Analyse et simulation numérique de modèles non-locaux en morphodynamique littorale

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Plan

1. Méthodes de splitting pour l'équation de Fowler

   Collaboration : Rémi Carles (CNRS & Univ. Montp2)

2. Les principes de minimisation appliqués à la dynamique littorale

   Collaboration : Bijan Mohammadi (CERFACS & Univ. Montp2)

Plan

1 Méthodes de splitting pour l'équation de Fowler

2 Les principes de minimisation appliqués à la dynamique littorale

Model for dune morphodynamics

A conservative nonlinear model

For all $t \in (0,T)$ et $x \in \mathbb{R}$,

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x (t,x) - u_{xx}(t,x) + \mathcal{I}[u(t,\cdot)](x) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Model for dune morphodynamics

A nonlinear and nonlocal conservative model (A.C Fowler, Oxford)

For all $t \in (0,T)$ and $x \in \mathbb{R}$,

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x(t,x) - u_{xx}(t,x) + \int_0^{+\infty} |\xi|^{-1/3} u_{xx}(t,x-\xi)\, d\xi = 0, \\ u(0,x) = u_0(x). \end{cases}$$

Remark : This nonlocal term also appears in the work of P.-Y. Lagrée (Paris VI).

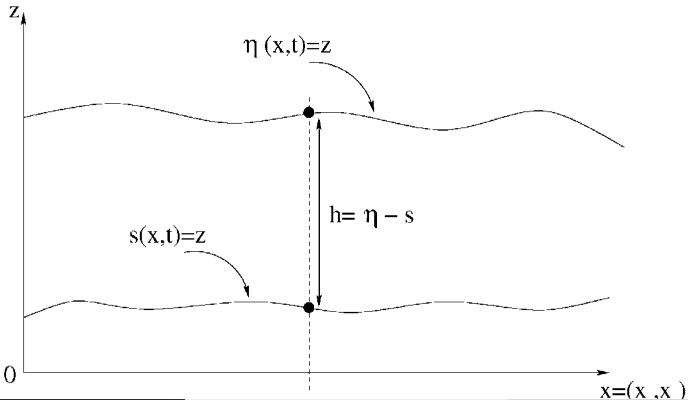
References :
P-Y Lagrée, *Asymptotic Methods in Fluid Mechanics : Survey and Recent Advances*, lecture notes 523, CISM International Centre for Mechanical Sciences Udine, H. STEINRÜCK Ed., Springer, (2010).
A.C. Fowler, *Mathematics and environment*, lecture note, 2006.

Exner equation

Exner equation

$$\frac{\partial s}{\partial t} + \frac{1}{1-\lambda_p}\frac{\partial q}{\partial x} = 0$$

where λ_p is the porosity of the bed and q the sediment transport flux.



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Exner equation

$$\frac{\partial s}{\partial t} + \frac{1}{1-\lambda_p}\frac{\partial q}{\partial x} = 0$$

where λ_p is the porosity of the bed and q the sediment transport flux.

- Meyer-Peter & Müller :

$$q \propto [\tau - \tau_c]_+^{3/2}$$

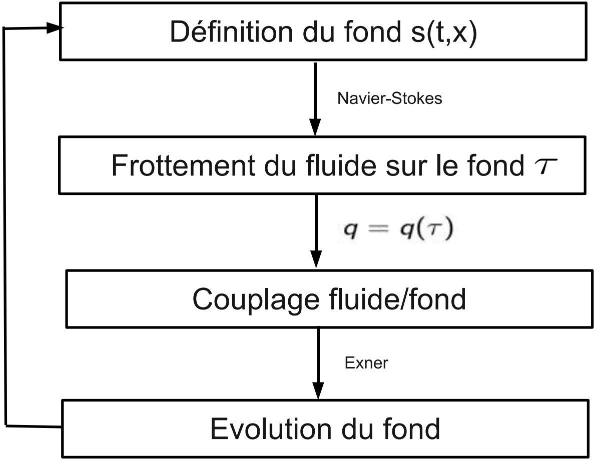
- Grass :

$$q = Au|u|^{m-1},$$

  où $1 \leq m \leq 4$.

- ...

Processus d'interaction



Kernel of $\mathcal{I} - \partial_{xx}^2$

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$K(t,\cdot) = \mathcal{F}^{-1}\left(e^{-t\psi_{\mathcal{I}}}\right)$ with $\psi_{\mathcal{I}}(\xi) = 4\pi^2\xi^2 - a_{\mathcal{I}}|\xi|^{4/3} + b_{\mathcal{I}} i\xi|\xi|^{1/3}$

Properties of K

"Nice" properties

- C^0**-semi-group :**

$$K(t) * K(s) = K(t+s)$$
$$\forall u_0 \in L^2(\mathbb{R}), \lim_{t\to 0} K(t) * u_0 = u_0$$

- **Regularity**

$$K \in C^\infty((0,+\infty)\times\mathbb{R})$$

- **Estimates for the gradient :**

$$||\partial_x K(t)||_{L^2} \leq Ct^{-3/4}$$
$$||\partial_x K(t)||_{L^1} \leq Ct^{-1/2}$$

"Bad" property

- $\forall t > 0, \quad K(t,\cdot)$ is not positive

Splitting method

$$\begin{cases} u_t(t,x) + \left(\frac{u^2}{2}\right)_x (t,x) - u_{xx}(t,x) + \mathcal{I}[u(t,\cdot)](x) = 0, \\ u(0,x) = u_0(x). \end{cases}$$

$Notation : u(t,\cdot) = S^t u_0$

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

$$\begin{cases} v_t + \left(\frac{v^2}{2}\right)_x - \epsilon\, v_{xx} = 0 \\ v(0,\cdot) = v_0, \end{cases}$$

Notation : $v(t,\cdot) = Y^t v_0$

$$\begin{cases} w_t + \mathcal{I}[w] - \eta\, w_{xx} = 0 \\ w(0,\cdot) = w_0, \end{cases}$$

Notation : $w(t,\cdot) = X^t w_0$

Lie method : $Z_L^t = X^t Y^t$

Splitting method

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Note : H. Holden, C. Lubich, N.-H. Risebro ; *Operator splitting for partial differential equations with Burgers nonlinearity*, to appear, Math. Comp (2012).

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Lie method : $Z_L^t = X^t Y^t$

$\Downarrow$ Finite difference method

$\Downarrow$ FFT

Numerical stability

- Stable under the CFL-Peclet condition

$$\Delta t = \min\left(\frac{\Delta x}{|u|}, \frac{\Delta x^2}{2\epsilon}\right)$$

Expressions for the flows

- Linear flow

$$X^t v_0 = D(t,\cdot) * v_0,$$

where $D(t,\cdot) = \mathcal{F}^{-1}\left(e^{-t\,\phi_{\mathcal{I}}}\right)$ with $\phi_{\mathcal{I}}(\xi) = 4\pi^2\eta\xi^2 - a_{\mathcal{I}}|\xi|^{4/3} + b_{\mathcal{I}}\xi|\xi|^{1/3}$

- Nonlinear flow (viscous Burgers' equation)

$$Y^t w_0 = G(t,\cdot) * w_0 - \frac{1}{2}\int_0^t \partial_x G(t-s,\cdot) * \left(Y^s w_0\right)^2\, ds,$$

where G is the heat kernel.

- Exact flow

$$S^t u_0 = K(t,\cdot) * u_0 - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * \left(S^s u_0\right)^2\, ds$$

- Splitting operator

$$Z_L^t u_0 = K(t) * u_0 - \frac{1}{2}\int_0^t D(t) * G(t-s) * \partial_x \left(Y^s u_0\right)^2 ds$$

L^2 local error estimate

Proposition

Let $u_0 \in H^3(\mathbb{R})$*. There exists* $C\left(\|u_0\|_{L^2(\mathbb{R})}\right)$ *such that for all* $t \in [0,1]$*,*

$$\|Z_L^t u_0 - S^t u_0\|_{L^2(\mathbb{R})} \leqslant C\left(\|u_0\|_{L^2(\mathbb{R})}\right) t^2 \|u_0\|^2_{H^3(\mathbb{R})}.$$

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Ingredients of the proof

- $Z_L^t u_0 - S^t u_0 = \frac{1}{2}\int_0^t \partial_x K(t-s) * \left((S^s u_0)^2 - (Z_L^s u_0)^2\right)\, ds + R(t)$,
- The remainder $R(t)$ is written as $R(t) = \frac{1}{2}\int_0^t R_1(s) ds$, with

$$R_1(s) = \partial_x K(t-s) * (Z_L^s u_0)^2 - D(t) * \partial_x G(t-s,\cdot) * (Y^s u_0)^2,$$

and satisfies :

$$\|R(t)\|_{L^2(\mathbb{R})} \leqslant C\left(\|u_0\|_{L^2(\mathbb{R})}\right)\ t^2 \|u_0\|^2_{H^3(\mathbb{R})}.$$

Modified fractional Gronwall Lemma

Lemma

Let $\phi : [0,T] \to \mathbb{R}_+$ *be a bounded measurable function and* P *be a polynomial with positive coefficients and no constant term. We assume there exists two positive constants* C *and* $\theta \in]0,1[$ *such that for all* $t \in [0,T]$,

$$0 \leqslant \phi(t) \leqslant \phi(0) + P(t) + C\frac{d^{-\theta}}{dt^{-\theta}}\phi(t).$$

Then there exists $C_T(\theta)$ *such that for all* $t \in [0,T]$,

$$\phi(t) \leqslant C_T(\theta)\,\phi(0) + C_T(\theta)\,P(t).$$

Riemann-Liouville operator :

$$\frac{d^{-\theta}}{dt^{-\theta}}\phi(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}\phi(s)\,ds.$$

We will also need the fact that the flow map S^t is uniformly Lipschitzean on balls of $H^2(\mathbb{R})$.

Proposition

Let $T, R > 0$*. There exists* $K = K(R,T) < \infty$ *such that if*

$$\|u_0\|_{H^2(\mathbb{R})} \leqslant R,\ \|v_0\|_{H^2(\mathbb{R})} \leqslant R,$$

then

$$\|S^t u_0 - S^t v_0\|_{L^2(\mathbb{R})} \leqslant K \|u_0 - v_0\|_{L^2(\mathbb{R})}, \quad \forall t \in [0,T].$$

Convergence

Theorem

For all $u_0 \in H^3(\mathbb{R})$ *and for all* $T > 0$, *there exist positive constants* c_1, c_2 *and* Δt_0 *such that for all* $\Delta t \in]0, \Delta t_0]$ *and for all* $n \in \mathbb{N}$ *such that* $0 \leqslant n\Delta t \leqslant T$,

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2(\mathbb{R})} \leqslant c_1 \, \Delta t \quad \text{and} \quad \|(Z_L^{\Delta t})^n u_0\|_{H^3(\mathbb{R})} \leqslant c_2.$$

Here, c_1, c_2 *and* Δt_0 *depend only on* T, $\rho = \max_{t\in[0,T]} \|S^t u_0\|_{H^2(\mathbb{R})}$, *and* $\|u_0\|_{H^3(\mathbb{R})}$.

B.A., Carles R., *Splitting methods for the nonlocal Fowler equation*, Math. Comp, to appear.

Proof

- The proof follows the same idea as in [1,2] for instance.
- We prove by induction that there exists $\gamma, \Delta t_0$ such that if $0 < \Delta t \leqslant \Delta t_0$, for all $n \in \mathbf{N}$ with $n\Delta t \leqslant T$,

$$\|(Z_L^{\Delta t})^n u_0\|_{L^2(\mathbb{R})} \leqslant 2\rho, \quad \|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2(\mathbb{R})} \leqslant \gamma \Delta t.$$

- The triangle inequality yields

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} \leqslant \sum_{k=0}^{n-1} \left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2},$$

with $u_k = (Z_L^{\Delta t})^k$

References :
[1] C. Besse, B. Bidegaray, S. Descombes; *Order estimates in time of splitting methods for the nonlinear Schrödinger equation*, SIAM J. Numer. Anal., 40 (2002).
[2] H. Holden, C. Lubich, N.-H. Risebro; *Operator splitting for partial differential equations with Burgers nonlinearity*, to appear, Math. Comp (2012).

Proof

- Lipschitz property of S^t yields

$$\left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2} \leq K \left\| Z_L^{\Delta t} u_k - S^{\Delta t} u_k \right\|_{L^2}$$

- From L^2 local error estimate, we infer

$$\left\| S^{(n-k-1)\Delta t}\left(Z_L^{\Delta t} u_k\right) - S^{(n-k-1)\Delta t}\left(S^{\Delta t} u_k\right)\right\|_{L^2} \leqslant CK(\Delta t)^2 \|u_0\|_{H^3}^2,$$

for some constant C.

- Therefore,

$$\|(Z_L^{\Delta t})^n u_0 - S^{n\Delta t} u_0\|_{L^2} \leqslant nCK(\Delta t)^2 \|u_0\|_{H^3}^2 \leqslant CTK\Delta t,$$

which yields the two estimates of the induction, provided one takes $\gamma = CTK$, which is uniform in n and Δt.

Numerical experiments : initial data

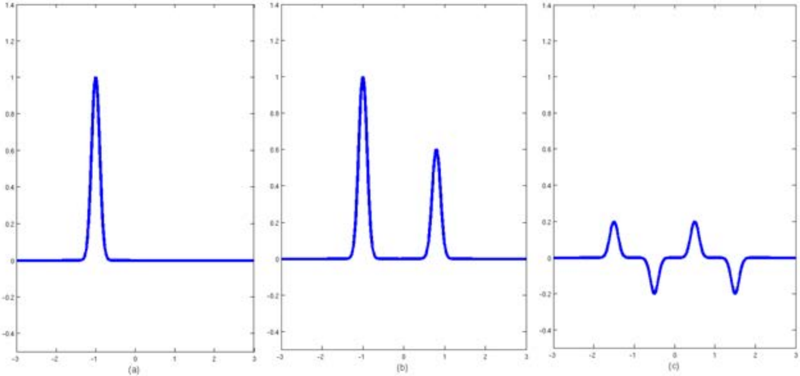


FIGURE: Initial data used for numerical experiments.

Numerical convergence for Lie operator

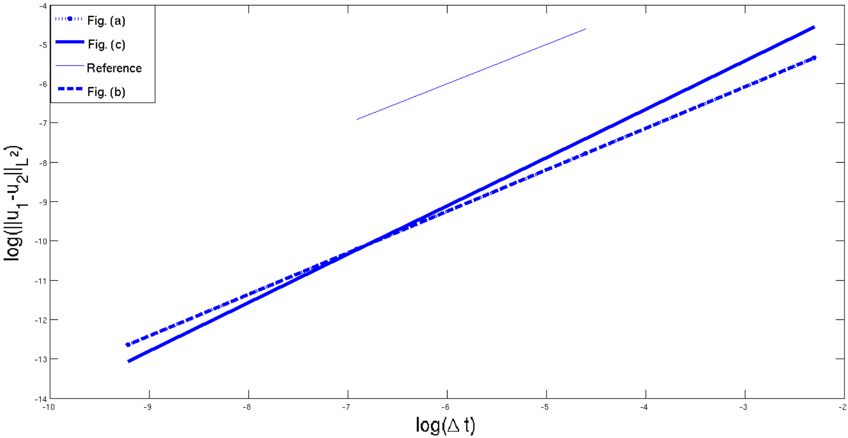


FIGURE: Lie method

Strang method

Strang operator

$$Z_S^t = X^{t/2} Y^t X^{t/2}$$

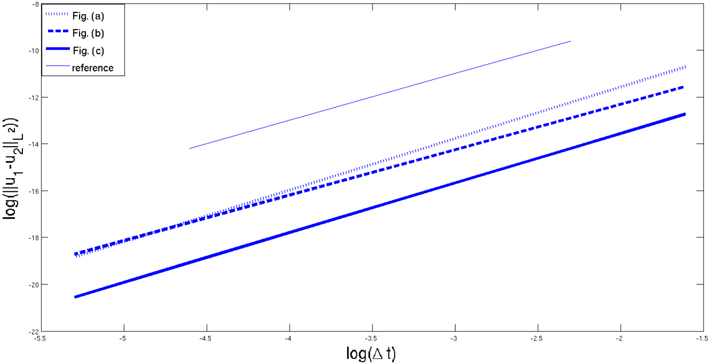


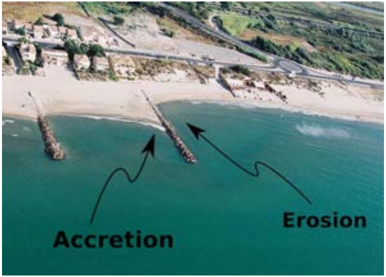
FIGURE: Strang method

Plan

1. Méthodes de splitting pour l'équation de Fowler

2. Les principes de minimisation appliqués à la dynamique littorale

Dans le passé

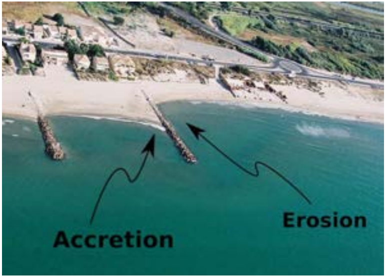


Référence :

ISÈBE, D., *Modélisation, simulation et optimisation en gènie côtier*, thèse (2007).

Dans le passé

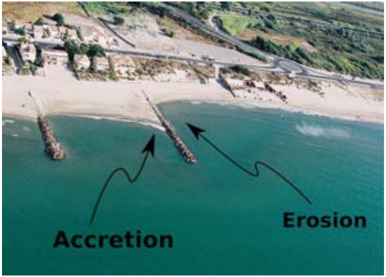
$\min_{x \in \Omega_{ad}} J(x)$, où $\Omega_{ad} \subset \mathbb{R}^N$.



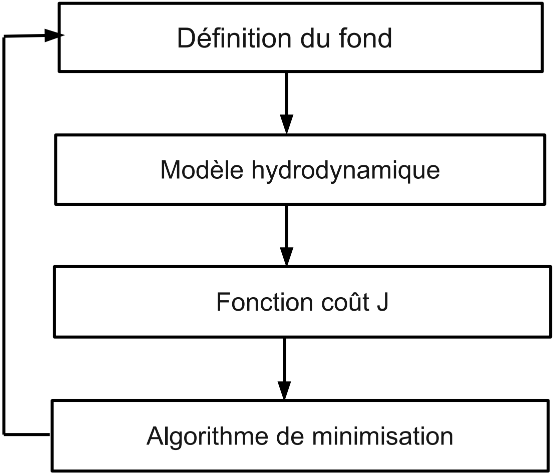
Dans le passé

$\min_{x \in \Omega_{ad}} J(x)$, où $\Omega_{ad} \subset \mathbb{R}^N$.

« le mouvement du fond décroît avec l'énergie de la houle ».



Stratégie pour la minimisation



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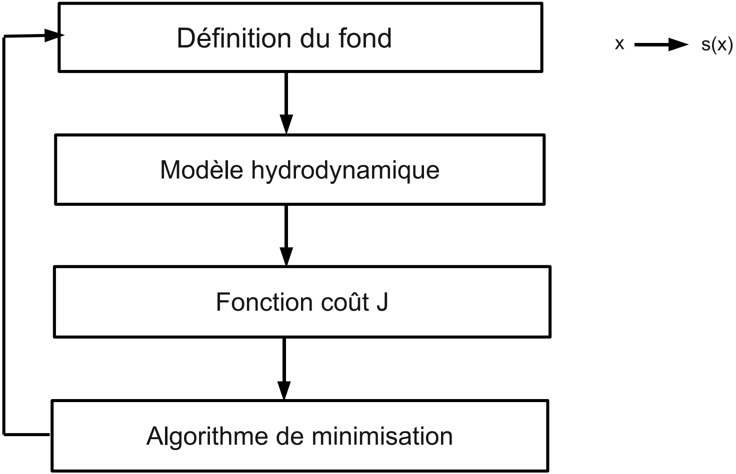
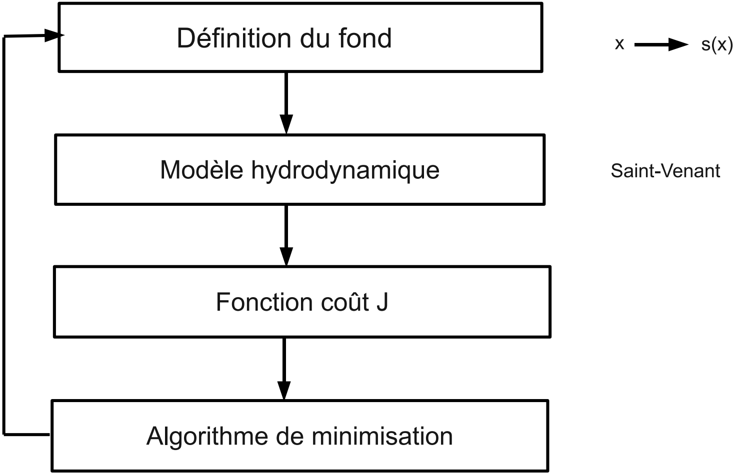


Stratégie pour la minimisation



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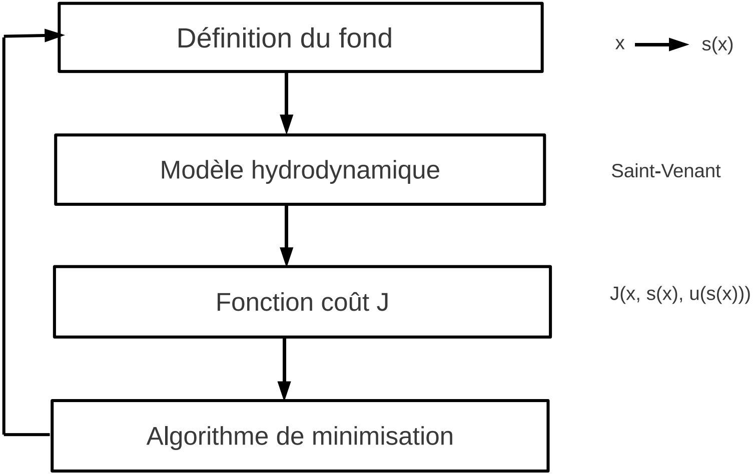


Shallow water equations

Equations de Saint-Venant

$$\begin{pmatrix} u \\ hu \\ hv \end{pmatrix}_t + \begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{pmatrix}_x + \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix}_y = \begin{pmatrix} 0 \\ -ghs_x \\ -ghs_y \end{pmatrix}$$

Fonction coût

La fonction coût J

$$J(s) = \int_{\Omega} \frac{1}{2}\rho_w g A^2 \; d\Omega + \int_{t-T}^{t} \int_{\Omega} \rho_s g(s(\tau) - s(t-T))^2 \; d\tau d\Omega$$

Données :

- $\Omega \subset \mathbb{R}^2$
- $s : \mathbb{R}^+ \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+$
- ρ_w (resp. ρ_s) densité de l'eau (resp. du sable)

Les algorithmes de minimisation

- Les algorithmes de minimisation :

$$x_{t+1} = x_t - \rho \nabla_x J(x_t)$$

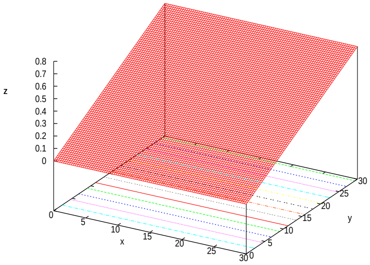
- Gradient :

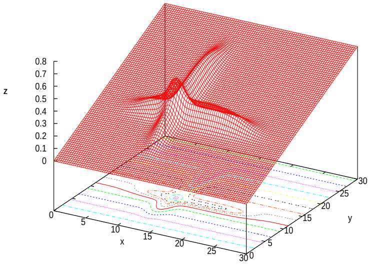
$$\frac{dJ}{dx} = \frac{\partial J}{\partial x} + \frac{\partial J}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial J}{\partial U}\frac{\partial U}{\partial s}\frac{\partial s}{\partial x}$$

- La méthode adjointe

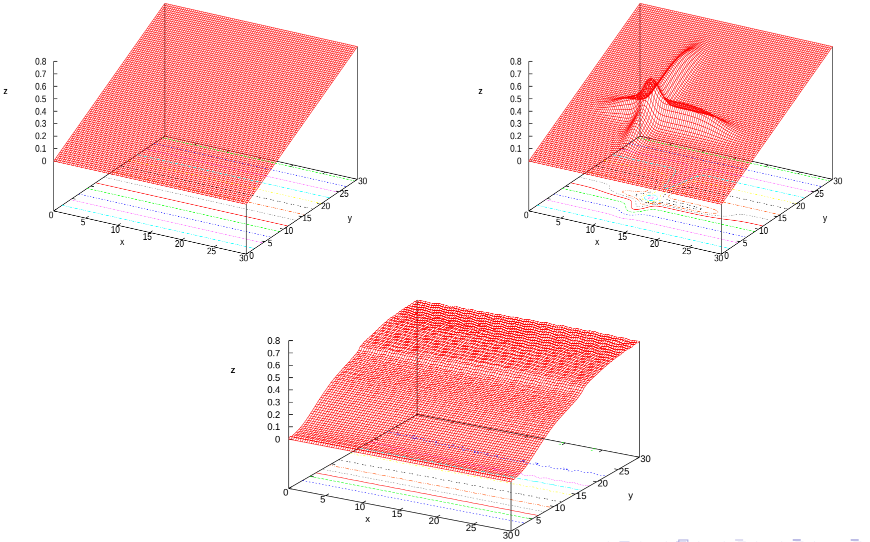
Tapenade, Equipe Tropics, INRIA Sophia-Antipolis

Simulation numérique

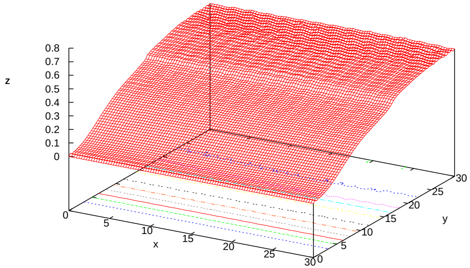


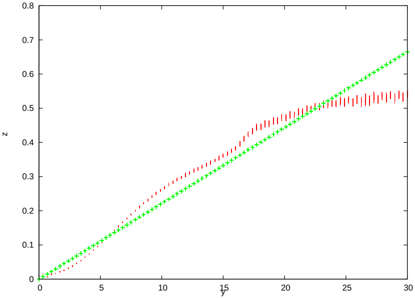


Simulation numérique



Simulation numérique

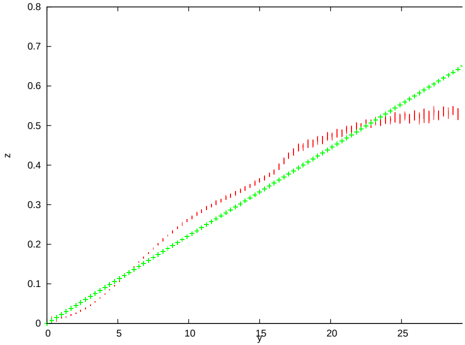




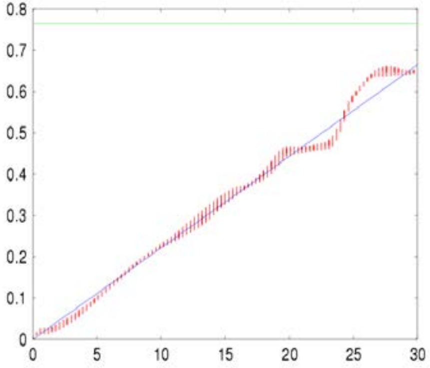
MOHAMMADI B. ; B. A., *Optimal dynamics of soft shapes in shallow waters*, Computers and Fluids, **40/1**, pp. 291-298 (2011).

B. A. ; MOHAMMADI B., *Minimization principles for the evolution of a soft sea bed interacting with a shallow sea*, IJCFD (2012).

Le rôle de ρ_s



$\rho_s = 1700\ kg/m^3$



$\rho_s = 2000\ kg/m^3$

Exner... le retour

- Minimiser $J \Leftrightarrow$ Résoudre le modèle suivant pour le fond s

$$s_t = -\rho \nabla_s J(\mathbf{U}),$$
$$s(t=0,\cdot) = s_0,$$

où $s : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$.

- Pour $\rho = 1/(1-\lambda_p)^n$ et $1 \leq n$,

Flux de matière transportée

$$q(t,x) = \int_{-\infty}^{x} (1-\lambda_p(\zeta))^{1-n} J_s(\zeta) d\zeta$$

Merci de votre attention