Course of Numerical Methods for Engineering
Lab 5

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MOX - Politecnico di Milano

PHYS-ENG, A.Y. 2020-21

12-13/10/2020

Topics of this session:

- **LU factorization**

LU factorization

Given the linear system $Ax = b$, we are interested in finding suitable matrices B and C such that if the matrix is factorized as $A = BC$, the solution procedure is cheaper than $O(n!)$.

Definition

*(**LU factorization**) An $n \times n$ square matrix* A *admits an LU factorization if* $\mathsf{A} = \mathsf{LU}$*, where* L *is a lower triangular matrix and* U *is an upper triangular matrix.*

Forward and backward substitution

Definition

(**Forward substitution algorithm**) *Consider a lower triangular system* $\mathrm{Lx} = \mathrm{b}$, *such that* $l_{i,i} \neq 0, i = 1, \ldots, n$. *Its solution can be computed by the following steps:*

1) *compute* $x_1 = b_1 / l_{1,1}$;
2) *for* $k = 2, \ldots, n$, *compute*

$$x_k = \frac{b_k - \sum_{s=1}^{k-1} l_{k,s} x_s}{l_{k,k}}.$$

Definition

(**Backward substitution algorithm**) *Consider an upper triangular system* $\mathrm{Ux} = \mathrm{b}$, *such that* $u_{i,i} \neq 0,\ i = 1, \ldots, n$. *Its solution can be computed by the following steps:*

1) *compute* $x_n = b_n / l_{n,n}$;
2) *for* $k = n-1, \ldots, 1$ *compute*

$$x_k = \frac{b_k - \sum_{s=k+1}^{n} u_{k,s} x_s}{u_{k,k}}.$$

Exercise 1

1. **Write two functions `fwsub.m` and `bksub.m` in Octave/MATLAB implementing the forward substitution and backward substitution algorithms.**
2. **Apply them to solve the linear systems with lower and upper triangular part corresponding to the lower and upper triangular part of matrix**

$$A = \begin{pmatrix} 10 & 2 & 3 & 4 & 5 \\ 2 & 10 & 2 & 3 & 4 \\ 3 & 2 & 10 & 2 & 3 \\ 4 & 3 & 2 & 10 & 2 \\ 5 & 4 & 3 & 2 & 10 \end{pmatrix}$$

and with right hand sides such that their exact solutions are both given by the vector $x_{ex} = [1, 1, 1, 1, 1]^T$.

Solution with LU

Definition

*(**General permutation matrix**)* *The $n \times n$ matrix* $\mathbf{P}$ *is a general permutation matrix if it is obtained from the $n \times n$ identity matrix* $\mathbf{I}$ *by a generic permutation of its columns.*

Definition

*(**Solution algorithm based on LU factorization with row pivoting**)* *Consider a linear system* $\mathrm{Ax} = \mathrm{b}$ *and assume that there is a permutation matrix* $\mathbf{P}$ *such that* $\mathrm{PA} = \mathrm{LU}$, *where* L, U *are lower and upper triangular matrices with nonzero elements on the main diagonal.*

The system $\mathrm{Ax} = \mathrm{b}$ *is then equivalent to the system* $\mathrm{PAx} = \mathrm{Pb}$, *whose solution* x *can be computed by solving in sequence the two triangular systems*

$$\mathrm{Ly} = \mathrm{Pb}$$

$$\mathrm{Ux} = \mathrm{y}$$

Exercise 2

1. **Compute the LU factorization of the matrix** A **in the previous exercise using the MATLAB command** `lu`**.**
2. **Compute the determinant of** A **by taking the product of the values on the main diagonal of its** U **factor and compare the result with that of the MATLAB command** `det`**.**

Exercise 3

Solve the linear systems with matrices defined in exercises 4-7 of session 4, using the LU decomposition method implemented in the command `lu`, and

1. **the `fwsub, bksub` functions implemented in Exercise 1;**
2. **the \ (backslash) command**

to solve the resulting lower triangular and upper triangular systems.

In each case,

(a) **Determine whether or not pivoting has been carried out checking the structure of the permutation matrix P.**

(b) **Compute the absolute and relative error of the computed solution with respect to x_{ex} in the l^2 and l^∞ norm.**