# Unconstrained sparse polynomial optimization

**Thémathique:** Algorithms, Complexity, Optimization

**Laboratoire et universitè:** Centre de Mathématiques Appliquées (CMAP), École Polytechnique

**Ville et Pays:** Lozère, France

## Context

The Unconstrained Polynomial Optimization Problem (UPOP) corresponds to optimizing a multivariate polynomial with real coefficients, i.e., $f \in \mathbb{R}[x_1, \ldots, x_n]$ over the real space $\mathbb{R}^n$, i.e. finding

$$\min_{x \in \mathbb{R}^n} f(x).$$

One way of solving this problem is to write our polynomial as a Sum of Squares (SOS), which allows us to solve or approximate a solution of a UPOP but also provides a certificate of feasibility or positivity; for example, for a real number $\lambda$, we have

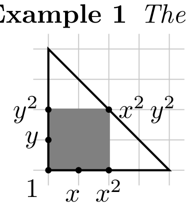
$$\text{If } f - \lambda = \sum_i \sigma_i^2, \text{ for } \sigma_i \in \mathbb{R}[x_1, \ldots, x_n], \qquad \text{then } f(x) > \lambda \text{ for any } x \in \mathbb{R}^n.$$

Even though these certificates do not exist in general [Hil88], they do exist for important cases. In this project, we concentrate in the case where the gradient ideal of $f(x)$, that is the ideal defined by the derivatives of $f$, is radical and $f(x)$ attains its minimum, say at $\lambda$. In this case, it is possible to write $f(x) - \lambda$ as a SOS modulo its gradient ideal $\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle$ [Par02, NDS06], that is, to write it as

$$f(x) - \lambda = \sum_j \sigma_j^2 + \sum \hat{\sigma}_i \frac{\partial f}{\partial x_i} \text{ for } \sigma_j, \hat{\sigma}_i \in \mathbb{R}[x_1, \ldots, x_n]. \tag{1}$$

To compute this representation exactly, that is to only employ polynomials with rational coefficients, the state of the art [MDV23] rely on computations of Gröbner bases (GB) and rational univariate representations (RUR) [Rou99] of the gradient ideal. In practice, the polynomial systems we often encounter exhibit specific structures. Therefore, it is of utmost importance to exploit this structure when dealing with larger inputs. One of the most common examples is sparsity, where polynomials involve only a few monomials

**Example 1** *The following polynomial is positive and sparse. We can visualize its sparsity by looking at its Newton polytope, that is, the convex hull in $\mathbb{R}^2$ of the set of exponents of every monomial appearing in the polynomial. The gray square in the image on the left is the Newton polytope of this polynomial.*




$$1 + x + y + x^2 + y^2 + x^2\,y^2 \in \mathbb{R}[x, y] \tag{2}$$

## Objective

The objective of this project is to improve [MDV23] to exploit the sparsity structure (Newton polytopes) of the input equations. For this, the successful candidate will have to get familiar with the classical tools in GB and RUR, as well as their generalisations for sparse inputs [MST17, BFT19].

**Contact:** Matías Bender (matias.bender@inria.fr) & Elias Tsigaridas (elias.tsigaridas@inria.fr).

**Gratification possible.**

# References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