



UNIVERSIDAD DE CHILE  
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS  
DEPARTAMENTO DE FÍSICA

DYNAMICS OF PARTICLE-LIKE SOLUTIONS IN NON-LOCAL SYSTEMS

TESIS PARA OPTAR AL GRADO DE  
MAGÍSTER EN CIENCIAS MENCIÓN FÍSICA

MARTIN LUKAS BATAILLE GONZALEZ

PROFESOR GUÍA:  
MARCEL CLERC GAVILAN

MIEMBROS DE LA COMISIÓN:  
MARCEL CLERC  
KARIN ALFARO  
OLEH OMEL'CHENKO  
IGNACIO BORDEU  
MUSTAPHA TLIDI

SANTIAGO DE CHILE  
2023

# Resumen

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

# Abstract

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Nulla malesuada porttitor diam. Donec felis erat, congue non, volutpat at, tincidunt tristique, libero. Vivamus viverra fermentum felis. Donec nonummy pellentesque ante. Phasellus adipiscing semper elit. Proin fermentum massa ac quam. Sed diam turpis, molestie vitae, placerat a, molestie nec, leo. Maecenas lacinia. Nam ipsum ligula, eleifend at, accumsan nec, suscipit a, ipsum. Morbi blandit ligula feugiat magna. Nunc eleifend consequat lorem. Sed lacinia nulla vitae enim. Pellentesque tincidunt purus vel magna. Integer non enim. Praesent euismod nunc eu purus. Donec bibendum quam in tellus. Nullam cursus pulvinar lectus. Donec et mi. Nam vulputate metus eu enim. Vestibulum pellentesque felis eu massa.

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetur adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

*My biggest enemy is me ever since day one.*  
- Stefani Germanotta.

# Acknowledgments

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetur id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminary concepts</b>	<b>3</b>
2.1	Dynamical System . . . . .	3
2.2	Bifurcations . . . . .	3
2.2.1	Saddle-Node bifurcation . . . . .	3
2.2.2	Pitchfork bifurcation . . . . .	3
2.2.3	Hopf bifurcation . . . . .	3
2.3	Localized Structure. . . . .	3
2.4	Chimera states . . . . .	3
<b>3</b>	<b>Numerical Continuation</b>	<b>4</b>
3.1	Natural parameter continuation . . . . .	5
3.2	Pseudo-arclength continuation . . . . .	6
3.3	Continuation of traveling states . . . . .	8
3.4	Continuation for periodic orbits . . . . .	10
<b>4</b>	<b>Moving Solitons in the Lugiato-Lefever equation.</b>	<b>12</b>
4.1	Lugiato-Lefever equation. . . . .	12
4.2	Raman effect. . . . .	13
4.3	Isolas and traveling solitons. . . . .	13
4.4	A reduced model. . . . .	15

4.5	Oscillatory bound states. . . . .	15
<b>5</b>	<b>Moving spiral wave chimeras</b>	<b>17</b>
<b>6</b>	<b>Conclusions</b>	<b>20</b>
	<b>Bibliography</b>	<b>22</b>
<b>Appendix A</b>	<b>Anexo</b>	<b>23</b>

# Chapter 1

## Introduction

Las partículas o corpúsculos han sido un concepto fundamental y transversal en la física. Clásicamente se describen como un punto material con una masa y posición bien definida. Sin embargo, con el desarrollo de la mecánica cuántica, hemos aprendido que las partículas microscópicas son soluciones localizadas de un campo correspondiente a la amplitud de la probabilidad. Por otro lado, en sistemas macroscópicos fuera del equilibrio, producto del balance entre la inyección y disipación de energía, estos sistemas pueden exhibir soluciones localizadas que, en analogía con el caso anterior, usualmente se denominan soluciones tipo partícula [?]. Estas estructuras localizadas han sido observadas en diversos sistemas de dinámica de fluidos, óptica, química e incluso ecología [?, ?]. Dependiendo del contexto físico en que se observan también reciben el nombre de solitones disipativos, patrones localizados, quimeras, entre otros.

Tradicionalmente, la descripción matemática de las estructuras localizadas se ha realizado usando modelos de reacción difusión [?, ?, ?]. Como el nombre sugiere el acoplamiento espacial ocurre mediante un término de difusión y, por lo tanto, es puramente local. No obstante, diversos sistemas ópticos, neuronales e incluso en vegetación presentan un acoplamiento más complejo y de largo alcance, usualmente llamado acoplamiento no local [?, ?, ?]. En estos casos se ha encontrado que el término no local es responsable de la estabilización de las soluciones tipo partícula [?, ?] y por tanto es fundamental en el entendimiento de estas soluciones.

**En esta tesis nos enfocaremos en la dinámica de las estructuras localizadas en sistemas no locales.** En particular, buscaremos entender los mecanismos que permiten la propagación de estas soluciones tipo partícula. Para lograrlo, estudiaremos dos sistemas diferentes: solitones disipativos en una cavidad de cristal de fibra fotónico [?], y quimeras espirales en redes bidimensionales de osciladores de fase heterogéneos.

En el primer caso, estudiaremos solitones brillantes y oscuros en resonadores de cavidad de cristal de fibra óptica. Los solitones disipativos temporales han recibido una enorme atención estas últimas décadas por su capacidad de generar *frequency combs* o peines de frecuencia que han revolucionado diversas áreas de la ciencia y tecnología, en particular la espectroscopía de alta precisión y metrología [?, ?]. La mayor parte de estos esfuerzos científicos se han centrado en la generación de solitones mediante un balance entre la no linealidad Kerr del



material y el acoplamiento local temporal por ejemplo debido a la dispersión [?]. No obstante, en materiales amorfos como lo son los cristales de fibra óptica emerge un acoplamiento no local temporal debido a una respuesta retardada del material a la excitación electromagnética que se conoce como scattering estimulado de Raman [?, ?]. Gracias al efecto Raman es posible la estabilización de estas estructuras localizadas [?]. En este trabajo, buscaremos estudiar cómo son afectadas las estructuras localizadas debido al efecto Raman y la no linealidad Kerr, y en particular caracterizar precisamente cómo se auto-organizan estas soluciones en función de los parámetros del sistema.

Recientemente hemos podido reducir el modelo paradigmático de Lugiato-Lefever en torno a la emergencia de la biestabilidad encontrando así la ecuación de Swift-Hohenberg. De forma preliminar, hemos encontrado solitones brillantes y oscuros en la ecuación de Swift-Hohenberg con efecto Raman, en la región de coexistencia del estado homogéneo y el estado patrón. Además, debido al acoplamiento no local por efecto Raman, la simetría del sistema se rompe, lo que nos permite encontrar una propagación de las estructuras localizadas y una desconexión de las ramas de solución dando origen a una cadena de isolas, ver figura.

En el segundo caso, analizaremos estados ligados de dos quimeras espirales en redes de osciladores acoplados espacialmente de forma no local. Las quimeras espirales se caracterizan por tener un núcleo incoherente donde los osciladores están desincronizados, rodeado por una estructura coherente en forma de espiral donde los osciladores están sincronizados. Estas soluciones fueron reportadas por primera vez por Kuramoto y Shima hace dos décadas al acoplar osciladores de forma no local [?]. Durante estas últimas dos décadas han sido ampliamente estudiadas en diversos sistemas neuronales, eléctricos e incluso en ecología [?, ?, ?], además de ser observadas experimentalmente en redes de osciladores químicos [?, ?]. Sin embargo, la mayor parte de estos estudios se han centrado en quimeras estacionarias y poco se conoce sobre las quimeras propagativas [?, ?]. En trabajos preliminares, encontramos un nuevo tipo de quimera espiral: un estado ligado de dos espirales propagativas que pueden moverse en una línea recta o siguiendo una trayectoria más compleja.

Mediante simulaciones preliminares hemos logrado encontrar 3 clases de quimeras espirales propagativas: simétricas, asimétricas y cicloidales, junto con su región de estabilidad correspondiente, ver figura ???. Las quimeras simétricas poseen simetría de reflexión y se propagan en la dirección de su eje de simetría mientras que en las quimeras asimétricas, una espiral se vuelve más grande que la otra y se propagan en una dirección inclinada. Por último las espirales cicloidales presentan, además de un movimiento de traslación, una oscilación de sus núcleos que también suele llamarse *meandering* [?].

# Chapter 2

## Preliminary concepts

### 2.1 Dynamical System

### 2.2 Bifurcations

#### 2.2.1 Saddle-Node bifurcation

[insert example]

#### 2.2.2 Pitchfork bifurcation

[insert example]

#### 2.2.3 Hopf bifurcation

[insert example]

### 2.3 Localized Structure.

### 2.4 Chimera states

# Chapter 3

## Numerical Continuation

As stated in the previous section, we will be interested in following the solution branches as it experiences several bifurcations. Since an analytical treatment is not usually available for the problems studied in this work, we must resort / are required to numerical methods. Namely, we will implement and develop a numerical continuation algorithm. These type of methods aim to solve a nonlinear equation or, more generally, a system of nonlinear equations in order to find the desired steady states of a dynamical system as parameters are changed. This task corresponds to finding the roots (zeros) of a vector function  $\mathbf{F}$ , as in Eq. (3.1). We will assume we previously know a solution  $\mathbf{u}_0$  for a certain parameter  $\eta_0$ , obtained for example through numerical simulation.

[It will be assumed ... + formalidad uso de that, we will assume that we previously know a solution... Borrar recall newton's method...]

$$0 = \mathbf{F}(\mathbf{u}, \eta) \quad (3.1)$$

Although there are several methods for finding the roots of a vector function, in this thesis we will only use Newton's method because of its fast (quadratic) convergence and simplicity. This method corresponds to an iterative algorithm that given an initial guess  $\mathbf{u}_0$  will perform successive iterations until a certain accuracy or tolerance is reached. Each iteration is computed using Eqs (3.2-3.3), where  $\mathbf{J}(\mathbf{u}_i, \eta)$  is the Jacobian of  $\mathbf{F}$ . Moreover, since we have access to the Jacobian in Newton's method, the stability of the solution can be tracked and bifurcation points can be detected by tracking changes in the sign of the determinant of the Jacobian.

$$\mathbf{J}(\mathbf{u}_i, \eta) \Delta \mathbf{u}_{i+1} = -\mathbf{F}(\mathbf{u}_i, \eta) \quad (3.2)$$

$$\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta \mathbf{u}_{i+1} \quad (3.3)$$

### 3.1 Natural parameter continuation

The simplest way to perform numerical continuation is to fix the value of the parameter, in this case  $\eta$ , and solve the equation (or system of equations) by means of Newton's method. Then, one can increase the parameter by a small step  $\eta = \eta_0 + \Delta\eta$  and find the new solution using the previous solution  $\mathbf{u}_0$  as initial guess for Newton's method. Finally, is repeated the process until the whole solution branch has been computed. This method is usually called *Natural Parameter Continuation* [2].

**Example 3.1.1.** In order to illustrate the method lets consider a simple example: finding the homogeneous solutions to the Swift-Hohenberg equation. More specifically, the stable and unstable equilibria of Eq. (3.4) will be computed.

$$\dot{u} = \eta + \varepsilon u - u^3 \quad (3.4)$$

This task corresponds to finding the roots of a cubic polynomial:  $F(u, \eta) = \eta + \varepsilon u - u^3 = 0$ . The derivative can be determined easily:  $J(u, \eta) = \varepsilon - 3u^2$ .

Fig ... illustrates / shows the case where  $\varepsilon = 0.1$ , for which the system is bistable. Indeed, if we start the algorithm at  $\eta = -0.02$  taking as initial guess  $u_0 = -0.4$  and perform a forward sweep (slowly increasing  $\eta$ ), and then repeat the process backwards we obtain Fig. (3.2).

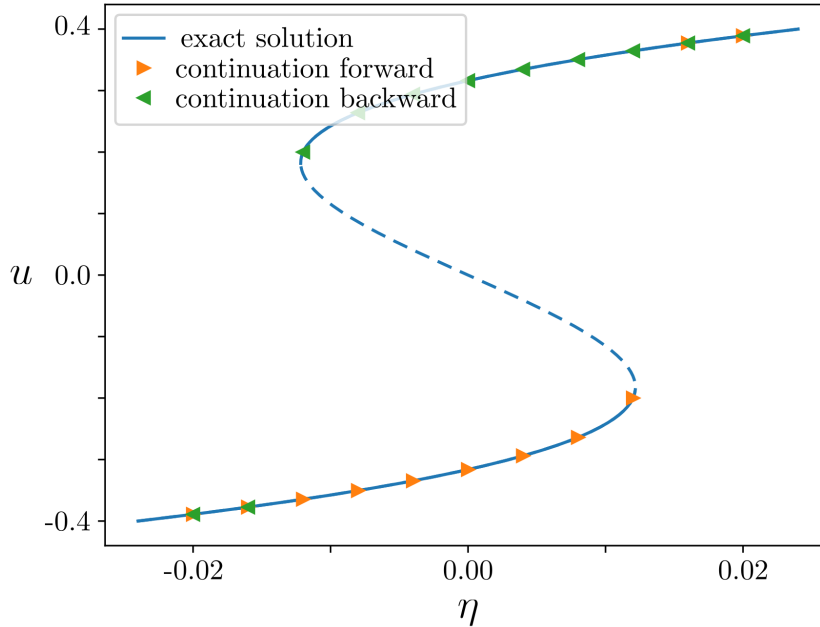


Figure 3.1: Solution of Eq. (3.4) as a function of the parameter  $\eta$  obtained through natural continuation (orange and green triangles) compared to the exact solution (blue curve).

Note that the natural continuation succeeds at finding the lower and upper branch of the solution in this example. However, it could not follow the branch past the fold (or saddle-node) bifurcation. The only way to access the middle branch using this algorithm would be to use an adequate initial guess close to the middle branch. Although in this case, it is not

difficult to find this guess, for a higher-dimensional system where the bifurcation scenario is more complicated, this quickly becomes impractical. To overcome this limitation, one can implement a more robust continuation scheme: the *pseudo-arclength continuation*.

## 3.2 Pseudo-arclength continuation

As shown in the previous example,  $\eta$  is not necessarily the good parameter to describe the curve, as it does not allow us to follow the branch through a fold point. A different approach can be taken where we parametrize the solution branch by a different parameter:  $s$ , which is somewhat similar to the arc-length. Therefore, our goal is to obtain a set of points  $\mathbf{y}(s) = (\mathbf{u}(s), \eta(s))$ . Then the *pseudo-arclength* algorithm [4] consists essentially of two steps that ensure that the branch is followed through folds.

1. *Predictor step.* Extrapolate a distance  $\Delta s$  along the tangent  $\boldsymbol{\tau}_0$  from a previously known point  $(\mathbf{u}_0, \eta_0)$  in the  $(\mathbf{u}, \eta)$  space, to obtain the predicted point (the point used as initial guess).

$$\mathbf{y} = \mathbf{y}_0 + \boldsymbol{\tau}_0 \Delta s$$

2. *Corrector step.* Force the solution to stay in the plane perpendicular to the tangent. Or, equivalently, that the solution projected onto the tangent has length  $\Delta s$ .

$$(\mathbf{y} - \mathbf{y}_0) \cdot \boldsymbol{\tau}_0 = \Delta s$$

A new object is introduced-... We have introduced in these steps a new object: the tangent  $\boldsymbol{\tau}$  of the solution curve  $\mathbf{y}(s)$ , which is defined as,

$$\boldsymbol{\tau} = \frac{d}{ds} \mathbf{y} = \left( \frac{d\mathbf{u}}{ds}, \frac{d\eta}{ds} \right) \quad (3.5)$$

An additional step must therefore be carried out in order to implement this method: computing the tangent vector. To do this, it is convenient to revisit Eq. (3.1) and write out the dependence on the new parameter  $s$  explicitly.

$$0 = \mathbf{F}(\mathbf{u}(s), \eta(s)) \quad (3.6)$$

We can take the derivative with respect to  $s$  on both sides of the above equation, and obtain the following,

$$0 = \mathbf{J}(\mathbf{u}(s), \eta(s)) \frac{d\mathbf{u}}{ds} + \mathbf{F}_\eta(\mathbf{u}(s), \eta(s)) \frac{d\eta}{ds} \quad (3.7)$$

Additionally, in order to uniquely define the tangent vector, we must impose a restriction on its length, the most reasonable choice being to normalize it. Consequently, another equation must be added,

$$\left\| \frac{d\mathbf{u}}{ds} \right\|^2 + \left( \frac{d\eta}{ds} \right)^2 = 1 \quad (3.8)$$

[averiguar such that]

Then, we can fix  $\frac{d\eta}{ds} = 1$  and solve Eq. (3.7) for  $\frac{d}{ds}\mathbf{u}$ . Since it constitutes a system of linear equations, it can be solved using a standard linear solver. Finally, the tangent vector must be normalized in order to satisfy Eq. (3.8) and its sign chosen such that it has the same orientation as the previously known tangent  $\boldsymbol{\tau}_0$  i.e. such that  $\boldsymbol{\tau} \cdot \boldsymbol{\tau}_0 > 0$ . In the very first step of the continuation method, the previous tangent is unknown. In that case, we can choose the orientation of  $\boldsymbol{\tau}$  such that its last element (corresponding to  $\frac{d\eta}{ds}$ ) is positive if we want to compute the solution branch for increasing values of the parameter  $\eta$  or negative for decreasing values of the parameter.

is positive to move forward or negative to move backward.

It is convenient to define an extended vector function  $\tilde{\mathbf{F}}$  that incorporates  $\mathbf{F}$  and the corrector step in the following manner,

$$\tilde{\mathbf{F}}(\mathbf{y}) = \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ (\mathbf{y} - \mathbf{y}_0) \cdot \boldsymbol{\tau}_0 - \Delta s \end{pmatrix}. \quad (3.9)$$

And its corresponding Jacobian  $\tilde{\mathbf{J}}$  reads

$$\tilde{\mathbf{J}} = \begin{pmatrix} \mathbf{J} & \mathbf{F}_\eta \\ \frac{d\mathbf{u}}{ds} & \frac{d\eta}{ds} \end{pmatrix}. \quad (3.10)$$

Notice that the last row of the extended Jacobian  $\tilde{\mathbf{J}}$  corresponds exactly to the tangent vector  $\boldsymbol{\tau}$ .

The pseudo-arclength continuation algorithm can be summarized in the following steps.

1. Compute a first point in the solution branch  $\mathbf{y}_0 = (\mathbf{u}_0, \eta_0)$ , typically through direct numerical simulations. Additionally, one could run Newton's method once while keeping the parameter fixed at  $\eta = \eta_0$  to obtain a more accurate approximation for  $\mathbf{u}_0$ .
2. Solve Eq. (3.7) and find the tangent at that point  $\boldsymbol{\tau}_0$ . Choose the orientation of  $\boldsymbol{\tau}_0$  such that it points in the desired direction on the  $\eta$ -axis.
3. Using  $\mathbf{y}_0 + \boldsymbol{\tau}_0 \Delta s$  as initial guess in Newton's method, solve Eq. (3.9) to find the next point in the solution branch  $\mathbf{y}_{i+1}$ .
4. Again, solve Eq. (3.7) and find the tangent at that point  $\boldsymbol{\tau}_{i+1}$ . Choose the orientation such that it matches the previous tangent,  $\boldsymbol{\tau}_{i+1} \cdot \boldsymbol{\tau}_i > 0$ .
5. Repeat steps 3-4 until the whole solution branch has been computed. One could also track changes in the sign of the determinant of  $\mathbf{J}$  in order to estimate the location of bifurcation points.

**Example 3.2.1.** In order to illustrate, it is useful to revisit the previous example and implement the pseudo-arclength continuation to the same problem. The extended function  $\tilde{\mathbf{F}}(u, \eta)$  can be written in the following form,

$$\tilde{\mathbf{F}}(u, \eta) = \begin{pmatrix} \eta + \varepsilon u - u^3 \\ (u - u_0) \frac{du}{ds} + (\eta - \eta_0) \frac{d\eta}{ds} - \Delta s \end{pmatrix}. \quad (3.11)$$

Therefore, the extended Jacobian  $\tilde{\mathbf{J}}$  reads,

$$\tilde{\mathbf{J}} = \begin{pmatrix} \varepsilon - 3u^2 & 1 \\ \frac{du}{ds} & \frac{d\eta}{ds} \end{pmatrix}.$$

The tangent vector  $\boldsymbol{\tau} = (\tau_u, \tau_\eta)$  can be computed by solving Eq. (3.7). We start by fixing  $\frac{d}{ds}\eta = 1$ , then  $\frac{d}{ds}u$  can be obtained directly,

$$\frac{du}{ds} = -\frac{F_\eta}{J} = -\frac{1}{\varepsilon - 3u^2}.$$

Finally, we normalize  $\boldsymbol{\tau}$  to obtain the tangent vector.

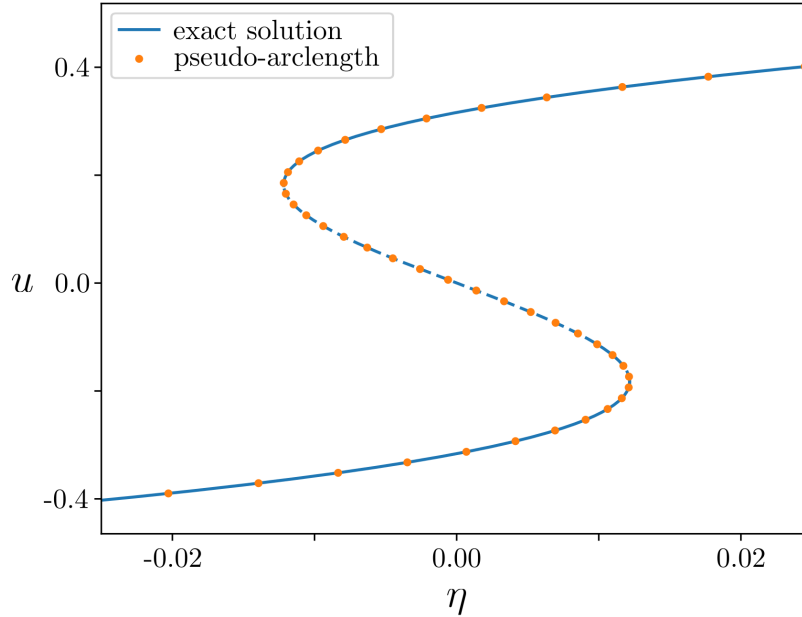


Figure 3.2: Solution of Eq. (3.11) as a function of the parameter  $\eta$  obtained through the pseudo-arclength continuation (orange dots) compared to the exact solution (blue curve).

### 3.3 Continuation of traveling states

In the particular case of following moving solutions with constant speed  $c$ , which is the core of this work, some difficulties arise. The first and most evident one, is that the desired solution

is not steady anymore. This can be solved rather easily by changing to the co-moving frame of reference, i.e. by inserting the traveling wave ansatz  $\mathbf{u}(x, t) = \mathbf{a}(x - ct)$ , where  $a$  is the solution profile in the co-moving frame, into Eq. (3.1). Due to the chain rule, an additional term in the form of a spatial derivative appears in the equation,

$$0 = \mathbf{F}(\mathbf{a}, \eta) + c\partial_x \mathbf{a} \quad (3.12)$$

The second problem is that usually the speed  $c$  will change as parameters are varied along the solution branch. Therefore, at each step, the speed will have to be determined by the algorithm. The solution to this problem is to add the speed as another unknown, that is to say, we will now be interested in solving for  $\mathbf{y} = (\mathbf{a}, c, \eta)$ . This leads to the third and final problem, which is that due to the additional unknown, we are missing an additional equation that will guarantee a unique solution to the linearized system. Moreover, we will be solving these systems considering periodic boundary conditions meaning that a translational invariance will appear. In order to deal with the translational symmetry and guarantee a unique solution, a *phase condition* or *pinning condition* must be established. Indeed, if we find a solution  $\mathbf{a}(x)$ , then  $\tilde{\mathbf{a}}_\theta(x) = \mathbf{a}(x + \theta)$  is also a solution for every  $\theta$ .

The most widely used condition is the *integral phase condition* [1] which takes a reference solution  $\mathbf{a}_0$  for a certain parameter value  $\eta_0$  close to the desired solution. The idea is to find the phase that minimizes the difference  $D$  between the desired solution  $\mathbf{a}$  and the reference solution  $\mathbf{a}_0$ . We can define the difference as follows,

$$D(\theta) = \int_0^L dx' \|\mathbf{a}(x' + \theta) - \mathbf{a}_0(x')\|^2 \quad (3.13)$$

In order to minimize the difference, we differentiate the above equation, set it equal to zero and then integrate by parts. Thus, we arrive at the following condition which is simpler to implement.

$$p(\mathbf{a}, \mathbf{a}_0) = \int_0^L dx' \mathbf{a}(x') \cdot \left. \frac{d\mathbf{a}_0}{dx} \right|_{x'} = 0. \quad (3.14)$$

We can re-define the extended vector function for which we want to find the root of in the following manner,

$$\mathbf{H}(\mathbf{y}) = \begin{pmatrix} \mathbf{F}(\mathbf{a}, \eta) + c\partial_x \mathbf{a} \\ p(\mathbf{a}, \mathbf{a}_0) \\ q(\mathbf{y}, \mathbf{y}_0) \end{pmatrix} \quad (3.15)$$

The derivative of the integral phase condition with respect to the state vector  $p_{\mathbf{a}}$  may differ depending on the chosen phase condition and implementation of the phase condition. In the simplest case, replacing the integral as a Riemann sum (which is the same as the trapezoidal rule in the case of periodic boundary conditions), the derivative reads



$$p_{\mathbf{a}} = \Delta x \frac{d\mathbf{a}_0}{dx}. \quad (3.16)$$

Therefore, the corresponding Jacobian of the extended function reads

$$\mathbf{J}_{\mathbf{H}}(\mathbf{y}) = \begin{pmatrix} \mathbf{J}(\mathbf{a}, \eta) + c\partial_x & \partial_x \mathbf{a} & \mathbf{F}_{\eta}(\mathbf{a}, \eta) \\ p_{\mathbf{a}}(\mathbf{a}_0) & 0 & 0 \\ \dot{\mathbf{a}} & \dot{T} & \dot{\eta} \end{pmatrix}. \quad (3.17)$$

Note that the last row corresponds, once again, exactly to the tangent vector  $\tau = (d\mathbf{u}/ds, dT/ds, d\eta/ds) = (\dot{\mathbf{u}}, \dot{T}, \dot{\eta})$ .

### 3.4 Continuation for periodic orbits

If we now wish to follow periodic solutions with the continuation method we must first derive a new set of equations to be solved. Mainly, periodic solutions are not steady solutions of the differential equation, however they satisfy the following boundary-value problem.

$$\frac{d\mathbf{u}}{dt} = \mathbf{F}(\mathbf{u}, \eta) \quad (3.18)$$

$$\mathbf{u}(t=0) = \mathbf{u}(t=T) \quad (3.19)$$

It is convenient to rescale the time  $t \rightarrow tT$ , therefore the condition for periodicity of the solution becomes  $\mathbf{u}(t=0) = \mathbf{u}(t=1)$ . Moreover, due to the rescaling in time, a factor  $T$  appears on the right-hand side of the dynamical equation. Therefore, the rescaled system becomes,

$$\frac{d\mathbf{u}}{dt} = T\mathbf{F}(\mathbf{u}, \eta) \quad (3.20)$$

$$\mathbf{u}(t=0) = \mathbf{u}(t=1) \quad (3.21)$$

Moreover, due to the additional time-dependence of  $\mathbf{u}$ , the parametrizing equation for the pseudo-arclength method needs to be modified accordingly. It now reads,

$$q(\mathbf{y}, \mathbf{y}_0) = \int_0^1 (\mathbf{u}(t) - \mathbf{u}_0(t)) \cdot \frac{d\mathbf{u}}{ds} dt + (T - T_0) \frac{dT}{ds} + (\eta - \eta_0) \frac{d\eta}{ds} - \Delta s = 0 \quad (3.22)$$

Additionally, one can add weights to the previous equation in order to tune the search direction in Newton's method "horizontally" (taking larger steps in the parameter  $\eta$ ) or "vertically" (smaller steps in  $\eta$ ), see [6] for a more detailed discussion.

Note that we have introduced the period  $T$  as another unknown which will be solved through Newton's method along with  $\mathbf{u}$  and  $\eta$ , i.e. we want to solve for  $\mathbf{y}(t) \equiv (\mathbf{u}(t), T, \eta)$ . Moreover, as in the previous section, a phase condition  $p(\mathbf{u}, \mathbf{u}_0) = 0$  must be satisfied in order to deal with the translational invariance (in time) and guarantee the uniqueness of the solution. Finally, we can re-define the extended vector function for which we want to find the root of in the following manner,

$$\mathbf{H}(\mathbf{y}) = \begin{pmatrix} T\mathbf{F}(\mathbf{u}, \eta) - \frac{d\mathbf{u}}{dt} \\ p(\mathbf{u}, \mathbf{u}_0) \\ q(\mathbf{y}, \mathbf{y}_0) \end{pmatrix} \quad (3.23)$$

In order to solve this system of equations subject to periodic boundary conditions, many strategies can be followed. Namely, orthogonal collocation methods (implemented in AUTO [1]), multiple shooting methods [3], and last but not least, finite difference methods (implemented in pde2path [6]). Although the latter has the least accuracy it is by far the simplest to implement. In the finite difference method, a possible approximation to the first equation in the above system is the trapezoidal rule (used for instance in the Crank-Nicolson scheme for simulating PDEs),

$$\left( \frac{F(\mathbf{u}_i) + F(\mathbf{u}_{i+1})}{2} \right) T - \frac{\mathbf{u}_{i+1} - \mathbf{u}_i}{t_{i+1} - t_i} = 0 \quad (3.24)$$

Note that the derivative of  $\mathbf{u}$  with respect to  $t$  can be written as a product of a matrix  $\nabla_t$  and the time-discretized vector  $\mathbf{u}_i$  ( $1 \leq i \leq n_t$ ), i.e. it can be written as  $\nabla_t \mathbf{u}$ , in which case the Jacobian  $\mathbf{J}_\mathbf{H}$  of  $\mathbf{H}$  reads

$$\mathbf{J}_\mathbf{H}(\mathbf{y}) = \begin{pmatrix} T\mathbf{J}(\mathbf{u}, \eta) - \nabla_t & \mathbf{F}(\mathbf{u}, \eta) & T\mathbf{F}_\eta(\mathbf{u}, \eta) \\ p_\mathbf{u}(\mathbf{u}_0) & 0 & 0 \\ \dot{\mathbf{u}} & \dot{T} & \dot{\eta} \end{pmatrix}. \quad (3.25)$$

# Chapter 4

## Moving Solitons in the Lugiato-Lefever equation.

In section 2.3 we introduced the concept of dissipative localized structures (LSs). Here, we will study the formation of such structures in nonlinear optical systems where they are often called optical or cavity solitons. More specifically, we will analyze the paradigmatic Lugiato-Lefever equation (LLE) [5] used to describe fiber resonators, and study the formation of LSs when a fourth order derivative and a non-local term are considered.

### 4.1 Lugiato-Lefever equation.

In 1987, Lugiato and Lefever proposed a simple yet extremely rich nonlinear partial differential equation to study the formation of patterns and localized states in the framework of nonlinear optics [5]. They considered a cavity filled with a nonlinear medium in the low transmission (or high quality) limit driven by a continuous wave. In order to keep the equation as simple as possible, they considered a cubic nonlinearity which is characteristic of Kerr media. Moreover, In virtue of the low dissipation limit, they originally neglected the longitudinal variable  $z$  (along which light propagates) and kept only the transversal plane  $x - y$  as spatial variables in the equation. In contrast, a longitudinal (or temporal) LLE was later formulated by Haelterman and his colleagues [ref], where only the longitudinal coordinate becomes relevant. The main difference between these two equations is that in the former, a transversal Laplacian appears due to diffraction of the light, whereas in the latter, a longitudinal Laplacian appears due to dispersion of the light. However, from a mathematical point of view, they are the same equations.

$$\frac{\partial E}{\partial t} = E_{in} - (1 + i\theta)E + i|E|^2E + i\nabla^2 E \quad (4.1)$$

In our case, we will consider the longitudinal LLE corresponding to Eq. (4.1) as a starting point and we will analyze the effect of adding a fourth order dispersion term and the Raman effect which will be explained in the following section.

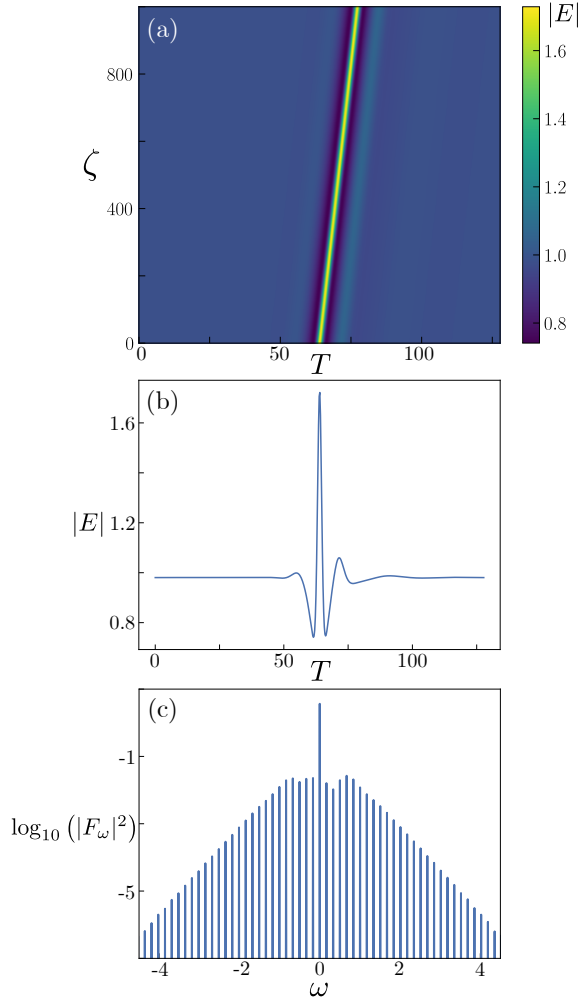


Figure 4.1: asdasd

## 4.2 Raman effect.

Raman effect frequently observed 26-31.

Stabilization of LSs by means of the Raman effect. 39, 41, 41-43 in normal dispersion and far from MI.

## 4.3 Isolations and traveling solitons.

In that case the LS is formed due to front locking between the two CW solutions (i.e. it requires bistability). However, in this case the LS arise due to coexistence between periodic state and CW, so even in the monostable can be observed.

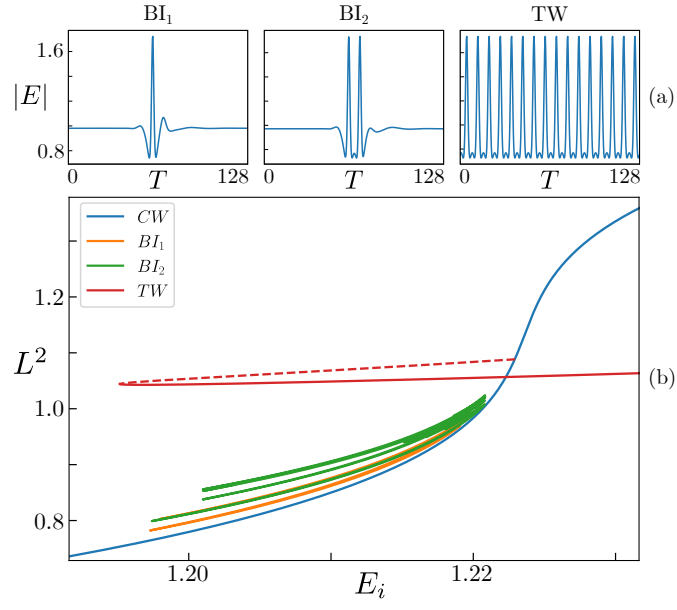


Figure 4.2: asd

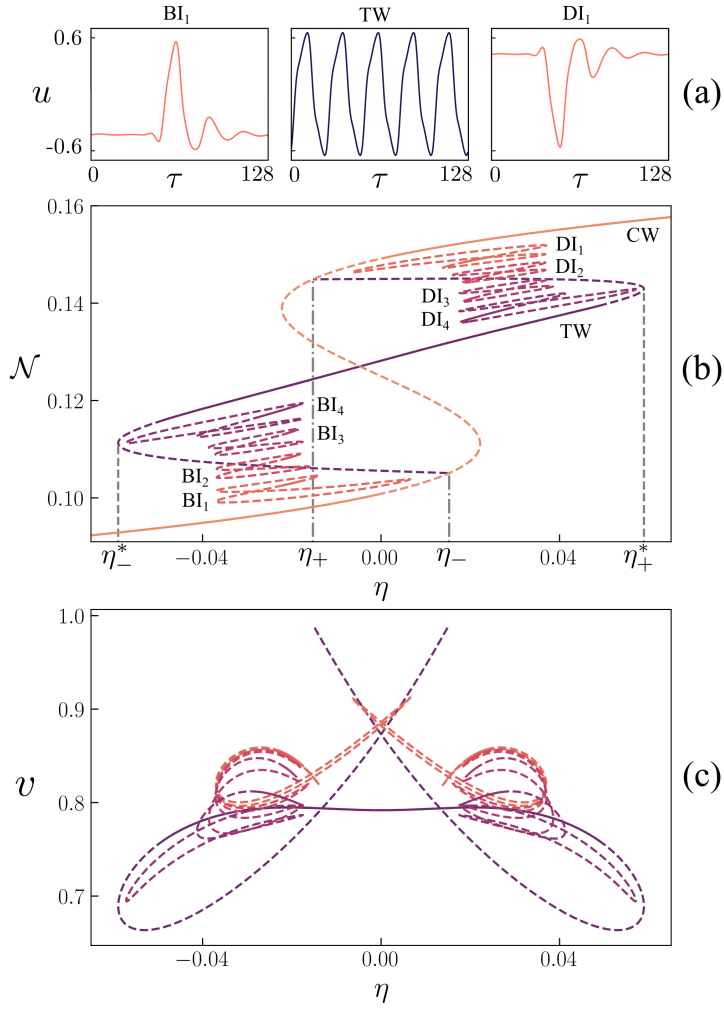


Figure 4.3: asdasd

#### 4.4 A reduced model.

#### 4.5 Oscillatory bound states.

1.

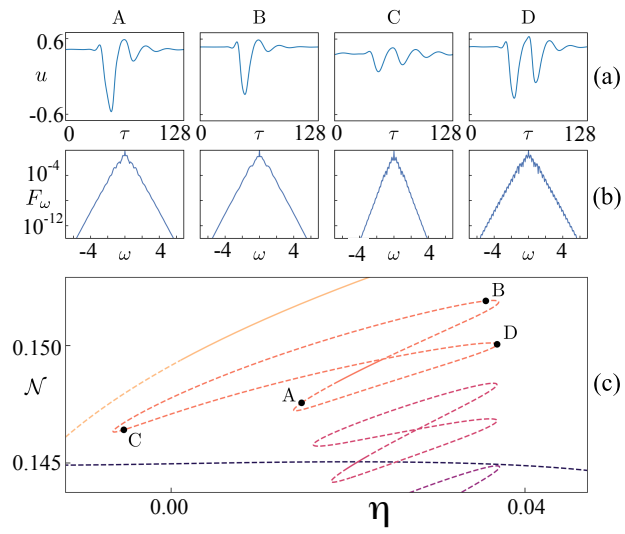


Figure 4.4: asd

## Chapter 5

# Moving spiral wave chimeras

Quisque facilisis auctor sapien. Pellentesque gravida hendrerit lectus. Mauris rutrum sodales sapien. Fusce hendrerit sem vel lorem. Integer pellentesque massa vel augue. Integer elit tortor, feugiat quis, sagittis et, ornare non, lacus. Vestibulum posuere pellentesque eros. Quisque venenatis ipsum dictum nulla. Aliquam quis quam non metus eleifend interdum. Nam eget sapien ac mauris malesuada adipiscing. Etiam eleifend neque sed quam. Nulla facilisi. Proin a ligula. Sed id dui eu nibh egetas tincidunt. Suspendisse arcu.

Maecenas dui. Aliquam volutpat auctor lorem. Cras placerat est vitae lectus. Curabitur massa lectus, rutrum euismod, dignissim ut, dapibus a, odio. Ut eros erat, vulputate ut, interdum non, porta eu, erat. Cras fermentum, felis in porta congue, velit leo facilisis odio, vitae consectetur lorem quam vitae orci. Sed ultrices, pede eu placerat auctor, ante ligula rutrum tellus, vel posuere nibh lacus nec nibh. Maecenas laoreet dolor at enim. Donec molestie dolor nec metus. Vestibulum libero. Sed quis erat. Sed tristique. Duis pede leo, fermentum quis, consectetur eget, vulputate sit amet, erat.

Donec vitae velit. Suspendisse porta fermentum mauris. Ut vel nunc non mauris pharetra varius. Duis consequat libero quis urna. Maecenas at ante. Vivamus varius, wisi sed egetas tristique, odio wisi luctus nulla, lobortis dictum dolor ligula in lacus. Vivamus aliquam, urna sed interdum porttitor, metus orci interdum odio, sit amet euismod lectus felis et leo. Praesent ac wisi. Nam suscipit vestibulum sem. Praesent eu ipsum vitae pede cursus venenatis. Duis sed odio. Vestibulum eleifend. Nulla ut massa. Proin rutrum mattis sapien. Curabitur dictum gravida ante.

Phasellus placerat vulputate quam. Maecenas at tellus. Pellentesque neque diam, dignissim ac, venenatis vitae, consequat ut, lacus. Nam nibh. Vestibulum fringilla arcu mollis arcu. Sed et turpis. Donec sem tellus, volutpat et, varius eu, commodo sed, lectus. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Quisque enim arcu, suscipit nec, tempus at, imperdiet vel, metus. Morbi volutpat purus at erat. Donec dignissim, sem id semper tempus, nibh massa eleifend turpis, sed pellentesque wisi purus sed libero. Nullam lobortis tortor vel risus. Pellentesque consequat nulla eu tellus. Donec velit. Aliquam fermentum, wisi ac rhoncus iaculis, tellus nunc malesuada orci, quis volutpat dui magna id mi. Nunc vel ante. Duis vitae lacus. Cras nec ipsum.



Morbi nunc. Aliquam consectetur varius nulla. Phasellus eros. Cras dapibus porttitor risus. Maecenas ultrices mi sed diam. Praesent gravida velit at elit vehicula porttitor. Phasellus nisl mi, sagittis ac, pulvinar id, gravida sit amet, erat. Vestibulum est. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Curabitur id sem elementum leo rutrum hendrerit. Ut at mi. Donec tincidunt faucibus massa. Sed turpis quam, sollicitudin a, hendrerit eget, pretium ut, nisl. Duis hendrerit ligula. Nunc pulvinar congue urna.

Nunc velit. Nullam elit sapien, eleifend eu, commodo nec, semper sit amet, elit. Nulla lectus risus, condimentum ut, laoreet eget, viverra nec, odio. Proin lobortis. Curabitur dictum arcu vel wisi. Cras id nulla venenatis tortor congue ultrices. Pellentesque eget pede. Sed eleifend sagittis elit. Nam sed tellus sit amet lectus ullamcorper tristique. Mauris enim sem, tristique eu, accumsan at, scelerisque vulputate, neque. Quisque lacus. Donec et ipsum sit amet elit nonummy aliquet. Sed viverra nisl at sem. Nam diam. Mauris ut dolor. Curabitur ornare tortor cursus velit.

Morbi tincidunt posuere arcu. Cras venenatis est vitae dolor. Vivamus scelerisque semper mi. Donec ipsum arcu, consequat scelerisque, viverra id, dictum at, metus. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut pede sem, tempus ut, porttitor bibendum, molestie eu, elit. Suspendisse potenti. Sed id lectus sit amet purus faucibus vehicula. Praesent sed sem non dui pharetra interdum. Nam viverra ultrices magna.

Aenean laoreet aliquam orci. Nunc interdum elementum urna. Quisque erat. Nullam tempor neque. Maecenas velit nibh, scelerisque a, consequat ut, viverra in, enim. Duis magna. Donec odio neque, tristique et, tincidunt eu, rhoncus ac, nunc. Mauris malesuada malesuada elit. Etiam lacus mauris, pretium vel, blandit in, ultricies id, libero. Phasellus bibendum erat ut diam. In congue imperdiet lectus.

Aenean scelerisque. Fusce pretium porttitor lorem. In hac habitasse platea dictumst. Nulla sit amet nisl at sapien egestas pretium. Nunc non tellus. Vivamus aliquet. Nam adipiscing euismod dolor. Aliquam erat volutpat. Nulla ut ipsum. Quisque tincidunt auctor augue. Nunc imperdiet ipsum eget elit. Aliquam quam leo, consectetur non, ornare sit amet, tristique quis, felis. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque interdum quam sit amet mi. Pellentesque mauris dui, dictum a, adipiscing ac, fermentum sit amet, lorem.

Ut quis wisi. Praesent quis massa. Vivamus egestas risus eget lacus. Nunc tincidunt, risus quis bibendum facilisis, lorem purus rutrum neque, nec porta tortor urna quis orci. Aenean aliquet, libero semper volutpat luctus, pede erat lacinia augue, quis rutrum sem ipsum sit amet pede. Vestibulum aliquet, nibh sed iaculis sagittis, odio dolor blandit augue, eget mollis urna tellus id tellus. Aenean aliquet aliquam nunc. Nulla ultricies justo eget orci. Phasellus tristique fermentum leo. Sed massa metus, sagittis ut, semper ut, pharetra vel, erat. Aliquam quam turpis, egestas vel, elementum in, egestas sit amet, lorem. Duis convallis, wisi sit amet mollis molestie, libero mauris porta dui, vitae aliquam arcu turpis ac sem. Aliquam aliquet dapibus metus.

Vivamus commodo eros eleifend dui. Vestibulum in leo eu erat tristique mattis. Cras at elit. Cras pellentesque. Nullam id lacus sit amet libero aliquet hendrerit. Proin placerat, mi non elementum laoreet, eros elit tincidunt magna, a rhoncus sem arcu id odio. Nulla

eget leo a leo egestas facilisis. Curabitur quis velit. Phasellus aliquam, tortor nec ornare rhoncus, purus urna posuere velit, et commodo risus tellus quis tellus. Vivamus leo turpis, tempus sit amet, tristique vitae, laoreet quis, odio. Proin scelerisque bibendum ipsum. Etiam nisl. Praesent vel dolor. Pellentesque vel magna. Curabitur urna. Vivamus congue urna in velit. Etiam ullamcorper elementum dui. Praesent non urna. Sed placerat quam non mi. Pellentesque diam magna, ultricies eget, ultrices placerat, adipiscing rutrum, sem.

# Chapter 6

## Conclusions

Mauris ac ipsum. Duis ultrices erat ac felis. Donec dignissim luctus orci. Fusce pede odio, feugiat sit amet, aliquam eu, viverra eleifend, ipsum. Fusce arcu massa, posuere id, nonummy eu, pulvinar ut, wisi. Sed dui. Vestibulum nunc nisl, rutrum quis, pharetra eget, congue sed, dui. Donec justo neque, euismod eget, nonummy adipiscing, iaculis eu, leo. Duis lectus. Morbi pellentesque nonummy dui.

Aenean sem dolor, fermentum nec, gravida hendrerit, mattis eget, felis. Nullam non diam vitae mi lacinia consectetur. Fusce non massa eget quam luctus posuere. Aenean vulputate velit. Quisque et dolor. Donec ipsum tortor, rutrum quis, mollis eu, mollis a, pede. Donec nulla. Duis molestie. Duis lobortis commodo purus. Pellentesque vel quam. Ut congue risus. Sed ligula. Aenean dictum pede vitae felis. Donec sit amet nibh. Maecenas eu orci. Quisque gravida quam sed massa.

Nunc euismod, mauris luctus adipiscing pellentesque, augue ligula pellentesque lectus, vitae posuere purus velit a pede. Phasellus leo mi, egestas imperdiet, blandit non, sollicitudin pharetra, enim. Nullam faucibus tellus non enim. Sed egestas nunc eu eros. Nunc euismod venenatis urna. Phasellus ullamcorper. Vivamus varius est ac lorem. In id pede eleifend nibh consectetur faucibus. Phasellus accumsan euismod elit. Etiam vitae elit. Integer imperdiet nibh. Morbi imperdiet orci euismod mi.

Donec tincidunt tempor metus. Aenean egestas cursus nulla. Fusce ac metus at enim viverra lacinia. Vestibulum in magna non eros varius suscipit. Nullam cursus nibh. Mauris neque. In nunc quam, convallis vitae, posuere in, consequat sed, wisi. Phasellus bibendum consectetur massa. Curabitur quis urna. Pellentesque a justo.



Figure 6.1: Logo de la Facultad

Table 6.1: Tabla 1

<b>Campo 1</b>	<b>Campo 2</b>	<b>Num</b>
Valor 1a	Valor 2a	3
Valor 1b	Valor 2b	3

In sit amet dui eget lacus rutrum accumsan. Phasellus ac metus sed massa varius auctor. Curabitur velit elit, pellentesque eget, molestie nec, congue at, pede. Maecenas quis tellus non lorem vulputate ornare. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Etiam magna arcu, vulputate egestas, aliquet ut, facilisis ut, nisl. Donec vulputate wisi ac dolor. Aliquam feugiat nibh id tellus. Morbi eget massa sit amet purus accumsan dictum. Aenean a lorem. Fusce semper porta sapien.

Curabitur sit amet libero eget enim eleifend lacinia. Vivamus sagittis volutpat dui. Suspendisse potenti. Morbi a nibh eu augue fermentum posuere. Curabitur elit augue, porta quis, congue aliquam, rutrum non, massa. Integer mattis mollis ipsum. Sed tellus enim, mattis id, feugiat sed, eleifend in, elit. Phasellus non purus sed elit viverra rhoncus. Vestibulum id tellus vel sem imperdiet congue. Aenean in arcu. Nullam urna justo, imperdiet eget, volutpat vitae, semper eu, quam. Sed turpis dui, porttitor ut, egestas ac, condimentum non, wisi. Fusce iaculis turpis eget dui. Quisque pulvinar est pellentesque leo. Ut nulla elit, mattis vel, scelerisque vel, blandit ut, justo. Nulla feugiat risus in erat.

# Bibliography

- [1] Eusebius J Doedel. Auto: A program for the automatic bifurcation analysis of autonomous systems. *Congr. Numer*, 30(265-284):25–93, 1981.
- [2] Eusebius J Doedel. Lecture notes on numerical analysis of nonlinear equations. *Numerical Continuation Methods for Dynamical Systems: Path following and boundary value problems*, pages 1–49, 2007.
- [3] H.B. Keller. *Numerical Methods for Two-point Boundary-value Problems*. Blaisdell book in numerical analysis and computer science. Blaisdell, 1968.
- [4] Herbert B Keller. Numerical solution of bifurcation and nonlinear eigenvalue problem. *Application of bifurcation theory*, 1977.
- [5] Luigi A Lugiato and René Lefever. Spatial dissipative structures in passive optical systems. *Physical review letters*, 58(21):2209, 1987.
- [6] Hannes Uecker. Hopf bifurcation and time periodic orbits with pde2path – algorithms and applications. *Communications in Computational Physics*, 25, 08 2017.

# Appendix A

## Anexo

Quisque facilisis auctor sapien. Pellentesque gravida hendrerit lectus. Mauris rutrum sodales sapien. Fusce hendrerit sem vel lorem. Integer pellentesque massa vel augue. Integer elit tortor, feugiat quis, sagittis et, ornare non, lacus. Vestibulum posuere pellentesque eros. Quisque venenatis ipsum dictum nulla. Aliquam quis quam non metus eleifend interdum. Nam eget sapien ac mauris malesuada adipiscing. Etiam eleifend neque sed quam. Nulla facilisi. Proin a ligula. Sed id dui eu nibh egetas tincidunt. Suspendisse arcu.

Maecenas dui. Aliquam volutpat auctor lorem. Cras placerat est vitae lectus. Curabitur massa lectus, rutrum euismod, dignissim ut, dapibus a, odio. Ut eros erat, vulputate ut, interdum non, porta eu, erat. Cras fermentum, felis in porta congue, velit leo facilisis odio, vitae consectetur lorem quam vitae orci. Sed ultrices, pede eu placerat auctor, ante ligula rutrum tellus, vel posuere nibh lacus nec nibh. Maecenas laoreet dolor at enim. Donec molestie dolor nec metus. Vestibulum libero. Sed quis erat. Sed tristique. Duis pede leo, fermentum quis, consectetur eget, vulputate sit amet, erat.

Donec vitae velit. Suspendisse porta fermentum mauris. Ut vel nunc non mauris pharetra varius. Duis consequat libero quis urna. Maecenas at ante. Vivamus varius, wisi sed egetas tristique, odio wisi luctus nulla, lobortis dictum dolor ligula in lacus. Vivamus aliquam, urna sed interdum porttitor, metus orci interdum odio, sit amet euismod lectus felis et leo. Praesent ac wisi. Nam suscipit vestibulum sem. Praesent eu ipsum vitae pede cursus venenatis. Duis sed odio. Vestibulum eleifend. Nulla ut massa. Proin rutrum mattis sapien. Curabitur dictum gravida ante.

Phasellus placerat vulputate quam. Maecenas at tellus. Pellentesque neque diam, dignissim ac, venenatis vitae, consequat ut, lacus. Nam nibh. Vestibulum fringilla arcu mollis arcu. Sed et turpis. Donec sem tellus, volutpat et, varius eu, commodo sed, lectus. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Quisque enim arcu, suscipit nec, tempus at, imperdiet vel, metus. Morbi volutpat purus at erat. Donec dignissim, sem id semper tempus, nibh massa eleifend turpis, sed pellentesque wisi purus sed libero. Nullam lobortis tortor vel risus. Pellentesque consequat nulla eu tellus. Donec velit. Aliquam fermentum, wisi ac rhoncus iaculis, tellus nunc malesuada orci, quis volutpat dui magna id mi. Nunc vel ante. Duis vitae lacus. Cras nec ipsum.

Morbi nunc. Aliquam consectetur varius nulla. Phasellus eros. Cras dapibus porttitor risus. Maecenas ultrices mi sed diam. Praesent gravida velit at elit vehicula porttitor. Phasellus nisl mi, sagittis ac, pulvinar id, gravida sit amet, erat. Vestibulum est. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Curabitur id sem elementum leo rutrum hendrerit. Ut at mi. Donec tincidunt faucibus massa. Sed turpis quam, sollicitudin a, hendrerit eget, pretium ut, nisl. Duis hendrerit ligula. Nunc pulvinar congue urna.

Nunc velit. Nullam elit sapien, eleifend eu, commodo nec, semper sit amet, elit. Nulla lectus risus, condimentum ut, laoreet eget, viverra nec, odio. Proin lobortis. Curabitur dictum arcu vel wisi. Cras id nulla venenatis tortor congue ultrices. Pellentesque eget pede. Sed eleifend sagittis elit. Nam sed tellus sit amet lectus ullamcorper tristique. Mauris enim sem, tristique eu, accumsan at, scelerisque vulputate, neque. Quisque lacus. Donec et ipsum sit amet elit nonummy aliquet. Sed viverra nisl at sem. Nam diam. Mauris ut dolor. Curabitur ornare tortor cursus velit.

Morbi tincidunt posuere arcu. Cras venenatis est vitae dolor. Vivamus scelerisque semper mi. Donec ipsum arcu, consequat scelerisque, viverra id, dictum at, metus. Lorem ipsum dolor sit amet, consectetur adipiscing elit. Ut pede sem, tempus ut, porttitor bibendum, molestie eu, elit. Suspendisse potenti. Sed id lectus sit amet purus faucibus vehicula. Praesent sed sem non dui pharetra interdum. Nam viverra ultrices magna.

Aenean laoreet aliquam orci. Nunc interdum elementum urna. Quisque erat. Nullam tempor neque. Maecenas velit nibh, scelerisque a, consequat ut, viverra in, enim. Duis magna. Donec odio neque, tristique et, tincidunt eu, rhoncus ac, nunc. Mauris malesuada malesuada elit. Etiam lacus mauris, pretium vel, blandit in, ultricies id, libero. Phasellus bibendum erat ut diam. In congue imperdiet lectus.

Aenean scelerisque. Fusce pretium porttitor lorem. In hac habitasse platea dictumst. Nulla sit amet nisl at sapien egestas pretium. Nunc non tellus. Vivamus aliquet. Nam adipiscing euismod dolor. Aliquam erat volutpat. Nulla ut ipsum. Quisque tincidunt auctor augue. Nunc imperdiet ipsum eget elit. Aliquam quam leo, consectetur non, ornare sit amet, tristique quis, felis. Vestibulum ante ipsum primis in faucibus orci luctus et ultrices posuere cubilia Curae; Pellentesque interdum quam sit amet mi. Pellentesque mauris dui, dictum a, adipiscing ac, fermentum sit amet, lorem.

Ut quis wisi. Praesent quis massa. Vivamus egestas risus eget lacus. Nunc tincidunt, risus quis bibendum facilisis, lorem purus rutrum neque, nec porta tortor urna quis orci. Aenean aliquet, libero semper volutpat luctus, pede erat lacinia augue, quis rutrum sem ipsum sit amet pede. Vestibulum aliquet, nibh sed iaculis sagittis, odio dolor blandit augue, eget mollis urna tellus id tellus. Aenean aliquet aliquam nunc. Nulla ultricies justo eget orci. Phasellus tristique fermentum leo. Sed massa metus, sagittis ut, semper ut, pharetra vel, erat. Aliquam quam turpis, egestas vel, elementum in, egestas sit amet, lorem. Duis convallis, wisi sit amet mollis molestie, libero mauris porta dui, vitae aliquam arcu turpis ac sem. Aliquam aliquet dapibus metus.

Vivamus commodo eros eleifend dui. Vestibulum in leo eu erat tristique mattis. Cras at elit. Cras pellentesque. Nullam id lacus sit amet libero aliquet hendrerit. Proin placerat, mi non elementum laoreet, eros elit tincidunt magna, a rhoncus sem arcu id odio. Nulla

eget leo a leo egestas facilisis. Curabitur quis velit. Phasellus aliquam, tortor nec ornare rhoncus, purus urna posuere velit, et commodo risus tellus quis tellus. Vivamus leo turpis, tempus sit amet, tristique vitae, laoreet quis, odio. Proin scelerisque bibendum ipsum. Etiam nisl. Praesent vel dolor. Pellentesque vel magna. Curabitur urna. Vivamus congue urna in velit. Etiam ullamcorper elementum dui. Praesent non urna. Sed placerat quam non mi. Pellentesque diam magna, ultricies eget, ultrices placerat, adipiscing rutrum, sem.