Course of Numerical Methods for Engineering
Lab 12

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Topic of this session:

- **Applications of the Singular Value Decomposition**

Frobenius norm of a matrix

Definition

For any $m \times n$ matrix A its Frobenius norm is defined by

$$\|\mathrm{A}\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{i,j}|^2}$$

Theorem

For any $m \times n$ matrix A with $m \geq n$, it holds

$$\|\mathrm{A}\|_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2(\mathrm{A})},$$

where $\sigma_i(\mathrm{A})$ denote the singular values of (A).

Theorem

Given a $m \times n$ matrix A with $m > n$ and with rank $r(A) = r \leq n$, let $A = U\Sigma V^T$ be its singular value decomposition and let $\hat{A} = U\hat{\Sigma}V^T$, where $\hat{\Sigma}$ is obtained from Σ setting $\sigma_{p+1}(A) = \cdots = \sigma_r(A) = 0$ for some $p < r$. One has then that

$$\|A - \hat{A}\|_F = \sqrt{\sum_{i=p+1}^{r} \sigma_i^2(A)}.$$

Application to ill-conditioned problems: when $\sigma_1/\sigma_n \gg 1$, take $\hat{A} = U\hat{\Sigma}V^T$, where $\hat{\Sigma}$ is obtained by dropping a sufficient number of singular values so as to make σ_1/σ_p sufficiently small, but such that

$$\|A - \hat{A}\|_F = \sqrt{\sum_{i=p+1}^{n} \sigma_i^2}$$

is still sufficiently small.

Exercise 1

Let $N = 400$ and consider the linear functional $f(\mathrm{y}) = \mathrm{v}^T\mathrm{y}$, where $\mathrm{y} \in \mathrm{R}^N$ and $\mathrm{v} = [1, 1, \ldots, 1, 1]^T \in \mathrm{R}^N$. Let the vectors e_i denote the canonical basis of R^N and consider the vector $\mathrm{b} \in \mathrm{R}^{2N-1}$ such that

$$b_i = f(\mathrm{e}_i) + \epsilon_\sigma, \quad i = 1, \ldots, N \qquad b_i = f(\mathrm{e}_i + \mathrm{e}_N) + \epsilon_\sigma \quad i = N+1, \ldots, 2N-1,$$

where ϵ_σ denotes a random variable with Gaussian distribution with zero mean and standard deviation σ. Build the matrix of the overdetermined system whose $i-$th equation is given by $\mathrm{e}_i^T\mathrm{x} = b_i \quad i = 1, \ldots, N \qquad (\mathrm{e}_i + \mathrm{e}_N)^T\mathrm{x} = b_i \quad i = N+1, \ldots, 2N-1$ in the case $\sigma = 0.01$ (use the Octave/MATLAB command `randn`).
Check that the associated normal equations have a unique solution. Compute the solution of the overdetermined system:

(a) **solving the normal equations by the Cholesky method;**

(b) **using the SVD decomposition approach.**

Compare all the numerical solutions obtained with the reference solution $\mathrm{x}_{ex} = \mathrm{v}$, by computing relative l_2 and l_∞ errors. Repeat the computation for $\sigma = 2$ and discuss the differences between the two cases.

Exercise 2

Repeat the previous exercise for the underdetermined system whose data are given by the vector $\mathrm{b} \in \mathrm{R}^{N/2}$ **such that**

$$b_i = f(\mathrm{e}_i) + \epsilon_\sigma, \quad i = 1, \dots, N/2.$$

Exercise 3

Consider the 100×100 **tridiagonal matrix** B **with values equal to 2 on the main diagonal and to -1 on the first sub and superdiagonal. Build the matrix**

$$\mathrm{A} = \exp\left(-5\mathrm{B}\right) = \sum_{k=0}^{\infty} \frac{\left(-5\mathrm{B}\right)^k}{k!}$$

using the MATLAB command `expm`**. Using the SVD of** A**, build an approximation** $\hat{\mathrm{A}}$ **of rank** 40 **and compute the relative error in the Frobenius norm. Display the singular values of** A **in a semi-logaritmic plot.**

Exercise 4

Consider the vector $\mathrm{v} = [0, 1/N, \ldots, (N-1)/N, 1]^T$ **with** $N = 50$ **and build the Vandermonde matrix** A **associated to** v **using the Octave/MATLAB command** `vander`**.**

(a) **Solve the linear system** $\mathrm{Ax} = \mathrm{b}$ **whose right-hand side is such that the exact solution is** $\mathrm{x}_{ex} = [1, 1, \ldots, 1, 1]^T \in \mathrm{R}^{N+1}$ **by the LU factorization approach with pivoting. Check that the** ***a priori*** **error estimate in the** l_2 **and** l_∞ **norms are verified.**

(b) **Using the SVD of** A**, build an approximation** $\hat{\mathrm{A}}$ **of rank** $p < N+1$ **such that** $\|\hat{\mathrm{A}} - \mathrm{A}\|_F / \|\mathrm{A}\|_F < 10^{-7}$ **and** $\sigma_1(\hat{\mathrm{A}})/\sigma_p(\hat{\mathrm{A}}) < 10^7$**. Display the singular values of** A **in a semi-logaritmic plot. Solve the resulting singular system in the minimum norm least square sense using the SVD of** $\hat{\mathrm{A}}$**. Compute the relative error in the** l_2 **norm.**

Exercise 5

Check whether it is possible to solve exercise 4 when starting from the vector

$$\mathsf{v} = [0, 1/N, \dots, (N-1)/N, 3]^T.$$

Exercise 6

Plot filled contours (Octave/MATLAB: `contourf`) of the function $f(x, y) = \sin(6\pi(x + y^2))$ in $[0, 1]x[0, 1]$ using an equispaced grid with $\Delta x = \Delta y = 0.01$. Encode the values of f in a matrix A.
Using the SVD of A, build an approximation $\hat{A}$ of rank 3, compute the relative error in the Frobenius norm, and visually compare the contour plots of A and $\hat{A}$.