Spectraltheory

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Sommersemester 2017

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Contents

Chapter I

Unbounded, adjoint and self-adjoint operators

Let H be a separable Hilbert space, i.e. a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product $\langle \cdot, \cdot \rangle$ on H.

A linear operator T in H is a linear map

$$u \mapsto Tu$$

defined on a subspace $\mathcal{D}(T)$ of H, and we call $\mathcal{D}(T)$ the **domain** of T. For $T \colon \mathcal{D}(T) \to H$ we denote the **range** of T with

$$\mathcal{R}(T) \coloneqq \mathrm{Image}(T).$$

We say that T is **bounded** if it is continuous from $\mathcal{D}(T)$ into H, with respect to the topology induced by H. If $\mathcal{D}(T) = H$ we recall the definition of bounded operators from the functional analysis course. From now on, if $\mathcal{D}(T) \neq H$ we will assume that $\mathcal{D}(T)$ is **dense** in H, i.e. $\overline{\mathcal{D}(T)} = H$.

In this case, if T is bounded then T has a unique continuous extension to all of H. As this simplifies many considerations some of the following theorems would be trivial, and hence, we won't focus on bounded operators during this lecture.

Recall: An operator is called **closed** if the graph

$$G(T) := \left\{(x, y) \in H \times H \;\middle|\; x \in \mathcal{D}(T), y = Tx\right\}$$

is closed in $H \times H$.

Definition I.1: *Let $T\colon \mathcal{D}(T) \to H$ be a (linear) operator with $\mathcal{D}(T)$ dense in H. Then T is called **closed** if for all*

$$u_n \in \mathcal{D}(T), \; u_n \to u \in H \text{ and } Tu_n \to v \in H$$

follows that

$$u \in \mathcal{D}(T), \; v = Tu$$

holds.

Example:

a) Let $H = L^2(\mathbb{R}^n)$, then $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is dense in H. Define the operator

$$T_0 = -\Delta,$$

and take $u \in W^{2,2}(\mathbb{R}^n) \setminus C_c^\infty(\mathbb{R}^n)$, s.t $u \in L^2(\mathbb{R}^n)$. Due to the density:

$$\exists \; (u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n) \colon \; u_n \to u \text{ in } W^{2,2}(\mathbb{R}^n).$$

As a result, $(u_n, -\Delta u_n) \in G(T_0)$ converges in $L^2 \times L^2$ to $(u, -\Delta u) \notin G(T_0)$.

b) Let $H = L^2(\mathbb{R}^n)$, and set $\mathcal{D}(T_1) = W^{2,2}(\mathbb{R}^n) \subseteq H$. Define the operator

$$T_1 = -\Delta.$$

For $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T_1)$ with

$$u_n \to u \in H \text{ and } (-\Delta u_n) \to v \in L^2$$

follows that $-\Delta u = v \in L^2(\mathbb{R}^n)$ weakly, i.e. for all $\varphi \in C_c^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} v\varphi \longleftarrow \int_{\mathbb{R}^n} (-\Delta u_n)\,\varphi = \int_{\mathbb{R}^n} u_n\,(-\Delta\varphi) \longrightarrow \int_{\mathbb{R}^n} u\,(-\Delta\varphi)\,.$$

$\xRightarrow{PDE} u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow T_1$ is closed.

Definition I.2: *An operator T is called* ***closable*** $\iff$ $\overline{G(T)}$ *is a graph.*

Remark: We call $\overline{T}$ the **closure** of T, and in such case we have

$$\mathcal{D}\left(\overline{T}\right) := \left\{x \in H \;\middle|\; \exists\, y \colon (x,y) \in \overline{G(R)}\right\}$$

For any $x \in \mathcal{D}\left(\overline{T}\right)$ the assumption that $\overline{G(T)}$ is a graph implies that y is unique and hence

$$\Rightarrow G(\overline{T}) = \overline{G(T)},\ \overline{T}x := y$$

Equivalently, $\mathcal{D}\left(\overline{T}\right)$ is the set of all $x \in H$ such that there exists a sequence $x_n \in \mathcal{D}(T)$ with $x_n \to x$ in H and Tx_n is a cauchy sequence. For such x we define

$$\overline{Tx} := \lim_{n\to\infty} Tx_n$$

Example: Let $T_0 := -\Delta$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is closable with $\overline{T_0} = T_1$.

Proof: Let $u \in L^2(\mathbb{R}^n)$ such that there exists $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ with $u_n \to u$ in L^2 and $-\Delta u_n \to u$ in L^2, as above:

$$-\Delta u = v \in L^2$$

For a given u the function v is unique, and hence, T_0 is closable. Let $\overline{T_0}$ be the closure with domain $\mathcal{D}\left(\overline{T_0}\right)$ and $u \in \mathcal{D}(T_0)$

$$\Rightarrow \Delta u \in L^2 \Rightarrow u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow \mathcal{D}\left(\overline{T_0}\right) \subseteq \mathcal{D}(T_1)$$

but $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}(T_1) = W_{2,2}(\mathbb{R}^n)$

$$\Rightarrow W^{2,2}(\mathbb{R}^n) \subseteq \mathcal{D}\left(\overline{T_0}\right) \Rightarrow \mathcal{D}\left(\overline{T_0}\right) = \mathcal{D}(T_1)$$

$\Rightarrow T_1 = \overline{T_0}$. $\square$

Remark: Assume for a second in the example above that $W^{2,2} \not\subseteq \mathcal{D}(T_0)$ holds:

$$\Rightarrow \exists u \in W^{2,2} \setminus D\left(\overline{T_0}\right),\ \exists (u_n)_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n) : u_n \to u \text{ in } W^{2,2}$$

same argument as in example 1) $\Rightarrow \overline{T_0}$ not closed!

Recall: If $T\colon H \to H$ is bounded then T^* is defined through

$$\langle u, T^* v\rangle = \langle Tu, v\rangle,\ \forall u, v \in H$$

$u \mapsto \langle Tu, v\rangle$ defines a continuous linear map on H $(\in H')$. Riesz' representation theorem then ensures the existence of T^*.

Definition I.3: *If T is an unbounded operator on H with dense domain we define*

$$\mathcal{D}\left(T^*\right) := \Big\{v \in H\colon \mathcal{D}\left(T\right) \ni u \mapsto \langle Tu, v\rangle \text{ can be extended as a linear continuous form on } H\Big\}$$

Using Riesz' representation theorem $\exists! f \in H$:

$$\langle u, f\rangle = \langle Tu, v\rangle,\ \forall u \in \mathcal{D}\left(T\right)$$

then define $T^* v = f$, where the uniqueness follows from the density of $\mathcal{D}\left(T\right)$ in H.

Remark: If $\mathcal{D}\left(T\right) = H$ and T is bounded then we recover the "old" adjoint.

Example: $T_0^* = T_1$,

$$\begin{aligned}\mathcal{D}\left(T_0^*\right) &= \Big\{v \in L^2(\mathbb{R}^n)\colon C_c^\infty(\mathbb{R}^n) \ni u \mapsto \langle -\Delta u, v\rangle \text{ can be extended as a linear} \\ &\qquad \text{continuous form on } L^2(\mathbb{R}^n)\Big\} \\ &= \Big\{v \in L^2(\mathbb{R}^n)\colon\ -\Delta v \in L^2(\mathbb{R}^n)\Big\} = W^{2,2}(\mathbb{R}^n) = \mathcal{D}\left(T_1\right)\end{aligned}$$

Damit ist

$$\langle T_1 u, v\rangle = \langle -\Delta u, v\rangle = \int v\left(-\Delta u\right) = -\int \left(-\Delta v\right) u = \langle u, T_1 v\rangle$$

Theorem I.1: *T^* is a closed operator.*

Proof: $v_n \in \mathcal{D}\left(T^*\right)$ such that $v_n \to v$ in H and $T^* v_n \to w^*$ in H for $(v, w^*) \in H \times H$. For all $u \in \mathcal{D}\left(T\right)$ we have

$$\langle Tu, v\rangle = \lim_{n\to\infty} \langle Tu, v_n\rangle = \lim_{n\to\infty} \langle u, T^* v_n\rangle = \langle u, w^*\rangle$$

$(H \ni u \mapsto \langle u, w^*\rangle$ is continuous$) \Rightarrow v \in \mathcal{D}\left(T^*\right)$ and $w^* = T^* v$ by definition. $\square$

Theorem I.2: *Let T be an operator in H with domain $\mathcal{D}(T)$. Then*

$$G(T^*) = \left(V\left(\overline{G(T)}\right)\right)^{\perp}$$

where $V\colon H \times H \to H \times H, V(x,y) = (y, -x)$ ($V^2 = -\mathbb{I}$).

Proof: Let $u \in \mathcal{D}(T), (v, w^*) \in H \times H$

$$\Rightarrow \langle V(u, Tu), (v, w^*)\rangle_{H\times H} = \langle Tu, v\rangle - \langle u, w^*\rangle$$

Considering the right-hand side it follows

$$\langle Tu, v\rangle - \langle u, w^*\rangle = 0\, \forall u \in \mathcal{D}(T) \iff v \in \mathcal{D}(T^*) \text{ and } w^* = T^*v \iff (v, w^*) \in G(T^*),$$

and considering the left-hand side:

$$\Rightarrow \langle V(u, Tu), (v, w^*)\rangle_{H\times H} = 0\ \forall u \in \mathcal{D}(T) \iff (v, w^*) \in V(G(T))^{\perp}$$

In general: $U^{\perp} = \overline{U}^{\perp}$, and hence

$$\Rightarrow V(G(T))^{\perp} = \left(\overline{V(G(T))}\right)^{\perp} = \left(V\left(\overline{G(T)}\right)\right)^{\perp}.$$

$\square$

Theorem I.3: *Let T be a closable operator. Then:*

a) $\mathcal{D}(T^*)$ is dense in H

b) $T^{**} \coloneqq (T^*)^* = \overline{T}$

Proof:

a) Proof through contradiction: $D(T^*)$ not dense in $H \to \exists w \neq 0 : \langle w, v\rangle = 0\ \forall v \in \overline{\mathcal{D}(T^*)}$

$$\begin{aligned}
&\Longrightarrow \langle (0, w), (T^*v, -v)\rangle_{H\times H} = 0\ \forall v \in \mathcal{D}(T^*)\\
&\Longrightarrow (0, w) \perp V(G(T^*))\\
&\xRightarrow[I.2]{Thm} V\left(\overline{G(T)}\right) = G(T^*)^{\perp}\\
&\Longrightarrow V\left(G(T^*)^{\perp}\right) = \overline{G(T)}
\end{aligned}$$

For any $M \subseteq H \times H$ we have $V\left(M^{\perp}\right) = V(M)^{\perp}$ since for $(u,v) \in V(M)^{\perp}$, $(x,y) \in M$

$\langle V(u,v), (x,y)\rangle_{H\times H} = -\langle (u,v), V(x,y)\rangle_{H\times h} \Rightarrow V(u,v) \in M^{\perp} \Rightarrow (u,v) \in V\left(M^{\perp}\right)$

$\Longrightarrow V\left(G\left(T^{*}\right)\right)^{\perp} = \overline{G(T)} = G\left(\overline{T}\right) \Longrightarrow (0,w) \in G\left(\overline{T}\right) \Longrightarrow w = 0$

b) $G\left(T^{**}\right) \overset{Thm}{\underset{I.2}{=}} V\left(\overline{G\left(T^{*}\right)}\right)^{\perp} \overset{Thm}{\underset{I.1}{=}} V\left(G\left(T^{*}\right)\right)^{\perp} \overset{(\perp)}{=} G\left(\overline{T}\right) \Longrightarrow \mathcal{D}\left(T^{**}\right) = \mathcal{D}\left(\overline{T}\right), T^{**} = \overline{T}$

□

Definition I.4: *We say $T\colon \mathcal{D}(T) \to H$ is* ***symmetric*** *if and only if*

$$\langle Tu, v\rangle = \langle u, Tv\rangle \quad \forall u, v \in \mathcal{D}(T)$$

Example: $T_0 = -\Delta$, $\mathcal{D}(T_0) = C_c^{\infty}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta u)\, v = \int_{\mathbb{R}^n} u\,(-\Delta v)$$

Remark: If T is symmetric $\Rightarrow \mathcal{D}(T) \subseteq \mathcal{D}(T^{*})$ and

$$Tu = T^{*}u \quad \forall u \in \mathcal{D}(T)$$

$\Rightarrow (T^{*}, \mathcal{D}(T^{*}))$ is an extension of $(T, \mathcal{D}(T))$.

Lemma I.1: *A symmetric operator T is closable.*

Proof: It suffice to show that for $u_n \in \mathcal{D}(T)$ with $u_n \to 0$ and $u_n \to x \in H$ we have $x = 0$

$$\langle x, v\rangle \leftarrow \langle Tu_n, v\rangle = \langle u_n, Tu\rangle \to \langle 0, Tu\rangle = 0 \quad \forall v \in \mathcal{D}(T)$$

$\Rightarrow x = 0$. □

Remark: The proof actually shows that if $\mathcal{D}(T^{*})$ is dense in H, then T is closable.

Definition I.5: *We call an operator T self-adjoint if*

$$T = T^{*} \text{ and } Tu = T^{*}u \quad \forall u \in \mathcal{D}(T),$$

note that the first property implies that $\mathcal{D}(T) = \mathcal{D}(T^{*})$.

Theorem I.4: *Every self-adjoint operator is closable.*

Proof: Lemma I.1 $\square$

Theorem I.5: *Let T be an invertible self-adjoint operator, then T^{-1} is also self-adjoint.*

Proof: For $T\colon \mathcal{D}(T) \to \mathcal{R}(T)$ consider

Step 1 $\mathcal{R}(T)$ is dense in H. We have to show that $\mathcal{R}(T)^{\perp} = \{0\}$.

Let $w \in H$ such that

$$\langle Tu, w \rangle = 0 \ \forall u \in \mathcal{D}(T)$$

$\Longrightarrow w \in \mathcal{D}(T^*)$ and $T^*w = 0 \overset{inj.}{\underset{s.a.}{\Longrightarrow}} w = 0.$

Step 2 Let $w\colon H \times H \to H \times H$, $w(x,y) = (y,x)$

$$\Longrightarrow G\left(T^{-1}\right) = \left\{ \left(x, T^{-1}x\right) : x \in \mathcal{D}(T) \right\} = w(G(T)) = \left\{ (Ty, y) : y \in \mathcal{D}(T) \right\}$$

$$\begin{aligned} G\left(T^{-1}\right) &= G\left(\left(T^*\right)^{-1}\right) \underset{Thm. I.2}{\overset{Proof}{=}} w\left(V\left(G(T)^{\perp}\right)\right) \\ &= V\left(w(G(T))^{\perp}\right) = V\left(w(G(T))\right)^{\perp} \\ &= V\left(G\left(T^{-1}\right)\right)^{\perp} \underset{I.2}{\overset{Thm.}{=}} G\left(\left(T^{-1}\right)^*\right) \end{aligned}$$

$\Rightarrow T^{-1} = \left(T^{-1}\right)^*$

$\square$

Chapter II

Representation Theorems

Theorem II.1 (Riesz)**:** *Let $u \mapsto F(u)$ be a linear continuous function on H. Then $\exists! w \in H$:*

$$F(u) = \langle u, w \rangle \quad \forall u \in H$$

Lax-Milgram: V Hilbertspace, sesquilinear form is defined on $V \times V$, $(u, v) \mapsto \alpha(u, v)$ continuous with

$$|\alpha(u, v)| \leq c\|u\|\|v\| \quad \forall u, v \in V$$

Riesz: $\exists$ linear map $A \colon V \to V$:

$$\alpha(u, v) = \langle Au, v \rangle$$

Definition II.1: *A bilinear form $a \colon V \times V \to \mathbb{R}$ is V***-coercive** *if there exists $\lambda > 0$ such that*

$$a\,(u, u) \geq \lambda\|u\|^2 \quad \forall u \in V$$

Theorem II.2: *Let a be a continuous sesquilinear and V-coercive on $V \times V$ then A is an isomorphism.*

Proof:

Step 1: A is injective:

$$\|fu\|\|u\| \overset{C.S.}{\geq} |\langle Au, u \rangle| = |u(u, u)| \geq \lambda\|u\|^2$$

$\Rightarrow \|Au\| \geq \lambda\|u\|$ for all $u \in V$.

Step 2: $A(V)$ is dense in V. Let $u \in V$ such that

$$\langle Au, v \rangle = 0 \quad \forall v \in V$$

take $v = u \Rightarrow a(u, u) \Longrightarrow u = 0$.

Step 3: $\mathbb{R}(A) = A(V)$ is closed. Let v_n be a sequence in $A(V)$ and let a_n be such that

$$Au_n = v_n$$

$$\overset{(+)}{\Longrightarrow} u_n \text{ is a Cauchy sequence} \Rightarrow u_n \to u \in V \text{ und } Au_n \to Au \Rightarrow v_n \to Au \in A(V)$$

Step 4: $u = A^{-1}v \overset{(+)}{\Longrightarrow} \|A^{-1}v\| \leq \lambda^{-1}\|v\| \ \forall v \in V$.

$\square$

Next we consider two Hilbert spaces V, H with $V \subset H$ (the inclusion is continuous), i.e.

$$\exists c < \infty\colon \quad \|u\|_H \leq c\|u\|_V \quad \forall u \in V$$

and we assume that V is dense in H.

Example: $V = W^{1,1}(\mathbb{R}^n)$, $H = L^2(\mathbb{R}^n)$

$$\|u\|_L^2 \leq \|u\|_{W^{1,2}}$$

There exists a natural injection from H into V'. Let $h \in H$ then $V \ni u \mapsto \langle u, h\rangle_H$ is continuous on $V \xRightarrow[\text{Thm. II.1}]{} \exists l_h \in V'$:

$$l_h(u) = \langle u, h\rangle_H \quad \forall u \in V$$

injectivity follows from density of V in H. $V \subset H$

Appendix A

Exercises

Exercise 1

a) H separable $\Rightarrow \exists (e_n)_{n\in\mathbb{N}} \subseteq H$ orthonormal basis of H.

Proof: H separable $\Rightarrow \exists (u_n)_{n\in\mathbb{N}} \subseteq H$: $\overline{\{u_n | n \in \mathbb{N}\}} = H$. Define:

$$H_n := \operatorname{lin}\{u_1, \dots, u_n\}, \quad \text{for } n \in \mathbb{N}$$

$H_n \subseteq H$ closed subspace of H as $\dim H_n \leq n < \infty$. By Projektionssatz there exists an orthogonal projection P_n on H_n. Set

$$g_n := u_n - P_{n-1}un \in H_n \cap H_{n-1}^{\perp}, N := \{n \in \mathbb{N} | g_n \neq 0\}$$

Now we define

$$e_n := \begin{cases} \frac{g_n}{\|g_n\}}, & n \in N \\ 0, & n \notin N \end{cases}$$

$\Rightarrow e_n \in H_n \cap H_{n-1}^{\perp}$, $\operatorname{lin}\{.e_1, \dots, e_n\} = H_n \Rightarrow (e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H.
□

b) $H = L^2(0,1)$, $\mathcal{D}\left(()\, T\right) = W^{1,2}(0,1) =: H^1(0,1)$, $Tf = if'$.

Proof: T abgeschlossen: $(x_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}\left(T\right)$ Cauchyfolge bezüglich $\|\cdot\|_{W^{1,2}}$. $\xRightarrow[\text{vollständig}]{L^2}$
$\exists x, y \in L^2(0,1) : x_n \to x, x_n' \to y$ in $L^2(0,1)$

$$\Rightarrow \int_0^1 x\varphi' dt = \lim_{n\to\infty} \int_0^1 x_n \varphi' dt = -\lim_{n\to\infty} \int_0^1 x_n'' \varphi dt = -\int_0^1 y\varphi dt \;\forall \varphi \in C_c^{\infty}(0,1)$$

$\Rightarrow x \in \mathcal{D}\left(T\right)$, $x' = y$, $Tx = ix' \Rightarrow T$ abgeschlossen.
We still have to show that T isn't symmetric:

$$\langle Tx, y\rangle_{L^2} = \int_0^1 ix'\overline{y} dt \overset{P.I.}{=} [ixy]_0^1 - \int_0^1 ix\overline{y}' dt = \underbrace{[ixy]_0^1}_{\neq 0 \text{ i. g.}} + \langle x, Ty\rangle_{L^2},$$

d.h. T is not symmetrich $\Rightarrow$ t nicht self-adjoint. T isn't halb-beschränkt nach unten;: siehe (iii). $\square$

c) $\mathcal{D}(T) = W_0^{1,2}(0,1)$, $H = L^2(0,1)$, $Tf = if'$.

Proof: T closed: as in (ii),
T symmetrisch:

$$\langle Tx, y\rangle_{L^2} = \ldots = \underbrace{[ixy]_0^1}_{=0} + \langle x, Ty\rangle_{L^2} = \langle x, Ty\rangle \ \forall x, y \in \mathcal{D}(T)$$

T not self-adjoint:

$$\mathcal{D}(T^*) = \Big\{ y \in L^2(0,1) \colon x \mapsto \langle Tx, y\rangle_{L^2} \text{ continuous on } \mathcal{D}(T) \Big\}$$

Vermutung: $W^{1,2}(0,10) \subseteq \mathcal{D}(T^*)$ (even "="). Let $x \in \mathcal{D}(T), y \in W^{1,2}(0,1)$:

$$\langle Tx, y\rangle_{L^2} = \ldots = \underbrace{[ixy]_0^1}_{=0} + \langle x, iy'\rangle_{L^2} = \langle x, iy'\rangle_{L^2}$$

continuous on $\mathcal{D}(T)$, i.e. $W^{1,2}(0,1) \subseteq \mathcal{D}(T^*)$, however $W^{1,2}(0,1) \not\subseteq W_0^{1,2}(0,1)$, i.e. $\mathcal{D}(T) \neq \mathcal{D}(T^*) \Rightarrow T \neq T^*$.
T is not halb-beschränkt nach unten:

$$\textit{Consider the comment: } \langle Tx, x\rangle_{L^2} = -2\int_0^2 1\,(\operatorname{Im} x)' \operatorname{Re} x dt \overset{\text{“?”}}{\geq} c\langle x, x\rangle_{L^2}$$

For $f_0 \in W_0^{1,2}(0,1)$ with $\langle f_0, f_0\rangle_{L^2} = 1$, $w \in \mathbb{R}$, $f_w(t) := e^{iwt} f_0(t) \Rightarrow \langle f_w, f_w\rangle_{L^2} = 1$

$$\begin{aligned} f_w'(t) &= iwe^{iwt} f_0(t) + e^{iwt} f_0'(t) \\ &= iwf_w(t) + e^{iwt} f_0'(t). \end{aligned}$$

$$\begin{aligned} \langle Tf_w, f_w\rangle &= \int_0^1 \Big(-wf_w(t) + ie^{iwt} f_0'(t) \Big) e^{-iwt} \overline{f_0(t)} dt \\ &= \underbrace{\int_0^1 -w|f_0|^2 dt}_{=-w} + \underbrace{\int_0^1 if_0'(t)\overline{f_0(t)} dt}_{\langle Tf_0, f_0\rangle_{L^2}} = -w + \underbrace{\langle Tf_0, f_0\rangle_{L^2}}_{\in \mathbb{R}} \to \pm\infty \end{aligned}$$

for $w \to \pm\infty \Rightarrow T$ ist not halb-beschränkt. In (iii) T has a self-adjoint Erweitunerung S:

$$\mathcal{D}(S) = \left\{x \in W^{1,2}(0,1) \colon x(0) = x(1)\right\},\ Sf = if'$$

$\square$

Definition A.1: *Sei $\Omega \subseteq \mathbb{C}$ offen, $r \colon \Omega \to X$ eine Funktion. Man definiert*

a) r ist schwach analytisch $\iff \forall \varphi \in X^* \colon \varphi \circ r$ analytisch auf Ω

b) r ist analytisch $\iff \frac{d}{dz} r(z_0) \coloneqq r'(z_0) \coloneqq \lim_{z \to z_0} (z - z_0)^{-1} [r(z) - r(z_0)]$ existiert in X $\forall z_0 \in \Omega$

c) Kurvenintegrale: Sein $\Gamma \coloneqq \left\{\gamma(t) \colon t \in [a,b]\right\}$ endlich-stückweise glatte Kurve in Ω, r stetig, dann:

$$\int_\Gamma r(\lambda) d\lambda \coloneqq \int_a^b r(\gamma(t)) \cdot \gamma'(t) dt \in X$$

Satz (Lemma von Dunford)**:** $r \colon \Omega \to X$ schwach analytisch $\iff$ r analytisch

Satz (Cuachy's Integralsatz und Formel)**:** Sei $\Omega \subseteq \mathbb{C}$ offen und konvex, $r \colon \Omega \to X$ analytisch. Dann gilt:

a) $\gamma \subseteq \Omega$ stückweise glatt und geschlossen $\Rightarrow \int_\Gamma r(\lambda) d\lambda 0$.

b) $\forall \lambda_0 \in \Omega$, $a > 0$ mit $\overline{B(\lambda_0, a)} \subseteq \Omega$:

$$r(\lambda) = \frac{1}{2\pi i} \int_{|\mu - \lambda_0| = a} \frac{1}{\mu - \lambda} r(\mu) d\mu \in X.$$

Proof: Sei $x^* \in X^*$, dann ist $x^* \circ r$ analytisch auf Ω. Nach Integralsatz bzw. -formel aus der Funktionentheorie folgt:

$$0 = \int_\Gamma x^*(r(\lambda)) d\lambda = x^* \left(\underbrace{\int_\Gamma r(\lambda) d\lambda}_{=: x_1} \right),$$

$$x^*(r(\lambda)) = \frac{1}{2\pi i} \int_{|\mu - \lambda_0| = a} \frac{1}{\mu - \lambda} x^*(r(\mu)) d\mu = x^* \left(\frac{1}{2\pi i} \int_{|\mu - \lambda_0| = a} \frac{1}{\mu - \lambda} r(\mu) d\mu \right)$$

$$\iff x^* \left(\underbrace{r(\lambda) - \frac{1}{2\pi i} \int_\Gamma \frac{1}{\mu - \lambda} r(\mu) d\mu}_{=: x_2} \right) = 0$$

$\Rightarrow x^*(x_1) = 0,\ x^*(x_2) = 0\ \forall x^* \in X^* \xRightarrow[Banach]{Hahn-} x_1 = 0,\ x_2 = 0.$ $\square$

Das Dunford-Kalkül

Definition A.2 (Kalkül für Polynome)**:** *Sei* $A \in L(X)$, $p\colon \lambda \mapsto \sum_{k=0}^{n} a_k \lambda^k$ *Polynom,* $a_k \in \mathbb{C}$ *für* $k = 0, \dots, n$. *Dann definiert man:*

$$p(A) = \sum_{k=0}^{n} a_k A^k \in L(X).$$

$\mathcal{P}$ *Vektorraum aller Polynome.*

Satz (Eigenschaften)**:** $p_1, p_2, p \in \mathcal{P}$, $p(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$, $A \in L(X)$, $\alpha, \beta \in \mathbb{C}$. Dann gilt:

(1) Linearität: $(\alpha p_1 + \beta p_2)(A) = \alpha p_1(A) + \beta p_2(A)$.

(2) Multiplikativität:: $(p_1 \cdot p_2)(A) = p_1(A) p_2(A) = p_2(A) \cdot p_1(A)$.

(3) Beschränktheit: $\|p(A)\|_{L(X)} \leq \sum_{k=0}^{n} |a_k| \|A\|_{L(X)}^k$.

(4) Spektrale Abbildungseigenschaft: $\sigma(p(A)) = p(\sigma(A))$.

Proof:

(1) - (3): klar.

(4) "$\supseteq$": Sei $\mu \in \sigma(A)$, dann hat das Polynom $\lambda \mapsto p(\mu) - p(\lambda) \in \mathcal{P}$ eine Nullstelle in $\lambda = \mu$. Somit folgt:

$$p(\mu) - p(\lambda) = (\mu - \lambda)\, q(\lambda),$$

für ein $q \in \mathcal{P}$ für alle $\lambda \in \mathbb{C}$.

$$\xRightarrow[\lambda=A]{(1),(2)} p(\mu) - p(A) = (\mu - A)\, q(A) = q(A)\, (\mu - A)$$

Da $\mu \in \sigma(A)$ ist $\mu - A$ nicht injektiv oder nicht surjektiv.

$$\Rightarrow p(\mu) - p(A)$$

kann nicht injektiv oder nicht surjektiv sein $\Rightarrow p(\mu) \subseteq \sigma(p(A))$. "$\subseteq$": Sei $\mu \in \sigma(p(A))$.Wähle Nullstellen $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ von $\lambda \mapsto \mu - p(\mu)$, d.h.

$$\mu - p(\lambda) = a\, (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_m),$$

$a \neq 0 \xRightarrow[\lambda)A]{(1),(2)} \mu - p(A) = a\,(A-\lambda_1)\cdot\ldots\cdot(A-\lambda_m)$. Angenommen: $\lambda_1, \ldots, \lambda_m \notin \sigma(A)$

$$\Rightarrow L(X) \ni (A-\lambda_m)^{-1} \cdot \ldots \cdot (A-\lambda_1)^{-1}\, a^{-1} = (\mu - p(A))^{-1},$$

was einen Widerspruch zu $\mu \in \sigma\,(p(A))$ darstellt $\Rightarrow \exists j_0 \in \{1, \ldots, m\}\colon\ \lambda_{j_0} \in \sigma(A)$.

$$\mu - p(\lambda_{j_0}) = 0 \iff \mu = p(\lambda_{j_0}),$$

d.h. $\mu \in p\,(\sigma(A))$.

$\square$

Bemerkung:

Kalkül für Polynome	
Verallgemeinerte	Approximiere:
Polynome = Potenzreihen	Satz von Weierstraß
	$\overline{\mathcal{P}}^{\|\cdot\|_{C^0[0,1]}} = C^0[0,1]$
$A \mapsto \sum_{k=0}^{\infty} a_k A^k$	
“Konvergenzradius”?	$\Rightarrow$ Kalkül für $C^0[0,1]$
$\Rightarrow$ analytische Funktionen	Stetige Funktionalkalkül
$\Rightarrow$ Dunford-Kalkül	

Definition A.3 (Kalkül für Potenzreihen)**:** *Sei $f(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$, $(a_k)_{k \in \mathbb{N}_0} \subseteq \mathbb{C}$, Potenzreihe mit Konvergenzradius $R > 0$. Zu $A \in L(X)$, $r(A) < R$ definieren wir:*

$$f(A) := \lim_{n \to \infty} \left(\sum_{k=0}^{n} a_k A^k \right) =: \sum_{k=0}^{\infty} a_k A^k$$

in $L(X)$.

Stichwortverzeichnis