Advanced Game Thoerie

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(inoffizielles Skript)

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Contents

Chapter 1

Noncooperative Games

1.1 Basic Elements of Noncooperative Games

Definition: *A* ***game*** *is a formal representation of a situation in which a number of individuals interact in a setting of strategic interdependence.*

- *The players: Who is involved?*
- *The rules: Who moves when? What do they know when they move? What can they do?*
- *The outcomes: For each possible set of actions by the players, what is the outcome of the game?*
- *The payoffs: What are the players' preferences over the possible outcomes?*

Example 1.1 (of simultaneous move games)**:**

a) Matching Pennies

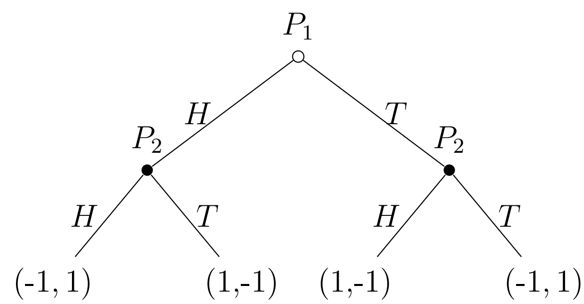
		Player 2	
		Heads	Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

b) Meeting in New York

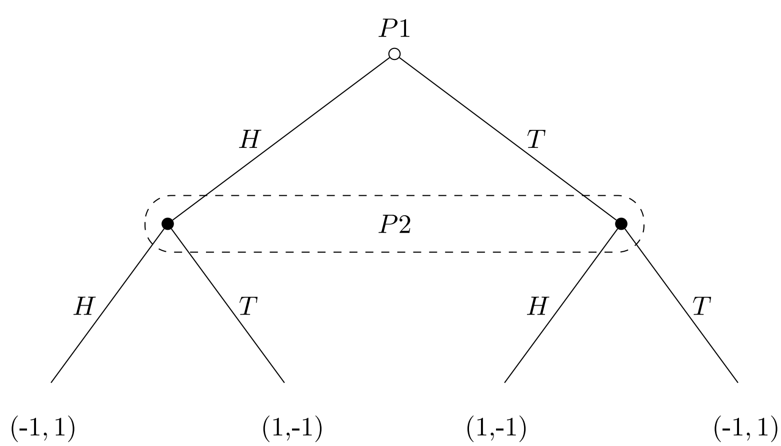
Player 1		Player 2: Empire State	Player 2: Grand Central
	Empire State	$100, 100$	$0, 0$
	Grand Central	$0, 0$	$100, 100$

c) Examples of (simple) dynamic games

Prisoner's Dilemma in Extensive-form



d) Matching Pennies Version C



Definition (Information)**:**

a) ***Information Set:*** *A player doesn't know which of the nodes in the information set she is actually at. Therefore, at any decision node in a player's information set, there must be the same possible actions.*

b) ***Perfect Information:*** *A game is said to be of perfect information if each information set contains a single decision node. Otherwise, it is a game of* ***imperfect information****.*

Definition (Extensive Form Game)**:** *A game in* ***extensive form*** *consists of:*

(i) *A finite set of nodes $\mathcal{X}$, a finite set of possible actions $\mathcal{A}$, and a finite set of players $\{1, \dots, l\}$.*

(ii) *A funktion $p\colon \mathcal{X} \Rightarrow \{\mathcal{X} \cup \emptyset\}$ specifying a single immediate predecessor of each node x; $p(x) \in \mathcal{X}$ expect for one element x_0, the* ***initial node****. The immediate* ***successor node*** *of x are $s(x) = p^{-1}(x)$.*
To have a tree structure, a predecessor can never be a successor and vice versa. The set of ***terminal nodes*** *$T = \{x \in \mathcal{X} \colon s(x) = \emptyset\}$. All other nodes $X \setminus T$ are* ***decision nodes****.*

(iii) *A function $\alpha\colon \mathcal{X} \setminus \{x_0\} \Rightarrow \mathcal{A}$ giving the action that leads to any non-initial node x from its immediate predecessor $p(x)$ with $x', x'' \in s(x); x' \neq x'' \Rightarrow \alpha(x') \neq \alpha(x'')$. The set of choices at decision node x is $c(x) = \{a \in \mathcal{A} \colon a = \alpha(x')$ for some $x' \in s(x)\}$.*

(iv) *A collection of information sets $\mathcal{H}$, and a function $H\colon \mathcal{X} \Rightarrow \mathcal{H}$ assigning each decision node x to an information set $H(x) \in \mathcal{H}$ with $c(x) = c(x')$ if $H(x) = H(x')$.*

The choices available at information set H can be written as

$$C(H) = \{a \in \mathcal{A} \colon a \in c(x) \text{ for } x \in H\}.$$

(v) *A function $\iota \colon \mathcal{H} \Rightarrow \{0, 1, \ldots, l\}$ assigning a player to each information set ($i = 0$ 'nature').*

The collection of player i's information set is denoted by

$$\mathcal{H}_i = \{H \in \mathcal{H} \colon i = \iota(H)\}.$$

(vi) *A function $\rho \colon \mathcal{H}_0 \times \mathcal{A} \Rightarrow [0, 1]$ assigning a probability to each action of nature with $\rho(H, a) = 0$ if $a \notin C(H)$ und $\sum_{a \in C(H)} \rho(H, a) = 1$ for all $H \in \mathcal{H}_0$.*

(vii) *A collection of payoff function $u = \{u_1(\cdot), \ldots, u_l(\cdot)\}$, where $u_i \colon T \Rightarrow \mathbb{R}$.*

A game in extensive form: $\Gamma_E = \{\mathcal{X}, \mathcal{A}, I, p(\cdot), \alpha(\cdot), \mathcal{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}$.

Comment: Restrictions of this definition:

a) Finite set of actions

b) Finite number of moves

c) Finite number of players

Definition (Strategy)**:** *Let $\mathcal{H}_i$ denote the collection of player i's information sets, $\mathcal{A}$ the set of possible actions in the game, and $C(H) \subset \mathcal{A}$ the set of actions possible at information set H. A* ***strategy*** *for player i is a function $s_i \colon \mathcal{H}_i \Rightarrow \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.*

Definition (Normal Form Representation)**:** *For a game with I players, the* ***normal form representation*** *Γ_N specifies for each player i a set of strategies $\mathcal{S}_i$ (with $s_i \in \mathcal{S}_i$) and a payoff function $u_i(s_1, \ldots, s_l)$, formally*

$$\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}].$$

Definition:

a) $s_i\colon \mathcal{H}_i \Rightarrow \mathcal{A}$ *describes deterministic choices at each* $H \in \mathcal{H}_i$ *and is called a* ***pure strategy***

b) *a* ***mixed strategy*** *is a probability distribution over all pure strategies* $\sigma_i\colon \mathcal{S}_i \Rightarrow [0,1]$*, with* $\sigma_i(s_i) \geq 0$ *and* $\sum_{s_i \in \mathcal{S}_i} \sigma_i(s_i) = 1$*.*

c) *player i's set of possible mixed strategies can be associated with the points of the simplex* $\Delta(\mathcal{S}_i)$*, called the* ***mixed extension*** *of* $\mathcal{S}_i$*.*

d) *since we assume that individuals are expected utility maximisers, player i's utility of a profile of mixed strategies* $\sigma = (\sigma_i, \dots, \sigma_l)$ *is given by*

$$u_i(\sigma) = \sum_{s \in \mathcal{S}} [\sigma_1(s_1) \cdot \sigma_2(s_2) \cdot \ldots \cdot \sigma_l(s_l)] \cdot u_i(s),$$

where $s = (s_1, \dots, s_l)$*.*

Definition (Behaviour Strategy)**:** *Given an extensive form game* Γ_E*, a* ***behaviour strategy*** *for player i specifies for every information set* $h \in \mathcal{H}_i$ *and action* $a \in C(H)$*, a probability* $\lambda_i(a, H) \geq 0$*, with*

$$\sum_{a \in C(H)} \lambda_i(a, H) = 1 \text{ for all } H \in \mathcal{H}_i.$$

Definition (Perfect Recall)**:** *A player has* ***perfect recall*** *if he doesn't "forget" what she once knew, including her own actions.*

Theorem 1.2: *If* Γ_E *is an extensive form game with perfect recall, then for any mixed strategy there is an outcome equivalent behaviour strategy and vice versa.*

1.2 Rationalisable Strategies

Central question of Game Theory: What should we expect to observe in a game played by rational players? Or more precisely: What should we expect to observe in a game played by rational players who are fully knowledgeable about the structure of the game and each others' rationality?

We first address the above question for simultaneous-move games, which we study using their normal form representation. We use the following notation:

- $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)]$ if we consider pure strategies only,
  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if we allow for mixed strategies
- $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_l) \in \mathcal{S}_{-i}$ where $\mathcal{S}_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \cdots \times S_l$
- $s = (s_i, s_{-i})$

Example 1.3 (Prisoners' Dilemma)**:**

		Player 2: don't confess	Player 2: confess
Player 1	don't confess	$-2, -2$	$-10, -1$
	confess	$-1, -10$	$-5, -5$

What should we expect to observe in the Prisoners' Dilemma?

Definition (Strictly Dominant Strategy)**:** *A strategy $s_i \in \mathcal{S}_i$ is **strictly dominant** for player i in game $\Gamma_N =]I, \{\mathcal{S}_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$:*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_i \in \mathcal{S}_{-i}$.

Applied to Prisoner's Dilemma: Confess is a strictly dominant strategy for each player.

Definition (Strictly Dominated Strategy)**:** *$s_i \in \mathcal{S}_i$ is **strictly dominated** for player i in game Γ_N if there exists another strategy $s_i' \in \mathcal{S}_i$ such that:*

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in \mathcal{S}_{-i}$. In this case we say that s_i' strictly dominates s_i.

Definition (Weakly Dominated Strategy)**:** *$s_i \mathcal{S}_i$ is **weakly dominated** for player i in game Γ_N if there exists another strategy $s_i' \in \mathcal{S}_i$ such that:*

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in \mathcal{S}_{-i}$, with strict inequality for at least one s_{-i}.

Example 1.4:

		Player 2	
		L	R
	U	$1, -1$	$-1, 1$
Player 1	M	$-1, 1$	$1, -1$
	D	$-2, 5$	$-3, 2$

$\Rightarrow D$ is strictly dominated by U and M.

		Player 2	
		L	R
	U	$5, 1$	$4, 0$
Player 1	M	$6, 0$	$3, 1$
	D	$6, 4$	$4, 4$

$\Rightarrow U$ and M are weakly dominated by D.

Example 1.5 (Prisoners' Dilemma – A Variation)**:** Assume Prisoner 1 is the district attorney's brother: If neither player confesses, player 1 is free

		Player 2	
		don't confess	confess
Player 1	don't confess	$0, -2$	$-10, -1$
	confess	$-1, -10$	$-5, -5$

$\Rightarrow$ D is strictly dominated by U and M.

$\Rightarrow$ Player 1 has no dominant strategy anymore.

In this game, the iterated elimination of strictly dominated strategies still leads to a unique prediction. In general, the order of elimination of strictly dominated strategies does not matter! How about iterated elimination of weakly dominated strategies?

Definition: *A strategy $\sigma_i \in \Delta(\mathcal{S}_i)$ is strictly dominated for i in game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that for all $\sigma_{-i} \in \Pi_{j \neq i}\Delta(\mathcal{S}_j)$:*

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

Proposition 1.6: Player i's pure strategy $s_i \in \mathcal{S}_i$ is strictly dominated in a game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in \mathcal{S}_{-i}.$$

Proof: This follows because we can write

$$u_i(\sigma_i', \sigma_{-i}) - u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in \mathcal{S}_{-i}} [\Pi_{k \neq i}\sigma_k(s_k)] \left[u_i(\sigma_i', s_{-i}) - u_i(s_i, s_{-i})\right].$$

And this expression is positive for all σ_{-i} if and only if $u_i(\sigma_i', s_{-i}) - u_i(s_i, s_{-i})$ is positive for all s_{-i}. □

Example 1.7:

		Player 2	
		L	R
Player 1	U	$10, 1$	$0, 4$
	M	$4, 2$	$4, 3$
	D	$0, 5$	$10, 2$

$$\Rightarrow \frac{1}{2}U + \frac{1}{2}D \text{ strictly dominates } M.$$

Definition (Best response)**:** *The strategy σ_i is a **best response** for player i to her rivals' strategies σ_{-i} if:*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(\mathcal{S}_i)$. Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is a best response.

Definition (Rationalisable Strategies)**:** *In game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(\mathcal{S}_i)$ that survive the iterated elimination of strategies that are never a best response are known as player i's **rationalisable strategies**.*

Example 1.8:

		Player 2			
		b_1	b_2	b_3	b_4
Player 1	a_1	$0, \underline{7}$	$2, 5$	$\underline{7}, 0$	$0, 1$
	a_2	$5, 2$	$\underline{3}, \underline{3}$	$5, 2$	$0, 1$
	a_3	$\underline{7}, 0$	$2, 5$	$0, \underline{7}$	$0, 1$
	a_4	$0, \underline{0}$	$0, -2$	$0, \underline{0}$	$\underline{10}, -1$

$$\Rightarrow \frac{1}{2}U + \frac{1}{2}D \text{ strictly dominates } M.$$

$\Rightarrow b_4$ is never best response for player 2 and *then* a_4 is never best response for player 1.

$\Rightarrow \{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are the rationalisable strategies in this game.

1.3 Nash Equilibrium

		Player 2		
		L	M	R
	U	$5,3$	$0,4$	$3,5$
Player 1	M	$4,0$	$\underline{5},\underline{5}$	$4,0$
	D	$3,5$	$0,4$	$5,3$

Example 1.9:

All strategies in this game are rationalisable, i.e. best responses to reasonable conjectures about other players' strategies. Yet only one strategy profile (namely (M, M)) contains best responses to correct conjectures about other players' strategies.

Definition (Nash Equilibrium)**:** *A strategy profile $s = (s_1, \ldots, s_l)$ constitutes a Nash equilibrium (NE) of game Γ_N if for every $i = 1, \ldots, l$*

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in \mathcal{S}_i$$

In the game on the previous slide, there is only one Nash equilibrium.

Example 1.10 (Meeting in New York)**:** a) Part 1

		Player 2	
		$EmpireState$	$GrandCentral$
Player 1	$EmpireState$	$\underline{100},\underline{100}$	$0,0$
	$GrandCentral$	$0,0$	$\underline{100},\underline{100}$

$\Rightarrow$ $(EmpireState, EmpireState)$ and $(GrandCentral, GrandCentral)$ are Nash equilibria.

		Player 2	
		EmpireState	*GrandCentral*
Player 1	*EmpireState*	$\underline{100}, \underline{100}$	$0, 0$
	GrandCentral	$0, 0$	$\underline{1000}, \underline{1000}$

$\Rightarrow$ Again, $(EmpireState, EmpireState)$ and $(GrandCentral, GrandCentral)$ are Nash equilibria.

b) Part 2

Why should we acre about Nash equilibria? Why should players' conjecture about each other's play be correct?

a) If there is a unique predict outcome to a game, it must be a Nash-Equilibrium.

b) Thus, a "focal point" (see example part 2) can be the unique predicted outcome to a game only if it is a Nash-Equilibrium.

c) An agreement between players is self-enforcing if it is a Nash-Equilibrium.

d) In a repeated game, a social convention to play the game might emerge. Only a Nash-Equilibrium can be maintained as a stable convention.

Definition (Mixed Strategy Nash Equilibrium)**:** *A mixed strategy profile* $\sigma = (\sigma_1, \ldots, \sigma_l)$ *constitutes a Nash equilibrium of game* Γ_N *if for every* $i = 1, \ldots, l$*,*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(\mathcal{S}_i).$$

Proposition 1.11: Let $\mathcal{S}_i^+ \subset \mathcal{S}$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_l)$. Strategy profile σ is a Nash equilibrium in game Γ_N if and only if for all $i = 1, \ldots, l$,

a) $u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i})$ for all $s_i, s'_i \in \mathcal{S}_i^+$

b) $u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i})$ for all $s_i \in \mathcal{S}_i^+$ and all $s'_i \notin \mathcal{S}_i^+$.

		Player 2	
		$EmpireState$	$GrandCentral$
Player 1	$EmpireState$	$\underline{100}, \underline{100}$	$0, 0$
	$GrandCentral$	$0, 0$	$\underline{1000}, \underline{1000}$

$\Rightarrow$ Again, $(EmpireState, EmpireState)$ and $(GrandCentral, GrandCentral)$ are Nash equilibria.

Example 1.12 (Meeting in New York 2)**:**

There are two pure strategy Nash-Equilibria, but are there also mixe strategy Nash-Equilibria? Let p denote the probability of (player $-i$) playing ES. Then for player i to play ES and GC with positive probability in a Nash-Equilibrium, we need

$$u_i(ES, p \cdot ES + (1-p) \cdot GC) = u_i(GC, p \cdot ES + (1-p) \cdot GC)$$

$$\Rightarrow 100 \cdot p = 1000 \cdot (1-p) \iff p = \frac{10}{11}$$

Thus, there is a mixed strategy Nash-Equilibrium with both players playing ES with probability $\frac{10}{11}$.

- In a Nash equilibrium each player's strategy is a best response to all other players' strategies.
- Let $b_i(s_{-i})$ denote the best response(s) of player i to the strategies s_{-i}
- Then $b_i \colon \mathcal{S}_{-i} \Rightarrow \mathcal{S}_i$ is a correspondence, called player i's best response correspondence.
- Define $b \colon \mathcal{S} \Rightarrow \mathcal{S}$ by $(s_1, \ldots, s_l) \Rightarrow b_1(s_{-1}) \times \ldots \times b_l(s_{-l})$

- A strategy profile $s \in \mathcal{S}$ is a Nash equilibrium if and only if $s \in b(s)$

- Thus, to prove existence of Nash equilibria, we have to show that a fixed point of the correspondence b exists

- To do so, we employ Kakutani's fixed-point theorem

Lemma 1.13: If $\mathcal{S}_1, \ldots, \mathcal{S}_l$ are nonempty, compact and convex and u_i is continuous in $\mathcal{S}_1, \ldots, \mathcal{S}_l$ and quasi-concave, then player $i's$ best response correspondence $b_i(\cdot)$ is nonempty-valued, convex-valued and upper hemicontinuous.

Definition (Quasi-Concave Function)**:** *The function $f\colon A \Rightarrow \mathbb{R}$, defined on the convex set $A \subset \mathbb{R}^N$, is quasi-concave if its upper contour sets $\{x \in A\colon f(x) \geq t\}$ are convex sets.*

Definition (Upper Hemicontinuous Correspondence)**:** *Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, the correspondence $f\colon A \Rightarrow Y$ is upper hemicontinuous if it has a closed graph and the images of compact sets are bounded.*

Theorem 1.14 (Kakutani's Fixed Point Theorem)**:** *Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, convex set, and that $f\colon A \Rightarrow A$ is a correspondence from A into itself that is nonempty-valued, convex-valued and upper hemicontinuous.*

Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x \in f(x)$.

Proposition 1.15: A Nash equilibrium exists in game γ_N if for all $i = 1, \ldots, l$,

a) $\mathcal{S}_i$ is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$

b) $u_i(s_1, \ldots, s_l)$ is continuous in $(s_1, \ldots, s_l)$ and quasi-concave in s_i.

Proof: By the lemma about strategy sets, $b(\cdot)$ is nonempty, convex-valued and upper hemicontinuous. By Kakutani's fixed point theorem there exists an $s \in \mathcal{S}$ such that $s \in b(s)$. By the definition of b: $s_i \in b_i(s_{-i})$ for all $i = 1, \ldots, l$. Thus s is a Nash equilibrium. $\square$

Proposition 1.16: Every Game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ in which the sets $\mathcal{S}_1, \ldots, \mathcal{S}_l$ have a finite number of elements has a mixed strategy Nash equilibrium.

This follows from the previous proposition on the existence of Nash equilibria because the set of mixed strategies $\Delta(\mathcal{S}_i)$ of a finite number if pure strategies is nonempty, convex, and compact.

Example 1.17:

		Player 2	
		L	R
Player 1	O	$-2, 1$	$1, -1$
	U	$1, -2$	$-1, 1$

Player 1's calculation:

$$\begin{aligned}
\Rightarrow & u_1(p \cdot O + (1-p) \cdot U, q \cdot L + (1-q) \cdot R) \\
& = pq(-2) + p(1-q)1 + (1-p)q1 + (1-p)(1-q)(-1) \\
& = -2pq + p - pq + q - pq - 1 + p + q - pq \\
& = -5pq + 2p + 2q - 1 \\
& = (2-5q)p + 2q - 1
\end{aligned}$$

$2 - 5q > 0 \Rightarrow p = 1$ optimal (pure strategy)

$2 - 5q = 0 \Rightarrow p \in [0, 1]$ optimal

$2 - 5q < 0 \Rightarrow p = 0$ optimal (pure strategy)

$$b_1(q) = \begin{cases} 1, & q < \frac{2}{5} \\ \in [0,1], & q = \frac{2}{5} \\ 0, & q > \frac{2}{5} \end{cases}$$

Player 2's calculation:

$$
\begin{aligned}
\Rightarrow & u_2(p \cdot O + (1-p) \cdot U, q \cdot L + (1-q) \cdot R) \\
&= pq1 + p(1-q)(-1) + (1-p)q(-2) + (1-p)(1-q)1 \\
&= pq - p + pq - 2q + 2pq - q + pq - p + 1 \\
&= 5pq - 2p - 3q + 1 \\
&= (5p-3)q - 2p + 1
\end{aligned}
$$

$5p - 3 > 0 \Rightarrow q = 1$ optimal (pure strategy)

$5p - 3 = 0 \Rightarrow q \in [0, 1]$ optimal

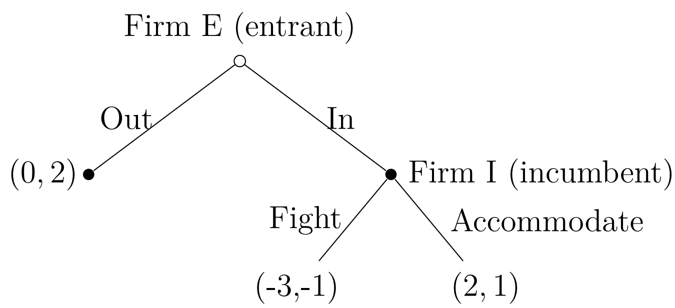
$5p - 3 < 0 \Rightarrow q = 0$ optimal (pure strategy)

$$
b_2(p) = \begin{cases} o, & p < \frac{3}{5} \\ \in [0,1], & p = \frac{3}{5} \\ 1, & p > \frac{3}{5} \end{cases}
$$

$\Rightarrow$ The only mixed strategy Nash-Equilibrium in this game $\left(\frac{3}{5}O + \frac{2}{5}U, \frac{2}{5}L + \frac{3}{5}R\right)$.

1.4 Subgame Perfection in Dynamic Games

Example 1.18 (Predation Game)**:**



Predation Game in normal form representation:

		Player 2	
		Fight	Accom.
Player 1	Out	$0, 2$	$0, 2$
	In	$-3, -1$	$2, 1$

$\Rightarrow$ two Nash-Equilibria in the normal form game: (Out, Fight if 'In'), (In, Accommodate if 'In')

But: Is the strategy Fight if 'In' credible?

Proposition 1.19 (Principle of sequential rationality)**:** A strategy should specify optimal actions at every point in the game tree given the opponents' strategies.

Proposition 1.20 (Backward induction)**:** Backward induction is an iterative procedure to identify Nash equilibria that satisfy the principle of sequential rationality in dynamic games:

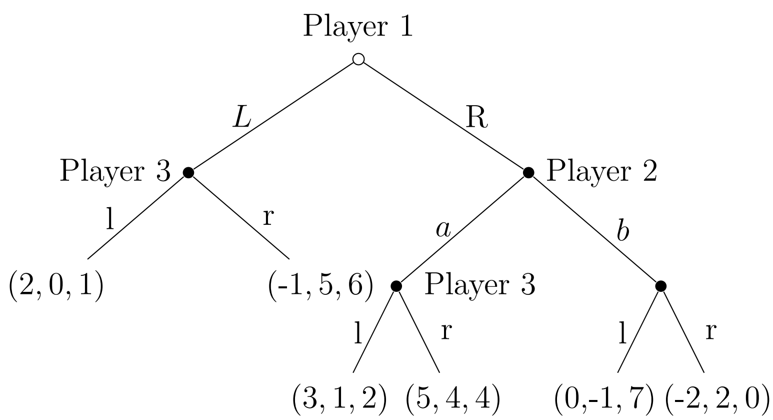
- Determine the optimal actions at the final decision nodes in the tree.

- Derive the reduced extensive form game by deleting the part of the game following these decision nodes and replacing them by the payoffs that result from the optimal play.

- Proceed to the next-to-last decision nodes and solve for the optimal actions to be taken there by players who correctly anticipate the actions that will follow at the final nodes.

- Continue in this way backwards through the game tree.

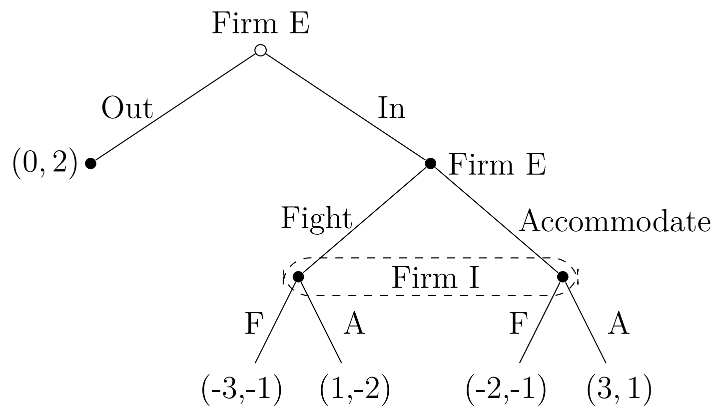
Example 1.21 (Predation Game)**:**



$\Rightarrow$ Equilibrium via backward induction: (R, a, rrl), other NEs: e.g. (L, b, rlr)

Theorem 1.22 (Zermelo's Theorem)**:** *Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.*

Example 1.23 (Predation Game Version B (Selten, 1865))**:**



$\Rightarrow$ Equilibrium via backward induction: (R, a, rrl), other NEs: e.g. (L, b, rlr)

Normal form representation:

		Firm I: A	Firm I: F
Firm E	Out, A if In	$0, 2$	$\underline{0, 2}$
	Out, F if In	$0, 2$	$\underline{0, 2}$
	In, A if In	$\underline{3, 1}$	-2, -1
	In, F if In	1, -2	-3, -1

$\Rightarrow$ Three NEs: $[(\text{Out, A if In}), F], [(\text{Out, F if In}), F], [(\text{In, A if In}), A]$

But: Principle of sequential rationality: A strategy should specify optimal actions at every point in the game tree given the opponents' strategies.

Definition (Subgame)**:** *A subgame of an extensive form game* Γ_E *is a subset of the game having the following properties:*

a) It begins with an information set containing a single decision node, contains all the decision nodes that are successors of this node, and contains only these nodes.

b) *If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x.*

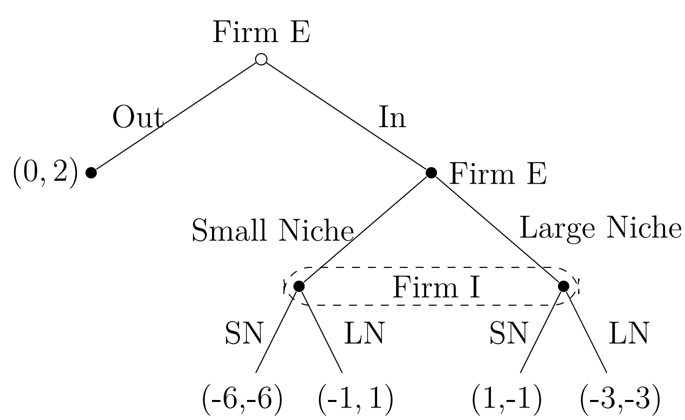
Definition (Subgame Perfect Nash Equilibrium)**:** *A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_l)$ in an l-player extensive form game Γ_E is a subgame perfect Nash equilibrium (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.*

Proposition 1.24: Consider an extensive form game Γ_E and some subgame G of Γ_E. Suppose that strategy profile γ^G is a SPNE in subgame G, and let Γ_E be the reduced game formed by replacing subgame G by a terminal node with payoffs equal to those arising from play of σ^G. Then:

a) In any SPNE σ of σ_E in which σ^G is the play in subgame G, players' moves at information sets outside subgame G must constitute a SPNE of reduced game Γ'_E.

b) If σ' is a SPNE of Γ'_E, then the strategy profile σ specifies the moves in σ^G information sets in subgame G and that specifies the moves in σ' at information sets not in G is a SPNE of Γ_E.

Example 1.25 (Niche Choice Game)**:**



Post-entry subgame: $\Rightarrow$ Two NEs in subgame: (SN, LN), (LN, SN)

		Firm I: SN	Firm I: LN
Firm E	SN	-6, -6	-1, 1
Firm E	LN	1, -1	-3, -3

Normal form representation of the whole game: $\Rightarrow$ Two SPNE in pure strategies:

		Firm I: SN	Firm I: LN
Firm E	Out, SN	0, 2	$\underline{0,\ 2}$
Firm E	Out, LN	0, 2	$\underline{0,\ 2}$
Firm E	In, SN	-6, -6	-1, 1
Firm E	In, LN	$\underline{1,\ -1}$	-3, -3

$[(Out, SN), LN]$, $[(In, LN), SN]$. Note: $[(Out, LN), LN]$ is not subgame perfect!

Period 1:

- Player 1 offers a split: $s^1 \in [0, v]$
- Player 2 can reject and the game continues in period 2, or accept and the split is implemented and the game ends immediately with $u_1 = s^1$, $u_2 = v - s^1$.

Period 2:

- Player 2 offers a split: $s^2 \in [0, v]$
- Player 1 can reject and the game continues in period 3, or accept and the split is implemented and the game ends immediately with $u_1 = \delta \cdot s^2$, $u_2 = \delta \cdot (v - s^2)$.

and so on ...

There is a unique SPNE:

Suppose T is odd:

Period T: Player 1 makes the offer in period T and player 2 is willing to accept any offer. Payoffs: $(\delta^{T-1} \cdot v, 0)$.

Period T-1: Player 2 makes the offer and player 1 will accept if and only if the payoff for player 1 is at least $\delta^{T-1} \cdot v$. Payoffs: $(\delta^{T-1} \cdot v, \delta^{T-2} \cdot v - \delta^{T-1} \cdot v)$.

Period T-2: Player 1 makes the offer and player 2 will accept if and only if the payoff for player 2 is at least $\delta^{T-2} \cdot v - \delta^{T-1} \cdot v$. Payoffs: $(\delta^{T-3} \cdot v - \delta^{T-2} \cdot v + \delta^{T-1} \cdot v, \delta^{T-2} \cdot v - \delta^{T-1} \cdot v)$.

...

$\Rightarrow$ The resulting SPNE for odd T is:

$$v_1^*(T) = v(1 - \delta + \delta^2 - \ldots + \delta^{T-1}) = v\left[(1-\delta)\left(\tfrac{1-\delta^{T-1}}{1-\delta^2}\right) + \delta^{T-1}\right]$$

$$v_2^*(T) = v - v_1^*(T)$$

$\Rightarrow$ The resulting SPNE for even T is:

$$v_1^*(T) = v - \delta v_1^*(T-1)$$

$$v_2^*(T) = v_1^*(T-1).$$

$\Rightarrow$ For large T, this converges to:

$\lim_{T\to\infty} v_1^*(T) = \frac{v}{1+\delta}$

$\lim_{T\to\infty} v_2^*(T) = \frac{\delta v}{1+\delta}$

Now consider the bilateral bargaining gam with infinite horizon:

Proposition 1.26 (Shaked & Sutton (1984))**:** The infinite horizon bargaining game has a unique SPNE in which the players reach an agreement in period 1 such that player 1 earns $\frac{v}{1+\delta}$ and player 2 $\frac{\delta v}{1+\delta}$.

Proof: Let $\overline{v_1}$ be the largest payoff that player 1 gets in any SPNE. Then player 1's payoff in any SPNE cannot be lower than $\underline{v_1} = v - \delta\overline{v_1}$. Also $overline v_1 \leq v - \delta\underline{v_1}$ because player 2 rejects any offer of less than $\delta\underline{v_1}$. And we have:

$$\overline{v_1} \leq v - \delta\underline{v_1} = \underline{v_1} + \delta\overline{v_1} - \delta\underline{v_1} \iff \overline{v_1}(1-\delta) \leq \underline{v_1}(1-\delta).$$

Which implies $\overline{v_1} = \underline{v_1}$, so player 1's SPNE is uniquely determined:

$$\Rightarrow v_1^0 = v - \delta v_1^0 = \frac{v}{1+\delta} \text{ and } v_2^0 = \frac{\delta v}{1+\delta}$$

□

1.5 Exercises

Advanced Game Theory - 1. Exercise

Exercise 1.1

In a game where player i has N information sets indexed $n = 1, \dots, N$ and M_n possible actions at information set n, how many strategies does player i have?

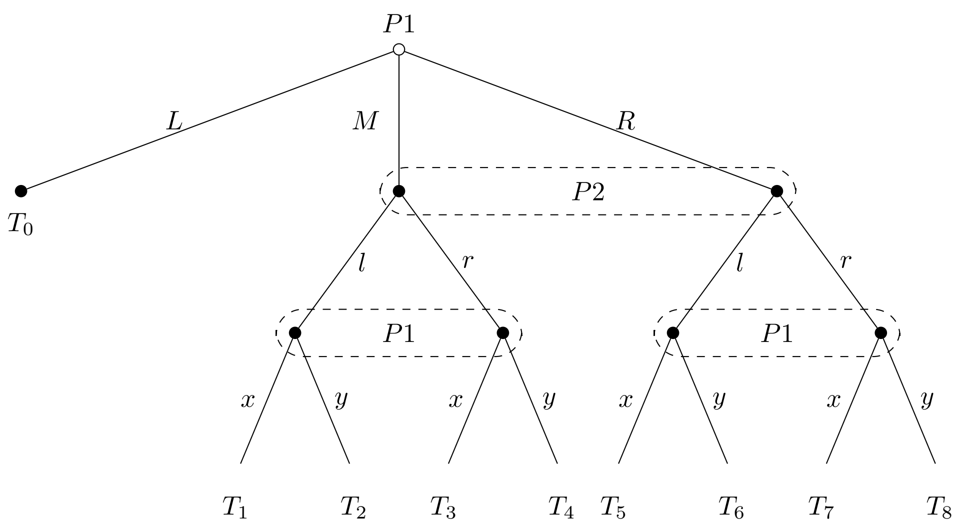
Proof: It holds that $|S| = \Pi_{n=1}^{N} M_n$, while:

$$S = M_1 \times \cdots \times M.$$

$\square$

Exercise 1.2

Consider the two-player game whose extensive form representation (excluding payoffs) is depicted below.



Extensive form game with imperfect information

a) What are the possible strategies of player 1 and player 2?

Proof: The possible strategies are:

$$\begin{aligned}
S_1 &= \Big\{(L,x,x),(L,x,y),(L,y,x),(L,y,y),\\
&\qquad (M,x,x),(M,x,y),(M,y,x),(M,y,y),\\
&\qquad (R,x,x),(R,x,y),(R,y,x),(R,y,y)\Big\}\\
&= \Big\{S_1^1,\ldots,S_1^{12}\Big\}\\
S_2 &= \{(l),(r)\}
\end{aligned}$$

□

b) Show that for any behaviour strategy of player i, there is a mixed strategy for that player that yields exactly the same distribution over outcomes for any strategies, mixed or behaviour, that might be played by i's rivals [this result is due to Kuhn (1953)].

Proof: It must hold for $p_i, q_i, r_i \geq 0$ that:

$$\begin{aligned}
p_1 + p_2 + p_3 &= 1\\
q_1 + q_2 &= 1\\
r_1 + r_2 &= 1
\end{aligned}$$

Example of a behaviour strategy: $(p_1 L + p_2 M + p_3 R, q_1 x + q_2 y, r_1 x + r_2 y)$
Example of a mixed strategy: $\sum_{i=1}^{12} p_i S_1^i$
For player 2 there is nothing to show.

Probability distribution of the outcomes:

$$p_1,\ p_2\sigma(l)q_1,\ p_2\sigma(l)q_2,\ p_2\sigma(r)q_1,\ p_2\sigma(r)q_2,\ldots$$

The following mixed strategy of player 1 is realisation equivalent

$$\left(p_1 S^1 + p_2 q_1 S_1^5 + p_2 q_2 S_1^7 + p_3 r_1 S_1^9 + p_3 r_2 S_1^{10}\right)$$

z.z.: $1 = p_1 + p_2 q_1 + p_2 q_2 + p_3 r_1 + p_3 r_3$. Beweis: klar. $\square$

c) Musterlösung im Ilias

Exercise 1.3

What was this - I can't find the corresponding exercise?

Player 2

		LL	L	M,	R
Player 1	U	$100, 2$	$-100, 1$	$0, 0$	$-100, -100$
	D	$-100, -100$	$100, -49$	$1, 0$	$100, 2$

Advanced Game Theory - 2. Exercise

Exercise 2.1

Prove that the order of removal does not matter for the set of strategies that survives a process of iterated deletion of strategies that are never a best response.

Proof: Let Σ_i^N be the set of strategies for player i that remain after N rounds of elimination of never best response strategies.

Suppose $s_1 \in \Sigma_i^N$ is never a best response to any strategy in Σ_{-i}^N. Suppose we do not delete s_1 in round $N+1$. Now s_1 will not be a best response to any strategy in $\Sigma_{-i}^{N+1} \subseteq \Sigma_{-i}^N$. In particular, it will never be a best response to any strategy in Σ_{-i}^{N+k} for $k \geq 1$.

$$\Rightarrow s_1 \text{ will be deleted in a later round.}$$

I think that this prove is incomplete: one has to show or at least mention that the eliminated strategy was never a best response. For example, if you have 3 strictly ordered strategies two of them are never a best response, however, if I eliminate the best response one of the two can't be deleted anymore. $\square$

Exercise 2.2

Prove that if pure strategy s_i is a strictly dominated strategy in game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ then so is any strategy that plays s_i with positive probability.

Proof: Suppose $s_i^* \in S_i$ is strictly dominated by $\sigma_i^* \in \Delta(S_i)$. Suppose further that $\sigma_i \in \Delta(S_i)$ plays s_i^* with positiv probability $\sigma_i(s_i^*)$

Claim: σ_i is strictly dominated by $\sigma_i' \in \Delta(S_i)$ which is equivalent to σ_i but assigns the $\sigma(s_i^*)$ to σ_i^* instead of s_i^*. Proof:

$$
\begin{aligned}
u_i(\sigma_i, \sigma_{-i}) &= \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, \sigma_{-i}) \\
&= \sum_{s_i \neq s_i^*} \sigma_i(s_i) u_i(s_i, \sigma_i) + \sigma_i(s_i^*) u_i(s_i^*, \sigma_{-i}) \\
&< \sum_{s_i \neq s_i^*} \sigma_i(s_i) u_i(s_i, \sigma_i) + \sigma_i(s_i^*) u_i(\sigma_i^*, \sigma_{-i}) = u_i(\sigma_i', \sigma_{-i}), \quad \forall \sigma_{-i} \in \Delta(S_{-i})
\end{aligned}
$$

$\square$

Exercise 2.3

a) Determine all strictly dominant strategies.

Proof: There are no strictly dominant strategies. □

b) Determine all weakly dominant strategies.

Proof: There are no weakly dominant strategies. □

c) Determine all strictly dominated strategies

Proof: s_1 (by e.g. s_2). By exercise 2.2 it holds further that any mixed strategy that contains s_1 is strictly dominated as well. □

d) Determine all weakly dominated strategies.

Proof: The strategies s_1, s_2 and s_6 are weakly dominated. For player 2, the strategies t_1 and t_2 are weakly dominated. Applying exercise 2.2 with the non-strict inequality yields: all mixed strategies that play a weakly dominated strategy with positive probability are also weakly dominated. □

e) Determine which strategies survive the iterative elimination of weakly dominated strategies.

Proof: The strategies s3 and t3 are the only strategies that survive the iterative elimination of weakly dominated strategies.

In this case it isn't, but in general is the iterative elimination of weakly dominated strategies dependent on the order of elimination, isn't it? □

f) Determine all rationalisable strategies.

Proof: The strategies s_3, s_4, s_5, t_3, t_4 and t_6 are the rationalisable strategies. □

Advanced Game Theory - 3. Exercise

Exercise 3.1

Show that if there is a unique profile of strategies that survives iterated removal of strictly dominated strategies, this profile is a Nash equilibrium.

Proof: First, we assume that there is only a finite number of pure strategies $\Rightarrow$ there exists at least one Nash-Equilibrium.

By exercise 2.2., when a strategy σ_i is eliminated then so is every strategy that plays σ_i with positive probability. We define S^∞ as the set of strategies that survive iterated elimination of strictly dominated strategies. Note that by assumption it holds:

$$|S^\infty| = 1.$$

Claim: If $(s_1^*, \ldots, s_I^*)$ is a Nash-Equilibrium, then $s^* \in S^\infty$. Proof:

Let $(s_1^*, \ldots, s_I^*)$ be a Nash-Equilibrium and assume $s^* \notin S^\infty$. Let i be the player whose strategy is eliminated first (in round k), i.e. $\exists \sigma_i, \sigma_i' \in \Delta(S_i)$:

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}) \quad \forall s_i \in S_{-i}^{k-1}$$

and σ_i' is played with positiv probability in s_i^*.

Let s_i' be derived from s_i^* by replacing σ_i' by σ_i.

$$\Rightarrow \quad u_i(s_i', s_{-i}^*) = u_i(s_i^*, s_{-i}^*) + \underbrace{s_i^*}_{>0}(\sigma_i') \underbrace{\left[u_i(\sigma_i, s_{-i}^*) - u_i(\sigma_i', s_{-i}^*)\right]}_{>0} > u_i(s_i^*, s_{-i}^*),$$

which contradicts the fact that s^* is a Nash-Equilibrium. $\square$

Exercise 3.2

Consider a bargaining situation in which two individuals are considering undertaking a business venture that will earn them 100 dollars in profit, but they must agree on how to split the 100 dollars. Bargaining works as follows: The two individuals each make a demand simultaneously. If their demands sum to more than 100 dollars, they fail to agree, and each gets nothing. If their demand sums to less than 100 dollars, they do the project, each gets his demand, and the rest goes to charity.

a) What are each player's strictly dominated strategies?

Proof: Musterlösung im Ilias. □

b) What are each player's weakly dominated strategies?

Proof: Musterlösung im Ilias. □

c) What are the pure strategy Nash equilibria of this game?

Proof: Musterlösung im Ilias. □

Exercise 3.3

		Player 2			
		LL	L	M,	R
Player 1	U	$100, 2$	$-100, 1$	$0, 0$	$-100, -100$
	D	$-100, -100$	$100, -49$	$1, 0$	$100, 2$

a) Play M, if one considers the max min-principle as status quo.

b) Pure Nash-Equilibria: (U, LL) and (D, R)

Mixed Equilibria:

(i) Player 1 mixes U and D with probabilities p and $1-p$ respectively.

(ii) Player 2 can mix between: $(LL, L), (LL, M), (LL, R), (L, M), (L, R),$

$$(M, R), (LL, L, M), (LL, L, R), (LL, M, R), (L, M, R), (LL, L, M, R)$$

Claim: Only (LL, L) will lead to a Nash-Equilibrium.

Proof (Using the Proposition after the Definition of Mixed Strategy NE):
Only (LL, L) will lead to a Nash Equilibrium

$$\begin{aligned} u_2(LL) = u_2(L) &\iff 2p - 100(1-p) = p - 49(1-p) \\ &\iff p = \frac{51}{52} \end{aligned}$$

Therefore: $u_2(LL) = u_2(L) = \frac{1}{26}$, $u_2(M) = 0$, $u_2(R) < 0$.

$$\begin{aligned} u_1(u) = u_1(D) &\iff 100q - 100(1-q) = -100q + 100(1-q) \\ &\iff q = \frac{1}{2} \end{aligned}$$

where q is the probability of Player 2 playing LL.

$$\Rightarrow \quad \text{Nash Equilibrium: } \left(\frac{51}{52}U + \frac{25}{26}D, \frac{1}{2}LL + \frac{1}{2}L\right).$$

Now we have proven that (LL, L) is a Nash Equilibrium. We will subsequently show that no other Nash Equilibrium exists:

- (LL, M): $u_2(LL) \overset{!}{=} u_2(M) = 0 \iff p = \frac{50}{51}$, but then $u_2(L) = \frac{1}{51} > 0$ and hence deviation would result in a higher payout. Therefore (LL, M) is no Nash Equilibrium.

- (LL, R): $u_2(LL) = u_2(R) \iff p = \frac{1}{2}$, but then $u_2(LL) = -49$ and $u_2(M) = 0 > -49$ and again a contradiction to the Nash Equilibrium (LL, R)
- (L, M), (L, R), (M, R), (M, L, R): choosing on of these strategies we can see in the Normalform representation that Player 1 will always play $D \Rightarrow$ Player 2 plays R without mixing it, hence there is no positiv probability in playing M or L.
- For the remaining cases four cases the proof follows analogously; we find the necessary probability and show that deviation is enlarging the utility.

$\square$

c) M is not part of any Nash Equilibrium. However, M is best response to $\frac{1}{2}U + \frac{1}{2}D$ and therefore rationalisable.

d) Whenever communication is possible, we can even expect (U, LL) or (D, R) as outcome as both players would profit.

Chapter 2

Kooperative Spiele

2.1 Der Kern

2.2 Der Shapley-Wert

2.3 Einfache Spiele

2.4 Konvexe Spiele

2.5 Übungen

Advanced Game Theory - 4. Exercise

Aufgabe 4.1

Gegeben sei ein Drei-Personen-Abstimmungsspiel $\Gamma_C = [N, v]$ mit $N = \{1, 2, 3\}$, in dem jeder Spieler genau eine Stimme hat und in dem anhand der Einfachen-Mehrheit-Regel über die Aufteilung x eines Kuchens auf die drei Personen entschieden werden soll, wobei $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_i \neq 0$ für alle $i \in N$ und $\sum_i x_i \leq i$. Der individuelle Nutzen eines jeden Spielers ist gleich dem Anteil am Kuchen, den er erhält, d.h. $u_i(x_i) = x_i$ für $i \in N$.

a) Bestimmen Sie die charakteristischen Funktionswerte v(K) aller Koaliationen $K \subseteq N$.

Proof:

$$v(\{1\}) = 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0, \quad v(\{1,2,3\}) = 1$$

$$v(\{1,2\}) = 1, \quad v(\{2,3\}) = 1, \quad v(\{1,3\}) = 1$$

□

b) Bestimmen Sie den Kern $C(\Gamma_C)$ und den Shapley-Wer $\Phi(\Gamma_C)$.

Proof: Den Kern $C(\Gamma_C)$ erhält man, indem man einer Aufteilung x_1, x_2, x_3 das folgende Gleichungssystem als Randbedingungen mitgibt:

$$x_1 + x_2 + x_3 = 1 = v(\{1,2,3\})$$
$$x_1 + x_3 \geq 1 = v(\{1,3\})$$
$$x_2 + x_3 \geq 1 = v(\{2,3\})$$
$$x_1 + x_2 \geq 1 = v(\{1,2\})$$
$$x_3 \geq 0 = v(\{3\})$$
$$x_2 \geq 0 = v(\{2\})$$
$$x_1 \geq 0 = v(\{1\})$$

Setzen wir die Gleichungen 2 - 4 ineinander ein, so erhalten wir:

$$x_1 \geq 1 - x_2, \quad x_3 \geq 1 - x_2$$
$$1 - x_2 + 1 - x_2 \geq 1 \iff x_2 \geq \frac{1}{2}$$

Aus Symmetrie (oder einfach Wiederholung der obigen Schritte für x_1 und x_2) erhalten wir:

$$x_1, x_2, x_3 \geq \frac{1}{2}.$$

Allerdings bedeutet dies:

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \leq x_1 + x_2 + x_3 = 1, \qquad \lightning$$

d.h. $C(\Gamma_C) = \emptyset$. Für den Shapley-Wert betrachten wir folgendes:

Reihenfolge/Marg. Beitrag	Sp. 1	Sp. 2	Sp. 3
$1,2,3$	0	1	0
$1,3,2$	0	0	1
$2,1,3$	1	0	0
$2,3,1$	0	0	1
$3,1,2$	1	0	0
$3,2,1$	0	1	0
$\phi_i(\Sigma_C) = \Sigma$	2	2	2

d.h. $\Phi(\Sigma_C) = \left(\frac{2}{6}, \frac{2}{6}, \frac{2}{6}\right)$. $\square$

c) Lösen Sie die Teilaufgaben a) und b) unter der Bedingung, dass Koalitionen, die sowohl Spieler 2 als auch Spieler 3 enthalten, nicht gebildet werden.

Proof: Die Randbedingung ändern sich wie folgender Maßen:

$$x_1 + x_2 + x_3 = 1 = v(\{1,2,3\})$$
$$x_1 + x_3 \geq 1 = v(\{1,3\})$$
$$x_2 + x_3 \geq 0 = v(\{2,3\})$$
$$x_1 + x_2 \geq 1 = v(\{1,2\})$$
$$x_3 \geq 0 = v(\{3\})$$
$$x_2 \geq 0 = v(\{2\})$$
$$x_1 \geq 0 = v(\{1\})$$

d.h. die eine/zwei Randbedingungen werden trivial. Der Kern besteht also aus

$$x_3 \geq 1 - x_1, \quad x_2 \geq 1 - x_1$$

$$\Rightarrow x_1 = 1$$

D.h. $C(\Gamma_C) = \{(1,0,0)\}$. Der Shapely-Wert lässt sich wieder über folgendes bestimmen

Reihenfolge/Marg. Beitrag	Sp. 1	Sp. 2	Sp. 3
1, 2, ~~3~~	0	1	0
1, 3, ~~2~~	0	0	1
2, 1, ~~3~~	1	0	0
2, ~~3, 1~~	0	0	0
3, 1, ~~2~~	1	0	0
3, ~~2, 1~~	0	0	0
$\phi_i(\Sigma_C) = \Sigma$	2	1	1

d.h. $\Phi(\Sigma_C) = \left(\frac{2}{c}, \frac{1}{c}, \frac{1}{c}\right)$; die Frage bleibt aber, welchen Wert c annehmen muss. Mein Tipp wäre $\frac{v(N)}{\sum \phi_i(\Sigma_C)}$[1]. Laut Musterlösung gilt $c = 4$ was konsistent mit meiner Vermutung wäre. □

d) Lösen Sie die Teilaufgaben a) und b) für das Drei-Personen-Abstimmungsspiel $\Gamma_C = [N, v]$ mit $N = \{1,2,3\}$ in dem Spieler 1 ein Stimmengewicht von 60% und Spieler 2 und 3 von jeweils 20% besitzen und Entscheidungen anhand der Zweidrittel-Mehrheit-Regel (qualifizierte Mehrheit) getroffen werden.

Proof:

(i) Bestimmen Sie die charakteristischen Funktionswerte v(K) aller Koaliationen $K \subseteq N$.

$$v(\{1\}) = 0, \quad v(\{2\}) = 0, \quad v(\{3\}) = 0, \quad v(\{1,2,3\}) = 1$$

$$v(\{1,2\}) = 1, \quad v(\{1,3\}) = 1, \quad v(\{2,3\}) = 0$$

[1]Scheint mir nicht ganz konsistent mit der Vorlesung ($1/n!$) zu sein

(ii) Bestimmen Sie den Kern $C(\Gamma_C)$ und den Shapley-Wer $\Phi(\Gamma_C)$.

Den Kern $C(\Gamma_C)$ erhält man, indem man einer Aufteilung x_1, x_2, x_3 das folgende Gleichungssystem als Randbedingungen mitgibt:

$$\begin{aligned} x_1 + x_2 + x_3 = 1 &= v(\{1,2,3\}) \\ x_1 + x_3 \geq 1 &= v(\{1,3\}) \\ x_1 + x_2 \geq 1 &= v(\{1,2\}) \\ x_2 + x_3 \geq 0 &= v(\{2,3\}) \\ x_3 \geq 0 &= v(\{3\}) \\ x_2 \geq 0 &= v(\{2\}) \\ x_1 \geq 0 &= v(\{1\}) \end{aligned}$$

d.h. $x_3 \geq 1 - x_1$, $x_2 \geq 1 - x_1$.

$$1 \geq 1 - x_3 + x_2 + x_3 \iff x_2 = 0$$

$$1 \geq 1 + x_3 + x_2 - x_2 \iff x_3 = 0$$

d.h. $\Phi(\Sigma_C) = (1, 0, 0)$. Schneller geht das auch, durch das Theorem, dass bei einem Veto-Spieler alle anderen die Auszahlung 0 erhalten müssen. Für den Shapley-Wert betrachten wir:

Reihenfolge/Marg. Beitrag	Sp. 1	Sp. 2	Sp. 3
$1,2,3$	0	1	0
$1,3,2$	0	0	1
$2,1,3$	1	0	0
$2,3,1$	1	0	0
$3,1,2$	1	0	0
$3,2,1$	1	0	0
$\phi_i(\Sigma_C) = \Sigma$	4	1	1

d.h. $\Phi(\Sigma_C) = \left(\frac{4}{c}, \frac{1}{c}, \frac{1}{c}\right)$, $c = 4(?)$.

$\square$

Aufgabe 4.2

Gegeben sei folgende Auszahlungstabelle eines Zwei-Personen-Spiels in Normalform

	s_{21}	s_{22}
s_{11}	$3,3$	$0,\alpha$
s_{12}	$\alpha,0$	$1,1$

Beschreiben Sie jeweils für $\alpha = 5$ und $\alpha = 7$ das korrespondierende Koalitionsspiel $\Gamma_C = [N, v]$ und bestimmen Sie den Kern $C(\Gamma_C)$.

Proof: Wir haben das Spiel gegeben durch $N = \{1, 2\}$ und $v\colon P(N) \Rightarrow \mathbb{R}$.

Angenommen v ist superadditiv, und da wir wissen, dass dieses Spile symmetrisch ist, gilt:

a) $\alpha = 5$

$$v(N) = \max_{i,j,k,j \in N} u(s_{ij}, s_{kj}) = \max_{i,j,k,j \in N} (u_1(s_{ij}, s_{kj}) + u_2(s_{ij}, s_{kj})) = 3 + 3 = 6,$$
$$v(\{1\}) = v(\{2\}) = \min_{i,j,k,j \in N} u_{1/2}(s_{ij}, s_{kj}) = 1$$

Um den Kern zu bestimmen, betrachte:

$$x_1 + x_2 = 6 = v(\{1, 2, 3\})$$
$$x_2 \geq 1 = v(\{2\})$$
$$x_1 \geq 1 = v(\{1\})$$

$$\Rightarrow C(\Gamma_C) = \{x_1, x_2 \colon x_1, x_2 \geq 1, x_1 + x_2 = 6\} \neq \emptyset$$

b) $\alpha = 6$

$$v(N) = \max_{i,j,k,j \in N} u(s_{ij}, s_{kj}) = \max_{i,j,k,j \in N} (u_1(s_{ij}, s_{kj}) + u_2(s_{ij}, s_{kj})) = 7 + 0 = 0 + 7 = 7,$$
$$v(\{1\}) = v(\{2\}) = \min_{i,j,k,j \in N} u_{1/2}(s_{ij}, s_{kj}) = 1$$

Um den Kern zu bestimmen, betrachte:

$$x_1 + x_2 = 7 = v(\{1, 2, 3\})$$
$$x_2 \geq 1 = v(\{2\})$$
$$x_1 \geq 1 = v(\{1\})$$

$$\Rightarrow C(\Gamma_C) = \{x_1, x_2 \colon x_1, x_2 \geq 1, x_1 + x_2 = 7\} \neq \emptyset$$

$\square$

Aufgabe 4.3

Ein Kleintierzüchterverein hat sieben Mitglieder: zwei Meerschweinchenzüchter M_1 und M_2, zwei Taubenzüchter T_1 und T_2 und drei Hasenzüchter H_1, H_2 und H_3. Entscheidungen werden mit einfacher Mehrheit gefällt.

a) Beschreiben Sie unter der Bedingung, dass die Mitglieder einer Zuchtgruppe stets einheitlich abstimmen, das Koalitionsspiel $\Gamma_C = [N, v]$ für die drei unabhängingen Spieler in Form der drei Zuchtgruppen $M = \{M_1, M_2\}$, $T = \{T_1, T_2\}$ und $H = \{H_1, H_2, H_3\}$, also $N = \{M, T, H\}$, und berechnen Sie die Shapley-Werte für M, T und H.

Proof: Es gilt $\Gamma_C = [N, v]$, wobei $N = \{M, T, H\}$ und $v\colon P(N) \Rightarrow \mathbb{N}$ mit:

$$v(N) = 1,\ v(\{M, T\}) = 1,\ v(\{T, H\}) = 1,\ v(\{M, H\}) = 1,$$

$$v(\{M\}) = v(\{T\}) = v(\{H\}) = 0.$$

Für den Shapley-Wert betrachten wir:

Reihenfolge/Marg. Beitrag	M	T	H
M, T, H	0	1	0
T, H, M	0	0	1
T, M, H	1	0	0
H, T, M	0	1	0
H, M, T	1	0	0
M, H, T	0	0	1
$\phi_i(\Sigma_C) = \Sigma$	2	2	2

d.h. $\Phi(\Sigma_C) = \left(\frac{2}{c}, \frac{2}{c}, \frac{2}{c}\right)$, $c = 6(?)$. $\square$

b) Eines Tages zerstreiten sich die drei Hasenzüchter, was dazu führt, dass sie die Hasenkoalition auflösen und in Abstimmungen einzeln auftreten. Die Meerschweinchenzüchter und Taubenzüchter stimmen weiterhin einheitlich ab. Wie lauten die Ergebnisse von Teilaufgabe a) für die fünf unabhängingen Spieler M, T, H_1, H_2 und H_3. Vergleichen Sie die Shapley-Werte mit denen von Teilaufgabe a). Was fällt auf?

Proof: Es gilt $\Gamma_C = [N, v]$, wobei $N = \{M, T, H_1, H_2, H_3\}$ und $v\colon P(N) \Rightarrow \mathbb{N}$ mit:

$$v(\{T, H_1, H_2, H_3\}) = 1,\ v(\{M, H_1, H_2, H_3\}) = 1,\ v(\{T, H_i, H_j\}) = 1,\ v(\{M, H_i, H_j\}) = 1,$$

$$v(\{M\}) = v(\{T\}) = v(\{H_i\}) = v(\{H_i, H_j\}) = v(\{H_1, H_2, H_3\}) = 0.$$

$$v(N) = 1,\ v(\{M, T\}) = 1,\ v(\{T, H_i\}) = 0,\ v(\{M, H_i\}) = 0$$

Aus Symmetrie-Gründen können wir den Shapley-Wert für z.B. T berechnen über:

Reihenfolge	Marg. Beitrag von T
$M, T, \pi(H_1, H_2, H_3)$	6
$M, H_i, T, \pi(H_j, H_k)$	$3 \cdot 2 = 6$
$H_i, M, T, \pi(H_j, H_k)$	$3 \cdot 2 = 6$
$\pi(H_i, H_j), T, M, H_k$	$3 \cdot 2 = 6$
$\pi(H_i, H_j), T, H_k, M$	$3 \cdot 2 = 6$
$\pi(H_1, H_2, H_3), M, T$	6
$\phi_T(\Sigma_C) = \Sigma$	36

d.h. $\Phi_T(\Sigma_C) = \frac{36}{n!} = \frac{36}{120} = \frac{3}{10}$. Eben aus Symmetrie-Gründen gilt: $\Phi_T(\Sigma_C) = \Phi_M(\Sigma_C)$.

Schließlich gilt wieder aus Symmetriegründen:

$$\Phi_{H_i}(\Sigma_C) = \frac{1 - \Phi_T(\Sigma_C) - \Phi_M(\Sigma_C)}{3} = \frac{1 - 0,\overline{3} - 0,\overline{3}}{3} = 0,1\overline{3}, \quad \forall i \in \{1, 2, 3\}.$$

Es fällt auf, dass in der Summe die Shapley Werte der Hasen höher ist, als in der a). Dies ist der Kritikpunkt am Shapley-Wert. □

Chapter 3

Evolutionäre Spieltheorie

3.1 Spiele in Normalform

Für symmetrische Spiele:

$$A = (a_{ij}) \quad i = 1, \ldots, m_i,\ j = 1, \ldots, m_j$$

d.h.

N: Spielermenge $|N| = n$

Σ_i: Menge der reinen Strategien von $i \in N$, $|\Sigma_i| = m_i$, $\sigma_i \in \mathcal{E}_i$.

S_i: Menge der gemischten Strategien von $i \in N$

$S_i = \{\}$.

$s_{ij} = \mathbb{P}(\sigma_{ij})$.

Definition (Trägermenge)**:** *Wir definieren die Trägermenge für jeden Spieler $i \in N$:*

$$C(S_i) = \{\sigma_{ij} \in \Sigma_i : s_{ij} > 0\},$$

als die Menge der Strategien die mit positiver Wahrscheinlichkeit gespielt werden.

Definition (Beste-Antwort-Menge)**:** *Sei*

$$B_i(s_{-i}) = \left\{\sigma_j \in \Sigma_i : H(\sigma_{ij}, s_{-i}) = \max_{\sigma_{ik} \in \Sigma_i} H(\sigma_{ik}, s_{-i})\right\}$$

H bezeichne pay-off-Funktion

$$\hat{H}(s_{-i}) \coloneqq \max_{\sigma_{il} \in \Sigma_i} H(\sigma_{ik}, s_{-i})$$

Beispiel: $\sigma_{ij} \in B_i(S_{-i})$ und $\sigma_{ik} \in B(S_{-i}) \Rightarrow$ alle $s_i \in S_i$ mit

$$C(S_i) = \{\sigma_{ij}, \sigma_{ik}\}$$

sind auch beste Antwort, denn

$$s_{ij} H(\sigma_j, s_{-i}) + s_{ik} H(\sigma_{ik}, s_{-i}) = (s_{ij} + s_{ik}) \hat{H}(s_{-i}) = \hat{H}(s_{-i}).$$

Sei $s^* = (s_1^*, \dots, s_n^*)$ ein Nash-Gleichgewicht. Mit $s_i^* = (s_{i1}^*, \dots s_{im_i}^*)$ gilt

$$C(s_i^*) \subseteq B_i(s_{-i}^*)$$

Hinreichend? Ja! Proposition Slide 36 (AGT Teil 1).

3.1 Grundannahmen der evolutionären Spieltheorie: a) große Population

b) Population ist monomorph

c) random matching

d) Wettstreit (Spiel) ist statisch und symmetrisch

$\Rightarrow$ symmetrisches Spiel in Normalform mit zwei Spielern.

e) Auszahlung entspricht der “biologischen Fitness” (ϕ Anzahl Nachkommen)

f) Reproduktion ist asexuell und die von den Eltern gewählt Strategie wird unverändert an die Nachkommen vererbt (nur Selektion, keine Mutation).

3.2 Symmetrisches 2-Personenspiel in Normalform: Spieler müssen nicht unterschieden werden $\Rightarrow$ Strategieraum:

$$S = \{s \in \mathbb{R}^m : \sum_{i=1}^{m} s_i = 1, s_i \geq 0, i = 1, \ldots, m\}$$

Definition (Evolutionär stabile Strategie, ESS)**:** *Eine Strategie $p \in S$ heißt evolutionär stabil, wenn*

a) $H(p,p) \geq H(q,p)$ für alle $q \in S$ (Gleichgewichtsbedingung)

b) Für alle $q \in S \setminus \{p\}$ mit $H(q,p) = H(p,p)$ gilt: $H(p,q) > H(q,q)$ (Stabilitätsbedingung)

3.3 Eigenschaften von evolutionär stabilen Strategien:

- Ist $p \in S$ eine evolutionär stabile Strategie, dann bildet (p,p) ein symmetrisches Nash-Gleichgewicht

- Jede 2×2-Matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ mit $H(p,p) = p'Ap$ sodass $a_{11} \neq a_{21}$ und / oder $a_{12} = a_{22}$, besitzt eine ESS Ist (p,p) ein striktes NGG, dann ist p eine ESS. Im strikten NGG (p,p) gilt $C(p) = B(p)$. Ein striktes NGG ist immer ein Gleichgewicht

  in reinen Strategien. Beispiel:

$3,3$	$2,0$
$0,2$	$4,4$

- Im Normalformspielen mit $m \times m$-Matrizen a mit $m \geq 3$ existieren entweder endlich viele ESS keine.

3.4 Allgemein gilt: Ist p ESS $\Rightarrow \neg\exists\sigma \in C(p)$ mit $\sigma \in C(S^*)$ für $s^* \neq q$ ist Nash-Gleichgewicht

$\Rightarrow$ # ESS $\leq |\Sigma|$ - Gleichheit nur, fall es kein ESS in gemischten Strategien gibt.

3.2 Übungen

Advanced Game Theory - 5. Exercise

Aufgabe 5.2

b) Gegeben sei:

	σ_1	σ_2
σ_1	0, 0	0, 0
σ_2	0, 0	1, 1

$$\Rightarrow A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ (Gegenbeispiel)}$$

$$\sigma^* = (\sigma_1, \sigma_1), \sigma^{**} = (\sigma_2, \sigma_2)$$

a) Angenommen $p \in S$ ist ESS und wird von $q \in S$ schwach dominiert

$$\Rightarrow H(q,z) \geq H(p,z) \quad \forall z \in S$$

$$\Rightarrow H(p,p) = H(q,p) \quad \text{Bedingung 1: ok}$$

$$\Rightarrow H(p,q) \leq H(q,q) \quad \text{Bedingung 2: verletzt}$$

c) klar!

Aufgabe 5.1 (Hawk-Dove-Game / Falke-Taube-Spiel)

	F	T
F	$\frac{v-c}{2}, \frac{v-c}{2}$	$v, 0$
T	$v, 0$	$\frac{v}{2}, \frac{v}{2}$

$A = \begin{pmatrix} \frac{v-c}{2} & v \\ 0 & \frac{v}{2} \end{pmatrix}, c > v > 0$

- es existiert keine dominante Strategie
- es existiert kein symmetrisch Nash-Gleichgewicht in reinen Strategien
- (F, T), (T, F) sind strikte Nash-Gleichgewichte

Interpretation: Recource v, Tauben teilen friedlich, Falken vertielgt Zaube, Falken kämpfen $\Rightarrow$ neg, outcome für beide.

$p = (p_F, p_T)$

$$H(F,p) \overset{!}{=} H(T,p) \overset{!}{=} H(p,p)$$

$$\left.\begin{aligned} H(F,p) &= p_F \tfrac{v-c}{2} + p_T v \\ H(T,p) &= p_F 0 + p_T \tfrac{v}{2} \end{aligned}\right\} \xRightarrow{p_F + p_T = 1} p_F = \frac{v}{c}, p_T = 1 - \frac{v}{c}$$

ist das einzige symmetrisch Nas-Gleichgewicht, kein triviales Spiel $\Rightarrow \exists ESS$

$$\Rightarrow \left(\frac{v}{2}, 1 - \frac{v}{2}\right) \text{ ist ESS}$$

oder man rechnet nach $H(F,p) = H(t,p) = H(p,p)$

$\Rightarrow z.z. H(p,F) > H(F,F)$, $H(p,T) > H(T,T)$

Aufgabe 5.3

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

a) Nash-Gleichgewicht in reinen Strategien:

$$(x,x),\ (x,y),\ (y,x),\ (y,z),\ (z,y),\ (z,z)$$

Trivial: $C(A) = \{A\}$, $A \in \{x, y, z\}$

$$B(x) = \{x, y\},\ B(y) = \{x, y, z\},\ B(z) = \{y, z\}$$

$$\Rightarrow C(\cdot) \underset{\neq}{\subset} B(\cdot)$$

Nash-Gleichgewicht in gemischten Strategien (nur sym.)

$$S^* = \{(s_x, s_y, 0) : s_x \in (0,1), s_y = 1 - s_x\} \quad C(S^*) = \{x, y\}$$

$$S^{**} = \{(0, s_y, s_z) : s_y \in (0,1), s_z = 1 - s_y\} \quad C(S^{**}) = \{y, z\}$$

$B(S^*) = \{x, y\}$, $B(S^{**}) = \{y, z\}$

b) Angenommen $p \in S$ mit $p_x \in [0, 1]$ und $p_y = 1 - p_x$ ist ein ESS

Bedingung 1: ✓

Bedingung 2: $H(x, p) = H(p, p) = 1$ mit $p_x < 1$

$$\Rightarrow H(p, x) > H(x, x) \text{ Widerspruch!}$$

analog in anderen Fällen $\Rightarrow$ ESS existiert nicht.

Erinnerung: Ist p ESS $\Rightarrow \neg\exists\sigma \in C(p)$ mit $\sigma \in C(S^*)$ für $s^* \neq q$ ist Nash-Gleichgewicht.
$\Rightarrow$ # ESS $\leq |\Sigma|$ - wobei Gleichheit nur gilt, fall es kein ESS in gemischten Strategien gibt.