Advanced Game Theory - Exercise 3

3.1

Assumption we make: finite number of pure strategies $\Rightarrow$ there exists a Nash-Equilibrium.

When a strategy σ_i is eliminated then so is every strategy that plays σ_i with positive probability.

S^∞ : set of strategies that survive iterated elimination of strictly dominated strategies.

$|S^\infty| = 1$.

Claim: If $(s_1^*, \dots, s_I^*)$ is a Nash-Equilibrium, then $s^* \in S^\infty$.

Proof: Let $(s_1^*, \dots, s_I^*)$ be a Nash-Equilibrium and assume $s^* \notin S^\infty$. Let i be the player whose strategy is eliminated first (in round k).

i.e. $\exists \sigma_i, \sigma_i' \in \Delta(S_i)$:

$$u_i(\sigma_i, s_{-i}) > u_i(\sigma_i', s_{-i}) \quad \forall s_i \in S_{-i}^{k-1}$$

and σ_i' is played with positiv probability in s_i^*.

Let s_i' be derived from s_i^* with replacing σ_i' by σ_i.

$$\begin{aligned} \Rightarrow \quad u_i(s_i', s_{-i}^*) &= u_i(s_i^*, s_{-i}^*) + \underbrace{s_i^*}_{>0}(\sigma_i') \underbrace{\left[u_i(\sigma_i, s_{-i}^*) - u_i(\sigma_i', s_{-i}^*)\right]}_{>0} \\ &> u_i(s_i^*, s_{-i}^*) \end{aligned}$$

which contradicts the fact that s^* is a Nash-Equilibrium. $\square$

3.3

		Player 2 LL	L	M,	R
Player 1	U	$100, 2$	$-100, 1$	$0, 0$	$-100, -100$
	D	$-100, -100$	$100, -49$	$1, 0$	$100, 2$

a) Play M, todo: explanation

b) Pure Nash-Equilibria: (U, LL) and (D, R)
Mixed Equilibria:

(i) Player 1 mixes U and D with probabilities p and $1-p$ respectively.

(ii) Player 2 can mix between: $(LL, L), (LL, M), (LL, R), (L, M), (L, R),$

$$(M, R), (LL, L, M), (LL, L, R), (LL, M, R), (L, M, R), (LL, L, M, R)$$

Claim: Only (LL, L) will lead to a Nash-Equilibrium.

Proof (Using the Proposition after the Definition of Mixed Strategy NE):
Only (LL, L) will lead to a Nash Equilibrium

$$\begin{aligned} u_2(LL) = u_2(L) \quad &\iff \quad 2p - 100(1-p) = p - 49(1-p) \\ &\iff \quad p = \frac{51}{52} \end{aligned}$$

Therefore: $u_2(LL) = u_2(L) = \frac{1}{26}$, $u_2(M) = 0$, $u_2(R) < 0$.

$$\begin{aligned} u_1(u) = u_1(D) \quad &\iff \quad 100q - 100(1-q) = -100q + 100(1-q) \\ &\iff \quad q = \frac{1}{2} \end{aligned}$$

where q is the probability of Player 2 playing LL.

$$\Rightarrow \quad \text{Nash Equilibrium: } \left(\frac{51}{52}U + \frac{25}{26}D, \frac{1}{2}LL + \frac{1}{2}L\right).$$

Now we have proven that (LL, L) is a Nash Equilibrium. We will subsequently show that no other Nash Equilibrium exists:

- (LL, M): $u_2(LL) \stackrel{!}{=} u_2(M) = 0 \iff p = \frac{50}{51}$, but then $u_2(L) = \frac{1}{51} > 0$ and hence deviation would result in a higher payout. Therefore (LL, M) is no Nash Equilibrium.
- (LL, R): $u_2(LL) = u_2(R) \iff p = \frac{1}{2}$, but then $u_2(LL) = -49$ and $u_2(M) = 0 > -49$ and again a contradiction to the Nash Equilibrium (LL, R)
- (L, M), (L, R), (M, R), (M, L, R): choosing on of these strategies we can see in the Normalform representation that Player 1 will always play $D \Rightarrow$ Player 2 plays R without mixing it, hence there is no positiv probability in playing M or L.
- For the remaining cases four cases the proof follows analogously; we find the necessary probability and show that deviation is enlarging the utility.

$\square$

c) M is not part of any Nash Equilibrium. However, M is best response to $\frac{1}{2}U + \frac{1}{2}D$ and therefore rationalisable.

d) Whenever communication is possible, we can even expect (U, LL) or (D, R) as outcome as both players would profit.