Spectraltheory

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Contents

Chapter I

Unbounded, adjoint and self-adjoint Operators

Let H be a separable Hilbert space, i.e. a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product $\langle \cdot, \cdot \rangle$ on H.

Recall: A mapping $\langle \cdot, \cdot \rangle \colon H \times H \to \mathbb{C}$ is called an **inner product**, if for all $x, y \in H$, $\lambda \in \mathbb{C}$ holds:

- $(S1)$ $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$, $\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle$
- $(S2)$ $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, $\langle x, \lambda y \rangle = \overline{\lambda} \langle x, y \rangle$
- $(S3)$ $\langle x, y \rangle = \overline{\langle y, x \rangle}$
- $(S4)$ $\langle x, x \rangle \geq 0, \quad \langle x, x \rangle = 0 \iff x = 0$

A linear operator T in H is a linear map

$$u \mapsto Tu$$

defined on a subspace $\mathcal{D}(T)$ of H, and we call $\mathcal{D}(T)$ the **domain** of T. For $T \colon \mathcal{D}(T) \to H$ we denote the **range** of T with

$$\mathcal{R}(T) \coloneqq \operatorname{Image}(T).$$

We say that T is **bounded** if it is continuous from $\mathcal{D}(T)$ into H, with respect to the topology induced by H. We recall that if $\mathcal{D}(T) = H$ holds, boundedness of a linear operator is equivalent to continuity in 0, boundedness of $T(U_{(X,\|\cdot\|)})$ in Y and that there $\exists c < \infty$ such that $\|Tx\| \leq c\|x\|$, for proof see theorem 3.3 in the functional analysis course.

From now on, if $\mathcal{D}(T) \neq H$ we will assume that $\mathcal{D}(T)$ is **dense** in H, i.e. $\overline{\mathcal{D}(T)} = H$. If in this case T would be bounded then T has a unique continuous extension to all of H, for proof see proposition 5.10 in the functional analysis course. As this simplifies many considerations some of the following theorems would be trivial, and hence, we won't focus on bounded operators during this lecture.

Recall: An operator is called **closed** if the graph

$$G(T) \coloneqq \left\{(x, y) \in H \times H \;\middle|\; x \in \mathcal{D}(T), y = Tx\right\}$$

is closed in $H \times H$.

Definition I.1: *Let $T\colon \mathcal{D}(T) \to H$ be a (linear) operator with $\mathcal{D}(T)$ dense in H. Then T is called **closed** if for all*

$$u_n \in \mathcal{D}(T),\ u_n \to u \in H \text{ and } Tu_n \to v \in H$$

follows that

$$u \in \mathcal{D}(T),\ v = Tu$$

holds.

Example:

a) Let $H = L^2(\mathbb{R}^n)$, then $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is dense in H. Define the operator

$$T_0 = -\Delta,$$

and take $u \in W^{2,2}(\mathbb{R}^n) \setminus C_c^\infty(\mathbb{R}^n)$, s.t $u \in L^2(\mathbb{R}^n)$. Due to the density:

$$\exists\ (u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n):\ u_n \to u \text{ in } W^{2,2}(\mathbb{R}^n).$$

As a result, $(u_n, -\Delta u_n) \in G(T_0)$ converges in $L^2 \times L^2$ to $(u, -\Delta u) \notin G(T_0)$.

b) Let $H = L^2(\mathbb{R}^n)$, and set $\mathcal{D}(T_1) = W^{2,2}(\mathbb{R}^n) \subseteq H$. Define the operator

$$T_1 = -\Delta.$$

For $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{D}(T_1)$ with

$$u_n \to u \in H \text{ and } (-\Delta u_n) \to v \in L^2$$

follows that $-\Delta u = v \in L^2(\mathbb{R}^n)$ weakly, i.e. for all $\varphi \in C_c^\infty(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} v\varphi \longleftarrow \int_{\mathbb{R}^n} (-\Delta u_n)\,\varphi = \int_{\mathbb{R}^n} u_n\,(-\Delta\varphi) \longrightarrow \int_{\mathbb{R}^n} u\,(-\Delta\varphi)\,.$$

$\xRightarrow{PDE} u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow T_1$ is closed.

Definition I.2: *An operator T is called **closable** $\iff$ $\overline{G(T)}$ is a graph.*

Remark: We call $\overline{T}$ the **closure** of T, and in such case we have

$$\mathcal{D}\left(\overline{T}\right) \coloneqq \left\{x \in H \;\middle|\; \exists\, y\colon (x,y) \in \overline{G(R)}\right\}$$

For any $x \in \mathcal{D}\left(\overline{T}\right)$ the assumption that $\overline{G(T)}$ is a graph implies that y is unique and hence

$$\Rightarrow G(\overline{T}) = \overline{G(T)},\ \overline{T}x \coloneqq y$$

Equivalently, $\mathcal{D}\left(\overline{T}\right)$ is the set of all $x \in H$ such that there exists a sequence $x_n \in \mathcal{D}(T)$ with $x_n \to x$ in H and Tx_n is a cauchy sequence. For such x we define

$$\overline{T}x \coloneqq \lim_{n\to\infty} Tx_n$$

Example: Let $T_0 \coloneqq -\Delta$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^n)$ is closable with $\overline{T_0} = T_1$.

Proof: Let $u \in L^2(\mathbb{R}^n)$ such that there exists $(u_n)_{n\in\mathbb{N}} \subseteq C_c^\infty(\mathbb{R}^n)$ with $u_n \to u$ in L^2 and $-\Delta u_n \to u$ in L^2, as above:

$$-\Delta u = v \in L^2$$

For a given u the function v is unique, and hence, T_0 is closable. Let $\overline{T_0}$ be the closure with domain $\mathcal{D}\left(\overline{T_0}\right)$ and $u \in \mathcal{D}(T_0)$

$$\Rightarrow \Delta u \in L^2 \Rightarrow u \in W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1) \Rightarrow \mathcal{D}\left(\overline{T_0}\right) \subseteq \mathcal{D}(T_1)$$

but $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{D}(T_1) = W_{2,2}(\mathbb{R}^n)$

$$\Rightarrow W^{2,2}(\mathbb{R}^n) \subseteq \mathcal{D}\left(\overline{T_0}\right) \Rightarrow \mathcal{D}\left(\overline{T_0}\right) = \mathcal{D}(T_1)$$

$\Rightarrow T_1 = \overline{T_0}$. $\square$

Remark: Assume for a second in the example above that $W^{2,2} \not\subseteq \mathcal{D}(T_0)$ holds:

$$\Rightarrow \exists u \in W^{2,2} \setminus D\left(\overline{T_0}\right),\ \exists (u_n)_{n\in\mathbb{N}} \in C_c^\infty(\mathbb{R}^n) : u_n \to u \text{ in } W^{2,2},$$

using the same arguments as in the example above $\Rightarrow \overline{T_0}$ not closed!

Recall: If $T\colon H \to H$ is bounded then T^* is defined through

$$\langle u, T^* v\rangle = \langle Tu, v\rangle,\ \forall u, v \in H$$

$u \mapsto \langle Tu, v\rangle$ defines a continuous linear map on H $(\in H')$. Riesz' representation theorem then ensures the existence of T^*.

Definition I.3: *If T is an unbounded operator on H with dense domain we define*

$\mathcal{D}(T^*) := \Big\{v \in H\colon \mathcal{D}(T) \ni u \mapsto \langle Tu, v\rangle$ *can be extended as a linear continuous form on* $H\Big\}$

Using Riesz' representation theorem $\exists! f \in H$:

$$\langle u, f\rangle = \langle Tu, v\rangle,\ \forall u \in \mathcal{D}(T)$$

then define $T^* v = f$, where the uniqueness follows from the density of $\mathcal{D}(T)$ in H.

Remark: If $\mathcal{D}(T) = H$ and T is bounded then we recover the "old" adjoint.

Example: $T_0^* = T_1$,

$$\begin{aligned}\mathcal{D}(T_0^*) &= \Big\{v \in L^2(\mathbb{R}^n)\colon C_c^\infty(\mathbb{R}^n) \ni u \mapsto \langle -\Delta u, v\rangle \text{ extendable as a lin. continuous form on } L^2(\mathbb{R}^n)\Big\} \\ &= \Big\{v \in L^2(\mathbb{R}^n)\colon -\Delta v \in L^2(\mathbb{R}^n)\Big\} = W^{2,2}(\mathbb{R}^n) = \mathcal{D}(T_1)\end{aligned}$$

Damit ist

$$\langle T_1 u, v\rangle = \langle -\Delta u, v\rangle = \int v\,(-\Delta u) = \int (-\Delta v)\, u = \langle u, T_1 v\rangle$$

Theorem I.1: *T^* is a closed operator.*

Proof: $v_n \in \mathcal{D}(T^*)$ such that $v_n \to v$ in H and $T^* v_n \to w^*$ in H for $(v, w^*) \in H \times H$. For all $u \in \mathcal{D}(T)$ we have

$$\langle Tu, v\rangle = \lim_{n\to\infty} \langle Tu, v_n\rangle = \lim_{n\to\infty} \langle u, T^* v_n\rangle = \langle u, w^*\rangle$$

$(H \ni u \mapsto \langle u, w^*\rangle$ is continuous$) \Rightarrow v \in \mathcal{D}(T^*)$ and $w^* = T^* v$ by definition. □

Theorem I.2: *Let T be an operator in H with domain $\mathcal{D}(T)$. Then*

$$G(T^*) = \left(V\left(\overline{G(T)}\right)\right)^{\perp}$$

where $V\colon H \times H \to H \times H, V(x, y) = (y, -x)$ $(V^2 = -\mathbb{1})$.

Proof: Let $u \in \mathcal{D}(T), (v, w^*) \in H \times H$

$$\Rightarrow \langle V(u, Tu), (v, w^*)\rangle_{H\times H} = \langle Tu, v\rangle - \langle u, w^*\rangle$$

Considering the right-hand side it follows

$$\langle Tu, v\rangle - \langle u, w^*\rangle = 0\, \forall u \in \mathcal{D}(T) \iff v \in \mathcal{D}(T^*) \text{ and } w^* = T^* v \iff (v, w^*) \in G(T^*),$$

and considering the left-hand side:

$$\Rightarrow \langle V(u, Tu), (v, w^*)\rangle_{H\times H} = 0\ \forall u \in \mathcal{D}(T) \iff (v, w^*) \in V(G(T))^{\perp}$$

In general: $U^{\perp} = \overline{U}^{\perp}$, and hence

$$\Rightarrow V(G(T))^{\perp} = \left(\overline{V(G(T))}\right)^{\perp} = \left(V\left(\overline{G(T)}\right)\right)^{\perp}.$$

□

Theorem I.3: *Let T be a closable operator. Then:*

a) $\mathcal{D}(T^*)$ *is dense in* H

b) $T^{**} := (T^*)^* = \overline{T}$

Proof:

a) Proof through contradiction: $D(T^*)$ not dense in $H \to \exists w \neq 0 : \langle w, v\rangle = 0\ \forall v \in \overline{\mathcal{D}(T^*)}$

$$\begin{aligned}
&\Longrightarrow \langle (0,w), (T^*v, -v)\rangle_{H\times H} = 0\ \forall v \in \mathcal{D}(T^*) \\
&\Longrightarrow (0,w) \perp V(G(T^*)) \\
&\overset{Thm}{\underset{I.2}{\Longrightarrow}} V\left(\overline{G(T)}\right) = G(T^*)^{\perp} \\
&\Longrightarrow V\left(G(T^*)^{\perp}\right) = \overline{G(T)}
\end{aligned}$$

For any $M \subseteq H \times H$ we have $V\left(M^{\perp}\right) = V(M)^{\perp}$ since for $(u,v) \in V(M)^{\perp}$, $(x,y) \in M$

$\langle V(u,v), (x,y)\rangle_{H\times H} = -\langle (u,v), V(x,y)\rangle_{H\times h} \Rightarrow V(u,v) \in M^{\perp} \Rightarrow (u,v) \in V\left(M^{\perp}\right)$

$\Longrightarrow V(G(T^*))^{\perp} = \overline{G(T)} = G\left(\overline{T}\right) \Longrightarrow (0,w) \in G\left(\overline{T}\right) \Longrightarrow w = 0$

b) $G(T^{**}) \overset{Thm}{\underset{I.2}{=}} V\left(\overline{G(T^*)}\right)^{\perp} \overset{Thm}{\underset{I.1}{=}} V(G(T^*))^{\perp} \overset{(\perp)}{=} G\left(\overline{T}\right) \Longrightarrow \mathcal{D}(T^{**}) = \mathcal{D}\left(\overline{T}\right), T^{**} = \overline{T}$

$\square$

Definition I.4: *We say $T\colon \mathcal{D}(T) \to H$ is* ***symmetric*** *if and only if*

$$\langle Tu, v\rangle = \langle u, Tv\rangle \quad \forall u, v \in \mathcal{D}(T)$$

Example: $T_0 = -\Delta$, $\mathcal{D}(T_0) = C_c^{\infty}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} (-\Delta u)\, v = \int_{\mathbb{R}^n} u\,(-\Delta v)$$

Remark: If T is symmetric $\Rightarrow \mathcal{D}(T) \subseteq \mathcal{D}(T^*)$ and

$$Tu = T^*u \quad \forall u \in \mathcal{D}(T)$$

$\Rightarrow (T^*, \mathcal{D}(T^*))$ is an extension of $(T, \mathcal{D}(T))$.

Lemma I.1: *A symmetric operator T is closable.*

Proof: It suffice to show that for $u_n \in \mathcal{D}(T)$ with $u_n \to 0$ and $Tu_n \to x \in H$ we have $x = 0$

$$\langle x, v\rangle \leftarrow \langle Tu_n, v\rangle = \langle u_n, Tv\rangle \to \langle 0, Tu\rangle = 0 \quad \forall v \in \mathcal{D}(T)$$

$\Rightarrow x = 0.$ $\square$

Remark: The proof actually shows that if $\mathcal{D}(T^*)$ is dense in H, then T is closable.

Definition I.5: *We call an operator T **self-adjoint** if*

$$T = T^* \text{ and } Tu = T^*u \quad \forall u \in \mathcal{D}(T),$$

note that the first property implies that $\mathcal{D}(T) = \mathcal{D}(T^*)$.

Theorem I.4: *Every self-adjoint operator is closable.*

Proof: Lemma I.1 $\square$

Theorem I.5: *Let T be an invertible self-adjoint operator, then T^{-1} is also self-adjoint.*

Proof: For $T\colon \mathcal{D}(T) \to \mathcal{R}(T)$ consider

Step 1 $\mathcal{R}(T)$ is dense in H. We have to show that $\mathcal{R}(T)^{\perp} = \{0\}$.
Let $w \in H$ such that

$$\langle Tu, w\rangle = 0 \ \forall u \in \mathcal{D}(T)$$

$\Longrightarrow w \in \mathcal{D}(T^*)$ and $T^*w = 0 \overset{inj.}{\underset{s.a.}{\Longrightarrow}} w = 0.$

Step 2 Let $w\colon H \times H \to H \times H$, $w(x,y) = (y,x)$

$$\Longrightarrow G\left(T^{-1}\right) = \left\{ \left(x, T^{-1}x\right) : x \in \mathcal{D}(T) \right\} = w\left(G(T)\right) = \left\{ (Ty, y) : y \in \mathcal{D}(T) \right\}$$

$$\begin{aligned} G\left(T^{-1}\right) = G\left((T^*)^{-1}\right) &\overset{Proof}{\underset{Thm.I.2}{=}} w\left(V\left(G(T)^{\perp}\right)\right) \\ &= V\left(w\left(G(T)\right)^{\perp}\right) = V\left(w\left(G(T)\right)\right)^{\perp} \\ &= V\left(G\left(T^{-1}\right)\right)^{\perp} \overset{Thm.}{\underset{I.2}{=}} G\left(\left(T^{-1}\right)^{*}\right) \end{aligned}$$

$\Rightarrow T^{-1} = \left(T^{-1}\right)^{*}$

$\square$

Chapter II

Representation Theorems

Theorem II.1 (Riesz)**:** *Let $u \mapsto F(u)$ be a linear continuous function on H. Then $\exists! w \in H$:*

$$F(u) = \langle u, w \rangle \quad \forall u \in H$$

Lax-Milgram: V Hilbertspace, sesquilinear form is defined on $V \times V$, $(u,v) \mapsto \alpha(u,v)$ continuous with

$$|\alpha(u,v)| \leq c\|u\|\|v\| \quad \forall u, v \in V$$

Riesz: $\exists$ linear map $A\colon V \to V$:

$$\alpha(u,v) = \langle Au, v \rangle$$

Definition II.1: *A bilinear form $a\colon V \times V \to \mathbb{R}$ is V**-coercive** if there exists $\lambda > 0$ such that*

$$a\,(u,u) \geq \lambda \|u\|^2 \quad \forall u \in V$$

Theorem II.2: *Let a be a continuous sesquilinear and V-coercive on $V \times V$ then A is an isomorphism.*

Proof:

Step 1: A is injective:

$$\|Au\|\|u\| \overset{C.S.}{\geq} |\langle Au, u \rangle| = |a(u,u)| \geq \lambda \|u\|^2 \tag{+}$$

$\Rightarrow \|Au\| \geq \lambda \|u\|$ for all $u \in V$.

Step 2: $A(V)$ is dense in V. Let $u \in V$ such that

$$\langle Au, v \rangle = 0 \quad \forall v \in V$$

take $v = u \Rightarrow a(u,u) = 0 \Rightarrow u = 0$.

Step 3: $\mathbb{R}(A) = A(V)$ is closed. Let v_n be a sequence in $A(V)$ and let u_n be such that

$$Au_n = v_n$$

$$\overset{(+)}{\Longrightarrow} u_n \text{ is a Cauchy sequence} \Rightarrow u_n \to u \in V \text{ und } Au_n \to Au \Rightarrow v_n \to Au \in A(V)$$

Step 4: $u = A^{-1}v \overset{(+)}{\Longrightarrow} \|A^{-1}v\| \leq \lambda^{-1}\|v\| \ \forall v \in V$.

$\square$

Next we consider two Hilbert spaces V, H with $V \subset H$ (the inclusion is continuous), i.e.

$$\exists c < \infty\colon \quad \|u\|_H \leq c\|u\|_V \quad \forall u \in V$$

and we assume that V is dense in H.

Example: $V = W^{1,2}(\mathbb{R}^n)$, $H = L^2(\mathbb{R}^n)$

$$\|u\|_L^2 \leq \|u\|_{W^{1,2}}$$

There exists a natural injection from H into V'. Let $h \in H$ then $V \ni u \mapsto \langle u, h\rangle_H$ is continuous on $V \xRightarrow[\text{Thm. II.1}]{} \exists l_h \in V'$:

$$l_h(u) = \langle u, h\rangle_H \quad \forall u \in V$$

injectivity follows from density of V in H. $V \subseteq H \subset V'$ cont. sesquilinear form a on $V \times V$ which is V-coercive $\to$ Associate an unbounded operator S with a

$$\mathcal{D}(S) := \Big\{u \in V\colon a(u, v) \text{ is cont. on } V \text{ with respect to the topology induced by } H\Big\}$$

skipped a part! Unreadable

Theorem II.3: *Let a be a continuous sesquilinear form on V which is V-coercive then S is bijective from $\mathcal{D}(S)$ into H and $S^{-1} \in L(H, \mathcal{D}(S))$. Moreover, $\mathcal{D}(S)$ is dense in H.*

Proof:

1) S injective: $\exists \alpha > 0$:

$$\alpha\|u\|_H^2 \leq C\alpha\|u\|_V^2 \leq C\,|a(u,u)| = c\,|\langle Su, u\rangle_H| \leq c\|Su\|_H\|u\|_H, \quad \forall u \in \mathcal{D}(S)$$

$\Rightarrow \alpha\|u\|_H \leq c\|Su\|_H,\ \forall u \in \mathcal{D}(S)\ \ (+)$.

2) S surjective:
   Let $h \in H$. Choose $w \in V$ such that

$$\langle h, v\rangle_H = \overline{\langle v, h\rangle_H} = \overline{l_h(v)} = \langle w, v\rangle \quad \forall v \in V$$

   where we used Riesz' representation theorem in the last step.

$$\text{(Note: } l_h \in V' \Rightarrow \overline{l_h} \in \text{continuous lineare form on } V\text{).}$$

is tis correct? Hard to read!

   Define $u \coloneqq A^{-1}w \in V \Rightarrow a(u, v) = \langle Au, v\rangle_V = \langle w, v\rangle_V = \langle h, v\rangle_H$

$$\Rightarrow u \in \mathcal{D}\left(S\right), \; Su = h$$

   (V dense in H). (+) implies that S^{-1} is continuous.

3) Density of $\mathcal{D}\left(S\right)$:
   Let $h \in H$ such that $\langle u, h\rangle_H = 0 \; \forall u \in \mathcal{D}\left(S\right)$. S surjective $\exists v \in DOS$: $Sv = h$

$$\Rightarrow \langle Sv, u\rangle = 0 \; \forall u \in \mathcal{D}\left(S\right)$$

$$\Rightarrow \langle Sv, v\rangle_H = 0 \Rightarrow a(v, v) = 0 \Rightarrow v = 0 \Rightarrow h = 0$$

   a hermitian iff

$$a(u, v) = \overline{a(v, u)} \quad \forall u, v \in V$$

□

Theorem II.4: *Under the assumptions of Theorem II.3 and a being hermitian it follows that*

a) S *is closed*

b) $S = S^*$

c) $\mathcal{D}\left(S\right)$ *dense in* V

Proof:

a) Theorem I.4

b) a hermitian

$$\Rightarrow \langle Su, v\rangle_H = a(u,v) = \overline{a(v,u)} = \overline{\langle Sv, u\rangle_H} = \langle u, Sv\rangle_H \quad \forall u, v \in \mathcal{D}(S)$$

$\Rightarrow S$ symmetric $\Rightarrow \mathcal{D}(S) \subset \mathcal{D}(S^*)$. Let $v \in \mathcal{D}(S^*)$, S surjective

$$\Rightarrow v_0 \in \mathcal{D}(S) : Sv_0 = S^*v.$$

For all $u \in \mathcal{D}(S)$ we get

$$\langle Su, v_0\rangle_H = \langle u, Sv_0\rangle_H = \langle u, S^*v\rangle_H = \langle Su, v\rangle_H$$

$\Rightarrow v = v_0 \Rightarrow \mathcal{D}(S) = \mathcal{D}(S^*)$, $Sv = S^*v \ \forall v \in \mathcal{D}(S)$.

c) follows from Theorem II.3

$\square$

Chapter III

Friedrichs Extension

Definition III.1: *Let T_0 be a symmetric unbounded operator with domain $\mathcal{D}(T_0)$ we say that T_0 is **semi-bounded** if $\exists c > 0$:*

$$\langle T_0 u, u\rangle_H \geq -c\|u\|_H^2 \quad \forall u \in \mathcal{D}(T_0)$$

Example:

a) Schrödinger Operator. $\mathbb{R}^m$, $H = L^2(\mathbb{R}^m)$, $\mathcal{D}(T_0) = C_c^\infty(\mathbb{R}^m)$

$$T_0 \coloneqq -\Delta + V(x),$$

$V \in C_0(\mathbb{R}^m)$ with $V(x) \geq -c\ \forall x \in \mathbb{R}^m$. For $u \in \mathcal{D}(T_0)$

$$\langle T_0 u, u\rangle_H = \int_{\mathbb{R}^m} (\Delta u + Vu)\, u = \underbrace{\int_{\mathbb{R}^m} |\nabla u|^2}_{\geq 0} + \underbrace{\int_{\mathbb{R}^m} V(x)\,|u(x)|^2}_{\geq -c\int |u|^2 = -c\|u\|_H^2}$$

b) $S_z \coloneqq -\Delta - \frac{z}{r}$, whereas $r = |x|, z \in \mathbb{R}$

Hardy inequality in $\mathbb{R}^3 (m = 3)$:

$$\int_{\mathbb{R}^3} |x|^{-2}|u(x)|^2 dx \leq 4\int_{\mathbb{R}^3} |\nabla u|^2 (x) dx \quad \forall u \in C_c^\infty(\mathbb{R}^m)$$

Proof: $\int_{\mathbb{R}^3} \left|\nabla u + \frac{1}{2}\frac{x}{|x|^2}u\right|^2 dx \geq 0$

$$\iff \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{4}\frac{|u|^2}{|x|^2} dx \geq -\int_{\mathbb{R}^3} \langle \nabla u(x), \frac{x}{|x|}\rangle u(x) dx$$

now

$$-2\int_{\mathbb{R}^3} \langle \nabla u, \frac{|x|^2}{|x|^2}\rangle u dx = -\int_{\mathbb{R}^3} \langle \nabla |u|^2, \frac{x}{|x|^2} dx = \int_{\mathbb{R}^3} |u|^2 \underbrace{\operatorname{div} \frac{x}{|x|^2}}_{=\frac{1}{|x|^2}} = \int_{\mathbb{R}^3} \frac{|u|^2}{|x|^2} dx$$

is tis correct? Hard to read!

$\Rightarrow \int |\nabla u|^2 \geq \frac{1}{4} \int \frac{|u|^2}{|x|^2}$. Now

$$\int_{\mathbb{R}^3} \frac{1}{r} |u(x)|^2 dx \leq \left(\int \frac{|u(x)|^2}{r^2} dx \right)^{\frac{1}{2}} \cdot \|u\|_L^2$$

$\int_{\mathbb{R}^3} \frac{1}{r^2} |u(x)|^2 dx \leq 4 \langle -\Delta u, u \rangle_L^2$

$$\Rightarrow \forall \epsilon > 0 : \quad \int_{\mathbb{R}^3} \frac{1}{r} |u(x)|^2 dx \leq \epsilon \cdot \langle -\Delta u, u \rangle_L^2 + \frac{1}{\epsilon} \|u\|_{L^2}^2$$

hence

$$\langle S_z u, u \rangle_{L^2} = \langle -\Delta u, u \rangle_{L^2} - z \langle \frac{u}{r}, u \rangle_{L^2} \geq (1 - \epsilon) \langle -\Delta u, u \rangle_{L^2} - \frac{z}{\epsilon} \|u\|_{L^2}^2$$

Choose $\epsilon = \frac{1}{z} \Rightarrow \langle S_z u, u \rangle_{L^2} \geq -z^2 \|u\|_{L^2}^2$ $\square$

Theorem III.1: *A symmetric semibounded operator T_0 on H with dense domain $\mathcal{D}(T_0)$ admits a self-adjoint extension, called **Friedrichs extension**.*

Proof: Replace T_0 by $T_0 + \lambda \mathbb{1}$ such that

$$\langle T_0 u, u \rangle_H \geq \|u\|_H^2 \quad \forall u \in \mathcal{D}(T_0)$$

$(u, v) \mapsto a_0(u, v) := \langle T_0 u, v \rangle_H$ on $\mathcal{D}(T_0) \times \mathcal{D}(T_0)$

$$\Rightarrow a_0(u, u) \geq \|u\|_H^2$$

Let V be the completion in H of $\mathcal{D}(T_0)$ for the norm $u \mapsto \rho_0(u) = \sqrt{a_0(u, u)} \iff u \in H$ belongs to V if $\exists u_n \in \mathcal{D}(T_0)$ s.t. $u_n \to u$ in H and $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to ρ_0

"Candidate Norm":

$$\|u\|_V = \lim_{n \to \infty} \rho_0(u_n)$$

where u_n is as above. $\square$

Lemma III.1: *Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{D}(T_0)$ with respect to ρ_0 such that $x_n \to 0$ in H. Then $p_0(x_n) \to 0$.*

Proof: Observe that $p_0(x_n)$ is a Cauchy sequence in $\mathbb{R}_+$, and hence, converges in $\overline{\mathbb{R}_+}$. Assume that $p_0(x_n) \to \alpha > 0$. Now $a_0(x_n, x_m) = a_0(x_n, x_n) + a_0(x_n, x_m - x_n)$

$$|a_0(x_n, x_m - x_n)| \leq \sqrt{a_0(x_n, x_n)}\sqrt{a_0(x_m - x_n, x_m - x_n)}$$

$\forall \epsilon > 0\ \exists N\ \forall n, m \geq N$:

$$\left|a_0(x_n, x_m) - \alpha^2\right| < \epsilon$$

$\epsilon = \frac{\alpha^2}{2} \Rightarrow |a_0(x_n, x_m)| = |\langle T_0 x_n, x_m\rangle| \geq \frac{1}{2}\alpha^2 > 0\ \forall n, m \geq N$. Let $m \to \infty \Rightarrow x_m \to 0$, which leads to the contradiction. $\square$

page 13 hard to read AND understand structure! skipped that completely

Theorem III.2 (Example: Dirichlet Realisation)**:** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ smooth, T_1 is defined by:*

$$\mathcal{D}(T_1) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),\ T_1 = -\Delta\colon \mathcal{D}(T_1) \to L^2(\Omega)$$

T_1 is self-adjoint; T_1 is called the Dirichlet realisation of $-\Delta$.

Proof: (with gaps)
Define

$$\mathcal{D}(T_0) := C_c^\infty(\Omega),\ T_0 := -\Delta,\ H = L^2(\Omega)$$

Through integration by parts respectively Green's formula:

$$T_0 \text{ is symmetric, non-negative (i.e. semi-bounded)}$$

Consider $\tilde{T_0} := T_0 + \mathbb{1}_H$. V: closure of $C_c^\infty(\Omega)$. in $W^{2,2}(\Omega)$

$$\xrightarrow[\substack{Friedrich\\ ext.}]{S} \mathcal{D}(S) = \left\{u \in W_0^{12}(\Omega) \mid -\Delta u \in L^2(\Omega)\right\}$$

$$\xrightarrow[Theory]{Regularity} \mathcal{D}(S) = W^{2,2} \cap W_0^{1,2}(\Omega)$$

for more details about the regularity theory see "2nd order elliptic operators", PDE Evans. $\square$

Example:

a) Harmonic oscillator
Define $\mathcal{D}(T_0) :- C_c^\infty(\mathbb{R}^n)$, $T_0 := -\Delta + |x|^2 + 1$, $= L^2(\mathbb{R}^2)$. Let V be the completion of C_c^∞ in L^2 with respect to the norm

$$u \mapsto \left(\langle \nabla u, \nabla u\rangle_{L^2} + \langle |x|u(x), |x|u(x)\rangle_{L^2} + \langle u, u\rangle_{L^2}\right)^{\frac{1}{2}} = \sqrt{\langle T_0 u, u\rangle_{L^2}}.$$

By calculation:

$$V = \left\{ u \in W^{1,2}(\mathbb{R}^n) \mid x_j u \in L^2(\mathbb{R}^n)\ \forall j = 1, \ldots, n \right\}$$

Domain of S:

$$\mathcal{D}(S) = \left\{ u \in V \mid T \circ u \in L^2(\mathbb{R}^n) \right\} = \left\{ u \in W^{2,2}(\mathbb{R}^n) \colon x^\alpha u \in L^2(\mathbb{R}^n)\ \forall \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq 2 \right\}$$

b) Schrödinger operator with a Coulomb potential
Define $\mathcal{D}(T_0) := C_c^\infty(\mathbb{R}^3)$, $T_0 = -\Delta - \frac{1}{r}$, $H = L^2(\mathbb{R}^3)$. We saw that T_0 is semi-bounded: $\langle T_0 u, u\rangle_{L^2} \geq -\|u\|_{L^2}^2$

$$\tilde{T}_0 := T_0 + 2 \cdot \mathbb{1}_H,\ \mathcal{D}\left(\tilde{T}_0\right) = \mathcal{D}(T_0)$$

satisfies the assumptions of the Friedrichs extension. Completion V of C_c^∞ in L^2 with respect to the norm

$$u \mapsto \left(\langle \nabla u, \nabla u\rangle_{L^2} + \int_{\mathbb{R}^3} \left(2 - \frac{1}{r}\right) |u(x)|^2 dx \right)^{\frac{1}{2}} = \sqrt{\langle \tilde{T}_0 u, u\rangle_{L^2}}$$

is $V = W^{1,2}(\mathbb{R}^3)$.

Proof: $C_c^\infty(\mathbb{R}^3)$ is dense in $W^{1,2}(\mathbb{R}^3)$. Therefore, we only need to check that the norm above and $\|\cdot\|_{W^{1,2}}$ are equivalent. By the proof of the analysis of the Schrödinger operator:

$$\int_{\mathbb{R}^3} \frac{1}{r}|u(x)|^2 dx \leq \epsilon\langle -\Delta u, u\rangle_{L^2} + \frac{1}{\epsilon}\|u\|_{L^2}^2 = \epsilon\|\nabla u\|_{L^2}^2 + \frac{1}{\epsilon}\|u\|_{L^2}^2 \quad \forall \epsilon > 0,\ u \in C_c^\infty(\mathbb{R}^3)$$

(See Bernhard Helfer, "Spectral theory and app.").

$$\|w\|_{W^{1,2}}^2 = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \overset{Hardy}{\leq} 5\|\nabla u\|_{L^2}^2 + \|\left(2 - \frac{1}{r}\right) u\|_{L^2}^2 \quad \forall u \in C_c^\infty(\mathbb{R}^3)$$

Domination of S: Hardy inequality $\Rightarrow \frac{1}{r}u \in L^2$ for $u \in W^{1,2}(\mathbb{R}^3)$.

$$\Rightarrow u \in D(S): \ \Delta u \in L^2(\mathbb{R}^3) \Rightarrow \mathcal{D}\left(S\right) = W^{2,2}(\mathbb{R}^3)$$

□

c) Neumann boundary conditions: on the half-plane $H = L^2\left((0,\infty)\right)$ define the form

$$a(u,v) := \int_0^\infty u'(x)v'(x)dx$$

for $u, v \in \mathcal{D}\left(a\right) = W^{1,2}(0,\infty) \Rightarrow a(u,u) = \|u'\|_{L^2}^2 \geq -\|u\|_{L^2}^2$. a is closed by completeness of $W^{1,2}(0,\infty)$.

Associated operator T: $v \in \mathcal{D}\left(T\right)\ \exists f_v \in L^2(0,\infty)$:

$$\int_0^\infty u'(x)v'(x)dx = \int_0^\infty u(x)f_v(x)dx \quad \forall u \in W^{1,2}(0,\infty).$$

$\Rightarrow f_v = -\left(v'\right)' = -v''$, therefore $v \in W^{2,2}(0,\infty), Tv = -v''$. Note for $v \in W^{2,2}(0,\infty)$, $u \in W^{1,2}(0,\infty)$:

$$\begin{aligned} a(u,v) &= \int_0^\infty u'(x)v'(x)dx \\ &= \left[u(x)v'(x)\right]_0^\infty - \int_0^\infty u(x)v''(x)dx \\ &= \underbrace{u(0)v'(0)}_{=0} + \int_0^\infty u(x)Tv(x)dx = \langle u, Tv\rangle_{L^2} \end{aligned}$$

Therefore the associated operator is $T_N := T$ acts as $T_N v = -v''$ on the domain

$$\mathcal{D}\left(T_N\right) = \left\{v \in W^{2,2}(0,\infty) \mid v'(0) = 0\right\}$$

T_N is called the Neumann Laplacian.

Chapter IV

Spectrum and Resolvent

Let X be a Banach space and H a Hilbert space.

Definition IV.1: *Let $T \colon \mathcal{D}(T) \subseteq\rightarrow X$ linear operator. We define*

- *We call the following set the **resolvent set**:*

$$\operatorname{res}(T) \coloneqq \rho(T) \coloneqq \Big\{\lambda \in \mathbb{C} \mid \lambda \mathbb{1} - T \textit{ is bijective with bounded inverse}\Big\}.$$

- *The set* $\operatorname{spec}(T) \coloneqq \sigma(T) \coloneqq \mathbb{C} \setminus \rho(T)$ *is called **spectrum**.*
- *The set* $\operatorname{spec}_p(T) \coloneqq \sigma_p(T) \coloneqq \Big\{\textit{Eigenvalues of } T\Big\}$ *is the **point spectrum**.*
- *The following set is called the **continuous spectrum**:* $\operatorname{spec}_c(T) \coloneqq \sigma_c(T)$

$$\sigma_c(T) \coloneqq \Big\{\lambda \in \mathbb{C} \mid \lambda \mathbb{1} - T \textit{ is inj., but not surj., } \operatorname{range}(\lambda \mathbb{1} - T) \textit{ is dense in } X\Big\}$$

- *The following set is called the **residual spectrum**:* $\operatorname{spec}_{res}(T) \coloneqq \sigma_{res}(T)$

$$\sigma_{res}(T) \coloneqq \Big\{\lambda \in \mathbb{C} \mid \lambda \mathbb{1} - T \textit{ is inj., but not surj., } \operatorname{range}(\lambda \mathbb{1} - T) \textit{ is not dense in } X\Big\}$$

- *The **resolvent function**:* $R_T \colon \rho(T) \to L(X, X) =: L(X)$

$$\lambda \mapsto R_T(\lambda) \coloneqq R(\lambda, T) \coloneqq (\lambda \mathbb{1} - T)^{-1}$$

Remarks:

- $\dim(X) < \infty : \sigma(T) = \sigma_p(T)$
- $\sigma(T) = \sigma_p(T) \,\dot\cup\, \sigma_c(T) \,\dot\cup\, \sigma_{res}(T)$

Theorem IV.1: *If $\rho(T) \neq \emptyset$ then T is closed.*

Proof: $\lambda \in \rho(T)$ then graph$(R(\lambda, T))$ is closed (by the closed graph theorem). For $x \in \mathcal{D}(T), y \in X$ with $R(\lambda, T)y = x$:

$$\|x\|_{\lambda \mathbb{1} - T} = \|(\lambda - T)x\|_X + \|x\|_X = \|y\|_X + \|R(\lambda, T)y\|_X = \|y\|_{R(\lambda, T)}.$$

Therefore, graph$(\lambda\mathbb{1} - T)$ and graph$(R(\lambda, T))$ are isometric, and so $\lambda\mathbb{1} - T$ is closed

$$\Rightarrow T \text{ is closed}$$

$\square$

Theorem IV.2: *For a closed operator T one has the equivalence*

$$\lambda \in \rho(T) \iff \begin{cases} \text{kern}(\lambda\mathbb{1} - T) = 0, & \text{“\textit{inj.}”} \\ \text{range}(\lambda\mathbb{1} - T) = X, & \text{“\textit{surj.}”} \end{cases}$$

Proof:

“$\Rightarrow$” By definition.

“$\Leftarrow$” Let $\lambda \in C$ with kern$(\lambda\mathbb{1} - T) = 0$, range$(\lambda\mathbb{1} - T) = X$. Then the inverse

$$(\lambda\mathbb{1} - T)^{-1} : X \to X$$

is defined everywhere and has a closed graph (as $\lambda\mathbb{1} - T$ is closed), see proof of Theorem IV. 1. By the closed graph theorem $(\lambda\mathbb{1} - T)^{-1}$ is bounded, i.e. $\lambda \in \rho(T)$.

$\square$

Theorem IV.3 (Properties of the resolvent)**:**

(i) *For $\lambda_0 \in \rho(T)$, $\lambda \in \mathbb{C}$ with $|\lambda_0 - \lambda| < \frac{1}{\|R(\lambda_0, T)\|_{L(X)}}$, we have*

$$R(\lambda, T) = \sum_{n=0}^{\infty} R(\lambda_0, T)^{n+1} (\lambda_0 - \lambda)^n$$

and $\lambda \in \rho(T)$, i.e. $R(\cdot, T)$ is locally holomorphic, $\rho(T)$ is open, $\sigma(T)$ is closed.

(ii) Resolvent equation

$$R(\lambda,T) - R(\mu,T) = (\mu-\lambda)\, R(\lambda,T)R(\mu,T) \quad \forall \lambda,\mu \in \rho(T)$$

Note $\frac{1}{\lambda-T} - \frac{1}{\mu-T} = \frac{(\mu-T)-(\lambda-T)}{(\lambda-T)(\mu-T)} = \frac{(\mu-\lambda)}{(\lambda-T)(\mu-T)}$

(iii) $R(\lambda,T)R(\mu,T) = R(\mu,T)R(\lambda,T)\ \forall \mu,\lambda \in \rho(T)$.

(iv) $\frac{\partial}{\partial\lambda} R(\lambda,T) = -R(\lambda,T)^2\ \forall \lambda \in \rho(T)$

(v) $TR(\lambda,T) = R(\lambda,T)T\ \forall \lambda \in \rho(T)$

Proof:

(i) $\lambda - T = (\lambda-\lambda_0) + (\lambda_0 - T) = (\lambda_0 - T)\left[\mathbb{1}_X - \frac{(\lambda_0-\lambda)}{(\lambda-\lambda_0)} R(\lambda_0,T)\right]$.
Define $S \coloneqq (\lambda_0-\lambda)\, R(\lambda_0,T) \in L(X)$ with $\|S\|_{L(X)} < 1$. By Neumann series:

$$R(\lambda,T) = (\mathbb{1}_X - S)^{-1}\, R(\lambda_0,T) = \left(\sum_{n=0}^{\infty} S^n\right) R(\lambda_0,T) = \sum_{n=0}^{\infty} R(\lambda_0,T)^{n+1}\,(\lambda_0-\lambda)^n$$

and $\lambda \in \rho(T)$.

(ii)
$$\begin{aligned} R(\lambda,T) - R(\mu,T) &= R(\lambda,T)\,[\mathbb{1}_X - (\lambda-T)\,R(\mu,T)] \\ &= R(\lambda,T)\,[(\mu-T)-(\mu-T)]\,R(\mu,T) \\ &= (\mu-\lambda)R(\lambda,T)R(\mu,T) \end{aligned}$$

(iii) $R(\lambda,T)R(\mu,T) \overset{(ii)}{=} \frac{R(\lambda,T)-R(\mu,T)}{\mu-\lambda} = \frac{R(\mu,T)-R(\lambda,T)}{\lambda-\mu} \overset{(ii)}{=} R(\mu,T)R(\lambda,T)$

(iv) $\lim_{\mu\to\lambda} \frac{R(\mu;T)-R(\lambda,T)}{\mu-\lambda} \overset{(ii)}{=} \lim_{\mu\to\lambda} \frac{(\lambda-\mu)R(\mu,T)R(\lambda,T)}{(\mu-\lambda)} = -R(\lambda,T)^2$

(v)
$$\begin{aligned} TR(\lambda,T) - R(\lambda,T)T &= (T - R(\lambda,T)T(\lambda-T))\,R(\lambda,T) \\ &= R(\lambda,T)\underbrace{((\lambda-T)\,T - T\,(\lambda-T))}_{=0} R(\lambda,T) = 0 \end{aligned}$$

□

Next, we look at some basic examples:

Definition IV.2 (essential range, dt. wesentlicher Bildbereich)**:** *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space* $f \colon \Omega \to \mathbb{C}$ *measurable. Define the essential range of* f *as:*

$$\operatorname{essrange}(f) := \Big\{\lambda \in \mathbb{C} \mid \mu\left(\{x \in \Omega \colon\ |\lambda - f(x)| < \epsilon\}\right) > 0\ \forall \epsilon > 0\Big\}$$

Theorem IV.4 (Example: Spectrum of the multiplication operator)**:** *Let* $f \in L^\infty_{loc}(\mathbb{R}^n)$*,* M_d *be the multiplication operator in* $L^2(\mathbb{R}^n)$*, i.e.*

$$\mathcal{D}\left(M_f\right) = \Big\{u \in L^2(\mathbb{R}^n) : fu \in L^2(\mathbb{R}^n)\Big\},\ M_f u = fu$$

Then there holds:

(i) $\sigma(M_f) = \operatorname{essrange}(f)$

(ii) $\sigma_p(M_f) = \Big\{\lambda \in \mathbb{C} \mid \mathcal{L}^d\left(\{x \mid f(x) = \lambda\}\right) > 0\Big\}$

Proof:

(i) **"⊆"** $\lambda \notin \operatorname{essrange}(f)$, i.e. $\exists c > 0$ s.t. $|\lambda - f(x)| \geq c$ for a.e. $x \in \mathbb{R}^n$.

$$\Rightarrow g := \frac{1}{\lambda - f} \in L^\infty(\mathbb{R}^n) \Rightarrow gu \in L^2\ \forall u \in L^2$$

$\Rightarrow M_g \in L(L^2)$ with

$$\begin{aligned} M_g(\lambda - M_f)u(x) &= \frac{1}{\mu - f(x)}\left(\lambda - f(x)\right)u(x) \\ &= u(x) = \left(\lambda - M_f\right) M_g u(x) \quad \forall u \in \mathcal{D}\left(M_f\right) \end{aligned}$$

i.e. $\lambda \in \rho(M_f)$ with $R(\lambda, M_f) = M_g$.

"⊇" $A \in \operatorname{essrange}(f)$, we denote for any $m \in \mathbb{N}$:

$$\tilde{S_m} := \Big\{x \in \mathbb{R}^n \colon\ |\lambda - f(x)| < 2^{-m}\Big\}.$$

Choose $S_m \subseteq \tilde{S_m}$ s.t. $\mathcal{L}^d(S_m) \in (0, \infty)$; define

$$\phi_m(x) := \begin{cases} 1, & x \in S_m \\ 0, & x \notin S_m \end{cases}$$

Then

$$\|(\lambda - M_f)\,\phi_m\|_{L^2}^2 = \int_{S_m} |\lambda - f(x)|^2\,|\phi_m(x)|^2\,dx$$
$$\leq 2^{-2m}\|\phi_m\|_{L^2}^2 \quad \forall m \in \mathbb{N}$$

$\Rightarrow$ If $\lambda \in \rho(M_f)$, but then

$$\left\|(\lambda - M_f)^{-1}\right\|_{L(L^2)} \geq \frac{\left\|(\lambda - M_f)^{-1}\,(\lambda - M_f)\,\phi_m\right\|_{L^2}}{\|(\lambda - M_f)\,\phi_m\|_{L^2}} \geq 2^{2m} \quad \forall m \in \mathbb{N}$$

$\Rightarrow \lambda \in \sigma(M_f)$.

(ii) "$\Rightarrow$" $\lambda \in \sigma_p(M_f) \iff \exists \phi \in L^2 \setminus \{0\}$ s.t. $(\lambda - f(x))\,\phi(x) = 0$ for a.e. x, i.e. $\phi = 0$ a.e. on $\{y \in \mathbb{R}^d \mid f(y) \neq \lambda\}$, and therefore

$$\sigma_p(M_f) = \emptyset \iff L^d\left(\{x \mid f(x) = \lambda\}\right) = 0$$

"$\Leftarrow$" $\lambda \in \mathbb{C}$ s.t. $\mathcal{L}^d\left(\{x \mid f(x) = \lambda\}\right) > 0$. Take $S \subseteq \{x \mid f(x) = \lambda\}$ with $\mathcal{L}^d(s) \in (0, \infty)$, define

$$\phi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

Then:

$$f(x)\phi(x) = \lambda\phi(x) \text{ for a.e. } x \in \mathbb{R}^n$$

$\Rightarrow \phi$ is Eigenfunction to the Eigenvalue $\lambda \Rightarrow \lambda \in \sigma_p(M_f)$.

□

Example:

a) $(\sigma(T) = \emptyset)$ Setting: $H = L^2(0,1)$, $\mathcal{D}\left(T\right) = \{f \in W^{1,2}(0,1) \mid f(0) = 0\}$, $Tf = f'$. Claim: $\sigma(T) = \emptyset, \sigma(T) = \mathbb{C}$.

Proof: For $g \in L^2(0,1)$, $\lambda \in \mathbb{C}$ the equation $(\lambda - T)f = g$ has the unique solution

$$f(x) = -\int_0^x e^{-\lambda(x-t)}g(t)dt,\ x \in (0,1), f \in \mathcal{D}\left(T\right)$$

since $f'(x) = \lambda \int_0^x e^{\lambda(x-t)} g(t)dt - g(t) = \lambda f(x) - g(x)$. For $\tilde{f} \in \mathcal{D}(T)$ with $(\lambda - T)\tilde{f} = g$ we get

$$\lambda(f - \tilde{f}) = \left(f - \tilde{f}\right)' \iff f - \tilde{f} = ce^{\lambda\cdot} \xRightarrow{\tilde{f}(0)=f(0)} c = 0 \Rightarrow f = \tilde{f}$$

So $\lambda \in \rho(T)$, since

$$\begin{aligned} \|f\|_{L^2}^2 &= \int_0^1 \left| - \int_0^x e^{-\lambda(x-t)} g(t)dt \right|^2 dx \\ &\leq \int_0^1 e^{2|\lambda|x} \left| \int_0^1 |g(t)|dt \right|^2 dx \overset{\text{Hölder}}{\leq} \underbrace{\int_0^1 e^{2|\lambda|x} dx}_{<\infty} \|g\|_{L^2}^2 \end{aligned}$$

$\Rightarrow \rho(T) = \mathbb{C} \Rightarrow \sigma(T) = \emptyset$. □

b) $(\sigma(T) = \mathbb{C})$ Setting: $H = L^2(0,1)$, $\mathcal{D}(T) = W^{1,2}(0,1)$, $Tf = f'$.
Claim: $\sigma(T) = \mathbb{C}, \sigma(T) = \emptyset$.

Proof: For $\lambda \in \mathbb{C}$: $\phi_\lambda(x) \coloneqq e^{\lambda x}$, $x \in \mathbb{R}$. $\phi_\lambda \in \mathcal{D}(T)$ with $T\phi_\lambda = \lambda\phi_\lambda$.

$$\Rightarrow \sigma_p(T) = \mathbb{C} \Rightarrow \sigma(T) = \mathbb{C},\ \rho(T) = \emptyset$$

□

Following are going to be some basic facts on the spectra of self-adjoint operators.

Lemma IV.1: *Let T be a closable operator in a Hilbert space H, $\lambda \in \mathbb{C}$, then*

a) $\operatorname{kern}\left(\overline{\lambda} - T^*\right) = \operatorname{range}(\lambda - T)^\perp$

b) $\overline{\operatorname{range}(\lambda - T)} = \operatorname{kern}(\overline{\lambda} - T^*)^\perp$

Proof:

a) $\mathcal{D}(T)$ is dense in H:

$$\begin{aligned} f \in \operatorname{kern}(\overline{\lambda} - T^*) &\iff \langle g, \left(\overline{\lambda} - T^*\right) f\rangle_H = 0 \quad \forall g \in \mathcal{D}(T) \\ &\iff \langle g, T^* f\rangle_H = \lambda\langle g, f\rangle \quad \forall g \in \mathcal{D}(T) \\ &\iff \langle Tgf\rangle_H = \langle \lambda g, f\rangle \quad \forall g \in \mathcal{D}(T) \\ &\iff \langle (\lambda - T)\, g, f\rangle = 0 \quad \forall g \in \mathcal{D}(T) \\ &\iff f \in \operatorname{range}(\lambda - T)^\perp \end{aligned}$$

b) $\operatorname{kern}\left(\overline{\lambda} - T^*\right)^{\perp} = \left(\operatorname{range}(\lambda - T)^{\perp}\right)^{\perp} = \overline{\operatorname{range}(\lambda - T)}$

$\square$

Theorem IV.5 (Spectrum of self-adjoint operator is real)**:** *thm:iv.5] Let T be a self-adjoint operator on H, then $\sigma(T) \subseteq \mathbb{R}$ and for $\lambda \in \mathbb{C} \setminus \mathbb{R}$:*

$$\|R(\lambda, T)\|_{L(H)} \leq \frac{1}{|\operatorname{Im}(\lambda)|}$$

Proof: Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $u \in \mathcal{D}(T)$:

$$\langle(\lambda - T)u, u\rangle = \lambda\langle u, u\rangle - \langle Tu, u\rangle = \operatorname{Re}(\lambda)\langle u, u\rangle + i\operatorname{Im}(\lambda)\langle u, u\rangle - \langle Tu, u\rangle,$$

$\langle Tu, u\rangle = \langle u, T^* u\rangle \overset{T \ s.a.}{=} \overline{\langle u, Tu\rangle} \Rightarrow \langle Tu, u\rangle \in \mathbb{R}$

$$\Rightarrow |\operatorname{Im}(\lambda)| \, \|u\|_H^2 \leq |\langle(\lambda - T)\, u, u\rangle| \overset{C.S.}{\leq} \|(\lambda - T)\, u\|_H \, \|u\|_H$$

$$\Rightarrow |\operatorname{Im}(\lambda)| \leq \|(\lambda - T)\, u\|_H\,, \tag{$*$}$$

$\operatorname{kern}(\lambda - T) = \{0\}$ and $\operatorname{range}(\lambda - T)$ is closed, since for $u \in \overline{\operatorname{range}(\lambda - T)}$, $(u_n)_n \subseteq \mathcal{D}(T)$ s.t. $(\lambda - T)u_n \to u$ in H.

$$\Rightarrow \left((\lambda - T)u_n\right)_n \text{ Cauchy in } H \overset{(*)}{\Longrightarrow} (u_n)_n \text{ Cauchy in } H$$

$$\xRightarrow[\text{complete}]{H} \exists u \in H \text{ s.t. } u_n \to u \text{ in } H \xRightarrow[\text{closed}]{\lambda - T} u \in \mathcal{D}(T) \, \textit{with} \, (\lambda - T)\, u = v.$$

$$\Rightarrow v \in \operatorname{range}(\lambda - T)$$

$$\xRightarrow{\text{Lem.IV.1(ii)}} \operatorname{range}(\lambda - T) = \{0\}^{\perp} = H$$

$\Rightarrow \lambda \in \rho(T), R(\lambda, T) \in L(H)$ with

$$\|R(\lambda, T)\|_{L(H)} = \sup_{u \neq 0} \frac{\|R(\lambda, T)u\|_H}{\|\,(\lambda - T)\, R(\lambda, T)u\|_H} \overset{(*)}{\leq} \sup_{u \neq 0} \frac{\|R(\lambda, T)u\|}{|\operatorname{Im}(\lambda)| \, \|R(\lambda, T)u\|} = \frac{1}{|\operatorname{Im}(\lambda)|}$$

$\square$

Lemma IV.2 (Spectrum of bounded operators)**:** *Let $T \in L(X)$ then:*

$$\emptyset \neq \sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|_{L(X)}\},$$

i.e. $\sigma(T)$ is compact.

Proof: See a FA course. □

Theorem IV.6 (Location of spectrum of self-adjoint bounded operators)**:** *Let $T \in L(H)$ be self-adjoint. Denote*

$$m := m(T) := \inf_{u \neq 0} \frac{\langle Tu, u\rangle}{\langle u, u\rangle}, \;\; M := M(T) := \sup_{u \neq 0} \frac{\langle Tu, u\rangle}{\langle u, u\rangle}$$

Then $\sigma(T) \subseteq [m, M], \{m, M\} \subseteq \sigma(T)$.

Proof: By Theorem IV.5: $\sigma(T) \subseteq \mathbb{R}$. For $\lambda \in (M, \infty)$:

$$|\langle (\lambda - T)\, u, u\rangle| = \lambda \langle u, u\rangle - \langle Tu, u\rangle \geq (\lambda - M)\, \langle u, u\rangle$$

$\Rightarrow (\lambda - T)^{-1} \in L(H)$ by Lax-Milgram Theorem, $\lambda \in \rho(T)$. For $\lambda \in (-\infty, m)$: same way

$$\Rightarrow \sigma(T) \subseteq [m, M].$$

$M \in \sigma(T)$: obtain $(u, v) \mapsto \langle (M - T)u, v\rangle_H$ is an inner product. For $u, v \in H$ by C.S.-inequality

$$|\langle (M - T)\, u, v\rangle|^2 \leq \langle (M - T)\, u, z\rangle \langle (M - T)\, v, v\rangle$$

$$\Rightarrow \|\, (M - T)\, u\|_H^2 = \left| \sup_{\|v\| \leq 1} \langle (M - T)\, u, v\rangle \right| \langle (M - T)\, u, u\rangle = \|M - T\|_{L(H)}\, \langle (M - T)\, u, u\rangle$$

By assumption construct $(u_n)_n \subseteq H$ s.t. $\|u_n\| = 1$.

$$\langle Tu_n, u_n\rangle \to M$$

$\Rightarrow (M - T)\, u_n \to 0 \Rightarrow M \notin \rho(T) \Rightarrow M \in \sigma(T)$. For m in the same way. □

Lemma IV.3: $T = T^* \in L(H, H$ *and* $\operatorname{spec}(T) = \{0\} \Rightarrow 0$.

Proof: Theorem IV.6 $\Rightarrow m = M = 0 \Rightarrow \langle Tx, x\rangle = 0\ \forall x \in H$

$$\xRightarrow[Forumla]{What?} \langle Tx, y\rangle = 0\ \forall x, y \in H \Rightarrow T = 0$$

□

Theorem IV.7: *The spectrum of a self-adjoint operator on a Hilbert space is a non-empty closed subset of $\mathbb{R}$.*

Proof: Assume $\operatorname{spec}(T) = \emptyset \Rightarrow T^{-1} \in L(H, H)$. Let $\lambda \in \mathbb{C} \setminus \{0\}$.

$$L_\lambda := -\frac{T}{\lambda}\left(ZT - \frac{1}{\lambda}\right)^{-1} = -\frac{1}{\lambda} - \frac{1}{\lambda^2}\left(T - \frac{1}{\lambda}\right)^{-1} \in L(H, H), \;\; L_\lambda^{-1} = \left(T^{-1} - \lambda\right)$$

$\Rightarrow \lambda \in \rho(T^{-1}) \Rightarrow \operatorname{spec}(T^{-1}) = \{0\}$. T^{-1} is self-adjoint by Theorem I.5 $\Rightarrow T^{-1} = 0$, which yields a contradiction. □

Chapter V

The Spectral Theorem

Let T be a self-adjoint operator on a Hilbert space H.

Statements:

1) T is unitarily equivalent to a multiplication operator on a suitable L^2-Space (Finite dimension: every self-adjoint $n \times n$-matrix is unitarily equivalent to a diagonal matrix, i.e. a multiplication operator on $L^2(\{1, \dots, n\})$.

2) There exists a functional calculus for T, i.e. for all bounded Borell-functions f, $f(T)$ is defined and $f \mapsto f(T)$ is a homeomorphismus of the algebra of Borell-functions

$$f\colon X \to Y,\ X, Y \text{cup?}\ VS \text{ s.t. } f^{-1}(U) \in B(X)\ \forall U \text{ open}$$

$$\left(\text{Finite dimensions: } A = U \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} U^{-1},\ f(A) := U \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix} U^{-1}\right)$$

3) T has a spectral representation. There exists a projection?-valued measure P_T on $\sigma(T)$, s.t.

$$T = \int_{\sigma(T)} \lambda dP_T(\lambda)$$

$$\text{(Finite dimension: } A = \sum_{j=1}^{n} \lambda_j P_j,\ P_j \text{ projection onto Eigenspace)}$$

How to define $f(T)$? Polynomials: $f(x) = \sum_{j=0}^{n} c_j x^j$

$$f(T) = \sum_{j=0}^{n} c_j T^j,\ D(f(t)) =?$$

Analytic $f(x) = \sum_{j=0}^{\infty} \frac{f^j(x_0)}{j!}(x - x_0)^j,\ f(T) = \sum_{j=0}^{\infty} \frac{f^j(x_0)}{j!}(T - x_0)^j$ for $\|T - x_0\| < \rho$

$$f(x) = (x - z)^{-1},\ z \in \mathbb{C} \setminus \mathbb{R} \quad f(T) = (T - z)^{-1} = \mathrm{Res}_T(z)$$

Cauchy integral Formula: Let f be holomorph.

$$f(x_0) = \frac{i}{2\pi} \int_\Gamma f(z) \left(x_0 - z\right)^{-1} dz = \frac{1}{2\pi i} \int_\Gamma f(z) \left(z - x_0\right)^{-1} dz$$

$w = f(z) \left(z_0 - z\right)^{-1} dz$, $\mathbb{C} = \mathbb{R}^2$

$$z = x + iy, \overline{z} = x - iy$$

$$dz = dx + idy, d\overline{z} = dx - idy$$

$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right)$, $\partial_{\overline{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right)$

$$\begin{aligned} dw &= \frac{\partial w}{\partial z} \underbrace{dz \wedge dz}_{=0} + \frac{\partial w}{\partial \overline{z}} d\overline{z} \wedge dz \\ &= \left(\partial_{\overline{z}} f(z)\right) \left(z_0 - z\right)^{-1} 2i dx \wedge dy \end{aligned}$$

$M \subset \mathbb{C}$ compact with smooth boundary Γ and $B_\delta(z_0) \subseteq \mathring{M} \hookrightarrow M \setminus B_\delta(z_0)$

$$\int_{\partial\left(\Gamma \setminus B_{\delta(z_0)}\right)} w = \int_{\partial \Gamma} w - \int_{\partial B_\delta(z_0)} w = \int_{\Gamma \setminus B_\delta(z_0)} dw$$

If f is continuous at z_0, then

$$\begin{aligned} \lim_{\delta \to 0} \int_{\partial B_\delta(z_0)} w &= \lim_{\delta \to 0} \int_{\partial B_\delta(z_0)} \frac{f(z)}{z_0 - z} \\ &= \lim_{\delta \to 0} \int_0^{2\pi} \frac{f(z_0 + \delta e^{it})}{-\delta e^{it}} i\delta e^{it} dt \\ &= -2\pi i f(z_0) \end{aligned}$$

$$\lim_{\delta \to 0} \int_{\Gamma \setminus B_{\delta(z_0)}} \frac{\partial f}{\partial \overline{z}} \left(z_0 - z\right)^{-1} 2i dx \wedge dy = 2i \int_\Gamma \frac{\partial f}{\partial \overline{z}} \left(z_0 - z\right)^{-1} dx \wedge dy$$

$$\Rightarrow f(z_0) = -\frac{1}{2\pi i} \int_{\partial \Gamma} w + \frac{1}{\pi} \int_\Gamma \frac{\partial f}{\partial \overline{z}}(z) \left(z_0 - z\right)^{-1} dx \wedge dy$$

If $f = 0$ on $\partial \Gamma$ then

$$f(z_0) = \frac{1}{\pi} \int_\Gamma \frac{\partial f}{\partial \overline{z}}(z) \left(z_0 - z\right)^{-1} dx \wedge dy$$

($f \colon \mathbb{C} \to \mathbb{C}$ for as $f \colon \mathbb{R} \to \mathbb{C}$). Now $\Gamma = \mathbb{C}$

$$f(T) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{2f}{2\overline{z}}(z) \left(T - z\right)^{-1} dx \wedge dy$$

However, there 2 Problems:

1) maybe $f\colon \mathbb{R} \to \mathbb{C}$

2) $\operatorname{spec}(T) \subset \mathbb{C}$

Definition V.1: *Let* $\tau \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{spec}(T(\tau)) \subset [-2,2]$, $\tau|_{[-1,1]} \equiv 1$.

$$\sigma(x,y) \coloneqq \tau\left(\frac{y}{\langle x\rangle}\right),\ \langle x\rangle \coloneqq \left(1+x^2\right)^{\frac{1}{2}}$$

For $f \in C^\infty(\mathbb{R})$ *and* $n \geq 1$ *we define the* n*-th* ***almost analytic extension*** $\tilde{f}_n\colon \mathbb{C} \to \mathbb{C}$ *by*

$$\tilde{f}_n(z)\colon \sigma(x,y)\sum_{j=0}^{n}\frac{f^j(x)}{j!}(iy)^j$$

Ruh:

$$\frac{\partial \tilde{f}}{\partial \overline{z}}(z) = \frac{1}{2}\left(\partial_x - \frac{1}{i}\partial_y\right)\tilde{f}_n(x,y) \tag{V.1}$$

$$= \frac{1}{2}\left(\sum_{j=0}^{n}\frac{f^{(j+1)}(x)}{j!}(iy)^j - \sum_{j=1}^{n}\frac{f^{(j)}(x)}{(j-1)!}(iy)^{i-1}\right)\sigma(x,y) + \frac{1}{2}\sum \ldots (\partial_x\sigma + i\partial_y\sigma) \tag{V.2}$$

$$= \frac{1}{2}\frac{f^{(n+1)}(x)}{n!}(iy)^n\sigma(x,y) + \ldots \tag{V.3}$$

$\sigma \equiv 1$ *on a strip of size* $\langle x\rangle$ *around* $\mathbb{R}$

$$\Rightarrow \left|\frac{\partial \tilde{f}_n(z)}{\partial \overline{z}}\right| = \mathcal{O}\left(|y|^n\right) \text{ for } y \to 0$$

Definition V.2: *Let* $T = T^*$ *be a map which associates to every element* $f\colon \mathbb{R} \to \mathbb{C}$ *of a subalgebra* $\mathcal{E}$ *of the Borelfunctions* $\mathcal{B}(\mathbb{R})$ *an operator* $f(T) \in L(H,H)$ *is called functional calculus for* T *if*

i) $f \mapsto f(T)$ *is an algebra homomorphism*

$$(f+\alpha g)(T) = f(T) + \alpha g(T),\ (f\cdot g)(T) = f(T)\cdot g(T) \quad \forall f,g \in \mathcal{E}$$

ii) $f(T)^* = \overline{f}(T)$

iii) $\|f(T)\| \leq \|f\|_{L^\infty}$

iv) *For* $z \in \mathbb{C} \setminus \mathbb{R}$ *and* $r_z(x) = (x-z)^{-1}$ *is*

$$r_z(T) = \mathrm{Res}_T(z)$$

v) *If* $f \in C_c^\infty(\mathbb{R})$ *vandities on* $\mathrm{spec}(T)$, *i.e.*

$$\mathrm{sp}\, T(f) \cap \mathrm{spec}(T) = \emptyset, \textit{then } f(t) = 0$$

Definition V.3: *For* $\beta \in \mathbb{R}$ *let*

$$S^\beta \coloneqq \left\{ f \in C^\infty(\mathbb{R}) :\ \forall n \in \mathbb{N}_0\ \exists c_n < \infty\ s.t.\ \left| f^{(n)}(x) \right| \le c_n \langle x \rangle^{\beta - n}\ \forall x \in \mathbb{R} \right\}$$

$\mathcal{A} \coloneqq \bigcup_{\beta<0} S^\beta$

$$\|f\|_n \coloneqq \sum_{j=0}^{n} \int_{-\infty}^{\infty} \underbrace{\left| f^{(j)}(x) \right| \langle x \rangle^{j-1}}_{\sim \langle x \rangle^{\beta-1}}$$

well-defined

Addendum

Theorem (Riesz' representation theorem, FA 17.2)**:** Let H be a Hilbert space, and let H' denote its dual space, consisting of all continuous linear functionals from H into the field ($\mathbb{C}$ or $\mathbb{R}$). For every element of $x' \in X'$there exists a unique $x \in X$ such that

$$x'(y) = \langle y, x \rangle,$$

for all $y \in X$, and $\|x'\|_{X'} = \|x\|_X$.

Theorem (Closed graph theorem, FA 12.6)**:** thm:acgt] If X and Y are Banach spaces, and $T\colon X \to Y$ is a linear operator, then T is continuous if and only if its graph is closed in $X \times Y$, with respect to the product topology.

Definition (Isometric)**:** Let X and Y be metric spaces with metrics d_X and d_Y. A map $f\colon X \to Y$ is called an isometry or distance preserving if for any $a, b \in X$ one has

$$d_Y\left(f(a), f(b)\right) = d_X\left(a, b\right).$$

X and Y are called isometric if there is a bijective isometry from X to Y.

Stichwortverzeichnis