Spectraltheory

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Contents

Chapter I

Unbounded operators, adjoint and self-adjoint operators

Let H be a separable Hilbert space, $\langle \cdot, \cdot \rangle$ denote the scalar product on H.

A linear operator T in H is a linear map $u \mapsto Tu$ defined on a subspace $\mathcal{D}(T)$ of H, and we call $\mathcal{D}(T)$ the domain of T.
For $T \colon \mathcal{D}(T) \to H$ we denote the range of T with

$$\mathcal{R}(T) \coloneqq \operatorname{Image}(T).$$

We say that T is **bounded** if it is continuous from $\mathcal{D}(T)$ into H, with respect to the topology induced by H.
If $\mathcal{D}(T) = H$ we recall the definition of bounded operators from the functional analysis course.

From now on, if $\mathcal{D}(T) \neq H$ we will assume that $\mathcal{D}(T)$ is **dense** in H, i.e. $\overline{\mathcal{D}(T)} = H$. In this case, if T is bounded then T has a unique continuous extension to all of H.

$$\Rightarrow T \text{ bounded is boring!}$$

Recall: An operator is called **closed** if the graph

$$G(T) \coloneqq \Big\{ (x, y) \in H \times H \colon x \in \mathcal{D}(T), y = Tx \Big\}$$

is closed in $H \times H$.

Definition I.1: *Let $T\colon \mathcal{D}(T) \to H$ be a (linear) operator with $\mathcal{D}(T)$ dense in H. Then T is called* ***closed*** *if the conditions*

$$\left.\begin{aligned} &u_n \in \mathcal{D}(T) \\ &u_n \to u \ in\ H \\ &Tu_n \to v \ in\ H \end{aligned}\right\} \Rightarrow u \in \mathcal{D}(T), v = Tu$$

hold.

Example:

a) Let $T_0 = -\Delta$, $H = \ell^2(\mathbb{R}^n)$ and $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ dense in H.
Take $u \in W^{2,2}(\mathbb{R}^n) \setminus C_c^\infty(\mathbb{R}^n)$

$$\xRightarrow{\text{densly}} \exists (u_n)_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n)\colon\ u_n \to u \text{ in } W^{2,2}(\mathbb{R}^n)$$

$(u_n, -\Delta u_n) \in G(T_0)$ converges in $\ell^2 \times L^2$ to $(u, -\Delta u) \notin G(T_0)$.

b) Let $T_1 = -\Delta$, $\mathcal{D}(T_1) = W^{2,2}(\mathbb{R}^n)$ and $H = L^2(\mathbb{R}^n)$. For $u_n \in \mathcal{D}(T_1)$ with

$$u_n \to u \text{ in } H \text{ and } (-\Delta u_n) \to u \text{ in } L^2$$

follows that $-\Delta u = v \in L^2(\mathbb{R}^n)$ weakly, i.e. $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} v\varphi \leftarrow \int_{\mathbb{R}^n} (-\Delta u_n)\,\varphi = \int_{\mathbb{R}^n} u_n\,(-\Delta\varphi) \to \int_{\mathbb{R}^n} u\,(-\Delta\varphi)\,.$$

$\xRightarrow{PDE} u \in W^{2,2}(\mathbb{R}^n), \mathcal{D}(T_1) \Rightarrow T_1$ is closed

Definition I.2: *An operator T is called* ***closable*** $\iff$ *$\overline{G(T)}$ is a graph.*

Remark: We call $\overline{T}$ the **closure** of T and we have

$$\mathcal{D}\left(\overline{T}\right) := \left\{x \in H\colon \exists y \text{ such that } (x,y) \in \overline{G(R)}\right\}$$

For any $x \in \mathcal{D}\left(\overline{T}\right)$ the assumption that $\overline{G(T)}$ is a graph implies that y is unique and hence

$$\Rightarrow G(\overline{T}) = \overline{G(T)},\ \overline{T}x := y$$

Equivalently, $\mathcal{D}\left(\overline{T}\right)$ is the set of all $x \in H$ such that there exists a sequence $x_n \in \mathcal{D}(T)$ with $x_n \to x$ in H and Tx_n is a cauchy sequence. For such x we define

$$\overline{Tx} := \lim_{n\to\infty} Tx_n$$

Example: Let $T_0 \coloneqq -\Delta$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is closable with $\overline{T_0} = T_1$.

Proof: Let $u \in L^2(\mathbb{R}^n)$ such that there exists $(u_n)_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n)$ with $u_n \to u$ in L^2 and $-\Delta u_n \to u$ in L^2, as above:

$$-\Delta u = v \in L^2$$

For a given u the function u is unique $\Rightarrow$ T_0 is closable. Let $\overline{T_0}$ be the closure with domain $\mathcal{D}\left(\overline{T_0}\right)$ and $u \in \mathcal{D}(T_0)$

$$\Rightarrow \Delta u \in L^2 \Rightarrow u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow \mathcal{D}\left(\overline{T_0}\right) \subseteq \mathcal{D}(T_1)$$

but $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}(T_1) = W_{2,2}(\mathbb{R}^n)$

$$\Rightarrow W^{2,2}(\mathbb{R}^n) \subseteq \mathcal{D}\left(\overline{T_0}\right) \Rightarrow \mathcal{D}\left(\overline{T_0}\right) = \mathcal{D}(T_1) \Rightarrow T_1 = \overline{T_0},$$

Assumption:

$$W^{2,2} \nsubseteq \mathcal{D}(T_0) \Rightarrow \exists u \in W^{2,2} \setminus D\left(\overline{T_0}\right),\ \exists (u_n)_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n) : u_n \to u \text{ in } W^{2,2}$$

same argument as in example 1) $\Rightarrow$ $\overline{T_0}$ not closed! $\square$

Recall: If $T\colon H \to H$ is bounded then T^* is defined through

$$\langle u, T^* v\rangle = \langle Tu, v\rangle,\ \forall u, v \in H$$

In the ... $u \mapsto \langle Tu, v\rangle$ deines a continuous linear map on H ($\in H'$), Riesz' representation theorem then ensures the existence of T^*.

Definition I.3: *If T is an unbounded operator on H with dense domain, we defined*

$$\mathcal{D}(T^*) \coloneqq \left\{ v \in H\colon \mathcal{D}(T) \ni u \mapsto \langle Tu, v\rangle \text{ can be extended as a linear continuous form on } H \right\}$$

Appendix A

Exercises

Exercise 1

a) H separable $\Rightarrow \exists (e_n)_{n\in\mathbb{N}} \subseteq H$ orthonormal basis of H.

Proof: H seperable $\Rightarrow \exists (u_n)_{n\in\mathbb{N}} \subseteq H$: $\overline{\{u_n | n \in \mathbb{N}\}} = H$. Define:

$$H_n := \operatorname{lin}\{u_1, \dots, u_n\}, \quad \text{for } n \in \mathbb{N}$$

$H_n \subseteq H$ closed subspace of H as $\dim H_n \leq n < \infty$. By Projektionssatz there exists an orthogonal projection P_n on H_n. Set

$$g_n := u_n - P_{n-1}un \in H_n \cap H_{n-1}^{\perp}, N := \{n \in \mathbb{N} | g_n \neq 0\}$$

Now we define

$$e_n := \begin{cases} \frac{g_n}{\|g_n\}}, & n \in N \\ 0, & n \notin N \end{cases}$$

$\Rightarrow e_n \in H_n \cap H_{n-1}^{\perp}$, $\operatorname{lin}\{.e_1, \dots, e_n\} = H_n \Rightarrow (e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H.
$\square$

b) $H = L^2(0,1)$, $\mathcal{D}\left(()\, T\right) = W^{1,2}(0,1) =: H^1(0,1)$, $Tf = if'$.

Proof: T abgeschlossen: $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T)$ Cauchyfolge bezüglich $\|\cdot\|_{W^{1,2}}$. $\xRightarrow[\text{vollständig}]{L^2}$
$\exists x, y \in L^2(0,1) : x_n \to x, x_n' \to y$ in $L^2(0,1)$

$$\Rightarrow \int_0^1 x\varphi' dt = \lim_{n\to\infty} \int_0^1 x_n \varphi' dt = -\lim_{n\to\infty} \int_0^1 x_n'' \varphi dt = -\int_0^1 y\varphi dt \ \forall \varphi \in C_c^\infty(0,1)$$

$\Rightarrow x \in \mathcal{D}(T)$, $x' = y$, $Tx = ix' \Rightarrow T$ abgeschlossen.
We still have to show that T isn't symmetric:

$$\langle Tx, y\rangle_{L^2} = \int_0^1 ix'\overline{y} dt \stackrel{P.I.}{=} [ixy]_0^1 - \int_0^1 ix\overline{y}' dt = \underbrace{[ixy]_0^1}_{\neq 0 \text{ i. g.}} + \langle x, Ty\rangle_{L^2},$$

d.h. T is not symmetrich $\Rightarrow$ t nicht self-adjoint. T isn't halb-beschränkt nach unten;: siehe (iii). $\square$

c) $\mathcal{D}(T) = W_0^{1,2}(0,1)$, $H = L^2(0,1)$, $Tf = if'$.

Proof: T closed: as in (ii),
T symmetrisch:

$$\langle Tx, y\rangle_{L^2} = \ldots = \underbrace{[ixy]_0^1}_{=0} + \langle x, Ty\rangle_{L^2} = \langle x, Ty\rangle \ \forall x, y \in \mathcal{D}(T)$$

T not self-adjoint:

$$\mathcal{D}(T^*) = \Big\{ y \in L^2(0,1) \colon x \mapsto \langle Tx, y\rangle_{L^2} \text{ continuous on } \mathcal{D}(T) \Big\}$$

Vermutung: $W^{1,2}(0,10) \subseteq \mathcal{D}(T^*)$ (even "="). Let $x \in \mathcal{D}(T), y \in W^{1,2}(0,1)$:

$$\langle Tx, y\rangle_{L^2} = \ldots = \underbrace{[ixy]_0^1}_{=0} + \langle x, iy'\rangle_{L^2} = \langle x, iy'\rangle_{L^2}$$

continuous on $\mathcal{D}(T)$, i.e. $W^{1,2}(0,1) \subseteq \mathcal{D}(T^*)$, however $W^{1,2}(0,1) \not\subseteq W_0^{1,2}(0,1)$, i.e. $\mathcal{D}(T) \neq \mathcal{D}(T^*) \Rightarrow T \neq T^*$.
T is not halb-beschränkt nach unten:

$$\textit{Consider the comment: } \langle Tx, x\rangle_{L^2} = -2\int_0^2 1\,(\operatorname{Im} x)' \operatorname{Re} x dt \overset{\text{“?”}}{\geq} c\langle x, x\rangle_{L^2}$$

For $f_0 \in W_0^{1,2}(0,1)$ with $\langle f_0, f_0\rangle_{L^2} = 1$, $w \in \mathbb{R}$, $f_w(t) := e^{iwt} f_0(t) \Rightarrow \langle f_w, f_w\rangle_{L^2} = 1$

$$\begin{aligned} f_w'(t) &= iwe^{iwt} f_0(t) + e^{iwt} f_0'(t) \\ &= iwf_w(t) + e^{iwt} f_0'(t). \end{aligned}$$

$$\begin{aligned} \langle Tf_w, f_w\rangle &= \int_0^1 \left(-wf_w(t) + ie^{iwt} f_0'(t)\right) e^{-iwt} \overline{f_0(t)} dt \\ &= \underbrace{\int_0^1 -w|f_0|^2 dt}_{=-w} + \underbrace{\int_0^1 if_0'(t)\overline{f_0(t)} dt}_{\langle Tf_0, f_0\rangle_{L^2}} = -w + \underbrace{\langle Tf_0, f_0\rangle_{L^2}}_{\in \mathbb{R}} \to \pm\infty \end{aligned}$$

for $w \to \pm\infty \Rightarrow T$ ist not halb-beschränkt. In (iii) T has a self-adjoint Erweitunerung S:

$$\mathcal{D}\left(S\right)=\left\{x\in W^{1,2}(0,1)\colon x(0)=x(1)\right\},\ Sf=if'$$

□

Definition A.1: *Sei $\Omega\subseteq\mathbb{C}$ offen, $r\colon\Omega\to X$ eine Funktion. Man definiert*

a) r ist schwach analytisch $\iff\forall\varphi\in X^*\colon\varphi\circ r$ analytisch auf Ω

b) r ist analytisch $\iff\frac{d}{dz}r(z_0)\coloneqq r'(z_0)\coloneqq\lim_{z\to z_0}\left(z-z_0\right)^{-1}\left[r(z)-r(z_0)\right]$ existiert in X $\forall z_0\in\Omega$

c) Kurvenintegrale: Sein $\Gamma\coloneqq\left\{\gamma(t)\colon t\in[a,b]\right\}$ endlich-stückweise glatte Kurve in Ω, r stetig, dann:

$$\int_\Gamma r(\lambda)d\lambda\coloneqq\int_a^b r\left(\gamma(t)\right)\cdot\gamma'(t)dt\in X$$

Satz (Lemma von Dunford)**:** $r\colon\Omega\to X$ schwach analytisch $\iff r$ analytisch

Satz (Cuachy's Integralsatz und Formel)**:** Sei $\Omega\subseteq\mathbb{C}$ offen und konvex, $r\colon\Omega\to X$ analytisch. Dann gilt:

a) $\gamma\subseteq\Omega$ stückweise glatt und geschlossen $\Rightarrow\int_\Gamma r(\lambda)d\lambda 0$.

b) $\forall\lambda_0\in\Omega$, $a>0$ mit $\overline{B(\lambda_0,a)}\subseteq\Omega$:

$$r(\lambda)=\frac{1}{2\pi i}\int_{|\mu-\lambda_0|=a}\frac{1}{\mu-\lambda}r(\mu)d\mu\in X.$$

Proof: Sei $x^*\in X^*$, dann ist $x^*\circ r$ analytisch auf Ω. Nach Integralsatz bzw. -formel aus der Funktionentheorie folgt:

$$0=\int_\Gamma x^*\left(r(\lambda)\right)d\lambda=x^*\left(\underbrace{\int_\Gamma r(\lambda)d\lambda}_{=:x_1}\right),$$

$$x^*\left(r(\lambda)\right)=\frac{1}{2\pi i}\int_{|\mu-\lambda_0|=a}\frac{1}{\mu-\lambda}x^*\left(r(\mu)\right)d\mu=x^*\left(\frac{1}{2\pi i}\int_{|\mu-\lambda_0|=a}\frac{1}{\mu-\lambda}r(\mu)d\mu\right)$$

$$\iff x^*\left(\underbrace{r(\lambda)-\frac{1}{2\pi i}\int_\Gamma\frac{1}{\mu-\lambda}r(\mu)d\mu}_{=:x_2}\right)=0$$

$\Rightarrow x^*(x_1)=0,\ x^*(x_2)=0\ \forall x^*\in X^*\xRightarrow[Banach]{Hahn-}x_1=0,\ x_2=0.$ □

Das Dunford-Kalkül

Definition A.2 (Kalkül für Polynome)**:** *Sei* $A \in L(X)$, $p\colon \lambda \mapsto \sum_{k=0}^{n} a_k \lambda^k$ *Polynom,* $a_k \in \mathbb{C}$ *für* $k = 0, \dots, n$. *Dann definiert man:*

$$p(A) = \sum_{k=0}^{n} a_k A^k \in L(X).$$

$\mathcal{P}$ *Vektorraum aller Polynome.*

Satz (Eigenschaften)**:** $p_1, p_2, p \in \mathcal{P}$, $p(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$, $A \in L(X)$, $\alpha, \beta \in \mathbb{C}$. Dann gilt:

(1) Linearität: $(\alpha p_1 + \beta p_2)(A) = \alpha p_1(A) + \beta p_2(A)$.

(2) Multiplikativität:: $(p_1 \cdot p_2)(A) = p_1(A) p_2(A) = p_2(A) \cdot p_1(A)$.

(3) Beschränktheit: $\|p(A)\|_{L(X)} \leq \sum_{k=0}^{n} |a_k| \|A\|_{L(X)}^k$.

(4) Spektrale Abbildungseigenschaft: $\sigma(p(A)) = p(\sigma(A))$.

Proof:

(1) - (3): klar.

(4) "$\supseteq$": Sei $\mu \in \sigma(A)$, dann hat das Polynom $\lambda \mapsto p(\mu) - p(\lambda) \in \mathcal{P}$ eine Nullstelle in $\lambda = \mu$. Somit folgt:

$$p(\mu) - p(\lambda) = (\mu - \lambda)\, q(\lambda),$$

für ein $q \in \mathcal{P}$ für alle $\lambda \in \mathbb{C}$.

$$\xrightarrow[\lambda = A]{(1),(2)} p(\mu) - p(A) = (\mu - A)\, q(A) = q(A)\, (\mu - A)$$

Da $\mu \in \sigma(A)$ ist $\mu - A$ nicht injektiv oder nicht surjektiv.

$$\Rightarrow p(\mu) - p(A)$$

kann nicht injektiv oder nicht surjektiv sein $\Rightarrow p(\mu) \subseteq \sigma(p(A))$. "$\subseteq$": Sei $\mu \in \sigma(p(A))$.Wähle Nullstellen $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ von $\lambda \mapsto \mu - p(\mu)$, d.h.

$$\mu - p(\lambda) = a\,(\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_m),$$

$a \neq 0 \xRightarrow[\lambda)A]{(1),(2)} \mu - p(A) = a\,(A-\lambda_1)\cdot\ldots\cdot(A-\lambda_m)$. Angenommen: $\lambda_1, \ldots, \lambda_m \notin \sigma(A)$

$$\Rightarrow L(X) \ni (A-\lambda_m)^{-1} \cdot \ldots \cdot (A-\lambda_1)^{-1}\, a^{-1} = (\mu - p(A))^{-1}\,,$$

was einen Widerspruch zu $\mu \in \sigma\,(p(A))$ darstellt $\Rightarrow \exists j_0 \in \{1, \ldots, m\}$: $\lambda_{j_0} \in \sigma(A)$.

$$\mu - p(\lambda_{j_0}) = 0 \iff \mu = p(\lambda_{j_0}),$$

d.h. $\mu \in p\,(\sigma(A))$.

$\square$

Bemerkung:

Kalkül für Polynome

Verallgemeinerte	Approximiere:
Polynome = Potenzreihen	Satz von Weierstraß
	$\overline{\mathcal{P}}^{\lVert\cdot\rVert_{C^0[0,1]}} = C^0[0,1]$
$A \mapsto \sum_{k=0}^{\infty} a_k A^k$	
"Konvergenzradius"?	$\Rightarrow$ Kalkül für $C^0[0,1]$
$\Rightarrow$ analytische Funktionen	Stetige Funktionalkalkül
$\Rightarrow$ Dunford-Kalkül	

Definition A.3 (Kalkül für Potenzreihen)**:** *Sei* $f(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$, $(a_k)_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$, *Potenzreihe mit Konvergenzradius* $R > 0$. *Zu* $A \in L(X)$, $r(A) < R$ *definieren wir:*

$$f(A) := \lim_{n\to\infty} \left(\sum_{k=0}^{n} a_k A^k \right) =: \sum_{k=0}^{\infty} a_k A^k$$

in $L(X)$.

Stichwortverzeichnis