# Bilateral Bargaining: An Infinite Sequential Game

Consider the following game: there is an amount $v$ of pie being sequentially proposed to split between two players; let $\delta \in (0, 1)$ be a discounting factor.

Period 1:

- Player 1 chooses a share $x_1 \in [0, v]$ of the pie for himself and offers the split.
- Player 2 can either reject the offer, the pie size is reduced by $\delta$ to $\delta v$ and the game continues in period 2, or accept, the split is implemented as proposed by player 1 and the game ends immediately with $u_1 = x_1, u_2 = v - x_1$.

Period 2:*

- Player 2 chooses a share $x_2 \in [0, \delta v]$ of the pie for himself and anon offers the split.
- Player 1 can either reject the offer, the pie size is again reduced by $\delta$ to $\delta^2 v$ and the game continues in period 2, or accept, the split is implemented as proposed by player 1 and the game ends immediately with $u_1 = \delta v - x_2, u_2 = x_2$.

and so on...

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Now consider the bilateral bargaining game with infinite horizon:

**Proposition (Shaked & Sutton (1984))**

The infinite horizon bargaining game has a unique SPNE in which the players reach an agreement in period 1 such that player 1 earns $\frac{v}{1+\delta}$ and player 2 $\frac{\delta v}{1+\delta}$.

*Proof:* Let $\underline{v}_1$ be the lowest payoff player 1 receives in any SPNE in a subgame when she makes the initial offer and let $\overline{v}_1$ be her highest SPNE payoff in such a subgame; likewise, we define $\underline{v}_2$ and $\overline{v}_2$.

To derive a sufficient condition, we set the property for $i, j = 1, 2$, $i \neq j$:

$$\tilde{v}_i = \delta(\tilde{v} - \tilde{v}_j), \tag{1}$$

where $\tilde{v}_k$ denotes an arbitrary proposal of player k for himself in the respective round $(k = i, j)$ and $\tilde{v}$ is the remaining pie size in player js turn, therewith $\tilde{v} \leq v$ as $\delta < 1$.

More specifically, equation (1) states that the value of a proposal for the player -i discounts each round with the factor $\delta$.

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*Proof (cont.):* By definition, player j won't accept an offer that is lower than $\delta \underline{v}_j$, hence, as $v$ is the maximum pie size and by (1):

$$\overline{v}_i \leq v - \delta \underline{v}_j. \tag{2}$$

Analogously, player $j$ will accept any offer that gives her at least $\delta_j \overline{v}_j$, so

$$\underline{v}_i \geq v - \delta \overline{v}_j. \tag{3}$$

Subtracting the two equations (2) and (3) from each other for both players yields

$$\overline{v}_1 - \underline{v}_1 \leq \delta \left( \overline{v}_2 - \underline{v}_2 \right) \tag{4}$$

and

$$\overline{v}_2 - \underline{v}_2 \leq \delta \left( \overline{v}_1 - \underline{v}_1 \right). \tag{5}$$

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*Proof (cont.):* Multiplying (5) with $\delta$ and using (4) results in

$$\overline{v}_1 - \underline{v}_1 \leq \delta\left(\overline{v}_2 - \underline{v}_2\right) \leq \delta^2\left(\overline{v}_1 - \underline{v}_1\right), \tag{6}$$

and analogously for player 2. Let us recall that $\overline{v}_i$ is the largest payoff in each SPNE for player i and $\underline{v}_i$ is the smallest. As $\delta < 1$ and $\overline{v}_i \geq \underline{v}_i$, (6) implies $\overline{v}_i = \underline{v}_i =: v_i$ for $i = 1, 2$. Hence, the SPNE payoffs are unique in the Bilateral Bargaining.

Using the fact of uniqueness in equations (2) and (3), we can state that

$$v_1 = v - \delta v_2 \quad \text{and} \quad v_2 = v - \delta v_1,$$

and direct substitution therefore yields

$$v_1 = (1-\delta)v + \delta^2 v_i \iff v_1 = \frac{(1-\delta)v}{(1-\delta^2)} = \frac{(1-\delta)v}{(1+\delta)(1-\delta)}$$

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*Proof (cont.):* Hence, the proposal in a SPNE is

$$v_1^* = \frac{v}{(1+\delta)}, \quad v_2^* = v - v_1^* = \frac{\delta v}{(1+\delta)},$$

where $v$ is the pie size available for split.

Interpretation:

- Efficiency: player 2 accepts player 1s first proposal, resulting in immediate agreement without delay; a delay is costly for both player due to discounting.
- This game contains a first mover advantage: whoever makes the initial proposal offers herself a higher share.
- A similar result can be made for individual discounting rates $(\delta_1, \delta_2)$, where each $\delta_k$ represents the patience of player $k$.