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4: Linearity and Superposition

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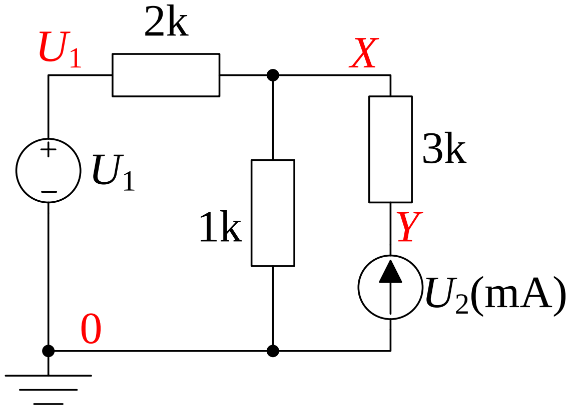
Linearity Theorem

Suppose we use variables instead of fixed values for all of the *independent* voltage and current sources. We can then use nodal analysis to find all node voltages in terms of the source values.



(1) Label all the nodes

(2) KCL equations:

$$\frac{X-U_1}{2}+\frac{X}{1}+\frac{X-Y}{3}=0 \qquad \frac{Y-X}{3}+(-U_2)=0$$

(3) Solve for the node voltages:

$$X=\frac{1}{2}U_1+\frac{2}{3}U_2, \quad Y=\frac{1}{3}U_1+\frac{11}{3}U_2$$

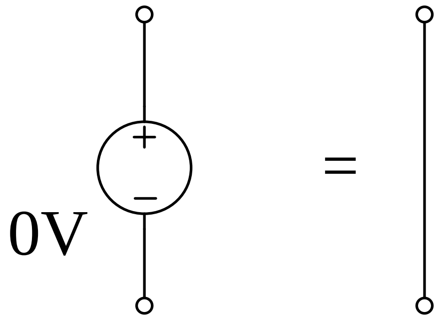
Steps (2) and (3) never involve multiplying two source values together, so:

Linearity Theorem: For any circuit containing resistors and independent voltage and current sources, every node voltage and branch current is a linear function of the source values and has the form $\sum a_i U_i$ where the U_i are the source values and the a_i are suitably dimensioned constants.

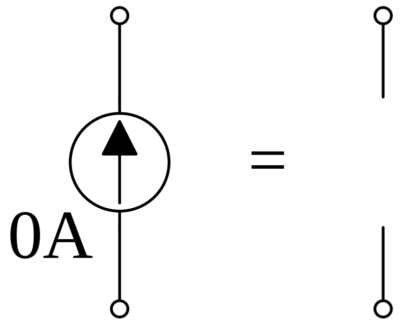
Also true for a circuit containing *dependent* sources whose values are proportional to voltages or currents elsewhere in the circuit.

Zero-value Sources

A zero-valued voltage source has zero volts between its terminals for any current. It is equivalent to a *short-circuit* or a piece of wire or a resistor of $0\,\Omega$ (or $\infty\, S$)



A zero-valued current source has no current flowing between its terminals. It is equivalent to an *open-circuit* or a broken wire or a resistor of $\infty\ \Omega$ (or $0\ S$)



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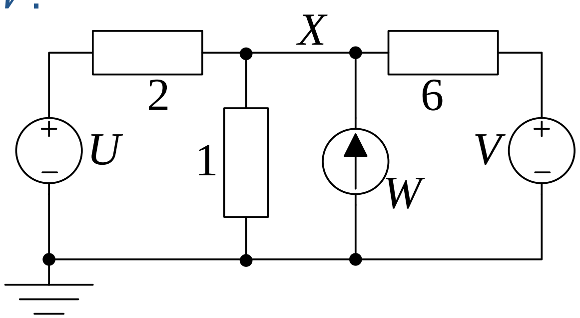
Superposition

We can use nodal analysis to find X in terms of U, V and W.

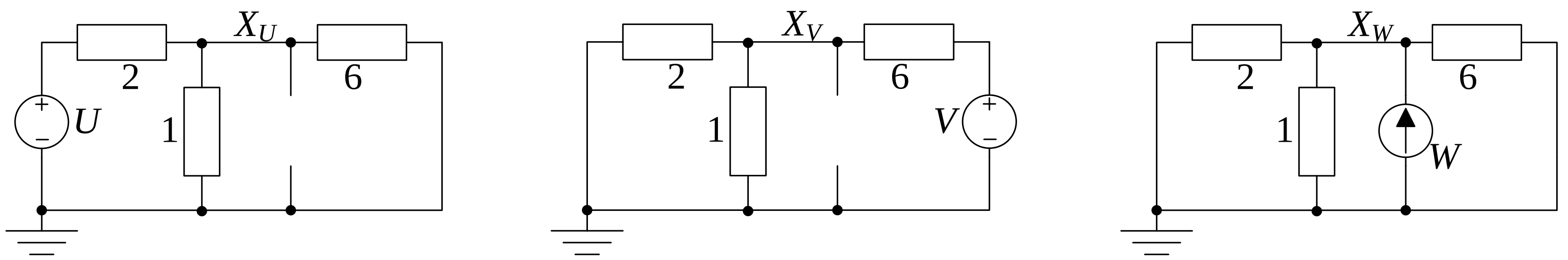
KCL: $\frac{X-U}{2}+\frac{X-V}{6}+\frac{X}{1}-W=0$

$10X-3U-V-6W=0$

$X=0.3U+0.1V+0.6W$



From the linearity theorem, we know anyway that $X=aU+bV+cW$ so all we need to do is find the values of a, b and c. We find each coefficient in turn by setting all the other sources to zero:



From the LH diagram we have $X_U=aU+b\times 0+c\times 0=aU$.

Similarly, $X_V=bV$ and $X_W=cW \Rightarrow X=X_U+X_V+X_W$.

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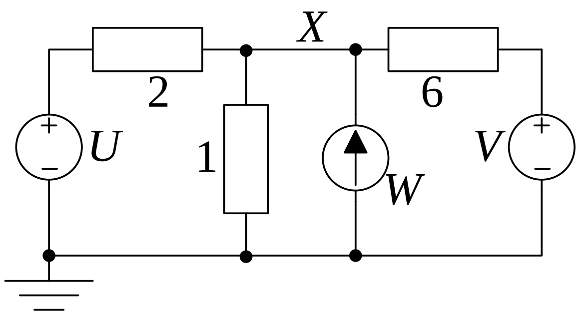
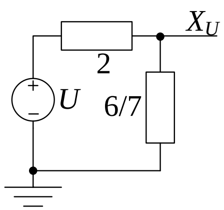
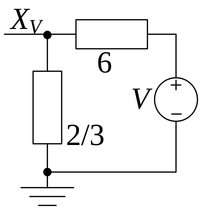
Superposition Calculation

Method: Find the effect of each source on its own by setting all other sources to zero. Then add up the results.

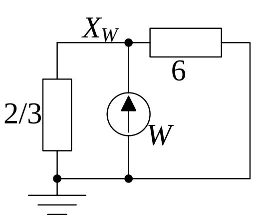




$$X_U = \frac{\frac{6}{7}}{2+\frac{6}{7}}U = \frac{6}{20}U = 0.3U$$



$$X_V = \frac{\frac{2}{3}}{6+\frac{2}{3}}V = \frac{2}{20}V = 0.1V$$



$$X_W = \frac{6}{6+\frac{2}{3}}W \times \frac{2}{3} = \frac{12}{20}W = 0.6W$$

Adding them up: $X = X_U + X_V + X_W = 0.3U + 0.1V + 0.6W$

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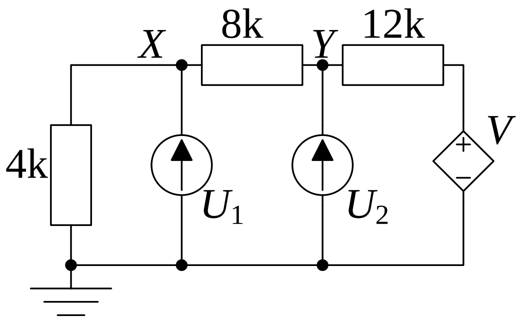
Superposition and Dependent Sources

A *dependent source* is one that is determined by the voltage and/or current elsewhere in the circuit via a known equation. Here $V \triangleq Y - X$.



Step 1: Pretend all sources are independent and use superposition to find expressions for the node voltages:

$$X = \frac{10}{3}U_1 + 2U_2 + \frac{1}{6}V \qquad \mathrm{Y} = 2U_1 + 6U_2 + \frac{1}{2}V$$

Step 2: Express the dependent source values in terms of node voltages:

$$V = Y - X$$

Step 3: Eliminate the dependent source values from the node voltage equations:

$$X = \frac{10}{3}U_1 + 2U_2 + \frac{1}{6}(Y - X) \quad \Rightarrow \quad \frac{7}{6}X - \frac{1}{6}Y = \frac{10}{3}U_1 + 2U_2 \quad \Rightarrow \quad X = 3U_1 + 3U_2$$

$$\mathrm{Y} = 2U_1 + 6U_2 + \frac{1}{2}(Y - X) \quad \Rightarrow \quad \frac{1}{2}X + \frac{1}{2}Y = 2U_1 + 6U_2 \quad \Rightarrow \quad \mathrm{Y} = U_1 + 9U_2$$

Note: This is an alternative to nodal analysis: you get the same answer.

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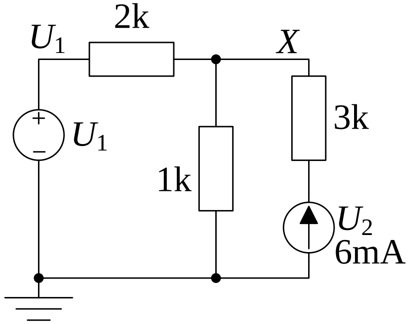
Single Variable Source

Any current or voltage can be written $X = a_1 U_1 + a_2 U_2 + a_3 U_3 + \ldots$

Using nodal analysis (slide 4-2) or else superposition:

$$X = \frac{1}{3}U_1 + \frac{2}{3}U_2$$



Suppose we know $U_2 = 6\ \mathrm{mA}$, then:

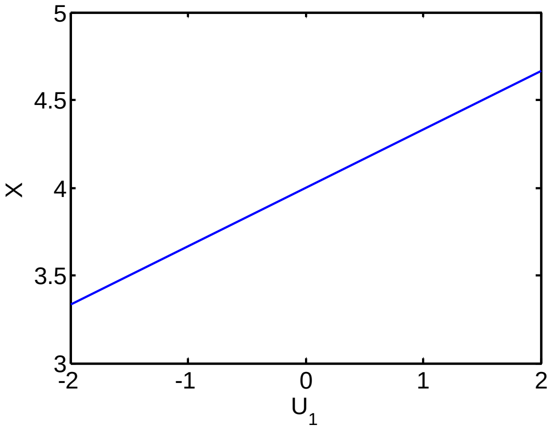
$$X = \frac{1}{3}U_1 + 4$$

If all the independent sources except for U_1 have known fixed values, then:

$$X = a_1 U_1 + b$$

where $b = a_2 U_2 + a_3 U_3 + \ldots$

This has a straight line graph.



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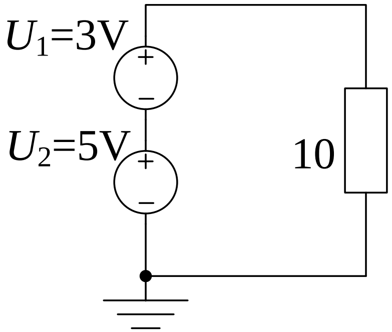
Superposition and Power

The power absorbed (or *dissipated*) by a component always equals VI where the measurement directions of V and I follow the passive sign convention.

For a resistor $VI = \frac{V^2}{R} = I^2 R$



Power in resistor is $P = \frac{(U_1+U_2)^2}{10} = 6.4\ \mathrm{W}$

Power due to U_1 alone is $P_1 = \frac{U_1^2}{10} = 0.9\ W$

Power due to U_1 alone is $P_2 = \frac{U_2^2}{10} = 2.5\ W$

$P \neq P_1 + P_2 \Rightarrow$ Power does not obey superposition.

You must use superposition to calculate the total V and/or the total I and then calculate the power.

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Proportionality

From the linearity theorem, all voltages and currents have the form $\sum a_i U_i$ where the U_i are the values of the independent sources.

If you multiply *all* the independent sources by the same factor, k, then all voltages and currents in the circuit will be multiplied by k.

The power dissipated in any component will be multiplied by k^2.

Special Case:
If there is only one independent source, U, then all voltages and currents are proportional to U and all power dissipations are proportional to U^2.

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Summary

- Linearity Theorem: $X = \sum_i a_i U_i$ over all independent sources U_i
- Superposition: sometimes simpler than nodal analysis, often more insight.
  - Zero-value voltage and current sources
  - Dependent sources - treat as independent and add dependency as an extra equation
- If all sources are fixed except for U_1 then all voltages and currents in the circuit have the form $aU_1 + b$.
- Power does not obey superposition.
- Proportionality: multiplying all sources by k multiplies all voltages and currents by k and all powers by k^2.

For further details see Hayt Ch 5.