Course of Numerical Methods for Engineering
Lab 4

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Topics of this session:

- **Matrix manipulation**
- **Numerical solution of linear systems**

Linear systems

A linear system can be written as

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,n}x_n &= b_2 \\ \dots & \\ a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,n}x_n &= b_n. \end{aligned}$$

In matrix notation, setting $\mathrm{x} = [x_1, \dots, x_n]^T$, $\mathrm{b} = [b_1, \dots, b_n]^T$, **and**

$$\mathrm{A} = \left[\begin{array}{cccc} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{array} \right],$$

The system can be rewritten in more compact form as

$$\mathrm{Ax} = \mathrm{b}$$

Exercise 1

Build the following matrices

$$A = \begin{bmatrix} 50 & 1 & 3 \\ 1 & 6 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 50 & 1 \\ 3 & 20 \\ 10 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}.$$

(a) **Compute** $D = I + BC$.

(b) **Check that** A **is different from** A^T. **Compute**

$$A_s = \frac{A + A^T}{2} \qquad A_{as} = \frac{A - A^T}{2}$$

and chech that $A_s = A_s^T$ **and** $A_{as} = -A_{as}^T$.

(c) **Check that** AD **is different from** DA. **and compute the commutator**

$$[A, D] = AD - DA.$$

(d) **Compute** $E = I + 2A^T A + 3A^3$. **Check that** E **is invertible by computing its determinant with the MATLAB function** `det` **and compute its inverse with the MATLAB function** `inv` **.**

(e) **Check that** E^{-1} **and** `inv`**(**E**) coincide up to roundoff errors.**

Exercise 2

Using the matrices defined in Exercise 1, build the matrices

$$A_1 = \begin{bmatrix} D & E \\ -E^T & D^{-1} \end{bmatrix}, \quad A_2 = \begin{bmatrix} I & 0 & A_1 \\ 2I & -A_1 & I \\ A_1^T & 0 & 3I \end{bmatrix},$$

where $I, 0$ **are the identity and the zero matrix of the appropriate dimension. Compute the dimensions of** A_1, A_2 **using the MATLAB function** `size`**.**

Exercise 3

(a) **Build the Toeplitz symmetric matrix**

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

a) using the MATLAB command `toeplitz` **b) applying repeatedly command** `diag` **c) using repeated for cycles.**

(b) **Extract the upper and lower triangular matrices contained in** A, **using the commands** `triu` **and** `tril`. **Check what happens with the commands** `tril(A,2)` **o** `triu(A,-2)`.

(c) **Compute the determinants of the upper and lower triangular matrices contained in** A. **Compute the eigenvalues of the upper and lower triangular matrices contained in** A **and check that the determinants are a) the product of the eigenvalues b) the product of the terms on the main diagonal.**

Exercise 4

Build a vector $v = [2, 4, 2, 4, \ldots, 2, 4]^T$, $v \in R^{100}$.

(a) **Build a matrix** A **that has** v **on the main diagonal, all components equal to -1 on the first superdiagonal and all components equal to 1 on the first subdiagonal.**

(b) **Compute**

$$B = \frac{-3A + 2A^2}{I + 4A - A^4}.$$

(c) **Check if** B **is invertible by computing its determinant with the MATLAB function** `det` **and compute its inverse with the MATLAB function** `inv`**.**

(d) **Compute the vector** $d = Bx_{ex}$, **where** $x_{ex} = [-1, -1, \ldots, -1, -1]^T$, $x_{ex} \in R^{100}$.

(e) **Solve the system** $Bx = d$ **using a) the inverse of** B **computed by** `inv` **b) the inverse of** B **computed as** B^{-1} **c) the command** $\backslash$ **(backslash).**

Exercise 5

Build the Hilbert matrix of dimension n, given by

$$A = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots & 1/n \\ 1/2 & 1/3 & 1/4 & \cdots & 1/(n+1) \\ 1/3 & 1/4 & 1/5 & \cdots & 1/(n+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 1/(n+1) & 1/(n+2) & \cdots & 1/(2n-1) \end{pmatrix}$$

using the command `hilb`, in the case $n = 7$.

(a) **Extract the third column of the matrix; substitute it with a column containing values equal to one. Repeat the procedure with the first three components of the fifth row of the matrix, using in each case a single MATLAB command.**

(b) **Extract all the diagonals of A using the command `diag`.**

(c) **Compute the determinant of matrix A.**

(d) **Compute the eigenvalues of matrix A.**

(e) **Compute the vector $b = Ax_{ex}$, where $x_{ex} = [1, 1, \ldots, 1, 1]^T$, $x_{ex} \in R^7$.**

(f) **Solve the system $Ax = b$ using a) the inverse of A computed by `inv` b) the inverse of A computed as A^{-1} c) the command \ (backslash). In each case, compute the absolute and relative error of the computed solution with respect to x_{ex} in the l^2 and l^∞ norm.**

Exercise 6

Build a matrix A **of dimension** 20×20 **such that:**

- **it has all the integers between 11 and 30 on the main diagonal**
- **it has all values equal to** π **on the second upper diagonal**
- **it has all values equal to 2 on the first lower diagonal**
- **it has all values equal to 5 on the tenth column (for the values that have not been defined yet)**
- **it has all values equal to zero elsewhere.**

Then:

1. **Compute the determinant of matrix** A. **Compute the eigenvalues of matrix** A.
2. **Compute the vector** $\mathrm{b} = \mathrm{A}\mathrm{x}_{ex}$, **where** $\mathrm{x}_{ex} = [1, -1, \ldots, 1, -1]^T$, $\mathrm{x}_{ex} \in \mathrm{R}^{20}$.
3. **Solve the system** $\mathrm{A}\mathrm{x} = \mathrm{b}$ **using a) the inverse of** A **computed by** `inv` **b) the inverse of** A **computed as** A^{-1} **c) the command** \ **(backslash). In each case, compute the absolute and relative error of the computed solution with respect to** x_{ex} **in the** l^2 **and** l^∞ **norm.**

Exercise 7

Build a block-diagonal matrix A **of dimension** 1000×1000 **using blocks of dimension** 5×5 **around the main diagonal all equal to**

$$\mathrm{B} = \begin{pmatrix} 10 & 3 & 4 & 1 & -1 \\ 1 & 30 & 3 & 4 & 5 \\ 1 & 2 & 50 & 4 & 5 \\ 1 & 2 & 3 & 30 & 5 \\ 1 & 2 & 3 & 4 & 10 \end{pmatrix}$$

Then:

- **Compute the determinant and the eigenvalues of the matrix** A.
- **Compute the vector** $\mathrm{b} = \mathrm{A}\mathrm{x}_{ex}$, **where** $\mathrm{x}_{ex} = [-2, 1, \ldots, -2, 1]^T$, $\mathrm{x}_{ex} \in \mathrm{R}^{1000}$.
- **Solve the system** $\mathrm{A}\mathrm{x} = \mathrm{b}$ **using a) the inverse of** A **computed by** `inv` **b) the inverse of** A **computed as** A^{-1} **c) the command** $\backslash$ **(backslash). In each case, compute the absolute and relative error of the computed solution with respect to** x_{ex} **in the** l^2 **and** l^∞ **norm.**