Routing games

The game is given by a directed graph $G = (V, E)$, a set of edge cost functions $(c_e)_{e\in\mathcal{E}}$, and a finite number of players, indexed by $k \in \{1, \dots, K\}$.

- The cost function of an edge e is a function $c_e : \mathbb{R}_+ \to \mathbb{R}_+$. It determines the cost of the edge given the total flow on that edge.
- A player k is given by: a source node $s_k \in V$, a destination node $d_k \in V$, and a total flow F_k (i.e. the total mass of traffic that this player is allocating). The action set of the player is the paths that connect s_k to d_k, denoted by $\mathcal{P}_k$.
- At iteration, each player k chooses a flow distribution $f_k \in \mathbb{R}_+^{\mathcal{P}_k}$, such that $\sum_{p\in\mathcal{P}_k} f_{k,p} = F_k$. The flow distributions of all players determine the edge flows as follows: for an edge e, the edge flow is

  $$\phi_e = \sum_k \sum_{p\in\mathcal{P}_k : e\in p} f_{k,p}$$

  Another way to write this is

  $$\phi = \sum_k M^k f_k$$

  where M^k is an incidence matrix for player k, defined as follows: $M^k \in \mathbb{R}^{\mathcal{E}\times\mathcal{P}_k}$, such that

  $$M^k_{e,p} = \begin{cases} 1 & \text{if } e \in p \\ 0 & \text{otherwise} \end{cases}$$

  Once we have the edge flows, we can compute the edge costs simply by applying the edge cost functions. Let y be the vector in $\mathbb{R}^E$ defined by

  $$y_e = c_e(\phi_e)$$

  Then we compute the path costs by summing the edge costs along the path. So for all k, and all $p \in \mathcal{P}_k$, the path cost is

  $$\ell^k_p = \sum_{e\in p} y_e$$

  so the path costs for player k can be written simply in terms of the incidence matrix

  $$\ell^k = (M^k)^T y$$

To summarize, when we construct the graph, we need: the node set V, the edge set E, the edge cost functions $c_e, e \in E$, and the player description (s_k, d_k, F_k) for each k. From this, we can compute the paths $\mathcal{P}_k$ and compute the incidence matrices M_k.

Then, at each iteration, each player chooses a path flow distribution f_k, which we use to compute:

1. the edge flows ϕ
2. the edge costs y
3. the path costs ℓ_k for each k

Once this is done we reveal to each player k the path cost vector ℓ_k and we start the next round.