\bar{j}=comp_{\bar{j}}\,\bar{a}\ \text{and}\ a_3=\bar{a}\cdot\bar{i}=comp_{\bar{k}}\,\bar{a}.}$$

Generally, the component of $\bar{a}$ on an arbitrary vector $\bar{b}$ is given by $comp_{\bar{b}}\,\bar{a}=|\bar{a}|\cos\theta=\frac{|\bar{a}||\bar{b}|\cos\theta}{|\bar{b}|}=\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|}=\bar{a}\cdot\left(\frac{1}{|\bar{b}|}\bar{b}\right)$. See Figure 10

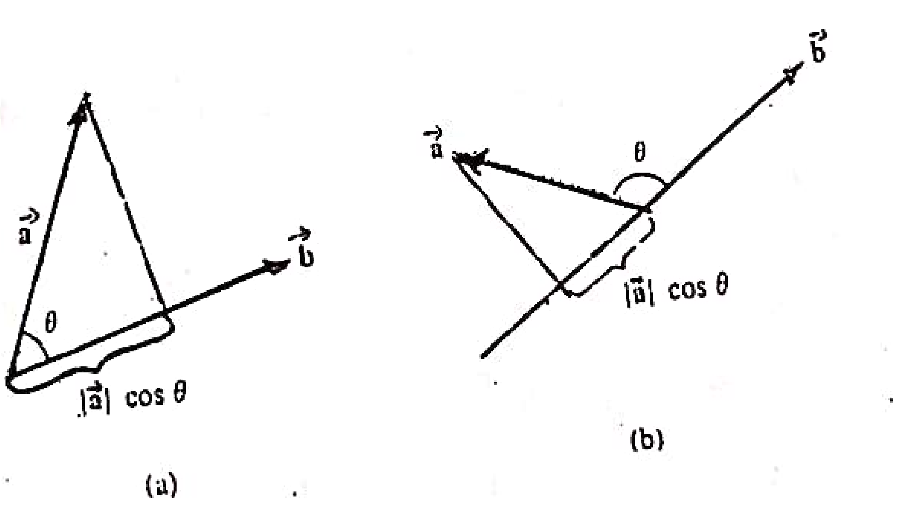

Figure 10

Example (6): Let $\vec{a} = 2\,\vec{i} + 3\vec{j} - 4\,\vec{k}$ and $\vec{b} = i + j + 2\,k$, find the component of $\vec{a}$ on $\vec{b}$ $\left(comp_{\vec{b}}\,\vec{a}\right)$ and the component of $\vec{b}$ on $\vec{a}$ $\left(comp_{\vec{a}}\,\vec{b}\right)$.

Solution: $comp_{\vec{b}}\,\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \dfrac{\left(2\vec{i}+3\vec{j}-4\vec{k}\right)\cdot\left(\vec{i}+\vec{j}+2\vec{k}\right)}{\sqrt{1+1+4}} = \dfrac{-3}{\sqrt{6}}$, similarly, we get $comp_{\vec{a}}\,\vec{a} = \dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|} = \dfrac{\left(2\vec{i}+3\vec{j}-4\vec{k}\right)\cdot\left(\vec{i}+\vec{j}+2\vec{k}\right)}{\sqrt{4+9+16}} = \dfrac{-3}{\sqrt{29}}$.

(2) Projection of $\vec{a}$ onto $\vec{b}$: if $\vec{a} = \langle a_1, a_2, a_3\rangle = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$, then $\boxed{proj_{\vec{i}}\,\vec{a} = \left(comp_{\vec{i}}\,\vec{a}\right)\vec{i} = \left(\vec{a}\cdot\vec{i}\right)\vec{i} = a_1\vec{i}}$, and so on.

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Figure 11 shows the general case of the projection of $\vec{a}$ onto $\vec{b}$:

Projection of $\vec{a}$ onto $\vec{b}$ = $proj_{\vec{b}}\,\vec{a} = \left(comp_{\vec{b}}\,\vec{a}\right)\left(\dfrac{1}{|\vec{b}|}\vec{b}\right)$.

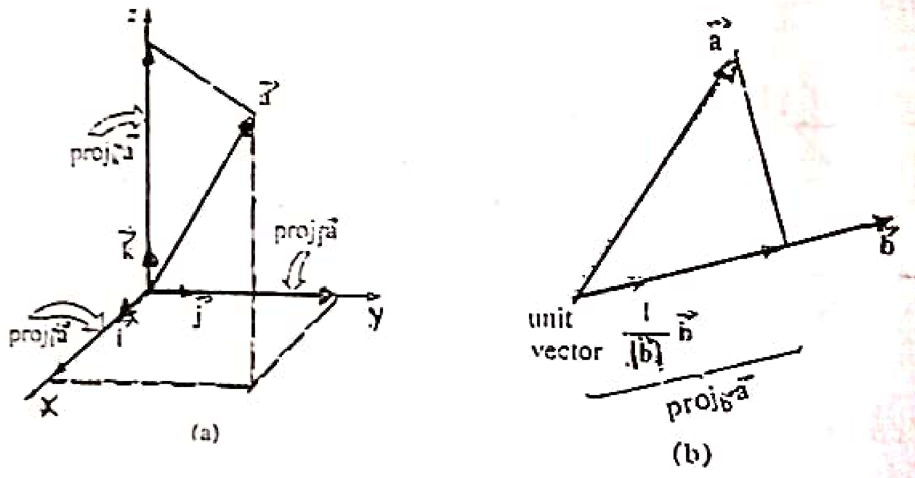

Figure 11

Example (7): Find the projection of $\vec{a} = 4\,\vec{i} + \vec{j}$ on the vector $\vec{b} = 2\vec{i} + 3\vec{j}$.

Solution: first, we find the component of $\vec{a}$ and $\vec{b}$,

$$comp_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{\left(4\vec{i}+\vec{j}\right)\cdot\left(2\vec{i}+3\vec{j}\right)}{\sqrt{4+9}} = \frac{11}{\sqrt{13}}.$$

Thus, we get $proj_{\vec{b}}\,\vec{a} = \left(comp_{\vec{b}}\,\vec{a}\right)\left(\dfrac{1}{|\vec{b}|}\vec{b}\right) = \left(\dfrac{11}{\sqrt{13}}\right)\left(\dfrac{1}{\sqrt{13}}\right)\left(2\vec{i}+3\vec{j}\right) = \dfrac{22}{13}\vec{i} + \dfrac{33}{13}\vec{j}$

(3) Projection of $\vec{a}$ onto $\vec{b}^{\perp}$: If $\vec{b}^{\perp}$ is orthogonal to $\vec{b}$, then from Figure 12 we see that a can be written as the sum of two projections $\boxed{proj_{\vec{b}}\vec{a} + proj_{\vec{b}^{\perp}}\vec{a} = \vec{a}}$ This equation enables us to define the projection of $\vec{a}$ onto $\vec{b}^{\perp}$. i.e $\quad proj_{\vec{b}^{\perp}}\vec{a} = \vec{a} - proj_{\vec{b}}\vec{a}$

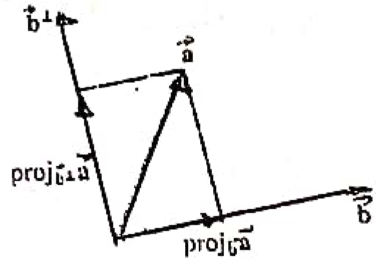

Figure 12

Example (8): Let $\vec{a} = 3\vec{i} - \vec{j} + 5\vec{k}$ and $\vec{b} = 2\vec{i} + \vec{j} + 2\vec{k}$, find $proj_{\vec{b}}\vec{a}$ and $proj_{\vec{b}^{\perp}}\vec{a}$.

Solution: Since $|\vec{b}| = 3$, we have

$$comp_{\vec{b}}\vec{a} = \left(3\vec{i} - \vec{j} + 5\vec{k}\right) \cdot \frac{\left(2\vec{i} + \vec{j} + 2\vec{k}\right)}{3} = \frac{15}{3} = 5$$

i.e $\quad proj_{\vec{b}}\vec{a} = (5)\frac{\left(2\vec{i} + \vec{j} + 2\vec{k}\right)}{3} = \frac{10\vec{i} + 5\vec{j} + 10\vec{k}}{3}$

i.e $\quad proj_{\vec{b}^{\perp}}\vec{a} = \vec{a} - proj_{\vec{b}}\vec{a} = \left(3\vec{i} - \vec{j} + 5\vec{k}\right) - \left(\frac{10}{3}\vec{i} + \frac{5}{3}\vec{j} + \frac{10}{3}\vec{k}\right) = -\frac{1}{3}\vec{i} - \frac{8}{3}\vec{j} + \frac{5}{3}\vec{k}$.

(4) The work done: If a constant force $\vec{F}$ applied to a body acts at an angle θ to the direction of motion, then the work done by $\vec{F}$ is defined to be the product of the component of $\vec{F}$ in the direction of the displacement and the distance $|d|$ that the body moves: $W = (|\vec{F}|\cos\theta)|d| = |\vec{F}||d|\cos\theta$.

Example (9): Find the work done by a constant force $\vec{F} = 2\vec{i} + 4\vec{j}$ if its point of application to a block moves from $P_1(1,1)$ to $P_2(4,6)$.

Solution: The displacement of the block is given by

$d = \overline{P_1P_2} = \overline{OP_2} - \overline{OP_1} = 3\vec{i} + 5\vec{j}$.

Then we get $\quad W = \left(2\vec{i} + 4\vec{j}\right) \cdot \left(3\vec{i} + 5\vec{j}\right) = 26$ work units.

The Cross (Vector) Product:

Definition4:

The cross product of two vectors $\bar{a}$ and $\bar{b}$ is the vector $\bar{a}\wedge\bar{b} = \left(|\bar{a}|\,|\bar{b}|\sin\theta\right)\bar{n}$

Where θ is the angle between the two vectors such that $0 \le \theta \le \pi$ and $\bar{n}$ is a unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$ with direction given by the right-hand rule.

The cross product possesses the following properties:

1) $\bar{a}\wedge\bar{b} = -\bar{b}\wedge\bar{a}$

2) $\bar{a}\wedge\bar{b} = 0$ If $\bar{a} = 0$ or $\bar{b} = 0$ or $\bar{a}$ and $\bar{b}$ are parallel.

3) $\bar{i}\wedge\bar{j} = \bar{k}$, $\bar{j}\wedge\bar{k} = \bar{i}$, $\bar{k}\wedge\bar{i} = \bar{j}$. See Figure 13

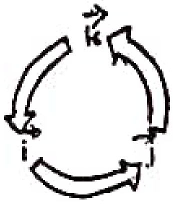

Figure 13

4) $\bar{a}\wedge\bar{a} = 0$, and from 4), we get $\bar{i}\wedge\bar{i} = 0$, $\bar{j}\wedge\bar{j} = 0$, $\bar{k}\wedge\bar{k} = 0$

Using 3) and 4) and if $\bar{a} = \langle a_1, a_2, a_3\rangle = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

and $\bar{b} = \langle b_1, b_2, b_3\rangle = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$, we get $\bar{a}\wedge\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

5) If θ is the angle between nonzero two vectors $\bar{a}$ and $\bar{b}$ in the space, then $|\bar{a}\wedge\bar{b}| = |\bar{a}|\,|\bar{b}|\sin\theta$, and $\bar{n} = \dfrac{\bar{a}\wedge\bar{b}}{|\bar{a}\wedge\bar{b}|}$

Example (9): Let $\bar{a} = 4\bar{i} - 2\bar{j} + 5\bar{k}$ and $\bar{b} = 3\bar{i} + \bar{j} - 2\bar{k}$, find $\bar{a}\wedge\bar{b}$.

Solution:

$$\bar{a}\wedge\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & -2 & 5 \\ 3 & 1 & \begin{smallmatrix}-1\\-2\end{smallmatrix} \end{vmatrix} = -3\bar{i} + 19\bar{j} + 10\bar{k}.$$

Special Products:

1) The scalar product between three vectors (or sometime is called mixed product) $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by $\vec{a}\cdot(\vec{b}\wedge\vec{c})$, where

$$\vec{a}\cdot(\vec{b}\wedge\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Furthermore, from the properties of determinates, we have

i) $\vec{a}\cdot(\vec{b}\wedge\vec{c})=\vec{b}\cdot(\vec{c}\wedge\vec{a})=\vec{c}\cdot(\vec{a}\wedge\vec{b})$

ii) $\vec{a}\cdot(\vec{a}\wedge\vec{c})=0$

2) The vector product between three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is given by

$\vec{a}\wedge(\vec{b}\wedge\vec{c})$, where $\quad \vec{a}\wedge(\vec{b}\wedge\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$

Applications of the cross product:

1) Areas: Two nonzero and nonparallel vectors $\vec{a}$ and $\vec{b}$ can be considered to be the sides of a parallelogram. The area A of a parallelogram is $A=|\vec{a}|\,|\vec{b}|\sin\theta=|\vec{a}\wedge\vec{b}|$. See Figure 14(a).

Also, we see that the area of a triangle with sides $\vec{a}$ and $\vec{b}$ is $A=\frac{1}{2}|\vec{a}|\,|\vec{b}|\sin\theta=\frac{1}{2}|\vec{a}\wedge\vec{b}|$. See Figure 14(b).

2) Volume: If the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ do not lie in the same plane, then the volume of the parallelepiped with edges $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

V= (area of base) (height) = $|\vec{a}\cdot(\vec{b}\wedge\vec{c})|$. See Figure 15

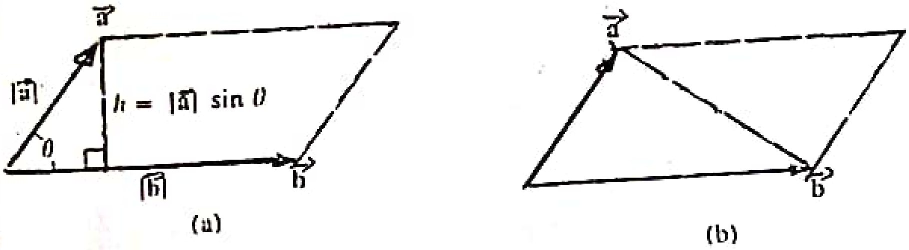

Figure 14

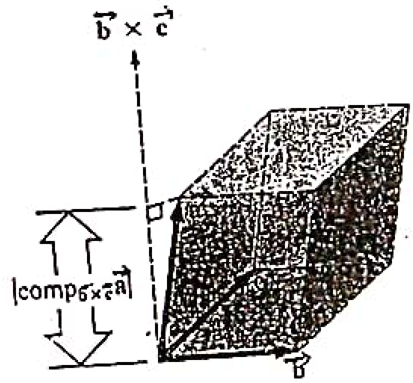

Figure 15

From 2), if $\left|\bar{a}\cdot\left(\bar{b}\wedge\bar{c}\right)\right|=0$, then the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are called coplanar vectors.

Example (10): Calculate the area of the triangle $\Delta\, PQR$, where $P(2,4,-7), Q(3,7,18)$ *and* $R(-5,12,8)$.

Solution: Let $\bar{v}=\overline{PQ}$ and $\bar{w}=\overline{PR}$, as in Figure 16. Then $\bar{v}=\langle 3-2,7-4,18+7\rangle=\langle 1,3,25\rangle$ and $\bar{w}=\langle -5-2,12-4,8+7\rangle=\langle -7,8,15\rangle$, so the area of the triangle $\Delta\, PQR$ is

$$A=\frac{1}{2}\left|\bar{v}\wedge\bar{w}\right|=\frac{1}{2}\left\|\begin{matrix}\bar{i} & \bar{j} & \bar{k}\\ 1 & 3 & 25\\ -7 & 8 & 15\end{matrix}\right\|=\frac{1}{2}\left|-155\,\bar{i}-190\,\bar{j}+29\,\bar{k}\right|$$

$$=\frac{1}{2}\sqrt{(-155)^2+(-190)^2+(29)^2}=\frac{1}{2}\sqrt{60966}$$

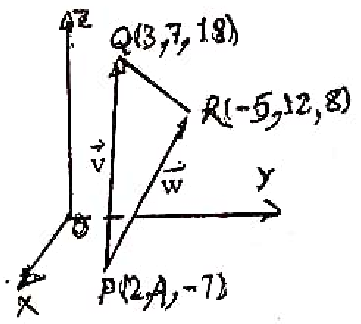

Figure 16

Example (11): Calculate the area of the parallelogram $PQRS$, where $P(1,1), Q(2,3), R(5,4)$ *and* $S(4,2)$.

Solution: Let $\bar{v}=\overline{SP}$ and $\bar{w}=\overline{SR}$, as in Figure 17. Then $\bar{v}=\langle 1-4,1-2\rangle=\langle -3,-1\rangle$ .and $\bar{w}=\langle 5-4,4-2\rangle=\langle 1,2\rangle$, so the area of the triangle $PQRS$ is

$$A=\left|\bar{v}\wedge\bar{w}\right|=\left\|\begin{matrix}\bar{i} & \bar{j} & \bar{k}\\ -3 & -1 & 0\\ 1 & 2 & 0\end{matrix}\right\|=\frac{1}{2}\left|0\bar{i}-0\bar{j}-5\,\bar{k}\right|=\sqrt{25}=5$$

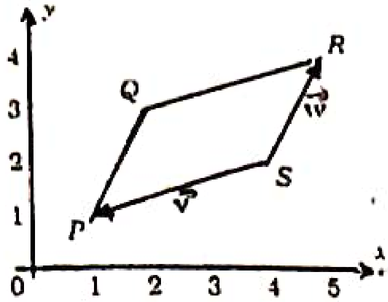

Figure 17

Example (12): Calculate the volume of the parallelepiped P for which the vectors $\bar{u} = 3i + \bar{j} + \bar{k}$, $\bar{v} = \bar{i} + 4\bar{j} + \bar{k}$ and $\bar{w} = \bar{i} + \bar{j} + 5\bar{k}$ are three edges.

Solution: The volume of the parallelepiped P.(See Figure 18) is

$$V = \bar{u} \cdot (\bar{v} \wedge \bar{w}) = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 5 \end{vmatrix} = 3(20-1) - 1(5-1) + 1(1-4) = 50.$$

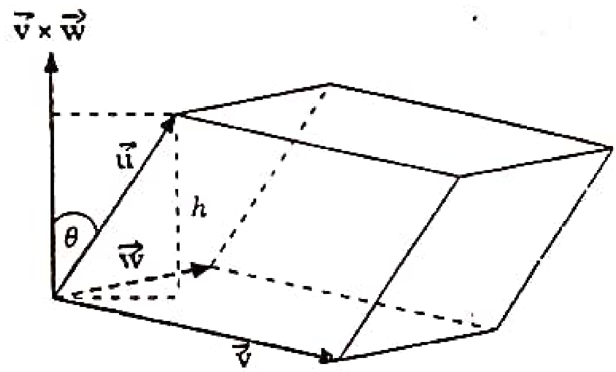

Figure 18

Example (13): Calculate $\bar{u} \wedge (\bar{v} \wedge \bar{w})$ for $\bar{u} = \bar{i} + 2\bar{j} + 4\bar{k}$, $\bar{v} = 2\bar{i} + 2\bar{j}$ and $\bar{w} = \bar{i} + 3\bar{j}$.

Solution: Since $\bar{u} \wedge (\bar{v} \wedge \bar{w}) = (\bar{u} \cdot \bar{w})\bar{v} - (\bar{u} \cdot \bar{v})\bar{w}$, and $\bar{u} \cdot \bar{v} = 6$, *and* $\bar{u} \cdot \bar{v} = 7$, then $\bar{u} \wedge (\bar{v} \wedge \bar{w}) = (\bar{u} \cdot \bar{w})\bar{v} - (\bar{u} \cdot \bar{v})\bar{w} = 7\langle 2,2,0\rangle - 6\langle 1,3,0\rangle = \langle 8,-4,0\rangle = 8\bar{i} - 4\bar{j}$.

Example (14): Prove $(\bar{u} \wedge \bar{v}) \cdot (\bar{w} \wedge \bar{z}) = \begin{vmatrix} \bar{u} \cdot \bar{w} & \bar{u} \cdot \bar{z} \\ \bar{v} \cdot \bar{w} & \bar{v} \cdot \bar{z} \end{vmatrix}$ for all vectors $\bar{u}, \bar{v}, \bar{w}, \bar{z}$ in the space.

Solution: Let $\bar{x} = \bar{u} \wedge \bar{v}$, then

$$
\begin{aligned}
(\bar{u} \wedge \bar{v}) \cdot (\bar{w} \wedge \bar{z}) = \bar{x} \cdot (\bar{w} \wedge \bar{z}) &= \bar{w} \cdot (\bar{z} \wedge \bar{x}) \\
&= \bar{w} \cdot \left[\bar{z} \wedge (\bar{u} \wedge \bar{v})\right] \\
&= \bar{w} \cdot \left[(\bar{z} \cdot \bar{v})\bar{u} - (\bar{z} \cdot \bar{u})\bar{v}\right] \\
&= (\bar{z} \cdot \bar{v})(\bar{w} \cdot \bar{u}) - (\bar{z} \cdot \bar{u})(\bar{w} \cdot \bar{v}) \\
&= (\bar{v} \cdot \bar{z})(\bar{u} \cdot \bar{w}) - (\bar{u} \cdot \bar{z})(\bar{v} \cdot \bar{w}) \\
&= \begin{vmatrix} \bar{u} \cdot \bar{w} & \bar{u} \cdot \bar{z} \\ \bar{v} \cdot \bar{w} & \bar{v} \cdot \bar{z} \end{vmatrix}
\end{aligned}
$$

Ordinary Derivative of Vectors:

Let $\bar{R}(u)$ be a vector depending on a single scalar variable u.

Then

$$\frac{\delta \bar{R}}{\delta u} = \frac{\bar{R}(u + \delta u) - \bar{R}(u)}{\delta u}$$

where δu denotes an increment in u

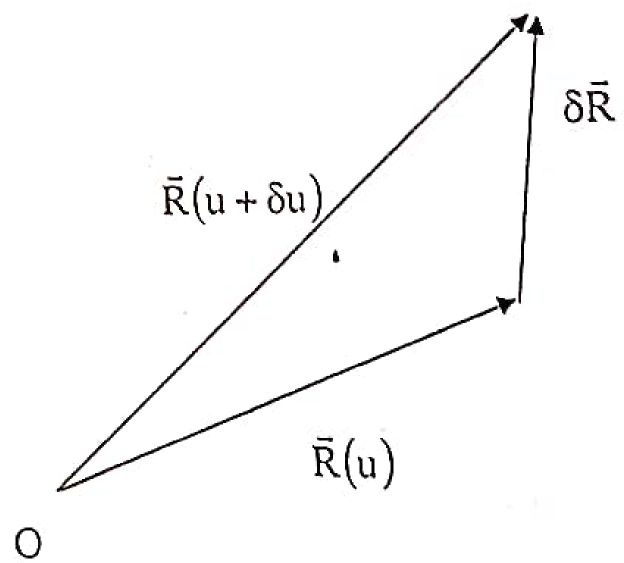

Figure 19

The ordinary derivative of the vector $\bar{R}(u)$ with respect to the scalar u is given by

$$\frac{d\bar{R}}{du} = \lim_{\delta u \to 0} \frac{\delta \bar{R}}{\delta u} = \lim_{\delta u \to 0} \frac{\bar{R}(u + \delta u) - \bar{R}(u)}{\delta u}$$

if it exists.

Since $\frac{d\bar{R}}{du}$ is itself a vector depending on u, we can consider its derivative with respect to u. If this derivative exists it is denoted by $\frac{d^2\bar{R}}{du^2}$. In this manner higher order derivatives are described.

Space Curve:

If in particular $\bar{R}(u)$ is the position vector $\bar{r}(u)$ joining the origin O of a coordinate system and any point (x, y, z), then

$\bar{r}(u) = x(u)\bar{i} + y(u)\bar{j} + z(u)\bar{k}$

and specification of the vector function $\bar{r}(u)$ defines x, y and z as functions of u.

As u changes, the terminal point of $\bar{r}$ describes a space curve having parametric equations

$$x = x(u), \qquad y = y(u), \qquad z = z(u)$$

Then $\frac{\delta\bar{r}}{\delta u} = \frac{\bar{r}(u+\delta u) - \bar{r}(u)}{\delta u}$ is a vector in the direction of $\delta\bar{r}$.

If $\lim_{\delta u \to 0} \frac{\delta\bar{r}}{\delta u} = \frac{d\bar{r}}{du}$ exists, the limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and is given by

$$\frac{d\bar{r}}{du} = \frac{dx}{du}\bar{i} + \frac{dy}{du}\bar{j} + \frac{dz}{du}\bar{k}$$

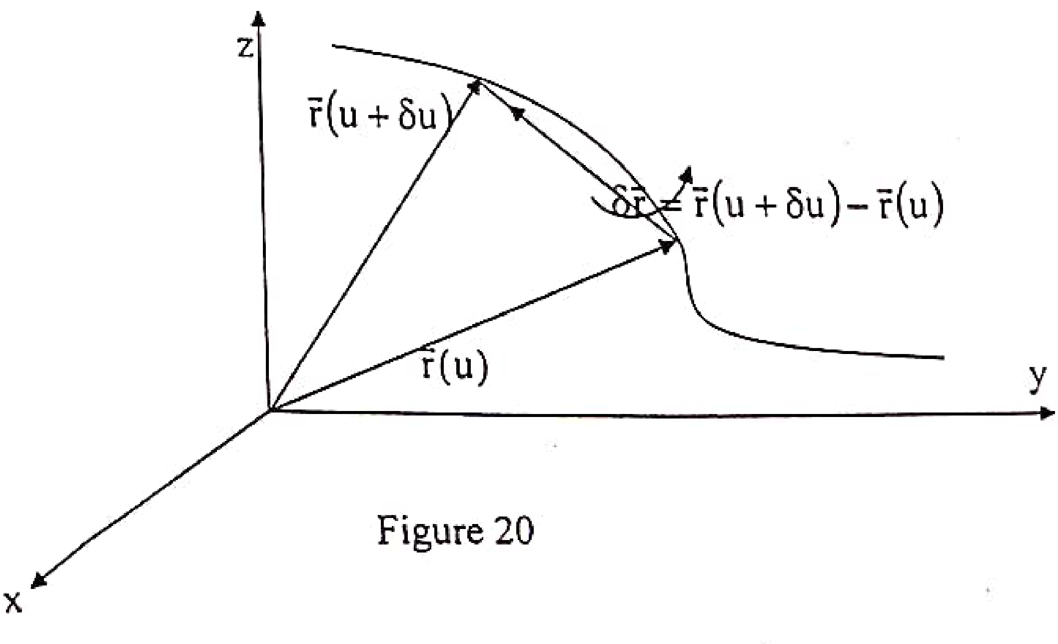

Figure 20

If u is the time t, $\frac{d\bar{r}}{dt}$ represents the velocity $\bar{v}$ with which the terminal point of $\bar{r}$ describes the curve. Similarly, $\frac{d\bar{v}}{dt}=\frac{d^2\bar{r}}{dt^2}$ represents its acceleration $\bar{a}$ along the curve.

Differentiation Formulae:

1- $\frac{d}{du}\left(\bar{A}+\bar{B}\right)=\frac{d\bar{A}}{du}+\frac{d\bar{B}}{du}$

2- $\frac{d}{du}\left(\bar{A}\cdot\bar{B}\right)=\bar{A}\cdot\frac{d\bar{B}}{du}+\frac{d\bar{A}}{du}\cdot\bar{B}$

3- $\frac{d}{du}\left(\bar{A}\wedge\bar{B}\right)=\bar{A}\wedge\frac{d\bar{B}}{du}+\frac{d\bar{A}}{du}\wedge\bar{B}$

4- $\frac{d}{du}\left(\phi\,\bar{A}\right)=\phi\frac{d\bar{B}}{du}+\frac{d\phi}{du}\bar{A}$

5- $\frac{d}{du}\left(\bar{A}\cdot\bar{B}\wedge\bar{C}\right)=\bar{A}\cdot\bar{B}\wedge\frac{d\bar{C}}{du}+\bar{A}\cdot\frac{d\bar{B}}{du}\wedge\bar{C}+\frac{d\bar{A}}{du}\cdot\bar{B}\wedge\bar{C}$

6- $\frac{d}{du}\left(\bar{A}\wedge\left(\bar{B}\wedge\bar{C}\right)\right)=\bar{A}\wedge\left(\bar{B}\wedge\frac{d\bar{C}}{du}\right)+\bar{A}\wedge\left(\frac{d\bar{B}}{du}\wedge\bar{C}\right)+\frac{d\bar{A}}{du}\wedge\left(\bar{B}\wedge\bar{C}\right)$

The order in these products may be important.

Differentials of Vectors:

Follow rules to those of elementary calculus. For example,

1- If $\bar{A}=A_1\bar{i}+A_2\bar{j}+A_3\vec{k}$. Then $d\bar{A}=dA_1\bar{i}+dA_2\bar{j}+dA_3\vec{k}$

2- $d\left(\bar{A}\cdot\bar{B}\right)=\bar{A}\cdot d\bar{B}+d\bar{A}\cdot\bar{B}$

3- $d\left(\bar{A}\wedge\bar{B}\right)=\bar{A}\wedge d\bar{B}+d\bar{A}\wedge\bar{B}$

4- If $\bar{A}=\bar{A}(x,y,z)$. Then $d\bar{A}=\frac{\partial\bar{A}}{\partial x}dx+\frac{\partial\bar{A}}{\partial y}dy+\frac{\partial\bar{A}}{\partial z}dz$, etc.

A study of forces on moving objects is considered in dynamics. Fundamentals to this study is Newton's famous law which states that if $\bar{F}$ is the net force acting on an object of mass m moving with velocity $\bar{v}$, then

$\bar{F} = \frac{d}{dt}(m\bar{v})$ where $m\bar{v}$ is the momentum of the object. If m is

constant this becomes $\bar{F} = m\frac{d}{dt}(\bar{v}) = m\bar{a}$, where $\bar{a}$ is the

acceleration of the object.

Ex. 15

Given $\bar{R} = \sin t\,\bar{i} + \cos t\,\bar{j} + t\bar{k}$, find $\frac{d\bar{R}}{dt}, \frac{d^2\bar{R}}{dt^2}, \left|\frac{d\bar{R}}{dt}\right|, \left|\frac{d^2\bar{R}}{dt^2}\right|$.

Sol.

$$\frac{d\bar{R}}{dt} = \frac{d}{dt}(\sin t)\bar{i} + \frac{d}{dt}(\cos t)\bar{j} + \frac{d}{dt}(t)\bar{k} = \cos t\;\bar{i} - \sin t\;\bar{j} + \bar{k},$$

$$\frac{d^2\bar{R}}{dt^2} = \frac{d}{dt}\left(\frac{d\bar{R}}{dt}\right) = \frac{d}{dt}(\cos t)\bar{i} - \frac{d}{dt}(\sin t)\bar{j} + \frac{d}{dt}(1)\bar{k} = -\sin t\;\bar{i} - \cos t\;\bar{j}$$

$$\left|\frac{d\bar{R}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{2}$$

$$\left|\frac{d^2\bar{R}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1.$$

Ex. 16

A particle moves along a curve whose parametric equations are

$$x = e^{-t},\ \ y = 2\cos 3t,\ z = 2\sin 3t,$$

where t is the time.

(a)Determine its velocity and acceleration at any time.

(b) Find the magnitude of the velocity and acceleration at $t = 0$.

Sol.

(a) the position vector $\bar{r}$ of the particle is

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} = e^{-t}\bar{i} + 2\cos 3t\ \bar{j} + 2\sin 3t\ \bar{k}$$

Then the velocity is $\bar{v} = \frac{d\bar{r}}{dt} = -e^{-t}\bar{i} - 6\sin 3t\ \bar{j} + 6\cos 3t\ \bar{k}$

and acceleration is $\bar{a} = \frac{d^2\bar{r}}{dt^2} = e^{-t}\bar{i} - 18\cos 3t\ \bar{j} - 18\sin 3t\ \bar{k}$

(b) At $t = 0, \bar{v} = \frac{d\bar{r}}{dt} = -\bar{i} + 6\bar{k}$ and $\bar{a} = \frac{d^2\bar{r}}{dt^2} = \bar{i} - 18\bar{j}$

Magnitude of velocity at $t = 0$ is $\sqrt{(-1)^2 + 6^2} = \sqrt{37}$

Magnitude of acceleration at $t = 0$ is $\sqrt{(1)^2 + (18)^2} = \sqrt{325}$.

Ex. 17

A particle moves along the curve

$$x = 2t^2,\ y = t^2 - 4t,\ z = 3t - 5$$

where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\vec{i} - 3\vec{j} + 2\vec{k}$.

Sol.

$$\text{Velocity} = \frac{d\bar{r}}{dt} = \frac{d}{dt}\left(2t^2\bar{i} + \left(t^2 - 4t\right)\bar{j} + (3t - 5)\bar{k}\right)$$

$$= 4t\vec{i} + (2t - 4)\vec{j} + 3\vec{k} = 4\vec{i} - 2\vec{j} + 3\vec{k}, \quad \text{at } t = 1$$

Unit vector in direction $\bar{i}-3\bar{j}+2\bar{k}$ is

$$\frac{\bar{i}-3\bar{j}+2\bar{k}}{\sqrt{1^2+(-3)^2+2^2}}=\frac{\bar{i}-3\bar{j}+2\bar{k}}{\sqrt{14}}$$

Then the component of the velocity in the given direction is

$$\frac{\left(4\bar{i}-2\bar{j}+3\bar{k}\right)\cdot\left(\bar{i}-3\bar{j}+2\bar{k}\right)}{\sqrt{14}}=\frac{4(1)+(-2)(-3)+(3)(2)}{\sqrt{14}}=\frac{16}{\sqrt{14}}=\frac{8\sqrt{14}}{7}$$

Acceleration

$$=\frac{d^2\bar{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\bar{r}}{dt}\right)=\frac{d}{dt}\left(4t\bar{i}+(2t-4)\bar{j}+3\bar{k}\right)=4\bar{i}+2\bar{j}$$

Then the component of the acceleration in the given direction is

$$\frac{\left(4\bar{i}+2\bar{j}\right)\cdot\left(\bar{i}-3\bar{j}+2\bar{k}\right)}{\sqrt{14}}=\frac{(4)(1)+(2)(-3)+(2)(0)}{\sqrt{14}}=-\frac{2}{\sqrt{14}}=-\frac{\sqrt{14}}{7}$$

Ex. 18

(a)Find the unit tangent vector to any point on the curve

$$x=t^2+1,\ \ y=4t-3,\ z=2t^2-6t.$$

(b) Determine the unit tangent at the point when $t=2$.

Sol.

(a) A tangent vector to the curve at any point is

$$\frac{d\bar{r}}{dt}=\frac{d}{dt}\left(\left(t^2+1\right)\bar{i}+(4t-3)\bar{j}+\left(2t^2-6t\right)\bar{k}\right)=2t\bar{i}+4\bar{j}+(4t-6)\bar{k}$$

The magnitude of the vector is $\left|\frac{d\bar{r}}{dt}\right|=\sqrt{(2t)+4^2+\left(4t-6^2\right)^2}$

Then the required unit vector is $\bar{T}=\dfrac{2t\bar{i}+4\bar{j}+(4t-6)\bar{k}}{\sqrt{(2t)^2+4^2+\left(4t-6^2\right)^2}}$.

Note that since $\left|\frac{d\bar{r}}{dt}\right| = \frac{ds}{dt}$, $\quad \bar{T} = \dfrac{\frac{d\bar{r}}{dt}}{\frac{ds}{dt}} = \frac{d\bar{r}}{ds}$.

(b) At $t = 2$, the unit tangent vector is

$$\bar{T} = \frac{4\bar{i} + 4\bar{j} + 2\bar{k}}{\sqrt{(4)^2 + 4^2 + 2^2}} = \tfrac{2}{3}\bar{i} + \tfrac{2}{3}\bar{j} + \tfrac{1}{2}\bar{k}.$$

$\frac{1}{3}$

Ex. 19

If $\bar{A} = 5t^2\bar{i} + t\bar{j} - t^3\bar{k}$ and $\bar{B} = \sin t\,\bar{i} - \cos t\,\bar{j}$,

Find (a) $\frac{d}{dt}(\bar{A}\cdot\bar{B})$, (b) $\frac{d}{dt}(\bar{A}\wedge\bar{B})$, (c) $\frac{d}{dt}(\bar{A}\cdot\bar{A})$.

Sol.

(a) $\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \bar{A}\cdot\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{B}$

$$= \left(5t^2\bar{i} + t\bar{j} - t^3\bar{k}\right)\cdot\left(\cos t\,\bar{i} + \sin t\,\bar{j}\right) + \left(10t\bar{i} + \bar{j} - 3t^2\bar{k}\right)\cdot\left(\sin t\,\bar{i} - \cos t\,\bar{j}\right)$$

$$= 5t^2\cos t + t\sin t + 10t\sin t - \cos t = \left(5t^2 - 1\right)\cos t + 11t\sin t.$$

Another Method: $\bar{A}\cdot\bar{B} = 5t^2\sin t - t\cos t$. Then

$$\frac{d}{dt}(\bar{A}\cdot\bar{B}) = \frac{d}{dt}\left(5t^2\sin t - t\cos t\right) = 5t^2\cos t + 10t\sin t + t\sin t - \cos t$$

$$= \left(5t^2 - 1\right)\cos t + 11t\sin t.$$

(b) $\frac{d}{du}(\bar{A}\wedge\bar{B}) = \bar{A}\wedge\frac{d\bar{B}}{du} + \frac{d\bar{A}}{du}\wedge\bar{B}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ \cos t & \sin t & 0 \end{vmatrix} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 10t & 1 & -3t^2 \\ \sin t & -\cos t & 0 \end{vmatrix}$$

$$= \left[t^3 \sin t\,\bar{i} - t^3 \cos t\,\bar{j} + \left(5t^2 \sin t - t\cos t\right)\bar{k}\right] +$$
$$\left[-3t^2 \cos t\,\bar{i} - 3t^2 \sin t\,\bar{j} + \left(-10t\cos t - \sin t\right)\bar{k}\right]$$
$$= \left(t^3 \sin t - 3t^2 \cos t\right)\bar{i} - \left(t^3 \cos t + 3t^2 \sin t\right)\bar{j} + \left(5t^2 \sin t - \sin t - 11t\cos t\right)\bar{k}$$

Another Method:

$$\bar{A} \wedge \bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = -t^3 \cos t\,\bar{i} - t^3 \sin t\bar{j} + \left(-5t^2 \cos t - t\sin t\right)\bar{k}.$$

Then

$$\frac{d}{dt}\left(\bar{A} \wedge \bar{B}\right) = \left(t^3 \sin t - 3t^2 \cos t\right)\bar{i} - \left(t^3 \cos t + 3t^2 \sin t\right)\bar{j}$$
$$+ \left(5t^2 \sin t - \sin t - 11t\cos t\right)$$

(c) $$\frac{d}{dt}\left(\bar{A} \cdot \bar{A}\right) = \bar{A} \cdot \frac{d\bar{A}}{dt} + \frac{d\bar{A}}{dt} \cdot \bar{A} = 2\bar{A} \cdot \frac{d\bar{A}}{dt}$$
$$= \left(5t^2\bar{i} + t\bar{j} - t^3\bar{k}\right) \cdot \left(10t\bar{i} + \bar{j} - 3t^2\bar{k}\right) = 100t^3 + 2t + 6t^5$$

Another Method:

$$\bar{A} \cdot \bar{A} = \left(5t^2\right) + t^2 + \left(-t^3\right)^2 = 25t^4 + t^2 + t^6.$$

Then

$$\frac{d}{dt}\left(25t^4 + t^2 + t^6\right) = 100t^3 + 2t + 6t^5.$$

Ex. 20

If $\bar{A}$ has constant magnitude show that $\bar{A}$ and $\dfrac{d\bar{A}}{dt}$ are perpendicular provided .

that $\left|\frac{d\bar{A}}{dt}\right| \neq 0$.

Sol.

Since $\bar{A}$ has constant magnitude, $\bar{A}\cdot\bar{A} = const.$

Then $\frac{d}{dt}\left(\bar{A}\cdot\bar{A}\right) = \bar{A}\cdot\frac{d\bar{A}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{A} = 2\bar{A}\cdot\frac{d\bar{A}}{dt} = 0$

Thus $\bar{A}\cdot\frac{d\bar{A}}{dt} = 0$ and $\bar{A}$ perpendicular to $\frac{d\bar{A}}{dt}$ provided that

$\left|\frac{d\bar{A}}{dt}\right| \neq 0$.

Ex. 21

Show that $\bar{A}\cdot\frac{d\bar{A}}{dt} = A\frac{dA}{dt}$

Sol.

Since $\bar{A}\cdot\bar{A} = A^2$, $\frac{d}{dt}\left(\bar{A}\cdot\bar{A}\right) = \frac{d}{dt}\left(A^2\right)$

$\frac{d}{dt}\left(\bar{A}\cdot\bar{A}\right) = \bar{A}\cdot\frac{d\bar{A}}{dt} + \frac{d\bar{A}}{dt}\cdot\bar{A} = 2\bar{A}\cdot\frac{d\bar{A}}{dt}$ and $\frac{d}{dt}\left(A^2\right) = 2A\frac{dA}{dt}$

Then $2\bar{A}\cdot\frac{d\bar{A}}{dt} = 2A\frac{dA}{dt} \Rightarrow \bar{A}\cdot\frac{d\bar{A}}{dt} = A\frac{dA}{dt}$.

Ex. 22

A particle moves so that its position vector is given by

$\bar{r} = \cos\omega t\,\bar{i} + \sin\omega t\,\bar{j}$ where ω is a constant. Show that

(a) the velocity $\bar{v}$ of the particle is perpendicular to $\bar{r}$,

(b) the acceleration $\bar{a}$ is directed toward the origin and has magnitude proportional to the distance from the origin,

(c) $\bar{r}\wedge\bar{v} =$ a constant vector.

Sol.

(a) $\bar{v} = \dfrac{d\bar{r}}{dt} = -\omega \sin \omega t\, \bar{i} + \omega \cos \omega t\, \bar{j}$

Then $\bar{r} \cdot \bar{v} = (\cos \omega t\, \bar{i} + \sin \omega t\, \bar{j}) \cdot (-\omega \sin \omega t\, \bar{i} + \omega \cos \omega t\, \bar{j})$

$= (\cos \omega t)(-\omega \sin \omega t) + (\sin \omega t)(\omega \cos \omega t) = 0$

and $\bar{r}$ and $\bar{v}$ are perpendicular.

(b)

$$\frac{d^2\bar{r}}{dt^2} = \frac{d\bar{v}}{dt} = -\omega^2 \cos \omega t\, \bar{i} - \omega^2 \sin \omega t\, \bar{j} = -\omega^2 (\cos \omega t\, \bar{i} + \sin \omega t\, \bar{j}) = -\omega^2 \bar{r}$$

Then the acceleration is opposite to the direction of $\bar{r}$, i.e. it is directed toward the origin. Its magnitude is proportional to $|\bar{r}|$ which is the distance from the origin.

(c)

$$\bar{r} \wedge \bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \cos \omega t & \sin \omega t & 0 \\ -\omega \sin \omega t & \omega \cos \omega t & 0 \end{vmatrix}$$
$$= \omega (\cos^2 \omega t + \sin^2 \omega t) \bar{k} = \omega \bar{k}$$

Physically, the motion is of a particle moving on the circumference of a circle with constant angular speed ω. The acceleration, directed toward the center of the circle, is the centripetal acceleration.

Ex. 23

Evaluate $\dfrac{d}{dt}\left(\bar{V} \cdot \left(\dfrac{d\bar{V}}{dt} \wedge \dfrac{d^2\bar{V}}{dt^2}\right)\right)$.

Sol.

We use the formula:

$$\frac{d}{dt}\left(\vec{A}\cdot\left(\vec{B}\wedge\vec{C}\right)\right)=\vec{A}\cdot\left(\vec{B}\wedge\frac{d\vec{C}}{dt}\right)+\vec{A}\cdot\left(\frac{d\vec{B}}{dt}\wedge\vec{C}\right)+\frac{d\vec{A}}{dt}\cdot\left(\vec{B}\wedge\vec{C}\right),$$

with $\vec{A}=\vec{V},\ \vec{B}=\frac{d\vec{V}}{dt},\ \ \vec{C}=\frac{d^2\vec{V}}{dt^2}$

Therefore

$$\frac{d}{dt}\left(\vec{V}\cdot\left(\frac{d\vec{V}}{dt}\wedge\frac{d^2\vec{V}}{dt^2}\right)\right)=\vec{V}\cdot\frac{d\vec{V}}{dt}\wedge\frac{d^3\vec{V}}{dt^3}+\vec{V}\cdot\frac{d^2\vec{V}}{dt^2}\wedge\frac{d^2\vec{V}}{dt^2}+\frac{d\vec{V}}{dt}\cdot\frac{d\vec{V}}{dt}\wedge\frac{d^2\vec{V}}{dt^2}$$

$$=\vec{V}\cdot\frac{d\vec{V}}{dt}\wedge\frac{d^3\vec{V}}{dt^3}+0+0$$

$$=\vec{V}\cdot\frac{d\vec{V}}{dt}\wedge\frac{d^3\vec{V}}{dt^3}.$$

Ex. 24 Prove That

$$\bar{A}\wedge\frac{d^2\vec{B}}{dt^2}-\frac{d^2\bar{A}}{dt^2}\wedge\vec{B}=\frac{d}{dt}\left(\bar{A}\wedge\frac{d\vec{B}}{dt}-\frac{d\bar{A}}{dt}\wedge\vec{B}\right).$$

Sol.

$$\frac{d}{dt}\left(\bar{A}\wedge\frac{d\vec{B}}{dt}-\frac{d\bar{A}}{dt}\wedge\vec{B}\right)=\frac{d}{dt}\left(\bar{A}\wedge\frac{d\vec{B}}{dt}\right)-\frac{d}{dt}\left(\frac{d\bar{A}}{dt}\wedge\vec{B}\right)$$

$$=\vec{A}\wedge\frac{d^2\vec{B}}{dt^2}+\frac{d\vec{A}}{dt}\wedge\frac{d\vec{B}}{dt}-\frac{d\vec{A}}{dt}\wedge\frac{d\vec{B}}{dt}-\frac{d^2\vec{A}}{dt^2}\wedge\vec{B}$$

$$=\vec{A}\wedge\frac{d^2\vec{B}}{dt^2}-\frac{d^2\vec{A}}{dt^2}\wedge\vec{B}.$$

Exercises

1) Let $\vec{a}=\vec{i}-3\vec{j}+2\vec{k}$, $\vec{b}=-\vec{i}+\vec{j}+\vec{k}$, *and* $\vec{c}=2\vec{i}+6\vec{j}+9\vec{k}$, find the indicated vector or scalar in the following problems:

i) $2\vec{a}-(\vec{b}-\vec{c})$ ii) $\vec{b}+2(\vec{a}-3\vec{c})$ iii) $-6\vec{b}+4(\vec{a}+2\vec{c})$

iv) $|2\vec{b}|\,|\vec{c}|$ v) $\left|\frac{\vec{a}}{|\vec{a}|}\right|+5\left|\frac{\vec{b}}{|\vec{b}|}\right|$ vi) $|\vec{b}|\,\vec{a}+|\vec{a}|\,\vec{b}$

2) For the points $P(1,-1,1)$, $Q(2,-2,2)$, $R(2,0,1)$ *and* $S(3,-1,2)$, does $\overline{PQ}=\overline{RS}$

3) Let $\vec{v}=a\vec{i}+b\vec{j}+c\vec{k}$, $\vec{w}=3a\vec{i}+3b\vec{j}+3c\vec{k}$, show that $|\vec{w}|=3|\vec{v}|$

4) Let $\vec{v}=-\vec{i}+5\vec{j}-2\vec{k}$, $\vec{w}=3\vec{i}+\vec{j}+\vec{k}$, find

i) $\vec{v}-\vec{w}$ ii) $\vec{v}+\vec{w}$ iii) $\frac{\vec{v}}{|\vec{v}|}$

iv) The vector $\vec{u}$ such that $\vec{u}+\vec{v}+\vec{w}=2\vec{j}+\vec{k}$

v) Is there a scalar m such that $m(\vec{v}+2\vec{w})=\vec{k}$? If so, find it.

vi) Is $|\vec{v}-\vec{w}|=|\vec{v}|-|\vec{w}|$?. If not, which quantity is large?

vii) Is $|\vec{v}+\vec{w}|=|\vec{v}|+|\vec{w}|$?. If not, which quantity is large?

5) Find a unit vector in the same direction as $\vec{a}=10\vec{i}-5\vec{j}+10\vec{k}$

6) Find a unit vector in the opposite direction of $\vec{a}=\vec{i}+3\vec{j}+2\vec{k}$

7) Find a vector $\vec{b}$ that is 4 times as long as $\vec{a}=\vec{i}-\vec{j}+\vec{k}$ in the same direction as $\vec{a}$.

8) Find a vector $\vec{b}$ for which $|\vec{b}|=\frac{1}{2}$ that is parallel to $\vec{a}=-6\vec{i}+3\vec{j}-2\vec{k}$ but has the opposite direction.

9) Let $\vec{a}=2\vec{i}-3\vec{j}+4\vec{k}$, $\vec{b}=-\vec{i}+2\vec{j}+5\vec{k}$, *and* $\vec{c}=3\vec{i}+6\vec{j}-\vec{k}$, find the indicated vector or scalar in the following problems:

i) $\vec{a}\cdot\vec{b}$ ii) $\vec{b}\cdot\vec{c}$ iii) $\vec{a}\cdot(\vec{b}+\vec{c})$

iv) $(2\vec{b})\cdot(3\vec{c})$ v) $(2\vec{a})\cdot(\vec{a}-2\vec{b})$ vi) $\left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$

10) Determine which of the following vectors are orthogonal:

i) $\vec{a}=2\vec{i}+\vec{k}$, $\vec{b}=3\vec{i}+2\vec{j}-\vec{k}$

ii) $\vec{a}=2\vec{i}-\vec{j}-\vec{k}$, $\vec{b}=\vec{i}-4\vec{j}+6\vec{k}$

iii) $\vec{a}=\vec{i}-\vec{j}+\vec{k}$, $\vec{b}=-4\vec{i}+3\vec{j}+8\vec{k}$

11) Determine a scalar c so that the following vectors are orthogonal:

i) $\vec{a} = 2\vec{i} - c\vec{j} + 3\vec{k}$, $\vec{b} = 3\vec{i} + 2\vec{j} + 4\vec{k}$

ii) $\vec{a} = c\vec{i} + \frac{1}{2}\vec{j} + c\vec{k}$, $\vec{b} = -3\vec{i} + 4\vec{j} + c\vec{k}$

12) Find a vector $\vec{v} = x_1\vec{i} + y_1\vec{j} + \vec{k}$ that is orthogonal to both $\vec{a} = 3\vec{i} + \vec{j} - \vec{k}$, $\vec{b} = -3\vec{i} + 2\vec{j} + 2\vec{k}$

13) Verify that the vector $\vec{c} = \vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}$ is orthogonal to the vector $\vec{a}$

14) Determine a scalar c so that the angle between $\vec{a} = \vec{i} + c\vec{j}$, $\vec{b} = \vec{i} + \vec{j}$ is 45°.

H 15) Find the angle θ between the following vectors:

i) $\vec{a} = 2\vec{i} + 4\vec{j}$, $\vec{b} = -\vec{i} - \vec{j} + 4\vec{k}$

ii) $\vec{a} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \frac{3}{2}\vec{k}$, $\vec{b} = 2\vec{i} - 4\vec{j} + 6\vec{k}$

16) Let $\vec{a} = \vec{i} - \vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + 6\vec{j} + 3\vec{k}$, find:

i) $comp_{\vec{b}}\vec{a}$

ii) $comp_{\vec{a}}\vec{b}$

iii) $comp_{\vec{b}}(\vec{b} - \vec{a})$

iv) $comp_{2\vec{k}}(\vec{a} + \vec{b})$

17) Find the $proj_{\vec{b}}\vec{a}$ and $proj_{\vec{b}^{\perp}}\vec{a}$ for the following problems: H·W

i) $\vec{a} = -\vec{i} - 2\vec{j} + 7\vec{k}$, $\vec{b} = 6\vec{i} - 3\vec{j} - 2\vec{k}$

ii) $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = -2\vec{i} + 2\vec{j} - \vec{k}$

18) Find the $proj_{(\vec{a}+\vec{b})}\vec{a}$ and $proj_{(\vec{a}-\vec{b})}\vec{b}$ for the following two vectors

$\vec{a} = 4\vec{i} - 3\vec{j}$, $\vec{b} = -\vec{i} + \vec{j}$

19) Find the work done if the point at which the constant force $\vec{F} = 4\vec{i} - 3\vec{j} + 5\vec{k}$ is applied to an object moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

20) Find $\vec{a} \wedge \vec{b}$ for the following problems:

i) $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j} - \vec{k}$

ii) $\vec{a} = 4\vec{i} + \vec{j} + 5\vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - \vec{k}$

iii) $\vec{a} = \langle \frac{1}{2}, 0, \frac{1}{2} \rangle$ $\vec{b} = \langle 4, 6, 0 \rangle$

iv) $\vec{a} = \langle 8, 1, 6 \rangle$ $\vec{b} = \langle 1, -2, 10 \rangle$

21) Find a unit vector that is perpendicular to both $\vec{a}$ and $\vec{b}$ for:

i) $\bar{a} = 2\dot{i} + 7\bar{j} - 4\bar{k}, \bar{b} = \dot{i} + \bar{j} - \bar{k}$

ii) $\bar{a} = \langle -1, -2, 4 \rangle \quad \bar{b} = \langle 4, -1, 0 \rangle$

22) Verify that $\bar{a} \cdot (\bar{a} \wedge \bar{b}) = 0$ and $\bar{b} \cdot (\bar{a} \wedge \bar{b}) = 0$ for:

i) $\bar{a} = 5\dot{i} - 2\bar{j} + \bar{k}, \bar{b} = 2\dot{i} - 7\bar{k}$

ii) $\bar{a} = \langle \frac{1}{2}, \frac{-1}{4}, 0 \rangle \quad \bar{b} = \langle 2, -2, 6 \rangle$

23) Let $\bar{a} = 2\dot{i} - 3\bar{j} + 4\bar{k}$, $\bar{b} = -\dot{i} + 2\bar{j} + 5\bar{k}$, *and* $\bar{c} = 3\dot{i} + 6\bar{j} - \bar{k}$, find the indicated vector or scalar in the following problems:

i) $\bar{a} \wedge \bar{b}$ ii) $\bar{b} \wedge \bar{c}$ iii) $\bar{a} \wedge (\bar{b} + \bar{c})$

iv) $a \cdot (\bar{b} \wedge \bar{c})$ v) $(2\bar{a}) \cdot (\bar{a} \wedge 2\bar{b})$

24) Verify that the given quadrilateral is a parallelogram, and find its area. See Figures 21 and 22.

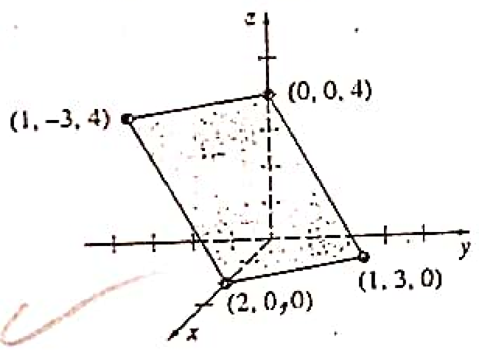

Figure 21

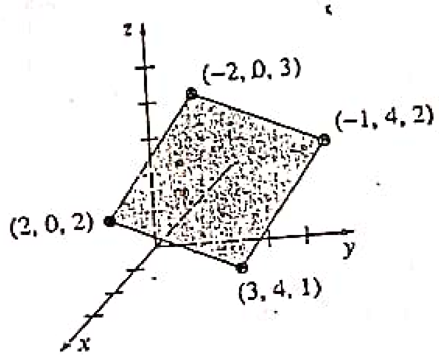

Figure 22

25) Find the area of the triangle determined by the given vertices:

H i) $A(1,1,1), B(1,2,1), C(1,1,2)$

ii) $A(1,2,4), B(1,-1,3), C(-1,-1,2)$

26) Find the volume of the parallelogram for which the given vectors are three edges:

H i) $\bar{a} = \dot{i} + \bar{j}, \bar{b} = -\dot{i} + 4\bar{j}$ *and* $\bar{c} = 2\dot{i} + 2\bar{j} + 2\bar{k}$

ii) $\bar{a} = 3\dot{i} + \bar{j} + \bar{k}, \bar{b} = \dot{i} + 4\bar{j} + \bar{k}$, *and* $\bar{c} = \dot{i} + \bar{j} + 5\bar{k}$

27) Determine whether the vectors $\bar{a} = 4\dot{i} + 6\bar{j}, \bar{b} = -2\dot{i} + 6\bar{j} - 6\bar{k}$ *and* $\bar{c} = \frac{5}{2}\dot{i} + 3\bar{j} + \frac{1}{2}\bar{k}$ are coplanar.

28) Determine whether the four points $A(1,1,-2)$, $B(4,0,-3)$, $C(1,-5,10)$ *and* $D(-7,2,4)$ lie in the same plane.

29) Prove or disprove $\vec{a}\wedge(\vec{b}\wedge\vec{c})=(\vec{a}\wedge\vec{b})\wedge\vec{c}$.

30) Prove $\vec{a}\cdot(\vec{b}\wedge\vec{c})=(\vec{a}\wedge\vec{b})\cdot\vec{c}$.

31) Prove $\vec{a}\wedge(\vec{b}\wedge\vec{c})+\vec{b}\wedge(\vec{c}\wedge\vec{a})+\vec{c}\wedge(\vec{a}\wedge\vec{b})=0$.

32) Prove $|\vec{a}\wedge\vec{b}|^2+(\vec{a}\cdot\vec{b})^2=|\vec{a}|^2|\vec{b}|^2$. or $|\vec{a}\wedge\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2-|\vec{a}\cdot\vec{b}|^2$

33) Does $\vec{a}\wedge\vec{c}=\vec{a}\wedge\vec{b}$ imply that $\vec{c}=\vec{b}$

34) Show that $(\vec{a}+\vec{b})\wedge(\vec{a}-\vec{b})=2\vec{b}\wedge\vec{a}$.

35) Calculate $\vec{a}\cdot(\vec{b}\wedge\vec{c})$ *and* $\vec{a}\wedge(\vec{b}\wedge\vec{c})$, where $\vec{a}=\vec{i}+\vec{j}+\vec{k}, \vec{b}=3\vec{i}+2\vec{k}$ *and* $\vec{c}=2\vec{i}+2\vec{j}+2\vec{k}$.

36) Consider the vector equation $\vec{a}\wedge\vec{x}=\vec{b}$ in the space, where $\vec{a}\neq 0$. Show that:

i) $\vec{a}\cdot\vec{b}=0$

ii) $\vec{x}=\dfrac{\vec{b}\wedge\vec{a}}{|\vec{a}|^2}+k\vec{a}$ is a solution to the equation, for any scalar k.

37) For all vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ in the space, show that:

$$(\vec{a}\wedge\vec{b})\wedge(\vec{c}\wedge\vec{d})=[\vec{d}\cdot(\vec{a}\wedge\vec{b})]\vec{c}-[\vec{c}\cdot(\vec{a}\wedge\vec{b})]\vec{d}$$

and that $(\vec{a}\wedge\vec{b})\wedge(\vec{c}\wedge\vec{d})=[\vec{a}\cdot(\vec{c}\wedge\vec{d})]\vec{b}-[\vec{b}\cdot(\vec{c}\wedge\vec{d})]\vec{a}$.

38) If $\bar{R}=e^t\,\bar{i}+\ln(t^2+1)\bar{j}-\tan t\,\bar{k}$, find

$$\frac{d\bar{R}}{dt},\ \frac{d^2\bar{R}}{dt^2},\ \left|\frac{d\bar{R}}{dt}\right|,\ \left|\frac{d^2\bar{R}}{dt^2}\right| \text{ at } t=0.$$

39) Find the velocity and acceleration of a particle which moves along the curve $x=2\sin 3t,\ y=2\cos 3t,\ z=8t$ at any time $t>0$. Find the magnitude of the velocity and acceleration.

40) Find a unit tangent vector to any point on the curve $x=a\cos\omega t,\ y=a\sin\omega t,\ z=bt$ where a,b,ω are constants.

41) If $\bar{A} = t^2\,\bar{i} - t\,\bar{j} + (2t+1)\bar{k}$ and $\bar{B} = (2t-3)\bar{i} - t\,\bar{j} - t\bar{k}$, find

$\frac{d}{dt}(\bar{A}\cdot\bar{B})$, $\frac{d}{dt}(\bar{A}\wedge\bar{B})$, $\frac{d}{dt}|\bar{A}+\bar{B}|$, $\frac{d}{dt}\left(\bar{A}\wedge\frac{d\bar{B}}{dt}\right)$ at $t=1$.

42) If $\bar{A} = \sin u\,\bar{i} + \cos u\,\bar{j} + u\bar{k}$, $\bar{B} = \cos u\,\bar{i} - \sin u\,\bar{j} - 3\bar{k}$ and

$\bar{C} = \sin u\,\bar{i} + \cos u\,\bar{j} - \bar{k}$, find $\frac{d}{du}(\bar{A}\wedge(\bar{B}\wedge\bar{C}))$, at $u=0$.

43) Find $\frac{d}{ds}\left(\bar{A}\cdot\frac{d\bar{B}}{ds} - \frac{d\bar{A}}{ds}\cdot\bar{B}\right)$ if $\bar{A}$ and $\bar{B}$ are differentiable functions of s .

44) If $\bar{A}(t) = 3t^2\,\bar{i} - (t+4)\bar{j} + (t^2-2t)\bar{k}$,

and $\bar{B}(t) = \sin t\,\bar{i} + 3e^{-t}\,\bar{j} - 3\cos t\,\bar{k}$, find $\frac{d^2}{dt^2}(\bar{A}\wedge\bar{B})$, at $t=0$.