Calculus I and Analytical Geometry

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Chapter 4

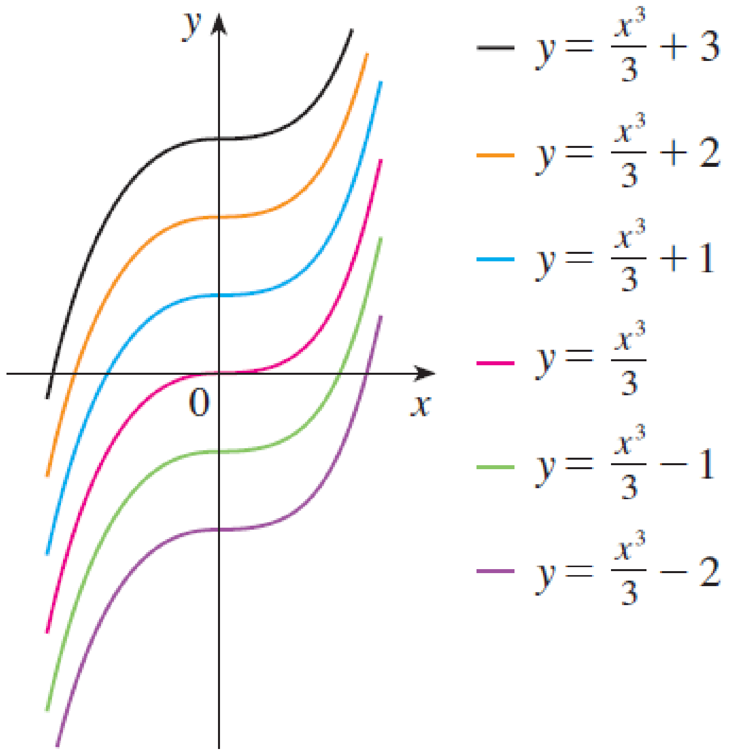
INTEGRATION

INDEFINITE INTEGRALS

DEFINITION A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

- For example, $f(x) = x^2$, then $F(x) = \frac{1}{3}x^3$.
- But the function $G(x) = \frac{1}{3}x^3 + 20$ also satisfies $G'(x) = x^2$.
- Therefore F, G are antiderivatives of f.
- Indeed, any function of the form $H(x) = \frac{1}{3}x^3 + C$, where C is a constant

  is an antiderivative of f.



1 THEOREM If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

The notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an **indefinite integral**. Thus

$$\int \boldsymbol{f(x)\,dx = F(x) + C}$$

The function $\boldsymbol{f}$ is the **integrand** of the integral , x is the variable of integration and C is a constant of integration.

➢ Properties of integration:

1) $\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$

2) $\int k\, f(x)dx = k \int f(x)\, dx$, , where k is a constant

➢ Basic rules of integration

1) $\int kdx = k \int dx = kx + C$, where k is a constant

2) $\int x^n dx = \frac{x^{n+1}}{n+1} + C \,, n \neq -1$

3) $\int \frac{1}{x} = \; ln|x| + C$

Example1:

$$1)\int x^4+x^{-2}+3x^5+\frac{5}{2}x^{\frac{1}{4}}\,dx=\frac{x^5}{5}+\frac{x^{-1}}{-1}+\frac{x^6}{2}+\frac{5}{2}*\frac{4}{5}x^{\frac{5}{4}}+C$$

$$2)\int\left(x^2+\frac{1}{x}\right)dx=\frac{x^3}{3}+\ln|x|+C$$

$$3)\int\ (x^2+5)^2\,dx=\int x^4+10x^2+25\ \ dx=\frac{1}{5}x^5+\frac{10}{3}x^3+25x+C$$

$$4)\int x^3+7x^8-\frac{4}{x^5}+\frac{1}{x\sqrt{x}}-4\sqrt[5]{x^8}+100\ \ dx=\int(x^3+7x^8-4x^{-5}+x^{-\frac{3}{2}}-4x^{\frac{8}{5}}+100\ )\ \ dx$$

$$=\frac{x^4}{4}+7\frac{x^9}{9}-4\frac{x^{-4}}{-4}+\frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}-\frac{4x^{\frac{13}{5}}}{\frac{13}{5}}+100x+C$$

Let $u = u(x)$, u is a function of x

4) $\int (u)^n \cdot u' dx = \frac{(u)^{n+1}}{n+1} + C \;, n \neq -1$

5) $\int \frac{u'}{\sqrt{u}} dx = 2\sqrt{u} + C$

6) $\int \frac{u'}{u} dx = \ln|u| + C$

7) $\int e^u \cdot u' dx = e^u + C$

8) $\int a^u \cdot u' dx = \frac{a^u}{\ln a} + C$

Example 2:

$$\int \boldsymbol{x\left(x^2-1\right)^3\,dx}$$

Solution

$$u = x^2 - 1,\; u' = 2x$$

$$\int x\,(x^2-1)^3\,dx = \frac{1}{2}\int 2x\,.\,(x^2-1)^3\,dx = \frac{1}{2}.\frac{(x^2-1)^4}{4} + C$$

Example 3:

$$\int \boldsymbol{\frac{8}{(3-2x)^5}\,dx}$$

Solution

$$\int \frac{8}{(3-2x)^5}\,dx = \frac{8}{-2}\int -2.\,(3-2x)^{-5}dx = -4.\frac{(3-2x)^{-4}}{-4} + C$$

Example 4:

$$\int \boldsymbol{cos^5(2x)\, sin(2x)\, dx}$$

Solution

$$\int \cos^5(2x)\sin(2x)\,dx = -\frac{1}{2}\int (\cos 2x)^5 \cdot -2\sin 2x\,dx$$

$$= -\frac{1}{2}.\frac{(\cos 2x)^6}{6} + C$$

Example 5:

$$\int \boldsymbol{e^{2x}\left(1 - e^{2x}\right)^3 dx}$$

Solution

$$\int e^{2x}(1 - e^{2x})^3\,dx = \frac{-1}{2}\int (-2).\,e^{2x}(1 - e^{2x})^3\,dx = \frac{-1}{2}\,.\frac{(1 - e^{2x})^4}{4} + c$$

Example 6:

$$\int \frac{\ln x}{x}\, dx$$

Solution

$$\int \frac{\ln x}{x}\, dx = \int \frac{1}{x}.(ln\ x)^1\ dx = \frac{1}{2}(\ln x)^2 + C$$

Example 7:

$$\int \frac{\left(1+\sin^{-1}\sqrt{x}\right)^4}{\sqrt{x-x^2}}\, dx$$

Solution

$$\int \frac{(1+\sin^{-1}\sqrt{x})^4}{\sqrt{x-x^2}}\, dx = 2\int (1+\sin^{-1}\sqrt{x})^4 \frac{1}{2\sqrt{x-x^2}}\, dx$$

$$= \frac{2}{5}(1+\sin^{-1}\sqrt{x})^5 + C$$

Example 8:

$$\int \frac{\boldsymbol{x^2}}{\sqrt{\boldsymbol{x^3+1}}}\,\boldsymbol{dx}$$

Solution

$$\int \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int \frac{3x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}.\,2\sqrt{x^3+1}+C$$

Example 9:

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{x\sqrt{2+\ln x}}}$$

Solution

$$\int \frac{dx}{x\sqrt{2+\ln x}} = \int \frac{\frac{1}{x}\,dx}{\sqrt{2+\ln x}} = 2\sqrt{2+\ln x}+C$$

Example 10:

$$\int \frac{\boldsymbol{x - \sqrt{sinh^{-1} x}}}{\boldsymbol{\sqrt{1 + x^2}}}\,\boldsymbol{dx}$$

Solution

$$\int \frac{x - \sqrt{\sinh^{-1} x}}{\sqrt{1 + x^2}}\,dx = \int \frac{x}{\sqrt{1 + x^2}}\,dx - \int (\sinh^{-1} x)^{\frac{1}{2}} \frac{1}{\sqrt{1 + x^2}}\,dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{1 + x^2}}\,dx - \int (\sinh^{-1} x)^{\frac{1}{2}} \frac{1}{\sqrt{1 + x^2}}\,dx = \frac{1}{2}\ .2\sqrt{1 + x^2} - \frac{(\sinh^{-1} x)^{3/2}}{3/2} + C$$

Example 11:

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{2x\sqrt{ln\,x}}}$$

Solution

$$\int \frac{dx}{2x\sqrt{\ln x}} = \frac{1}{2}\int \frac{\frac{1}{x}}{\sqrt{\ln x}}\,dx = \frac{1}{2}.\,2\sqrt{\ln x} + C = \sqrt{\ln x} + C$$

Example 12:

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{x + x\, ln\, x}}$$

Solution

$$\int \frac{dx}{x + x\ln x} = \int \frac{dx}{x(1 + \ln x)} = \int \frac{\frac{1}{x}\, dx}{1 + \ln x} = \ln|1 + \ln x| + c$$

Example 13:

$$\int \frac{(\boldsymbol{e^{2x} + x})}{(\boldsymbol{e^{2x} + x^2})}\, \boldsymbol{dx}$$

Solution

$$\int \frac{(e^{2x} + x)}{(e^{2x} + x^2)}\, dx = \frac{1}{2}\int \frac{2(e^{2x} + x)}{(e^{2x} + x^2)} = \frac{1}{2}\ln|e^{2x} + x^2| + C$$

Example 14:

$$\int \frac{\boldsymbol{x - \sqrt{\tan^{-1} x}}}{\boldsymbol{1 + x^2}} \boldsymbol{dx}$$

Solution

$$\int \frac{x - \sqrt{\tan^{-1} x}}{1 + x^2} dx = \int \left(\frac{x}{1 + x^2} - \frac{(\tan^{-1} x)^{\frac{1}{2}}}{1 + x^2} \right) dx = \frac{1}{2} \int \frac{2x}{1 + x^2} dx - \int \frac{(\tan^{-1} x)^{\frac{1}{2}}}{1 + x^2} dx$$

$$= \frac{1}{2} ln(1 + x^2) - \frac{2}{3} (\tan^{-1} x)^{\frac{3}{2}} + C$$

Example 15:

$$\int \frac{\boldsymbol{(x + 3)}}{\boldsymbol{x^2 + 6x + 4}} \boldsymbol{dx}$$

Solution

$$\int \frac{(x + 3)}{x^2 + 6x + 4} dx = \frac{1}{2} \int \frac{2(x + 3)}{x^2 + 6x + 4} dx = \frac{1}{2} \ln |x^2 + 6x + 4| + C$$

Example 15:

$$\int \frac{e^{4\tan x}}{1-\sin^2 x}\,dx$$

Solution

$$\int \frac{e^{4\tan x}}{1-\sin^2 x}\,dx = \int \frac{e^{4\tan x}}{\cos^2 x}\,dx = \frac{1}{4}\int 4\sec^2 x\; e^{4\tan x}\; dx = \frac{1}{4}e^{4\tan x} + c$$

Example 16:

$$\int 3^{\sin x}.\cos x\,dx$$

Solution

$$\int 3^{\sin x}.\cos x\,dx = \frac{3^{\sin x}}{\ln 3} + C$$

Example 17:

$$\int \frac{e^{\frac{1}{x}}}{x^2}\, dx$$

Solution

$$\int \frac{e^{\frac{1}{x}}}{x^2}\, dx = \int e^{x^{-1}} . x^{-2} dx = -\int e^{x^{-1}} . (-\, x^{-2}) dx = -e^{x^{-1}} + C$$

Example 18:

$$\int \left(e^{3x+2} + 5^{2x+3}\right) dx$$

Solution

$$\int (e^{3x+2} + 5^{2x+3})\, dx = \frac{1}{3} \int 3e^{3x+2}\, dx + \frac{1}{2} \int 2 . 5^{2x+3}\, dx = \frac{1}{3}\, e^{3x+2} + \frac{1}{2} . \frac{5^{2x+3}}{\ln 5} + C$$

Example 19:

$$\int \frac{e^{\cos^{-1} 2x}}{\sqrt{1-4x^2}} dx$$

Solution

$$\int \frac{e^{\cos^{-1} 2x}}{\sqrt{1-4x^2}} dx = -\frac{1}{2}\int e^{\cos^{-1} 2x} \cdot \frac{-2}{\sqrt{1-4x^2}} dx \; = -\frac{1}{2} e^{\cos^{-1} 2x} + C$$

Example 20:

$$\int \frac{\left(5+4\cot^{-1}(3x)\right)^5}{1+9x^2} dx$$

Solution

$$\int \frac{(5+4\cot^{-1}(3x))^5}{1+9x^2} dx = \int (5+4\cot^{-1}(3x))^5 \cdot \frac{1}{1+9x^2} dx \; = -\frac{1}{12}\int (5+4\cot^{-1}(3x))^5 \cdot 4 . \frac{-3}{1+9x^2} dx$$

$$= -\frac{1}{12} . \frac{(5+4\cot^{-1}(3x))^6}{6} + C$$

Example 21:

$$\int \frac{6x^2 + x}{3x + 2}\, dx$$

Solution

$$\therefore \int \frac{6x^2 + x}{3x + 2}\, dx = \int \left[(2x - 1) + \frac{2}{3x + 2} \right] dx$$

$$= \int (2x - 1)dx + \frac{2}{3} . \int \frac{3dx}{3x + 2}$$

$$= \frac{2x^2}{2} - x + \frac{2}{3} \ln|3x + 2| + c$$

Long Division

$$\begin{array}{r|l} & 2x - 1 \\ \hline 3x + 2 & 6x^2 + x \\ & \underline{6x^2 + 4x} \\ & -3x \\ & \underline{-3x - 2} \\ & \boxed{2} \end{array}$$

Example 22:

$$\int \boldsymbol{x}\sqrt{\frac{\mathbf{2}}{\boldsymbol{x}^{\mathbf{2}}}-\frac{\mathbf{3}}{\boldsymbol{x}}}\,\boldsymbol{dx}$$

Solution

$$\int x\sqrt{\frac{2}{x^2}-\frac{3}{x}}\,dx = \int \sqrt{x^2\left(\frac{2}{x^2}-\frac{3}{x}\right)}\,dx = \int \sqrt{2-3x}\,dx$$

$$= \frac{-1}{3}\int (-3).(2-3x)^{\frac{1}{2}}\,dx = \frac{-1}{3} * \frac{2}{3}(2-3x)^{\frac{3}{2}} + C$$

Example 23:

$$\int \frac{\boldsymbol{e}^{\boldsymbol{x}}}{(\mathbf{1}+\mathbf{tan}\,\boldsymbol{e}^{\boldsymbol{x}})\,\mathbf{cos}^{\mathbf{2}}\,\boldsymbol{e}^{\boldsymbol{x}}}\,\boldsymbol{dx}$$

Solution

$$\int \frac{e^x}{(1+\tan e^x)\cos^2 e^x}\,dx = \int \frac{e^x.\sec^2(e^x)}{(1+\tan e^x)}\,dx = \ln|1+\tan(e^x)| + C$$

Integration of trigonometric functions

Let $u = u(x)$, u is a function of x

$$9)\ \int \sin(u) \cdot u'\, dx = -\cos(u) + c$$

$$10)\ \int \cos(u) \cdot u'\, dx = \sin(u) + c$$

$$11)\ \int \sec^2(u) \cdot u'\, dx = \tan(u) + c$$

$$12)\ \int \csc^2(u) \cdot u'\, dx = -\cot(u) + c$$

$$13)\ \int \sec(u)\tan(u) \cdot u'\, dx = \sec(u) + c$$

$$14)\ \int \csc(u)\cot(u)\cdot u'\,dx = -\csc(u) + c$$

$$15)\ \int \tan(u)\cdot u'\,dx = \ln|\sec(u)| + c$$

$$16)\ \int \cot(u)\cdot u'\,dx = \ln|\sin(u)| + c$$

$$17)\ \int \sec(u)\cdot u'\,dx = \ln|\sec(u) + \tan(u)| + c$$

$$18)\ \int \csc(u)\cdot u'\,dx = \ln|\csc(u) - \cot(u)| + c$$

Example 24:

$$\int \left(\boldsymbol{x^5 + \frac{1}{\sqrt{x}} + \sin 2x + \cos\left(\frac{x}{2}\right) + \sqrt{x}}\right) \boldsymbol{dx}$$

Solution

$$\int \left(x^5 + \frac{1}{\sqrt{x}} + \sin 2x + \cos\left(\frac{x}{2}\right) + \sqrt{x}\right) dx = \frac{x^6}{6} + 2\sqrt{x} - \frac{\cos(2x)}{2} + 2\sin\left(\frac{x}{2}\right) + \cdot \frac{2}{3} x^{\frac{3}{2}} + C$$

Example 25:

$$\int \boldsymbol{[(7x-2)^7 + sec^2(5x-1) + \frac{1}{(x+1)^2} + 10]\, dx}$$

Solution

$$\int [(7x-2)^7 + \sec^2(5x-1) + \frac{1}{(x+1)^2} + 10]\, dx = \int (7x-2)^7 dx + \int \sec^2(5x-1) dx + \int (x+1)^{-2} dx + \int 10\, dx$$

$$= \frac{1}{7} \int (7).(7x-2)^7 dx + \frac{1}{5} \int 5.\sec^2(5x-1) dx + \int (x+1)^{-2} dx + \int 10\, dx$$

$$= \frac{1}{7} \cdot \frac{(7x-2)^8}{8} + \frac{1}{5} \tan(5x-1) + \frac{(x+1)^{-1}}{-1} + 10x + C$$

Example 26:

$$\int (\mathbf{2} + \mathbf{3}\sec \mathbf{t})^{\mathbf{2}} \mathbf{dt}$$

Solution

$$\int (2 + 3\sec t)^2 dt = \int 4 + 12\sec t + 9\sec^2 t\, dt = 4t + 12\ln|\sec t + \tan t| + 9\tan t + C$$

Example 27:

$$\int \frac{\boldsymbol{cos(2x)}}{\boldsymbol{cos\,x + sin\,x}}\, \boldsymbol{dx}$$

Solution

$$\int \frac{\cos(2x)}{\cos x + \sin x}\, dx = \int \frac{cos^2 x - sin^2 x}{\cos x + \sin x} + dx = \int \frac{(\cos x - \sin x)(\cos x + \sin x)}{\cos x + \sin x}\, dx$$

$$= \int (\cos x - \sin x)\, dx \qquad = \sin x - (-\cos x) + C = \sin x + \cos x + C$$

Example 28:

$$\int \tan x \,.\, \ln(\cos x)\, dx$$

Solution

$$\int \tan x \,.\, \ln(\cos x)\, dx = -\int -\frac{\sin x}{\cos x} \,.\, [\ln(\cos x)]^1\, dx = -\frac{[\ln(\cos x)]^2}{2} + C$$

Example 29:

$$\int \frac{1+\cos^2 x}{1-\sin^2 x}\, dx$$

Solution

$$\int \frac{1+\cos^2 x}{1-\sin^2 x}\, dx = \int \frac{1+\cos^2 x}{\cos^2 x}\, dx = \int (\sec^2 x + 1)\, dx = \tan x + x + C$$

Example 30:

$$\int \frac{\boldsymbol{x}}{\mathbf{tan}\,\boldsymbol{x}^{2}}\,\boldsymbol{dx}$$

Solution

$$\int \frac{x}{\tan x^2}\,dx = \int x.\cot x^2\,dx = \frac{1}{2}\int (2x).\cot x^2\,dx = \frac{1}{2}\ln|\sin x^2| + C$$

Example 31:

$$\int \boldsymbol{tan^2 x\,dx}$$

Solution

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)dx = \tan x - x + \mathrm{C}$$

Example 32:

$$\int \frac{\boldsymbol{cos^2 x}}{\mathbf{1+\sin} \boldsymbol{x}} \, \boldsymbol{dx}$$

Solution

$$\int \frac{\cos^2 x}{1+\sin x}\, dx = \int \frac{1-\sin^2 x}{1+\sin x}\, dx = \int \frac{(1-\sin x)\,(1+\sin x)}{1+\sin x}\, dx = \int (1-\sin x)\, dx = x+\cos x + C$$

Example 33:

$$\int \frac{\mathbf{1+\cos 2}\boldsymbol{x}}{\mathbf{sin^2}\, \boldsymbol{x}} \, \boldsymbol{dx}$$

Solution

$$\int \frac{1+\cos 2x}{\sin^2 x}\, dx = \int \frac{2\cos^2 x}{\sin^2 x}\, dx = 2\int \cot^2 x\, dx = 2\int csc^2\, x - 1\, dx = 2(-\cot x - x) + C$$

Integral formulae for inverse trigonometric functions

Let $u = u(x), u$ is a function of x

19) $\displaystyle \int \frac{u'}{\sqrt{a^2 - u^2}}\, dx = \sin^{-1}\left(\frac{u}{a}\right) + C$

20) $\displaystyle \int \frac{u'}{a^2 + u^2}\, dx = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + C$

21) $\displaystyle \int \frac{u'}{u\sqrt{u^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C$

Example 34:

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{x^2+9}}$$

Solution

$$\int \frac{dx}{x^2+9} = \frac{1}{3} tan^{-1}\left(\frac{x}{3}\right) + C$$

Example 35:

$$\int \frac{\boldsymbol{dx}}{\boldsymbol{x\sqrt{1-4\,ln^2 x}}}$$

Solution

$$\int \frac{dx}{x\sqrt{1-4\ln^2 x}} = \int \frac{\left(\frac{1}{x}\right)}{\sqrt{1^2-(2\ln x)^2}}dx = \frac{1}{2}\int \frac{2\left(\frac{1}{x}\right)}{\sqrt{1^2-(2\ln x)^2}}dx = \frac{1}{2}\sin^{-1}\left(\frac{2\ln x}{1}\right) + C$$

Example 36:

$$\int \frac{\cos x}{\sqrt{4 - sin^2 x}} dx$$

Solution

$$\int \frac{\cos x}{\sqrt{4 - sin^2 x}} dx = \int \frac{\cos x}{\sqrt{(2)^2 - (\sin x)^2}} dx = \sin^{-1}\left(\frac{\sin x}{2}\right) + C$$

Example 37:

$$\int \frac{dx}{x\sqrt{25\, x^2 - 2}}$$

Solution

$$\int \frac{dx}{x\sqrt{25\, x^2 - 2}} dx = \int \frac{5dx}{(5x).\sqrt{(5x)^2 - \left(\sqrt{2}\right)^2}} dx = \frac{1}{\sqrt{2}} \sec^{-1}\left(\frac{5x}{\sqrt{2}}\right) + C$$

Example 38:

$$\int \frac{\boldsymbol{e^{x} + e^{2x}}}{\boldsymbol{e^{2x} + 3}} \boldsymbol{dx}$$

Solution

$$\int \frac{e^{x} + e^{2x}}{e^{2x} + 3} dx = \int \frac{e^{x}}{e^{2x} + 3} dx + \int \frac{e^{2x}}{e^{2x} + 3} dx = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{e^{x}}{\sqrt{3}}\right) + \frac{1}{2} \ln|e^{2x} + 3| + C$$

Example 39:

$$\int \frac{\boldsymbol{1}}{\boldsymbol{4e^{x} + 5e^{-x}}} \boldsymbol{dx}$$

Solution

$$\int \frac{1}{4e^{x} + 5e^{-x}} dx = \int \frac{e^{x}}{4e^{2x} + 5} dx = \frac{1}{2} \int \frac{2e^{x}}{\left(\sqrt{5}\right)^{2} + (2e^{x})^{2}} dx = \frac{1}{2} . \frac{1}{\sqrt{5}} tan^{-1}\left(\frac{2e^{x}}{\sqrt{5}}\right) + \mathrm{C}$$

Example 40:

$$\int \frac{\boldsymbol{\sin 2\theta + \sin \theta}}{\boldsymbol{\sqrt{1 - 4\cos^2 \theta}}}\, \boldsymbol{d\theta}$$

Solution

$$\int \frac{\sin 2\theta + \sin \theta}{\sqrt{1 - 4\cos^2 \theta}}\, d\theta = \int \frac{2\sin \theta \cos \theta + \sin \theta}{\sqrt{1 - 4\cos^2 \theta}}\, d\theta$$

$$= \int \frac{2\sin \theta \cos \theta}{\sqrt{1 - 4\cos^2 \theta}} d\theta + \int \frac{\sin \theta}{\sqrt{1 - 4\cos^2 \theta}}\, d\theta$$

$$= \frac{1}{4}\int \frac{8\sin \theta \cos \theta}{\sqrt{1 - 4\cos^2 \theta}} d\theta - \frac{1}{2}\int \frac{-2\sin \theta}{\sqrt{1^2 - (2\cos \theta)^2}}\, d\theta$$

$$= \frac{1}{4}.\,2\sqrt{1 - 4\cos^2 \theta} - \frac{1}{2}\sin^{-1}\left(\frac{2\cos \theta}{1}\right) + C$$

Integral formulae for hyperbolic functions

22) $\int u' \sinh u \, dx = \cosh u + C$

23) $\int u' \cosh u \, dx = \sinh u + C$

24) $\int u' \operatorname{sech}^2 u \, dx = \tanh u + C$

25) $\int u' \operatorname{csch}^2 u \, dx = -\coth u + C$

26) $\int u' \operatorname{sech} u \,.\, \tanh u \, dx = -\operatorname{sech} u + C$

27) $\int u' \operatorname{csch} u \coth u \, dx = -\operatorname{csch} u + C$

Example 41:

$$\int \frac{sech^2(1 - ln\, t)}{t} dt$$

Solution

$$\int \frac{sech^2(1 - ln\, t)}{t} dt = \int sech^2(1 - ln\, t) \,.\, \frac{1}{t}\, dt = -\int sech^2(1 - ln\, t) \,.\, \frac{-1}{t}\, dt = -\, tanh(1 - ln\, t) + C$$

Example: 42

$$\int \frac{cosh\, x}{cosh^2\, x - 1}\, dx$$

Solution

$$\int \frac{cosh\, x}{cosh^2\, x - 1}\, dx = \int \frac{cosh\, x}{sinh^2\, x}\, dx = \int \coth x \operatorname{csch} x\, dx = -\operatorname{csch} x + C$$

Example 43:

$$\int \frac{\operatorname{sech}^2 x}{1 + \tanh x}\, dx$$

Solution

$$\int \frac{\operatorname{sech}^2 x}{1 + \tanh x}\, dx = \ln|1 + \tanh x| + C$$

Example 44:

$$\int \frac{\operatorname{csch}^2 (\sec^{-1} x)}{x\sqrt{x^2 - 1}}\, dx$$

Solution

$$\int \frac{\operatorname{csch}^2 (\sec^{-1} x)}{x\sqrt{x^2 - 1}}\, dx = \int \operatorname{csch}^2(\sec^{-1} x) \frac{1}{x\sqrt{x^2 - 1}}\, dx = -\coth(\sec^{-1} x) + C$$

Example 45:

$$\int sinh^2 x\, dx$$

Solution

$$\int \sinh^2 x\, dx = \frac{1}{2}\int (\cosh 2x - 1)dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right] + C$$

Example 46:

$$\int \frac{e^{\tanh x}}{\cosh^2 x} dx$$

Solution

$$\int \frac{e^{tanh\, x}}{cosh^2 x} dx = \int e^{\tanh x}\, sech^2 x\, dx = e^{\tanh x} + C$$

Integral formulae for inverse hyperbolic functions

$$28)\ \int \frac{u'}{\sqrt{u^2 + a^2}}\,dx = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

$$29)\ \int \frac{u'}{\sqrt{u^2 - a^2}}\,dx = cosh^{-1}\left(\frac{u}{a}\right) + C$$

$$30)\ \int \frac{u'}{a^2 - u^2}\,dx = \frac{1}{a}\,tanh^{-1}\left(\frac{u}{a}\right) + C$$

Example 47:

$$\int \frac{e^x}{\sqrt{e^{2x} - 16}} dx$$

Solution

$$\int \frac{e^x}{\sqrt{e^{2x} - 16}} dx = \int \frac{e^x}{\sqrt{(e^x)^2 - (4)^2}} dx = \cosh^{-1}\left(\frac{e^x}{4}\right) + C$$

Example 48:

$$\int \frac{\cos x (\sin x + 1)}{\sqrt{9 + \sin^2 x}} dx$$

Solution

$$\int \frac{\cos x (\sin x + 1)}{\sqrt{9 + \sin^2 x}} dx = \int \frac{\cos x \sin x + \cos x)}{\sqrt{9 + \sin^2 x}} dx = \int \frac{\cos x \sin x}{\sqrt{9 + \sin^2 x}} dx + \int \frac{\cos x}{\sqrt{9 + \sin^2 x}} dx$$

$$= \frac{1}{2} * 2\sqrt{9 + \sin^2 x} + \sinh^{-1}\left(\frac{\sin x}{3}\right) + C$$

Example 49:

$$\int \frac{\ln x + 2}{x(4 - \ln^2 x)}\, dx$$

Solution

$$\int \frac{\ln x + 2}{x(4 - \ln^2 x)}\, dx = \int \frac{\frac{1}{x}(\ln x + 2)}{4 - \ln^2 x}\, dx$$

$$= \int \frac{\frac{\ln x}{x}}{4 - \ln^2 x}\, dx + \int \frac{\frac{2}{x}}{4 - \ln^2 x}\, dx$$

$$= -\frac{1}{2}\int \frac{-2\frac{\ln x}{x}}{4 - \ln^2 x}\, dx + 2\int \frac{\frac{1}{x}}{(2)^2 - (\ln x)^2}\, dx$$

$$= -\frac{1}{2}\ln|4 - \ln^2 x| + 2.\frac{1}{2}\tanh^{-1}\left(\frac{\ln x}{2}\right) + C$$

Example 50:

$$\int \frac{4e^{4x} + e^{2x}}{\sqrt{25e^{4x} - 16}}\,dx$$

Solution

$$\int \frac{4\,e^{4x} + \ e^{2x}}{\sqrt{25\,e^{4x} - 16}}\,dx = \int \frac{4e^{4x}}{\sqrt{25e^{4x} - 16}}\,dx + \int \frac{e^{2x}}{\sqrt{25e^{4x} - 16}}\,dx$$

$$= \frac{1}{25}\int \frac{100\,e^{4x}}{\sqrt{25\,e^{4x} - 16}}\,dx + \frac{1}{10}\int \frac{10e^{2x}}{\sqrt{(5e^{2x}) - 16}}\,dx$$

$$= \frac{1}{25}.\,2\sqrt{25\,e^{4x} - 16} + \frac{1}{10}\cosh^{-1}\left(\frac{5e^{2x}}{4}\right) + C$$

Homework:

$$\int \frac{\mathbf{1}}{\mathbf{3}e^{x} + \mathbf{2}e^{-x}}\,dx$$