Advanced Game Thoerie

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Contents

Chapter 1

Noncooperative Games

1.1 Basic Elements of Noncooperative Games

Definition: *A **game** is a formal representation of a situation in which a number of individuals interact in a setting of strategic interdependence.*

- *The players: Who is involved?*
- *The rules: Who moves when? What do they know when they move? What can they do?*
- *The outcomes: For each possible set of actions by the players, what is the outcome of the game?*
- *The payoffs: What are the players' preferences over the possible outcomes?*

Example 1.1 (of simultaneous move games)**:**

a) Matching Pennies

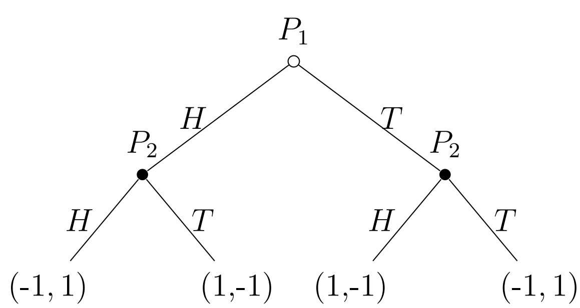
		Player 2: Heads	Player 2: Tails
Player 1	Heads	$-1, 1$	$1, -1$
	Tails	$1, -1$	$-1, 1$

b) Meeting in New York

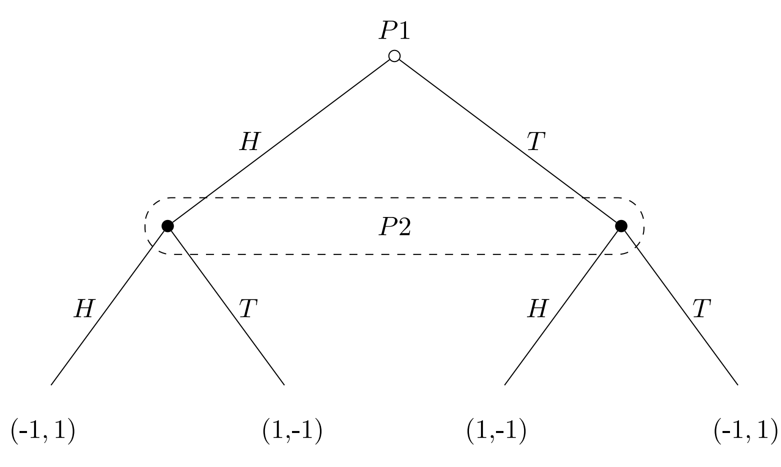
		Player 2	
		Empire State	Grand Central
Player 1	Empire State	100, 100	0, 0
	Grand Central	0, 0	100, 100

c) Examples of (simple) dynamic games

Prisoner's Dilemma in Extensive-form



d) Matching Pennies Version C



Definition (Information)**:**

a) ***Information Set:*** *A player doesn't know which of the nodes in the information set she is actually at. Therefore, at any decision node in a player's information set, there must be the same possible actions.*

b) ***Perfect Information:*** *A game is said to be of perfect information if each information set contains a single decision node. Otherwise, it is a game of* ***imperfect information****.*

Definition (Extensive Form Game)**:** *A game in* ***extensive form*** *consists of:*

(i) *A finite set of nodes $\mathcal{X}$, a finite set of possible actions $\mathcal{A}$, and a finite set of players $\{1, \dots, l\}$.*

(ii) *A funktion $p\colon \mathcal{X} \to \{\mathcal{X} \cup \emptyset\}$ specifying a single immediate predecessor of each node x; $p(x) \in \mathcal{X}$ expect for one element x_0, the* ***initial node****. The immediate* ***successor node*** *of x are $s(x) = p^{-1}(x)$.*
To have a tree structure, a predecessor can never be a successor and vice versa. The set of ***terminal nodes*** *$T = \{x \in \mathcal{X} \colon s(x) = \emptyset\}$. All other nodes $X \setminus T$ are* ***decision nodes****.*

(iii) *A function $\alpha\colon \mathcal{X} \setminus \{x_0\} \to \mathcal{A}$ giving the action that leads to any non-initial node x from its immediate predecessor $p(x)$ with $x', x'' \in s(x); x' \neq x'' \Rightarrow \alpha(x') \neq \alpha(x'')$. The set of choices at decision node x is $c(x) = \{a \in \mathcal{A} \colon a = \alpha(x')$ for some $x' \in s(x)\}$.*

(iv) *A collection of information sets $\mathcal{H}$, and a function $H\colon \mathcal{X} \to \mathcal{H}$ assigning each decision node x to an information set $H(x) \in \mathcal{H}$ with $c(x) = c(x')$ if $H(x) = H(x')$.*

The choices available at information set H can be written as

$$C(H) = \{a \in \mathcal{A}\colon a \in c(x) \text{ for } x \in H\}.$$

(v) *A function $\iota\colon \mathcal{H} \to \{0, 1, \ldots, l\}$ assigning a player to each information set ($i = 0$ 'nature').*

The collection of player i's information set is denoted by

$$\mathcal{H}_i = \{H \in \mathcal{H}\colon i = \iota(H)\}.$$

(vi) *A function $\rho\colon \mathcal{H}_0 \times \mathcal{A} \to [0, 1]$ assigning a probability to each action of nature with $\rho(H, a) = 0$ if $a \notin C(H)$ und $\sum_{a \in C(H)} \rho(H, a) = 1$ for all $H \in \mathcal{H}_0$.*

(vii) *A collection of payoff function $u = \{u_1(\cdot), \ldots, u_l(\cdot)\}$, where $u_i\colon T \to \mathbb{R}$.*

A game in extensive form: $\Gamma_E = \{\mathcal{X}, \mathcal{A}, I, p(\cdot), \alpha(\cdot), \mathcal{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}$.

Comment: Restrictions of this definition:

a) Finite set of actions

b) Finite number of moves

c) Finite number of players

Definition (Strategy)**:** *Let $\mathcal{H}_i$ denote the collection of player i's information sets, $\mathcal{A}$ the set of possible actions in the game, and $C(H) \subset \mathcal{A}$ the set of actions possible at information set H. A **strategy** for player i is a function $s_i\colon \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.*

Definition (Normal Form Representation)**:** *For a game with I players, the **normal form representation** Γ_N specifies for each player i a set of strategies $\mathcal{S}_i$ (with $s_i \in \mathcal{S}_i$) and a payoff function $u_i(s_1, \ldots, s_l)$, formally*

$$\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}].$$

Definition:

a) $s_i \colon \mathcal{H}_i \to \mathcal{A}$ *describes deterministic choices at each* $H \in \mathcal{H}_i$ *and is called a* ***pure strategy***

b) a ***mixed strategy*** *is a probability distribution over all pure strategies* $\sigma_i \colon \mathcal{S}_i \to [0,1]$, *with* $\sigma_i(s_i) \geq 0$ *and* $\sum_{s_i \in \mathcal{S}_i} \sigma_i(s_i) = 1$.

c) player i's set of possible mixed strategies can be associated with the points of the simplex $\Delta(\mathcal{S}_i)$, *called the* ***mixed extension*** *of* $\mathcal{S}_i$.

d) since we assume that individuals are expected utility maximisers, player i's utility of a profile of mixed strategies $\sigma = (\sigma_i, \ldots, \sigma_l)$ *is given by*

$$u_i(\sigma) = \sum_{s \in \mathcal{S}} [\sigma_1(s_1) \cdot \sigma_2(s_2) \cdot \ldots \cdot \sigma_l(s_l)] \cdot u_i(s),$$

where $s = (s_1, \ldots, s_l)$.

Definition (Behaviour Strategy)**:** *Given an extensive form game* Γ_E, *a* ***behaviour strategy*** *for player i specifies for every information set* $h \in \mathcal{H}_i$ *and action* $a \in C(H)$, *a probability* $\lambda_i(a, H) \geq 0$, *with*

$$\sum_{a \in C(H)} \lambda_i(a, H) = 1 \text{ for all } H \in \mathcal{H}_i.$$

Definition (Perfect Recall)**:** *A player has* ***perfect recall*** *if he doesn't "forget" what she once knew, including her own actions.*

Theorem 1.2: *If* Γ_E *is an extensive form game with perfect recall, then for any mixed strategy there is an outcome equivalent behaviour strategy and vice versa.*

1.2 Rationalisable Strategies

Central question of Game Theory: What should we expect to observe in a game played by rational players? Or more precisely: What should we expect to observe in a game played by rational players who are fully knowledgeable about the structure of the game and each others' rationality?

We first address the above question for simultaneous-move games, which we study using their normal form representation. We use the following notation:

- $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)]$ if we consider pure strategies only,
  $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if we allow for mixed strategies
- $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_l) \in \mathcal{S}_{-i}$ where $\mathcal{S}_{-i} = S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \cdots \times S_l$
- $s = (s_i, s_{-i})$

Example 1.3 (Prisoners' Dilemma)**:**

		Player 2: don't confess	Player 2: confess
Player 1	don't confess	$-2, -2$	$-10, -1$
	confess	$-1, -10$	$-5, -5$

What should we expect to observe in the Prisoners' Dilemma?

Definition (Strictly Dominant Strategy)**:** *A strategy $s_i \in \mathcal{S}_i$ is strictly dominant for player i in game $\Gamma_N =]I, \{\mathcal{S}_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$:*

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

for all $s_i \in \mathcal{S}_{-i}$.

Applied to Prisoner's Dilemma: Confess is a strictly dominant strategy for each player.

Definition (Strictly Dominated Strategy)**:** *$s_i \in \mathcal{S}_i$ is **strictly dominated** for player i in game Γ_N if there exists another strategy $s'_i \in \mathcal{S}_i$ such that:*

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in \mathcal{S}_{-i}$. In this case we say that s'_i strictly dominates s_i.

Definition (Weakly Dominated Strategy)**:** *$s_i \mathcal{S}_i$ is weakly dominated for player i in game Γ_N if there exists another strategy $s'_i \in \mathcal{S}_i$ such that:*

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

for all $s_{-i} \in \mathcal{S}_{-i}$, with strict inequality for at least one s_{-i}.

Example 1.4:

		Player 2	
		L	R
	U	$1, -1$	$-1, 1$
Player 1	M	$-1, 1$	$1, -1$
	D	$-2, 5$	$-3, 2$

$\Rightarrow$ D is strictly dominated by U and M.

		Player 2	
		L	R
	U	$5, 1$	$4, 0$
Player 1	M	$6, 0$	$3, 1$
	D	$6, 4$	$4, 4$

$\Rightarrow$ U and M are weakly dominated by D.

Example 1.5 (Prisoners' Dilemma – A Variation)**:** Assume Prisoner 1 is the district attorney's brother: If neither player confesses, player 1 is free

		Player 2	
		don't confess	confess
Player 1	don't confess	$0, -2$	$-10, -1$
	confess	$-1, -10$	$-5, -5$

$\Rightarrow$ D is strictly dominated by U and M.

$\Rightarrow$ Player 1 has no dominant strategy anymore.

In this game, the iterated elimination of strictly dominated strategies still leads to a unique prediction. In general, the order of elimination of strictly dominated strategies does not matter! How about iterated elimination of weakly dominated strategies?

Definition: *A strategy $\sigma_i \in \Delta(\mathcal{S}_i)$ is strictly dominated for i in game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that for all $\sigma_{-i} \in \Pi_{j \neq i}\Delta(\mathcal{S}_j)$:*

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

Proposition 1.6: Player i's pure strategy $s_i \in \mathcal{S}_i$ is strictly dominated in a game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in \mathcal{S}_{-i}.$$

Proof: This follows because we can write

$$u_i(\sigma_i', \sigma_{-i}) - u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in \mathcal{S}_{-i}} [\Pi_{k \neq i}\sigma_k(s_k)] \left[u_i(\sigma_i', s_{-i}) - u_i(s_i, s_{-i})\right].$$

And this expression is positive for all σ_{-i} if and only if $u_i(\sigma_i', s_{-i}) - u_i(s_i, s_{-i})$ is positive for all s_{-i}. □

Example 1.7:

		Player 2	
		L	R
	U	$10, 1$	$0, 4$
Player 1	M	$4, 2$	$4, 3$
	D	$0, 5$	$10, 2$

$$\Rightarrow \frac{1}{2}U + \frac{1}{2}D \text{ strictly dominates } M.$$

Definition (Best response)**:** *The strategy σ_i is a **best response** for player i to her rivals' strategies σ_{-i} if:*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(\mathcal{S}_i)$. Strategy σ_i is never a best response if there is no σ_{-i} for which σ_i is a best response.

Definition (Rationalisable Strategies)**:** *In game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(\mathcal{S}_i)$ that survive the iterated elimination of strategies that are never a best response are known as player i's **rationalisable strategies**.*

Example 1.8:

		Player 2			
		b_1	b_2	b_3	b_4
Player 1	a_1	$0, \underline{7}$	$2, 5$	$\underline{7}, 0$	$0, 1$
	a_2	$5, 2$	$\underline{3}, \underline{3}$	$5, 2$	$0, 1$
	a_3	$\underline{7}, 0$	$2, 5$	$0, \underline{7}$	$0, 1$
	a_4	$0, \underline{0}$	$0, -2$	$0, \underline{0}$	$\underline{10}, -1$

$$\Rightarrow \frac{1}{2}U + \frac{1}{2}D \text{ strictly dominates } M.$$

$\Rightarrow$ b_4 is never best response for player 2 and *then* a_4 is never best response for player 1.

$\Rightarrow$ $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are the rationalisable strategies in this game.

1.3 Nash Equilibrium

1.4 Subgame Perfection in Dynamic Games

1.5 Excercises

Chapter 2

Kooperative Spiele

2.1 Der Kern

2.2 Der Shapley-Wert

2.3 Einfache Spiele

2.4 Konvexe Spiele

2.5 Übungen

Chapter 3

Evolutionäre Spieltheorie