Solving bihomogeneous polynomial systems with a 0-dimensional projection

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joint work with Laurent Busé, Carles Checa[1], and Elias Tsigaridas

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[1]Thanks for some slides!

Joint work with L. Busé, C. Checa, E. Tsigaridas.

- "Solving bihomogeneous polynomial systems with a zero-dimensional projection" ISSAC 2025 [arXiv:2502.07048]

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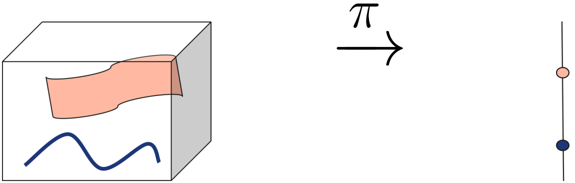

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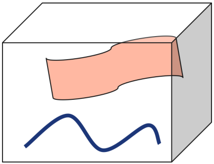

$\xrightarrow{\pi}$

Result Extension of eigenvalue method and complexity.

Example: Eigenvalue problem

Let $A \in k^{n \times n}$ be a matrix. Find the eigenvalues, i.e. eliminate the **y** variables in the algebraic set

$$V := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{P}^1 \times \mathbb{P}^{n-1} : (x_0 A - x_1 Id) \cdot \boldsymbol{y}^\top = 0\}$$

If an eigenspace is of dimension > 1, this variety is not finite.

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1. Characteristic polynomial of A

Nonlinear algebra

1. Determinantal ideals

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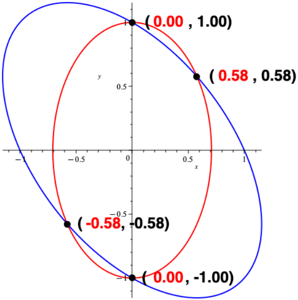
- Gröbner bases
- Linear recurrence of $\{A^i\}_i$ à la FGLM

Eigenvalue methods for solving 0-dimensional systems

$I = \langle 2x^2 + y^2 - 1, x^2 + xy + y^2 - 1 \rangle$

- If $I \subset R = k[x_1, \ldots, x_n]$ is 0-dim,
  R/I is finite dim k-vector space



Eigenvalue methods for solving 0-dimensional systems

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- Linear form $\ell \to$ multiplication map

$$M_\ell : R/I \xrightarrow{\times \ell} R/I$$

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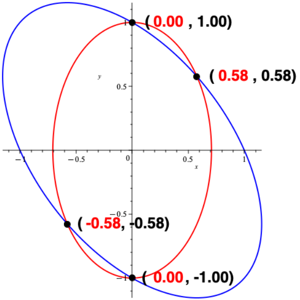

$$M_x = \begin{array}{c|cccc} \times x & 1 & x & y & y^2 \\ \hline 1 & & \frac{1}{2} & \frac{1}{2} & \\ x & 1 & & & \frac{1}{3} \\ y & & & & \\ y^2 & & -\frac{1}{2} & -\frac{1}{2} & \end{array}$$

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- Linear form $\ell \to$ multiplication map

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- Eigenvalues of $M_\ell \to$ points in $V(I)$,

$$det(M_\ell - \lambda id) = \prod_{\xi \in V(I)} (\ell(\xi) - \lambda)^{\mu_\xi}$$

where μ_ξ "multiplicity" of $\xi \in V(I)$.

[Lazard'81], [Auzinger-Stetter'88], [Marinari-Moeller-Mora'96], [Mourrain'02], ...

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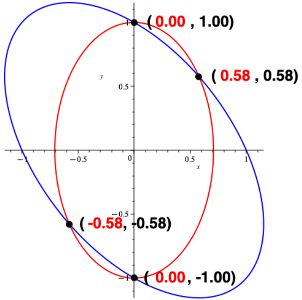

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$$det(M_x - \lambda id) = \lambda^2(\lambda - \tfrac{1}{\sqrt{3}})(\lambda + \tfrac{1}{\sqrt{3}})$$

Eigenvalue methods (projective)

Setting $I \subset R = k[x_0, \ldots, x_n]$ is homogeneous and 0-dimensional.

Hilbert function stabilises, that is, there is $d \gg 0$ st

$$\dim_k((R/I)_d) = \dim_k((R/I)_a), \text{ for any } a \geq d$$

Moreover, $\dim_k((R/I)_d) = \#V(I)$ (counting multiplicities).

Example $I = \langle x^3 + y^3 - 5z^3, 5x^3 - y^3 - 3z^3 \rangle \subset k[x, y, z]$

d	0	1	2	3	4	5	6	$\cdots$
$(R/I)_d$	1	3	6	8	9	9	9	$\cdots$

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Moreover, $\dim_k((R/I)_d) = \#V(I)$ (counting multiplicities).

If f, h are linear forms st $V(h) \cap V(I) = \emptyset$, then, for $a \gg 0$ consider

$$M_{\frac{f}{h},a} : (R/I)_a \xrightarrow{\times f} (R/I)_{a+1} \xrightarrow{(\times h)^{-1}} (R/I)_a$$

The eigenvalues of **multiplication map** are $\frac{f}{h}(\xi)$ for all $\xi \in V(I)$,

$$det\left(M_{\frac{f}{h},a} - \lambda id\right) = \prod_{\xi \in V(I)} \left(\frac{f}{h}(\xi) - \lambda\right)^{\mu_\xi}.$$

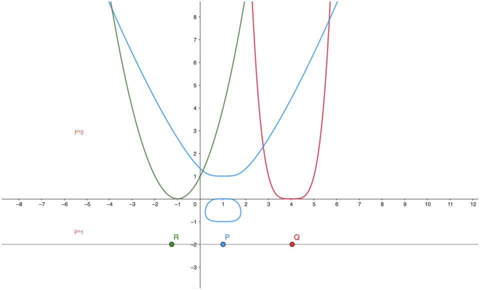
Smallest degree $a \to$ **Castelnuovo-Mumford regularity**

General framework

(Bi)-hom. polynomial systems $I \subset R = k[x_0, \dots, x_n, y_0, \dots, y_n]$ such that $V(I) \subset \mathbb{P}^n \times \mathbb{P}^m$ and $\pi(V(I))$ **is finite** where:

$$\pi_x : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$$

is the projection. **For instance,** over $\mathbb{P}^1 \times \mathbb{P}^2$.

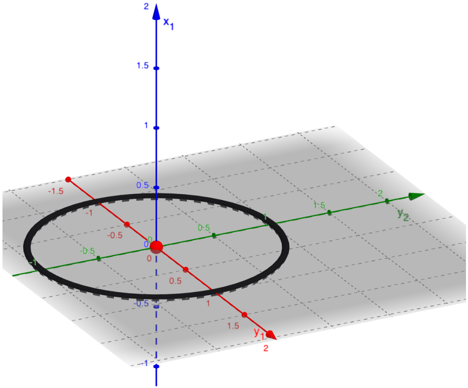


Problem: Find $\pi_x(V(I))$, i.e. $(I : \mathfrak{m}_y^\infty) \cap R_X$.

Example

Consider the bihomogeneous ideal $I \subset k[x_0, x_1] \otimes k[y_0, y_1, y_2]$,

$$I = (x_0 x_1^2, y_0^2 x_1^2, y_1^2 x_1^2, y_2^2 x_1^2, (-y_0^2 + y_1^2 - y_2^2) x_1 x_0 + (\boldsymbol{-y_0^2 + y_1^2 + y_2^2}) x_0^2).$$



If we consider $V(I) \subset \mathbb{P}^1 \times \mathbb{P}^2$, then $\pi_x(V(I)) = \{[1:0]\}$.

Example

$$I=(x_0x_1^2,y_0^2x_1^2,y_1^2x_1^2,y_2^2x_1^2,(-y_0^2+y_1^2-y_2^2)x_1x_0+(\boldsymbol{-y_0^2+y_1^2+y_2^2})x_0^2).$$

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$\dim_k((R/I)_{(i,j)}) =$

(i,j)	0	1	2	3	4	5
0	1	2	3	3	3	3
1	3	6	9	9	9	9
2	6	12	14	13	13	13
3	10	20	18	15	15	15
4	15	30	24	18	18	18
5	21	42	32	22	22	22
6	28	56	41	26	26	26

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For $b \gg 0$, we define multiplication matrices. Given $a \geq r(b)$,

$$M_{\frac{f}{h},a,b} : (R/I)_{a,b} \xrightarrow{\times f} (R/I)_{a+1,b} \xrightarrow{(\times h)^{-1}} (R/I)_{a,b}$$

where h is a generic linear form and f a linear form in $k[x_0, ..., x_n]$.

Admissible degrees $\rightarrow$ degs (a, b) where we can construct $M_{\frac{f}{h},a,b}$.

Geometry: what are the eigenvalues?

If we consider admissible degree $(3,2)$ and consider $h = x_0 + x_1$, the eigenvalues of $M_{\frac{x1}{x_0+x_1},3,2}$ are 0 and 1.

$$\mathrm{CharcPol}(M_{\frac{x1}{x_0+x_1},3,2})(\lambda) = \left(\lambda - \frac{x1}{x_0+x_1}(1,0)\right)^{10}(\lambda - 0)^3$$

The first eigenvalue correspond to the point $[1:0]$, but the second?

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Associated primes of I are

$$(x_0, x_1), (x_1, y_0^2 - y_1^2 - y_2^2), (x_0, y_0, y_1, y_2)$$

Primary decomposition of I contains

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At degree $(3,4)$, no more issues...

$$\mathrm{CharcPol}(M_{\frac{x1}{x_0+x_1},3,4})(\lambda) = \left(\lambda - \frac{x1}{x_0, x_1}(1,0)\right)^{18}$$

Geometry: what are the eigenvalues?

Lemma [B-Buse-Checa-Tsigaridas'25] Given (a, b) admissible degree, define

$$J_b := I : \langle y_0, \ldots, y_m \rangle^b \cap k[x_0, \ldots, x_n] \subset k[x_0, \ldots, x_n].$$

The variety $V(J_b) \in \mathbb{P}^n$ is zero-dimensional and

$$V(J_b) \subseteq V(J_{b+1}) \subseteq V(J_{b+2}) \subseteq \cdots \subseteq V(J_\infty) = \pi_x(V(I))$$

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Module-theoretic interpretation,

We have that $(R/I)_{*,b} := \bigoplus_d (R/I)_{d,b}$ is a $k[x_0, \ldots, x_n]$-module.

$$Ann_{k[x_0,\ldots,x_n]}((R/I)_{*,b}) = J_b \text{ and } V(J_b) = Supp((R/I)_{*,b})$$

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Theorem [B-Buse-Checa-Tsigaridas'25] Given (a, b) admissible degree and h st $V(x_0) \cap V(J_b) = \emptyset$,

$$\text{Eigenvalues}(M_{\frac{f}{h},a,b}) = \left\{ \frac{f}{h}(\alpha) : \alpha \in V(J_b) \subset \mathbb{P}^n \right\}$$

We understand algebraic multiplicity.

Geometry: Minimal polynomials and Gröbner basis

$$\mathrm{MinPol}(M_{\frac{x1}{x_0+x_1},3,4})(\lambda) = \mathrm{MinPol}(M_{\frac{x1}{x_0+x_1},3,5})(\lambda) = \left(\lambda - \frac{x1}{x_0,x_1}(1,0)\right)^2$$

Theorem [B-Buse-Checa-Tsigaridas'25] If (a,b) adm. and $h \in k[x_0,\dots,x_n]_1$ st $V(h)\cap V(J_b)=\emptyset$,

$$Proj(k[x_0,\dots,x_n]/J_b) \simeq Spec\left(k\left[M_{\frac{x_i}{h},a,b} : i \in \{0,\dots,n\}\right]\right)$$

Corollary [B-Buse-Checa-Tsigaridas'25] If $V(x_0)\cap V(J_b)=\emptyset$, we can recover the GBs of dehomogenization of J_b in $k[x_1,\dots,x_n]$, ie $x_0 \to 1$, by applying FGLM on $M_{\frac{x_1}{x_0},a,b},\dots,M_{\frac{x_n}{x_0}a,b}$.

Additional results

Admissible degrees

- Complete characterisation + algorithm to discover them
- Related to multigraded regularity [Chardin & Holanda, '22]

Complexity bounds

- Bounds on admissible degs, no assumptions (better for 0-dim)
- Single exponential algorithm on wrt input.
  (but exponential wrt to output, ie $\#\pi_x(V(I))$)

Criterion to discard extra points in $V(J_b)$

- Given $\xi \in V(J_b)$, we can check if $\xi \in \pi_x(V(I))$.
  (but not its multiplicity)

Relation of our approach and GBs

- Admissible degs related to complexity of GB of I wrt GRevLex.
- However, not GRevLex used to eliminate $\boldsymbol{y}$-variables. (why?)

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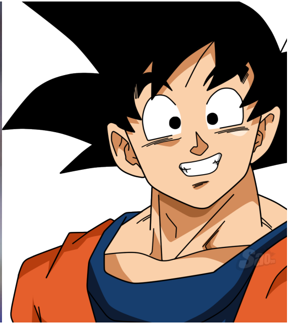

Merci beaucoup

The Macaulay bound

We can construct multiplication matrices via Gröbner bases or using Macaulay matrices.

In either case, degrees involved related to Castelnuovo-Mumford regularity.

Theorem ([Macaulay], [Jouanolou], [Chardin]) If $I \subset R = k[x_0, \dots, x_n]$ is 0-dimensional and generated by m polynomials of degree $\leq d$,

$$reg(I) \leq md - n$$

Corollary At degrees greater than $a \geq md - n$, the eigenvalues of $M_{\frac{f}{h},a}$ are $\frac{f}{h}(\xi)$ for all $\xi \in V(I)$.

Regularity: when can we construct multiplication maps?

Reminder In classical setting $\rightarrow$ Castelnuovo-Mumford regularity

Proposition [B-Buse-Checa-Tsigaridas'25] we can construct multiplication maps at admissible degree (a, b) if we there is (generic) linear h st

- Stabilisation $\dim_k((R/I)_{(a,b)}) = \dim_k((R/I)_{(a+1,b)})$
- Vanishing $\dim_k((R/(I,h))_{(a+1,b)}) = 0$

Moreover, if (a, b) admissible, then $(a+1, b)$ admissible.

Proof based on [Bayer & Stillman, '87].

This is NOT the multigraded regularity

(but regularity implies admissibility [Chardin & Holanda, '22])