1 On the spectra of Schrödinger operator with periodic delta potential

Let A denote the one-dimensional Schrödinger operator with a periodic delta potential, i.e. $\exists (x_k)_{k\geq 1}$ periodically distributed such that

$$A := -\Delta + c \cdot \sum_{i\geq 1} \delta_{x_i}$$

and define the domain of A as follows

$$\mathcal{D}(A) := \Big\{ u \in \bigcap_{i\geq 1} \left(H^2(x_i - 1/2, x_i) \cap H^2(x_i, x_i + 1/2) \right), \\ \nabla u(x_i - 0) - \nabla u(x_i + 0) + c u(x_i) = 0, \ \forall i \geq 1 \Big\} \quad (1.1)$$

Moreover, we identify with Ω_k the periodicity cell containing delta point x_k and let w.o.l.g. $|\Omega_k| = 1 \ \forall k \geq 1$.

1.1 The domain

First we will show that A is in this sense well-defined. For every fixed $k \geq 1$ and for a $v \in C^\infty(\mathbb{R})$ such that $\operatorname{supp} v = (x_k - 1/2, x_k)$ equation 1.2 yields

$$\int_{x_k - 1/2}^{x_k} \nabla u \overline{\nabla v} dx = \int_{x_k - 1/2}^{x_k} A u \overline{v} \iff \int_{x_k - 1/2}^{x_k} u \overline{\Delta v} dx = \int_{x_k - 1/2}^{x_k} -A u \overline{v} dx$$

So $\Delta u = -Au \in L^2$ on $(x_k - 1/2, x_k)$ and analogously on $(x_k, x_k + 1/2)$. As $k \geq 1$ was arbitrary one can therefore fix

$$\mathcal{D}(A) \subset \Big\{ u \in \bigcap_{i\geq 1} \left(H^2(x_i - 1/2, x_i) \cap H^2(x_i, x_i + 1/2) \right) \Big\}$$

Next, from choosing a $v \in C^\infty(\mathbb{R})$ such that for a $k \geq 1$ $\operatorname{supp} v = (x_k - 1/2, x_k + 1/2)$ in 1.2 and integrating on both sides of x_k by parts follows

$$-\left(\int_{x_k - 1/2}^{x_k} + \int_{x_k}^{x_k + 1/2} \right) \Delta u \cdot \overline{v} + \left(\nabla u(x_k - 0) \overline{v(x_k - 0)} - \nabla u(x_k + 0) \overline{v(x_k + 0)} \right)$$

$$+cu(x_k)\overline{v(x_k)} = -\int_{x_k-1/2}^{x_k} \Delta u\overline{v} - \int_{x_k}^{x_k+1/2} \Delta u\overline{v}$$

But this is equivalent to

$$\nabla u(x_k - 0) - \nabla u(x_k + 0) + cu(x_k) = 0$$

Such that

$$\mathcal{D}(A) \subset \Big\{u \in \bigcap_{i\geq 1} \big(H^2(x_i - 1/2, x_i) \cap H^2(x_i, x_i + 1/2)\big)\,, \\ \nabla u(x_i - 0) - \nabla u(x_i + 0) + cu(x_i) = 0,\ \forall i \geq 1\Big\} =: B$$

The opposite inclusion one shows, as[1] $\mathcal{R}(R_\mu) = \mathcal{D}(A)$, by proving $u \in \mathcal{R}(R_\mu)$. More specifically, as $\mathcal{D}(R_\mu) = L^2$ define for $u \in B$

$$f := Au = -\Delta u \text{ on } \bigcup_{i\geq 1} ((x_i - 1/2, x_i) \cap (x_i, x_i + 1/2))$$

Now, left to show is that $u = R_\mu(f - \mu u)$:

$$\sum_{i\geq 1} \int_{\Omega_i} (f - \mu u)\overline{v} = \sum_{i\geq 1} \int_{\Omega_i} \nabla u\overline{\nabla v} + cu(x_i)\overline{v(x_i)} - \mu \int_{\Omega_i} u\overline{v}$$

$$\iff -\sum_{i\geq 1} \int_{x_i-1/2}^{x_i} \Delta u\overline{v} + \int_{x_i}^{x_i+1/2} \Delta u\overline{v} = \sum_{i\geq 1} \int_{\Omega_i} \nabla u\overline{\nabla v} + cu(x_i)\overline{v(x_i)}$$

For each $k \geq 1$ partial integration for a v with $\operatorname{supp} v = (x_k - 1/2, x_k + 1/2)$ yields

$$\int_{x_k-1/2}^{x_k} \nabla u\overline{\nabla v} + \int_{x_k}^{x_k+1/2} \nabla u\overline{\nabla v} - \nabla u(x_k - 0)\overline{v(x_k)} + \nabla u(x_k + 0)\overline{v(x_k)} \\ = \int_{\Omega_k} \nabla u\overline{\nabla v} + cu(x_k)\overline{v(x_k)}$$

$$\iff (\nabla u(x_k + 0) - \nabla u(x_k - 0) - cu(x_k))\,\overline{v(x_k)} = 0$$

which holds for $u \in B$.

[1] notice, $R_\mu := (A - \mu I)^{-1}$ denotes the resolvent

Furthermore, for $u, v \in D(A)$ we can on top of that estimate for arbitrary $\tilde{x}_i \in \Omega_i$

$$
\begin{aligned}
\left|\sum_{i\geq 1} u(x_i)\overline{v(x_i)}\right|^2 &\leq \sum_{i\geq 1} |u(x_i)|^2 \sum_{i\geq 1} |\overline{v(x_i)}|^2 \\
&\leq \sum_{i\geq 1} \left| u(\tilde{x}_i) + \int_{\tilde{x}_i}^{x_i} \nabla u(\tau) d\tau \right|^2 \sum_{i\geq 1} \left| \overline{v(\tilde{x}_i) + \int_{\tilde{x}_i}^{x_i} \nabla v(\tau) d\tau} \right|^2 \\
&\leq \left(\sum_{i\geq 1} 2|u(\tilde{x}_i)|^2 + 2\int_{\tilde{x}_i}^{x_i} |\nabla u(\tau)|^2 \, d\tau \cdot (x_i - \tilde{x}_i) \right) \\
&\quad \cdot \left(\sum_{i\geq 1} 2|v(\tilde{x}_i)|^2 + 2\int_{\tilde{x}_i}^{x_i} |\nabla v(\tau)|^2 \, d\tau \cdot (x_i - \tilde{x}_i) \right) \\
&\leq 4 \cdot \left(\sum_{i\geq 1} \int_{\Omega_i} |u(\tilde{x}_i)|^2 d\tilde{x}_i + \int_{\Omega_i} |\nabla u(\tau)|^2 \, d\tau \right) \\
&\quad \cdot \left(\sum_{i\geq 1} \int_{\Omega_i} |v(\tilde{x}_i)|^2 d\tilde{x}_i + \int_{\Omega_i} |\nabla v(\tau)|^2 \, d\tau \right) \\
&= 4 \cdot \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) \cdot \left(\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \right) \\
&\leq 4 \cdot \|u\|_{H^1(\mathbb{R})}^2 \|v\|_{H^1(\mathbb{R})}^2
\end{aligned}
$$

The consequence being the operator A is well-defined on its domain.

1.2 The spectral problem

For fixed $k \geq 1$, we consider the eigenvalue problem

$$
\int \nabla u \overline{\nabla v} - \mu \int u\overline{v} + c \cdot \sum_{i\geq 1} u(x_i)\overline{v(x_i)} = \int f\overline{v} \tag{1.2}
$$

$H_k^1 := \{a\}$ For a bounded $f \in L^2$ we are interested in the spectral problem of the weak formulation to the corresponding differential equation with quasi periodic . For all $v \in D(A)$ we obtain therefore
Next, we show that R_μ is well-defined, which means that for all $u \in \mathcal{D}(A)$ there exists a unique solution. Lets assume that f, as the righthand-side of the given differential equation, is a bounded linear functional.

Lax-Milgram's theorem[2] would then guarantee the existence and uniqueness of a solution if $\mathcal{D}(A)$ is a Hilbert space and if the left-hand side as bilinear

$$B(u,\varphi) \coloneqq \langle \nabla u, \nabla \varphi \rangle + cu(x_0)\varphi(x_0) - \mu \langle u, \varphi \rangle$$

is bounded and $B[u,u]$ is coercive.

Theorem 1.1. *Existence of a unique solution of 1.2*

Proof.
*Todo: $\mathcal{D}(A)$ **is a Hilbert space***
H^1_k is evidently a subspace of the Hilbert space $H^1(\Omega)$, but additionally H^1_k is also closed, as for an arbitrary sequence $(\psi_j)_{j\geq 1} \in H_{1,k}$ the value on the boundary coincides. Define $f \coloneqq \psi_j - \lim \psi_j$ and then

$$\begin{aligned}
|f(-1/2)|^2 &= 2|f(x)|^2 + 2\left(\int_{-1/2}^{x} f'(\tau)d\tau\right)^2 \\
&\leq 2|f(x)|^2 + 2\int_{-1/2}^{1/2} |f'|^2 d\tau \\
&\leq 2\|f\|^2_{H^1(-1/2,1/2)}
\end{aligned}$$

With that $\psi \in H^1_k$ as

$$\psi(-1/2) = \lim_{j\to\infty} \psi_j(-1/2) = \lim_{j\to\infty} e^{ik}\psi_j(1/2) = e^{ik}\psi(1/2)$$

The bilinear form B is bounded

$$\begin{aligned}
|B(u,\varphi)| &\coloneqq |\langle \nabla u, \nabla \varphi \rangle + cu(x_0)\varphi(x_0) - \mu\langle u, \varphi\rangle| \\
&\underset{Inequality}{\overset{Schwarz's}{\leq}} \big|\|\nabla u\| \cdot \|\nabla \varphi\| + cu(x_0)\varphi(x_0) - \mu\|u\| \cdot \|\varphi\|\big|
\end{aligned}$$

Again we require $H^1(\mathbb{R}) \Subset C(\mathbb{R})$, we can estimate the modulus of

[2] formulation and prove in appendix A

$v(x_0) \in \{u(x_0), \varphi(x_0)\}$ over the periodicity cell I_k:

$$
\begin{aligned}
|v(x_0)|^2 &= \left| v(x) + \int_x^{x_0} \nabla v(\tau) d\tau \right|^2 \quad \textit{for an arbitrary } x \in (\inf I, x_0) \\
&\overset{convexity}{\leq} 2|v(x)|^2 + 2 \left| \int_x^{x_0} \nabla v(\tau) d\tau \right|^2 \\
&\overset{trace}{\leq} 2|v(x)|^2 + 2 \int_I |\nabla v(\tau)|^2 \, d\tau \cdot (x_0 - x)
\end{aligned}
$$

Integrating both sides over the interval I yields:

$$
|v(x_0)|^2 \cdot |I| = 2 \int_I |v(x)|^2 dx + 2 \int_I |\nabla v(\tau)|^2 \, d\tau \cdot |I| \cdot (x_0 - x)
$$

$$
\Rightarrow |v(x_0)|^2 = \frac{2}{|I|} \int_I |v(x)|^2 dx + 2 \int_I |\nabla v(\tau)|^2 \, d\tau \cdot \underbrace{(x_0 - x)}_{\leq |I|}
$$

and results in the following

$$
\begin{aligned}
|B(u, \varphi)| &\leq \big| \|\nabla u\| \cdot \|\nabla \varphi\| + c \cdot u(x_0)\varphi(x_0) - \mu \|u\| \cdot \|\varphi\| \big| \\
&\leq \big| \|\nabla u\| \cdot \|\nabla \varphi\| + c \left(u(x_0)^2 \varphi(x_0)^2 \right)^{1/2} - \mu \|u\| \cdot \|\varphi\| \big| \\
&= \big| \|\nabla u\| \cdot \|\nabla \varphi\| + 2c \left(\frac{1}{|I|} \|u\|^2 + \|\nabla u\|^2 \cdot |I| \right)^{1/2} \\
&\qquad \cdot \left(\frac{1}{|I|} \|\varphi\|^2 + \|\nabla \varphi\|^2 \cdot |I| \right)^{1/2} - \mu \|u\| \cdot \|\varphi\| \big| \\
&= \big| (1 + 2c \cdot |I|) \cdot \|\nabla u\| \cdot \|\nabla \varphi\| + (\frac{2c}{|I|} - \mu) \cdot \|u\| \cdot \|\varphi\| \\
&\qquad + 2c \left(\|u\| \cdot \|\nabla \varphi\| + \|\nabla u\| \cdot \|\varphi\| \right) \big| \\
&\leq \alpha \cdot \|u\|_{H^1} \cdot \|\varphi\|_{H^1} \qquad \square
\end{aligned}
$$

Next, the coercivity for $c \geq 0$ and as assumed at the start μ is small *enough, here* $\mu < -1$

$$\begin{aligned} B(u,u) &= \langle \nabla u, \nabla u \rangle + cu(x_0)^2 - \mu \langle u, u \rangle \\ &\geq \langle \nabla u, \nabla u \rangle - \mu \langle u, u \rangle \\ &\geq \langle \nabla u, \nabla u \rangle + \langle u, u \rangle \\ &= \|u\|^2_{H^1} \end{aligned}$$

and for $c < 0$

$$\begin{aligned} B(u,u) &= \langle \nabla u, \nabla u \rangle + c|u(x_0)|^2 - \mu \langle u, u \rangle \\ &= \langle \nabla u, \nabla u \rangle + c \left(\frac{2}{I} \int_I |u(x)|^2 dx + 2I \int_I |\nabla u(\tau)|^2 d\tau \right) - \mu \langle u, u \rangle \\ &= (1 + 2cI) \|\nabla u\|^2 + (-1 + c\frac{2}{I}) \|u\|^2 \\ &\geq \beta \|u\|^2_{H^1} \qquad \square \end{aligned}$$

All in all, Lax-Milgram's theorem now guarantees a unique element $u \in H$ *such that*

$$B(u,v) = l(\varphi)$$

for all $\varphi \in H^1_k$

1.3 A_k is a self-adjoint operator

Last but not least, to show that A_k is self-adjoint, we focus first on $R_{\mu,k}^{-1}$ which is given by

$$R_{\mu,k}(A)^{-1} = (A - \lambda I)$$

First one has to notice that $R_{\mu,k}^{-1}$ is symmetric, as $\forall v \in H_k^1$:

$$\begin{aligned}
\langle R_{\mu,k}^{-1}u, v\rangle &= \langle (A - \lambda I)u, v\rangle \\
&= \int (A - \lambda I)(u)v dx \\
&= \int u'v'dx - \int \lambda uv dx + cu(x_0)v(x_0) \\
&= \int u(A - \lambda I)(v)dx \\
&= \langle u, (A - \lambda I)v\rangle = \langle u, R_{\mu,k}^{-1}v\rangle
\end{aligned}$$

Now as $domR_{\mu,k} = L^2(\mathbb{R})$ and $rangeR_{\mu,k} = domR_{\mu,k}^{-1}$, we want to show that for each $f, g \in L^2$

$$\langle R_{\mu,k}f, g\rangle - \langle f, R_{\mu,k}g\rangle = \gamma$$

$\gamma = 0$. Now there are $u, v \in domA_k$ with $Rf = u, Rg = v$ applying to A_k to u, v one gets for all $\varphi, \psi \in H_k^1$

$$\begin{aligned}
\int u'\varphi' + cu(0)\varphi(0) - \mu \int u\varphi = \int f\varphi \\
\int v'\psi' + cv(0)\psi(0) - \mu \int v\psi = \int g\psi
\end{aligned}$$

As it has to hold for all $\varphi, \psi \in H_k^1$ the special choice of $\varphi = v$ and $\psi = u$ yields $\gamma = 0$ and $R_{\mu,k}$ is therefore symmetric.
All in all we can use this to show that $\mathbb{R}_{\mu,k}$ is self-adjoint, as we get for an arbitrary $v^* \in domainR_{\mu,k}^{-1}$ there exists a $v \in domR_{\mu,k}$:

$$\begin{aligned}
\langle u, v^*\rangle &= \langle R_{\mu,k}^{-1}R_{\mu,k}u, v^*\rangle \\
&= \langle R_{\mu,k}u, (R_{\mu,k}^{-1})v^*\rangle \\
&= \langle R_{\mu,k}u, v\rangle = \langle u, R_{\mu,k}v\rangle
\end{aligned}$$

So $v^* \in rangeR_{\mu,k} = domR_{\mu,k}^{-1}$ with that also $R_{\mu,k}^{-1}$ is self-adjoint and as A_k is simply $R_{\mu,k}^{-1}$ shifted by the real constant μ, $R_{\mu,k}^{-1}$ is self-adjoint as well.

$\square$

1.4 A_k being compact

Let $B_{H_k^1} = \{f \in H_k^1(\Omega) : \|f\| \leq 1\}$. We want to show that $\forall \epsilon > 0\ \exists g_1, \ldots, g_{n_\epsilon}$:

$$\forall f \in B\ \exists g \in \{g_1, \ldots, g_{n_\epsilon}\} : \quad \|f - g\| \leq \epsilon$$

Together with the closure of H_k^1 this yields the compact embedding. Now, as $H^1(\Omega) \subset C(\Omega)$:

$$|f(x) - f(y)| \leq c|x - y|^{1/2} \text{ for some } c > 0 \tag{$*$}$$

For a $f \in B_{H^1}$ follows from $(*)$ that

$$|f(x)|^2 \leq 2\|f\|_{L^2}^2 + 2 \leq 4 \quad \forall x \in \Omega$$

And with that we can approximate a $f \in B$ by simple functions through partitioning Ω into n_ϵ equidistant intervals. As our simple function is constant on each subinterval, we chose this constant c_k such that

$$|f(\frac{k}{n}) - c_{k+1}| < \frac{1}{n}$$

such that

$$\begin{aligned}
\|f - g\|_{L^2}^2 &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f - c_{k+1}|^2 dx \\
&= 2\sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f - f(\frac{k}{n})|^2 dx + 2\sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(\frac{k}{n} - c_{k+1}|^2 dx \\
&\leq 2\sum_{n=0}^{n-1} \frac{1}{n^2} + 2\sum_{n=0}^{n-1} \frac{1}{n^3} = \frac{2}{n} + \frac{2}{n^2} < \epsilon^2 \text{ for } n \text{ small enough.}
\end{aligned}$$

2 Appendix A

2.1 The inverse of a self-adjoint operator

If $T \in B(X, Y)$ is invertible, where X, Y are Hilbert spaces, then T^* has an inverse and $(T^*)^{-1} = (T^{-1})^*$

Proof. *Let $T \in B(X, Y)$ be invertible, notice that $\langle Tv, u\rangle = \langle v, T^* u\rangle$ for all $v \in X, u \in Y$. Then $\langle T^*(T^{-1})^* v, u\rangle = \langle (T^{-1})^* v, Tu\rangle = \langle v, T^{-1}Tu\rangle = \langle u, v\rangle$.*
Therefore $T^*(T^{-1})^* = I$, hence $(T^{-1})^* = (T^*)^{-1}$

2.2 Lax-Milgram

Let H be a real Hilbert space, with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ as well as the pairing of H with its dual space. Assume that

$$B \colon H \times H \to R$$

is a bilinear mapping, for which there exist constant $\alpha, \beta > 0$ such that

$$|B[u, v]| \leq \alpha \|u\| \|v\| \quad (u, v \in H)$$

and

$$\beta \|u\|^2 \leq B[u, u] \quad (u \in H)$$

Finally, let $f \colon H \to \mathbb{R}$ be a bounded linear functional on H.

Then there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v\rangle$$

for all $v \in H$.

Proof. *For each fixed element $u \in H$, the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H; whence the Riesz Representation Theorem asserts the existence of a unique element $w \in H$ satisfying*

$$B[u, v] = \langle w, v\rangle \quad (*)$$

Let us write $Au = w$ whenever $(*)$ holds; so that

$$B[u,v] = \langle Au, v\rangle \quad (u, v \in H)$$

We first claim $A\colon H \to H$ is a bounded linear operator. Indeed if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$, we see for each $v \in H$ that

$$\begin{aligned}
\langle A(\lambda_1 u_1 + \lambda_2 u_2), v\rangle &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \quad (by\ (*)) \\
&= \lambda_1 B[u_1, v] + \lambda_2 B u_2, v] \\
&= \lambda_1 \langle Au_1, v\rangle + \lambda_2 \langle Au_2, v\rangle \quad (by\ (*)\ again) \\
&= \langle \lambda_1 Au_1 + \langle_2 Au_2, v\rangle.
\end{aligned}$$

This equality obtains for each $v \in H$, and so A is linear. Furthermore

$$\|Au\|^2 = \langle Au, Au\rangle = B[u, Au] \leq \alpha\|u\|\|Au\|.$$

Consequently $\|Au\| \leq \alpha\|u\|$ for all $u \in H$, and so A is bounded.
Next we assert

$$\begin{cases} A\ is\ one\text{-}to\text{-}one,\ and \\ R(A),\ the\ range\ of\ A,\ is\ close\ in\ H. \end{cases} \qquad (+)$$

To prove this, let us compute

$$\beta\|u\|^2 \leq B[u,u] = \langle Au, u\rangle \leq \|Au\|\|u\|$$

Hence $\beta\|u\| \leq \|Au\|$. This inequality easily implies (+).
We demonstrate now

$$R(A) = H \quad (-)$$

For if not, then, since $R(A)$ is closed, there would exist a nonzero element $w \in H$ with $w \in R(A)^{\perp}$. But this fact in turn implies the contradiction $\beta\|w\|^2 \leq B[w,w] = \langle Aw, w\rangle = 0$.
Next, we observe once more from the Riesz' Representation Theorem that

$$\langle f, v\rangle = \langle w, v\rangle \ for\ all\ v \in H$$

for some element $w \in H$. We then utilise (+) and (-) to find $u \in H$ satisfying $Au = w$. Then

$$B[u,v] = \langle Au, v\rangle = \langle w, v\rangle = \langle f, v\rangle (v \in H)$$

and this is the claim.
Finally, we show there is at most one element $u \in H$ verifying the claim. For if both $B[u,v] = \langle f, v\rangle$ and $B[\tilde{u}, v] = \langle f, v\rangle$, then $B[u - \tilde{u}, v] = 0$ $(v \in H)$. We set $v = u - \tilde{u}$ to find $\beta\|u - \tilde{u}\|^2 \leq B[u - \tilde{u}, u - \tilde{u}] = 0$.

2.3 Sobolev Embedding

For $s > d/2$, the follow holds

$$H^s \subset C_b(\mathbb{R}^d)$$

Proof. *For $u \in S(\mathbb{R}^d)$.*

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^d} e^{2i\pi x\xi}(1+|\xi|^2)^{-s/2}\hat{u}(1+|\xi|^2)^{s/2}d\xi \\ &\leq \left(\int (1+|\xi|^2)^{-s}d\xi\right)^{1/2}\left(\int |\hat{u}|^2(1+|\xi|^2)^{s}d\xi\right)^{1/2} \end{aligned}$$

with $\left(\int (1+|\xi|^2)^{-s}d\xi\right)^{1/2} \leq \infty$ *und* $\left(\int |\hat{u}|^2(1+|\xi|^2)^{s}d\xi\right)^{1/2} = ||u||_{H^s}$.

2.4 dom A_k = range $\mathsf{R}_{\mu,k}$

Proof. *As we introduced the variational problem*

$$\forall v \in H^1(\mathbb{R}): \quad \int \nabla u \overline{\nabla v} dx - \mu \int u\overline{v}dx + \alpha u(x_0)v(x_0) = \int f\overline{v}dx \quad (1)$$

$\exists_1 u \in H^1(\mathbb{R})$ *satisfying* (1)

$$L^2(\mathbb{R}) \ni f \mapsto u =: R_\mu f$$

For $f_1 \neq f_2 \Rightarrow u_1 \neq u_2$, since:

$$\textit{Suppose } u_1 = u_2 \Rightarrow \int (f_1 - f_2)\overline{v} = 0 \quad \forall \underbrace{v \in H^1(\mathbb{R})}_{\substack{\textit{and therefore} \\ \forall v \in L^2(\mathbb{R})}} \Rightarrow f_1 = f_2$$

Since H^1 is dense in L^2 $\Longrightarrow f_1 = f_2$

$$\Rightarrow \left.\begin{array}{l} f = R_\mu^{-1} u \\ Au - \mu u \end{array}\right\} Au = R_\mu^{-1} u + \mu u$$

$\Rightarrow domA = rangeR_{mu}$