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FUZZY GOAL PROGRAMMING – AN ADDITIVE MODEL

R.N. TIWARI, S. DHARMAR and J.R. RAO
Department of Mathematics, Indian Institute of Technology, Kharagpur, India 721302

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1. Introduction

Goal Programming (GP) is a good decision aid in modelling real world decision problems. Goal programming extends linear programming to problems which involve multiple objectives. It is necessary to specify aspiration levels for the objectives and aims to reduce the deviations from aspiration levels. In the case of a problem with nonequivalent goals the weight or priority of the goal is reflected through its deviation variables. Often, in real world problems the aspiration levels and/or priority factors of the DM, and sometimes even the weights to be assigned to the goals, are imprecise in nature. In such situations the use of fuzzy set theory (Zadeh [16]) comes in handy.

The use of fuzzy set theory in GP was first considered by Narasimhan [8]. Hannan [2, 3, 4], Narasimhan [9], Ignizio [6]. Rubin and Narasimhan [11] and Tiwari et al. [14, 15] have investigated various aspects of decision problems using FGP. An extensive review of these papers is given in [15]. The main difference between FGP and GP is that the GP requires the DM to set definite aspiration values for each objective that he wishes to achieve, whereas in FGP these are specified in an imprecise manner. Throughout this paper a fuzzy goal is considered as a goal with imprecise aspiration level.

In the present work we investigate a particular modelling which is additive (weighted and preemptive) in FGP by employing the usual addition as an operator to aggregate the fuzzy goals. In conventional GP the simple additive model for m goals $G_i(x)$ with deviational variables d_i^+, d_i^- is defined as

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} (d_i^+ + d_i^-) \\ \text{subject to} \quad & G_i(x) + d_i^- - d_i^+ = g_i, \\ & d_i^+ \cdot d_i^- = 0, \\ & d_i^+, d_i^-, x \geq 0, \;\; i = 1, 2, \ldots, m, \end{aligned}$$

where g_i is the aspiration level of the i-th goal. Here we develop a similar model using membership functions instead of deviational variables.

We formulate and discuss the simple and weighted additive models in Section 2, and preemptive priority in the additive model in Section 3. The solution procedures are illustrated with numerical examples. Concluding remarks are made in Section 4.

2. The simple and weighted additive model

2.1. *The simple additive model*

Consider the FGP problem:

$$\begin{aligned} &\text{Find} && X \\ &\text{to satisfy} && G_i(X) \gtrsim g_i, \quad i = 1, 2, \ldots, m, \\ &\text{subject to} && AX \leqslant b, \\ &&& X \geqslant 0, \end{aligned} \tag{1}$$

where X is an n-vector with components $x_1, x_2, \ldots, x_n$ and $AX \leqslant b$ are system constraints in vector notation. The symbol '$\gtrsim$' refers to the fuzzification of the aspiration level (i.e., approximately greater than or equal to). The i-th fuzzy goal $G_i(X) \gtrsim g_i$ in (1) signifies that the DM is satisfied even if less than the g_i upto certain tolerance limit is attained. A linear membership function μ_i for the i-th fuzzy goal $G_i(X) \gtrsim g_i$ can be expressed according to Zimmermann [17, 18] as

$$\mu_i = \begin{cases} 1 & \text{if } G_i(X) \geqslant g_i, \\ \dfrac{G_i(X) - L_i}{g_i - L_i} & \text{if } L_i \leqslant G_i(X) \leqslant g_i, \\ 0 & \text{if } G_i(X) \leqslant L_i, \end{cases} \tag{2.1}$$

where L_i is the lower tolerance limit for the fuzzy goal $G_i(X)$. In case of the goal $G_i(X) \lesssim g_i$, the membership function is defined as

$$\mu_i = \begin{cases} 1 & \text{if } G_i(X) \leqslant g_i, \\ \dfrac{U_i - G_i(X)}{U_i - g_i} & \text{if } g_i \leqslant G_i(X) \leqslant U_i, \\ 0 & \text{if } G_i(X) \geqslant U_i, \end{cases} \tag{2.2}$$

where U_i is the upper tolerance limit.

The additive model of the FGP problem (1) is formulated by adding the membership functions together as

$$\text{maximize} \quad V(\mu) = \sum_{i=1}^{m} \mu_i \tag{3}$$

$$\text{subject to} \quad \mu_i = \frac{G_i(X) - L_i}{g_i - L_i},$$
$$AX \leqslant b,$$
$$\mu_i \leqslant 1,$$
$$X, \mu_i \geqslant 0, \quad i = 1, 2, \ldots, m,$$

where $V(\mu)$ is called the fuzzy achievement function or fuzzy decision function. This is a single objective optimization problem which can be solved by employing a suitable classical technique. Because the goals are fuzzy, unlike conventional GP (minimizing the deviations) the fuzzy decision function consisting of μ_i's is to be maximized here.

2.2. *Numerical example*

Let us consider a mathematical model of the problem having 5 fuzzy goals with 4 variables and 4 system constraints as follows:

Find X to satisfy the following fuzzy goals:

$$\begin{aligned}
4x_1 + 2x_2 + 8x_3 + x_4 &\leqslant 35, \\
4x_1 + 7x_2 + 6x_3 + 2x_4 &\geqslant 100, \\
x_1 - 6x_2 + 5x_3 + 10x_4 &\geqslant 120, \\
5x_1 + 3x_2 + \quad + 2x_4 &\geqslant 70, \\
4x_1 + 4x_2 + 4x_3 &\geqslant 40,
\end{aligned} \tag{4}$$

subject to

$$\begin{aligned}
7x_1 + 5x_2 + 3x_3 + 2x_4 &\leqslant 98, \\
7x_1 + x_2 + 6x_3 + 6x_4 &\leqslant 117, \\
x_1 + x_2 + 2x_3 + 6x_4 &\leqslant 130, \\
9x_1 + x_2 + \quad + 6x_4 &\leqslant 105, \\
x_1, x_2, x_3, x_4 &\geqslant 0.
\end{aligned}$$

Let the tolerance limits of the 5 goals be (55, 40, 70, 30, 10) respectively. Now the fuzzy goals are converted into crisp ones by using membership functions μ_i as defined in (2). Thus the problem (4) reduces to

$$\text{maximize} \quad V(\mu) = \sum_{i=1}^{5} \mu_i \tag{5a}$$

$$\text{subject to} \quad \mu_1 = \frac{55 - (4x_1 + 2x_2 + 8x_3 + x_4)}{20},$$

$$\mu_2 = \frac{4x_1 + 7x_2 + 6x_3 + 2x_4 - 40}{60},$$

$$\mu_3 = \frac{x_1 - 6x_2 + 5x_3 + 10x_4 - 70}{50}, \tag{5b}$$

$$\mu_4=\frac{5x_1+3x_2+2x_4-30}{40},$$

$$\mu_5=\frac{4x_1+4x_2+4x_3-10}{30},$$

$$\begin{aligned}
7x_1+5x_2+3x_3+2x_4&\leqslant 98,\\
7x_1+\ x_2+6x_3+6x_4&\leqslant 117,\\
x_1+\ x_2+2x_3+6x_4&\leqslant 130,\\
9x_1+\ x_2+\qquad +6x_4&\leqslant 105,\\
\mu_i&\leqslant 1,\\
x_j,\ \mu_i\geqslant 0,\quad i=1,2,\ldots,5,\ j&=1,2,3,4.
\end{aligned}$$

This problem is solved by using the simplex method and the results obtained are

$$x_1=0,\quad x_2=9.75,\quad x_3=0,\quad x_4=15.875,$$

with achieved goal values

$$G_1=35.375,\quad G_2=100.0,\quad G_3=100.25,\quad G_4=61.0,\quad G_5=39.0,$$

and membership values

$$\mu_1=0.981,\quad \mu_2=1.00,\quad \mu_3=0.605,\quad \mu_4=0.775,\quad \mu_5=0.967.$$

2.3. *The weighted additive model*

The weighted additive model is widely used in GP and multiobjective optimization techniques to reflect the relative importance of the goals/objectives. In this approach the DM assigns differential weights as coefficients of the individual terms in the simple additive fuzzy achievement function to reflect their relative importance, i.e., the objective function is formulated by multiplying each membership of the fuzzy goal with a suitable weight and then adding them together. This leads to the following formulation, corresponding to (3):

$$\begin{aligned}
\text{maximize}\quad & V(\mu)=\sum_{i=1}^{m} w_i\mu_i \qquad\qquad (6)\\
\text{subject to}\quad & \mu_i=\frac{G_i(X)-L_i}{g_i-L_i},\\
& AX\leqslant b,\\
& \mu_i\leqslant 1,\\
& X,\ \mu_i\geqslant 0,\ i=1,2,\ldots,m.
\end{aligned}$$

where w_i is the relative weight of the i-th fuzzy goal.

The major difficulty of this method is the DM's task to assess the relative importance of the goals correctly. The phrase 'relative importance' is a fuzzy concept whose various levels can be stated only imprecisely. However, there are some good approaches in the literature to assess these weights. We may mention in this regard the eigenvector method of Saaty [12], a geometric averaging procedure for constructing super-transitive approximations to binary comparison matrices by Narasimhan [10], the entropy method of Jaynes [7] and the weighted least squares method of Chu et al. [1]. These can be used to suitably specify the weights.

2.4. *Numerical example*

The example which was considered in the previous section is reformulated using relative weights $w = (0.49,\ 0.131,\ 0.153,\ 0.114,\ 0.112)$. Therefore the objective function (5a) of problem (5) becomes

$$\text{Maximize} \quad V(\mu) = 0.49\mu_1 + 0.131\mu_2 + 0.153\mu_3 + 0.114\mu_4 + 0.112\mu_5.$$

The results obtained with the same constraints (5b) as in (5) are

$$x_1 = 0, \quad x_2 = 9.545, \quad x_3 = 0, \quad x_4 = 14.909,$$

with achieved goal values

$$G_1 = 35, \quad G_2 = 98.633, \quad G_3 = 101.82, \quad G_4 = 60.453, \quad G_5 = 38.18,$$

and membership values

$$\mu_1 = 1.0, \quad \mu_2 = 0.977, \quad \mu_3 = 0.636, \quad \mu_4 = 0.761, \quad \mu_5 = 0.939.$$

It may be noted that, as compared to the previous solution of the simple additive model (which corresponds to equal importance of goals), in the present formulation the achievements of G_1 and G_3 (μ_1 and μ_3) have increased, and those of the remaining goals have decreased according to the weighting structure.

3. Preemptive priority in the additive model

In many decision problems the goals are not commensurable (not in the same measurable unit). Further, sometimes the goals are such that unless a particular goal or a subset of goals is achieved, the other goals should not be considered. In such situations the weighting scheme of the previous section is not an appropriate method. The preemptive priority structure may be stated as $p_i \ggg p_{i+1}$ which means that the goals in the i-th priority level have higher priority than the goals in the $(i+1)$-th priority level, i.e., however large N (a number) may be, p_i cannot be equal to Np_{i+1} [5, 13].

For the present investigation the problem is subdivided into k subproblems, where k is the number of priority levels. In the first subproblem the fuzzy goals belonging to the first priority level have only been considered and solved using the simple additive model as descibed in Section 1. But at other priority levels the membership values achieved earlier for higher priority levels are imposed as additional constraints. In general the i-th subproblem becomes

$$\begin{aligned}
\text{maximize} \quad & \sum_{s} (\mu_s)_{p_i} \\
\text{subject to} \quad & \mu_s = \frac{G_s - L_s}{g_s - L_s}, \\
& AX \leqslant b, \\
& (\mu)_{p_r} = (\mu^*)_{p_r}, \quad r = 1, 2, \ldots, j-1, \\
& \mu_s \leqslant 1, \\
& X, \mu_i \geqslant 0, \quad i = 1, 2, \ldots, m,
\end{aligned}$$

where $(\mu_s)_{p_i}$ refers to the membership functions of the goals in the i-th priority level and $(\mu^*)_{p_r}$ is the achieved membership value in the r-th ($r \leqslant j-1$) priority level.

3.1. *Numerical example*

Let the example discussed in Section 1 with five fuzzy goals have the following three priority levels:
priority level 1: G_1 and G_3,
priority level 2: G_2,
priority level 3: G_4 and G_5.
The subproblems are formulated accordingly as defined above. The solution found for the first two subproblems are $\mu_1 = 1$, $\mu_3 = 1$ and $\mu_3 = 0.795$. Now the 3rd and last subproblem to solve is

$$\begin{aligned}
\text{maximize} \quad & V(\mu) = \mu_4 + \mu_5 \\
\text{subject to} \quad & \mu_1 = 1, \\
& \mu_2 = 0.795, \\
& \mu_3 = 1, \\
& \mu_4 \leqslant 1, \\
& \mu_5 \leqslant 1, \\
& \text{and (5b) excluding } \mu_i \leqslant 1.
\end{aligned}$$

The results are

$$x_1 = 0.02, \quad x_2 = 7.479, \quad x_3 = 0.473, \quad x_4 = 16.251,$$

with achieved goal value

$$G_1 = 35.000, \quad G_2 = 87.70, \quad G_3 = 120.000, \quad G_4 = 54.949, \quad G_5 = 31.816,$$

and membership value

$$\mu_1 = 1.00, \quad \mu_2 = 0.795, \quad \mu_3 = 1.00, \quad \mu_4 = 0.624, \quad \mu_5 = 0.727.$$

Hence, the first priority goals G_1 and G_3 are achieved fully whereas the 2nd priority goal G_2 with $\mu_2 = 0.795$ and 3rd priority goals $\mu_4 = 0.624$, $\mu_5 = 0.727$ are achieved partially.

4. Concluding remarks

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