Stability of a General Preissmann Scheme

By Dennis A. Lyn,[1] S. M. ASCE, and Peter Goodwin,[2] S. M. ASCE

Abstract: The stability and convergence characteristics of a four-point implicit finite-difference scheme due to Preissmann, which has been widely used in open-channel flow modeling, are examined. The analysis is made for a general linear hyperbolic system of n first-order equations, but is restricted to the homogeneous or frictionless case. In particular, the effect of a weighting factor in space, as well as in time, is considered. The specific case of unsteady sediment-transport modeling, which conventionally results in a third-order system, is discussed with particular reference to its singularly perturbed nature. Recommendations for practical computations are made.

Introduction

Finite-difference schemes for fixed-bed flood routing have received considerable attention, and standard schemes have become prominent. As such, their stability and convergence properties are well known. Extensions of these schemes to the more complex problem of unsteady sediment routing have been more recent, and fewer results are known. Only the case of the constant-discharge approximation (3,9), where only two of the three governing differential equations are solved simultaneously, seems to have been studied. The present work examines the stability and convergence of a general linear implicit difference scheme from Preissmann (10) applied to a general linear homogeneous hyperbolic system of n equations. The method of analysis uses the standard Fourier series analysis (11). The results are then applied to the problem of unsteady sediment routing, and qualitative conclusions regarding the effects of numerical dissipation and dispersion are drawn. The singularly perturbed nature of the general movable-bed problem and its numerical consequences are emphasized. In particular, the artificial oscillations encountered by Lyn (7) are explained. Recommendations for computational strategies are given.

Unsteady Sediment-Routing in Open Channels

The governing equations describing sediment transport in a straight, infinitely wide, open channel may be stated in nondimensional form as (7)

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \qquad (1)$$

[1]Grad. Student, Dept. of Environmental Engrg. Sci., W. M. Keck Lab. of Hydr. & Water Resources, California Inst. of Tech., Pasadena, CA 91125.

[2]Grad. Student, Dept. of Civ. Engrg., Univ. of California, Berkeley, CA.

Note.—Discussion open until June 1, 1987. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on April 5, 1985. This paper is part of the *Journal of Hydraulic Engineering*, Vol. 113, No. 1, January, 1987. ©ASCE, ISSN 0733-9429/87/0001-0016/$01.00. Paper No. 21135.

J. Hydraul. Eng., 1987, 113(1): 16-28

$$\beta\left[\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(h+\frac{u^2}{2}\right)\right]+\frac{\partial z}{\partial x}+S_e=0 \quad (2)$$

$$\frac{1}{\beta\epsilon}\frac{\partial z}{\partial t}+\frac{\partial(Ch)}{\partial t}+\frac{\partial}{\partial x}(Cuh)=0 \quad (3)$$

expressing respectively conservation of fluid, fluid momentum, and sediment. A definition sketch is given in Lyn (7). Here, h = the depth of flow; z = the bed elevation; S_e = the energy slope; and C = the mean concentration. The nondimensional parameters are defined as $\beta = (H/L)/S_r$ and $\epsilon = C_r/(1-p)$, where H and L = characteristic vertical and longitudinal length scales; and S_r and C_r = characteristic slope and concentration scales. Eqs. 1–3 may be more succinctly expressed in vector form as

$$\frac{\partial \mathbf{g}}{\partial t}+A\frac{\partial \mathbf{g}}{\partial x}+\mathbf{b}=0 \quad (4)$$

where $\mathbf{g} = (u,h,z)$; A = a matrix of coupling coefficients; and $\mathbf{b}$ = a vector of inhomogeneous terms. For the purposes of this analysis, the simplifying assumption will be made that $\mathbf{b} = 0$. The possible consequences of this assumption will be discussed later.

The important difference between the fixed-bed problem and the movable-bed problem, at least in terms of numerical modeling, will be seen to be related to the presence of the parameter ϵ, which, typically, is very small. Lyn (7) showed that, as a consequence, the problem posed by Eqs. 1–3 is, in general, singularly perturbed. This is manifested in the large disparity in wave speeds that are associated with the free surface and with the movable bed and lead to the existence of two disparate time (or length) scales. Because of the singularly perturbed nature of the problem, it may be expected that the direct application of conventional numerical schemes will give rise to difficulties.

Preissmann Four-Point Linear Implicit Scheme

Among finite-difference schemes, the linear implicit four point scheme, first proposed by Preissmann (10), is one of the most widely used in the numerical modeling of unsteady processes in open channels (1). The scheme replaces the continuous function f, its time derivative and its space derivative by the difference formulae

$$f=\theta[\phi f_{j+1}^{n+1}+(1-\phi)f_j^{n+1}]+(1-\theta)[\phi f_{j+1}^{n}+(1-\phi)f_j^{n}] \quad (5)$$

$$\frac{\partial f}{\partial t}=\frac{1}{\Delta t}[\phi(f_{j+1}^{n+1}-f_{j+1}^{n})+(1-\phi)(f_j^{n+1}-f_j^{n})] \quad (6)$$

$$\frac{\partial f}{\partial x}=\frac{1}{\Delta x}[\theta(f_{j+1}^{n+1}-f_j^{n+1})+(1-\theta)(f_{j+1}^{n}-f_j^{n})] \quad (7)$$

where f_j^n = the value of f at the point $(x,t) = (j\Delta x, n\Delta t)$, Δt and Δx being the time step and mesh size of the grid (often constant). The weighting factors, ϕ for space and θ for time, range in value from zero to unity,

J. Hydraul. Eng., 1987, 113(1): 16-28

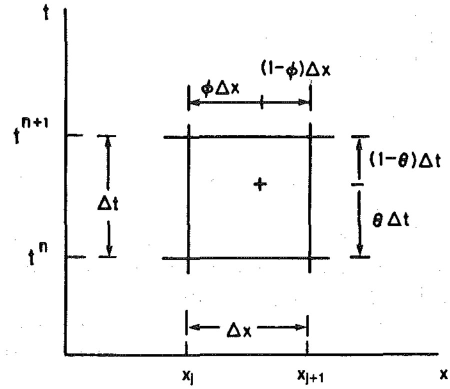


FIG. 1.—Sketch of Grid and Weighting Factors

and define the point about which the discretization is made. These may significantly affect the stability and convergence of the scheme. It has become standard to choose a scheme centered in space, $\phi = 1/2$, since this yields second-order accuracy in time. The more general case will be considered since it has implications for the movable-bed problem. A schematic picture of the grid is shown in Fig. 1.

Stability and Convergence Analysis

The standard technique for the analysis of the stability and convergence of a difference scheme is the Fourier series method attributed to von Neumann. It is noted, however, that, despite its wide use in computational hydraulics, the method can only be rigorously justified for the class of linear, pure initial-value problems with periodic initial data (11). Conclusions about the non-linear, mixed initial value-boundary value open channel problem with arbitrary initial and boundary data must be, at best, regarded as qualitative. To proceed, the system, Eq. 4, is linearized about a locally uniform state, $\mathbf{g} = \mathbf{g}_0$, such that $A = A_o(\mathbf{g}_o)$, and the difference scheme (Eqs. 5–7) is applied to give

$$\phi(\mathbf{g}_{j+1}^{n+1} - \mathbf{g}_{j+1}^{n}) + (1-\phi)(\mathbf{g}_{j}^{n+1} - \mathbf{g}_{j}^{n}) + rA_o[\theta(\mathbf{g}_{j+1}^{n+1} - \mathbf{g}_{j}^{n+1}) + (1-\theta)(\mathbf{g}_{j+1}^{n} - \mathbf{g}_{j}^{n})] \quad (8)$$

The growth of a Fourier component, $\mathbf{g} = \mathbf{g}_* e^{-i(\omega\Delta t - \sigma\Delta x)}$ is governed by

$$\xi = 1 - \frac{rc_i}{\phi + \dfrac{1}{\eta - 1} + rc_i\theta} \quad (9)$$

where $\xi = e^{-i\omega\Delta t}$; $\eta = e^{i\sigma\Delta x}$; $r = \Delta t/\Delta x$; and c_i = a characteristic wave speed of the system. This assumes that the same values of ϕ and θ are applied uniformly to the system. Eq. 9 is, in fact, valid for a general, homogeneous, linear hyperbolic system of n first-order equations. In particular, it is valid for a three-equation system, as in the present case, and also for a one-equation system. Thus, the effect of the difference scheme on a single characteristic wave is independent of all other characteristic waves, a result that may have been expected from the assumption of linearity. This result implies also that the behavior of the difference scheme with regard to each characteristic wave is determined by the dimensionless parameter, the Courant number, $Cr_i = rc_i$.

For the homogeneous problem, the von Neumann condition for the numerical stability of the scheme is that $|\xi| \leq 1$, which implies that

$$\frac{\phi - \frac{1}{2}}{Cr_i} + \left(\theta - \frac{1}{2}\right) \geq 0 \quad \text{(10)}$$

This result is shown schematically in Fig. 2, assuming $Cr_i > 0$. The conventional choice, $\phi = 1/2$, leads to the familiar conclusion, that, for unconditional stability, i.e., stable for all Courant numbers, it is necessary that $\theta \geq 1/2$. For a non-standard choice, $\phi \neq 1/2$, the stability will depend on the sign of Cr_i, or equivalently, on the direction of travel of a characteristic wave. Because of this, the use of $\phi \neq 1/2$ should be used with caution in situations where characteristics travel in both directions, particularly where characteristic directions may change, e.g., in a transition from subcritical to supercritical flow. The von Neumann condition is only a necessary condition for stability. Additional arguments, given in Appendix I, show that, for the present problem, it is also a sufficient condition.

Although a given difference scheme may be stable, its convergence may be poor. This is usually discussed in terms of its dissipative and dispersive characteristics. A measure of the former is the amplitude of

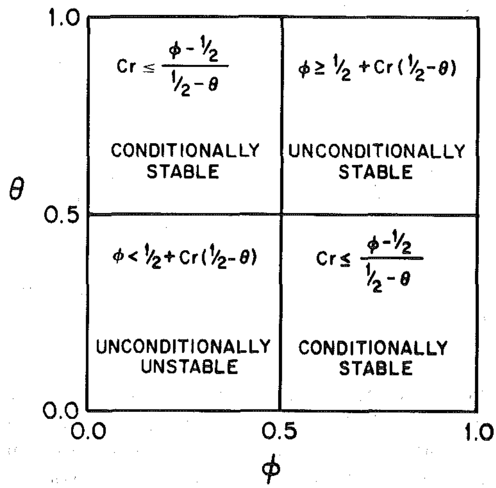
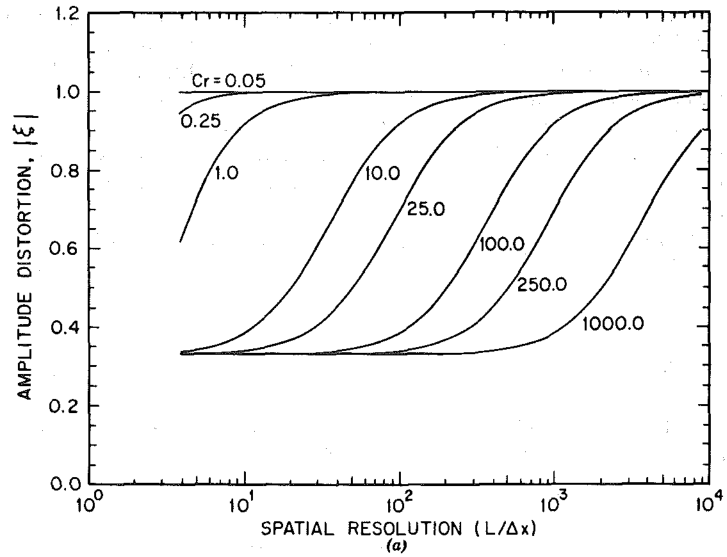


FIG. 2.—Regions of Stability in ϕ-θ Plane (for $Cr > 0$)



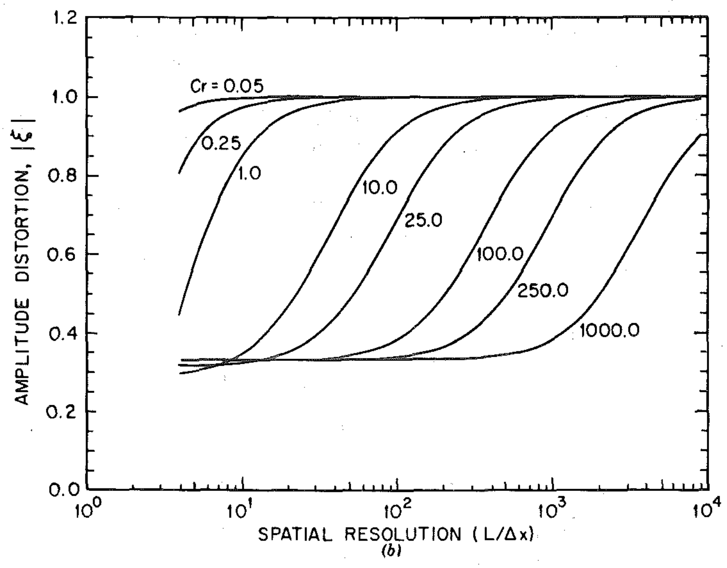
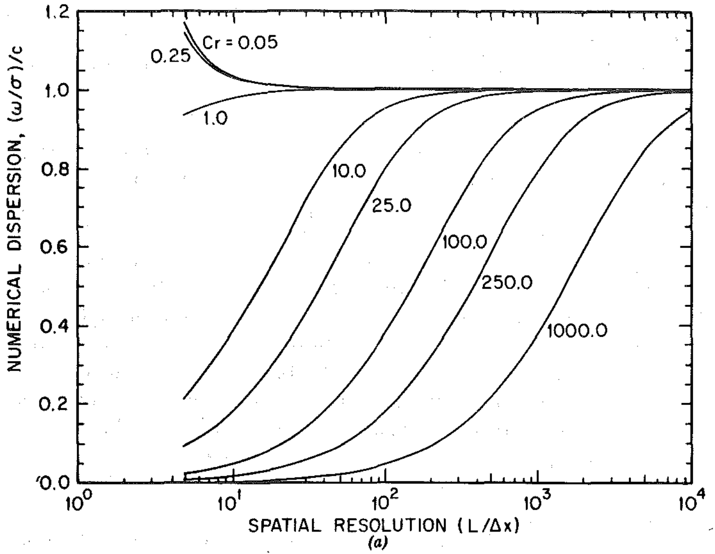


FIG. 3.—Variation of Numerical Amplitude Distortion with Cr **and** $L/\Delta x$**: (a)** $\phi = 0.5$, $\theta = 0.75$; **(b)** $\phi = 0.75$, $\theta = 0.75$

ξ, given by its real part, $|\xi|$, which ideally should be unity; deviations from unity may be attributed to the numerical scheme. Fig. 3 shows the extent of this numerical amplitude distortion for two values of ϕ. The dispersive characteristics are discussed in terms of the ratio of the numerical wave speed, ω/σ, to the true wave speed, c, as shown in Fig. 4. As shown, both dissipation and dispersion depend on the spatial resolution, $L/\Delta x$, where L = a characteristic wavelength and on the Courant number, Cr, as well as on the weighting factors. The effect on dissipation and dispersion of the change in ϕ from its conventional value of 1/2 is seen to be particularly marked for small $L/\Delta x$, the implications of which will be explored later.



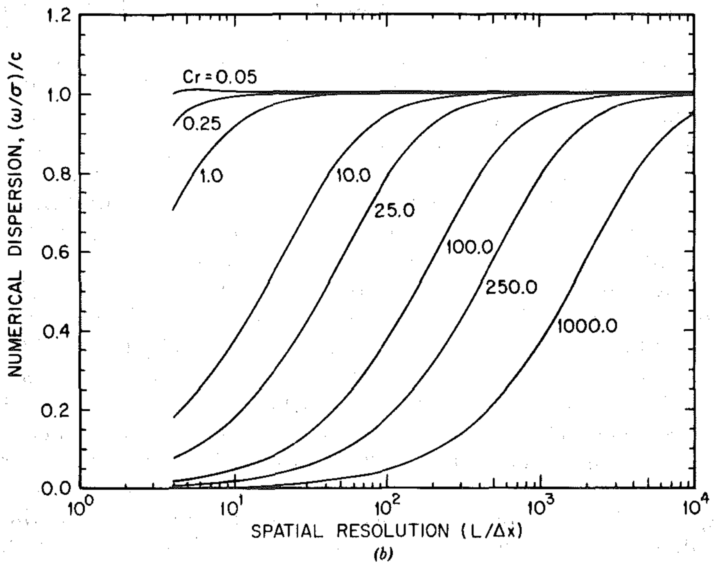


FIG. 4.—Variation of Numerical Dispersion with Cr and $L/\Delta x$: (a) $\phi = 0.5$, $\theta = 0.75$; (b) $\phi = 0.75$, $\theta = 0.75$

Implications for Numerical Modeling of Unsteady Sediment Transport

Problem of Disparate Scales.—The uniform application of the standard scheme ($\phi = 1/2$, $\theta > 1/2$) to the movable-bed problem was found to give rise to persistent oscillations in the bed profile (7). In the analogous fixed-bed problem, it is often chosen to increase the value of the time weighting factor, θ, towards unity in order to dampen parasitic oscillations, incurring some inaccuracy (see Ref. 2). This strategy was found ineffective in dealing with bed-profile oscillations. This behavior may be explained in terms of the results of the Fourier analysis.

The occurrence and persistence of artificial oscillations in the bed profile arise from the singularly perturbed nature of the movable-bed problem. This is associated with the presence of disparate scales. Numerically, this disparity is expressed in Figs. 3 and 4 in terms of Courant numbers and degrees of spatial resolution. As an example, consider a uniform grid with a mesh and time step chosen such that $L/\Delta x = L_w/\Delta x = 100$; and $Cr_w = 10$. The subscript w indicates that both parameters are based on the water wave. Parasitic oscillations stem from the dispersive character of the difference scheme (6). Figs. 3 and 4 show that, with these choices, dispersion is negligible in the solution of the water wave. Further, should oscillations occur, they comprise relatively short waves and will be strongly dissipated. In the case of the general movable-bed problem, the bed wave, which has a much smaller celerity than the water wave, must also be considered. A typical celerity ratio may result, for example, in $Cr_b/Cr_w = L_b/L_w = 0.005$, where the subscript b refers to bed wave scales. It is shown in Figs. 3 and 4 that, despite the adequacy of the mesh size for the water wave, the bed wave is poorly resolved. Indeed, $L_b/\Delta x$ may be inadvertently chosen to be less than two, thereby incurring aliasing to longer wavelengths (13). The poor resolution results in substantial numerical dispersion and a susceptibility to parasitic oscillations. Moreover, since $Cr_b << 1$, these oscillations are generated as longer waves, which are less strongly dissipated and, therefore, more persistent than those generated on the free surface. This analysis also shows that reducing the size of the time step, useful in the fixed-bed case, may, for a given mesh size, aggravate the problem. If Cr_w is reduced from 10 to 5 by halving the time step, then, for the water wave, dispersion will be reduced. At the same time, however, Cr_b will be reduced from 0.05 to 0.025, with a resulting increase in dispersion.

The numerical difficulties encountered by Lyn (7) in the use of the standard Preissmann scheme, which are exemplified by the results shown in Fig. 5 for $\theta = 0.75$, occurred near the boundary. This is explained by the Fourier analysis in terms of the small length scale induced by the imposed boundary condition, but without explicit reference to the boundary. This reflects the inability of the Fourier analysis to consider directly the effect of boundary conditions. Insight into this effect may be gained from an alternate analysis, based on the matrix method (12) of a model one-equation system, outlined in Appendix I. The effect of the boundary condition is shown to depend on the factor, $R = [-(1 - a)/a]^j$, where $a = (\phi + \theta Cr)$; and $j = 1, 2, \ldots$, indicating the position of a grid point ($j = 1$ at the boundary). For the conventional scheme, where

J. Hydraul. Eng., 1987, 113(1): 16-28

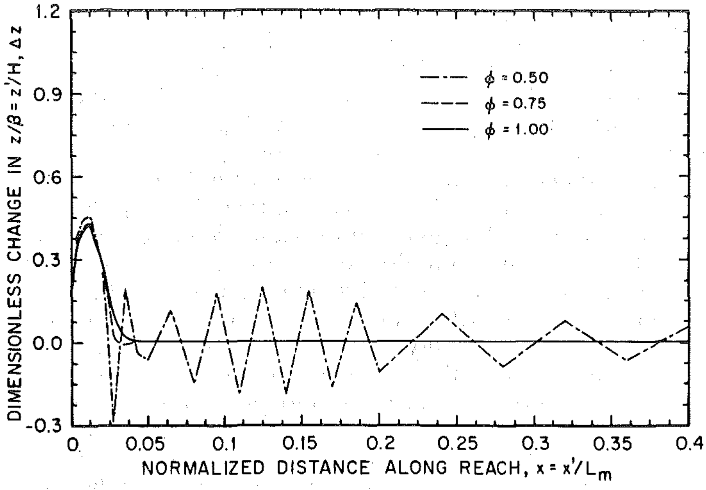


FIG. 5.—Effect of Variation in Spatial Weighting Factor, ϕ, on Bed Oscillations

$\phi = 1/2$, interior points far from the boundary, i.e., $j >> 1$, will not be affected by boundary conditions, i.e., $R << 1$, provided $Cr = 0(1)$. This is the case of the fixed-bed problem. In the general movable-bed problem, where $Cr << 1$, then, even for $j >> 1$, $R = 0(1)$, and so interior points far from the boundary may still be affected. Moreover, R will alternate in sign so as to produce artificial oscillations.

Strategies for Inhibiting Oscillations

To suppress these artificial oscillations, it is necessary to reduce dispersion or increase dissipation (or both). The latter introduces inaccuracies into the solution, which may or may not be tolerable. If the choice of a uniform mesh size and time step is based on the bed-wave scales rather than the water-wave scales, then the choice, $Cr_b > 1$ and $L_b/\Delta x = 50$ should give satisfactory results. Artificial oscillations would be avoided without introducing distortions due to numerical dissipation. This would, however, entail $Cr_w = 0(10^2)$ and $L_w/\Delta x = 0(10^4)$. Such a strategy may require substantial computational effort, in which machine precision may constrain the obtainable resolution. It is noted that, from this perspective, the constant-discharge method (3) is seen as implicitly accepting this conclusion. In that method, $Cr_b = 0(1)$ and $L_b = 0(10)$, while Cr_w and L_w are infinite. Ponce, et al. (9) in their analysis of the constant-discharge method explicitly recommended that Cr_b be chosen greater than unity. The use of a uniform mesh is inefficient if the regions of significant bed-wave changes are highly localized. This will be the case when relatively rapid changes are imposed on the system, e.g., through boundary conditions. A non-uniform mesh, with a fine mesh in the region of sharp gradients and a coarse mesh outside, would then be a more efficient solution.

J. Hydraul. Eng., 1987, 113(1): 16-28

An alternative strategy for inhibiting bed oscillations was used by Lyn (7) increasing the value of the spatial weighting factor from its conventional value of 1/2. This may be considered as an upwinding approach, which has been used in other contexts for singularly perturbed problems (5). The result of the Fourier series method, Eq. 10, assumed, however, that a single value of ϕ was applied uniformly in the discretization of the system of equations and so cannot be directly used. It is argued, nevertheless, that because of the small parameter, ϵ, a perturbation approach may be taken. A regular perturbation solution can be obtained and thus the sediment continuity equation can be decoupled from the hydraulic equations. This is permissible only if the problem is a pure initial-value problem, which is implicitly assumed in applying the Fourier method. Thus, the stability and convergence properties of the difference scheme applied to the sediment continuity equation may be discussed, in an approximate fashion, independently of the hydraulic equations. Since it is a single equation, Eq. 10 is applicable.

The effect of an increase in ϕ on the dissipative and dispersive characteristics is shown in Figs. 3 and 4. As pointed out earlier, the effect is most pronounced for small Cr and $L/\Delta x$, precisely the region of interest necessary to deal with bed-wave phenomena. Dissipation is increased, while dispersion may be reduced or increased depending on the value of ϕ. What is perhaps most important with regard to the persistence of bed oscillations is that the direction of dispersion is changed, in that shorter, rather than longer, waves tend to be generated if $\phi > 1/2$. These will be less persistent and, therefore, of less concern. However, this upwinding approach should not be used blindly. Gresho and Lee (5) emphasized the danger of using an upwinding scheme where it is inappropriate. The use of a non-centered value of ϕ does sacrifice second-order accuracy in time and so will require additional computation to achieve the same level of accuracy as the centered scheme, assuming that the results of the centered scheme are meaningful. Caution must be exercised in applying the non-centered scheme to situations where the direction of travel of the bed wave may change.

The above recommendations may also be equivalently interpreted using the results of the matrix analysis referred to in the previous section. Boundary effects may be localized by adjusting the factor R. Increased spatial resolution will mean that, for a given point in physical space, the exponent j will be larger, with the result that R will be reduced. In order to avoid oscillations, it is necessary that $(1 - a)/a < 0$, or

$$Cr \geq \frac{1 - \phi}{\theta} \quad \text{(11)}$$

If the conventional choice, $\phi = 1/2$, is made, a minimum Courant number exists below which oscillations will occur regardless of the value of θ. In particular, if the conventional choice is made, $\phi = 1/2$, together with $\theta = 1/2$, then in order to avoid oscillations altogether, it is necessary to have $Cr > 1$, as was recommended above, and also by Ponce, et al. (9). Only $\phi = 1$ will result in an oscillation-free solution regardless of how small (but positive) the Courant number is.

Effect of Friction.—This analysis has assumed an homogeneous system. A more realistic analysis would include the inhomogeneous term, which, for the present problem, describes the effect of friction. Since friction is a physical damping mechanism, its inclusion will not destabilize the system. It may also be argued that, if the dissipation and dispersion introduced by the numerical scheme is tolerable in the frictionless case, then this will also be true for the case with friction under many circumstances. This is so because the physical friction should then be significantly larger than the numerical friction. Where the effects of friction are essential to the physical phenomenon, however, such as the instability leading to roll waves at high Froude numbers, the results of the present analysis may be inapplicable. Some idea of the effects of friction may, in such cases, be obtained from studies in fixed-bed modeling where these effects are included (4,8).

Summary and Conclusions

Because of the singularly perturbed nature of the general movable-bed problem, a direct application of the conventional Preissman implicit scheme may give rise to artificial oscillations of the solution for the bed elevation. This difficulty may be explained through a Fourier series analysis of the difference equations by the large dispersion and its direction (towards longer waves) associated with small values of the bed Courant number and poor spatial resolution. In order to avoid these numerically induced oscillations, it is recommended that:

1. The computational grid be refined so that the local bed Courant number, Cr_b, is greater than unity, and the local bed spatial resolution, $L_b/\Delta x$, is of order 10 and preferably greater. A non-uniform mesh is recommended to reduce computational time and storage.
2. If numerical dissipation can be tolerated, the use of a non-standard value of the spatial weighting factor, $\phi > 1/2$, (for positive Courant numbers) may be used as a practical compromise.

Acknowledgments

The writers would like to thank Drs. R. C. Y. Koh and R. J. Sobey who commented on an earlier draft, and also Dr. N. H. Brooks for comments on the final draft.

This work was supported by the National Science Foundation under Grant CEE79-20311 A02. D. Lyn was also supported by a fellowship from the National Sciences and Engineering Research Council of Canada. P. Goodwin was supported by a Science and Engineering Research Council Grant, British Graduate Fellowship.

Appendix I.—Mathematical Details

Sufficiency of von Neumann Condition.—The von Neumann condition is sufficient for numerical stability only for a limited class of amplication matrices. Evans (4) transformed the discretized Euler equations in such a way that the Fourier decomposition resulted in a normal amplification matrix, for which the von Neumann condition is sufficient. A simpler yet more general argument, which does not require any transformation, relies on the following theorem, adapted from Richtmyer and Morton (11):

> If the amplification matrix, A^*, possesses a complete set of eigenvectors which are bounded independently of the numerical parameters, Δx and Δt, then the von Neumann condition is necessary and sufficient for stability.

The amplification matrix associated with Eq. 8 may be expressed as

$$A^* = \{[\phi(\eta - 1) + 1]I + r(\eta - 1)\theta A_0\}^{-1}\{[\phi(\eta - 1) + 1]I + r(\eta - 1)(\theta - 1)A_0\} \quad (12)$$

where I = the identity matrix. The eigenvalues of A^* are identical to those of A_0, since A^* is a rational matrix function of A_0. Since A_0 describes a constant-coefficient hyperbolic system, it possesses a complete set of eigenvectors, independent of Δx and Δt. It follows, then, that, for the general Preissmann scheme applied to the problem under consideration, the von Neumann condition is necessary and sufficient for stability within the context of Fourier stability analysis.

Matrix Stability Analysis.—Although stability analysis via the matrix method includes boundary effects directly, results for general problems are difficult to obtain. Nevertheless, the investigation of the simple first-order case gives insight to the direct effects of a boundary condition. If the condition $f(0,t) = f^*(t)$ is imposed on the scalar homogeneous equivalent of Eq. 4 (where f is used instead of $\mathbf{g}$), the resulting difference equation for $f^n = (f_1^n, \ldots, f_N^n)$, arising from the Preissmann discretization may be written in matrix form as

$$Tf^{n+1} = T'f^n - (1 - \alpha)f^{*n+1} + [1 - (\alpha - rc)]f^{*n} \quad (13)$$

where $T = aI + (1 - a)\mathbf{I}_{-1}$; $T' = (a - rc)I + [1 - (a - rc)]\mathbf{I}_{-1}$; I_{-1} being a matrix with ones on the first subdiagonal and zero elsewhere, and f^{*n+1} and f^{*n} are given by the boundary conditions. The stability of the scheme depends only on the matrix, $\mathbf{G} = T^{-1}\,T'$, which plays the role of an amplification matrix. The eigenvalues of $\mathbf{G}$ are all identically $(a - rc)/a$. In this case, however, the von Neumann condition is not sufficient for stability because $\mathbf{G}$ does not possess a complete set of eigenvectors. The fundamental requirement, that the matrix norm of $\mathbf{G}$, $\|\mathbf{G}\|$, be not greater than unity, must be used. A convenient and natural norm is the infinity norm, $\|\mathbf{G}\|_\infty = \max_i \Sigma_j |g_{ij}|$, which is the maximum of the sums of the absolute values of a row over all rows. It can be shown that, with this definition,

$$\|\mathbf{G}\|_\infty = \left|\frac{a - rc}{a}\right| + \frac{rc}{a^2}\sum_{k=0}^{N-2}\left|\frac{1 - a}{a}\right|^k \quad (14)$$

In order that $\|\mathbf{G}\|_\infty \leq 1$, it necessary that $|(a - rc)/a| \leq 1$, which reproduces the von Neumann condition for $\mathbf{G}$. It is evident, however, that this is not sufficient. Further analysis of the sum gives a more stringent condition, which is identical to the result obtained by the Fourier analysis, as should be expected. A closed-form solution to Eq. 13 can be expressed as

$$f_J^{n+1} = \left(\frac{a - rc}{a}\right) f_J^n + \frac{rc}{a^2} \sum_{k=0}^{J-2} \left(-\frac{1-a}{a}\right)^k f_{J-1-k}^n$$
$$+ \left(-\frac{1-a}{a}\right)^{J-1} \left\{\frac{[1-(a-rc)]f_0^{*n} - (1-a)f_0^{*n+1}}{a}\right\} \quad (15)$$

for $J > 2$. The relation between Eqs. 14 and 15 is apparent. The direct effect of the boundary condition, given by f_0^n and f_0^{n+1}, is transmitted through the factor $[-(1 - a)/a)]$.

Appendix II.—References

1. Abbott, M. B., *Computational Hydraulics*, Pitman, London, U.K., 1979.
2. Cunge, J. A., Discussion of "Implicit Numerical Modelling of Unsteady Flows," by M. Amein and H. L. Chu, *Journal of the Hydraulics Division*, ASCE, Vol. 102, No. HY1, Jan., 1976, pp. 120–122.
3. Cunge, J. A., and Perdreau, N., "Mobile Bed Fluvial Mathematical Model," *La Houille Blanche*, no. 7 1973, pp. 561–580.
4. Evans, E. P., "The Behaviour of a Mathematical Model of Open Channel," in *Proceedings*, 17th Congress of the International Association for Hydraulic Research on Hydraulic Engineering for Improved Water Management, Vol. 2, Aug. 15–19, 1977, Baden-Baden, West Germany, pp. 173–180.
5. Gresho, P. M., and Lee, R. L., "Don't Suppress the Wiggles, They're Telling You Something," *Computers and Fluids*, Vol. 9, 1981, pp. 223–253.
6. Liggett, J. A., and Cunge, J. A., "Numerical Methods of Solution of the Unsteady Flow Equations in Open Channels," in *Unsteady Flows in Open Channels*, K. Mahmood and V. Yevjevich, Eds., Vol. 1, Water Resources Publications, Fort Collins, Colo., 1975, pp. 89–182.
7. Lyn, D. A., "Unsteady Sediment-Transport Modeling," *Journal of Hydraulic Engineering*, ASCE, Vol. 113, No. 1, Jan., 1987, pp. 1–15.
8. Ponce, V. M., Indlekofer, H., and Simons, D. B., "Convergence of Four-Point Implicit Water-Wave Models," *Journal of the Hydraulics Division*, ASCE, Vol. 104, No. HY7, July, 1978, pp. 947–958.
9. Ponce, V. M., Indlekofer, H., and Simons, D. B., "The Convergence of Implicit Bed Transient Models," *Journal of the Hydraulics Division*, ASCE, Vol. 105, No. HY4, Apr., 1979, pp. 351–364.
10. Preissmann, A., "Propagation des intumescences dans les canaux et Les Rivieres," *1e Congres de l'Association Francaise de Calcule*, Grenoble, France, 1961, pp. 433–442.
11. Richtmyer, R. D., and Morton, K. W., *Difference Methods for Initial-Value Problems*, 2nd ed. Interscience Publishers, John Wiley & Sons, New York, N.Y., 1967.
12. Smith, G. D., *Numerical Solution of Partial Differential Equations: Finite-Difference Methods*, 2nd ed., Clarendon Press, Oxford, U.K., 1978.
13. Sobey, R. J., "Numerical Alternatives in Transient Stream Response," *Journal of Hydraulic Engineering*, ASCE, Vol. 110, No. 6, June, 1984, pp. 749–771.

Appendix III.—Notation

The following symbols are used in this paper:

A, A_0 = matrix of coupling coefficients and its linearization;
a = $(\phi + \theta\ Cr)$;

J. Hydraul. Eng., 1987, 113(1): 16-28

$\mathbf{b}$ = vector of inhomogeneous terms for first-order hyperbolic system;
C = flux-weighted depth-averaged concentration;
C_r = reference concentration;
Cr_i = Courant number for the ith characteristic wave, $Cr_i = c_i \Delta t/\Delta x$;
Cr_b, Cr_w = Courant number based on the bed and the water wave;
c_i = characteristic wave speed of ith wave;
f = arbitrary function to be discretized;
f_j^n = value of discretized function f_1 at the point $(x, t) = (j\Delta x, n\Delta t)$;
$\mathbf{g}, \mathbf{g}_0$ = vector of functions for which a solution is sought, and its linearization about a locally uniform state;
$\mathbf{g}^*$ = arbitrary vector of amplitudes of Fourier modes;
H = characteristic depth;
h = non-dimensional flow depth;
i = imaginary unit;
L = characteristic horizontal length scale;
L_b, L_w = horizontal length scale characteristic of a bed wave and a water wave;
p = porosity of bed;
r = ratio of time step to grid spacing, $r = \Delta t/\Delta x$;
S_e = normalized energy or friction slope;
S_r = reference slope;
t = time variable;
u = non-dimensional mean flow velocity;
x = distance variable;
z = non-dimensional bed elevation;
β = non-dimensional parameter, $\beta = H/(LS_r)$;
$\Delta t, \Delta x$ = time step and grid spacing;
ϵ = non-dimensional parameter, $\epsilon = C_r/(1 - p)$;
η = space-varying part of a Fourier mode;
θ = time-weighting factor in general Preissmann scheme;
ξ = time-varying part of a Fourier mode;
σ = complex wave number;
ϕ = space-weighting factor in Preissmann scheme; and
ω = complex frequency.

J. Hydraul. Eng., 1987, 113(1): 16-28