1 Introduction

Let A denote the one-dimensional Schrödinger operator with a periodic delta potential at $(x_k)_{k\geq 1}$ and let Ω be a fundamental domain of periodicity associated with the periodic delta potential. Now assuming w.l.o.g. that $x_0 = 0 \in \Omega$ and $|\Omega| = 1$ yields

$$A_0 u \coloneqq A|_\Omega u = -\Delta u + \delta(x_0)u, \quad \text{for } u \in domA_0$$

For fixed $k \in \mathbb{R}$, we are interested in the spectral problem of the weak formulation of our differential equation such that the main focus point is going to be the following equation

$$\int_\Omega u'\varphi' - \mu \int_\Omega u\varphi + cu(x_0)\varphi(x_0) = \int_\Omega f\varphi \forall \varphi \in H_k^1$$

for a $u \in H_k^1 \coloneqq \left\{u \in H^1(\Omega) : u(\frac{1}{2}) = e^{ik}u(-\frac{1}{2})\right\}$ and for all $\varphi \in H_k^1$

$$\begin{aligned} domA_0 = \Big\{ & u \in H^2(-\frac{1}{2}, 0) \cap H^2(0, \frac{1}{2}), u(-0) = u(+0), \\ & u'(-0) - u'(+0) + cu(0) = 0, u(\frac{1}{2}) = e^{ik}u(-\frac{1}{2}), u'(\frac{1}{2}) = e^{ik}u'(-\frac{1}{2})\Big\} \end{aligned}$$

The boundary conditions on $\partial\Omega$, so-called semi-periodic boundary conditions, are stated here to later extended eigenfunctions on the whole of $\mathbb{R}$ by those conditions.

Subsequently it is shown that A_k is well defined on its domain, that given the differential equation with a bounded $f \in L^2$ there exists a unique solution u in the domain for all $f \in L^\infty$ and that there is an orthonormal basis of eigenfunction to the corresponding eigenvalue problem.

2 The operator A

Lemma 2.1. *The operator A is well-defined on its domain.*

Proof. *To show that A_0 is well-defined on its domain consider $u \in domA$ and an arbitrary $v \in H^1_k$*

$$\int_\Omega (A_0 u - \mu u)\,\overline{v} = \int_\Omega u'\overline{v}' - \mu \int_\Omega u\overline{v} + cu(x_0)\overline{v(x_0)} \quad (*)$$

Now choosing a specific v such that $\operatorname{supp} v = (-\frac{1}{2}, x_0)$ alters $(*)$ to

$$\int_{-\frac{1}{2}}^{x_0} u'v'dx = \int_{-\frac{1}{2}}^{x_0} Au\overline{v} \iff \int_{-\frac{1}{2}}^{x_0} u\overline{v}''dx = \int_{-\frac{1}{2}}^{x_0} -Au\overline{v}dx$$

$\Rightarrow u'' = -Au \in L^2$ on $(-\frac{1}{2}, x_0)$ and an analogously on $(x_0, \frac{1}{2})$. One can therefore fix

$$domA \subset \{u \in H^2(-\frac{1}{2}, x_0), u \in H^2(x_0, \frac{1}{2})\}$$

Next, integrating $(*)$ on both sides of x_0 by parts yields

$$-\left(\int_{-\frac{1}{2}}^{x_0} + \int_{x_0}^{\frac{1}{2}}\right) u''\overline{v} + \big(u'(x_0-0)v(x_0-0) - u'(x_0+0)v(x_0+0)\big) + cu(x_0)\overline{v}(x_0)$$

$$= -\int_{-\frac{1}{2}}^{x_0} u''v - \int_{x_0}^{\frac{1}{2}} u''v$$

But as $v \in H^1_k \subseteq C$ this is equivalent

$$u'(x_0 - 0) - u'(x_0 + 0) + cu(x_0) = 0$$

Such that

$$domA \subset \{u \in H^2(-\infty, x_0) \cap H^2(x_0, \infty), u'(x_0-0) - u'(x_0+0) + cu(x_0) = 0\} =: B$$

The opposite inclusion one shows by taking an arbitrary $u \in B$ and proving that u is in the domain of A_0. However[1], as $rangeR_{\mu,0} = domA_0$ one can also show that $u = \mathcal{R}(R_{\mu,0})$. As $domR_{\mu,0} = L^2$ define

$$f := A_0 u = \begin{cases} -u'', & (-\frac{1}{2}, x_0) \\ -u'', & (x_0, \frac{1}{2}) \end{cases}$$

[1] notice, $R_{\mu,0} := (A_0 - \mu I)^{-1}$ denotes the resolvent

and show that $u = R_\mu(f - \mu u)$*:*

$$\int_\Omega (f - \mu u)v = \int_\Omega u'v' + cu(x_0)v(x_0) - \mu \int_\Omega uv$$

$$\iff -\int_{-\frac{1}{2}}^{x_0} u''v - \int_{x_0}^{\frac{1}{2}} u''v = \int_\Omega u'v' + cu(x_0)v(x_0)$$

Partial integration with $\operatorname{supp} v = (-\frac{1}{2}, \frac{1}{2})$ *yields*

$$\int_{-\frac{1}{2}}^{x_0} u'v' + \int_{x_0}^{\frac{1}{2}} u'v' - u'(x_0 - 0)v(x_0) + u'(x_0 + 0)v(x_0) = \int_\Omega u'v' + cu(x_0)v(x_0)$$

$$\iff u(x_0 + 0)v(x_0) - u(x_0 - 0)v(x_0) - cu(x_0)v(x_0) = 0$$

Such that $u \in B$*.*

2.1 A_k is a self-adjoint operator

Last but not least, to show that A_k is self-adjoint, we focus first on $R_{\mu,k}^{-1}$ which is given by

$$R_{\mu,k}(A)^{-1} = (A - \lambda I)$$

First one has to notice that $R_{\mu,k}^{-1}$ is symmetric, as $\forall v \in H_k^1$:

$$\begin{aligned}
\langle R_{\mu,k}^{-1} u, v \rangle &= \langle (A - \lambda I)u, v \rangle \\
&= \int (A - \lambda I)(u) v dx \\
&= \int u'v' dx - \int \lambda u v dx + cu(x_0)v(x_0) \\
&= \int u(A - \lambda I)(v) dx \\
&= \langle u, (A - \lambda I)v \rangle = \langle u, R_{\mu,k}^{-1} v \rangle
\end{aligned}$$

Now as $domR_{\mu,k} = L^2(\mathbb{R})$ and $rangeR_{\mu,k} = domR_{\mu,k}^{-1}$, we want to show that for each $f, g \in L^2$

$$\langle R_{\mu,k} f, g \rangle - \langle f, R_{\mu,k} g \rangle = \gamma$$

$\gamma = 0$. Now there are $u, v \in domA_k$ with $Rf = u, Rg = v$ applying to A_k to u, v one gets for all $\varphi, \psi \in H_k^1$

$$\begin{aligned}
\int u'\varphi' + cu(0)\varphi(0) - \mu \int u\varphi = \int f\varphi \\
\int v'\psi' + cv(0)\psi(0) - \mu \int v\psi = \int g\psi
\end{aligned}$$

As it has to hold for all $\varphi, \psi \in H_k^1$ the special choice of $\varphi = v$ and $\psi = u$ yields $\gamma = 0$ and $R_{\mu,k}$ is therefore symmetric.
All in all we can use this to show that $\mathbb{R}_{\mu,k}$ is self-adjoint, as we get for an arbitrary $v^* \in domainR_{\mu,k}^{-1}$ there exists a $v \in domR_{\mu,k}$:

$$\begin{aligned}
\langle u, v^* \rangle &= \langle R_{\mu,k}^{-1} R_{\mu,k} u, v^* \rangle \\
&= \langle R_{\mu,k} u, (R_{\mu,k}^{-1}) v^* \rangle \\
&= \langle R_{\mu,k} u, v \rangle = \langle u, R_{\mu,k} v \rangle
\end{aligned}$$

So $v^* \in rangeR_{\mu,k} = domR_{\mu,k}^{-1}$ with that also $R_{\mu,k}^{-1}$ is self-adjoint and as A_k is simply $R_{\mu,k}^{-1}$ shifted by the real constant μ, $R_{\mu,k}^{-1}$ is self-adjoint as well.

$\square$

2.2 A_k being compact

Let $B_{H_k^1} = \{f \in H_k^1(\Omega) : \|f\| \leq 1\}$. We want to show that $\forall \epsilon > 0\ \exists g_1, \ldots, g_{n_\epsilon}$:

$$\forall f \in B\ \exists g \in \{g_1, \ldots, g_{n_\epsilon}\} : \quad \|f - g\| \leq \epsilon$$

Together with the closure of H_k^1 this yields the compact embedding. Now, as $H^1(\Omega) \subset C(\Omega)$:

$$|f(x) - f(y)| \leq c|x - y|^{\frac{1}{2}} \text{ for some } c > 0 \tag{$*$}$$

Now, for a $f \in B_{H^1}$ follows from $(*)$ that

$$|f(x)|^2 \leq 2\|f\|_{L^2}^2 + 2 \leq 4 \quad \forall x \in \Omega$$

And with that we can approximate a $f \in B$ by simple functions through partitioning Ω into n_ϵ equidistant intervals. As our simple function is constant on each subinterval, we chose this constant c_k such that

$$|f(\frac{k}{n}) - c_{k+1}| < \frac{1}{n}$$

such that

$$\begin{aligned}
\|f - g\|_{L^2}^2 &= \sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f - c_{k+1}|^2 dx \\
&= 2\sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f - f(\frac{k}{n})|^2 dx + 2\sum_{k=0}^{n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(\frac{k}{n} - c_{k+1}|^2 dx \\
&\leq 2\sum_{n=0}^{n-1} \frac{1}{n^2} + 2\sum_{n=0}^{n-1} \frac{1}{n^3} = \frac{2}{n} + \frac{2}{n^2} < \epsilon^2 \text{ for } n \text{ small enough.}
\end{aligned}$$

2.3 Existence of a unique solution

Next, one has to show that $R_{\mu,k}$ is well-defined, which means that for all $u \in dom(A_k)$ there exists a unique solution. Lets assume that f, as the righthand-side of the given differential equation, is a bounded linear functional. Lax-Milgram's theorem[2] would indeed guarantee the existence and uniqueness to prove the existence of a unique solution and whereby $R_{\mu,k}$ being well-defined, but one has to show that H^1_k is a Hilbert space, for

$$\begin{aligned} B(u,\varphi) &\coloneqq \langle \nabla u, \nabla \varphi \rangle + cu(x_0)\varphi(x_0) - \mu \langle u, \varphi \rangle \\ l(\varphi) &\coloneqq \langle f, \varphi \rangle \end{aligned}$$

B is bounded and $B[u,u]$ is coercive.

2.3.1 H^1_k being a Hilbert space

H^1_k is evidently a subspace of the Hilbert space $H^1(\Omega)$, but additionally H^1_k is also closed, as for an arbitrary sequence $(\psi_j)_{j\geq 1} \in H_{1,k}$ the value on the boundary coincides. Define $f \coloneqq \psi_j - \lim \psi_j$ and then

$$\begin{aligned} |f(-\frac{1}{2})|^2 &= 2|f(x)|^2 + 2\left(\int_{-\frac{1}{2}}^{x} f'(\tau)d\tau\right)^2 \\ &\leq 2|f(x)|^2 + 2\int_{-\frac{1}{2}}^{\frac{1}{2}} |f'|^2 d\tau \\ &\leq 2\|f\|^2_{H^1(-\frac{1}{2},\frac{1}{2})} \end{aligned}$$

With that $\psi \in H^1_k$ as

$$\psi(-\frac{1}{2}) = \lim_{j\to\infty} \psi_j(-\frac{1}{2}) = \lim_{j\to\infty} e^{ik}\psi_j(\frac{1}{2}) = e^{ik}\psi(\frac{1}{2})$$

2.3.2 The bilinear form B is bounded

$$\begin{aligned} |B(u,\varphi)| &\coloneqq |\langle \nabla u, \nabla \varphi \rangle + cu(x_0)\varphi(x_0) - \mu \langle u, \varphi \rangle| \\ &\underset{Inequality}{\overset{Schwarz's}{\leq}} \big| \|\nabla u\| \cdot \|\nabla \varphi\| + cu(x_0)\varphi(x_0) - \mu \|u\| \cdot \|\varphi\| \big| \end{aligned}$$

[2]formulation and prove in appendix A

Again we require $H^1(\mathbb{R}) \Subset C(\mathbb{R})$, we can estimate the modulus of $v(x_0) \in \{u(x_0), \varphi(x_0)\}$ over the periodicity cell I_k:

$$
\begin{aligned}
|v(x_0)|^2 &= \left| v(x) + \int_x^{x_0} \nabla v(\tau) d\tau \right|^2 \quad \text{for an arbitrary } x \in (\inf I, x_0) \\
&\overset{convexity}{\leq} 2|v(x)|^2 + 2 \left| \int_x^{x_0} \nabla v(\tau) d\tau \right|^2 \\
&\overset{trace}{\leq} 2|v(x)|^2 + 2 \int_I |\nabla v(\tau)|^2 \, d\tau \cdot (x_0 - x)
\end{aligned}
$$

Integrating both sides over the interval I yields:

$$
|v(x_0)|^2 \cdot |I| = 2 \int_I |v(x)|^2 dx + 2 \int_I |\nabla v(\tau)|^2 \, d\tau \cdot |I| \cdot (x_0 - x)
$$

$$
\Rightarrow |v(x_0)|^2 = \frac{2}{|I|} \int_I |v(x)|^2 dx + 2 \int_I |\nabla v(\tau)|^2 \, d\tau \cdot \underbrace{(x_0 - x)}_{\leq |I|}
$$

and results in the following

$$
\begin{aligned}
|B(u, \varphi)| &\leq \big| \|\nabla u\| \cdot \|\nabla \varphi\| + c \cdot u(x_0)\varphi(x_0) - \mu \|u\| \cdot \|\varphi\| \big| \\
&\leq \big| \|\nabla u\| \cdot \|\nabla \varphi\| + c \left(u(x_0)^2 \varphi(x_0)^2 \right)^{\frac{1}{2}} - \mu \|u\| \cdot \|\varphi\| \big| \\
&= \Big| \|\nabla u\| \cdot \|\nabla \varphi\| + 2c \left(\frac{1}{|I|} \|u\|^2 + \|\nabla u\|^2 \cdot |I| \right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\frac{1}{|I|} \|\varphi\|^2 + \|\nabla \varphi\|^2 \cdot |I| \right)^{\frac{1}{2}} - \mu \|u\| \cdot \|\varphi\| \Big| \\
&= \Big| (1 + 2c \cdot |I|) \cdot \|\nabla u\| \cdot \|\nabla \varphi\| + (\frac{2c}{|I|} - \mu) \cdot \|u\| \cdot \|\varphi\| \\
&\qquad + 2c \left(\|u\| \cdot \|\nabla \varphi\| + \|\nabla u\| \cdot \|\varphi\| \right) \Big| \\
&\leq \alpha \cdot \|u\|_{H^1} \cdot \|\varphi\|_{H^1} \qquad \square
\end{aligned}
$$

2.3.3 $B[u,u]$ is coercive

Next, the coercivity for $c \geq 0$ and as assumed at the start μ is small enough, here $\mu < -1$

$$\begin{aligned} B(u,u) &= \langle \nabla u, \nabla u \rangle + cu(x_0)^2 - \mu \langle u, u \rangle \\ &\geq \langle \nabla u, \nabla u \rangle - \mu \langle u, u \rangle \\ &\geq \langle \nabla u, \nabla u \rangle + \langle u, u \rangle \\ &= \|u\|_{H^1}^2 \end{aligned}$$

and for $c < 0$

$$\begin{aligned} B(u,u) &= \langle \nabla u, \nabla u \rangle + c|u(x_0)|^2 - \mu \langle u, u \rangle \\ &= \langle \nabla u, \nabla u \rangle + c\left(\frac{2}{I}\int_I |u(x)|^2 dx + 2I\int_I |\nabla u(\tau)|^2 d\tau\right) - \mu \langle u, u \rangle \\ &= (1 + 2cI)\|\nabla u\|^2 + (-1 + c\frac{2}{I})\|u\|^2 \\ &\geq \beta \|u\|_{H^1}^2 \end{aligned}$$

□

All in all, Lax-Milgram's theorem now guarantees a unique element $u \in H$ such that

$$B(u,v) = l(\varphi)$$

for all $\varphi \in H_k^1$

3 Appendix A

3.1 The inverse of a self-adjoint operator

If $T \in B(X, Y)$ is invertible, where X, Y are Hilbert spaces, then T^* has an inverse and $(T^*)^{-1} = (T^{-1})^*$

Proof. *Let $T \in B(X, Y)$ be invertible, notice that $\langle Tv, u\rangle = \langle v, T^*u\rangle$ for all $v \in X, u \in Y$. Then $\langle T^*(T^{-1})^*v, u\rangle = \langle (T^{-1})^*v, Tu\rangle = \langle v, T^{-1}Tu\rangle = \langle u, v\rangle$.*
Therefore $T^*(T^{-1})^* = I$, hence $(T^{-1})^* = (T^*)^{-1}$

3.2 Lax-Milgram

Let H be a real Hilbert space, with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ as well as the pairing of H with its dual space. Assume that

$$B\colon H \times H \to R$$

is a bilinear mapping, for which there exist constant $\alpha, \beta > 0$ such that

$$|B[u, v]| \leq \alpha\|u\|\|v\| \quad (u, v \in H)$$

and

$$\beta\|u\|^2 \leq B[u, u] \quad (u \in H)$$

Finally, let $f\colon H \to \mathbb{R}$ be a bounded linear functional on H.

Then there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v\rangle$$

for all $v \in H$.

Proof. *For each fixed element $u \in H$, the mapping $v \mapsto B[u, v]$ is a bounded linear functional on H; whence the Riesz Representation Theorem asserts the existence of a unique element $w \in H$ satisfying*

$$B[u, v] = \langle w, v\rangle \quad (*)$$

Let us write $Au = w$ whenever $(*)$ holds; so that

$$B[u, v] = \langle Au, v\rangle \quad (u, v \in H)$$

We first claim $A\colon H \to H$ is a bounded linear operator. Indeed if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $u_1, u_2 \in H$, we see for each $v \in H$ that

$$\begin{aligned}\langle A(\lambda_1 u_1 + \lambda_2 u_2), v\rangle &= B[\lambda_1 u_1 + \lambda_2 u_2, v] \quad (by\ (*))\\ &= \lambda_1 B[u_1, v] + \lambda_2 Bu_2, v]\\ &= \lambda_1\langle Au_1, v\rangle + \lambda_2\langle Au_2, v\rangle \quad (by\ (*)\ again)\\ &= \langle \lambda_1 Au_1 + \langle_2 Au_2, v\rangle.\end{aligned}$$

This equality obtains for each $v \in H$, and so A is linear. Furthermore

$$\|Au\|^2 = \langle Au, Au\rangle = B[u, Au] \leq \alpha\|u\|\|Au\|.$$

Consequently $\|Au\| \leq \alpha\|u\|$ for all $u \in H$, and so A is bounded.
Next we assert

$$\begin{cases} A \text{ is one-to-one, and} \\ R(A), \text{ the range of } A, \text{ is close in } H. \end{cases} \qquad (+)$$

To prove this, let us compute

$$\beta\|u\|^2 \leq B[u, u] = \langle Au, u\rangle \leq \|Au\|\|u\|$$

Hence $\beta\|u\| \leq \|Au\|$. This inequality easily implies (+).
We demonstrate now

$$R(A) = H \quad (\text{-})$$

For if not, then, since $R(A)$ is closed, there would exist a nonzero element $w \in H$ with $w \in R(A)^{\perp}$. But this fact in turn implies the contradiction $\beta\|w\|^2 \leq B[w, w] = \langle Aw, w\rangle = 0$.
Next, we observe once more from the Riesz' Representation Theorem that

$$\langle f, v\rangle = \langle w, v\rangle \text{ for all } v \in H$$

for some element $w \in H$. We then utilise (+) and (-) to find $u \in H$ satisfying $Au = w$. Then

$$B[u, v] = \langle Au, v\rangle = \langle w, v\rangle = \langle f, v\rangle (v \in H)$$

and this is the claim.
Finally, we show there is at most one element $u \in H$ verifying the claim. For if both $B[u, v] = \langle f, v\rangle$ and $B[\tilde{u}, v] = \langle f, v\rangle$, then $B[u - \tilde{u}, v] = 0$ $(v \in H)$. We set $v = u - \tilde{u}$ to find $\beta\|u - \tilde{u}\|^2 \le B[u - \tilde{u}, u - \tilde{u}] = 0$.

3.3 Sobolev Embedding

For $s > d/2$, the follow holds

$$H^s \subset C_b(\mathbb{R}^d)$$

Proof. *For $u \in S(\mathbb{R}^d)$.*

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^d} e^{2i\pi x\xi}(1 + |\xi|^2)^{-s/2}\hat{u}(1 + |\xi|^2)^{s/2} d\xi \\ &\le \left(\int (1 + |\xi|^2)^{-s} d\xi\right)^{1/2} \left(\int |\hat{u}|^2 (1 + |\xi|^2)^s d\xi\right)^{1/2} \end{aligned}$$

with $\left(\int (1 + |\xi|^2)^{-s} d\xi\right)^{1/2} \le \infty$ *und* $\left(\int |\hat{u}|^2 (1 + |\xi|^2)^s d\xi\right)^{1/2} = ||u||_{H^s}$.

3.4 dom A_k = range $\mathsf{R}_{\mu,k}$

Proof. *As we introduced the variational problem*

$$\forall v \in H^1(\mathbb{R}): \quad \int \nabla u \overline{\nabla v} dx - \mu \int u\overline{v} dx + \alpha u(x_0)v(x_0) = \int f\overline{v} dx \quad (1)$$

$\exists_1 u \in H^1(\mathbb{R})$ *satisfying* (1)

$$L^2(\mathbb{R}) \ni f \mapsto u =: R_\mu f$$

For $f_1 \neq f_2 \Rightarrow u_1 \neq u_2$, since:

$$\textit{Suppose } u_1 = u_2 \Rightarrow \int (f_1 - f_2)\overline{v} = 0 \quad \forall \underbrace{v \in H^1(\mathbb{R})}_{\substack{\textit{and therefore} \\ \forall v \in L^2(\mathbb{R})}} \Rightarrow f_1 = f_2$$

Since H^1 is dense in $L^2 \Longrightarrow f_1 = f_2$

$$\Rightarrow \left.\begin{array}{l} f = R_\mu^{-1} u \\ Au - \mu u \end{array}\right\} Au = R_\mu^{-1} u + \mu u$$

$\Rightarrow domA = rangeR_{mu}$