

Spectraltheory

Prof. Dr. Tobias Lamm

Sommersemester 2017

Karlsruher Institut für Technologie

Contents

I	Unbounded, adjoint and self-adjoint operators	2
II	Representation Theorems	9
III	Friedrichs extension	13
IV	Spectrum and Resolvent	18

Chapter I

Unbounded, adjoint and self-adjoint operators

Let H be a separable Hilbert space, i.e. a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product $\langle \cdot, \cdot \rangle$ on H .

Recall: A mapping $\langle \cdot, \cdot \rangle: H \times H \rightarrow \mathbb{C}$ is called an **inner product**, if for all $x, y \in H$, $\lambda \in \mathbb{C}$ holds:

$$(S1) \quad \langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle, \quad \langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$$

$$(S2) \quad \langle \lambda x, y \rangle = \lambda \langle x, y \rangle, \quad \langle x, \lambda y \rangle = \bar{\lambda} \langle x, y \rangle$$

$$(S3) \quad \langle x, y \rangle = \overline{\langle y, x \rangle}$$

$$(S4) \quad \langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \iff x = 0$$

A linear operator T in H is a linear map

$$u \mapsto Tu$$

defined on a subspace $\mathcal{D}(T)$ of H , and we call $\mathcal{D}(T)$ the **domain** of T . For $T: \mathcal{D}(T) \rightarrow H$ we denote the **range** of T with

$$\mathcal{R}(T) := \text{Image}(T).$$

We say that T is **bounded** if it is continuous from $\mathcal{D}(T)$ into H , with respect to the topology induced by H . We recall that if $\mathcal{D}(T) = H$ holds, boundedness of a linear operator is equivalent to continuity in 0, boundedness of $T(U_{(X, \|\cdot\|)})$ in Y and that there $\exists c < \infty$ such that $\|Tx\| \leq c\|x\|$, for proof see theorem 3.3 in the [functional analysis](#) course.

From now on, if $\mathcal{D}(T) \neq H$ we will assume that $\mathcal{D}(T)$ is **dense** in H , i.e. $\overline{\mathcal{D}(T)} = H$. If in this case T would be bounded then T has a unique continuous extension to all of H , for proof see proposition 5.10 in the [functional analysis](#) course. As this simplifies many considerations some of the following theorems would be trivial, and hence, we won't focus on bounded operators during this lecture.

Recall: An operator is called **closed** if the graph

$$G(T) := \left\{ (x, y) \in H \times H \mid x \in \mathcal{D}(T), y = Tx \right\}$$

is closed in $H \times H$.

Definition I.1: Let $T: \mathcal{D}(T) \rightarrow H$ be a (linear) operator with $\mathcal{D}(T)$ dense in H . Then T is called **closed** if for all

$$u_n \in \mathcal{D}(T), \quad u_n \rightarrow u \in H \quad \text{and} \quad Tu_n \rightarrow v \in H$$

follows that

$$u \in \mathcal{D}(T), \quad v = Tu$$

holds.

Example:

a) Let $H = L^2(\mathbb{R}^n)$, then $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is dense in H . Define the operator

$$T_0 = -\Delta,$$

and take $u \in W^{2,2}(\mathbb{R}^n) \setminus C_c^\infty(\mathbb{R}^n)$, s.t $u \in L^2(\mathbb{R}^n)$. Due to the density:

$$\exists (u_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n): \quad u_n \rightarrow u \text{ in } W^{2,2}(\mathbb{R}^n).$$

As a result, $(u_n, -\Delta u_n) \in G(T_0)$ converges in $L^2 \times L^2$ to $(u, -\Delta u) \notin G(T_0)$.

b) Let $H = L^2(\mathbb{R}^n)$, and set $\mathcal{D}(T_1) = W^{2,2}(\mathbb{R}^n) \subseteq H$. Define the operator

$$T_1 = -\Delta.$$

For $(u_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(T_1)$ with

$$u_n \rightarrow u \in H \text{ and } (-\Delta u_n) \rightarrow v \in L^2$$

follows that $-\Delta u = v \in L^2(\mathbb{R}^n)$ weakly, i.e. for all $\varphi \in C_c^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} v \varphi \leftarrow \int_{\mathbb{R}^n} (-\Delta u_n) \varphi = \int_{\mathbb{R}^n} u_n (-\Delta \varphi) \longrightarrow \int_{\mathbb{R}^n} u (-\Delta \varphi).$$

$$\xrightarrow{PDE} u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow T_1 \text{ is closed.}$$

Definition I.2: An operator T is called **closable** $\iff \overline{G(T)}$ is a graph.

Remark: We call \overline{T} the **closure** of T , and in such case we have

$$\mathcal{D}(\overline{T}) := \{x \in H \mid \exists y: (x, y) \in \overline{G(T)}\}$$

For any $x \in \mathcal{D}(\overline{T})$ the assumption that $\overline{G(T)}$ is a graph implies that y is unique and hence

$$\Rightarrow G(\overline{T}) = \overline{G(T)}, \quad \overline{T}x := y$$

Equivalently, $\mathcal{D}(\overline{T})$ is the set of all $x \in H$ such that there exists a sequence $x_n \in \mathcal{D}(T)$ with $x_n \rightarrow x$ in H and Tx_n is a Cauchy sequence. For such x we define

$$\overline{T}x := \lim_{n \rightarrow \infty} Tx_n$$

Example: Let $T_0 := -\Delta$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is closable with $\overline{T_0} = T_1$.

Proof: Let $u \in L^2(\mathbb{R}^n)$ such that there exists $(u_n)_{n \in \mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ with $u_n \rightarrow u$ in L^2 and $-\Delta u_n \rightarrow v$ in L^2 , as above:

$$-\Delta u = v \in L^2$$

For a given u the function v is unique, and hence, T_0 is closable. Let $\overline{T_0}$ be the closure with domain $\mathcal{D}(\overline{T_0})$ and $u \in \mathcal{D}(T_0)$

$$\Rightarrow \Delta u \in L^2 \Rightarrow u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow \mathcal{D}(\overline{T_0}) \subseteq \mathcal{D}(T_1)$$

but $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}(T_1) = W_{2,2}(\mathbb{R}^n)$

$$\Rightarrow W^{2,2}(\mathbb{R}^n) \subseteq \mathcal{D}(\overline{T_0}) \Rightarrow \mathcal{D}(\overline{T_0}) = \mathcal{D}(T_1)$$

$$\Rightarrow T_1 = \overline{T_0}. \quad \square$$

Remark: Assume for a second in the example above that $W^{2,2} \not\subseteq \mathcal{D}(T_0)$ holds:

$$\Rightarrow \exists u \in W^{2,2} \setminus \mathcal{D}(\overline{T_0}), \exists (u_n)_{n \in \mathbb{N}} \in C_c^\infty(\mathbb{R}^n) : u_n \rightarrow u \text{ in } W^{2,2},$$

using the same arguments as in the example above $\Rightarrow \overline{T_0}$ not closed!

Recall: If $T : H \rightarrow H$ is bounded then T^* is defined through

$$\langle u, T^*v \rangle = \langle Tu, v \rangle, \quad \forall u, v \in H$$

$u \mapsto \langle Tu, v \rangle$ defines a continuous linear map on H ($\in H'$). Riesz' representation theorem then ensures the existence of T^* .

Definition I.3: If T is an unbounded operator on H with dense domain we define

$$\mathcal{D}(T^*) := \left\{ v \in H : \mathcal{D}(T) \ni u \mapsto \langle Tu, v \rangle \text{ can be extended as a linear continuous form on } H \right\}$$

Using Riesz' representation theorem $\exists! f \in H$:

$$\langle u, f \rangle = \langle Tu, v \rangle, \quad \forall u \in \mathcal{D}(T)$$

then define $T^*v = f$, where the uniqueness follows from the density of $\mathcal{D}(T)$ in H .

Remark: If $\mathcal{D}(T) = H$ and T is bounded then we recover the “old” adjoint.

Example: $T_0^* = T_1$,

$$\begin{aligned} \mathcal{D}(T_0^*) &= \left\{ v \in L^2(\mathbb{R}^n) : C_c^\infty(\mathbb{R}^n) \ni u \mapsto \langle -\Delta u, v \rangle \text{ extendable as a lin. continuous form on } L^2(\mathbb{R}^n) \right\} \\ &= \left\{ v \in L^2(\mathbb{R}^n) : -\Delta v \in L^2(\mathbb{R}^n) \right\} = W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \end{aligned}$$

Damit ist

$$\langle T_1 u, v \rangle = \langle -\Delta u, v \rangle = \int v(-\Delta u) = \int (-\Delta v) u = \langle u, T_1 v \rangle$$

Theorem I.1: T^* is a closed operator.

Proof: $v_n \in \mathcal{D}(T^*)$ such that $v_n \rightarrow v$ in H and $T^*v_n \rightarrow w^*$ in H for $(v, w^*) \in H \times H$. For all $u \in \mathcal{D}(T)$ we have

$$\langle Tu, v \rangle = \lim_{n \rightarrow \infty} \langle Tu, v_n \rangle = \lim_{n \rightarrow \infty} \langle u, T^*v_n \rangle = \langle u, w^* \rangle$$

$(H \ni u \mapsto \langle u, w^* \rangle \text{ is continuous}) \Rightarrow v \in \mathcal{D}(T^*)$ and $w^* = T^*v$ by definition. \square

Theorem I.2: *Let T be an operator in H with domain $\mathcal{D}(T)$. Then*

$$G(T^*) = \left(V \left(\overline{G(T)} \right) \right)^\perp$$

where $V: H \times H \rightarrow H \times H, V(x, y) = (y, -x)$ ($V^2 = -\mathbb{1}$).

Proof: Let $u \in \mathcal{D}(T), (v, w^*) \in H \times H$

$$\Rightarrow \langle V(u, Tu), (v, w^*) \rangle_{H \times H} = \langle Tu, v \rangle - \langle u, w^* \rangle$$

Considering the right-hand side it follows

$$\langle Tu, v \rangle - \langle u, w^* \rangle = 0 \quad \forall u \in \mathcal{D}(T) \iff v \in \mathcal{D}(T^*) \text{ and } w^* = T^*v \iff (v, w^*) \in G(T^*),$$

and considering the left-hand side:

$$\Rightarrow \langle V(u, Tu), (v, w^*) \rangle_{H \times H} = 0 \quad \forall u \in \mathcal{D}(T) \iff (v, w^*) \in V(G(T))^\perp$$

In general: $U^\perp = \overline{U}^\perp$, and hence

$$\Rightarrow V(G(T))^\perp = \left(\overline{V(G(T))} \right)^\perp = \left(V \left(\overline{G(T)} \right) \right)^\perp.$$

\square

Theorem I.3: *Let T be a closable operator. Then:*

a) $\mathcal{D}(T^*)$ is dense in H

b) $T^{**} := (T^*)^* = \overline{T}$

Proof:

a) Proof through contradiction: $D(T^*)$ not dense in $H \rightarrow \exists w \neq 0 : \langle w, v \rangle = 0 \ \forall v \in \overline{\mathcal{D}(T^*)}$

$$\implies \langle (0, w), (T^*v, -v) \rangle_{H \times H} = 0 \ \forall v \in \mathcal{D}(T^*)$$

$$\implies (0, w) \perp V(G(T^*))$$

$$\xrightarrow[\text{I.2}]{\text{Thm}} V(\overline{G(T)}) = G(T^*)^\perp$$

$$\implies V(G(T^*)^\perp) = \overline{G(T)}$$

For any $M \subseteq H \times H$ we have $V(M^\perp) = V(M)^\perp$ since for $(u, v) \in V(M)^\perp$, $(x, y) \in M$

$$\langle V(u, v), (x, y) \rangle_{H \times H} = -\langle (u, v), V(x, y) \rangle_{H \times H} \Rightarrow V(u, v) \in M^\perp \Rightarrow (u, v) \in V(M^\perp)$$

$$\implies V(G(T^*)^\perp) = \overline{G(T)} = G(\overline{T}) \implies (0, w) \in G(\overline{T}) \implies w = 0$$

$$\text{b) } G(T^{**}) \xrightarrow[\text{I.2}]{\text{Thm}} V(\overline{G(T^*)})^\perp \xrightarrow[\text{I.1}]{\text{Thm}} V(G(T^*)^\perp) \stackrel{(\perp)}{=} G(\overline{T}) \implies \mathcal{D}(T^{**}) = \mathcal{D}(\overline{T}), T^{**} = \overline{T}$$

□

Definition I.4: We say $T: \mathcal{D}(T) \rightarrow H$ is **symmetric** if and only if

$$\langle Tu, v \rangle = \langle u, Tv \rangle \quad \forall u, v \in \mathcal{D}(T)$$

Example: $T_0 = -\Delta$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta u) v = \int_{\mathbb{R}^n} u (-\Delta v)$$

Remark: If T is symmetric $\Rightarrow \mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and

$$Tu = T^*u \quad \forall u \in \mathcal{D}(T)$$

$\Rightarrow (T^*, \mathcal{D}(T^*))$ is an extension of $(T, \mathcal{D}(T))$.

Lemma I.1: A symmetric operator T is closable.

Proof: It suffice to show that for $u_n \in \mathcal{D}(T)$ with $u_n \rightarrow 0$ and $Tu_n \rightarrow x \in H$ we have $x = 0$

$$\langle x, v \rangle \leftarrow \langle Tu_n, v \rangle = \langle u_n, Tv \rangle \rightarrow \langle 0, Tv \rangle = 0 \quad \forall v \in \mathcal{D}(T)$$

$\Rightarrow x = 0$.

□

Remark: The proof actually shows that if $\mathcal{D}(T^*)$ is dense in H , then T is closable.

Definition I.5: We call an operator T self-adjoint if

$$T = T^* \text{ and } Tu = T^*u \quad \forall u \in \mathcal{D}(T),$$

note that the first property implies that $\mathcal{D}(T) = \mathcal{D}(T^*)$.

Theorem I.4: Every self-adjoint operator is closable.

Proof: [Lemma I.1](#) □

Theorem I.5: Let T be an invertible self-adjoint operator, then T^{-1} is also self-adjoint.

Proof: For $T: \mathcal{D}(T) \rightarrow \mathcal{R}(T)$ consider

Step 1 $\mathcal{R}(T)$ is dense in H . We have to show that $\mathcal{R}(T)^\perp = \{0\}$.

Let $w \in H$ such that

$$\langle Tu, w \rangle = 0 \quad \forall u \in \mathcal{D}(T)$$

$$\implies w \in \mathcal{D}(T^*) \text{ and } T^*w = 0 \xrightarrow[s.a.]{inj.} w = 0.$$

Step 2 Let $w: H \times H \rightarrow H \times H$, $w(x, y) = (y, x)$

$$\implies G(T^{-1}) = \left\{ (x, T^{-1}x) : x \in \mathcal{D}(T) \right\} = w(G(T)) = \left\{ (Ty, y) : y \in \mathcal{D}(T) \right\}$$

$$\begin{aligned} G(T^{-1}) &= G((T^*)^{-1}) \xrightarrow[\text{Thm. I.2}]{\text{Proof}} w(V(G(T)^\perp)) \\ &= V(w(G(T))^\perp) = V(w(G(T)))^\perp \\ &= V(G(T^{-1}))^\perp \xrightarrow[\text{I.2}]{\text{Thm.}} G((T^{-1})^*) \end{aligned}$$

$$\implies T^{-1} = (T^{-1})^*$$

□

Chapter II

Representation Theorems

Theorem II.1 (Riesz): *Let $u \mapsto F(u)$ be a linear continuous function on H . Then $\exists! w \in H$:*

$$F(u) = \langle u, w \rangle \quad \forall u \in H$$

Lax-Milgram: V Hilbertspace, sesquilinear form is defined on $V \times V$, $(u, v) \mapsto \alpha(u, v)$ continuous with

$$|\alpha(u, v)| \leq c \|u\| \|v\| \quad \forall u, v \in V$$

Riesz: \exists linear map $A: V \rightarrow V$:

$$\alpha(u, v) = \langle Au, v \rangle$$

Definition II.1: A bilinear form $a: V \times V \rightarrow \mathbb{R}$ is ***V-coercive*** if there exists $\lambda > 0$ such that

$$a(u, u) \geq \lambda \|u\|^2 \quad \forall u \in V$$

Theorem II.2: *Let a be a continuous sesquilinear and V -coercive on $V \times V$ then A is an isomorphism.*

Proof:

Step 1: A is injective:

$$\|Au\| \|u\| \stackrel{C.S.}{\geq} |\langle Au, u \rangle| = |a(u, u)| \geq \lambda \|u\|^2 \quad (+)$$

$$\Rightarrow \|Au\| \geq \lambda \|u\| \text{ for all } u \in V.$$

Step 2: $A(V)$ is dense in V . Let $u \in V$ such that

$$\langle Au, v \rangle = 0 \quad \forall v \in V$$

$$\text{take } v = u \Rightarrow a(u, u) = 0 \Rightarrow u = 0.$$

Step 3: $\mathbb{R}(A) = A(V)$ is closed. Let v_n be a sequence in $A(V)$ and let u_n be such that

$$Au_n = v_n$$

$$\stackrel{(+)}{\implies} u_n \text{ is a Cauchy sequence } \Rightarrow u_n \rightarrow u \in V \text{ und } Au_n \rightarrow Au \Rightarrow v_n \rightarrow Au \in A(V)$$

Step 4: $u = A^{-1}v \stackrel{(+)}{\implies} \|A^{-1}v\| \leq \lambda^{-1}\|v\| \quad \forall v \in V.$

□

Next we consider two Hilbert spaces V, H with $V \subset H$ (the inclusion is continuous), i.e.

$$\exists c < \infty: \quad \|u\|_H \leq c\|u\|_V \quad \forall u \in V$$

and we assume that V is dense in H .

Example: $V = W^{1,2}(\mathbb{R}^n)$, $H = L^2(\mathbb{R}^n)$

$$\|u\|_L^2 \leq \|u\|_{W^{1,2}}$$

There exists a natural injection from H into V' . Let $h \in H$ then $V \ni u \mapsto \langle u, h \rangle_H$ is continuous on $V \xrightarrow[\text{Thm. II.1}]{\implies} \exists l_h \in V'$:

$$l_h(u) = \langle u, h \rangle_H \quad \forall u \in V$$

injectivity follows from density of V in H . $V \subseteq H \subset V'$ cont. sesquilinear form a on $V \times V$ which is V -coercive \rightarrow Associate an unbounded operator S with a

$$\mathcal{D}(S) := \left\{ u \in V : a(u, v) \text{ is cont. on } V \text{ with respect to the topology induced by } H \right\}$$

Theorem II.3: *Let a be a continuous sesquilinear form on V which is V -coercive then S is bijective from $\mathcal{D}(S)$ into H and $S^{-1} \in L(H, \mathcal{D}(S))$. Moreover, $\mathcal{D}(S)$ is dense in H .*

Proof:

1) S injective: $\exists \alpha > 0$:

$$\begin{aligned} \alpha \|u\|_H^2 &\leq C\alpha \|u\|_V^2 \leq C |a(u, u)| = c |\langle Su, u \rangle_H| \leq c \|Su\|_H \|u\|_H, \quad \forall u \in \mathcal{D}(S) \\ \Rightarrow \alpha \|u\|_H &\leq c \|Su\|_H, \quad \forall u \in \mathcal{D}(S) \quad (+). \end{aligned}$$

2) S surjective:

Let $h \in H$. Choose $w \in V$ such that

$$\langle h, v \rangle_H = \overline{\langle v, h \rangle_H} = \overline{l_h(v)} = \langle w, v \rangle \quad \forall v \in V$$

where we used Riesz' representation theorem in the last step.

(Note: $l_h \in V' \Rightarrow \overline{l_h} \in \text{continuous linear form on } V$).

Define $u := A^{-1}w \in V \Rightarrow a(u, v) = \langle Au, v \rangle_V = \langle w, v \rangle_V = \langle h, v \rangle_H$

$$\Rightarrow u \in \mathcal{D}(S), \quad Su = h$$

(V dense in H). (+) implies that S^{-1} is continuous.

3) Density of $\mathcal{D}(S)$:

Let $h \in H$ such that $\langle u, h \rangle_H = 0 \quad \forall u \in \mathcal{D}(S)$. S surjective $\exists v \in \mathcal{D}(S)$: $Sv = h$

$$\Rightarrow \langle Sv, u \rangle = 0 \quad \forall u \in \mathcal{D}(S)$$

$$\Rightarrow \langle Sv, v \rangle_H = 0 \Rightarrow a(v, v) = 0 \Rightarrow v = 0 \Rightarrow h = 0$$

a hermitian iff

$$a(u, v) = \overline{a(v, u)} \quad \forall u, v \in V$$

□

Theorem II.4: Under the assumptions of [Theorem II.3](#) and a being hermitian it follows that

a) S is closed

b) $S = S^*$

c) $\mathcal{D}(S)$ dense in V

Proof:

a) [Theorem I.4](#)

b) a hermitian

$$\Rightarrow \langle Su, v \rangle_H = a(u, v) = \overline{a(v, u)} = \overline{\langle Sv, u \rangle_H} = \langle u, Sv \rangle_H \quad \forall u, v \in \mathcal{D}(S)$$

$\Rightarrow S$ symmetric $\Rightarrow \mathcal{D}(S) \subset \mathcal{D}(S^*)$. Let $v \in \mathcal{D}(S^*)$, S surjective

$$\Rightarrow v_0 \in \mathcal{D}(S) : Sv_0 = S^*v.$$

For all $u \in \mathcal{D}(S)$ we get

$$\langle Su, v_0 \rangle_H = \langle u, Sv_0 \rangle_H = \langle u, S^*v \rangle_H = \langle Su, v \rangle_H$$

$$\Rightarrow v = v_0 \Rightarrow \mathcal{D}(S) = \mathcal{D}(S^*), Sv = S^*v \quad \forall v \in \mathcal{D}(S).$$

c) follows from [Theorem II.3](#)

□

Chapter III

Friedrichs extension

Definition III.1: Let T_0 be a symmetric unbounded operator with domain $\mathcal{D}(T_0)$ we say that T_0 is **semi-bounded** if $\exists c > 0$:

$$\langle T_0 u, u \rangle_H \geq -c \|u\|_H^2 \quad \forall u \in \mathcal{D}(T_0)$$

Example:

a) Schrödinger Operator. \mathbb{R}^m , $H = L^2(\mathbb{R}^m)$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^m)$

$$T_0 := -\Delta + V(x),$$

$V \in C_0(\mathbb{R}^m)$ with $V(x) \geq -c \quad \forall x \in \mathbb{R}^m$. For $u \in \mathcal{D}(T_0)$

$$\langle T_0 u, u \rangle_H = \int_{\mathbb{R}^m} (\Delta u + V u) u = \underbrace{\int_{\mathbb{R}^m} |\nabla u|^2}_{\geq 0} + \underbrace{\int_{\mathbb{R}^m} V(x) |u(x)|^2}_{\geq -c \int |u|^2 = -c \|u\|_H^2}$$

b) $S_z := -\Delta - \frac{z}{r}$, whereas $r = |x|$, $z \in \mathbb{R}$

Hardy inequality in $\mathbb{R}^3 (m = 3)$:

$$\int_{\mathbb{R}^3} |x|^{-2} |u(x)|^2 dx \leq 4 \int_{\mathbb{R}^3} |\nabla u|^2(x) dx \quad \forall u \in C_c^\infty(\mathbb{R}^m)$$

Proof: $\int_{\mathbb{R}^3} \left| \nabla u + \frac{1}{2} \frac{x}{|x|^2} u \right|^2 dx \geq 0$

$$\iff \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{4} \frac{|u|^2}{|x|^2} dx \geq - \int_{\mathbb{R}^3} \langle \nabla u(x), \frac{x}{|x|} \rangle u(x) dx$$

now

$$-2 \int_{\mathbb{R}^3} \langle \nabla u, \frac{x}{|x|^2} \rangle u dx = - \int_{\mathbb{R}^3} \langle \nabla |u|^2, \frac{x}{|x|^2} \rangle dx = \int_{\mathbb{R}^3} |u|^2 \underbrace{\operatorname{div} \frac{x}{|x|^2}}_{=\frac{1}{|x|^2}} = \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} dx$$

$\Rightarrow \int |\nabla u|^2 \geq \frac{1}{4} \int \frac{|u|^2}{|x|^2}$. Now

$$\int_{\mathbb{R}^3} \frac{1}{r} |u(x)|^2 dx \leq \left(\int \frac{|u(x)|^2}{r^2} dx \right)^{\frac{1}{2}} \cdot \|u\|_L^2$$

$$\int_{\mathbb{R}^3} \frac{1}{r^2} |u(x)|^2 dx \leq 4 \langle -\Delta u, u \rangle_L^2$$

$$\Rightarrow \forall \epsilon > 0 : \quad \int_{\mathbb{R}^3} \frac{1}{r} |u(x)|^2 dx \leq \epsilon \cdot \langle -\Delta u, u \rangle_L^2 + \frac{1}{\epsilon} \|u\|_{L^2}^2$$

hence

$$\langle S_z u, u \rangle_{L^2} = \langle -\Delta u, u \rangle_{L^2} - z \left\langle \frac{u}{r}, u \right\rangle_{L^2} \geq (1 - \epsilon) \langle -\Delta u, u \rangle_{L^2} - \frac{z}{\epsilon} \|u\|_{L^2}^2$$

$$\text{Choose } \epsilon = \frac{1}{z} \Rightarrow \langle S_z u, u \rangle_{L^2} \geq -z^2 \|u\|_{L^2}^2 \quad \square$$

Theorem III.1: *A symmetric semibounded operator T_0 on H with dense domain $\mathcal{D}(T_0)$ admits a self-adjoint extension, called **Friedrichs extension**.*

Proof: Replace T_0 by $T_0 + \lambda \mathbf{1}$ such that

$$\langle T_0 u, u \rangle_H \geq \|u\|_H^2 \quad \forall u \in \mathcal{D}(T_0)$$

$$(u, v) \mapsto a_0(u, v) := \langle T_0 u, v \rangle_H \text{ on } \mathcal{D}(T_0) \times \mathcal{D}(T_0)$$

$$\Rightarrow a_0(u, u) \geq \|u\|_H^2$$

Let V be the completion in H of $\mathcal{D}(T_0)$ for the norm $u \mapsto \rho_0(u) = \sqrt{a_0(u, u)} \iff u \in H$ belongs to V if $\exists u_n \in \mathcal{D}(T_0)$ s.t. $u_n \rightarrow u$ in H and $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to ρ_0

“Candidate Norm”:

$$\|u\|_V = \lim_{n \rightarrow \infty} \rho_0(u_n)$$

where u_n is as above. \square

Lemma III.1: *Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}(T_0)$ with respect to ρ_0 such that $x_n \rightarrow 0$ in H . Then $p_0(x_n) \rightarrow 0$.*

Proof: Observe that $p_0(x_n)$ is a Cauchy sequence in \mathbb{R}_+ , and hence, converges in $\overline{\mathbb{R}_+}$. Assume that $p_0(x_n) \rightarrow \alpha > 0$. Now $a_0(x_n, x_m) = a_0(x_n, x_n) + a_0(x_n, x_m - x_n)$

$$|a_0(x_n, x_m - x_n)| \leq \sqrt{a_0(x_n, x_n)} \sqrt{a_0(x_m - x_n, x_m - x_n)}$$

$\forall \epsilon > 0 \exists N \forall n, m \geq N$:

$$|a_0(x_n, x_m) - \alpha^2| < \epsilon$$

$\epsilon = \frac{\alpha^2}{2} \Rightarrow |a_0(x_n, x_m)| = |\langle T_0 x_n, x_m \rangle| \geq \frac{1}{2} \alpha^2 > 0 \forall n, m \geq N$. Let $m \rightarrow \infty \Rightarrow x_m \rightarrow 0$, which leads to the contradiction. \square

Theorem III.2 (Example: Dirichlet Realisation): *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ smooth, T_1 is defined by:*

$$\mathcal{D}(T_1) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad T_1 = -\Delta: \mathcal{D}(T_1) \rightarrow L^2(\Omega)$$

T_1 is self-adjoint; T_1 is called the Dirichlet realisation of $-\Delta$.

Proof: (with gaps)

Define

$$\mathcal{D}(T_0) := C_c^\infty(\Omega), \quad T_0 := -\Delta, \quad H = L^2(\Omega)$$

Through integration by parts respectively Green's formula:

T_0 is symmetric, non-negative (i.e. semi-bounded)

Consider $\tilde{T}_0 := T_0 + \mathbf{1}_H$. V : closure of $C_c^\infty(\Omega)$. in $W^{2,2}(\Omega)$

$$\xrightarrow[\text{Friedrich ext.}]{S} \mathcal{D}(S) = \left\{ u \in W_0^{1,2}(\Omega) \mid -\Delta u \in L^2(\Omega) \right\}$$

$$\xrightarrow[\text{Theory}]{\text{Regularity}} \mathcal{D}(S) = W^{2,2} \cap W_0^{1,2}(\Omega)$$

for more details about the regularity theory see “2nd order elliptic operators”, PDE Evans. \square

Example:

a) Harmonic oscillator

Define $\mathcal{D}(T_0) := C_c^\infty(\mathbb{R}^n)$, $T_0 := -\Delta + |x|^2 + 1$, $= L^2(\mathbb{R}^2)$. Let V be the completion of C_c^∞ in L^2 with respect to the norm

$$u \mapsto (\langle \nabla u, \nabla u \rangle_{L^2} + \langle |x|u(x), |x|u(x) \rangle_{L^2} + \langle u, u \rangle_{L^2})^{\frac{1}{2}} = \sqrt{\langle T_0 u, u \rangle_{L^2}}.$$

By calculation:

$$V = \left\{ u \in W^{1,2}(\mathbb{R}^n) \mid x_j u \in L^2(\mathbb{R}^n) \ \forall j = 1, \dots, n \right\}$$

Domain of S :

$$\mathcal{D}(S) = \left\{ u \in V \mid T_0 u \in L^2(\mathbb{R}^n) \right\} = \left\{ u \in W^{2,2}(\mathbb{R}^n) : x^\alpha u \in L^2(\mathbb{R}^n) \ \forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq 2 \right\}$$

b) Schrödinger operator with a Coulomb potential

Define $\mathcal{D}(T_0) := C_c^\infty(\mathbb{R}^3)$, $T_0 = -\Delta - \frac{1}{r}$, $H = L^2(\mathbb{R}^3)$. We saw that T_0 is semi-bounded: $\langle T_0 u, u \rangle_{L^2} \geq -\|u\|_{L^2}^2$

$$\tilde{T}_0 := T_0 + 2 \cdot \mathbf{1}_H, \quad \mathcal{D}(\tilde{T}_0) = \mathcal{D}(T_0)$$

satisfies the assumptions of the Friedrichs extension. Completion V of C_c^∞ in L^2 with respect to the norm

$$u \mapsto \left(\langle \nabla u, \nabla u \rangle_{L^2} + \int_{\mathbb{R}^3} \left(2 - \frac{1}{r} \right) |u(x)|^2 dx \right)^{\frac{1}{2}} = \sqrt{\langle \tilde{T}_0 u, u \rangle_{L^2}}$$

is $V = W^{1,2}(\mathbb{R}^3)$.

Proof: $C_c^\infty(\mathbb{R}^3)$ is dense in $W^{1,2}(\mathbb{R}^3)$. Therefore, we only need to check that the norm above and $\|\cdot\|_{W^{1,2}}$ are equivalent. By the proof of the analysis of the Schrödinger operator:

$$\int_{\mathbb{R}^3} \frac{1}{r} |u(x)|^2 dx \leq \epsilon \langle -\Delta u, u \rangle_{L^2} + \frac{1}{\epsilon} \|u\|_{L^2}^2 = \epsilon \|\nabla u\|_{L^2}^2 + \frac{1}{\epsilon} \|u\|_{L^2}^2 \quad \forall \epsilon > 0, \ u \in C_c^\infty(\mathbb{R}^3)$$

(See Bernhard Helffer, “Spectral theory and app.”).

$$\|u\|_{W^{1,2}}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \stackrel{\text{Hardy}}{\leq} 5 \|\nabla u\|_{L^2}^2 + \left\| \left(2 - \frac{1}{r} \right) u \right\|_{L^2}^2 \quad \forall u \in C_c^\infty(\mathbb{R}^3)$$

Domination of S : Hardy inequality $\Rightarrow \frac{1}{r}u \in L^2$ for $u \in W^{1,2}(\mathbb{R}^3)$.

$$\Rightarrow u \in D(S) : \Delta u \in L^2(\mathbb{R}^3) \Rightarrow \mathcal{D}(S) = W^{2,2}(\mathbb{R}^3)$$

□

c) Neumann boundary conditions: on the half-plane $H = L^2((0, \infty))$ define the form

$$a(u, v) := \int_0^\infty u'(x)v'(x)dx$$

for $u, v \in \mathcal{D}(a) = W^{1,2}(0, \infty) \Rightarrow a(u, u) = \|u'\|_{L^2}^2 \geq -\|u\|_{L^2}^2$. a is closed by completeness of $W^{1,2}(0, \infty)$.

Associated operator T : $v \in \mathcal{D}(T) \exists f_v \in L^2(0, \infty)$:

$$\int_0^\infty u'(x)v'(x)dx = \int_0^\infty u(x)f_v(x)dx \quad \forall u \in W^{1,2}(0, \infty).$$

$\Rightarrow f_v = -(v')' = -v''$, therefore $v \in W^{2,2}(0, \infty)$, $Tv = -v''$. Note for $v \in W^{2,2}(0, \infty)$, $u \in W^{1,2}(0, \infty)$:

$$\begin{aligned} a(u, v) &= \int_0^\infty u'(x)v'(x)dx \\ &= [u(x)v'(x)]_0^\infty - \int_0^\infty u(x)v''(x)dx \\ &= \underbrace{u(0)v'(0)}_{=0} + \int_0^\infty u(x)Tv(x)dx = \langle u, Tv \rangle_{L^2} \end{aligned}$$

Therefore the associated operator is $T_N := T$ acts as $T_N v = -v''$ on the domain

$$\mathcal{D}(T_N) = \{v \in W^{2,2}(0, \infty) \mid v'(0) = 0\}$$

T_N is called the Neumann Laplacian.

Chapter IV

Spectrum and Resolvent

Let X be a Banach space and H a Hilbert space.

Definition IV.1: Let $T: \mathcal{D}(T) \subseteq \rightarrow X$ linear operator. We define

- We call the following set the **resolvent set**:

$$\text{res}(T) := \rho(T) := \left\{ \lambda \in \mathbb{C} \mid \lambda \mathbb{1} - T \text{ is bijective with bounded inverse} \right\}.$$

- The set $\text{spec}(T) := \sigma(T) := \mathbb{C} \setminus \rho(T)$ is called **spectrum**.
- The set $\text{spec}_p(T) := \sigma_p(T) := \left\{ \text{Eigenvalues of } T \right\}$ is the **point spectrum**.
- The following set is called the **continuous spectrum**: $\text{spec}_c(T) := \sigma_c(T)$

$$\sigma_c(T) := \left\{ \lambda \in \mathbb{C} \mid \lambda \mathbb{1} - T \text{ is inj., but not surj., range}(\lambda \mathbb{1} - T) \text{ is dense in } X \right\}$$

- The following set is called the **residual spectrum**: $\text{spec}_{res}(T) := \sigma_{res}(T)$

$$\sigma_{res}(T) := \left\{ \lambda \in \mathbb{C} \mid \lambda \mathbb{1} - T \text{ is inj., but not surj., range}(\lambda \mathbb{1} - T) \text{ is not dense in } X \right\}$$

- The **resolvent function**: $R_T: \rho(T) \rightarrow L(X, X) =: L(X)$

$$\lambda \mapsto R_T(\lambda) := R(\lambda, T) := (\lambda \mathbb{1} - T)^{-1}$$

Remarks:

- $\dim(X) < \infty : \sigma(T) = \sigma_p(T)$
- $\sigma(T) = \sigma_p(T) \dot{\cup} \sigma_c(T) \dot{\cup} \sigma_{res}(T)$

Theorem IV.1: *If $\rho(T) \neq \emptyset$ then T is closed.*

Proof: $\lambda \in \rho(T)$ then $\text{graph}(R(\lambda, T))$ is closed (by the closed graph theorem). For $x \in \mathcal{D}(T)$, $y \in X$ with $R(\lambda, T)y = x$:

$$\|x\|_{\lambda\mathbb{1}-T} = \|(\lambda - T)x\|_X + \|x\|_X = \|y\|_X + \|R(\lambda, T)y\|_X = \|y\|_{R(\lambda, T)}.$$

Therefore, $\text{graph}(\lambda\mathbb{1} - T)$ and $\text{graph}(R(\lambda, T))$ are isometric, and so $\lambda\mathbb{1} - T$ is closed

$\Rightarrow T$ is closed

□

Theorem IV.2: *For a closed operator T one has the equivalence*

$$\lambda \in \rho(T) \iff \begin{cases} \text{kern}(\lambda\mathbb{1} - T) = 0, & \text{“inj.”} \\ \text{range}(\lambda\mathbb{1} - T) = X, & \text{“surj.”} \end{cases}$$

Proof:

“ \Rightarrow ” By definition.

“ \Leftarrow ” Let $\lambda \in C$ with $\text{kern}(\lambda\mathbb{1} - T) = 0$, $\text{range}(\lambda\mathbb{1} - T) = X$. Then the inverse

$$(\lambda\mathbb{1} - T)^{-1} : X \rightarrow X$$

is defined everywhere and has a closed graph (as $\lambda\mathbb{1} - T$ is closed), see proof of Theorem IV. 1. By the closed graph theorem $(\lambda\mathbb{1} - T)^{-1}$ is bounded, i.e. $\lambda \in \rho(T)$.

□

Addendum

Theorem (Riesz' representation theorem, FA 17.2): To all $x' \in X'$ there exists a unique $x \in X$ such that

$$x'(y) = \langle y, x \rangle,$$

for $y \in X$ and $\|x'\|_{X'} = \|x\|_X$.

Theorem (Closed graph theorem, FA 12.6): thm:acgt] If X and Y are Banach spaces, and $T: X \rightarrow Y$ is a linear operator, then T is continuous if and only if its graph is closed in $X \times Y$, with respect to the product topology.

Stichwortverzeichnis

bounded, 3
 semi-, 13

closable, 4
closed, 3
closure, 4
coercive, 9

dense, 3
domain, 2

Friedrichs extension, 14

Graph, 3

Hardy inequality, 13

inner product, 2

Operator
 bounded, 3
 closable, 4
 closed, 3
 closure of, 4

range, 2
resolvent function, 18
resolvent set, 18

semi-bounded, 13
spectrum, 18
 continuous, 18
 point, 18
 residual, 18