

Bachelor Thesis

# On the spectra of Schrödinger operator with periodic delta potential

Martin Belica

13.09.2016

Supervisor: Prof. Dr. Michael Plum,  
Dr. Andrii Khrabustovskyi

Faculty for Mathematics

Karlsruhe Institute of Technology

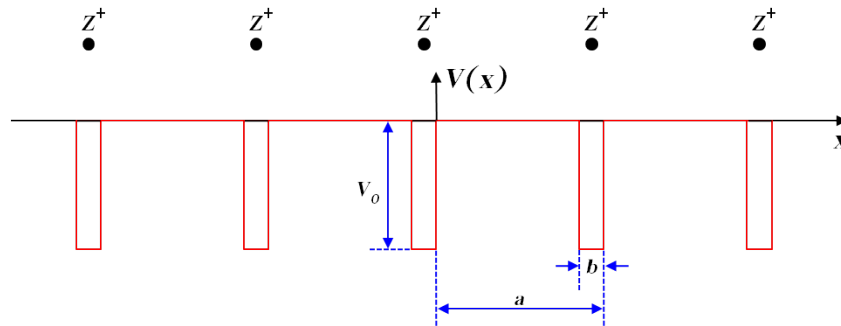
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# Chapter 1

## Introduction

The problem considered in this thesis arises from the Kronig-Penney model, see for example [7, Chap. 3], which describes an idealised quantum-mechanical system that models a quantum particle behaving as a matter wave moving in one-dimension through an infinite periodic array of rectangular potential barriers, i.e. through a space area in which a potential attains a local maximum. Such an array commonly occurs in models of periodic crystal lattices where the potential is caused by ions in the crystal structure. Those charged molecules create an electromagnetic field around themselves. Hence, any particle moving through such a crystal would be subject to a recurrent electromagnetic potential. Although a solid particle, simplified as a point mass, would be reflected at such a barrier, there is a possibility that the quantum particle, as it behaves like a wave, penetrates the barrier and continues its movement beyond. Assuming the spacing between all ions is equidistant the potential function  $V(x)$  in the lattice can be approximated by a rectangular potential like this:



where  $b$  is the “support” and  $\rho$  the magnitude of the potential.

In this thesis we will examine the spectrum of the operator describing the special case of the Kronig-Penney model when the particle moves through periodically distributed, singular potentials. Analytically speaking this means, taking the limit  $b \rightarrow 0$  while  $V_0$  remains of order  $\rho b$  in the above described situation.

## Mathematical Basics

For the upcoming analysis we need some basic concepts from functional analysis and spectral theory I wish to briefly review:

Let  $C_c^\infty$  denote the linear space containing all smooth function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with compact support, i.e. for  $f \in C_c^\infty$  there exists a compact interval  $I \subseteq \mathbb{R}$  such that  $f(x) = 0$  for all  $x \notin I$ . And will hereafter  $\langle x, x \rangle$  denote the scalar product in  $L^2(\mathbb{R})$ .

**Definition** (Weak derivative): Let  $\Omega \subseteq \mathbb{R}$  be an open set and let  $u$  be a function in the Lebesgue space  $L^1(\Omega)$ . Then  $v$  in  $L^1(\Omega)$  is a weak derivative of  $u$  if,

$$\int_{\Omega} u(t)\varphi'(t)dt = - \int_{\Omega} v(t)\varphi(t)dt$$

for all  $\varphi \in C_0^\infty(\Omega)$ .

Now, an important example for a Hilbert space is the Sobolev space  $H^k(\Omega)$ , which is defined to be the set of functions  $f$  in  $L^2(\Omega)$  such that the function  $f$  and its weak derivatives up to the order  $k$  have a finite  $L^2(\Omega)$  norm, by admitting the inner product in terms of the  $L^2(\Omega)$  inner product for all derivatives up to order  $k$ :

$$\langle u, v \rangle_{H^k(\Omega)} = \sum_{i=0}^k \langle D^i u, D^i v \rangle_{L^2(\Omega)}.$$

**Definition** (Distributions): On  $C_0^\infty$  a sequence  $(f_n)$  converges against  $f \in C_0^\infty$  if the support of all members of the sequence is in a compact interval  $I \subset \mathbb{R}$ , i.e.

$$\text{supp}(f_n) \subseteq I \quad \forall n \in \mathbb{N},$$

and on this interval  $f_n$  and all its derivatives converge uniformly against  $f$ , i.e.

$$\|f_n^{(i)} - f^{(i)}\|_\infty \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

for all  $i \in \mathbb{N}_0$ . One can proof that this concept of convergence generates a topology on  $C_0^\infty$  and one usually denoted with  $\mathcal{D}(\mathbb{R})$  the space  $C_0^\infty$  equipped with this topology. From now on, we denote with  $D'(\mathbb{R})$  the space of all linear functionals on  $C_0^\infty$  that are continuous with respect to this topology and call those functionals distributions.

An important example for a distribution is the Dirac delta function  $\delta_{x_0}$  where  $x_0 \in \mathbb{R}$ . It is defined as the limit of a weakly converging sequence of functionals over normed symmetric around  $x_0$  cumulative distribution functions  $\delta_\epsilon$ , whereas the support of those cumulative distributions converges to zero. It holds  $\delta_{x_0} = \lim_{\epsilon \rightarrow 0} \delta_\epsilon$  in  $D'(\mathbb{R})$ . An example for such a sequence is

$$\delta_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{x^2}{2\epsilon^2}}. \quad (1.1)$$

Which implies the definition

$$\delta_{x_0}(f) := \int_{-\infty}^{\infty} \delta_{x_0} f(x) dx := \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \delta_\epsilon(x - x_0) f(x) dx.$$

Moreover, is easily seen that  $\delta_{x_0}(f) = \lim_{\epsilon \rightarrow 0} \delta_\epsilon(f) = f(x_0)$ , for a proof see [9].

**Definitions:** Let  $X, Y$  be Banach spaces and let  $A: \mathcal{D}(A) \rightarrow Y$  be a linear operator with domain  $\mathcal{D}(A) \subset X$ .

- a) We call  $A$  closed if  $\text{graph}(A) := \{(x, Ax) : x \in \mathcal{D}(A)\} \subseteq X \times Y$  is a closed set.

Are  $X, Y$  Hilbert spaces and  $A$  a bounded operator we call

- b)  $A$  symmetric, if  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for all  $x, y \in \mathcal{D}(A)$ , and
- c)  $A$  self-adjoint, if  $A$  is densely defined on  $X$  and coincides with its adjoint.

Furthermore, let  $I$  denote the identity operator on  $X$  and  $A$  be a linear, bounded and closed operator.

- d)  $\lambda \in \mathbb{C}$  belongs in the resolvent set of  $A$ ,  $\lambda \in \rho(A)$ , if

$A - \lambda I: \mathcal{D}(A) \rightarrow X$  bijective, i.e.  $(A - \lambda I)^{-1}: X \rightarrow \mathcal{D}(A)$  is a bounded linear operator,

- e)  $\sigma(A) = \mathbb{C} \setminus \rho(A)$  is called the spectrum of  $A$ , and
- f)  $\lambda \in \rho(A) \rightarrow R(\lambda, A) = (A - \lambda I)^{-1}$  is the resolvent function of  $A$ .

**Theorem 1.1:** *The resolvent set  $\rho(A) \subseteq \mathbb{C}$  of a bounded linear operator  $A$  is an open set.*

*Proof:* See [4, Chap. 6]

□

## Chapter 2

# The one-dimensional Schrödinger operator $A$

The mathematical representation of the above introduced problem can be done by introducing a one-dimensional Schrödinger operator  $A$  where the potential is given by a periodic delta-distribution. Formally the operation of  $A$  is defined by

$$-\frac{d^2}{dx^2} + \rho \sum_{i \in \mathbb{Z}} \delta_{x_i} \quad (2.1)$$

on the whole of  $\mathbb{R}$ , where  $\delta_{x_i}$  denotes the Dirac delta distribution supported at the point  $x_i$ .  $\Omega_k$  will hereafter identify the periodicity cell containing point  $x_k$  and w.l.o.g. let  $x_0 = 0$  and  $|\Omega_i| = 1$  for all  $i \in \mathbb{Z}$ .

In general, one cannot say given a right-hand side  $f \in L^2(\mathbb{R})$  in which sense a solution exists to the problem  $Au = f$ , since the potential is given by a singular distribution. Suppose for the moment the problem is smooth, more specifically if the potential is given by (1.1), then formally multiplying by a test function and integrating by parts yields the so called weak-formulation to the problem which requires less regularity. Hence, motivated by this we consider, by taking the limit of the potential for  $\mu \in \mathbb{R}$ , the problem to find  $u \in H^1(\mathbb{R})$  such that

$$\int_{\mathbb{R}} u' \overline{v'} + \rho \sum_{i \in \mathbb{Z}} u(x_i) \overline{v(x_i)} - \mu \int u \overline{v} = \int f \overline{v} \quad \forall v \in H^1(\mathbb{R}), \quad (2.2)$$

holds.

We should first note that left-hand side of problem (2.2) is actually well-defined and finite, as for any  $h \in (0, 1]$  we can estimate

$$\begin{aligned} \sum_{i \in \mathbb{Z}} |u(x_i)|^2 &\leq \sum_{i \in \mathbb{Z}} \left( 2|u(x_i + h)|^2 + 2h \int_{x_i}^{x_i+h} |u'(\tau)|^2 d\tau \right) \\ &\leq 2 \sum_{i \in \mathbb{Z}} \left( \frac{1}{h} \int_{\Omega_i} |u(x)|^2 dx + h \int_{\Omega_i} |u'(\tau)|^2 d\tau \right). \end{aligned} \quad (2.3)$$

The particularly choice of  $h = 1$  yields hence the estimation

$$\sum_{i \in \mathbb{Z}} |u(x_i)|^2 \leq 2 \|u\|_{H^1(\mathbb{R})}^2. \quad (2.4)$$

We will now show that for each  $f \in L^2(\mathbb{R})$  the equation (2.2) has a unique solution.

**Definition:** Given  $f \in L^2(\mathbb{R})$ , we define a functional  $l: H^1(\mathbb{R}) \rightarrow \mathbb{R}$  through

$$l(v) := \int_{\mathbb{R}} f v$$

and a sesquilinear form  $B_\mu: H^1(\mathbb{R}) \times H^1(\mathbb{R}) \rightarrow \mathbb{C}$  for  $\mu \in \mathbb{R}$  through

$$B_\mu[u, v] := \int_{\mathbb{R}} u' \overline{v'} + \rho \sum_{i \in \mathbb{Z}} u(x_i) \overline{v(x_i)} - \mu \int_{\mathbb{R}} u \overline{v}.$$

Then (2.2) is equivalent to finding  $u \in H^1(\mathbb{R})$  such that

$$B_\mu[u, v] = l(v) \quad \forall v \in H^1(\mathbb{R}). \quad (2.5)$$

The existence of a unique  $u \in H^1(\mathbb{R})$  satisfying (2.5) follows from Lax-Milgram's Theorem if the sesquilinear form  $B$  is bounded and coercive, which we will prove in the next theorem.

**Theorem 2.1:** *The sesquilinear form  $B_\mu$  is (for small enough  $\mu \in \mathbb{R}$ )*

*i) bounded, i.e. there exists a constant  $\alpha > 0$  such that*

$$|B_\mu[u, v]| \leq \alpha \|u\| \|v\|$$

*holds for all  $u, v \in H^1(\mathbb{R})$ .*

*ii) coercive, i.e. there exists a constant  $\beta > 0$  such that*

$$\beta \|u\|^2 \leq B_\mu[u, u]$$

for all  $u \in H^1(\mathbb{R})$ .

*Proof:*

i) The boundedness follows from (2.4) as for an arbitrary  $\rho \in \mathbb{R}$  there exists  $\alpha \in \mathbb{R}$  such that

$$\begin{aligned}
|B(u, \varphi)|^2 &\leq \|u'\| \|v'\| + 2|\rho| \sum_{i \in \mathbb{Z}} |u(x_i)|^2 |v(x_i)|^2 - \mu \|u\| \|v\| \\
&\leq \|u'\| \|v'\| + 8|\rho| \|u\|_{H^1(\mathbb{R})}^2 \|v\|_{H^1(\mathbb{R})}^2 - \mu \|u\| \|v\| \\
&= (8|\rho| - \mu) \|u\| \|v\| + 8|\rho| (\|u\| \|v'\| + \|u'\| \|v\|) + (8|\rho| + 1) \|u'\| \|v'\| \\
&\leq \alpha \|u\|_{H^1(\mathbb{R})} \|\varphi\|_{H^1(\mathbb{R})}
\end{aligned}$$

where  $\alpha = \max \{8|\rho| - \mu, 8|\rho| + 1\}$ .

ii) For the coercivity we first assume  $\rho \geq 0$ . Now, if  $\mu < -1$  we get

$$\begin{aligned}
B(u, u) &= \langle u', u' \rangle + \rho \sum_{i \in \mathbb{Z}} u(x_i)^2 - \mu \langle u, u \rangle \\
&\geq \langle u', u' \rangle + \langle u, u \rangle \\
&= \|u\|_{H^1(\mathbb{R})}^2.
\end{aligned}$$

Analogously for  $\rho < 0$ , using (2.3) we can choose  $h < \frac{1}{2|\rho|}$  and with that if  $\mu < -\frac{2|\rho|}{h}$  there exists  $\beta \in \mathbb{R}$  such that

$$\begin{aligned}
B(u, u) &= \langle u', u' \rangle + \rho \sum_{i \in \mathbb{Z}} |u(x_i)|^2 - \mu \langle u, u \rangle \\
&\geq \langle u', u' \rangle + 2\rho \sum_{i \in \mathbb{Z}} \left( \frac{1}{h} \int_{\Omega_i} |u(x)|^2 dx + h \int_{\Omega_i} |u'(\tau)|^2 d\tau \right) - \mu \langle u, u \rangle \\
&= (2\rho h + 1) \|u'\|^2 + (2\rho \frac{1}{h} - \mu) \|u\|^2 \\
&\geq \beta \|u\|_{H^1(\mathbb{R})}^2,
\end{aligned}$$

where  $\beta = \min \{2\rho h + 1, 2\rho \frac{1}{h} - \mu\}$ . □

Thus, there exists a unique solution  $u \in H^1(\mathbb{R})$  to the problem (2.5) for fixed  $f \in L^2(\mathbb{R})$ , and the operator  $R_\mu: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), f \mapsto u$  is well-defined for  $\mu \in \mathbb{R}$  small enough, taking in mind that  $\mathcal{R}(R_\mu) \subseteq H^1(\mathbb{R})$ . This mapping is injective since for  $u_1 = u_2$

$$0 = B_\mu[u_1, v] - B_\mu[u_2, v] = \int (f_1 - f_2) \bar{v} \quad \forall v \in H^1(\mathbb{R}). \quad (2.6)$$



Since further  $H^1(\mathbb{R})$  is dense in  $L^2(\mathbb{R})$  this yields that the equality (2.6) holds also for all  $v \in L^2(\mathbb{R})$  and therefore  $f_1 = f_2$  almost everywhere. Consequently  $R_\mu$  is invertible and as a result we can now define the Schrödinger operator as follows

$$A := R_\mu^{-1} + \mu I$$

from which additionally follows that  $R_\mu$  is the resolvent of  $A$ .

## 2.1 The domain of $A$

First, we define

$$H^2\left(\mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} x_i\right) := \left\{ u \in L^2(\mathbb{R}) : u \in \bigcap_{i \in \mathbb{Z}} H^2(x_i, x_{i+1}), \sum_{i \in \mathbb{Z}} \|u\|_{H^2(x_i, x_{i+1})}^2 < \infty \right\}$$

If we now consider in (2.2) any fixed  $k \in \mathbb{Z}$  and an arbitrary test function  $v \in C^\infty(\mathbb{R})$  with  $\text{supp } v \subseteq [x_k, x_{k+1}]$  we get furthermore

$$\int_{x_k}^{x_{k+1}} u'(x) \overline{v'(x)} dx = \int_{x_k}^{x_{k+1}} Au \overline{v} \iff \int_{x_k}^{x_{k+1}} -u(x) \overline{v''(x)} dx = \int_{x_k}^{x_{k+1}} Au \overline{v},$$

whence  $-u'' \in L^2(x_k, x_{k+1})$  and  $Au = -u''$  and since  $k \in \mathbb{Z}$  was arbitrary we obtain

$$\mathcal{D}(A) \subseteq \left\{ u \in \bigcap_{i \in \mathbb{Z}} H^2(x_i, x_{i+1}) \right\}.$$

Next, a test function  $v \in C^\infty(\mathbb{R})$  such that  $\text{supp } v = \Omega_k$  will yield for an arbitrary  $k \in \mathbb{Z}$  from (2.2) through integration by parts on both sides of  $x_k$  that

$$\begin{aligned} & - \left( \int_{x_k - \frac{1}{2}}^{x_k} + \int_{x_k}^{x_k + \frac{1}{2}} \right) u'' \overline{v} + \left( u'(x_k - 0) \overline{v(x_k)} - u'(x_k + 0) \overline{v(x_k)} \right) \\ & + \rho u(x_k) \overline{v(x_k)} = - \int_{x_k - \frac{1}{2}}^{x_k} u'' \overline{v} - \int_{x_k}^{x_k + \frac{1}{2}} u'' \overline{v}. \end{aligned}$$

But as  $v \in C^\infty(\mathbb{R})$  was arbitrary, choosing one that is non-zero in  $x_k$  yields

$$u'(x_k - 0) - u'(x_k + 0) + \rho u(x_k) = 0,$$

and therefore

$$\mathcal{D}(A) \subseteq \left\{ u \in \bigcap_{i \in \mathbb{Z}} H^2(x_i, x_{i+1}) : u'(x_i - 0) - u'(x_i + 0) + \rho u(x_i) = 0 \ \forall i \in \mathbb{Z} \right\} \quad (2.7)$$

Using both these properties, choosing a function  $v \in C_0^\infty(x_{-n}, x_{n+1})$  in (2.2) yields with partial integration on every interval  $(x_i, x_{i+1})$  with for every  $i \in \mathbb{Z}$  that

$$\begin{aligned} \sum_{i=-n}^{n-1} - \int_{x_i}^{x_{i+1}} u'' \bar{v} + \sum_{i=-n}^{n-1} u' v \Big|_{x_i}^{x_{i+1}} + \rho \sum_{i=-n}^{n-1} u(x_i) \overline{v(x_j)} - \mu \int_{x_{-n}}^{x_n} u \bar{v} &= \int_{x_{-n}}^{x_n} f \bar{v} \\ \iff \sum_{i=-n}^{n-1} \int_{x_i}^{x_{i+1}} u'' \bar{v} &= - \int_{x_{-n}}^{x_n} f \bar{v} - \mu \int_{x_{-n}}^{x_n} u \bar{v} \end{aligned} \quad (2.8)$$

By defining  $w_n := \sum_{i=-n}^{n-1} u'' \mathbb{1}_{[x_i, x_{i+1}]}$  we can estimate the left-hand side of (2.8) by

$$\begin{aligned} |\langle w_n, v \rangle| &\leq \left| \mu \int_{x_{-n}}^{x_n} u \bar{v} \right| + \left| \int_{x_{-n}}^{x_n} f \bar{v} \right| \\ &\leq |\mu| \|u\|_{L^2(x_{-n}, x_n)} \|v\|_{L^2(x_{-n}, x_n)} + \|f\|_{L^2(x_{-n}, x_n)} \|v\|_{L^2(x_{-n}, x_n)} \\ &\leq c \|v\|_{L^2(x_{-n}, x_n)}, \end{aligned} \quad (2.9)$$

for some  $c \in \mathbb{R}$ . Since  $c$  is independent from  $n$  we have by (2.9) that

$$\sum_{i \in \mathbb{Z}} \|u''\|_{L^2(x_i, x_{i+1})}^2 < \infty.$$

Since  $u \in H^1(\mathbb{R})$  this yields

$$\mathcal{D}(A) \subseteq \left\{ u \in H^1(\mathbb{R}) : u \in H^2\left(\mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} x_i\right), u'(x_j - 0) - u'(x_j + 0) - \rho u(x_j) = 0 \ \forall j \right\} =: B$$

Hence, for an arbitrary  $u \in \mathcal{D}(A)$  it follows

$$Au = \begin{cases} -u'' & \text{on } (x_k - \frac{1}{2}, x_k) \\ -u'' & \text{on } (x_k, x_k + \frac{1}{2}), \end{cases} \quad \forall k \in \mathbb{Z}$$

We can further prove the opposite inclusion for (2.1). Since  $\mathcal{R}(R_\mu) = \mathcal{D}(A)$ , we proceed by proving each  $u \in B$  is also in the range of  $R_\mu$ . More specifically, as  $\mathcal{D}(R_\mu) = L^2(\mathbb{R})$  define  $f := -u''$  on  $(x_k, x_{k+1})$  for all  $i \in \mathbb{Z}$ . As we know that  $u \in H^2\left(\mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} x_i\right)$  we also know

that  $f \in L^2$  and hence we have to show  $u = R_\mu(f - \mu u)$  or equivalently

$$\begin{aligned} \int_{\mathbb{R}} u' \overline{v'} + \rho \sum_{i \in \mathbb{Z}} u(x_i) \overline{v(x_i)} - \mu \int_{\mathbb{R}} u \overline{v} &= \int_{\mathbb{R}} (f - \mu u) \overline{v} \\ \iff \sum_{i \in \mathbb{Z}} \int_{\Omega_i} u' \overline{v'} + \rho u(x_i) \overline{v(x_i)} &= - \sum_{i \in \mathbb{Z}} \int_{x_i}^{x_{i+1}} u'' \overline{v}. \end{aligned}$$

For each  $k \in \mathbb{Z}$  partial integration with a function  $v$  having  $\text{supp } v = (x_k - \frac{1}{2}, x_k + \frac{1}{2})$  yields

$$\begin{aligned} \int_{\Omega_k} u' \overline{v'} + \rho u(x_k) \overline{v(x_k)} &= \left( \int_{x_k + \frac{1}{2}}^{x_k} - \int_{x_k}^{x_k + \frac{1}{2}} \right) u' \overline{v'} - u'(x_k - 0) \overline{v(x_k)} + u'(x_k + 0) \overline{v(x_k)} \\ \iff u'(x_k - 0) - u'(x_k + 0) - \rho u(x_k) &= 0 \end{aligned}$$

such that we conclude

$$\mathcal{D}(A) = \left\{ u \in H^1(\mathbb{R}) : u \in H^2\left(\mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} x_i\right), u'(x_j - 0) - u'(x_j + 0) - \rho u(x_j) = 0 \ \forall j \right\}.$$

## 2.2 The operator $A$ is self-adjoint

In chapter 5, we will further utilise the fact that the operator  $A$  is self-adjoint. A self-adjoint operator is always closed, symmetric and has a completely real spectrum which narrows the analysis of its spectrum down.

**Theorem 2.2:**  $R_\mu$  and  $R_\mu^{-1}$  are symmetric operator.

*Proof:* First, focus on  $R_\mu^{-1} = (A - \mu I)$ . As for all  $v \in \mathcal{D}(A)$ :

$$\begin{aligned} \langle R_\mu^{-1} u, v \rangle &= \langle (A - \mu I) u, v \rangle \\ &= \int u' \overline{v'} - \mu \int u \overline{v} + \rho \sum_{i \in \mathbb{Z}} u(x_i) \overline{v(x_i)} \\ &= \langle u, (A - \mu I) v \rangle = \langle u, R_\mu^{-1} v \rangle, \end{aligned}$$

thus  $R_\mu^{-1}$  is symmetric. Now, as  $\mathcal{D}(R_\mu) = L^2(\mathbb{R})$  and  $\mathcal{R}(R_\mu) = \mathcal{D}(R_\mu^{-1})$  for each  $f, g \in L^2(\mathbb{R})$  it follows

$$\langle R_\mu f, g \rangle = \langle R_\mu f, R_\mu^{-1} R_\mu g \rangle = \langle f, R_\mu g \rangle$$

such that  $R_\mu$  is also symmetric. □

Now, using both symmetries we can show that  $A$  is self-adjoint:

**Theorem 2.3:**  *$A$  is a self-adjoint operator.*

*Proof:* As we already know that  $R_\mu$  and  $R_\mu^{-1}$  are symmetric, showing that  $R_\mu^{-1}$  is self-adjoint is equivalent to showing that if  $v \in \mathcal{D}(R_\mu^{-1*})$  and  $v^* \in L^2(\mathbb{R})$  are such that

$$\langle R_\mu^{-1}u, v \rangle = \langle u, v^* \rangle, \quad \forall u \in \mathcal{D}(R_\mu^{-1}) \quad (2.10)$$

then  $v \in \mathcal{D}(R_\mu^{-1})$  and  $R_\mu^{-1}v = v^*$ . In (2.10) we define  $u := R_\mu f$  for any  $f \in L^2(\mathbb{R})$  and use the fact that  $R_\mu$  is symmetric and defined on the whole of  $L^2(\mathbb{R})$ :

$$\langle f, v \rangle = \langle R_\mu f, v^* \rangle = \langle f, R_\mu v^* \rangle,$$

which means that  $v \in \mathcal{R}(R_\mu) = \mathcal{D}(R_\mu^{-1})$  and  $R_\mu^{-1}v = v^*$ , i.e.  $R_\mu^{-1}$  is self-adjoint. As the operator  $A$  is simply  $R_\mu^{-1}$  shifted by  $\mu \in \mathbb{R}$ ,  $A$  is self-adjoint as well.  $\square$

## Chapter 3

# Fundamental domain of periodicity and the Brillouin zone

Let  $\Omega$  be the fundamental domain of periodicity associated with (6.1), for simplicity let  $\Omega := \Omega_0$  and thus  $x_0 = 0$  being contained in  $\Omega$ . As commonly used in literature the reciprocal lattice for  $\Omega$  is  $[-\pi, \pi]$ , the so called one-dimensional Brillouin zone  $B$ . For fixed  $k \in \overline{B}$ , in this chapter we consider the operator  $A_k$  on  $\Omega$  formally defined by the operation

$$-\frac{d^2}{dx^2} + \rho\delta_{x_0}.$$

**Definition:** For every  $k$  we define for brevity the set of quasi-periodic functions

$$H_k^1 := \left\{ \psi \in H^1(\Omega) : \psi\left(\frac{1}{2}\right) = e^{ik}\psi\left(-\frac{1}{2}\right) \right\}. \quad (3.1)$$

**Remark:** Due to the fact that convergence in  $H_k^1$  implies the convergence on the boundary of  $\Omega$ ,  $H_k^1$  is a closed subspace of  $H^1(\mathbb{R})$ .

Analogous to before, we define  $A_k$  by considering the problem of finding for  $f \in L^2(\Omega)$  a function  $u \in H_k^1$  such that the equation

$$\int_{\Omega} u' \overline{v'} + \rho u(x_0) \overline{v(x_0)} - \mu \int_{\Omega} u \overline{v} = \int_{\Omega} f \overline{v}$$

holds for all  $v \in H_k^1$ . One can apply the same arguments as above to prove that  $R_{\mu,k} : L^2(\Omega) \rightarrow H_k^1$ ,  $f \mapsto u$  is well-defined and injective. Consequently, we define now

$$A_k := R_{\mu,k}^{-1} + \mu I.$$

This chapter is going to provide additional information about the operator  $R_{\mu,k}$ . We will see that the eigenfunctions of  $A_k$  form a complete and orthonormal system in  $H_k^1$ . Using this fact we can then deduce additional properties about the spectrum of  $A_k$  and  $A$  in chapter 5.

**Theorem 3.1:** *The operator  $R_{\mu,k}$  is compact.*

*Proof:* Let  $(f_j)_j \in L^2(\Omega)$  be a bounded sequence. We will show that

$$u_j := R_{\mu,k} f_j \quad \text{for all } j \geq 1.$$

is bounded a bounded sequence as well. Each such  $u_j$  is by definition in  $H_k^1$  and has to satisfies

$$\int_{\Omega} u_j' \overline{v'} + \rho u_j(x_0) \overline{v(x_0)} - \mu \int_{\Omega} u_j \overline{v} = \int_{\Omega} f_j \overline{v} \quad \forall v \in H_k^1. \quad (3.2)$$

Now, the particular choice of  $v = u_j$  in (3.2) yields with (2.4) for  $\mu$  small enough

$$\|u_j\|_{H^1(\Omega)} \leq \|f_j\|_{L^2(\Omega)} \|u_j\|_{L^2(\Omega)} \leq c \sqrt{\text{vol}(\Omega)}.$$

The compact embedding follows from the subsequently show estimation. As  $H^1(\Omega) \subseteq C^{\frac{1}{2}}(\overline{\Omega})$  we can further estimate

$$|f(x) - f(y)| \leq c|x - y|^{\frac{1}{2}} \quad \text{for some } c > 0, \quad (3.3)$$

from which for  $f \in B_{H_k^1} := \{f \in H_k^1(\Omega) : \|f\|_{H^1(\Omega)} \leq 1\}$  it follows that

$$|f(x)|^2 \leq 2\|f\|_{L^2}^2 + 2 \leq 4 \quad \forall x \in \Omega.$$

For an arbitrary  $\epsilon > 0$  we now partition  $\Omega$  into  $n_{\epsilon}$  equidistant, disjoint intervals  $I_k$ , i.e.  $\Omega = \bigcup_{j=1}^{n_{\epsilon}} I_j$ . As all  $f \in B_{H_k^1}$  are uniformly bounded on  $\Omega$  by (2.4), there exist for each subinterval  $I_k$  a finite number of constants  $c_{1,k}, \dots, c_{\nu_{\epsilon},k}$  such that

$$\forall f \in B_{H_k^1} \exists j \in \{1, \dots, \nu_{\epsilon}\} : \left| f\left(\frac{k}{n_{\epsilon}}\right) - c_{j,k} \right| < \epsilon \quad \forall k \in \{1, \dots, n_{\epsilon}\}.$$

Hence, there are finitely many step functions such that for any  $f \in L^2(\Omega)$  one of those step functions, let's call it  $g \in L^2(\Omega)$  with function value  $c_k$  on sub interval  $I_k$  for each

$k \in \{1, \dots, n_\epsilon\}$ , such that

$$\begin{aligned}
\|f - g\|_{L^2(\Omega)}^2 &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(x) - c_{k+1}|^2 dx \\
&= 2 \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(x) - f(\frac{k}{n})|^2 dx + 2 \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(\frac{k}{n}) - c_{k+1}|^2 dx \\
&\leq 2 \sum_{n=0}^{n-1} \frac{c}{n^2} + 2 \sum_{n=0}^{n-1} \frac{1}{n^3} = \frac{2}{n} \left( c + \frac{1}{n} \right) < \epsilon^2 \text{ for } n \text{ large enough.}
\end{aligned}$$

This means in conclusion that  $B_{H_k^1}$  is totally bounded in  $L^2(\Omega)$  and in return  $H_k^1$  can be compactly embedded in  $L^2(\Omega)$ . Thus, the operator  $R_{\mu,k}$  is compact.  $\square$

### 3.1 The spectrum of the operator $A_k$

As from now, consider the eigenvalue problem to find  $\psi \in H_k^1$  such that

$$A_k \psi = \lambda \psi \text{ on } \Omega. \quad (3.4)$$

In writing the boundary condition in (3.1), we understand  $\psi$  extended to the whole of  $\mathbb{R}$ . In fact, (3.1) forms boundary conditions on  $\partial\Omega$ , so-called semi-periodic boundary conditions. Obviously,  $A_k$  has the same sequence of eigenfunctions as  $R_{\mu,k}$ , and if  $\tilde{\lambda}$  is an eigenvalue for the eigenfunction  $\psi$  of  $R_{\mu,k}$  then respectively is

$$\lambda = \frac{1}{\tilde{\lambda}} - \mu$$

an eigenvalue for the same eigenfunction  $\psi$  for the operator  $A$ , for proof see [8]. Since  $\Omega$  is bounded, and  $R_{\mu,k}$  is a compact and symmetric operator,  $A_k$  has moreover a purely discrete spectrum satisfying

$$\lambda_1(k) \leq \lambda_2(k) \leq \dots \leq \lambda_s(k) \rightarrow \infty \text{ as } s \rightarrow \infty.$$

and the corresponding eigenfunction form a  $\langle \cdot, \cdot \rangle$ -orthonormal and complete system  $(\psi_s(\cdot, k))_{s \in \mathbb{N}}$  of eigenfunctions for (3.1). We transform the eigenvalue problem (3.4) such that the boundary condition is independent from  $k$ , define

$$\varphi_s(x, k) := e^{-ikx} \psi_s(x, k).$$

Then,

$$\begin{aligned}
A_k \psi_s(x, k) &= \frac{d^2}{dx^2} \psi_s(x, k)|_{(x_0 - \frac{1}{2}, x_0)} \cdot \mathbb{1}_{(x_0 - \frac{1}{2}, x_0)} + \frac{d^2}{dx^2} \psi_s(x, k)|_{(x_0, x_0 + \frac{1}{2})} \cdot \mathbb{1}_{(x_0, x_0 + \frac{1}{2})} \\
&= e^{ikx} \left( \frac{d}{dx} + ik \right)^2 \varphi_s(x, k)|_{(x_0 - \frac{1}{2}, x_0)} \cdot \mathbb{1}_{(x_0 - \frac{1}{2}, x_0)} \\
&\quad + e^{ikx} \left( \frac{d}{dx} + ik \right)^2 \varphi_s(x, k)|_{(x_0, x_0 + \frac{1}{2})} \cdot \mathbb{1}_{(x_0, x_0 + \frac{1}{2})}.
\end{aligned}$$

Defining the operator  $\tilde{A}_k: \mathcal{D}(A_k) \rightarrow L^2(\mathbb{R})$  through

$$\tilde{A}_k \varphi_s(x, k) := \begin{cases} \left( \frac{d}{dx} + ik \right)^2 \varphi_s(x, k)|_{(x_0 - \frac{1}{2}, x_0)} & \text{for } x \in (x_0 - \frac{1}{2}, x_0) \\ \left( \frac{d}{dx} + ik \right)^2 \varphi_s(x, k)|_{(x_0, x_0 + \frac{1}{2})} & \text{for } x \in (x_0, x_0 + \frac{1}{2}) \end{cases}$$

and using (3.4) and (3.1), hence yields

$$\varphi_s(x - \frac{1}{2}, k) = e^{-ik(x - \frac{1}{2})} \psi_s(x - \frac{1}{2}, k) = e^{-ik(x + \frac{1}{2})} \psi_s(x + \frac{1}{2}, k) = \varphi_s(x + \frac{1}{2}, k).$$

Which shows that  $(\varphi_s(\cdot, k))_{s \in \mathbb{N}}$  is an orthonormal and complete system of eigenfunctions of the periodic eigenvalue problem

$$\tilde{A}_k \varphi = \lambda_s(k) \varphi \text{ on } \Omega, \tag{3.5}$$

$$\varphi(x - \frac{1}{2}) = \varphi(x + \frac{1}{2}). \tag{3.6}$$

with the identical eigenvalue sequence  $(\lambda_s(s))_{s \in \mathbb{N}}$  as in (3.4).



## Chapter 4

# The Floquet transformation

In the next chapter we are going to show that the spectrum of the operator  $A$  can be constructed through the eigenvalue sequences  $(\lambda_s(k))_{s \in \mathbb{N}}$  by varying  $k$  over the Brillouin zone  $B$ . For that we need two results involving the Floquet transformation, which transfers the problem from  $L^2(\mathbb{R})$  to  $L^2(\Omega \times B)$  whereas  $\Omega \times B$  is by assumption compact. Even though the following two results do not differ in both the statement or the proof from standard theory, as in [1, Chap. 3], I still want to list them here for completeness.

**Theorem 4.1:** *The Floquet transformation  $U: L^2(\mathbb{R}) \rightarrow L^2(\Omega \times B)$*

$$(Uf)(x, k) := \frac{1}{\sqrt{|B|}} \sum_{n \in \mathbb{Z}} f(x - n) e^{ikn} \quad (x \in \Omega, k \in B). \quad (4.1)$$

*is an isometric isomorphism, with inverse given by*

$$(U^{-1}g)(x - n) = \frac{1}{\sqrt{|B|}} \int_B g(x, k) e^{-ikn} dk \quad (x \in \Omega, n \in \mathbb{Z}). \quad (4.2)$$

*If  $g(\cdot, k)$  is extended to the whole of  $\mathbb{R}$  by the semi-periodicity condition (3.1), the inverse simplifies to*

$$U^{-1}g = \frac{1}{\sqrt{|B|}} \int_B g(\cdot, k) dk. \quad (4.3)$$

*Proof:* For  $f \in L^2(\mathbb{R})$ ,

$$\int_{\mathbb{R}} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} \int_{\Omega} |f(x - n)|^2 dx, \quad (4.4)$$

where we used Beppo Levi's Theorem to exchange summation and integration. This shows that

$$\sum_{n \in \mathbb{Z}} |f(x - n)|^2 < \infty \text{ for almost every } x \in \Omega.$$

Thus,  $(Uf)(x, k)$  is well-defined by (4.1) (as a Fourier series with variable  $k$ ) for almost every  $x \in \Omega$ , and Parseval's equality gives for these  $x$

$$\int_B |(Uf)(x, k)|^2 dk = \sum_{n \in \mathbb{Z}} |f(x - n)|^2.$$

This expression is in  $L^2(\Omega)$  by (4.4), and

$$\|Uf\|_{L^2(\Omega \times B)} = \|f\|_{L^2(\mathbb{R})}.$$

We still haven't shown that  $U$  is onto, and that  $U^{-1}$  is given by (4.2) or (4.3). Let  $g \in L^2(\Omega \times B)$ , then define

$$f(x - n) := \frac{1}{\sqrt{|B|}} \int_B g(x, k) e^{-ikn} dk \quad (x \in \Omega, n \in \mathbb{Z}). \quad (4.5)$$

Parseval's Theorem states for fixed  $x \in \Omega$  that  $\sum_{n \in \mathbb{Z}} |f(x - n)|^2 = \int_B |g(x, k)|^2 dk$ . Integrating this equality over  $\Omega$  then yields

$$\int_{\Omega \times B} |g(x, k)|^2 dx dk = \int_{\Omega} \sum_{n \in \mathbb{Z}} |f(x - n)|^2 dx = \sum_{n \in \mathbb{Z}} \int_{\Omega} |f(x - n)|^2 dx = \int_{\mathbb{R}} |f(x)|^2 dx,$$

which means  $f \in L^2(\mathbb{R})$ . For almost every  $x \in \Omega$  (4.1) gives

$$f(x - n) = \frac{1}{\sqrt{|B|}} \int_B (Uf)(x, k) e^{-ikn} dk \quad (n \in \mathbb{Z}),$$

whence (4.5) implies  $Uf = g$  and (4.2). Now (4.3) follows from (4.2) and exploiting  $g(x + n, k) = e^{ikn} g(x, k)$ .  $\square$

## 4.1 Completeness of the Bloch waves

Using the Floquet transformation  $U$ , we can now prove the property of completeness of the Bloch waves  $\psi_s(\cdot, k)$  in  $L^2(\Omega)$  when we vary  $k$  over the Brillouin zone  $B$ .

**Theorem 4.2:** *For each  $f \in L^2(\mathbb{R})$  and  $l \in \mathbb{N}$ , define*

$$f_l(x) := \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_B \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) dk \quad (x \in \mathbb{R}). \quad (4.6)$$

*Then,  $f_l \rightarrow f$  in  $L^2(\mathbb{R})$  as  $l \rightarrow \infty$ .*

*Proof:* The last theorem tells us that  $Uf \in L^2(\Omega \times B)$ , which in return means that  $(Uf)(\cdot, k) \in L^2(\Omega)$  for almost all  $k \in B$  by Fubini's Theorem. As  $(\psi_s(\cdot, k))_{s \in \mathbb{N}}$  is an orthonormal and complete system of eigenfunctions in  $L^2(\Omega)$  for each  $k \in B$ , we derive

$$\lim_{l \rightarrow \infty} \|(Uf)(\cdot, k) - g_l(\cdot, k)\|_{L^2(\Omega)} = 0 \text{ for almost every } k \in B$$

where

$$g_l(x, k) := \sum_{s=1}^l \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k). \quad (4.7)$$

Moreover, we get by Bessel's inequality

$$\|(Uf)(\cdot, k) - g_l(\cdot, k)\|_{L^2(\Omega)}^2 \leq \|(Uf)(\cdot, k)\|_{L^2(\Omega)}^2$$

for all  $l \in \mathbb{N}$  and almost every  $k \in B$ . Next,  $\|(Uf)(\cdot, k)\|_{L^2(\Omega)}^2 \in L^1(B)$  as a function of  $k$  by Theorem 4.1, thus by Lebesgue's Dominated Convergence theorem

$$\lim_{l \rightarrow \infty} \int_B \|(Uf)(\cdot, k) - g_l(\cdot, k)\|_{L^2(\Omega)}^2 dk = \int_B \lim_{l \rightarrow \infty} \|(Uf)(\cdot, k) - g_l(\cdot, k)\|_{L^2(\Omega)}^2 dk = 0.$$

All in all, this means

$$\|Uf - g_l\|_{L^2(\Omega \times B)} \rightarrow 0 \text{ as } l \rightarrow \infty \quad (4.8)$$

If  $g(\cdot, k)$  is extended to the whole of  $\mathbb{R}$  by the semi-periodicity condition (3.1), using (4.6), (4.7) and (4.3), we find that  $f_l = U^{-1}g_l$ , whence (4.8) gives

$$\|U(f - f_l)\|_{L^2(\Omega \times B)} \rightarrow 0 \text{ as } l \rightarrow \infty,$$

□

## Chapter 5

# The spectrum of $A$

We will now prove the principal result, stating that for the operator  $A$  it holds

$$\sigma(A) = \bigcup_{s \in \mathbb{N}} I_s \quad (5.1)$$

where  $I_s := \{\lambda_s(k) : k \in \overline{B}\}$  ( $s \in \mathbb{N}$ ).

**Theorem 5.1:** *For all  $s \in \mathbb{N}$  the function  $\lambda_s(k)$  is continuous in  $k \in \overline{B}$ .*

*Proof:* By assumption our potential is bounded and we define

$$H_{per}^1 := \{v \in H^1(\Omega) : v(x - \frac{1}{2}) = v(x + \frac{1}{2})\}.$$

In the transformed eigenvalue problem (3.5) the boundary conditions (3.6) are periodic and independent from  $k$ . By Poincaré's min-max principle for eigenvalues we have

$$\lambda_s(k) = \min_{\substack{U \subseteq H_{per}^1(\Omega) \\ \dim U = s}} \max_{v \in U \setminus \{0\}} \frac{\langle A_k v, v \rangle_{L^2(\Omega)}}{\langle v, v \rangle_{L^2(\Omega)}}.$$

Now, let  $k \in B$  be fixed. For all  $\tilde{k} \in B$  and all  $v \in H_{per}^1(\Omega)$  using triangular inequality we can estimate for  $J \in \{(x_0 - \frac{1}{2}, x_0), (x_0, x_0 + \frac{1}{2})\}$ :

$$\begin{aligned} \frac{\langle (\frac{d}{dx} + i\tilde{k})v, (\frac{d}{dx} + i\tilde{k})v \rangle_{L^2(J)}}{\langle v, v \rangle_{L^2(J)}} & \begin{cases} \leq \\ \geq \end{cases} \frac{\langle (\frac{d}{dx} + ik)v, (\frac{d}{dx} + ik)v \rangle_{L^2(J)}}{\langle v, v \rangle_{L^2(J)}} \\ & \begin{cases} + \\ - \end{cases} \frac{2|k - \tilde{k}| \|v'\|_{L^2(J)} \|v\|_{L^2(J)}}{\|v\|_{L^2(J)}^2} \begin{cases} + \\ - \end{cases} \left| |k|^2 - |\tilde{k}|^2 \right| \quad (5.2) \end{aligned}$$

Moreover, we can estimate

$$\begin{aligned}
2\|v'\|_{L^2(J)}\|v\|_{L^2(J)} &\leq 2\left\|\left(\frac{d}{dx} + ik\right)v\right\|_{L^2(J)}\|v\| + 2|k|\|v\|_{L^2(J)}^2 \\
&\leq \left\|\left(\frac{d}{dx} + ik\right)v\right\|_{L^2(J)}^2 + \|v\|_{L^2(J)}^2 + 2|k|\|v\|_{L^2(J)}^2 \\
&\leq \left\langle\left(\frac{d}{dx} + ik\right)v, \left(\frac{d}{dx} + ik\right)v\right\rangle_{L^2(J)} + (1 + 2|k|)\|v\|_{L^2(J)}^2.
\end{aligned}$$

Hence (5.2) yields

$$\begin{aligned}
\frac{\left\langle\left(\frac{d}{dx} + i\tilde{k}\right)v, \left(\frac{d}{dx} + i\tilde{k}\right)v\right\rangle_{L^2(J)}}{\langle v, v \rangle_{L^2(J)}} \left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\} (1 \left\{ \begin{matrix} + \\ - \end{matrix} \right\} |k - \tilde{k}|) \frac{\left\langle\left(\frac{d}{dx} + ik\right)v, \left(\frac{d}{dx} + ik\right)v\right\rangle_{L^2(J)}}{\langle v, v \rangle_{L^2(J)}} \\
\left\{ \begin{matrix} + \\ - \end{matrix} \right\} \left( |k - \tilde{k}|(1 + 2|k|) + \left| |k|^2 - |\tilde{k}|^2 \right| \right).
\end{aligned}$$

Thus the min-max-principle gives (for  $|k - \tilde{k}| < 1$ )

$$\lambda_s(\tilde{k}) \left\{ \begin{matrix} \leq \\ \geq \end{matrix} \right\} (1 \left\{ \begin{matrix} + \\ - \end{matrix} \right\} |k - \tilde{k}|) \lambda_s(k) \left\{ \begin{matrix} + \\ - \end{matrix} \right\} \left( |k - \tilde{k}|(1 + 2|k|) + \left| |k|^2 - |\tilde{k}|^2 \right| \right),$$

Which means ultimately

$$|\lambda_s(\tilde{k}) - \lambda_s(k)| \leq |k - \tilde{k}| \left( \lambda_s(k) + 1 + 2|k| + |k| + |\tilde{k}| \right).$$

Now, the eigenvalue  $\lambda_s(k)$  is also an eigenvalue of the problem (3.4), where the operator is dependent on  $k$  and not the boundary conditions. However, all eigenvalues of (3.4) are by the min-max-principle dominated by eigenvalues of the eigenvalue problem of  $A_k$  with Dirichlet boundary conditions and as the eigenvalues for the Dirichlet boundary condition are independent from  $k$ ,  $\lambda_s(k)$  is uniformly bounded and hence continuous.  $\square$

**Remark:** As  $B$  is compact, connected and  $\lambda_s(k)$  is a continuous function of  $k \in B$  we derive for (5.1)

$$I_s \text{ is a compact real interval for each } s \in \mathbb{N}. \tag{5.3}$$

Through (5.3) we get moreover that  $\mu_s \leq \lambda_s(k)$  for all  $s \in \mathbb{N}$ ,  $k \in \overline{B}$  with  $(\mu_s)_{s \in \mathbb{N}}$  denoting the sequence of eigenvalues of problem (3.4) with Neumann boundary conditions. Since  $\mu_s \rightarrow \infty$  as  $s \rightarrow \infty$ , we obtain  $\min I_s \rightarrow \infty$  as  $s \rightarrow \infty$ , which together with (5.3) implies that

$$\bigcup_{s \in \mathbb{N}} I_s \text{ is closed.} \tag{5.4}$$

The first part of the statement (5.1) is

**Theorem 5.2:**  $\sigma(A) \supset \bigcup_{s \in \mathbb{N}} I_s$ .

*Proof:* Let  $\lambda \in \bigcup_{s \in \mathbb{N}} I_s$ , i.e.  $\lambda = \lambda_s(k)$  for some  $s \in \mathbb{N}$  and some  $k \in \overline{B}$ , and

$$A_k \psi_s(\cdot, k) = \lambda \psi_s(\cdot, k) \quad (5.5)$$

We regard  $\psi_s(\cdot, k)$  as extended to the whole of  $\mathbb{R}$  by the boundary condition (3.1), whence, due to the periodic structure of  $A$ ,  $\psi_s$  satisfies

$$A \psi_s = \lambda \psi_s$$

“locally”, i.e.

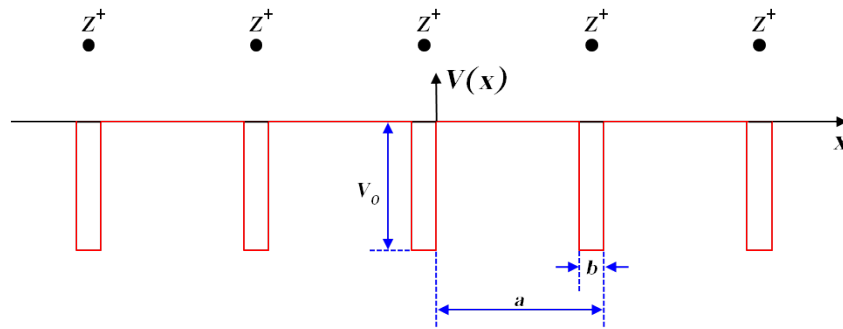
$$\psi_s \in \left\{ \psi \in H_{loc}^1(\mathbb{R}) : \psi \in H^2\left(\mathbb{R} \setminus \bigcup_{i \in \mathbb{Z}} x_i\right), \psi'(x_j - 0) - \psi'(x_j + 0) + \rho \psi(x_j) = 0 \ \forall j \right\},$$

thus  $\psi_s \in \mathcal{D}(A)$  and  $-\psi_s'' = \lambda \psi_s$  on each  $\Omega_j \setminus \{x_j\}$ . Now, if we choose a function  $\eta \in H^2(\mathbb{R})$  such that

$$\eta(x) = 1 \text{ for } |x| \leq \frac{1}{4}, \quad \eta(x) = 0 \text{ for } |x| \geq \frac{1}{2}, \quad (5.6)$$

and define, for each  $l \in \mathbb{N}$ ,

$$u_l(x) := \eta\left(\frac{|x|}{l}\right) \psi_s(x, k).$$



Then

$$\begin{aligned}
(A - \lambda I)u_l &= \sum_{i \in \mathbb{N}} \left[ \left( -\frac{d^2}{dx^2} - \lambda \right) u_l|_{(x_i, x_{i+1})} \cdot \mathbf{1}_{(x_i, x_{i+1})} \right] \\
&= \sum_{i \in \mathbb{N}} \left[ \left( -\frac{d^2}{dx^2} - \lambda \right) \left( \eta \left( \frac{|\cdot|}{l} \right) \psi_s(\cdot, k) \right) \Big|_{(x_i, x_{i+1})} \cdot \mathbf{1}_{(x_i, x_{i+1})} \right] \\
&= \sum_{i \in \mathbb{N}} \left[ \eta \left( \frac{|\cdot|}{l} \right) \left( -\frac{d^2}{dx^2} - \lambda \right) \psi_s(\cdot, k) \Big|_{(x_i, x_{i+1})} \cdot \mathbf{1}_{(x_i, x_{i+1})} \right] + R
\end{aligned} \tag{5.7}$$

where  $R$  is a sum of products of derivatives of order  $\geq 1$  of  $\eta \left( \frac{|\cdot|}{l} \right)$ , and derivatives of order  $\leq 1$  of  $\psi_s(\cdot, k)$ . Thus, note that  $\psi_s(\cdot, k) \in H_{loc}^2(\mathbb{R})$ , the semi-periodic structure of  $\psi_s(\cdot, k)$  implies

$$\|R\|_{L^2(\mathbb{R})} \leq \frac{c}{l} \|\psi_s(\cdot, k)\|_{H^1((\frac{l}{2}, \frac{l}{2}))} \leq c \frac{1}{\sqrt{l}}. \tag{5.8}$$

Together with (5.4), (5.5) and (5.7), this gives

$$\frac{1}{\|u_l\|} \|(A - \lambda I)u_l\| \leq \frac{c}{l}$$

Now, as moreover  $u_l \in \mathcal{D}(A)$  this results in

$$\frac{1}{\|u_l\|} \|(A - \lambda I)u_l\| \rightarrow 0 \text{ as } l \rightarrow \infty$$

Thus, either  $\lambda$  is an eigenvalue of  $A$ , or  $(A - \lambda I)^{-1}$  exists but is unbounded. In both cases,  $\lambda \in \sigma(A)$ .  $\square$

The other inclusion can be shown by using the previous shown properties of the Floquet transformation and the completeness of the Bloch waves. Hence, this prove doesn't differ from the one for an arbitrary m-th order linear differential operator with periodic coefficients. Again, due to completeness I still want to list the proof here, even though one can find it for example in [1, Chap. 3]

**Theorem 5.3:**  $\sigma(A) \subset \bigcup_{s \in \mathbb{N}} I_s$ .

*Proof:* Let  $\lambda \in \mathbb{R} \setminus \bigcup_{s \in \mathbb{N}} I_s$ . Hence, due to (5.4), there exists some  $\delta > 0$  such that

$$|\lambda_s(k) - \lambda| \geq \delta \quad \text{for all } s \in \mathbb{N}, k \in B \tag{5.9}$$

We have to prove that  $\lambda \in \rho(A)$ , i.e. for each  $f \in L^2(\mathbb{R})$  there exists some  $u \in \mathcal{D}(A)$

satisfying  $(A - \lambda I)u = f$ . For an arbitrary  $f \in L^2(\mathbb{R})$  and  $l \in \mathbb{N}$ , we define

$$f_l(x) := \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_B \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) dk$$

and

$$u_l := \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_B \frac{1}{\lambda_s(k) - \lambda} \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) dk \quad (5.10)$$

As  $\lambda$  is chosen to be outside of the spectrum the operator  $A_k - \lambda I$  is invertible, therefore the following equation has for every  $f \in L^2(\mathbb{R})$  and  $k \in \bar{B}$  a unique solution  $v \in \mathcal{D}(A_k)$

$$(A_k - \lambda I)v(\cdot, k) = (Uf)(\cdot, k) \quad \text{on } \Omega. \quad (5.11)$$

Due to (5.11), both  $v(\cdot, k)$  and  $\psi_s(\cdot, k)$  satisfy semi-periodic boundary conditions. Hence, (3.4), (5.9) and Parseval's identity yield

$$\begin{aligned} \|(Uf)(\cdot, k)\|_{L^2(\Omega)}^2 &= \sum_{s=1}^{\infty} |\langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)}|^2 \\ &= \sum_{s=1}^{\infty} |\langle (A_k - \lambda I)v(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)}|^2 \\ &= \sum_{s=1}^{\infty} |\lambda_s(k) - \lambda|^2 |\langle v(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)}|^2 \\ &\geq \delta^2 \|v(\cdot, k)\|_{L^2(\Omega)}^2. \end{aligned}$$

By Theorem 4.1 we know that  $f \in L^2(\Omega \times B)$ , this implies  $v \in L^2(\Omega \times B)$ , and we can define  $u := U^{-1}v \in L^2(\mathbb{R})$ . Thus, (5.11) gives

$$\begin{aligned} \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} &= \langle (A_k - \lambda I)(Uu)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \\ &= \langle (Uu)(\cdot, k), (A_k - \lambda I)\psi_s(\cdot, k) \rangle_{L^2(\Omega)} \\ &= (\lambda_s(k) - \lambda) \langle Uu(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)}. \end{aligned}$$

Now, we are able to apply Theorem 4.2 this yields for (5.10) that

$$u_l(x) = \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_B \langle (Uu)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) dk,$$



and whence Theorem 4.2 gives

$$u_l \rightarrow u, \quad f_l \rightarrow f \quad \text{in } L^2(\mathbb{R}) \text{ as } l \rightarrow \infty. \quad (5.12)$$

We will now prove that

$$\langle u_l, (A - \lambda I)v \rangle = \langle f_l, v \rangle \text{ for all } l \in \mathbb{N}, v \in \mathcal{D}(A); \quad (5.13)$$

As  $A$  is self-adjoint, by Theorem 2.3, this implies  $u_l \in \mathcal{D}(A)$ , and  $(A - \lambda I)u_l = f_l$  for all  $l \in \mathbb{N}$ . As furthermore every self-adjoint operator is also closed, (5.12) now implies

$$u \in \mathcal{D}(A) \text{ and } (A - \lambda I)u = f,$$

which is the desired result.

Eventually, we are left to prove (5.13). So, let  $\varphi \in C_0^\infty(\mathbb{R})$  be fixed, and let  $K \subseteq \mathbb{R}$  denote an open interval containing  $\text{supp}(\varphi)$  in its interior. By Fubini's Theorem we know that

$$r_s(x, k) := \frac{1}{\lambda_s(k) - \lambda} \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) \overline{(A - \lambda I)\varphi(x)},$$

and

$$t_s(x, k) := \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) \overline{\varphi(x)}$$

are in  $L^2(K \times B)$ , since (5.9),  $(A_k - \lambda I)\varphi \in L^\infty(K)$  and  $\varphi \in L^\infty(K)$  imply

$$\|r_s\|_{L^2(K \times B)} \leq c \|(Uf)(\cdot, k)\|_{L^2(\Omega)}^2 \|\psi_s(\cdot, k)\|_{L^2(K)}^2$$

and analogously for  $t_s$ . Now, as  $K$  is bounded there exists a finite number of copies of  $\Omega$  such that they cover  $K$ , hence  $\psi_s(\cdot, k)$  is in  $L^2(K)$  as a function of  $k$ , and  $(Uf)(\cdot, k)$  is in  $L^1(B)$  by Theorem 4.1. Since  $B$  is equally bounded,  $r$  and  $t$  are also in  $L^1(K \times B)$ . Therefore, Fubini's Theorem implies that the order of integration with respect to  $x$  and  $l$  may be exchanged for  $r$  and  $t$ . Thus, by (5.10), the fact that  $\varphi$  has compact support in the

interior of  $K$  and (3.4) we conclude

$$\begin{aligned}
\int_K u_l(x) \overline{(A - \lambda I)\varphi(x)} dx &= \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_K \left( \int_B r_s(x, k) dk \right) dx \\
&= \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_B \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \langle \psi_s(\cdot, k), \varphi \rangle_{L^2(K)} dk \\
&= \int_K \left[ \frac{1}{\sqrt{|B|}} \sum_{s=1}^l \int_B \langle (Uf)(\cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega)} \psi_s(x, k) dk \right] \overline{\varphi(x)} dx \\
&= \int_K f_l(x) \overline{\varphi(x)} dx,
\end{aligned}$$

i.e. (5.13).

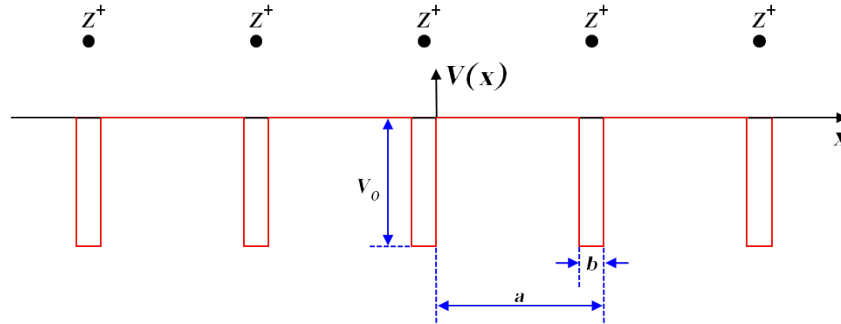
□

## Chapter 6

# The multi-dimensional case

In this last chapter, we are going to examine a more complex situation. We now want to model the movement of a particle in  $\mathbb{R}^n$  with periodically distributed, smooth,  $n - 1$ -dimensional surfaces inducing a potential. To show that the basic concepts presented in the previous chapters hold also in the new setting, we will merely give a formal justification of applicability of the one-dimensional proofs to the multi-dimensional case in a series of theorems, as below.

To start with, let  $Y$  denote a periodicity cell in  $\mathbb{R}^n$  and  $B^n$  the corresponding Brillouin zone, for simplicity assume  $Y$  being the unit cube  $Y = [0, 1]^n$  and hence  $B = [-\pi, \pi]^n$ . Contained in  $Y$  let  $S$  be a smooth surface, without a boundary with the conditions  $\dim S = n - 1$  and  $S \subseteq \overset{\circ}{Y}$ . Furthermore, let  $B \subseteq Y$  denote the set enclosed by  $S$ , such that  $S = \partial B$ . We will denote for any  $j \in \mathbb{Z}^n$  with  $Y_j = Y + j$  the  $j$ th copy of  $Y$ , which results through translation of the periodicity cell by  $j$ , and analogously for  $S_j = S + j$  and  $B_j = B + j$ .



The mathematical representation of the above is a multi-dimensional Schrödinger operator  $A^n$  whose operation is formally defined by

$$-\Delta + \rho \sum_{i \in \mathbb{Z}} \delta_{S_i} \quad (6.1)$$

on the whole of  $\mathbb{R}^n$ , where  $\delta_{S_i}$  denotes the Dirac delta distribution on hypersurface  $S_i$ , that integrates any function  $u \in L^2(\mathbb{R})$  over the compact set  $S_i$ .

Again motivated by the weak-formulation, given a right-hand side  $f \in L^2(\mathbb{R}^n)$  we consider for some  $\mu \in \mathbb{R}$  the problem to find  $u \in H^1(\mathbb{R}^n)$  such that

$$\int_{\mathbb{R}^n} \nabla u \overline{\nabla v} + \rho \sum_{i \in \mathbb{Z}^n} \int_{S_j} u \overline{v} ds - \mu \int_{\mathbb{R}^n} u \overline{v} = \int_Y f \overline{v} \quad (6.2)$$

holds for all  $v \in H^1(\mathbb{R}^n)$ , where  $s$  is the hypersurface measure associated to all  $S_j$  [?]

**Remark:** *The term originating from the potential is finite as*

$$\left| \sum_{j \in \mathbb{Z}^n} \int_{S_j} u \overline{v} \right|^2 \leq \left( \sum_{j \in \mathbb{Z}^n} \|u\|_{L^2(S_j)}^2 \right) \left( \sum_{j \in \mathbb{Z}^n} \|v\|_{L^2(S_j)}^2 \right).$$

Both terms on the right-hand side can be finitely estimated by the Trace theorem [3, Chap. 5] as

$$\|u\|_{L^2(S_j)}^2 \leq 2 \left( \frac{1}{h} \|u\|_{L^2(B_j)}^2 + h \|\nabla u\|_{L^2(B_j)}^2 \right)$$

for some  $h > 0$ .

We can once again exert Lax-Migram's Theorem to prove in (6.2) the existence of a unique solution  $u \in H^1(\mathbb{R}^n)$  in (6.2) for any  $f \in L^2(\mathbb{R}^n)$  if  $\mu$  is small enough, define the operator injective  $R_\mu^n: f \mapsto u$  and define  $A$  by means of  $R_\mu^n$ .

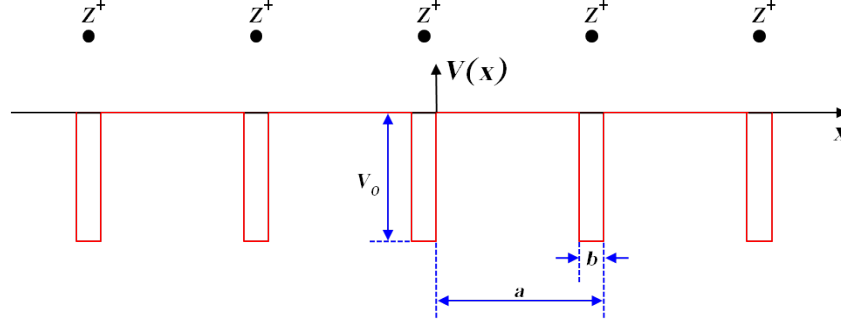
**Remark:** *The operator  $A$  is self-adjoint.*

**Theorem 6.1** (Characterisation of  $\mathcal{D}(A)$ ): *Let  $\Omega := \mathbb{R}^n \setminus \overline{\bigcup_{j \in \mathbb{Z}^n} B_j}$ . By choosing similar to above different functions  $v \in C^\infty(\mathbb{R}^n)$  in (6.2) we can further characterise such  $u \in \mathcal{D}(A)$ , namely for all  $j \in \mathbb{Z}^n$  it holds:*

1.  $\Delta u \in L^2(B_j)$ ,  $\Delta u \in L^2(\Omega)$  and  $\sum_{j \in \mathbb{Z}^n} \|\Delta u\|_{L^2(B_j)}^2 < \infty$
2.  $u|_{S_j-0} = u|_{S_j+0}$

3.  $\frac{\partial u}{\partial \eta_j} \big|_{S_j-0} - \frac{\partial u}{\partial \eta_j} \big|_{S_j+0} - \rho u \big|_{S_j} = 0$  where  $\eta_j$  denotes the normal on  $S_j$

Now, restricting again this problem to a fundamental domain of periodicity, for example  $Y$ ,



yields an operator  $A_k^n$  which we can define by considering analogous to before the problem to find  $u \in H_{k,n}^1 := \{w \in H^1(Y) : w|_{S_j^+} = w|_{S_j^-} e^{ik_j} \text{ for } k \in [-\pi, \pi]^2, j = 1, 2\}$  such that

$$\int_Y \nabla u \overline{\nabla v} + \rho \int_S u \overline{v} ds - \mu \int_Y u \overline{v} = \int_Y f \overline{v}. \quad (6.3)$$

for for all  $v \in H_{k,n}^1$ .

Again, Lax-Milgram's Theorem ensures the existence of a unique solution  $u \in H_{k,n}^1$  if  $\mu$  is small enough, and the operator  $R_{\mu,k} : f \mapsto u$  is well-defined and injective. In return, we are able to define

$$A_k^n := R_{\mu,k}^n + \mu I$$

The semi-periodic boundary conditions on  $H_{k,n}^1$  require a solution  $u \in H_{k,n}^1$  to (6.3) to satisfy even

$$\frac{\partial u}{\partial x_j} \big|_{S_j^+} = e^{ik_j} \frac{\partial u}{\partial x_j} \big|_{S_j^-} \quad j = 1, 2$$

**Theorem 6.2:** *The operator  $R_{\mu,k}^n$  on the fundamental domain of periodicity is compact.*

*Proof:* As in chapter 3 used, the compact embedding yields the desired result, for a multi-dimensional proof see [6, Chap. 4].  $\square$

By similar transformations of the problem (6.3) as in (3.5) and (3.6) we are able to show that the eigenvalues of  $A_k^n$  are continuous functions of  $k \in \overline{B}$ , and thus  $I_s = \{\lambda_s(k) : k \in \overline{B}\}$  is a compact real interval for each  $s \in \mathbb{N}$ .

Ultimately, using the same arguments as in Chapter 4 based on Bloch waves, Floquet

transform and a similar cut-off function  $\eta$  as in (5.6), we are able to prove the main result even for this multi-dimensional case, namely that the spectrum of a self-adjoint Schrödinger operator with periodic delta-potential on a hypersurface is the union of compact intervals, i.e.

$$\sigma(A) = \bigcup_{s \in \mathbb{N}} I_s.$$

## 6.1 Outlook

We have to notice that we haven't determined the nature of the spectrum that is if there are possible gaps in the spectrum. However,...

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## **Decleration**

I declare that I have developed and written the enclosed thesis completely by myself, have not used sources or means without declaration in the text and designated the included passage from other works, whether in substance or in principle, as such and that I adhered the statute of the Karlsruhe Institute of Technology for good scientific practice in their currently valid version.

Karlsruhe, den 13. September 2016