On Encoding LF in a Predicate Logic over Simply-Typed Lambda Terms

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Abstract

Felty and Miller have described what they claim to be a faithful encoding of the dependently typed λ -calculus LF in the logic of hereditary Harrop formulas, a sublogic of an intuitionistic variant of Church's Simple Theory of Types. Their encoding is based roughly on translating object expressions in LF into terms in a simply typed λ -calculus by erasing dependencies in typing and then recapturing the erased dependencies through the use of predicates. Unfortunately, this idea does not quite work. In particular, we provide a counterexample to the claim that the described encoding is faithful. The underlying reason for the falsity of the claim is that the mapping from dependently typed λ -terms to simply typed ones is not one-to-one and hence the inverse transformation is ambiguous. This observation has a broad implication for other related encodings.

A faithful encoding of a dependently typed λ -calculus within a predicate logic can be useful for a variety or reasons: it can help us understand the relative expressive power of the different systems, it can provide a means for implementing type checking in the λ -calculus, and it can be the basis for a framework for reasoning about specifications developed in the λ -calculus. An encoding that is motivated by such considerations has been provided by Felty and Miller [1] for the Edinburgh Logical Framework (LF) [2] in the logic of hereditary Harrop formulas [3]. This encoding has been claimed to be faithful: specifically, this claim is the content of Theorem 5.2 in [1]. Unfortunately, the mentioned theorem is false. We show this to be the case in this note by providing a counterexample to it.

Our counterexample is based on an LF signature that includes the following type-level constructors: nat: Type and $num: \Pi x: nat. Type$. Further, the signature includes the following object-level constants: z: nat and $c: \Pi w: (\Pi x: nat. \Pi y: (num\ x). nat). nat$. Given this signature, we may construct the object-level expression $(c\ (\lambda x: nat. \lambda y: num\ z.z))$. Our counterexample focuses on the typeability of this expression. More specifically, it considers the derivability of the LF judgement $\cdot \vdash (c\ (\lambda x: nat. \lambda y: num\ z.z)): nat$. It is easily seen that this judgement is in fact not derivable in LF but, as we show below, the encoding of Felty and Miller leads to a different conclusion.

The encoding of LF derivability questions in the logic of hereditary Harrop formulas works in three steps. In the first step, LF type and object expressions are translated into terms of two distinguished types in a simply typed λ -calculus. Specifically, the type ty is used for type expressions and tm is used for object expressions. To support this translation, the LF signature must be reflected into a suitable signature in the target language. In the example under consideration, this results in a signature with the following constants: $nat: ty, num: tm \to ty, z: tm$, and $c: (tm \to tm \to tm) \to tm$. Using this signature, the LF object-level expression under consideration would be represented by the simpy-typed λ -term $(c (\lambda x: tm. \lambda y: tm. z))$.

The translation carried out in the first step loses important typing information; in particular information about dependencies in typing is erased.¹ The second step of the encoding tries to capture the lost information through the use of a unary predicate *istype* that is intended to identify encodings of well-formed types and a binary predicate *hastype* that relates encodings of object expressions to those of (dependent) types. The realization of this step requires the LF signature to be translated into a collection of formulas that define these two predicates. Focusing only on the *hastype* predicate, something that suffices for our counterexample, this step yields the following formulas for this predicate in the context under consideration:

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hastype z nat, and \forall_{tm \to tm \to tm} w. (\forall_{tm} x. \text{hastype } x \text{ nat} \supset \\ \forall_{tm} y. \text{hastype } y \text{ (num } x) \supset \text{hastype (} w \text{ } x \text{ } y) \text{ nat)} \\ \supset \text{hastype (} c \text{ } w) \text{ nat}
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In what follows, we will use the notation of [1] in denoting the collection of formulas obtained by translating the signature Σ in our example by $\llbracket \Sigma \rrbracket$.

The last step in the encoding consists of posing the validity of an LF typing judgement as the derivability of a translated form of the typing judgement from the formulas obtained from translating the signature in the logic of hereditary Harrop formulas. In the particular situation under consideration, this reduces to considering the derivability of the formula

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hastype (c (\lambda x : tm. \lambda y : tm. z)) nat from the assumption formulas [\![\Sigma]\!].
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We now have all the pieces in place for our counterexample. It is easy to see that the formula identified by the translation is in fact derivable from $[\![\Sigma]\!]$, contrary to the earlier observation that the LF judgement that it is supposed to encode is not derivable. More specifically, this example indicates that Theorem 5.2 in [1] is false in the "if" direction: derivability in the encoded version does not imply derivability in LF. The reason for the mismatch should also be evident: the many-to-one nature of the encoding allows us to conclude that some LF typing judgement from which the translated version is obtained is valid but not that the specific LF judgement that is of interest is valid. A closer examination of the counterexample allows us to trace the problem even more specifically to the fact that some of the dependency information that is lost in the transformation of LF object expressions to simply typed λ -terms is not recovered by the predicate-level encoding.

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References

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¹The specific translation described loses more information than just dependencies in typing since it collapses all object types into the single type tm. However, the counterexample we present will remain one even under a translation that avoids this defect.

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