1 Theoretical Basics

We start with the set of test functions

$$\mathcal{D} = C_0^\infty(\mathbb{R}) = \{f \in C^\infty : \operatorname{supp}(f) \text{ is compact in } \mathbb{R}\}$$

and endow it with the topology: a sequence $(\varphi_j)_{j\in\mathbb{N}}$ with $\varphi_j \in \mathcal{D}$ converges against φ, if there is a compact set $K \subset \mathbb{R}$ with $\operatorname{supp}(\varphi_j) \subset K$ for all j and

$$\lim_{j\to\infty} \sup_{x\in K} \left| \frac{\partial^\alpha}{\partial x^\alpha} \left(\varphi_j(x) - \varphi(x)\right) \right| = 0$$

for all multi-indizies $\alpha \in \mathbb{N}^n$. The set $\mathcal{D}$ is – endowed with this convergence concept – a complete locally convex topological vector space satisfying the Heine–Borel property (Rudin 1991, §6.4–5).

The set of linear functionals from $\mathcal{D}$ to $\mathbb{R}$ we call the set of distributions

$$\mathcal{D}' = \{f \colon D \to \mathbb{R} : f \text{ is linear }\}$$

That is, a distribution T assigns to each test function φ a real (or complex) scalar $T(\varphi)$ such that

$$T(c_1\varphi_1 + c_2\varphi_2) = c_1T(\varphi_1) + c_2T(\varphi_2)$$

for all test functions φ_1, φ_2 and scalars c_1, c_2. Moreover, T is continuous if and only if

$$\lim_{k\to\infty} T(\varphi_k) = T\left(\lim_{k\to\infty} \varphi_k\right)$$

for every convergent sequence $\varphi_k \in \mathcal{D}$. (Even though the topology of $\mathcal{D}$ is not metrisable, a linear functional on $\mathcal{D}$ is continuous if and only if it is sequentially continuous.) Equivalently, T is continuous if and only if for every compact subset K of $\mathbb{R}$ there exists a positive constant C_K and a non-negative integer N_K such that

$$|T(\varphi)| \le C_K \sup_K |\partial^\alpha \varphi|$$

for all test functions φ with support contained in K and all multi-indices α with $|\alpha| \le N_K$ (Grubb 2009, p. 14).

The duality pairing between a distribution $T \in \mathcal{D}'$ and a test function $\varphi \in \mathcal{D}$ is denoted using angle brackets by

$$\begin{cases} \mathcal{D}' \times \mathcal{D} \to \mathbb{R} \\ (T, \varphi) \mapsto \langle T, \varphi \rangle \end{cases}$$

so that $\langle T, \varphi \rangle = T(\varphi)$. One interprets this notation as the distribution T acting on the test function φ to give a scalar, or symmetrically as the test function φ acting on the distribution T.

Now taking at distributions a closer look, we distinguish between two kinds: we call a distribution

- **regular** if there is locally integrable function

  $$f \in L^1_{loc}(\mathbb{R}) = \{f\colon \Omega \to \mathbb{C} \text{ measurable} : f|_K \in L_1(K)\ \forall K \subset \mathbb{R},\ K \text{ compact}\}$$

  such that the distribution T can be written as

  $$\langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(t)\varphi(t)dt \quad \forall \varphi \in D$$

  Sometimes, one abuses notation by identifying T_f with f, provided no confusion can arise, and thus the pairing between T_f and φ is often written

  $$\langle f, \varphi \rangle = \langle T_f, \varphi \rangle = \int_{\mathbb{R}} f(t)\varphi(t)dt \quad \forall \varphi \in D$$

- **singular** if the distribution is not regular, e.g. if there is no integral representation with a locally integrable function.

1.1 The dirac delta distribution

$$f = \delta_{x_0},\ \langle \delta_{x_0}, \varphi \rangle = \varphi(x_0)$$

It is easy to see that $f^\epsilon(x) \to 0$ almost everywhere and

$$\int f(y)dy = \alpha \neq 0$$

but

$$\int f^\epsilon dx = \frac{1}{\epsilon} \int f(\underbrace{\frac{x}{\epsilon}}_{=:y})dx = \int f(y)dy$$

Definition 1.1
For $f \in \mathcal{D}'$ we say $f^\epsilon \rightharpoonup f \in \mathcal{D}'$ if

$$\langle f^\epsilon, \varphi \rangle \to \langle f, \varphi \rangle \quad \forall \varphi \in \mathcal{D}$$

Example 1.2
Let $f^\epsilon(x) = \frac{1}{\epsilon} f(\frac{x - x_0}{\epsilon}$, $f \in C_c^\infty(\mathbb{R})$ with

$$\int_{\mathbb{R}} f(y)dy = \alpha < \infty$$

$f^\epsilon \rightharpoonup \alpha\delta_{x_0}$

1.2 The Schrödinger Operator

An Operator A is defined by three properties:

- space, i.e. a Hilbertspace H
- domain, $domA \subset H$
- operation, Au with $u \in domA$

The Schrödinger Operator is definied on the space $H = L^2(\mathbb{R})$ by

$$Au = -u'' + Vu$$

Now the question what is the domain of A arises.

Example 1.3

$$i \cdot \frac{\partial \psi}{\partial t}) A\psi$$

For $\Omega \subset \mathbb{R}$ the probability of an particle being with Ω in t can be described as

$$\int_\Omega |\psi(x,t)|^2$$

One now often is interested in quantum states of such particle and we therefore formulate the spectral problem

$$Au = \lambda u$$

Now assume V is rather good, e.g. $V \in L^\infty(\mathbb{R})$ and $domA = H^2(\mathbb{R})$.

Definition 1.4
We call an operator A symmetric if

$$(Au, v)_H = (u, Av) \quad \forall u, v \in domH$$

Example 1.5
Let $A = -\frac{d^2}{dx^2} + V$ for $V\colon \mathbb{R} \to \mathbb{R}$

$$\begin{aligned}(Au, v) &= \int -u''\overline{v}dx + \int Vu\overline{v}dx \\ &= \int u'\overline{v}'dx + \int Vu\overline{v}dx\end{aligned}$$

$$\begin{aligned}(u, Av) &= \int -u\overline{v''}dx + \int u\overline{V}\overline{v}dx \\ &= \int u'\overline{v}'dx + \int Vu\overline{v}dx\end{aligned}$$

And therefore A is symmetric.

Definition 1.6
For an operator A the adjoint A^* is the unique operator such that $\forall v, v^* \in H$ where

$$(Au, v)_H = (u, v*) \quad \forall u \in domA$$

Then define $domA^* = \{v \in H\}$ and $A^* v = v^*$.

Definition 1.7
An operator A is self-adjoint if $A = A*$.

If A is a self-adjoint operator then A is already symmetric but not the other way round. But as soon as A is bounded and symmetric then the operator is self-adjoint.

Example 1.8
Let $H = L^2(0,1)$ and define $A = -\frac{d^2}{dx^2}$ on $domA = C_0^\infty(0,1)$. As A is symmetric A^* is an extension of A, e.g. $A^* \supset A$.
Now for $v, v^* \in L^2$

$$\int -u''\overline{v}dx = \int u\overline{v^*}\forall u \in C_0^\infty(0,1)$$

Let $v \in C^\infty(0,1)$: $v^* := -v''$

$$\int -u''\overline{v}dx = -\int u'\overline{v'}dx = -\int u\overline{v''}dx = \int u\overline{v^*}dx$$

$\Rightarrow H^2(0,1) \cap H_0^1(0,1) \Rightarrow A$ is self-adjoint.

Further, we are going to examine small or even singular particles more closely. The potential V is a vector such that $gradV = F$ whereas F denotes the force acting upon a particle.

As in this case V has only a small support one could approximate V with a single-point potential.

But the operator itself is harder to understand

$$Au = -u'' + \delta_{x_0} u$$

For f, g:

$$\int (f \cdot g)\,\varphi dx = \int f\,(g \cdot \varphi)\,dx$$

Now suppose $f \in \mathcal{D}$, $g \in C^\infty(\mathbb{R})$:

$$\langle g \cdot f, \varphi \rangle \overset{def}{=} \langle f, \underbrace{g \cdot \varphi}_{\in \mathcal{D}} \rangle$$

1.3 Main Problem

For a differential equation we distinguish between three different solution concepts

- classical $u \in C^2$
- strong $u \in H^2$
- weak $u \in H^1$

With some conditions to the potential those terms can be equivalent.

1.3.1 I:

Define A^ϵ on $domA^\epsilon = H^2(\mathbb{R})$ with

$$A^\epsilon u = -u'' + \frac{1}{\epsilon} f(\frac{x}{\epsilon})u,\ f \in C_c^\infty(\mathbb{R})$$

while $\int_\mathbb{R} f(x)dx = \alpha \neq 0$. Now the question arises: $A^\epsilon \xrightarrow[\epsilon \to 0]{} ?$

1.3.2 II:

$Au - \mu u = f \in L^2, u \in \mathbb{C} \setminus \mathbb{R}$ with $A = -u'' + V$ the resolvent.

$$(A - \mu I)^{-1}$$

todo check if A is self-adjoint $\Rightarrow$ has solution for $\mathbb{C} \setminus \mathbb{R}$ defined and bounded arbetrary f
Partial integration yields

$$\int u'\overline{v'}dx + \int Vu\overline{v}dx + \mu \int u\overline{v}dx = \int f\overline{v}dx \quad (*)$$

which holds for an arbitrary $v \in C_0^\infty(\mathbb{R})$<

We say, u is a weak solution if $u \in H^1(\mathbb{R})$ and $(*)$ holds. If the potential is even in L^∞ then the solution is also a strong solution (i.e. $u \in H^2$) *todo proof*

For $f \in L^2(\mathbb{R})$, $Au - \mu u = f$ If we tale a potential V, then $\int Vu\overline{v}dx$ could be also written as $V(u\overline{v})$.
Now suppose $H = L^2(\mathbb{R}), A = -\frac{d^2}{dx^2} + \alpha\delta_{x_0}$

$$\int u'\overline{v}'dx + \alpha u(x_0)\overline{v}(x_0) + \mu \int u\overline{v}dx = \int f\overline{v}dx \quad \forall v \in C_0^\infty(\mathbb{R})(1)$$

Since this formular only evaluations $vandu$ in $x_0 \Rightarrow \forall v \in H^1(\mathbb{R}), u \in H^1(\mathbb{R})$ is enough.

For $d = 1$: $H^1(\mathbb{R}) \subset C(\mathbb{R})$ *todo proof*

2 The definition of our operator

2.0.1 I:

Now, we want to show that (1) has a unique solution using the Lax-Milgram theorem:

Let $\mathcal{H} = H^1(\mathbb{R})$ and define $a[u,v]$ as LHS and $\langle f, v\rangle$ as RHS of (1):

$$\begin{aligned} a[u,v] &\coloneqq \int u'\overline{v}'dx + \alpha u(x_0)\overline{v}(x_0) + \mu \int u\overline{v}dx \\ \langle f, v\rangle &\coloneqq \int f\overline{v}dx \end{aligned}$$

If $\mu \in \mathbb{R}$ *todo check if that is really necessary since Lax-Milgram needs it, (and usse trace? inequality)*

since $|u(x_0)|^2 = |u(x) + \int_x^{x_0} u'(\tau)d\tau|^2$ we can write with labech formula *todo which formular is this?*

$$\begin{aligned} a[u,v] &= \int |u'|^2 + \alpha|u(x_0)|^2 - \mu \int |u|^2 \\ &\leq 2|u(x)|^2 + 2|\int_{x_0}^{x} u'(\tau)d\tau|^2 \\ &\leq 2|u(x)|^2 + 2\int_{x_0}^{x} |u'(\tau)|^2 d\tau (x_1 - x_0)^2 \\ &\leq 2|u(x)|^2 + 2\int_{x_0}^{x_1} |u'(\tau)|^2 d\tau (x_1 - x_0)^2 \end{aligned}$$

$$\xRightarrow{Integr.} (x_1 - x_0)|u(x_0)|^2 \leq 2\int_{x_0}^{x_1} |u(x)|^2 dx + 2(x_1 - x_0)^2 \int_{x_0}^{x_1} u'(\tau)d\tau$$

todo is there something missing? can't read what I've written there

$$\Rightarrow |u(x_0)|^2 \leq \underbrace{\frac{2}{x_1 - x_0}}_{=:a} \int_{\mathbb{R}} |u(x)|^2 dx + \underbrace{2(x_1 - x_0)}_{=:b} \int_{\mathbb{R}} |u'(\tau)| d\tau$$

Now a is not independent from b, a small a results in a large b and vice versa.

For $\alpha \geq 0$:

$$\begin{aligned} a[u,u] &\overset{perdef}{\geq} \int |u'|^2 - \mu|u|^2 \\ &\overset{\mu<-1}{\geq} \int |u'|^2 + \int |u|^2 \\ &= \|u\|_{\mathcal{H}}^2 \end{aligned}$$

For $\alpha < 0$

$$\begin{aligned} a[u,u] &\geq \int |u'|^2 - \mu \int |u| + \alpha a \int |u(x)|^2 dx + \alpha b \int_{\mathbb{R}} |u'(\tau)| d\tau \\ &= (1+\alpha b) \int |u'|^2 + (\alpha a - \mu) \int |u|^2 \\ &= c\|u\|_{\mathcal{H}}^2 \end{aligned}$$

As we want that both coefficients in front of the integrals to be positive => we chose x_n correspondent *is it really*x_n We therefore generally choose μ 'relatively' close to $-\infty$.

2.0.2 II:

Now, we have to proof:

$$\begin{aligned} |a[u,v]| &\underset{ford=1}{\overset{Cauchy-Schwarz}{\leq}} \|u'\|_{L^2}\|v'\|_{L^2} + |\alpha| \left(a\|u\|_{L^2}^2 + b\|u'\|_{L^2}^2\right)^{\frac{1}{2}} (\text{...same vor v?})^{\frac{1}{2}} + |\mu| \|u\|_{L^2}\|v\|_{L^2} \\ &\leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \end{aligned}$$

2.0.3 III:

$$|\langle f, v\rangle| \leq c\|v\|_{\mathcal{H}}$$

Which follows from (1) RHS , Hölder => L^2 Norm of both $\leq H^1$ Norm! *todo proof*

Lax-Milgram gives as now the unique solution for the given problem.

2.1 Now to my homework

For $f \overset{R_\mu}{\mapsto} u$ solution of (1) and since (1) is the weak formulation of

$$\underbrace{-u'' + \alpha\delta_{x_0} u}_{=:Au} - \mu u = f \quad (2)$$

the mapping $f \mapsto u$ gives us also a weak solution of (2).

Now we focus on this equation: $Au - \mu = f$:

Definition 2.1
$domA = rangeR_\mu \subset H^1(\mathbb{R})$

$$Au := \underbrace{f}_{\in L^2} + \mu \underbrace{u}_{\in H^1} \subset L^2$$

So the next steps would be to

- describe domain of A explicitly
- show that $A = A^*$

Homework: Let $u \in domA$ and $u \in C^2(-\infty, x_0]$ and also $u \in C^2[x_0, \infty)$

$$\Rightarrow \text{ find conditions on } u \text{ at } x_0 such that our assumption holds$$

by the way $u \in H^1(\mathbb{R})$.

Interesting facts:

If we take the closure of$C_0^\infty(\mathbb{R})$ but the Closure in H^1!!! we get:

$$\bar{[}t]C_0^\infty(\mathbb{R}) = H^1$$

We want to construct the operator$A$ in a smart way with

$$A = -\frac{d^2}{dx^2} + \alpha d_{x_0}, \quad \mathcal{H} = L^2(\mathbb{R})$$

Then we introduced the variational problem

$$\forall v \in H^1(\mathbb{R}) : \quad \int \nabla u \overline{\nabla v} dx - \mu \int u\overline{v} dx + \alpha u(x_0)v(x_0) = \int f\overline{v} dx \quad (1)$$

$\exists_1 u \in H^1(\mathbb{R})$ satisfying (1)

$$L^2(\mathbb{R}) \ni f \mapsto u =: R_\mu f$$

For $f_1 \neq f_2 \Rightarrow u_1 \neq u_2$, since:

$$\text{Suppose } u_1 = u_2 \Rightarrow \int (f_1 - f_2)\overline{v} = 0 \quad \forall \underbrace{v \in H^1(\mathbb{R})}_{\substack{\text{and therefore} \\ \forall v \in L^2(\mathbb{R})}} \Rightarrow f_1 = f_2$$

Since H^1 is dense in $L^2 \implies f_1 = f_2$

$$\Rightarrow \left.\begin{array}{l} f = R_\mu^{-1} u \\ Au - \mu u \end{array}\right\} Au = R_\mu^{-1} u + \mu u$$

$\Rightarrow domA = rangeA\mathbb{R}_{mu}$
$R_\mu^{-1} u = Au - \mu u =: g$ or $u = R_\mu g$

$$\int u'v' - \mu \int u\overline{v} + \alpha u(x_0)v(x_0) = \int (Au - \mu u)\,\overline{v} \quad (2)$$

Lets take $v \in C_0^\infty(-\infty, x_0)$:

$$\Leftrightarrow \int_{-\infty}^{x_0} u'v'dx = \int_{-\infty}^{x_0} Au\overline{v}$$
$$\Leftrightarrow -\int_{-\infty}^{x_0} u\overline{v}''dx = \int_{-\infty}^{x_0} Au\overline{v}dx$$

for $u \in D'$:

$$\langle u^{(m)}, v \rangle = (-1)^m \langle u, v^{(m)} \rangle \quad v \in C^\infty$$

$$\Rightarrow \langle u^{(m)}, v \rangle = \langle u, v'' \rangle = \int_{-\infty}^{x_0} u\overline{v}'' = -\int_{-\infty}^{x_0} Au\overline{v}$$

$\Rightarrow u'' = \underbrace{-Au}_{\in L^2}$ on .$(-\infty, x_0)$ Analogous argument on (x_0, ∞)

Therefore we can fix the statement:

$$\boxed{domA \supset \{u \in H^1(\mathbb{R}), u \in H^2(-\infty, x_0), u \in H^2(x_0, \infty)}$$

for an arbitrary $b \in C_0^\infty(\mathbb{R})$, therefore with the help of (2) since $u \in H^2$ only on these two subintervals we integrate twice by parts on both sides of x_0

$$-\left(\int_{-\infty}^{x_0} + \int_{x_0}^{\infty}\right) u''\overline{v} + \big(u'(x_0-0)v_(x_0-0) - u'(x_0+0)v(x_0+0)\big) + \alpha u(x_0)\overline{v}(x_0)$$

$$= -\int_{-\infty}^{x_0} u''v - \int_{x_0}^{\infty} su''v$$

which we can rewrite with the fact that v is continous and $v(x_0+0) = v(x_0-0)$, after all we know that $v \in C_0^\infty$

$$\Leftrightarrow u'(x_0-0) - u'(x_0+0) + \alpha u(x_0) = 0$$

$\Rightarrow dom \subset \{u \in H^1(\mathbb{R}), u \in H^2(-\infty, x_0), u \in H^2(x_0,\infty), u'(x_0-0) - u'(x_0+0) + \alpha u(x_0) = 0\} =: B$ And the action of the operator is defined by

$$Au = \begin{cases} -u'', & (-\infty, x_0) \\ -u'', & (x_0, \infty) \end{cases}$$

Now lets show " $\supset$ ":

Let $u \in B$, and since for $u \in B$ it holds $u \in H^2$ for both sides $f := \begin{cases} -u'', & (-\infty, x_0) \\ -u'' & (x_0,\infty) \end{cases}$

Now we have to show that u is in Range of R_μ.

Idea: $Au = R_\mu^{-1}u + \mu u$

$$\begin{aligned} \Rightarrow u &= R_\mu A u - \mu R_\mu u \\ &= R_\mu(\underbrace{Au}_{f} - \mu u) \end{aligned}$$

We have to show $u \in domA = rangeR_\mu$

guess take $u \in B$ construct $f = \begin{cases} -u'', & (-\infty, x_0) \\ -u'', (x_0,\infty) \end{cases}$ and further to show::

So we have to show $u = R_\mu(f - \mu u)$:

$$\int u'v' - \mu \int uv + \alpha u(x_0)v(x_0) = \int (f - \mu u)v$$

$$\int u'v' + \alpha u(x_0)v(x_0) = -\int_{-\infty}^{x_0} u''v - \int_{x_0} u''v$$

$$\Rightarrow \int u'v' + \alpha u(x_0)v(x_0) = \int_{-\infty}^{x_0} u'v' + \int_{x_0}^{\infty} u'v' - u'(x_0-0)v(x_0) + u'(x_0+0)v(x_0)$$

$$\alpha u(x_0)v(x_0) = u(x_0+0)v(x_0) - u(x_0-0)v(x_0)$$

$\Rightarrow$ holds for $B \Rightarrow domA = B$.

Now we are going to show the self-adjointness:
We know that $A = R_\mu^{-1}u + \mu u$. We are going to show that R_μ^{-1} is symmetric and then A is of course symmetric as it is simply its shift.

As $domR_\mu = L^2(\mathbb{R})$ and $rangeR_\mu = domR_\mu^{-1}$ we are first going to focus on R_μ, and proof that this operator is symmetric:

$(\underbrace{R_\mu f}_{=u}, g) - (f, \underbrace{R_\mu g}_{=:v}) = \gamma$ We want to show that $\gamma = 0$:

$$\int u'\phi' - \mu u\overline{\phi} + \alpha u(x_0)\overline{\phi(x_0)} = \underbrace{\int f\overline{\phi}}_{v}$$

$$\int v'\psi - \mu \int v\overline{\psi} + \alpha u(x_0)\overline{\psi(x_0)} = \underbrace{\int g\overline{\psi}}_{u}$$

Summing over lines yields: $0 - 0 + 0 = \gamma$.

Now as we know that R_μ is symmetric we show that R_μ^{-1} is also symmetric:

$$(R_\mu^{-1}u, v) = (u, v^*) \quad u \in domR_\mu^{-1}$$

$u = R_\mu f$ for some f since $domR_\mu^{-1} = rangeR_\mu$. Now we have to show that $v \in domR_\mu^{-1}$ and since self-sadjoint and operator is definied on whole space

$$(f, v) = (R_\mu f, v^*) = (f, R_\mu v*) \quad \text{for arbitrary } f \in C^2$$

$v = R_\mu v^* \Rightarrow v \in rangeR_\mu = domR_\mu^{-1}$???