Distributed Second Order Methods with Fast Rates and Compressed Communication

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The Problem

$$\min_{x\in\mathbb{R}^d}\left[P(x) := f(x) + \frac{\lambda}{2}\|x\|^2\right]. \tag{1}$$

Function f is convex, and has an "average of averages" structure:

$$f(x) := \frac{1}{n}\sum_{i=1}^n f_i(x), \quad f_i(x) := \frac{1}{m}\sum_{j=1}^m f_{ij}(x), \tag{2}$$

and $\lambda \geq 0$ is a regularization parameter. Each f_{ij} is a function of the form: $f_{ij}(x) := \varphi_{ij}(a_{ij}^\top x)$. The Hessian of f_{ij} at point x is

$$\mathbf{H}_{ij}(x) := h_{ij}(x) a_{ij} a_{ij}^\top, \quad h_{ij}(x) := \varphi''_{ij}(a_{ij}^\top x). \tag{3}$$

The Hessian $\mathbf{H}_i(x)$ of local functions $f_i(x)$ and the Hessian $\mathbf{H}(x)$ of f can be represented as linear combination of one-rank matrices.

Assumptions

We assume that Problem (1) has at least one optimal solution x^*. For all i and j, φ''_{ij} is γ-smooth, twice differentiable, and its second derivative φ''_{ij} is ν-Lipschitz continuous.

Main goal

Our goal is to develop a communication efficient Newton-type method for distributed optimization.

Naive distributed implementation of Newton's method

Newton's step: $x^{k+1} \overset{(1)}{=} x^k - \left(\mathbf{H}(x^k) + \lambda\mathbf{I}\right)^{-1}\nabla P(x^k)$.
Each node: computes the local Hessian $\mathbf{H}_i(x^k)$ and gradient $\nabla f_i(x^k)$, then sends them to the server.
Server: averages the local Hessians and gradients to produce $\mathbf{H}(x^k)$ and $\nabla f(x^k)$, respectively, adds $\lambda\mathbf{I}$ to $\mathbf{H}(x^k)$ and λx^k to $\nabla f(x^k)$, then performs Newton step. Next, it sends x^{k+1} back to the nodes.

Pros:
- Fast local quadratic convergence rate
- Rate is independent on the condition number

Cons:
- Requires $\mathcal{O}(d^2)$ floats to be communicated by each worker to the server, where d is typically very large

NEWTON-STAR (NS)

Assume that the server has access to coefficients $h_{ij}(x^*)$ for all i and j, i.e access to the Hessian $\mathbf{H}(x^*)$.
Step of NEWTON-STAR: $x^{k+1} = x^k - \left(\mathbf{H}(x^*) + \lambda\mathbf{I}\right)^{-1}\nabla P(x^k)$.

Theorem 1 (Convergence of NS)

Assume that $\mathbf{H}(x^*) \succeq \mu^*\mathbf{I}$ for some $\mu^* \geq 0$ and that $\mu^* + \lambda > 0$. Then for any starting point $x^0 \in \mathbb{R}^d$, the iterates of NEWTON-STAR satisfy the following inequality:

$$\left\|x^{k+1} - x^*\right\| \leq \frac{\nu}{2(\mu^*+\lambda)} \cdot \left(\frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m \|a_{ij}\|^3\right) \cdot \left\|x^k - x^*\right\|^2.$$

Pros:
- Fast local quadratic convergence rate
- Rate is independent on the condition number
- Communication cost is $\mathcal{O}(d)$ per-iteration

Cons:
- Cannot be implemented in practice

NEWTON-LEARN

How to address the communication bottleneck?
- Compressed communication
- Taking advantage of the structure of the problem

In NEWTON-LEARN we maintain a sequence of vectors

$$h_i^k = (h_{i1}^k, \dots, h_{im}^k) \in \mathbb{R}^m, \tag{4}$$

for all $i = 1, \dots, n$ throughout the iterations $k \geq 0$, with the goal of learning the values $h_{ij}(x^*)$ for all i, j:

$$h_{ij}(x^k) \to h_{ij}(x^*) \quad \text{as} \quad k \to +\infty. \tag{5}$$

Using $h_{ij}^k \approx h_{ij}(x^*)$, we can estimate the Hessian $\mathbf{H}(x^*)$ via

$$\mathbf{H}(x^*) \approx \mathbf{H}^k := \frac{1}{n}\sum_{i=1}^n \mathbf{H}_i^k, \quad \mathbf{H}_i^k := \frac{1}{m}\sum_{j=1}^m h_{ij}^k a_{ij} a_{ij}^\top. \tag{6}$$

Compressed learning

Compression operator: A randomized map $\mathcal{C} : \mathbb{R}^m \to \mathbb{R}^m$ is a *compression operator (compressor)* if there exists a constant $\omega \geq 0$ such that for all $x \in \mathbb{R}^m$

$$\mathbb{E}\left[\mathcal{C}(x)\right] = x, \quad \mathbb{E}\left[\|\mathcal{C}(x)\|^2\right] \leq (\omega+1)\|x\|^2. \tag{7}$$

Random sparsification (random-r) [1]**:** Compressor defined as

$$\mathcal{C}(x) := \frac{m}{r}\cdot \xi \circ x, \tag{8}$$

where $\xi \in \mathbb{R}^m$ is a random vector distributed uniformly at random on the discrete set $\{y \in \{0,1\}^m : \|y\|_0 = r\}$. The variance parameter associated with this compressor is $\omega = \frac{m}{r} - 1$.

NEWTON-LEARN: NL1

Assumption: We assume that each $\varphi_{ij}(x)$ is convex, and $\lambda > 0$.

Learning the coefficients: the idea

We design a learning rule for vectors h_i^k via the **DIANA trick** [2] :

$$h_i^{k+1} = \left[h_i^k + \eta\mathcal{C}_i^k\left(h_i(x^k) - h_i^k\right)\right]_+, \tag{9}$$

where $\eta > 0$ is a learning rate, and $\mathcal{C}_i^k$ is a freshly sampled compressor by node i at iteration k.

Main properties:
- $h_{ij}^k \geq 0$ for all i, j
- update is sparse: $\|h_i^{k+1} - h_i^k\|_0 \leq s$, where $s = \mathcal{O}(1)$
- $\mathbf{H}^k \succeq \mathbf{0}$

Each node: Computes update $h_i^{k+1} = \left[h_i^k + \eta\mathcal{C}_i^k\left(h_i(x^k) - h_i^k\right)\right]_+$ and gradient $\nabla f_i(x^k)$. Then the node broadcasts the gradient, update $h_i^{k+1} - h_i^k$ and data points a_{ij} for which $h_{ij}^{k+1} - h_{ij}^k \neq 0$.
Server: averages the local gradients to produce $\nabla f(x^k)$ and constructs $\mathbf{H}^k$ via (6). Then it performs a Newton-like step:

$$x^{k+1} = x^k - \left(\mathbf{H}^k + \lambda\mathbf{I}\right)^{-1}\left(\nabla f(x^k) + \lambda x^k\right), \tag{10}$$

and finally broadcasts x^{k+1} back to the nodes.

Pros
- Local linear and superlinear rates
- Rates are **independent on the condition number**
- Communication cost $\mathcal{O}(d)$ per iteration

Algorithm 1 NL1: NEWTON-LEARN ($\lambda > 0$ case)

Parameters: learning rate $\eta > 0$
Initialization: $x^0 \in \mathbb{R}^d$; $h_1^0, \dots, h_n^0 \in \mathbb{R}_+^m$;
$\mathbf{H}^0 = \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m h_{ij}^0 a_{ij} a_{ij}^\top \in \mathbb{R}^{d\times d}$
for $k = 0, 1, \dots$ **do**
  Broadcast x^k to all workers
  for *each node* $i = 1, \dots, n$ **do**
    Compute local gradient $\nabla f_i(x^k)$
    $h_i^{k+1} = [h_i^k + \eta\mathcal{C}_i^k(h_i(x^k) - h_i^k)]_+$
    Send $\nabla f_i(x^k)$, $h_i^{k+1} - h_i^k$ and corresponding a_{ij} to server
  end
  $x^{k+1} = x^k - \left(\mathbf{H}^k + \lambda\mathbf{I}\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n \nabla f_i(x^k) + \lambda x^k\right)$
  $\mathbf{H}^{k+1} = \mathbf{H}^k + \frac{1}{nm}\sum_{i=1}^n\sum_{j=1}^m (h_{ij}^{k+1} - h_{ij}^k) a_{ij} a_{ij}^\top$
end

Convergence theory

The analysis relies on the Lyapunov function

$$\Phi_1^k = \left\|x^k - x^*\right\|^2 + \frac{1}{\eta n m \nu^2 R^2}\mathcal{H}^k, \quad \mathcal{H}^k = \sum_{i=1}^n \left\|h_i^k - h_i(x^*)\right\|^2,$$

where $R = \max_{i,j}\|a_{ij}\|$.

Theorem 2 (convergence of NL1)

Theorem 2. Let each φ_{ij} is convex, $\lambda > 0$, and $\eta \leq \frac{1}{\omega+1}$. Assume that $\|x^k - x^*\|^2 \leq \frac{\lambda^2}{12\nu^2 R^6}$ for all $k \geq 0$. Then for Algorithm 1 we have the inequalities

$$\mathbb{E}[\Phi_1^k] \leq \theta_1^k \Phi_1^0,$$

$$\mathbb{E}\left[\frac{\|x^{k+1} - x^*\|^2}{\|x^k - x^*\|^2}\right] \leq \theta_1^k\left(6\eta + \frac{1}{2}\right)\frac{\nu^2 R^6}{\lambda^2}\Phi_1^0,$$

where $\theta_1 = 1 - \min\left\{\frac{\eta}{2}, \frac{5}{8}\right\}$, which is independent on the condition number.

Assumption on $\|x^k - x^*\|$ can be relaxed using the following lemma:

Lemma 1

Assume h_{ij}^k is a convex combination of $\{h_{ij}(x^0), \dots, h_{ij}(x^k)\}$ for all i, j and k. Assume $\|x^0 - x^*\|^2 \leq \frac{\lambda}{12\nu^2 R^6}$. Then

$$\|x^k - x^*\|^2 \leq \frac{\lambda^2}{12\nu^2 R^6} \text{for all } k > 0.$$

It is easy to verify that if we choose $h_{ij}^0 = h_{ij}(x^0)$, use the random sparsification compressor (8) and $\eta \leq \frac{1}{\omega+1}$, then h_{ij}^k is always a convex combination of $\{h_{ij}(x^0), \dots, h_{ij}(x^k)\}$ for $k > 0$.

NEWTON-LEARN: NL2

We additionally develop a modified method (NL2) which handles the case where P is μ-strongly convex, $|h_{ij}^k| \leq \gamma$, and $\lambda \geq 0$.

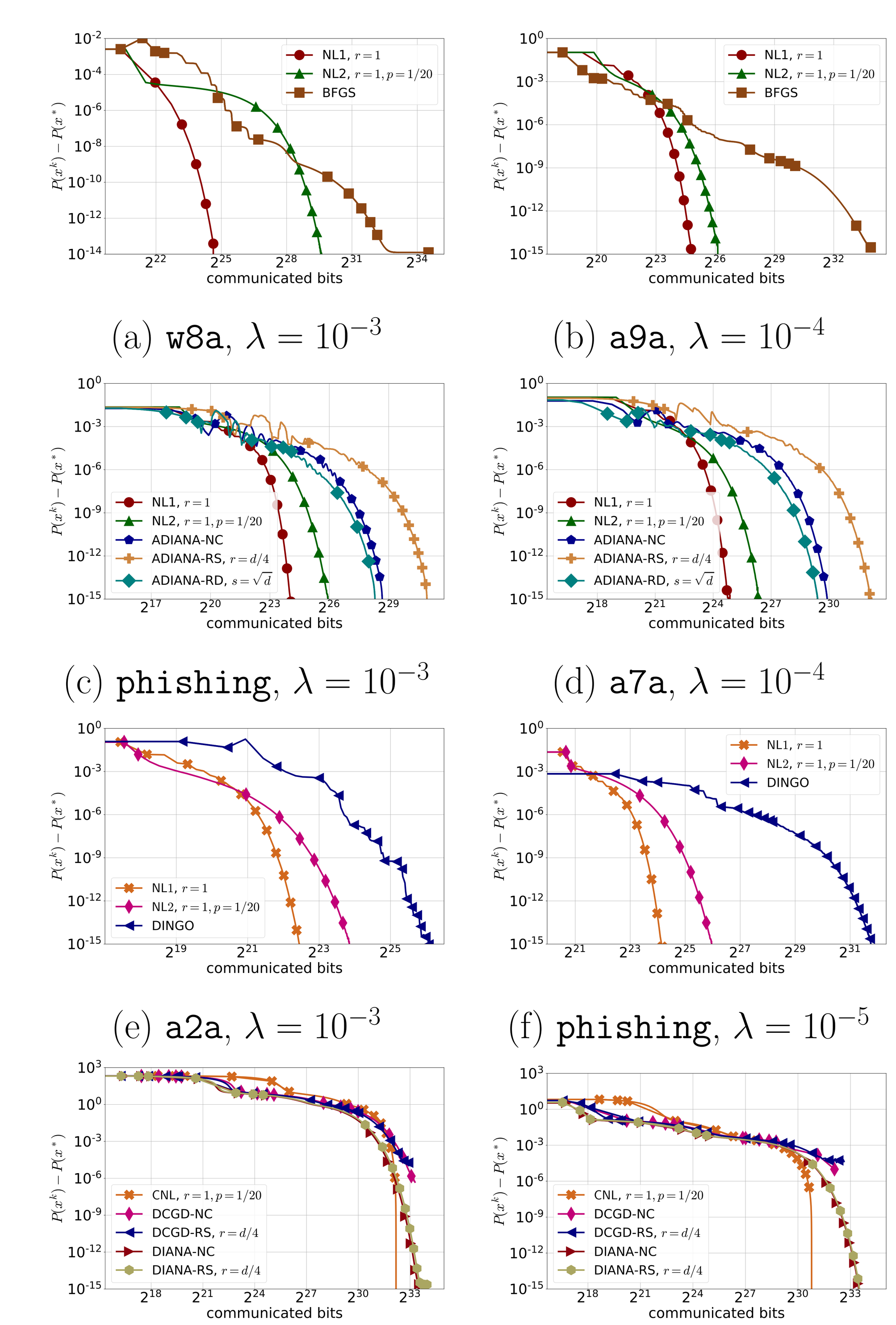
Pros:
- Local linear and superlinear rates
- Rates are **independent on the condition number**
- $\mathcal{O}(d)$ bits are communicated per iteration

CUBIC-NEWTON-LEARN

We also constructed a method (CNL) with global convergence guarantees using cubic regularization [3].

Pros:
- Local linear and superlinear rates
- Global linear rate in the strongly convex case and global sublinear rate in the convex case
- Rates are **independent on the condition number**
- $\mathcal{O}(d)$ bits are communicated per iteration

Experiments



(a) w8a, $\lambda = 10^{-3}$ (b) a9a, $\lambda = 10^{-4}$
(c) phishing, $\lambda = 10^{-3}$ (d) a7a, $\lambda = 10^{-4}$
(e) a2a, $\lambda = 10^{-3}$ (f) phishing, $\lambda = 10^{-5}$
(g) a2a, $\lambda = 10^{-3}$ (h) a7a, $\lambda = 10^{-4}$

Figure 1: Comparison of NL1, NL2 with (a), (b) BFGS; (c), (d) ADIANA; (e), (f) DINGO in terms of communication complexity. Comparison of CNL with (g), (h) DIANA and DCGD in terms of communication complexity.

References

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