Advanced Game Theory — Part 1

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Prof. Dr. Johannes Brumm

Institute of Economics (ECON)
Karlsruhe Institute of Technology (KIT)

Outline

Section 1

Basic Elements of Noncooperative Games

Outline

What is a game?

A game is a formal representation of a situation in which a number of individuals interact in a setting of *strategic interdependence*.

- The players: Who is involved?
- The rules: Who moves when? What do they know when they move? What can they do?
- The outcomes: For each possible set of actions by the players, what is the outcome of the game?
- The payoffs: What are the players' preferences over the possible outcomes?

What is a game?

Examples of simultaneous move games:

- *Matching Pennies*

		Player 2	
		Heads	Tails
Player 1	Heads	-1, 1	1, -1
	Tails	1, -1	-1, 1

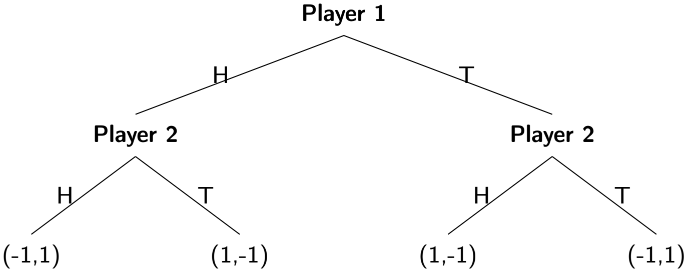
- *Meeting in New York*

		Player 2	
		Empire State	Grand Central
Player 1	Empire State	100, 100	0, 0
	Grand Central	0,0	100, 100

The Extensive Form Representation of a Game

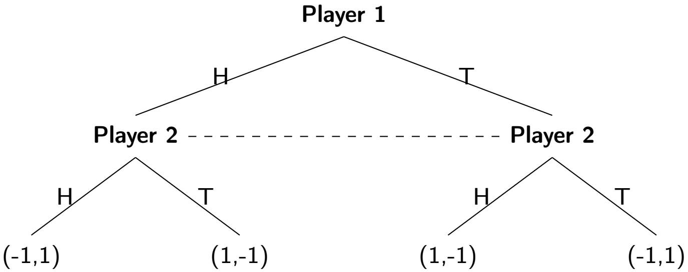
Examples of (simple) dynamic games:

- *Matching Pennies Version B*



The Extensive Form Representation of a Game

- Matching Pennies Version C



Information Set: A player doesn't know which of the nodes in the information set she is actually at. Therefore, at any decision node in a player's information set, there must be the same possible actions.

Perfect Information: A game is said to be of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of imperfect information.

The Extensive Form Representation: Definition (1/2)

Definition (Extensive Form Game)

A game in *extensive form* consists of:

(i) A finite set of nodes $\mathcal{X}$, a finite set of possible actions $\mathcal{A}$, and a finite set of players $\{1, ..., I\}$.

(ii) A function $p : \mathcal{X} \to \{\mathcal{X} \cup \emptyset\}$ specifying a single immediate predecessor of each node x; $p(x) \in \mathcal{X}$ except for one element x_0, the <u>initial node</u>. The immediate successor nodes of x are $s(x) = p^{-1}(x)$.
To have a tree structure, a predecessor can never be a succesor and vice versa. The set of <u>terminal nodes</u> is $T = \{x \in \mathcal{X} : s(x) = \emptyset\}$. All other nodes $\mathcal{X} \backslash T$ are <u>decision nodes</u>.

(iii) A function $\alpha : \mathcal{X} \backslash \{x_0\} \to \mathcal{A}$ giving the action that leads to any non-initial node x from its immediate predecessor $p(x)$ with $x', x'' \in s(x); x' \neq x'' \Rightarrow \alpha(x') \neq \alpha(x'')$. The set of choices at decision node x is $c(x) = \{a \in \mathcal{A} : a = \alpha(x') \text{ for some } x' \in s(x)\}$

The Extensive Form Representation: Definition (2/2)

(iv) *A collection of information sets* $\mathcal{H}$*, and a function* $H : \mathcal{X} \to \mathcal{H}$*, assigning each decision node* x *to an information set* $H(x) \in \mathcal{H}$ *with* $c(x) = c(x')$ *if* $H(x) = H(x')$*.*
The choices available at information set H *can be written as* $C(H) = \{a \in \mathcal{A} : a \in c(x) \text{ for } x \in H\}$*.*

(v) *A function* $\iota : \mathcal{H} \to \{0, 1, ..., I\}$ *assigning a player to each information set (*$i = 0$ *'nature').*
The collection of player i's information set is denoted by $\mathcal{H}_i = \{H \in \mathcal{H} : i = \iota(H)\}$*.*

(vi) *A function* $\rho : \mathcal{H}_0 x \mathcal{A} \to [0, 1]$ *assigning a probability to each action of nature with* $\rho(H, a) = 0$ *if* $a \notin C(H)$ *and* $\sum_{a \in C(H)} \rho(H, a) = 1$ *for all* $H \in \mathcal{H}_0$*.*

(vii) *A collection of payoff functions* $u = \{u_1(\cdot), ..., u_I(\cdot)\}$*, where* $u_i : T \to \mathbb{R}$

A game in extensive form: $\Gamma_E = \{\mathcal{X}, \mathcal{A}, I, p(\cdot), \alpha(\cdot), \mathcal{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}$

The Extensive Form Representation of a Game

Restrictions of this definition:

1. Finite set of *actions*
2. Finite number of *moves*
3. Finite number of *players*

Strategies and the Normal Form Representation of a Game

Definition (Strategy)

Let $\mathcal{H}_i$ denote the collection of player i's information sets, $\mathcal{A}$ the set of possible actions in the game, and $C(H) \subseteq \mathcal{A}$ the set of actions possible at information set H. A strategy for player i is a function $s_i : \mathcal{H}_i \to \mathcal{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathcal{H}_i$.

Definition (Normal Form Representation)

For a game with I players, the normal form representation Γ_N specifies for each player i a set of strategies $\mathcal{S}_i$ (with $s_i \in \mathcal{S}_i$) and a payoff function $u_i(s_1, ..., s_I)$, formally

$$\Gamma_N = [I, \{\mathcal{S}_i\}, \{u_i(\cdot)\}].$$

Randomized Choices

- $s_i : \mathcal{H}_i \to \mathcal{A}$ describes deterministic choices at each $H \in \mathcal{H}_i$ and is called a pure strategy
- a mixed strategy is a probability distribution over all pure strategies $\sigma_i : \mathcal{S}_i \to [0,1]$, with $\sigma_i(s_i) \geq 0$ and $\sum_{s_i \in \mathcal{S}_i} \sigma_i(s_i) = 1$
- player i's set of possible mixed strategies can be associated with the points of the simplex $\Delta(\mathcal{S}_i)$, called the mixed extension of $\mathcal{S}_i$
- since we assume that individuals are expected utility maximizers, player i's utility of a profile of mixed strategies $\sigma = (\sigma_1, ..., \sigma_I)$ is given by $u_i(\sigma) = \sum_{s \in \mathcal{S}} [\sigma_1(s_1) \cdot \sigma_2(s_2) \cdot ... \cdot \sigma_I(s_I)] \cdot u_i(s)$, where $s = (s_1, ..., s_I)$

Randomized Choices and Behavior Strategies

Definition (Behavior Strategy)

Given an extensive form game Γ_E, a behavior strategy for player i specifies, for every information set $H \in \mathcal{H}_i$ and action $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with

$$\sum_{a \in C(H)} \lambda_i(a, H) = 1 \text{ for all } H \in \mathcal{H}_i.$$

Theorem (Kuhn)

If Γ_E is an extensive form game with perfect recall, then for any mixed strategy there is an outcome equivalent behavior strategy and vice versa.

Perfect Recall: A player does not 'forget' what she once knew, including her own actions.

Section 2

Rationalizable Strategies

Outline

Motivation

Central question of Game Theory:

What should we expect to observe in a game played by rational players?

Or more precisely:

What should we expect to observe in a game played by rational players who are fully knowledgeable about ...

> ... *the structure of the game and*
>
> ... *each others' rationality?*

Simultaneous-Move Games

We first address the above question for simultaneous-move games, which we study using their normal form representation.

We use the following **notation**:

- $\Gamma_N = [I, \{\mathcal{S}_i\}, \{u_i(\cdot)\}]$ if we consider pure strategies only,
  $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if we allow for mixed strategies
- $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_I) \in \mathcal{S}_{-i}$,
  where $\mathcal{S}_{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_I$
- $s = (s_i, s_{-i})$

Prisoners' Dilemma

Prisoners' Dilemma

		Player 2	
		don't confess	confess
Player 1	don't confess	-2, -2	-10, -1
	confess	-1, -10	-5, -5

What should we expect to observe in this game?

Dominant Strategies

Definition (Strictly Dominant Strategy)

A strategy $s_i \in \mathcal{S}_i$ is *strictly dominant* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$:

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \text{ for all } s_{-i} \in \mathcal{S}_{-i}.$$

Applied to Prisoner's Dilemma:
Confess is a strictly dominant strategy for each player.

Dominated Strategies

Definition (Strictly Dominated Strategy)

$s_i \in \mathcal{S}_i$ is *strictly dominated* for player i in game Γ_N if there exists another strategy $s_i' \in \mathcal{S}_i$ such that

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in \mathcal{S}_{-i}.$$

We say that s_i' strictly dominates s_i.

Definition (Weakly Dominated Strategy)

$s_i \in \mathcal{S}_i$ is *weakly dominated* for player i in game Γ_N if there exists another strategy $s_i' \in \mathcal{S}_i$ such that:

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in \mathcal{S}_{-i},$$

with strict inequality for at least one s_{-i}.

Dominated Strategies

Examples:

Player 1 (rows), Player 2 (columns)

	L	R
U	1, -1	-1, 1
M	-1, 1	1, -1
D	-2, 5	-3, 2

$\Rightarrow$ D is strictly dominated by U and M.

Player 1 (rows), Player 2 (columns)

	L	R
U	5, 1	4, 0
M	6, 0	3, 1
D	6, 4	4, 4

$\Rightarrow$ U and M are weakly dominated by D.

Dominated Strategies

Prisoners' Dilemma – A Variation

Assume Prisoner 1 is the district attorney's brother: If neither player confesses, player 1 is free

		Player 2	
		don't confess	confess
Player 1	don't confess	0, -2	-10, -1
	confess	-1, -10	-5, -5

$\Rightarrow$ Player 1 has no dominant strategy anymore.

Iterated Elimination of Strictly Dominated Strategies

- In this game, the iterated elimination of strictly dominated strategies still leads to a unique prediction.
- In general, the order of elimination of strictly dominated strategies does not matter!
- How about iterated elimination of weakly dominated strategies?

Allowing for Mixed Strategies

Definition

A strategy $\sigma_i \in \Delta(\mathcal{S}_i)$ is strictly dominated for i in game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that for all $\sigma_{-i} \in \prod_{j \neq i} \Delta(\mathcal{S}_j)$:

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}).$$

Allowing for Mixed Strategies

Proposition

Player i's pure strategy $s_i \in \mathcal{S}_i$ is strictly dominated in a game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(\mathcal{S}_i)$ such that

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i} \in \mathcal{S}_{-i}.$$

This follows because we can write

$$u_i(\sigma_i', \sigma_{-i}) - u_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in \mathcal{S}_{-i}} \left[\prod_{k \neq i} \sigma_k(s_k) \right] [u_i(\sigma_i', s_{-i}) - u_i(s_i, s_{-i})].$$

And this expression is positive for all σ_{-i} if and only if $u_i(\sigma_i', s_{-i}) - u_i(s_i, s_{-i})$ is positive for all s_{-i}.

Allowing for Mixed Strategies

		Player 2	
		L	R
Player 1	U	10,1	0,4
	M	4,2	4,3
	D	0,5	10,2

$\Rightarrow \frac{1}{2}U + \frac{1}{2}D$ strictly dominates M.

Rationalizable Strategies: Definition

Definition (Best Response)

The strategy σ_i is a *best response* for player i to her rivals' strategies σ_{-i} if: $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all $\sigma_i' \in \Delta(\mathcal{S}_i)$.
Strategy σ_i is *never a best response* if there is no σ_{-i} for which σ_i is a best response.

Definition (Rationalizable Strategies)

In game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(\mathcal{S}_i)$ that survive the iterated elimination of strategies that are never a best response are known as player i's *rationalizable strategies*.

Rationalizable Strategies: Examples

Example:

Player 2

		b_1	b_2	b_3	b_4
	a_1	0 , $\underline{7}$	2 , 5	$\underline{7}$, 0	0 , 1
Player 1	a_2	5 , 2	$\underline{3}$, $\underline{3}$	5 , 2	0 , 1
	a_3	$\underline{7}$, 0	2 , 5	0 , $\underline{7}$	0 , 1
	a_4	0 , $\underline{0}$	0 , -2	0 , $\underline{0}$	$\underline{10}$, -1

$\Rightarrow$ b_4 is never best response for player 2 and *then* a_4 is never best response for player 1.

$\Rightarrow$ $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are the rationalizable strategies in this game.

Section 3

Nash Equilibrium

Outline

Motivation

Example:

		Player 2		
		L	M	R
Player 1	U	5,3	0,4	3,5
	M	4,0	**5,5**	4,0
	D	3,5	0,4	5,3

- All strategies in this game are rationalizable, i.e. best responses to *reasonable* conjectures about other players' strategies
- Yet only one strategy profile (namely (M, M)) contains best responses to *correct* conjectures about other players' strategies

Nash Equilibrium: Definition

Definition (Nash Equilibrium)

A strategy profile $s = (s_1, \dots, s_I)$ constitutes a *Nash equilibrium* (NE) of game Γ_N if for every $i = 1, \dots, I$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for all } s_i' \in \mathcal{S}_i.$$

In the game on the previous slide, there is only one Nash equilibrium.

Nash Equilibrium: Examples

Meeting in New York 1:

Player 2

Player 1		Empire State	Grand Central
	Empire State	**100,100**	0, 0
	Grand Central	0, 0	**100,100**

$\Rightarrow$ (Empire State, Empire State) and (Grand Central, Grand Central) are Nash equilibria.

Meeting in New York 2:

Player 2

Player 1		Empire State	Grand Central
	Empire State	**100,100**	0, 0
	Grand Central	0, 0	**1000,1000**

$\Rightarrow$ Again, (Empire State, Empire State) and (Grand Central, Grand Central) are Nash equilibria.

Nash Equilibrium: Discussion

Why should we care about Nash equilibria?

Why should players' conjectures about each other's play be correct?

- If there is a unique predicted outcome to a game, it must be a NE.
- Thus, a "focal point" (see example 2) can be the unique predicted outcome to a game only if it is a NE.
- An agreement between players is self-enforcing if it is a NE.
- In a repeated game, a social convention to play the game might emerge. Only a NE can be maintained as a stable convention.

Mixed Strategy Nash Equilibrium: Definition

Definition (Mixed Strategy Nash Equilibrium)

A mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ constitutes a *Nash equilibrium* of game Γ_N if for every $i = 1, \ldots, I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(\mathcal{S}_i).$$

Proposition

Let $\mathcal{S}_i^+ \subset \mathcal{S}$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$. Strategy profile σ is a Nash equilibrium in game Γ_N if and only if for all $i = 1, \ldots, I$,

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ *for all* $s_i, s_i' \in \mathcal{S}_i^+$;

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ *for all* $s_i \in \mathcal{S}_i^+$ *and all* $s_i' \notin \mathcal{S}_i^+$.

Mixed Strategy Nash Equilibrium: Example 1

Meeting in New York 2:

Player 2

Player 1		Empire State	Grand Central
	Empire State	**100,100**	0, 0
	Grand Central	0, 0	**1000,1000**

There are two pure strategy NE, but are there also mixed strategy NE?

Let p denote the probability of (player $-i$) playing ES.

Then for player i to play ES and GC with positive probability in a NE, we need:

$$u_i(ES, p \cdot ES + (1-p) \cdot GC) = u_i(GC, p \cdot ES + (1-p) \cdot GC)$$

$$\Rightarrow 100p = 1000(1-p)$$

$$\Rightarrow p = 10/11$$

Thus, there is a mixed strategy NE with both players playing ES with prob. 10/11.

Existence of Nash Equilibria: The Idea

- In a Nash equilibrium each player's strategy is a best response to all other players' strategies.
- Let $b_i(s_{-i})$ denote the best response(s) of player i to the strategies s_{-i}
- Then $b_i : \mathcal{S}_{-i} \to \mathcal{S}_i$ is a correspondence, called player i's best response correspondence
- Define $b : \mathcal{S} \to \mathcal{S}$ by $(s_1, \ldots, s_I) \longmapsto b_1(s_{-1}) \times \cdots \times b_I(s_{-I})$
- A strategy profile $s \in \mathcal{S}$ is a Nash equilibrium if and only if $s \in b(s)$
- Thus, to prove existence of Nash equilibria, we have to show that a fixed point of the correspondence b exists
- To do so, we employ Kakutani's fixed-point theorem ...

Existence of Nash Equilibria (1/3)

Lemma

If $\mathcal{S}_1, \ldots, \mathcal{S}_I$ are nonempty, compact and convex and u_i is continuous in $\mathcal{S}_1, \ldots, \mathcal{S}_I$ and quasi-concave, then player i's best response correspondence $b_i(\cdot)$ is nonempty-valued, convex-valued and upper hemicontinuous.

Definition (Quasi-Concave Function)

The function $f : A \longrightarrow \mathbb{R}$, defined on the convex set $A \subset \mathbb{R}^N$, is quasi-concave if its upper contour sets $\{x \in A : f(x) \geq t\}$ are convex sets.

Definition (Upper Hemicontinuous Correspondence)

Given $A \subset \mathbb{R}^N$ and the closed set $Y \subset \mathbb{R}^K$, the correspondence $f : A \longrightarrow Y$ is upper hemicontinuous if it has a closed graph and the images of compact sets are bounded.

Existence of Nash Equilibria (2/3)

Theorem (Kakutani's Fixed Point Theorem)

Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, convex set, and that $f : A \rightarrow A$ is a correspondence from A into itself that is nonempty-valued, convex-valued and upper hemicontinuous.
Then $f(\cdot)$ has a fixed point; that is, there is an $x \in A$ such that $x \in f(x)$.

Existence of Nash Equilibria (3/3)

Proposition

A Nash equilibrium exists in game Γ_N if for all $i = 1, \ldots, I$,

(i) *$\mathcal{S}_i$ is a nonempty, convex, and compact subset of some Euclidean space $\mathbb{R}^M$*

(ii) *$u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i*

Proof.

By the lemma, $b(\cdot)$ is nonempty, convex-valued and upper hemicontinuous.
By Kakutani's fixed point theorem there exists an $s \in \mathcal{S}$ such that $s \in b(s)$.
By the definition of b: $s_i \in b_i(s_{-i})$ for all $i = 1, \ldots, I$.
Thus s is a Nash equilibrium. $\square$

Existence of Mixed-Strategy Nash Equilibria

Proposition

Every game $\Gamma_N = [I, \{\Delta(\mathcal{S}_i)\}, \{u_i(\cdot)\}]$ in which the sets $\mathcal{S}_1, \ldots, \mathcal{S}_I$ have a finite number of elements has a mixed strategy Nash equilibrium.

This follows from the previous proposition on the existence of Nash equilibria because the set of mixed strategies $\Delta(\mathcal{S}_i)$ of a finite number of pure strategies is nonempty, convex, and compact.

Mixed Strategy Nash Equilibrium: Example 2 (1/3)

Example:

		Player 2	
		L	R
Player 1	O	-2, 1	1, -1
	U	1, -2	-1, 1

Player 1's calculation:

$$\Rightarrow \; u_1(p \cdot \mathrm{O} + (1-p) \cdot \mathrm{U}, q \cdot \mathrm{L} + (1-q) \cdot \mathrm{R})$$
$$= pq(-2) + p(1-q)1 + (1-p)q1 + (1-p)(1-q)(-1)$$
$$= -2pq + p - pq + q - pq - 1 + p + q - pq$$
$$= -5pq + 2p + 2q - 1$$
$$= (2-5q)p + 2q - 1$$

Mixed Strategy Nash Equilibrium: Example 2 (2/3)

$2 - 5q > 0 \Rightarrow p = 1$ optimal (pure strategy)
$2 - 5q = 0 \Rightarrow p \in [0, 1]$ optimal
$2 - 5q < 0 \Rightarrow p = 0$ optimal (pure strategy)

$$b_1(q) = \begin{cases} 1 & q < \frac{2}{5} \\ \in [0,1] & q = \frac{2}{5} \\ 0 & q > \frac{2}{5} \end{cases}$$

Player 2's calculation:

$$\begin{aligned} \Rightarrow \quad & u_2(p \cdot \mathrm{O} + (1-p) \cdot \mathrm{U}, q \cdot \mathrm{L} + (1-q) \cdot \mathrm{R}) \\ & = pq1 + p(1-q)(-1) + (1-p)q(-2) + (1-p)(1-q)1 \\ & = \dots \\ & = (5p-3)q - 2p + 1 \end{aligned}$$

Mixed Strategy Nash Equilibrium: Example 2 (3/3)

$5p - 3 > 0 \Rightarrow q = 1$ optimal (pure strategy)
$5p - 3 = 0 \Rightarrow q \in [0, 1]$ optimal
$5p - 3 < 0 \Rightarrow q = 0$ optimal (pure strategy)

$$b_2(p) = \begin{cases} 0 & p < \frac{3}{5} \\ \in [0, 1] & p = \frac{3}{5} \\ 1 & p > \frac{3}{5} \end{cases}$$

$\Rightarrow$ The only mixed strategy NE in this game is $(\frac{3}{5}O + \frac{2}{5}U, \frac{2}{5}L + \frac{3}{5}R)$

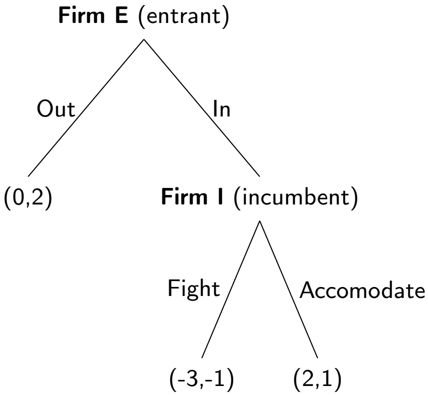
Section 4

Subgame Perfection in Dynamic Games

Outline

Dynamic Games of Perfect Information – Example (1/2)

Predation Game:



Dynamic Games of Perfect Information – Example (2/2)

Predation Game in normal form representation:

		Firm I: Fight	Firm I: Accom.
Firm E	Out	0 , 2	0 , 2
	In	-3 , -1	2 , 1

$\Rightarrow$ Two NEs in the normal form game:
(Out, Fight if 'In'), (In, Accomodate if 'In')

But: Is the strategy *Fight if 'In'* credible?

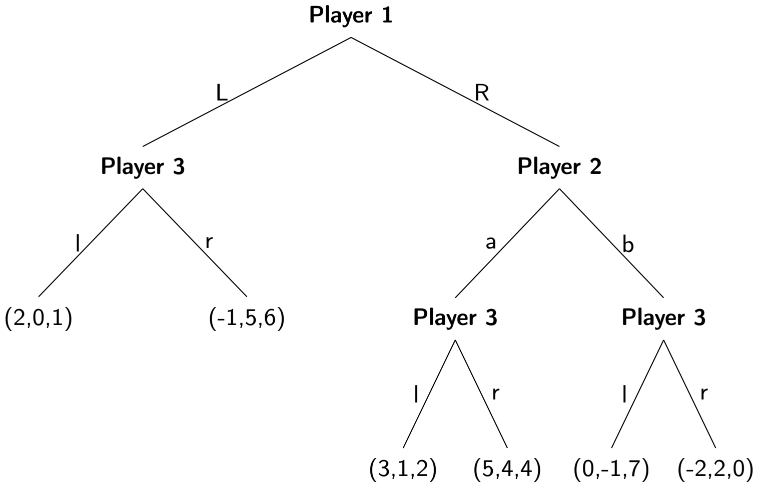
Principle of sequential rationality: A strategy should specify optimal actions at every point in the game tree given the opponents' strategies.

Backward Induction in Finite Games of Perfect Information

Backward induction is an iterative procedure to identify Nash equilibria that satisfy the principle of sequential rationality in dynamic games:

- Determine the optimal actions at the final decision nodes in the tree.
- Derive the reduced extensive form game by deleting the part of the game following these decision nodes and replacing them by the payoffs that result from the optimal play.
- Proceed to the next-to-last decision nodes and solve for the optimal actions to be taken there by players who correctly anticipate the actions that will follow at the final nodes.
- Continue in this way backwards through the game tree.

Backward Induction in Finite Games of Perfect Information



$\Rightarrow$ Equilibrium via backward induction: *(R,a,rrl)*,
other NEs: e.g. *(L,b,rlr)*

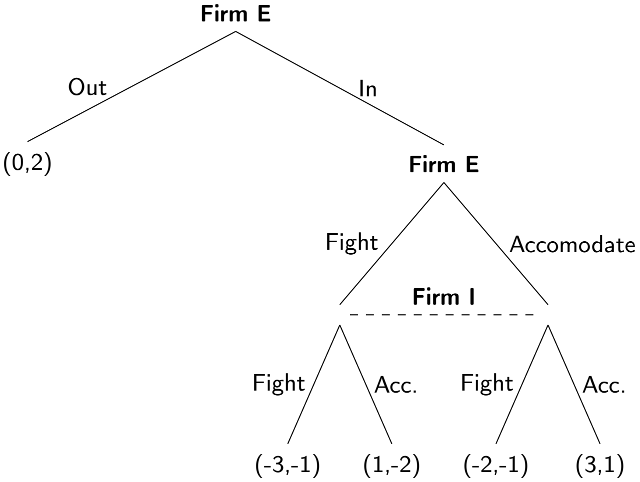
Backward Induction in Finite Games of Perfect Information

Theorem (Zermelo's Theorem)

Every finite game of perfect information Γ_E *has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.*

Dynamic Games of Imperfect Information – Example (1/2)

Predation Game Version B (Selten, 1965):



Dynamic Games of Imperfect Information – Example (2/2)

Normal form representation:

Firm E \ Firm I	A	F
Out, A if In	0 , 2	**0 , 2**
Out, F if In	0 , 2	**0 , 2**
In, A if In	**3 , 1**	-2 , -1
In, F if In	1 , -2	-3 , -1

$\Rightarrow$ Three NEs:

$$[(Out,\ A \text{ if } In),\ F]$$
$$[(Out,\ F \text{ if } In),\ F]$$
$$[(In,\ A \text{ if } In),\ A]$$

But:

Principle of sequential rationality: A strategy should specify optimal actions at every point in the game tree given the opponents' strategies.

Subgame Perfect Nash Equilibria

Definition (Subgame)

A *subgame* of an extensive form game Γ_E is a subset of the game having the following properties:

(i) It begins with an information set containing a single decision node, contains all the decision nodes that are successors of this node, and contains only these nodes.

(ii) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x.

Definition (Subgame Perfect Nash Equilibrium)

A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an I-player extensive form game Γ_E is a *subgame perfect Nash equilibrium (SPNE)* if it induces a Nash equilibrium in every subgame of Γ_E.

Subgame Perfect Nash Equilibria

Proposition

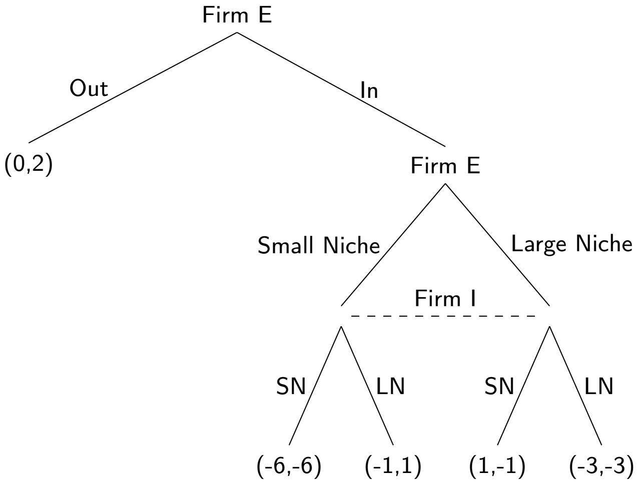
Consider an extensive form game Γ_E and some subgame G of Γ_E. Suppose that strategy profile σ^G is a SPNE in subgame G, and let Γ_E' be the reduced game formed by replacing subgame G by a terminal node with payoffs equal to those arising from play of σ^G. Then:

(i) *In any SPNE σ of Γ_E in which σ^G is the play in subgame G, players' moves at information sets outside subgame G must constitute a SPNE of reduced game Γ_E'.*

(ii) *If σ' is a SPNE of Γ_E', then the strategy profile σ that specifies the moves in σ^G at information sets in subgame G and that specifies the moves in σ' at information sets not in G is a SPNE of Γ_E.*

Backward Induction in Games of Imperfect Info. (1/2)

Example: Niche Choice Game



Backward Induction in Games of Imperfect Info. (2/2)

Post-entry subgame:

		Firm I	
		SN	LN
Firm E	SN	-6 , -6	-1 , 1
	LN	1 , -1	-3 , -3

$\Rightarrow$ Two NEs in subgame: *(SN,LN), (LN,SN)*

Normal form representation of the whole game:

		Firm I	
		SN	LN
Firm E	Out, SN	0 , 2	**0 , 2**
	Out, LN	0 , 2	**0 , 2**
	In, SN	-6 , -6	-1 , 1
	In, LN	**1 , -1**	-3 , -3

$\Rightarrow$ Two SPNE in pure strategies: [(Out, SN),LN], [(In, LN),SN]
Note: [(Out, LN),LN] is not subgame perfect!

Bilateral Bargaining – Finite Sequential Game (1/3)

Period 1:

- Player 1 offers a split: $s^1 \in [0, v]$
- Player 2 can reject and the game continues in period 2, or accept and the split is implemented and the game ends immediately with $u_1 = s^1, u_2 = v - s^1$.

Period 2:

- Player 2 offers a split: $s^2 \in [0, v]$
- Player 1 can reject and the game continues in period 3, or accept and the split is implemented and the game ends immediately with $u_1 = \delta \cdot s^2, u_2 = \delta \cdot (v - s^2)$.

and so on ...

Bilateral Bargaining – Finite Sequential Game (2/3)

There is a unique SPNE:

Suppose T is odd:
<u>Period T:</u> Player 1 makes the offer in period T and player 2 is willing to accept any offer.
Payoffs: $(\delta^{T-1} \cdot v, 0)$.

<u>Period T-1:</u> Player 2 makes the offer and player 1 will accept if and only if the payoff for player 1 is at least $\delta^{T-1} \cdot v$.
Payoffs: $(\delta^{T-1} \cdot v, \delta^{T-2} \cdot v - \delta^{T-1} \cdot v)$.

<u>Period T-2:</u> Player 1 makes the offer and player 2 will accept if and only if the payoff for player 2 is at least $\delta^{T-2} \cdot v - \delta^{T-1} \cdot v$.
Payoffs: $(\delta^{T-3} \cdot v - \delta^{T-2} \cdot v + \delta^{T-1} \cdot v, \delta^{T-2} \cdot v - \delta^{T-1} \cdot v)$.

...

Bilateral Bargaining – Finite Sequential Game (3/3)

$\Rightarrow$ The resulting SPNE for **odd T** is:
$v_1^*(T) = v(1 - \delta + \delta^2 - \ldots + \delta^{T-1}) = v\left[(1-\delta)(\frac{1-\delta^{T-1}}{1-\delta^2}) + \delta^{T-1}\right]$
$v_2^*(T) = v - v_1^*(T)$.

$\Rightarrow$ The resulting SPNE for **even T** is:
$v_1^*(T) = v - \delta v_1^*(T-1)$
$v_2^*(T) = v_1^*(T-1)$.

$\Rightarrow$ For **large T**, this converges to:
$\lim_{T \to \infty} v_1^*(T) = \frac{v}{1+\delta}$
$\lim_{T \to \infty} v_2^*(T) = \frac{\delta v}{1+\delta}$.

Bilateral Bargaining – Infinite Sequential Game

Now consider the bilateral bargaining game with *infinite horizon*:

Proposition (Shaked & Sutton (1984))

The infinite horizon bargaining game has a unique SPNE in which the players reach an agreement in period 1 such that player 1 earns $\frac{v}{1+\delta}$ *and player 2* $\frac{\delta v}{1+\delta}$.

Let $\bar{v}_1$ be the largest payoff that player 1 gets in any SPNE. Then player 1's payoff in any SPNE cannot be lower than $\underline{v}_1 = v - \delta \bar{v}_1$. Also $\bar{v}_1 \leq v - \delta \underline{v}_1$ because player 2 rejects any offer of less than $\delta \underline{v}_1$.
And we have:

$$\begin{aligned} & \bar{v}_1 \leq v - \delta \underline{v}_1 = \underline{v}_1 + \delta \bar{v}_1 - \delta \underline{v}_1 \\ \Leftrightarrow \quad & \bar{v}_1 (1-\delta) \leq \underline{v}_1 (1-\delta). \end{aligned}$$

Which implies $\bar{v}_1 = \underline{v}_1$, so player 1's SPNE is uniquely determined:
$\Rightarrow v_1^0 = v - \delta v_1^0 = \frac{v}{1+\delta}$ and $v_2^0 = \frac{\delta v}{1+\delta}$.