Spectraltheory

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Contents

Chapter I

Unbounded operators, adjoint and self-adjoint operators

Let H be a separable Hilbert space, $\langle \cdot, \cdot \rangle$ denote the scalar product on H.

A linear operator T in H is a linear map $u \mapsto Tu$ defined on a subspace $\mathcal{D}(T)$ of H, and we call $\mathcal{D}(T)$ the domain of T.
For $T\colon \mathcal{D}(T) \to H$ we denote the range of T with

$$\mathcal{R}(T) \coloneqq \operatorname{Image}(T).$$

We say that T is **bounded** if it is continuous from $\mathcal{D}(T)$ into H, with respect to the topology induced by H.
If $\mathcal{D}(T) = H$ we recall the definition of bounded operators from the functional analysis course.

From now on, if $\mathcal{D}(T) \neq H$ we will assume that $\mathcal{D}(T)$ is **dense** in H, i.e. $\overline{\mathcal{D}(T)} = H$.
In this case, if T is bounded then T has a unique continuous extension to all of H.

$$\Rightarrow T \text{ bounded is ...}$$

Recall: An operator is called **closed** if the graph

$$G(T) \coloneqq \{(x, y) \in H \times H \colon x \in \mathcal{D}(T), y = Tx\}$$

is closed in $H \times H$.

Definition I.1: *Let $T\colon \mathcal{D}(T) \to H$ be a (linear) operator with $\mathcal{D}(T)$ dense in H. Then T is called* ***closed*** *if the conditions*

$$\left.\begin{aligned} &u_n \in \mathcal{D}(T) \\ &u_n \to u \text{ in } H \\ &Tu_n \to v \text{ in } H \end{aligned}\right\} \Rightarrow u \in \mathcal{D}(T), v = Tu$$

hold.

Example:

a) Let $T_0 = -\Delta$, $H = \ell^2(\mathbb{R}^n)$ and $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ dense in H.
Take $u \in W^{2,2}(\mathbb{R}^n) \setminus C_c^\infty(\mathbb{R}^n)$

$$\xRightarrow{\text{densly}} \exists (u_n)_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n)\colon\; u_n \to u \text{ in } W^{2,2}(\mathbb{R}^n)$$

$(u_n, -\Delta u_n) \in G(T_0)$ converges in $\ell^2 \times L^2$ to $(u, -\Delta u) \notin G(T_0)$.

b) Let $T_1 = -\Delta$, $\mathcal{D}(T_1) = W^{2,2}(\mathbb{R}^n)$ and $H = L^2(\mathbb{R}^n)$. For $u_n \in \mathcal{D}(T_1)$ with

$$u_n \to u \text{ in } H \text{ and } (-\Delta u_n) \to u \text{ in } L^2$$

follows that $-\Delta u = v \in L^2(\mathbb{R}^n)$ weakly, i.e. $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} v\varphi \longleftarrow \int_{\mathbb{R}^n} (-\Delta u_n)\,\varphi = \int_{\mathbb{R}^n} u_n\,(-\Delta\varphi) \longrightarrow \int_{\mathbb{R}^n} u\,(-\Delta\varphi)\,.$$

Stichwortverzeichnis