UNIVERSIDAD DE CHILE
FACULTAD DE CIENCIAS FÍSICAS Y MATEMÁTICAS
DEPARTAMENTO DE FÍSICA

# DYNAMICS OF PARTICLE-LIKE SOLUTIONS IN NON-LOCAL SYSTEMS

TESIS PARA OPTAR AL GRADO DE
MAGÍSTER EN CIENCIAS MENCIÓN FISICA

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# Resumen

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# Abstract

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*My biggest enemy is me ever since day one.*
*- Stefani Germanotta.*

# Acknowledgments

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# Contents

# Chapter 1

# Introduction

Las partículas o corpúsculos han sido un concepto fundamental y transversal en la física. Clásicamente se describen como un punto material con una masa y posición bien definida. Sin embargo, con el desarrollo de la mecánica cuántica, hemos aprendido que las partículas microscópicas son soluciones localizadas de un campo corrrespondiente a la amplitud de la probabilidad. Por otro lado, en sistemas macroscópicos fuera del equilibrio, producto del balance entre la inyección y disipación de energía, estos sistemas pueden exhibir soluciones localizadas que, en analogía con el caso anterior, usualmente se denonominan soluciones tipo partícula [?]. Estas estructuras localizadas han sido observadas en diversos sistemas de dinámica de fluidos, óptica, química e incluso ecología [?, ?]. Dependiendo del contexto físico en que se observan también reciben el nombre de solitones disipativos, patrones localizados, quimeras, entre otros.

Tradicionalmente, la descripción matemática de las estructuras localizadas se ha realizado usando modelos de reacción difusión [?, ?, ?]. Como el nombre sugiere el acoplamiento espacial ocurre mediante un término de difusión y, por lo tanto, es puramente local. No obstante, diversos sistemas ópticos, neuronales e incluso en vegetación presentan un acoplamiento más complejo y de largo alcance, usualmente llamado acoplamiento no local [?, ?, ?]. En estos casos se ha encontrado que el término no local es responsable de la estabilización de las soluciones tipo partícula [?, ?] y por tanto es fundamental en el entendimiento de estas soluciones.

**En esta tesis nos enfocaremos en la dinámica de las estructuras localizadas en sistemas no locales**. En particular, buscaremos entender los mecanismos que permiten la propagación de estas soluciones tipo partícula. Para lograrlo, estudiaremos dos sistemas diferentes: solitones disipativos en una cavidad de cristal de fibra fotónico [?], y quimeras espirales en redes bidimensionales de osciladores de fase heterogéneos.

En el primer caso, estudiaremos solitones brillantes y oscuros en resonadores de cavidad de cristal de fibra óptica. Los solitones disipativos temporales han recibido una enorme atención estas últimas décadas por su capacidad de generar *frequency combs* o peines de frecuencia que han revolucionado diversas áreas de la ciencia y tecnología, en particular la espectroscopía de alta precisión y metrología [?, ?]. La mayor parte de estos esfuerzos científicos se han centrado en la generación de solitones mediante un balance entre la nolinearidad Kerr del material y el acoplamiento local temporal por ejemplo debido a la dispersión [?]. No obstante, en materiales amorfos como lo son los cristales de fibra óptica emerge un acoplamiento no local temporal debido a una respuesta retardada del material a la excitación electromagnética que se conoce como scattering estimulado de Raman [?, ?]. Gracias al efecto Raman es posible la estabilización de estas estructuras localizadas [?]. En este trabajo, buscaremos estudiar cómo son afectadas las estructuras localizadas debido al efecto Raman y la nolinearidad Kerr, y en particular caracterizar precisamente cómo se auto-organizan estas soluciones en función de los parámetros del sistema.

Recientemente hemos podido reducir el modelo paradigmático de Lugiato-Lefever en torno a la emergencia de la biestabilidad encontrando así la ecuación de Swift-Hohenberg. De forma preliminar, hemos encontrado solitones brillantes y oscuros en la ecuación de Swift-Hohenberg con efecto Raman, en la región de coexistencia del estado homogéneo y el estado patrón. Además, debido al acoplamiento no local por efecto Raman, la simetría del sistema se rompe, lo que nos permite encontrar una propagación de las estructuras localizadas y una desconexión de las ramas de solución dando origen a una cadena de isolas, ver figura.

En el segundo caso, analizaremos estados ligados de dos quimeras espirales en redes de osciladores acoplados espacialmente de forma no local. Las quimeras espirales se caracterizan por tener un núcleo incoherente donde los osciladores están desincronizados, rodeado por una estructura coherente en forma de espiral donde los osciladores están sincronizados. Estas soluciones fueron reportadas por primera vez por Kuramoto y Shima hace dos décadas al acoplar osciladores de forma no local [?]. Durante estas últimas dos décadas han sido ampliamiente estudiadas en diversos sistemas neuronales, eléctricos e incluso en ecología [?, ?, ?], además de ser observadas experimentalmente en redes de osciladores químicos [?, ?]. Sin embargo, la mayor parte de estos estudios se han centrado en quimeras estacionarias y poco se conoce sobre las quimeras propagativas [?, ?]. En trabajos preliminares, encontramos un nuevo tipo de quimera espiral: un estado ligado de dos espirales propagativas que pueden moverse en una línea recta o siguiendo una trayectoria más compleja.

Mediante simulaciones preliminares hemos logrado encontrar 3 clases de quimeras espirales propagativas: simétricas, asimétricas y cicloidales, junto con su región de estabilidad correspondiente, ver figura **??**. Las quimeras simétricas poseen simetría de reflexión y se propagan en la dirección de su eje de simetría mientras que en las quimeras asimétricas, una espiral se vuelve más grande que la otra y se propagan en una dirección inclinada. Por último las espirales cicloidales presentan, además de un movimiento de traslación, una oscilación de sus núcleos que también suele llamarse *meandering* [?].

# Chapter 2

# Preliminary concepts

## 2.1 Dynamical System

## 2.2 Bifurcations

### 2.2.1 Saddle-Node bifurcation

[insert example]

### 2.2.2 Pitchfork bifurcation

[insert example]

### 2.2.3 Hopf bifurcation

[insert example]

## 2.3 Localized Structure.

## 2.4 Chimera states

# Chapter 3

# Numerical Continuation

As stated in the previous section, we will be interested in following the solution branches as they experience several bifurcations. Since an analytical treatment is not usually available for the problems studied in this work, we must resort to numerical methods. Namely, we will implement and develop a numerical continuation algorithm. This type of method aims to solve a nonlinear equation or, more generally, a system of nonlinear equations to find the desired steady states of a dynamical system as parameters are changed. This task corresponds to finding the roots (zeros) of a vector function $\boldsymbol{F}$, as in Eq. (3.1).

$$0 = \boldsymbol{F}(\boldsymbol{u}, \eta) \tag{3.1}$$

Although there are several methods for finding the roots of a vector function, in this thesis we will only use Newton's method because of its fast (quadratic) convergence and simplicity. This method corresponds to an iterative algorithm that given an initial guess $\boldsymbol{u_0}$, will perform successive iterations until a certain accuracy or tolerance is reached. Typically, an adequate initial guess can be obtained through direct numerical simulations. Each iteration is computed using Eqs (3.2-3.3), where $\mathbf{J}(\boldsymbol{u}_i, \eta)$ is the Jacobian of $\boldsymbol{F}$. Moreover, since the Jacobian is known in each step of Newton's algorithm, the stability of the solution and the bifurcation points can be directly determined by analyzing the sign of the determinant of the Jacobian.

$$\mathbf{J}(\boldsymbol{u}_i, \eta)\Delta\boldsymbol{u}_{i+1} = -\boldsymbol{F}(\boldsymbol{u}_i, \eta) \tag{3.2}$$

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i + \Delta\boldsymbol{u}_{i+1} \tag{3.3}$$

## 3.1 Natural parameter continuation

The simplest way to perform numerical continuation is to fix the value of the parameter, in this case $\eta$, and solve the equation (or system of equations) through Newton's method. Then, one can increase the parameter by a small step $\eta = \eta_0 + \Delta\eta$ and find the new solution using the previous solution $\boldsymbol{u_0}$ as the initial guess for Newton's method. Finally, the process is repeated until the whole solution branch has been computed. This method is usually called *Natural Parameter Continuation* [2].

**Example 3.1.1.** To illustrate the method, consider a simple [imperfect pitchfork] example: finding the homogeneous solutions to the Swift-Hohenberg equation. More specifically, the stable and unstable equilibria of Eq. (3.4) will be computed.

$$\dot{u} = \eta + \varepsilon u - u^3 \tag{3.4}$$

This task corresponds to finding the roots of a cubic polynomial: $F(u, \eta) = \eta + \varepsilon u - u^3 = 0$. The derivative can be determined easily: $J(u, \eta) = \varepsilon - 3u^2$.

Fig (3.1) illustrates the case where $\varepsilon = 0.1$, in which the system is bistable. Indeed, if the algorithm is started at $\eta = -0.02$ taking as initial guess $u_0 = -0.4$ and a forward sweep is performed (slowly increasing $\eta$), and then repeat the process backward we obtain Fig. (3.1).

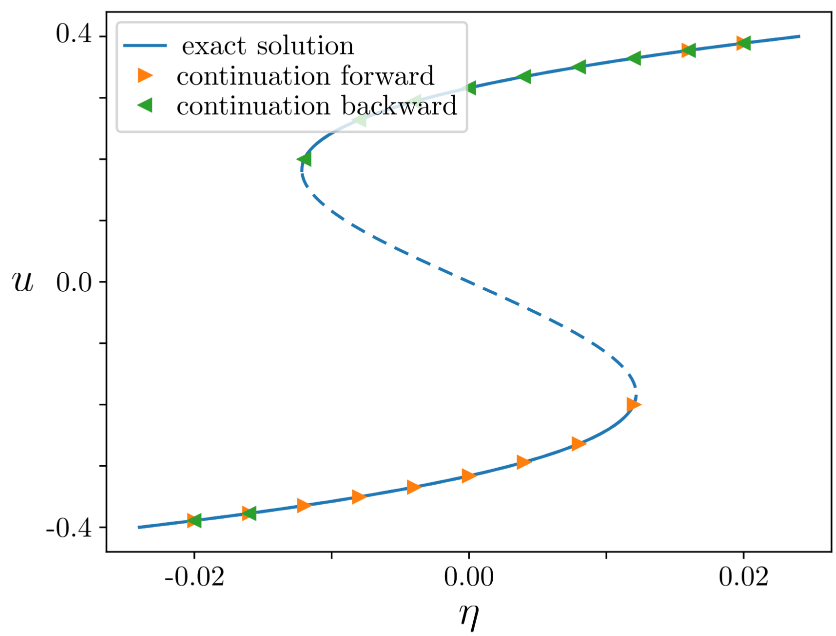


Figure 3.1: Solution of Eq. (3.4) as a function of the parameter $\eta$ obtained through natural continuation (orange and green triangles) compared to the exact solution (blue curve).

Note that the natural continuation succeeds at finding the lower and upper branches of the solution in this example. However, it could not follow the branch past the fold (or saddle-node) bifurcation. The only way to access the middle branch using this algorithm would be to use an adequate initial guess close to the middle branch. Although in this case, it is not difficult to find a suitable initial guess that will converge to the middle branch, for a higher-dimensional system where the bifurcation scenario is more complicated, this quickly becomes impractical. To overcome this limitation, one can implement a more robust continuation scheme: the *pseudo-arclength continuation*.

## 3.2 Pseudo-arclength continuation

As shown in the previous example, $\eta$ is not necessarily a good parameter to describe the curve, as it does not allow us to follow the branch through a fold point. A different approach can be taken where we parametrize the solution branch by a different parameter: $s$, which is somewhat similar to the arclength. Therefore, our goal is to obtain a set of points $\boldsymbol{y}(s) = (\boldsymbol{u}(s), \eta(s))$. Then the *pseudo-arclength* algorithm [4] consists essentially of two steps that ensure that the branch is followed through folds.

1. *Predictor step.* Extrapolate a distance $\Delta s$ along the tangent $\boldsymbol{\tau}_0$ from a previously known point $(\boldsymbol{u}_0, \eta_0)$ in the $(\boldsymbol{u}, \eta)$ space, to obtain the predicted point (the point used as an initial guess).
$$\boldsymbol{y} = \boldsymbol{y}_0 + \boldsymbol{\tau}_0 \Delta s$$
2. *Corrector step.* Force the solution to stay in the plane perpendicular to the tangent. Or, equivalently, that the solution projected onto the tangent has length $\Delta s$.
$$(\boldsymbol{y} - \boldsymbol{y_0}) \cdot \boldsymbol{\tau}_0 = \Delta s$$

In these steps, a new object is introduced: the tangent $\boldsymbol{\tau}$ of the solution curve $\boldsymbol{y}(s)$, which is defined as,

$$\boldsymbol{\tau} = \frac{d}{ds}\boldsymbol{y} = \left(\frac{d\boldsymbol{u}}{ds}, \frac{d\eta}{ds}\right) \tag{3.5}$$

An additional step must therefore be carried out: computing the tangent vector. To do this, it is convenient to revisit Eq. (3.1) and write out the dependence on the new parameter $s$ explicitly.

$$0 = \boldsymbol{F}(\boldsymbol{u}(s), \eta(s)) \tag{3.6}$$

We can take the derivative with respect to $s$ on both sides of the above equation, and obtain the following,

$$0 = \mathbf{J}(\boldsymbol{u}(s), \eta(s))\frac{d\boldsymbol{u}}{ds} + \boldsymbol{F}_{\boldsymbol{\eta}}(\boldsymbol{u}(s), \eta(s))\frac{d\eta}{ds} \tag{3.7}$$

Subsequently, in order to uniquely define the tangent vector, we must impose a restriction on its length, the most reasonable choice being to normalize it. Consequently, another equation must be added,

$$\left|\left|\frac{d\boldsymbol{u}}{ds}\right|\right|^2 + \left(\frac{d\eta}{ds}\right)^2 = 1 \tag{3.8}$$

[averiguar such that]

Then, we can fix $\frac{d\eta}{ds} = 1$ and solve Eq. (3.7) for $\frac{d}{ds}\boldsymbol{u}$. Since it constitutes a system of linear equations, it can be solved using a standard linear solver. The tangent vector must be normalized to satisfy Eq. (3.8) and its sign must be chosen such that it has the same orientation as the previously known tangent $\boldsymbol{\tau}_0$ i.e. such that $\boldsymbol{\tau} \cdot \boldsymbol{\tau}_0 > 0$. In the very first step of the continuation method, the previous tangent is unknown. In that case, one can choose the orientation of $\boldsymbol{\tau}$ such that its last element (corresponding to $\frac{d\eta}{ds}$) is positive to move forward (increasing the parameter) or negative to move backward (decreasing the parameter).

It is convenient to define an extended vector function $\tilde{\boldsymbol{F}}$ that incorporates $\boldsymbol{F}$ and the corrector step in the following manner,

$$\tilde{\boldsymbol{F}}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{F}(\boldsymbol{y}) \\ (\boldsymbol{y} - \boldsymbol{y}_0) \cdot \boldsymbol{\tau}_0 - \Delta s \end{pmatrix}. \tag{3.9}$$

And its corresponding Jacobian $\tilde{\mathbf{J}}$ reads

$$\tilde{\mathbf{J}} = \begin{pmatrix} \mathbf{J} & \boldsymbol{F}_\eta \\ \dfrac{d\boldsymbol{u}}{ds} & \dfrac{d\eta}{ds} \end{pmatrix}. \tag{3.10}$$

Notice that the last row of the extended Jacobian $\tilde{\mathbf{J}}$ corresponds exactly to the tangent vector $\boldsymbol{\tau}$.

Finally, the pseudo-arclength continuation algorithm can be summarized in the following steps.

1. Compute a first point in the solution branch $\boldsymbol{y}_0 = (\boldsymbol{u}_0, \eta_0)$, typically through direct numerical simulations. Additionally, one could run Newton's method once while keeping the parameter fixed at $\eta = \eta_0$ to obtain a more accurate approximation for $\boldsymbol{u}_0$.
2. Solve Eq. (3.7) and find the tangent at that point $\boldsymbol{\tau}_0$. Choose the orientation of $\boldsymbol{\tau}_0$ such that it points in the desired direction on the $\eta$-axis.
3. Using $\boldsymbol{y}_0 + \boldsymbol{\tau}_0 \Delta s$ as initial guess in Newton's method, solve Eq. (3.9) to find the next point in the solution branch $\boldsymbol{y}_{i+1}$.
4. Solve Eq. (3.7) and find the tangent at that point $\boldsymbol{\tau}_{i+1}$. Choose the orientation such that it matches the previous tangent, $\boldsymbol{\tau}_{i+1} \cdot \boldsymbol{\tau}_i > 0$.
5. Repeat steps 3-4 until the whole solution branch has been computed. One could also track changes in the sign of the determinant of $\mathbf{J}$ to estimate the location of bifurcation points.

**Example 3.2.1.** To illustrate the algorithm, it is useful to revisit the previous example and implement the pseudo-arclength continuation to the same problem. The extended function $\tilde{\boldsymbol{F}}(u, \eta)$ can be written in the following form,

$$\tilde{\boldsymbol{F}}(u, \eta) = \begin{pmatrix} \eta + \varepsilon u - u^3 \\ (u - u_0)\dfrac{du}{ds} + (\eta - \eta_0)\dfrac{d\eta}{ds} - \Delta s \end{pmatrix}. \tag{3.11}$$

Therefore, the extended Jacobian $\tilde{\mathbf{J}}$ reads,

$$\tilde{\mathbf{J}} = \begin{pmatrix} \varepsilon - 3u^2 & 1 \\ \dfrac{du}{ds} & \dfrac{d\eta}{ds} \end{pmatrix}.$$

The tangent vector $\boldsymbol{\tau} = (\tau_u, \tau_\eta)$ can be computed by solving Eq. (3.7).

We start by fixing $\frac{d}{ds}\eta = 1$, then $\frac{d}{ds}u$ can be obtained directly,

$$\frac{du}{ds} = -\frac{F_\eta}{J} = -\frac{1}{\varepsilon - 3u^2}.$$

Finally, we normalize $\boldsymbol{\tau}$ to obtain the tangent vector.

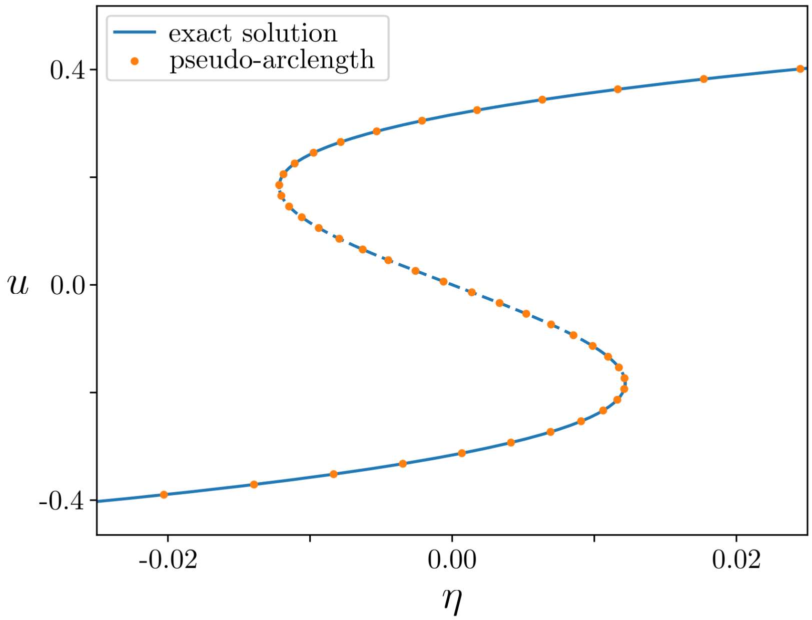


Figure 3.2: Solution of Eq. (3.11) as a function of the parameter $\eta$ obtained through the pseudo-arclength continuation (orange dots) compared to the exact solution (blue curve).

## 3.3 Continuation of traveling states

In the particular case of following moving solutions with constant speed $c$, which is the core of this work, some difficulties arise. The first and most evident one is that the desired solution is not steady anymore. This can be solved rather easily by changing to the co-moving frame of reference, i.e. by inserting the traveling wave ansatz $\boldsymbol{u}(x, t) = \boldsymbol{a}(x - ct)$, where $a$ is the solution profile in the co-moving frame, into Eq. (3.1). Due to the chain rule, an additional term in the form of a spatial derivative appears in the equation,

$$0 = \boldsymbol{F}(\boldsymbol{a}, \eta) + c\partial_x \boldsymbol{a} \tag{3.12}$$

The second problem is that usually the speed $c$ will change as parameters are varied along the solution branch. Therefore, at each step, the speed will have to be determined by the algorithm. The solution to this problem is to add the speed as another unknown, that is to say, we will now be interested in solving for $\boldsymbol{y} = (\boldsymbol{a}, c, \eta)$. This leads to the third and final problem, which is that due to the additional unknown, we are missing an additional equation that will guarantee a unique solution to the linearized system. Moreover, we will be solving these systems considering periodic boundary conditions meaning that a translational invariance will appear. To deal with the translational symmetry and guarantee a unique solution, a *phase condition* or *pinning condition* must be established. Indeed, if one finds a solution $\boldsymbol{a}(x)$, then $\tilde{\boldsymbol{a}}_\theta(x) = \boldsymbol{a}(x+\theta)$ is also a solution for every $\theta$.

The most widely used condition is the *integral phase condition* [1] which takes a reference solution $\boldsymbol{a}_0$ for a certain parameter value $\eta_0$ close to the desired solution. The idea is to find the phase that minimizes the difference $D$ between the desired solution $\boldsymbol{a}$ and the reference solution $\boldsymbol{a}_0$. The difference can be defined as follows,

$$D(\theta) = \int_0^L dx' \, ||\boldsymbol{a}(x'+\theta) - \boldsymbol{a}_0(x')||^2 \tag{3.13}$$

To minimize the difference, the above equation must be differentiated, set equal to zero and then integrated by parts. After carrying out these steps, the following condition is established which is simpler to implement.

$$p(\boldsymbol{a}, \boldsymbol{a}_0) = \int_0^L dx' \boldsymbol{a}(x') \cdot \left.\frac{d\boldsymbol{a}_0}{dx}\right|_{x'} = 0. \tag{3.14}$$

It is more convenient to re-define the extended vector function for which we want to find the root of in the following manner,

$$\boldsymbol{H}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{F}(\boldsymbol{a}, \eta) + c\partial_x \boldsymbol{a} \\ p(\boldsymbol{a}, \boldsymbol{a}_0) \\ q(\boldsymbol{y}, \boldsymbol{y}_0) \end{pmatrix} \tag{3.15}$$

The derivative of the integral phase condition with respect to the state vector $p_{\boldsymbol{a}}$ may differ depending on the chosen phase condition and implementation of the phase condition. In the simplest case, replacing the integral as a Riemann sum (which is the same as the trapezoidal rule in the case of periodic boundary conditions), the derivative reads

$$p_{\boldsymbol{a}} = \Delta x \frac{d\boldsymbol{a}_0}{dx}. \tag{3.16}$$

Therefore, the corresponding Jacobian of the extended function reads

$$\mathbf{J_H}(\boldsymbol{y}) = \begin{pmatrix} \mathbf{J}(\boldsymbol{a}, \eta) + c\partial_x & \partial_x \boldsymbol{a} & \boldsymbol{F}_\eta(\boldsymbol{a}, \eta) \\ p_{\boldsymbol{a}}(\boldsymbol{a}_0) & 0 & 0 \\ \dot{\boldsymbol{a}} & \dot{T} & \dot{\eta} \end{pmatrix}. \tag{3.17}$$

Note that the last row corresponds, once again, exactly to the tangent vector $\tau = (d\boldsymbol{u}/ds, dT/ds, d\eta/ds) = (\dot{\boldsymbol{u}}, \dot{T}, \dot{\eta})$.

## 3.4 Continuation for periodic orbits

In order to follow periodic solutions with the continuation method, a new set of equations must be satisfied. Mainly, periodic solutions are not steady solutions of the differential equation. However, they satisfy the following boundary-value problem.

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{F}(\boldsymbol{u}, \eta) \tag{3.18}$$

$$\boldsymbol{u}(t=0) = \boldsymbol{u}(t=T) \tag{3.19}$$

It is convenient to rescale the time $t \to tT$, therefore the condition for periodicity of the solution becomes $\boldsymbol{u}(t=0) = \boldsymbol{u}(t=1)$. Moreover, due to the rescaling in time, a factor $T$ appears on the right-hand side of the dynamical equation. Therefore, the rescaled system becomes,

$$\frac{d\boldsymbol{u}}{dt} = T\boldsymbol{F}(\boldsymbol{u}, \eta) \tag{3.20}$$

$$\boldsymbol{u}(t=0) = \boldsymbol{u}(t=1) \tag{3.21}$$

Moreover, due to the additional time-dependence of $\boldsymbol{u}$, the parametrizing equation for the pseudo-arclength method needs to be modified accordingly. It now reads,

$$q(\boldsymbol{y}, \boldsymbol{y}_0) = \int_0^1 (\boldsymbol{u}(t) - \boldsymbol{u}_0(t)) \cdot \frac{d\boldsymbol{u}}{ds} dt + (T - T_0)\frac{dT}{ds} + (\eta - \eta_0)\frac{d\eta}{ds} - \Delta s = 0 \tag{3.22}$$

Additionally, one can add weights to the previous equation in order to tune the search direction in Newton's method "horizontally" (taking larger steps in the parameter $\eta$) or "vertically" (smaller steps in $\eta$), see [6] for a more detailed discussion.

Note that the period $T$ has been introduced as another unknown which will be determined through Newton's method along with $\boldsymbol{u}$ and $\eta$, i.e. the method will solve for $\boldsymbol{y}(t) \equiv (\boldsymbol{u}(t), T, \eta)$. Moreover, as in the previous section, a phase condition $p(\boldsymbol{u}, \boldsymbol{u_0}) = 0$ must be satisfied in order to deal with the translational invariance (in time) and guarantee the uniqueness of the solution. Finally, one can re-define the extended vector function for which one wants to find the root in the following manner,

$$\boldsymbol{H}(\boldsymbol{y}) = \begin{pmatrix} T\boldsymbol{F}(\boldsymbol{u}, \eta) - \frac{d\boldsymbol{u}}{dt} \\ p(\boldsymbol{u}, \boldsymbol{u}_0) \\ q(\boldsymbol{y}, \boldsymbol{y}_0) \end{pmatrix} \tag{3.23}$$

In order to solve this system of equations subject to periodic boundary conditions, many strategies can be followed. Namely, orthogonal collocation methods (implemented in AUTO

[1]), multiple shooting methods [3], and finite difference methods (implemented in pde2path [6]). Although the latter has the least accuracy it is by far the simplest to implement. In the finite difference method, a possible approximation to the first equation in the above system is the trapezoidal rule (used for instance in the Crank-Nicolson scheme for simulating PDEs),

$$\left(\frac{F(\boldsymbol{u}_i) + F(\boldsymbol{u}_{i+1})}{2}\right) T - \frac{\boldsymbol{u}_{i+1} - \boldsymbol{u}_i}{t_{i+1} - t_i} = 0 \tag{3.24}$$

Note that the derivative of $\boldsymbol{u}$ with respect to $t$ can be written as a product of a matrix $\nabla_t$ and the time-discretized vector $\boldsymbol{u}_i$ ($1 \leq i \leq n_t$). In other words, it can be written as $\nabla_t \boldsymbol{u}$, in which case the Jacobian $\mathbf{J_H}$ of $\boldsymbol{H}$ reads

$$\mathbf{J_H}(\boldsymbol{y}) = \begin{pmatrix} T\mathbf{J}(\boldsymbol{u}, \eta) - \nabla_t & \boldsymbol{F}(\boldsymbol{u}, \eta) & T\boldsymbol{F}_\eta(\boldsymbol{u}, \eta) \\ p_{\boldsymbol{u}}(\boldsymbol{u}_0) & 0 & 0 \\ \dot{\boldsymbol{u}} & \dot{T} & \dot{\eta} \end{pmatrix}. \tag{3.25}$$

# Chapter 4

# Moving Solitons in the Lugiato-Lefever equation.

In section 2.3 we introduced the concept of dissipative localized structures (LSs). Here, we will study the formation of such structures in nonlinear optical systems where they are often called optical or cavity solitons. More specifically, we will analyze the paradigmatic Lugiato-Lefever equation (LLE) [5] used to describe fiber resonators, and study the formation of LSs when a fourth-order derivative and a non-local term are considered.

## 4.1 Lugiato-Lefever equation.

In 1987, Lugiato and Lefever proposed a simple yet extremely rich nonlinear partial differential equation to study the formation of patterns and localized states in the framework of nonlinear optics [5]. They considered a cavity filled with a nonlinear medium in the low transmission (or high quality) limit driven by a continuous wave. In order to keep the equation as simple as possible, they considered a cubic nonlinearity which is characteristic of Kerr media. Moreover, In virtue of the low dissipation limit, they originally neglected the longitudinal variable $z$ (along which light propagates) and kept only the transversal plane $x - y$ as spatial variables in the equation. In contrast, a longitudinal (or temporal) LLE was later formulated by Haelterman and his colleagues [ref], where only the longitudinal coordinate becomes relevant. The main difference between these two equations is that in the former, a transversal Laplacian appears due to diffraction of the light, whereas in the latter, a longitudinal Laplacian appears due to dispersion of the light. However, from a mathematical point of view, they are the same equations.

$$\frac{\partial E}{\partial t} = E_{in} - (1 + i\theta)E + i|E|^2E + i\nabla^2 E \tag{4.1}$$

In our case, we will consider the longitudinal LLE corresponding to Eq. (4.1) as a starting point and we will analyze the effect of adding a fourth-order dispersion term and the Raman effect which will be explained in the following section.¿ Indeed, the dispersion curve can be highly controlled using photonic crystal fibers and, when operating close to the zero dispersion wavelength, higher-order dispersion must be taken into account.

## 4.2 Raman effect.

Raman effect frequently observed 26-31.

Stabilization of LSs by means of the Raman effect. 39, 41, 41-43 in normal dispersion and far from MI.

## 4.3 Isolas and traveling solitons.

We will consider in the following sections a ring resonator operating close to the zero dispersion wavelength and pumped with short pulses. Therefore, higher-order dispersion terms must be included along with the stimulated Raman scattering. Taking into account these terms, the slowly varying envelope of the electric field within the resonator is described by the following generalized Lugiato-Lefever equation.

$$\frac{\partial E}{\partial \zeta} = E_i - (1+i\Delta)E + i(1-f_r)|E|^2 E + i\beta_2 \frac{\partial^2 E}{\partial T^2} + i\beta_4 \frac{\partial^4 E}{\partial T^4} + i f_r E \int_{-\infty}^{T} \phi(T-T')|E(T')|^2 dT' \tag{4.2}$$

Here, $E = E(\zeta, T)$ is the normalized mean-field cavity electric field, $E_i$ is the amplitude of the pumping field, $Delta$ is the detuning between the pump frequency and that of the cavity, and losses are normalized to unity. The time $\zeta$ represents the slow time scale along consecutive round trips and $T$ represents the fast time scale in the frame of reference moving with the group velocity of the light within the resonator. The parameters $\beta_2$ and $\beta_4$ correspond to the second and fourth-order dispersion, respectively. Lastly, the Raman-Kerr effect appears as the cubic local and non-local term, characterized by the strength fraction of the Raman effect $f_r$ and the kernel function,

$$\phi(T) = a \exp(-T/\tau_2) \sin(T/\tau_1) \tag{4.3}$$

with $a = \tau_0(\tau_1^2 + \tau_2^2)/(\tau_1 \tau_2^2)$. Typical values of these parameters for fused-silica-based fibers [48 from 164] are $f_r = 0.18$, $\tau_1 = 12.2$ fs, and $\tau_2 = 32$ fs.

Close to the nascent optical bistability a Swift-Hohenberg equation with an additional nonlocal term due to the Raman effect can be deduced for the deviation $u$ of the electric field from its value at the onset of the bistability.

$$\partial_t u = \eta + \mu u - u^3 + \beta \partial_\tau^2 u - \partial_\tau^4 u + \int_{-\infty}^{\tau} \phi(\tau - \tau') u(\tau') d\tau' \tag{4.4}$$

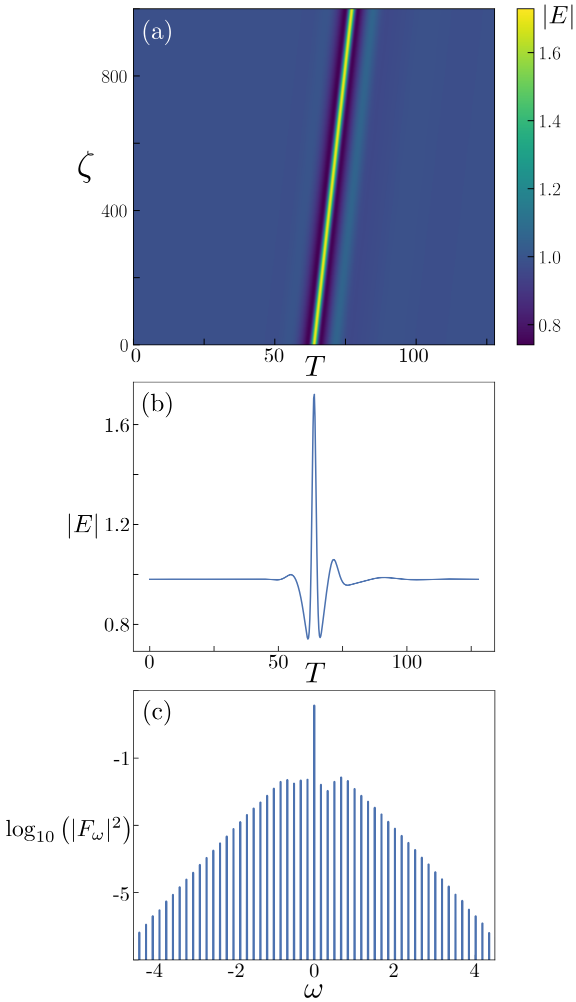


Figure 4.1: Moving bright soliton obtained by numerical simulation of Eq. (4.1). Panel (a) shows the temporal evolution in both the slow ($\zeta$) and fast ($T$) time scales. Panels (b) and (c) show the temporal profile for $\zeta = 0$ and Fourier spectrum, respectively. Parameters are $\Delta = 1.7$, $E_{in} = 1.219$, $f_r = 0.05$, $\beta_2 = 1$, $\beta_4 = 0.01$, $\tau_0 = 1$, $\tau_1 = 3$, $\tau_2 = 10$, $N = 512$, $\Delta x = 0.25$ and $\Delta t = 0.001$.

Note that for the numerical computation of the nonlocal term corresponding to the SRS we will exploit

In that case the LS is formed due to front locking between the two CW solutions (i.e. it requires bistability). However, in this case the LS arise due to coexistence between periodic state and CW, so even in the monostable can be observed.

## 4.4 A reduced model.

## 4.5 Oscillatory bound states.

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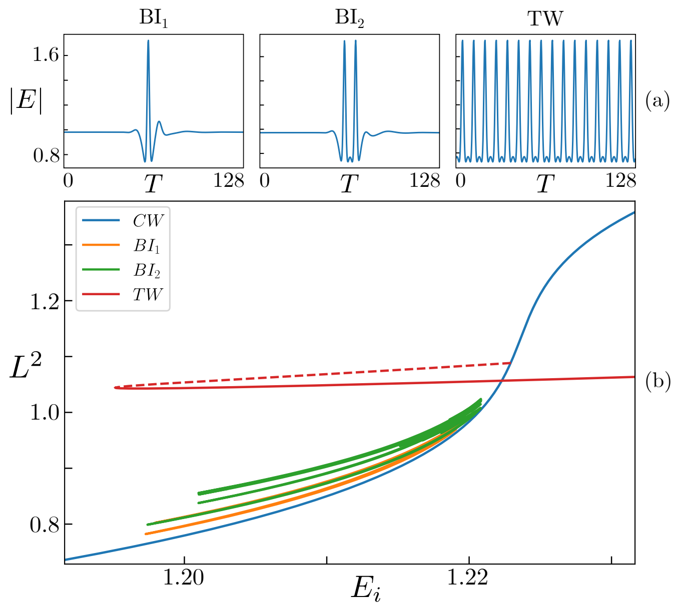


Figure 4.2: asd

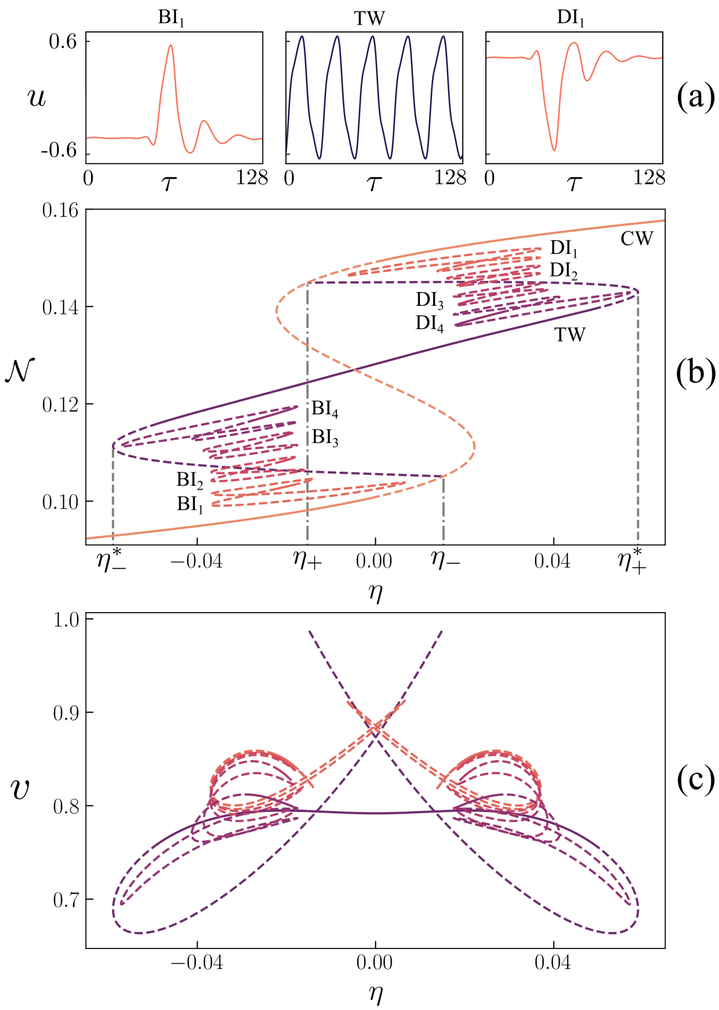


Figure 4.3: asdasd

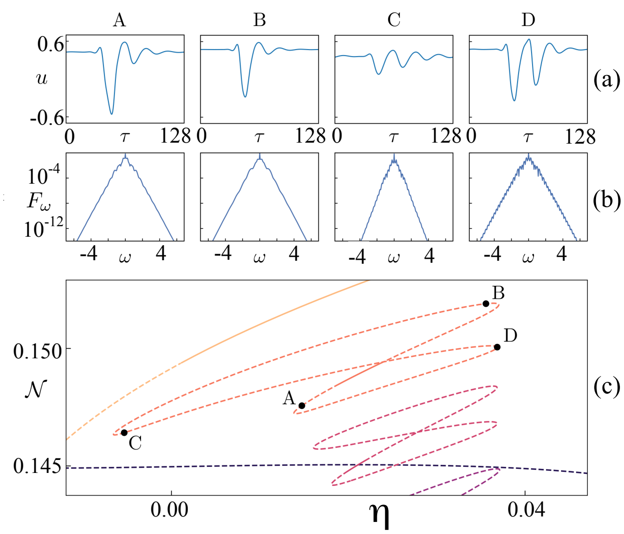


Figure 4.4: asd

# Chapter 5

# Moving spiral wave chimeras

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# Chapter 6

# Conclusions

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Figure 6.1: Logo de la Facultad

Table 6.1: Tabla 1

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# Appendix A

# Anexo

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