MEI Conference 2019

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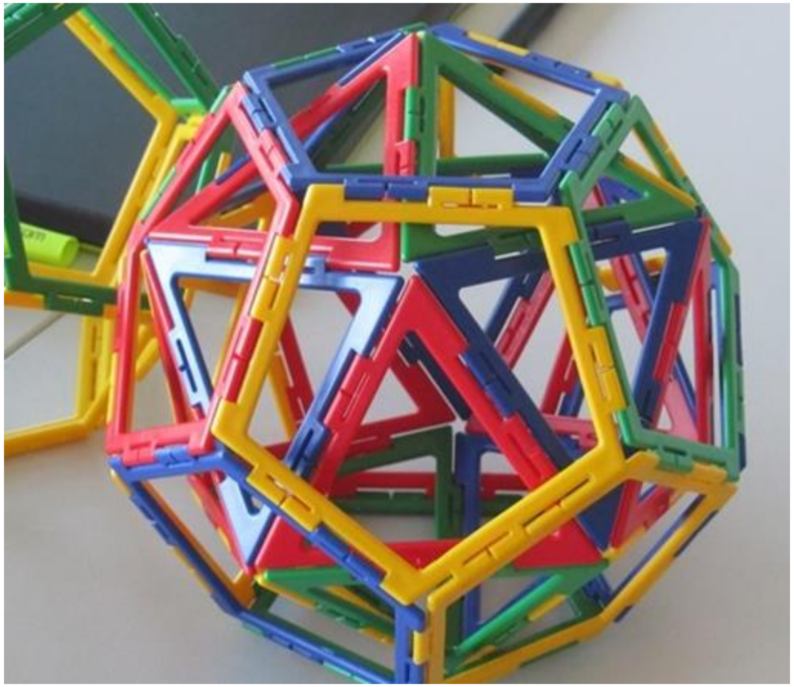


Planarity and Kuratowski’s Theorem

Graph Theory Topic Mapping

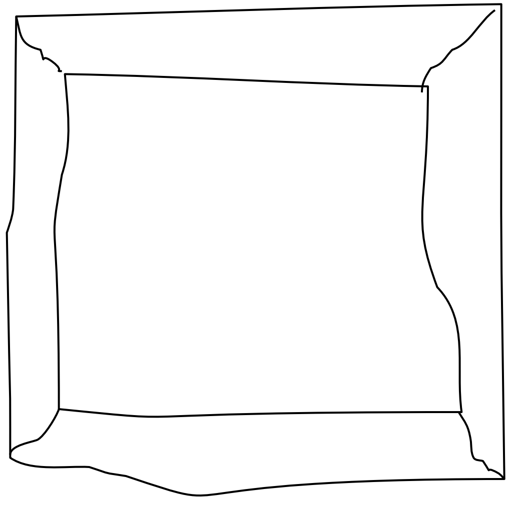
Discrete/Decision Mathematics Topics	AQA	Edexcel	MEI	OCR A
Nodes/vertices, degree/order	**AS**	**AS D1**	**MwA**	**AS**
Arcs/edges, simple, connected, related vocabulary	**AS**	**AS D1**	**MwA**	**AS**
Trees	**AS**		**MwA**	**AS**
Euler's relation: V - E + F = 2	**AS**			A Level
Bipartite graphs, $K_{m,n}$	**AS**		**MwA**	A Level
Walk, trail, path, cycle	**AS**			**AS**
Eulerian, semi-Eulerian graphs	**AS**	**AS D1**		**AS**
Hamiltonian cycles	**AS**	A Level D1		A Level
Complete graphs, K_n	**AS**	**AS D1**		**AS**
Isomorphic Equivalence	A Level	**AS D1**		**AS**
Planar graphs	**AS**	**AS D1**		A Level
Subdivision and contraction	A Level			A Level
Planarity Algorithm		A Level D1		
Kuratowski's Theorem	A Level			A Level
Thickness				A Level
Complement of a graph	A Level			
Ore's theorem				A Level

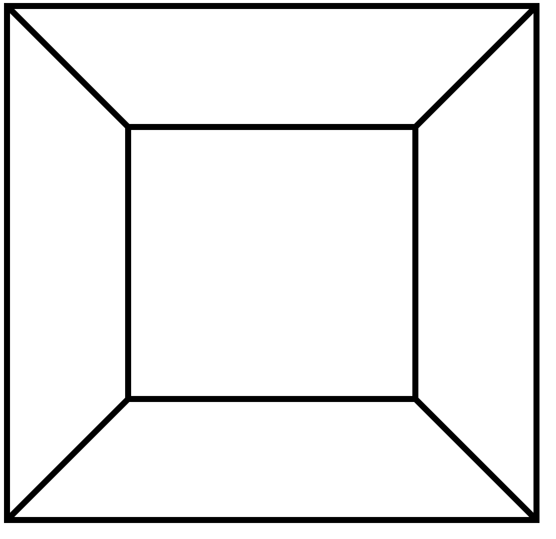
In this session:

- Definitions
- Planarity
- Kuratowski’s Theorem
- Subdivision
- Contraction
- Thickness

Imagineering

What did you see?





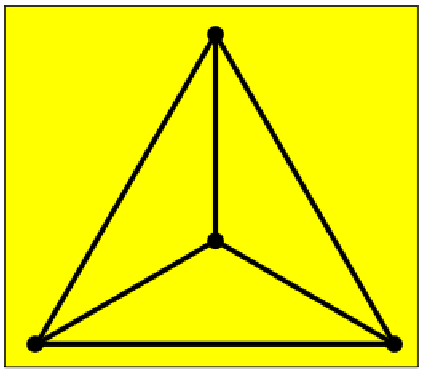
Can you do the same for a triangular prism?

Victor Schlegel (1843-1905)

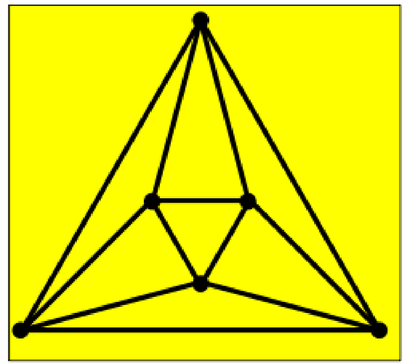
"It is always possible by suitable choice of the centre of projection to make the projection of one face completely contain the projections of all the other faces. This is called a *Schlegel diagram* of the polyhedron."

Duncan Sommerville, writing in 1929, described this approach, which was introduced by Victor Schlegel in 1886.

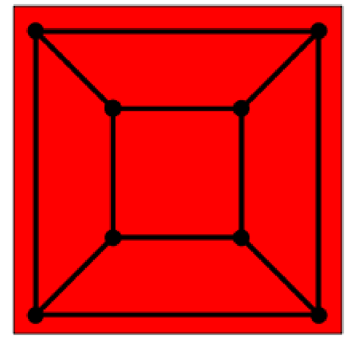
Schlegel Diagrams



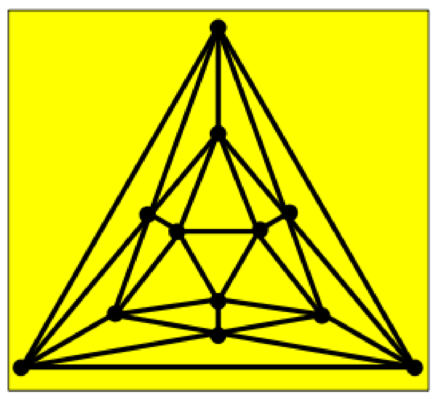
Tetrahedron



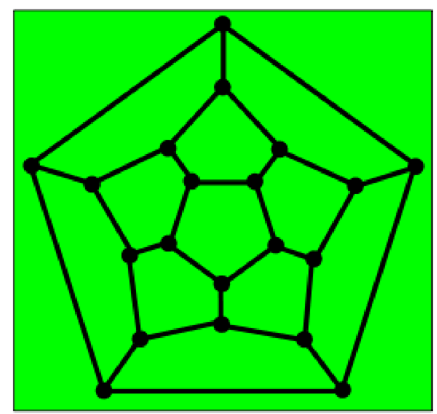
Octahedron



Hexahedron **(a.k.a. Cube!)**



Icosahedron



Dodecahedron

Euler’s Relation:

Name	Faces	Vertices	Edges
Cube	6	8	12
Tetrahedron	4	4	6
Octahedron	8	6	12
Dodecahedron	12	20	30
Icosahedron	20	12	30

Euler's Relation: $F + V = E + 2$

Name	Faces	Vertices	Edges
Cube	6	8	12
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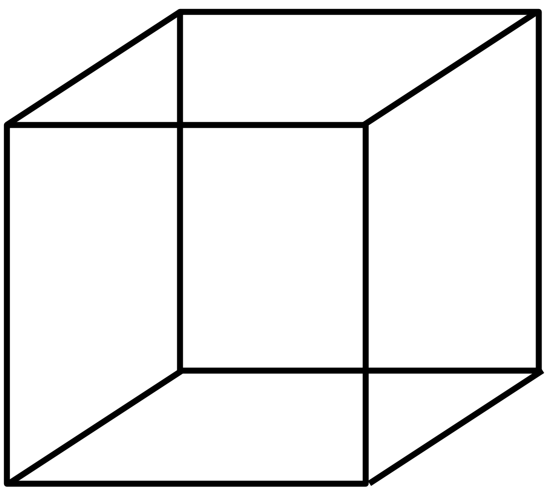
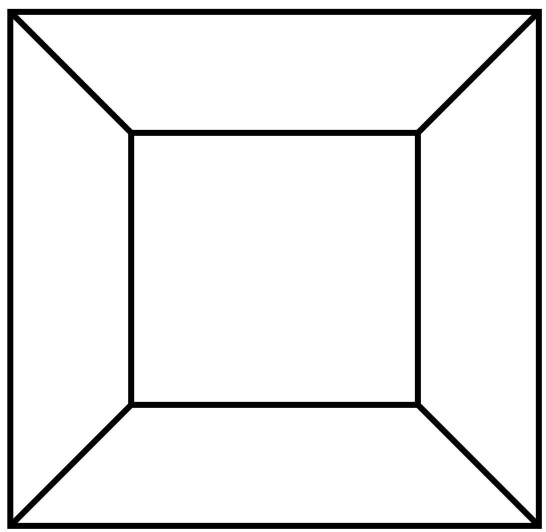
Isomorphic Equivalence

Two graphs are **isomorphically equivalent** if one can be stretched, distorted, or by repositioning the vertices transformed into the other.
(An isomorphism is a one-to-one matching.)





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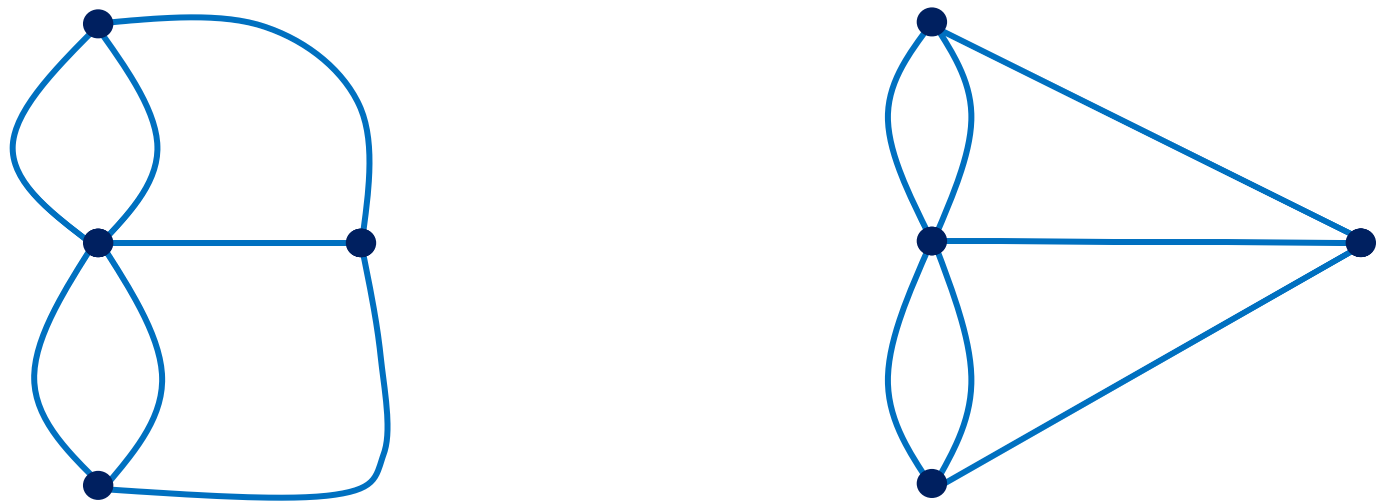

You might recognise this graph from the Bridges of Königsberg problem.

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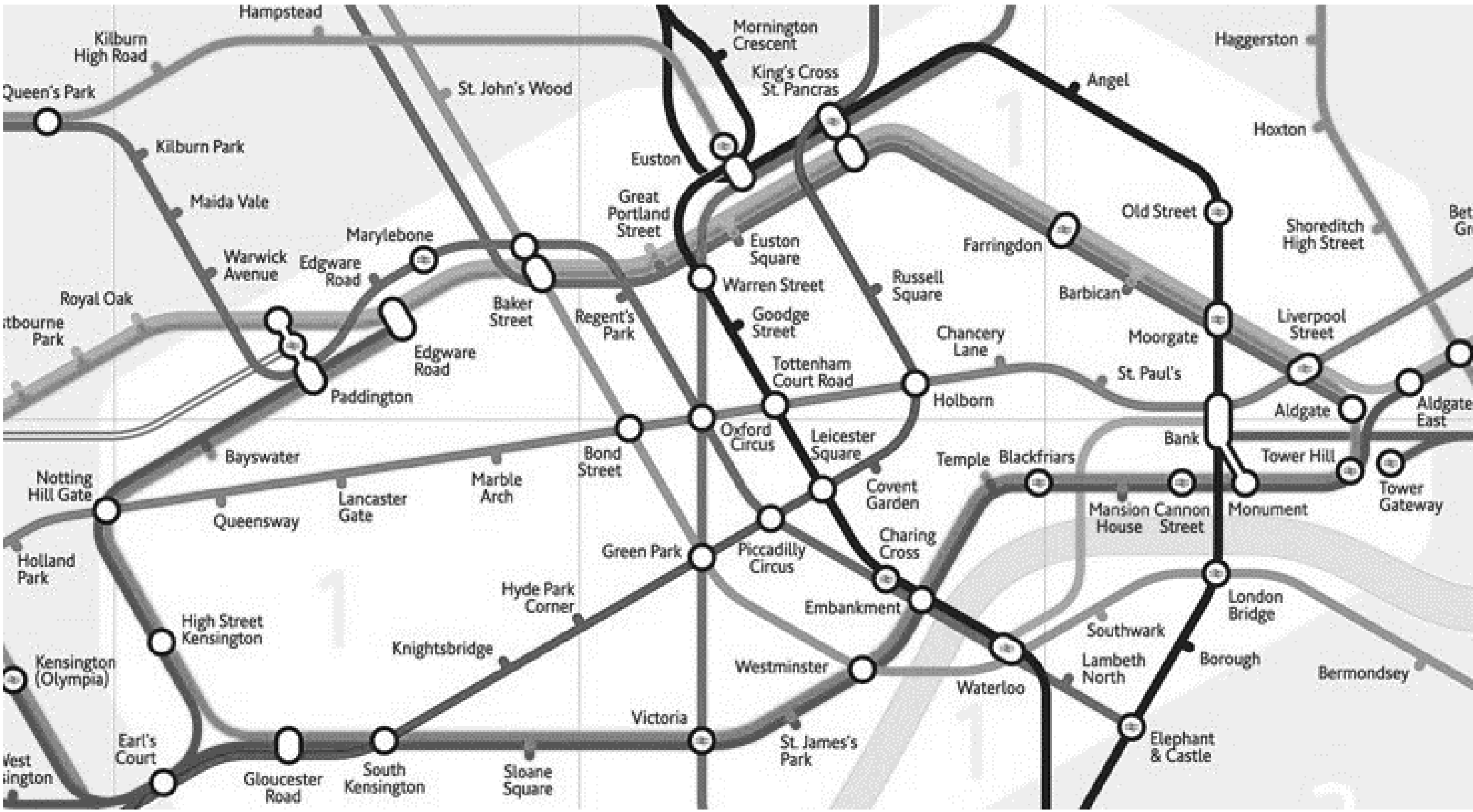
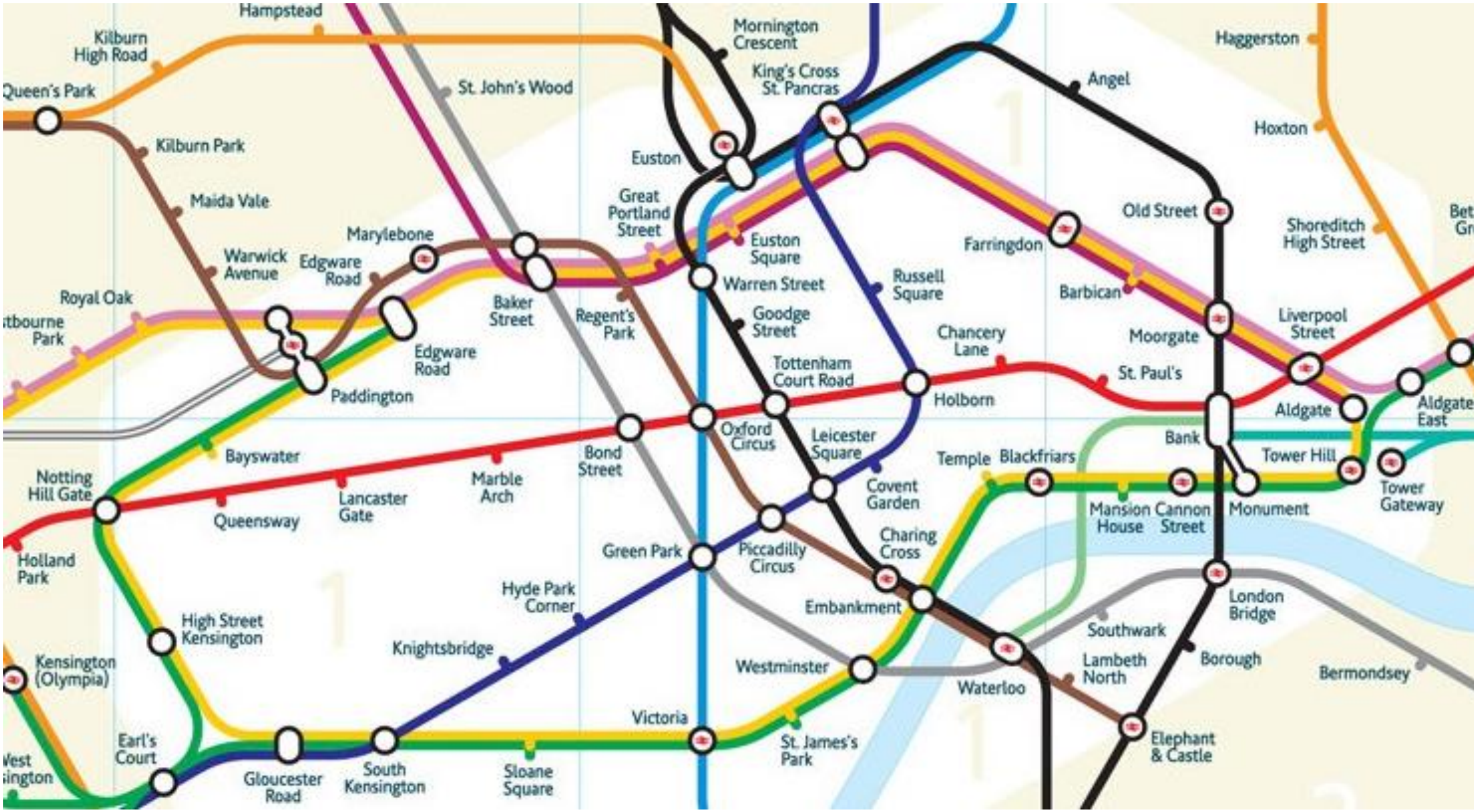
Isomorphic Equivalence



Isomorphic Equivalence



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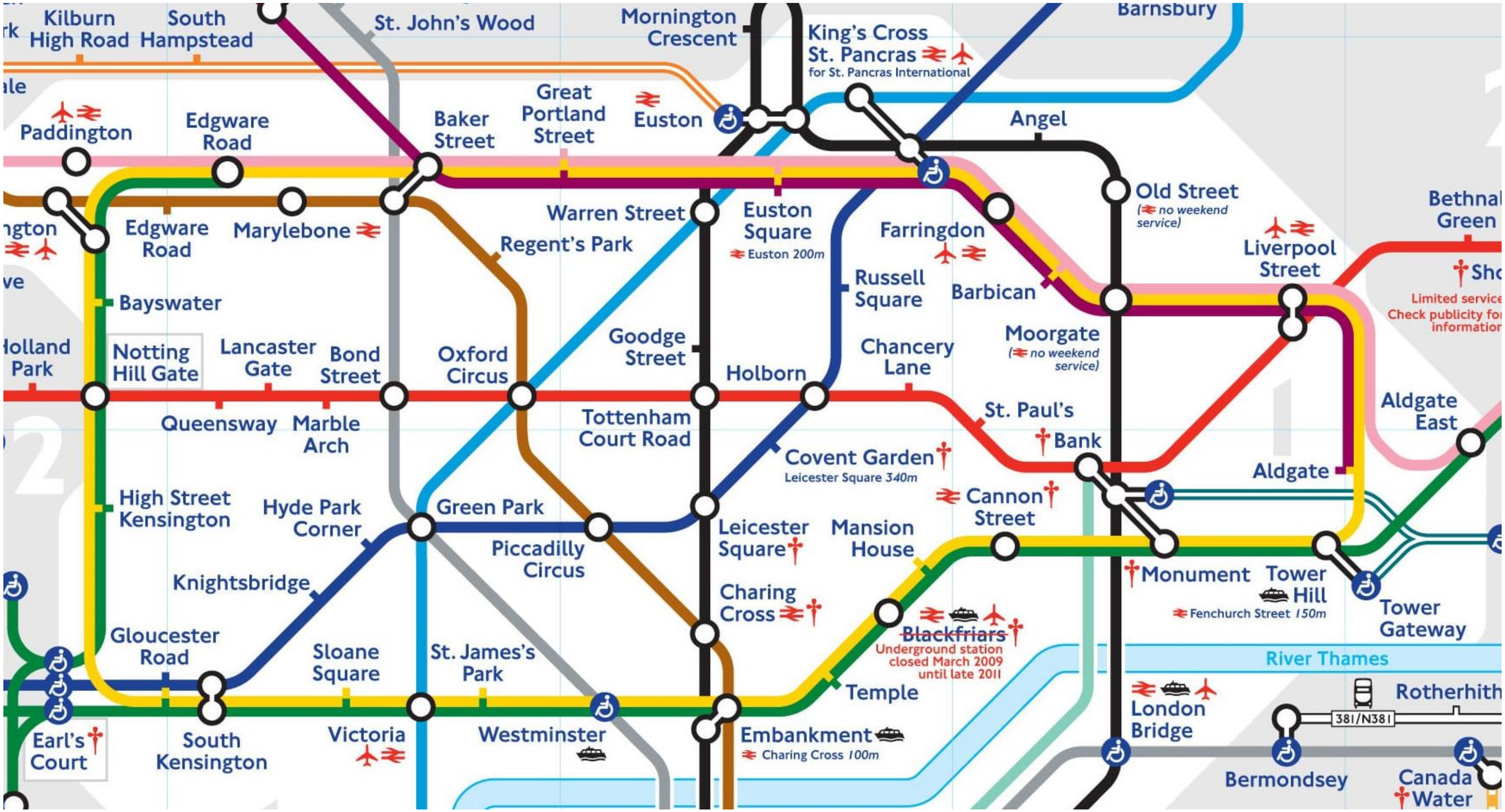
Isomorphic Equivalence



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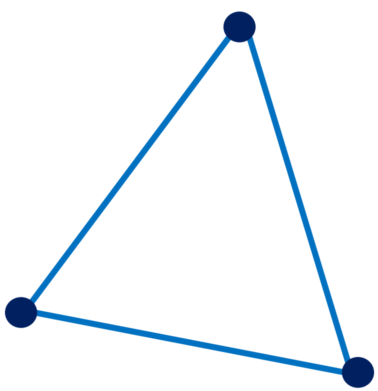
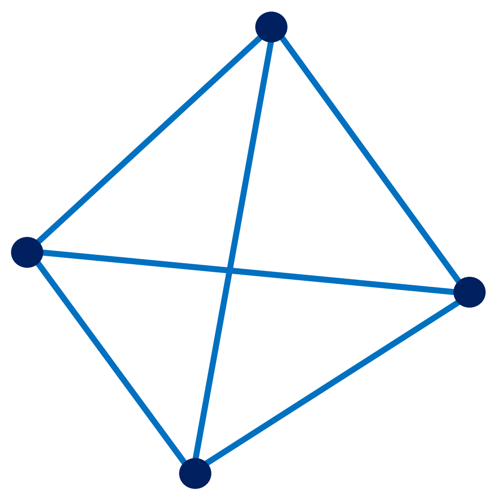
Isomorphic Equivalence

Complete graphs, K_n

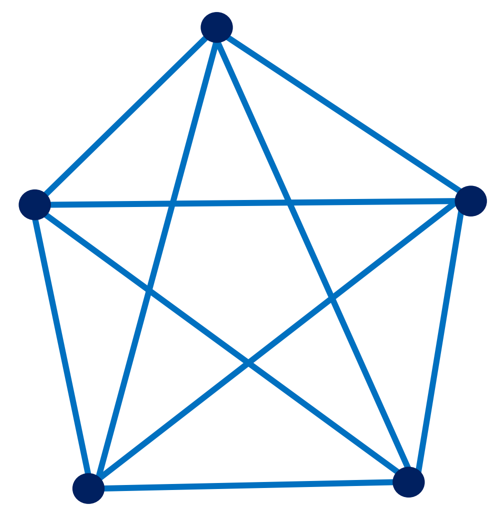
K_n denotes the complete graph on *n* vertices;
in this graph every pair of vertices is joined directly by one edge.



K_3

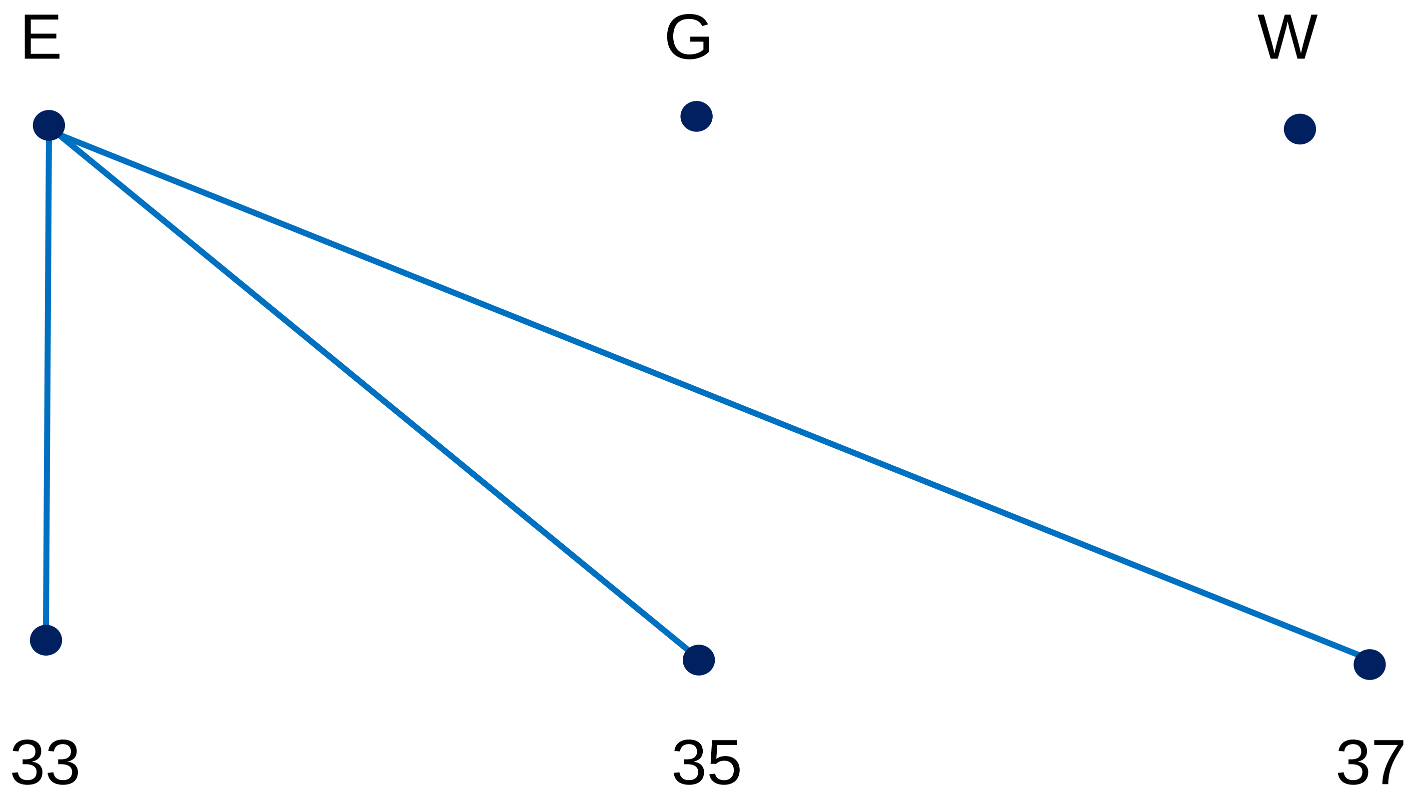


K_4



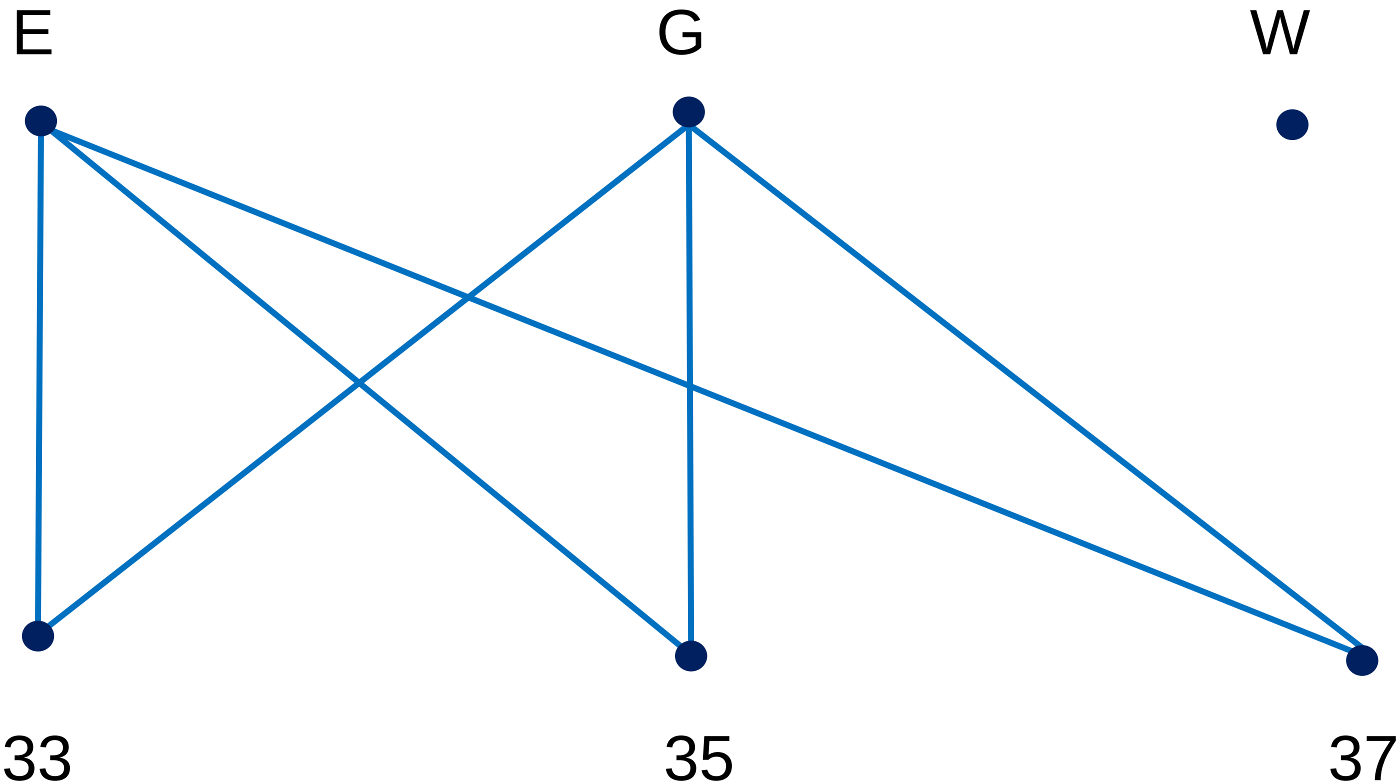
K_5

Bipartite graphs, $K_{3,3}$



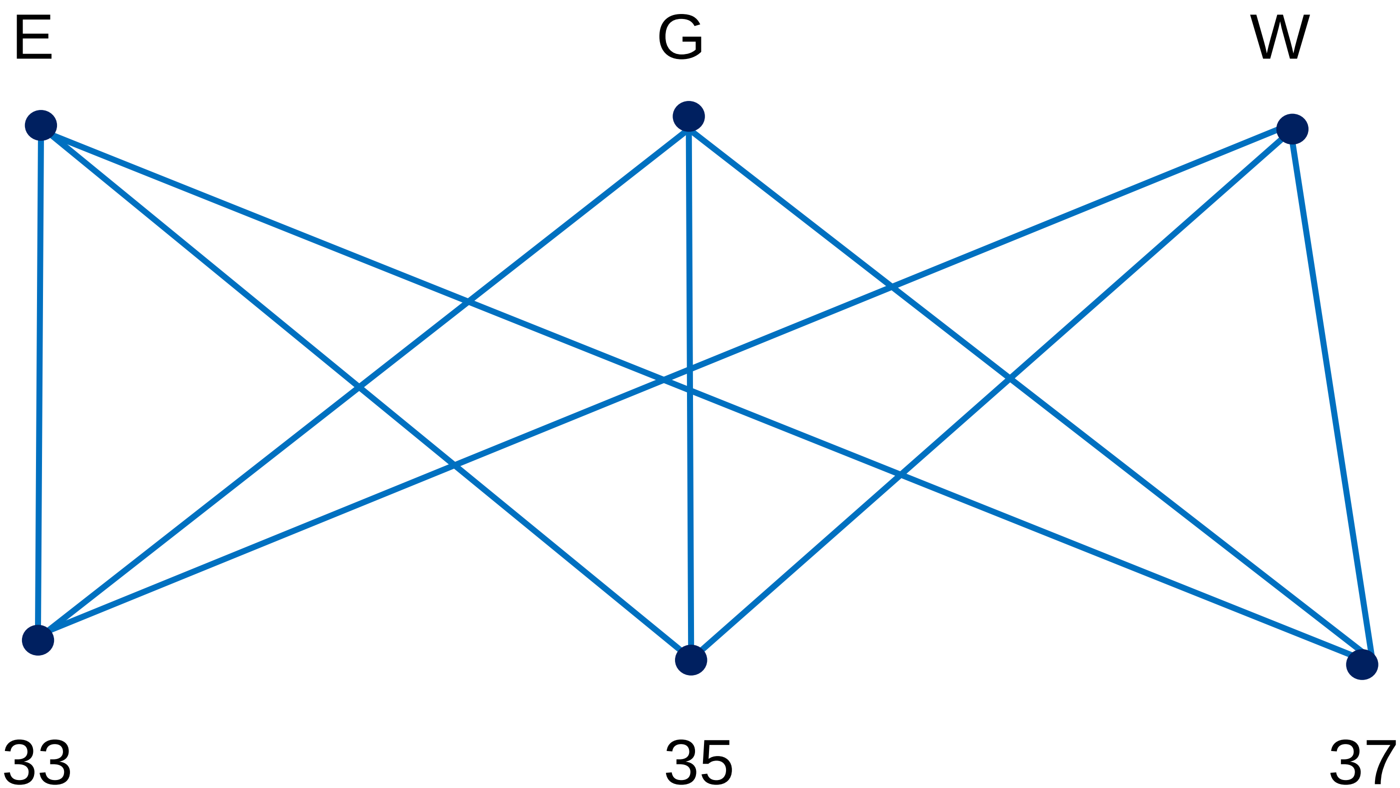
(At the moment, this graph is $K_{1,3}$)

Bipartite graphs, $K_{3,3}$



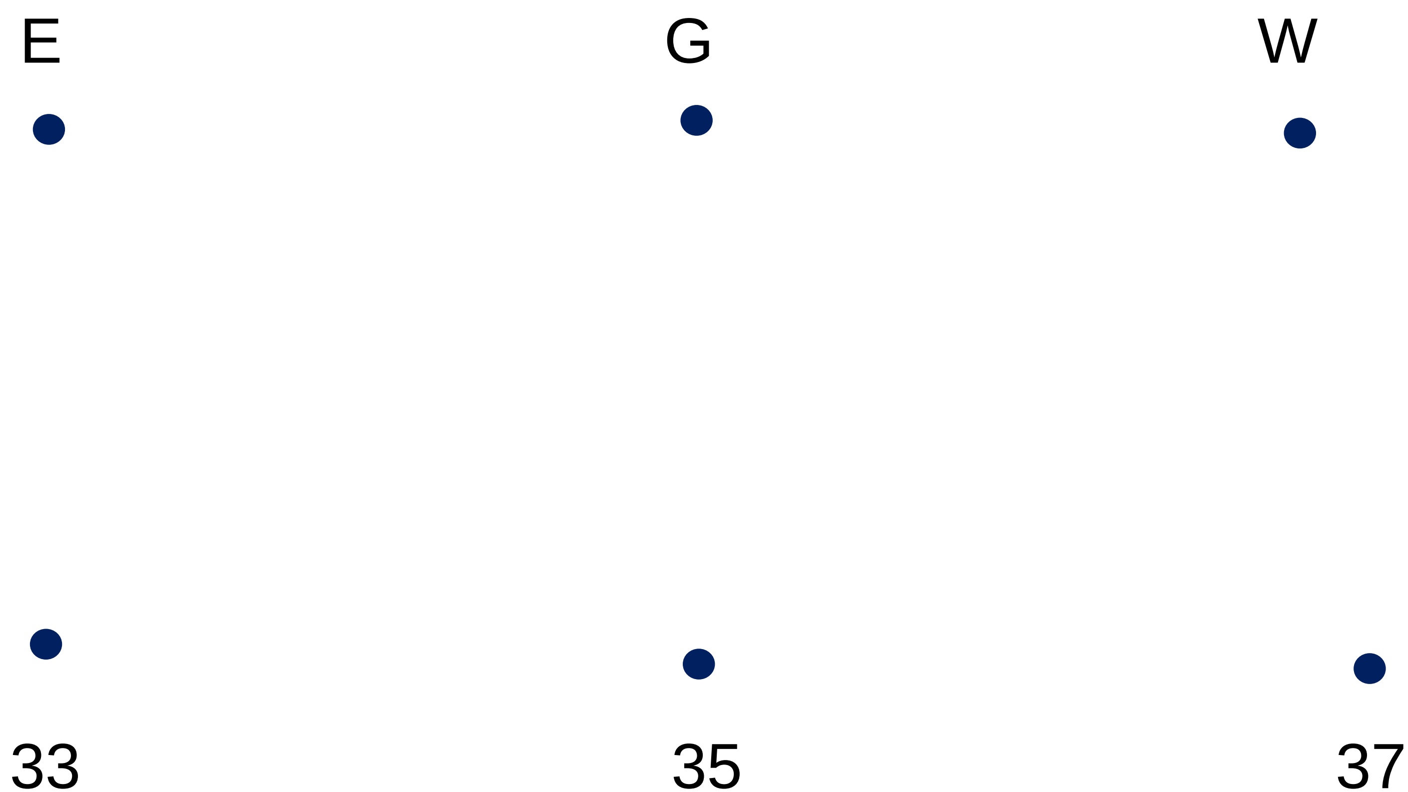
(This graph is now $K_{2,3}$)

Bipartite graphs, $K_{3,3}$



(This is now the finished graph of $K_{3,3}$)

Bipartite graphs, $K_{3,3}$

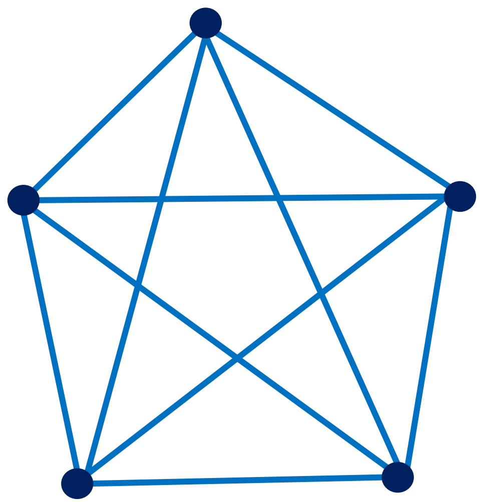


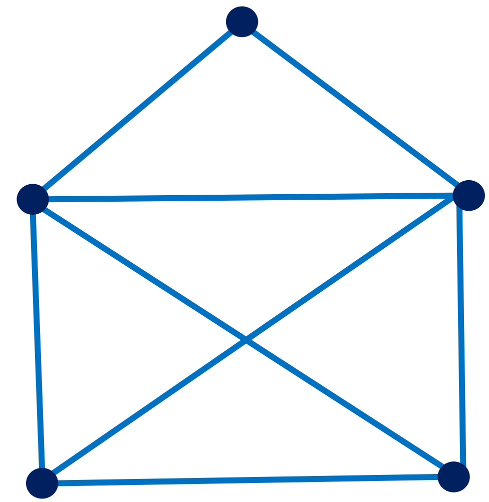
Could this graph be drawn without any edges crossing?

No – $K_{3,3}$ is non-planar.

Cycles

Are either of these graphs traversable?
(Is there an Eulerian path which traverses each edge once?)



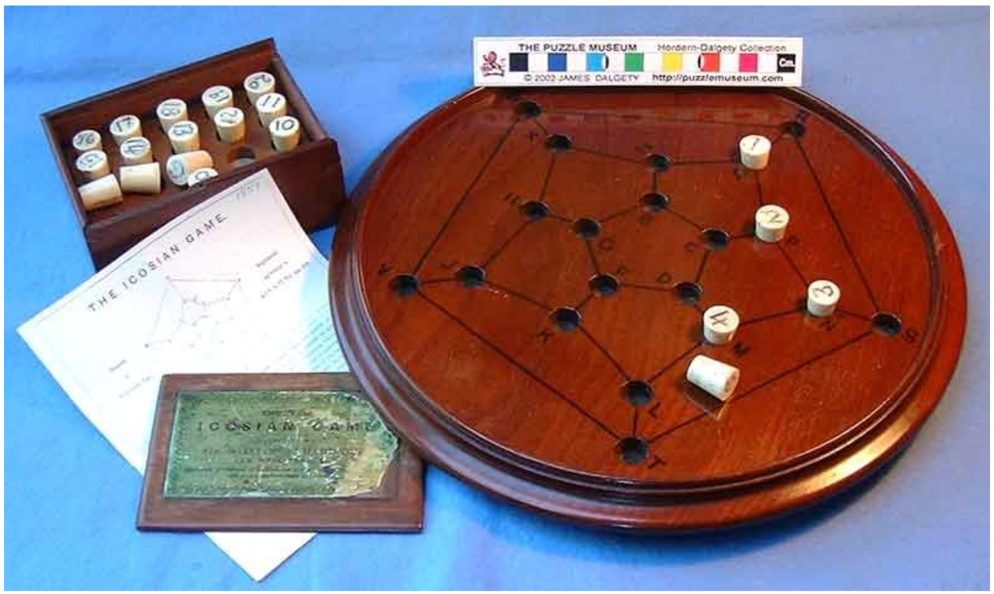


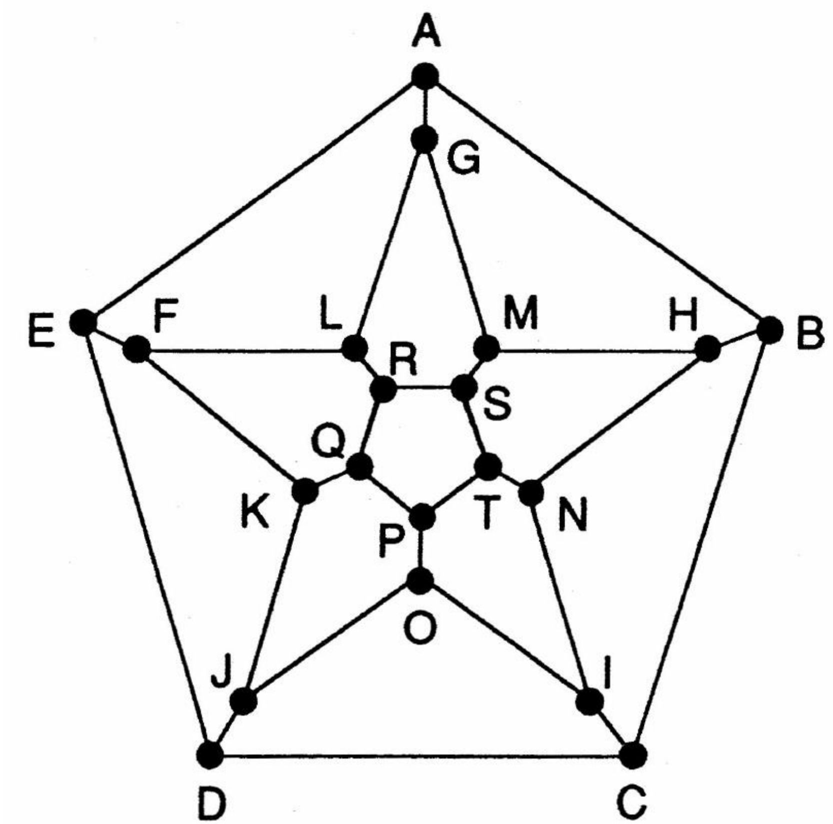
Eulerian graph
contains an Eulerian cycle

Semi-Eulerian graph
contains an Eulerian path

Cycles: The Icosian Game

Is there a path which visits every vertex once?





Hamiltonian graph
contains a Hamiltonian cycle

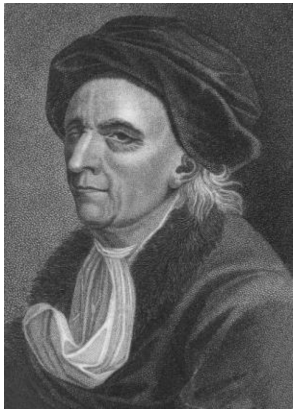
Cycles: Summary

A closed path is called a **cycle**. The sequence of vertices visited begins and ends at the same vertex.

A **Hamiltonian cycle** is a closed path which visits each vertex once and only once.

An **Eulerian cycle** is a closed path that travels along every edge once.

William Rowan Hamilton

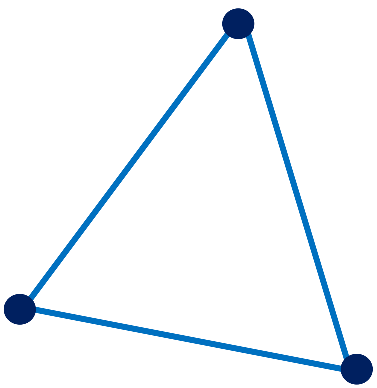


Leonhard Euler

Planarity

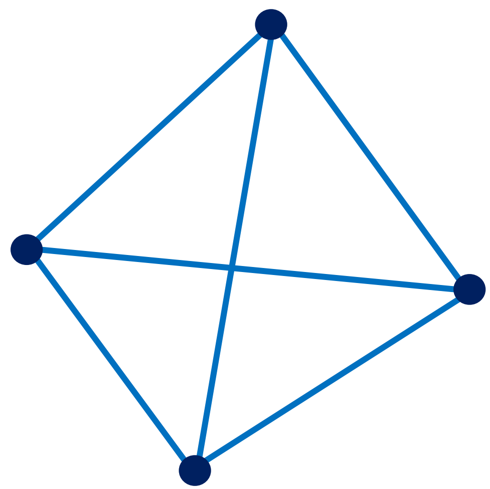
A **planar graph** can be drawn on a plane surface without any edges crossing.

For what values of *n* is K_n planar?



K_3

✓



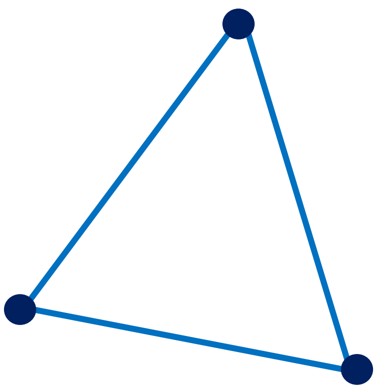
K_4

?

Planarity

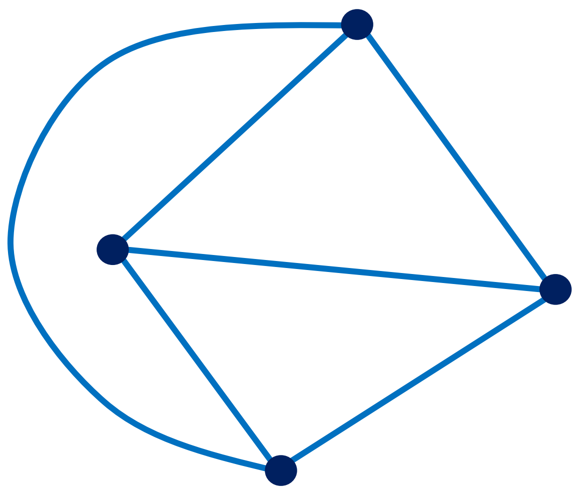
A **planar graph** can be drawn on a plane surface without any edges crossing.

For what values of n is K_n planar?



K_3

✓



K_4

✓

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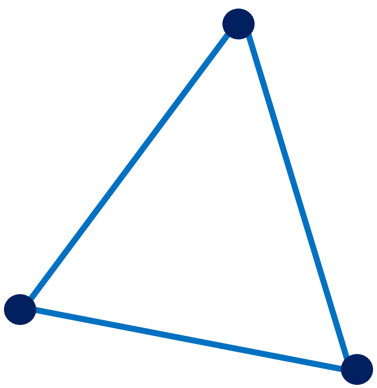
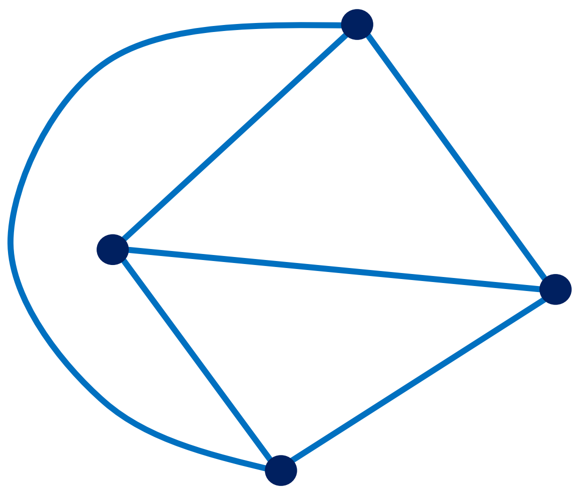


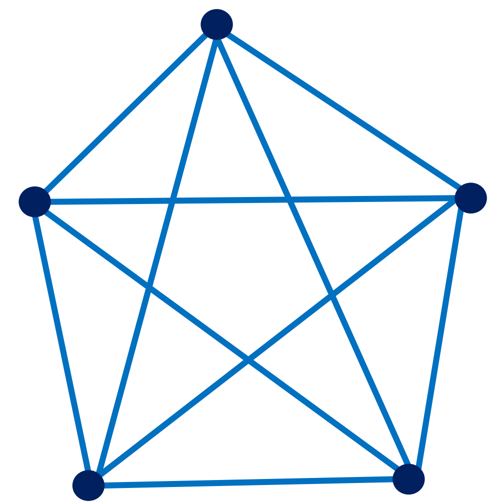
K_3

✓



K_4

✓



K_5

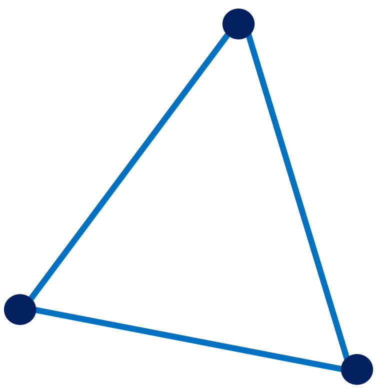
?

Planarity

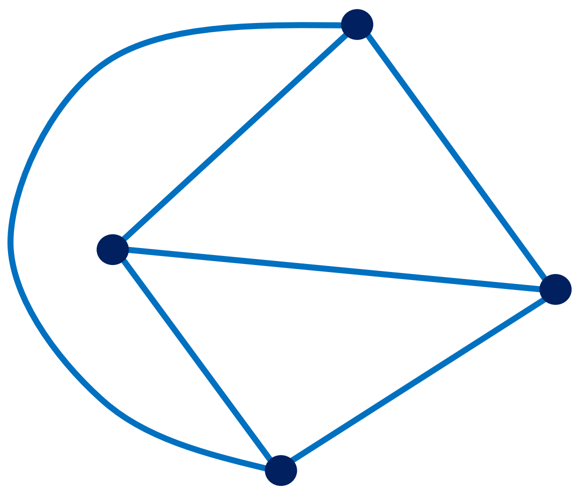
A **planar graph** can be drawn on a plane surface without any edges crossing.

For what values of *n* is K_n planar?

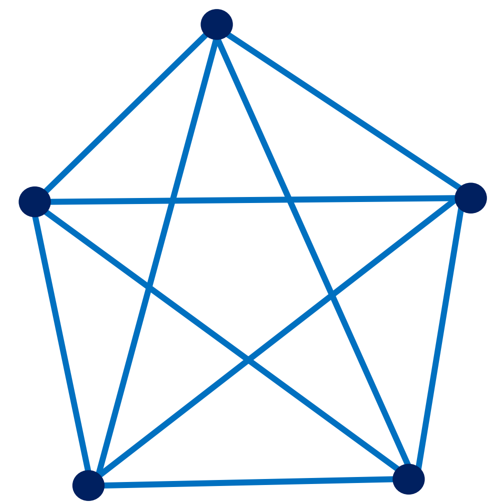
Only for $n \leq 4$



K_3

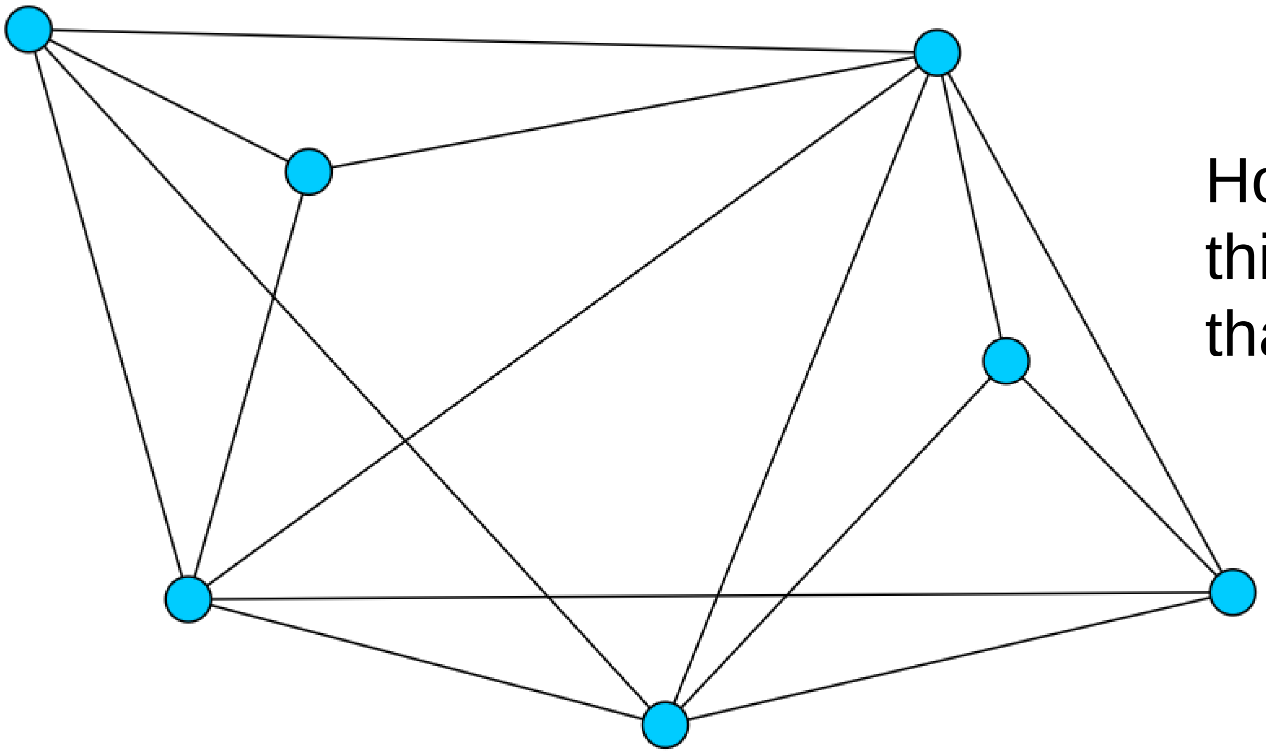


K_4



K_5

Planar representations



How can we redraw this graph to show that it is planar?

Image taken from www.jasondavies.com/planarity

Planar representations

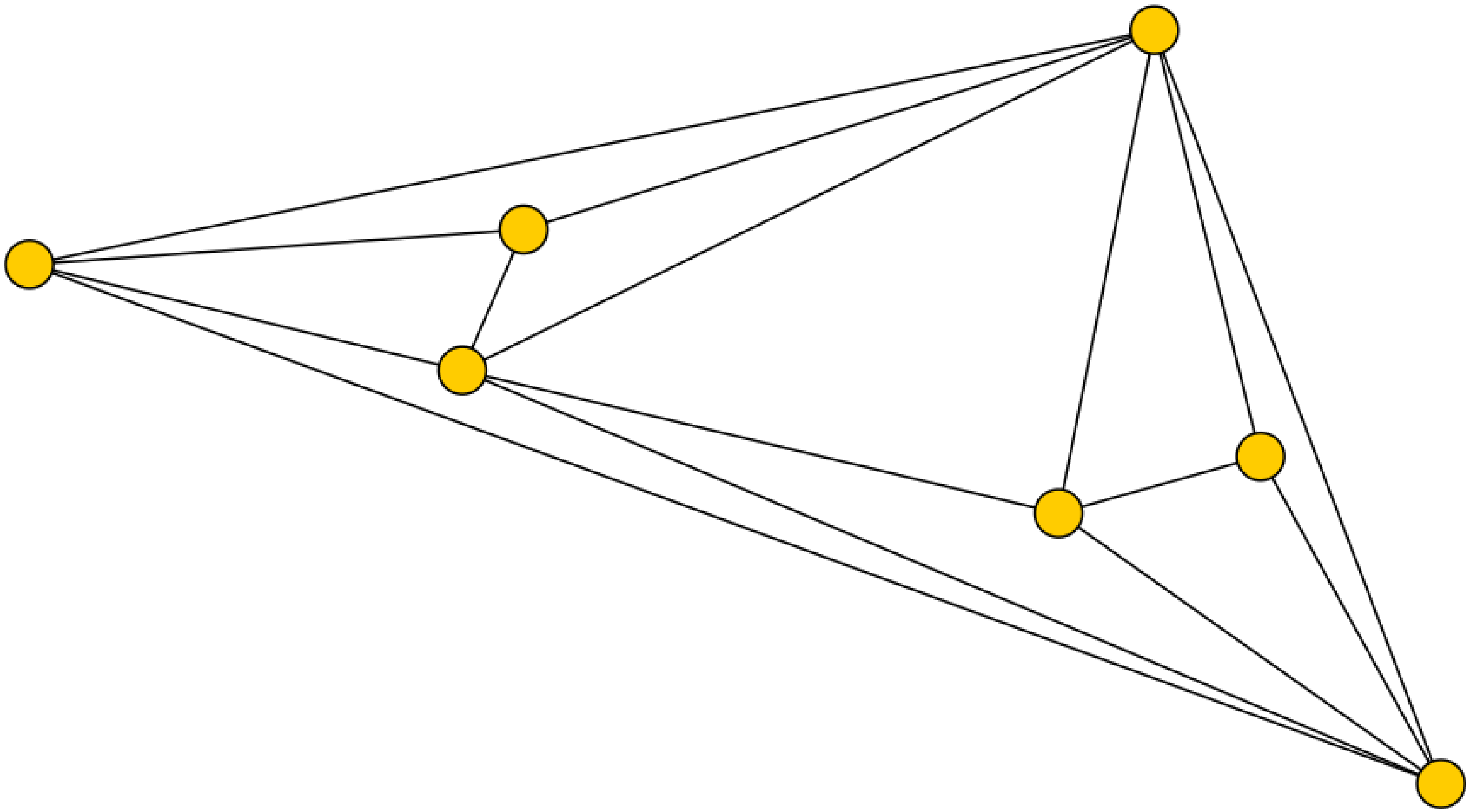


Image taken from www.jasondavies.com/planarity

Planarity Algorithm (Edexcel)

This process can be used to re-draw a graph with crossing edges into an obviously planar graph:

- Find a Hamiltonian cycle containing all vertices
- Re-draw the graph with this Hamiltonian cycle as an outer polygon with all other edges inside
- Select any inside edge and assign this to set P
- Assign any edge that crosses the first edge to set Q
- Assign any edge that any edge in Q crosses to set P, etc.
- Re-draw the graph with edges in set P drawn inside the cycle and edges in set Q drawn outside (or vice versa).

If it is impossible to allocate all edges to two distinct sets, the graph must *not* be planar.

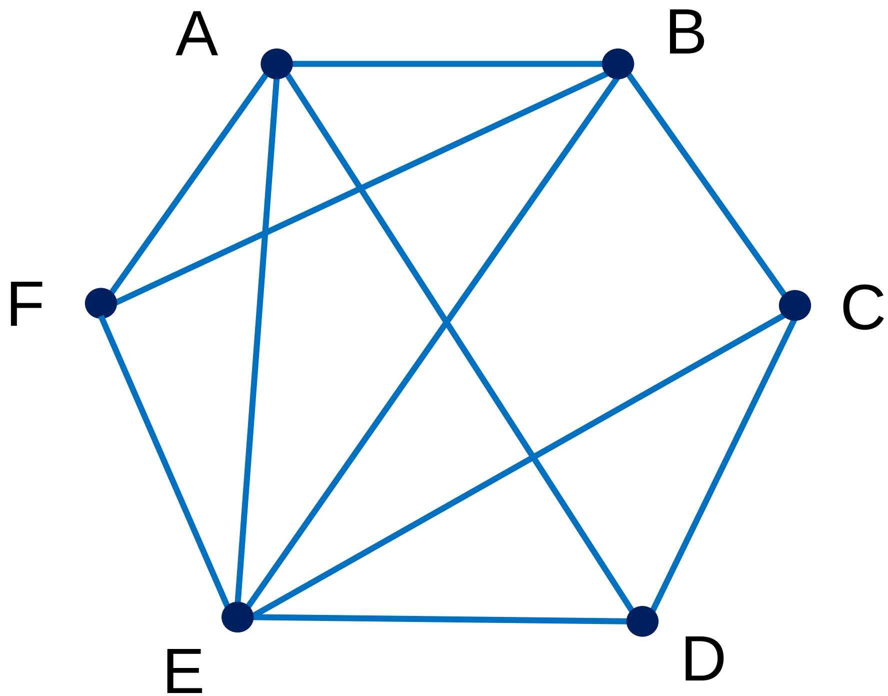
Kuratowski's Theorem (AQA/OCR)

Kuratowski's Theorem states that a graph is planar *if and only if* it does not contain either K_5 or $K_{3,3}$ or a subdivision or of either K_5 or $K_{3,3}$ as a subgraph.

A sufficient proof that a graph is *non-planar* would be to show that it contains K_5 or $K_{3,3}$ as a subgraph.

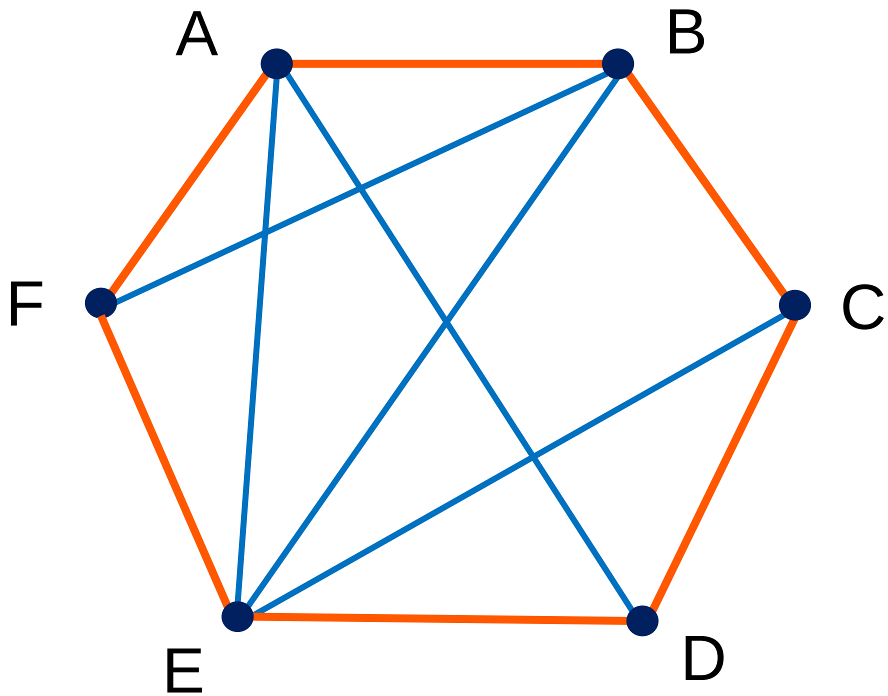
Example 1

Use the planarity algorithm to re-draw the graph shown so that no edges cross.



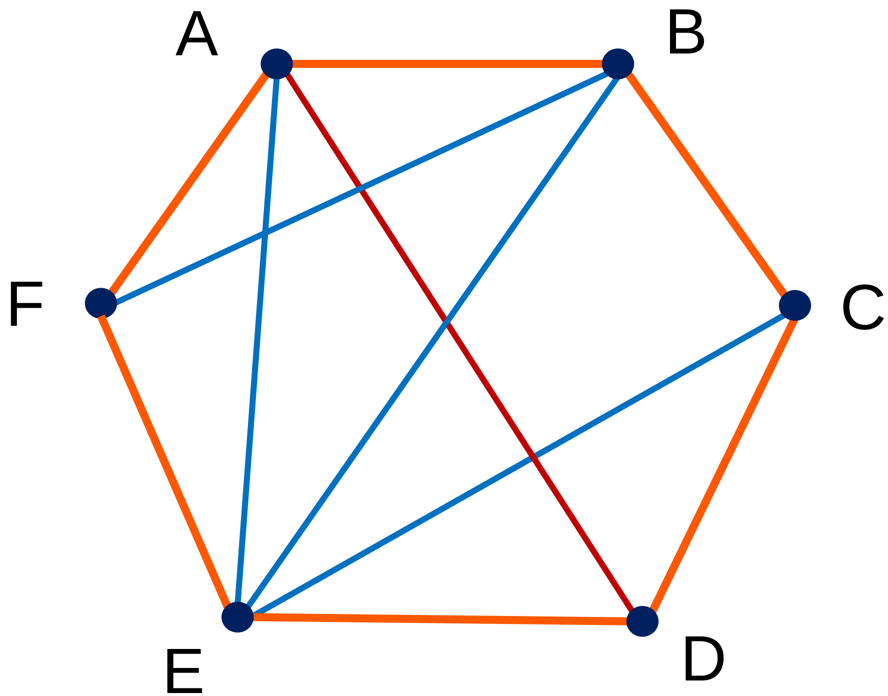
Example 1

An obvious Hamiltonian cycle already exists as an outside polygon: A-B-C-D-E-F-(A).



Example 1

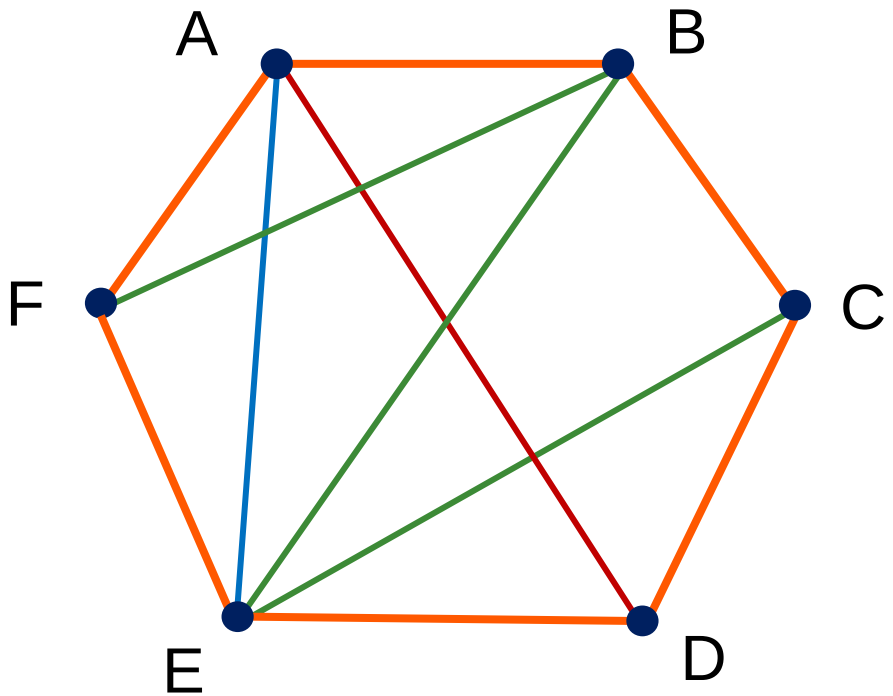
Assign edge AD to set P.



Set P	Set Q
AD	

Example 1

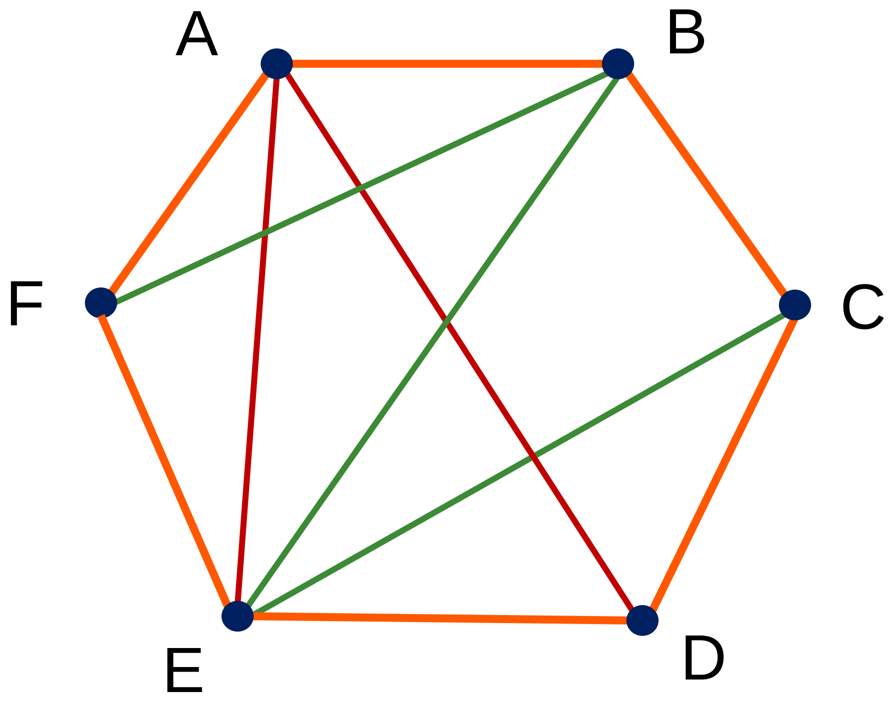
As edge AD crosses BF, BE and CE, assign these to set Q.



Set P	Set Q
AD	
	BF
	BE
	CE

Example 1

As edge BF crosses AE, assign AE to set P.

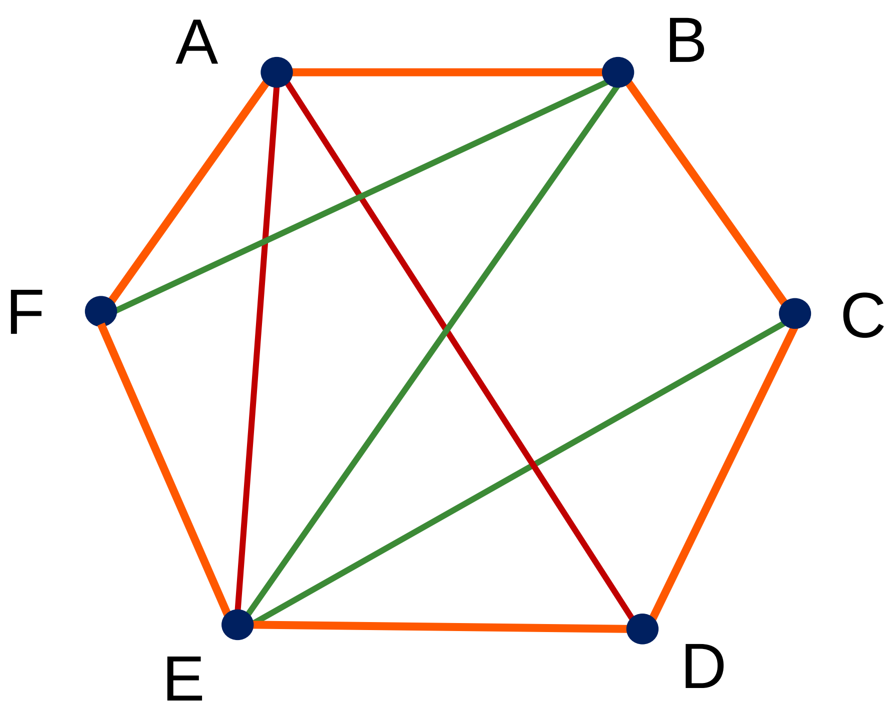


Set P	Set Q
AD	
	BF
	BE
	CE
AE	

All edges have been assigned to two distinct sets.

Example 1

Re-draw the graph with edges in set P inside and edges in set Q outside (or vice versa).



Set P	Set Q
AD	
	BF
	BE
	CE
AE	

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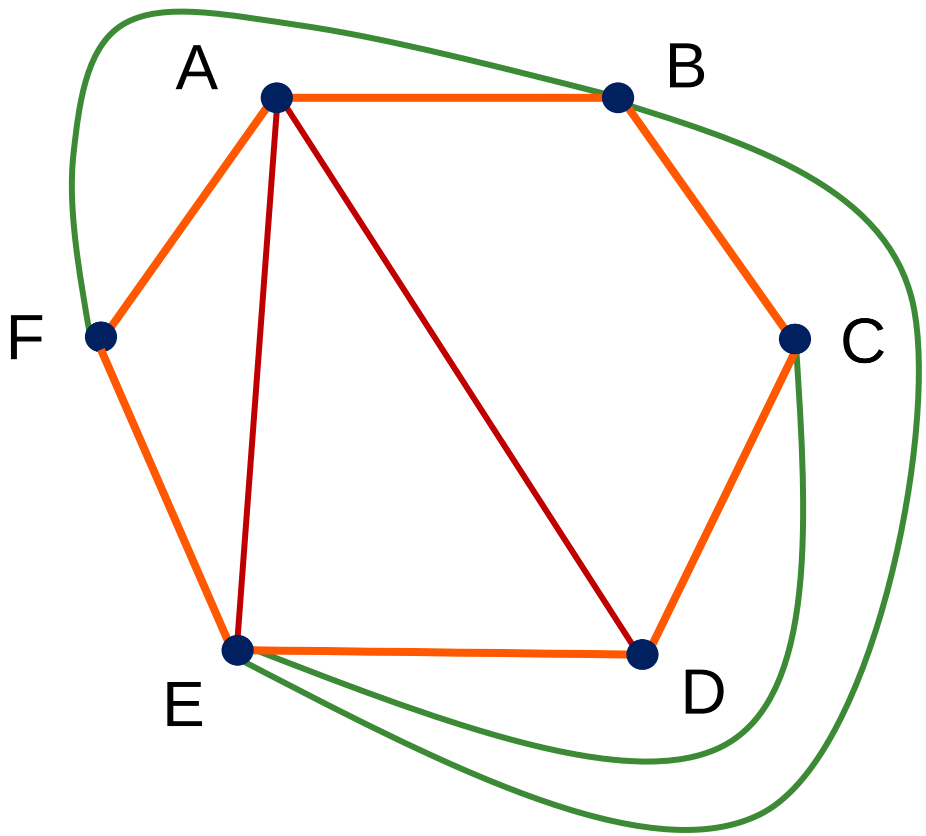
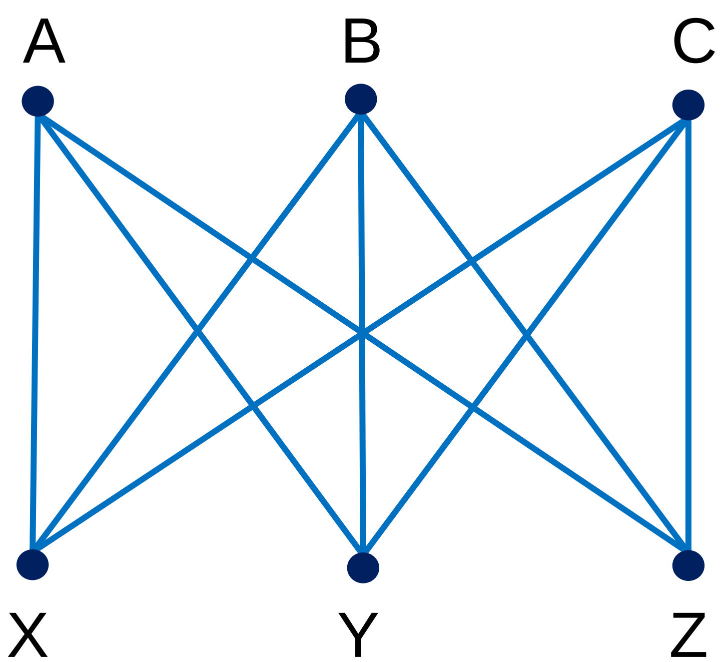


Set P	Set Q
AD	
	BF
	BE
	CE
AE	

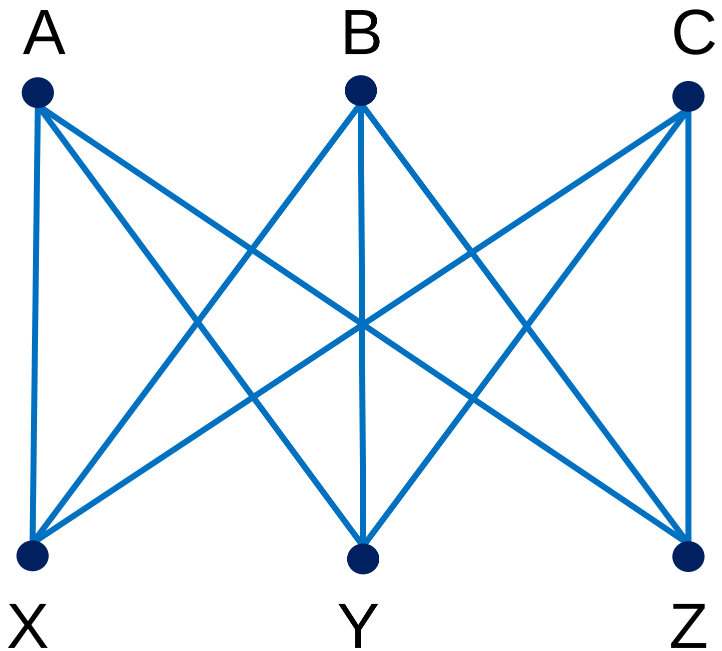
Example 2

By attempting the planarity algorithm, show that $K_{3,3}$ is non-planar.



Example 2

First, identify a Hamiltonian cycle.



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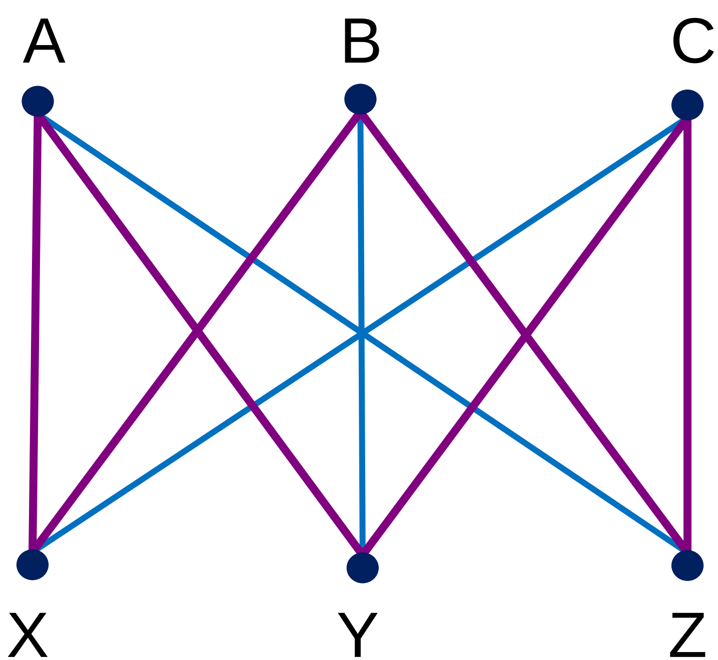
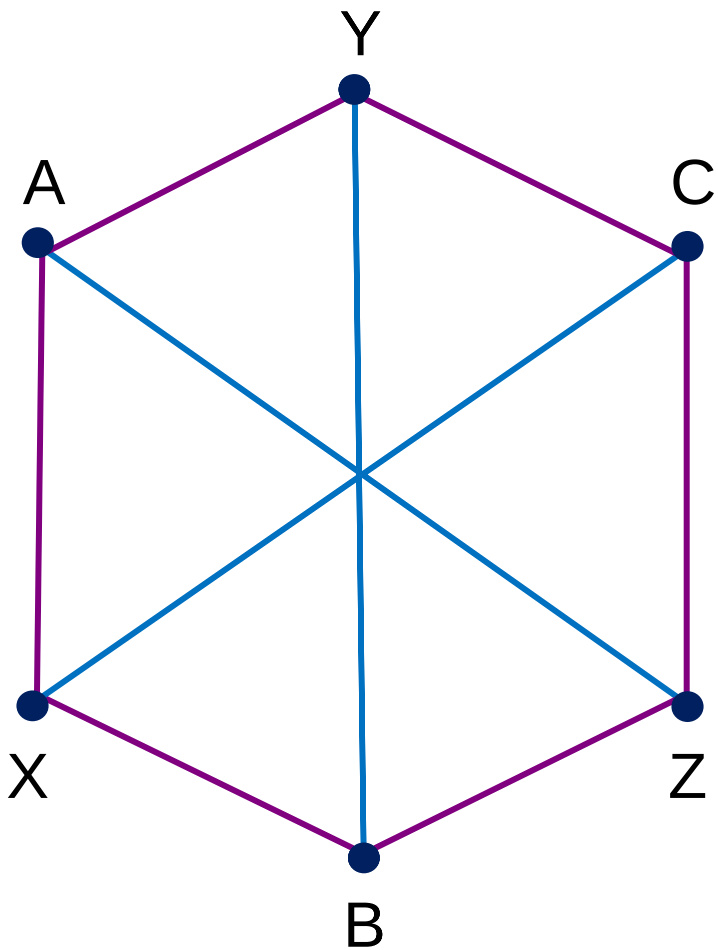


One possible Hamiltonian cycle is A-Y-C-Z-B-X-(A).

Example 2

Redraw the graph with the Hamiltonian cycle on the outside.

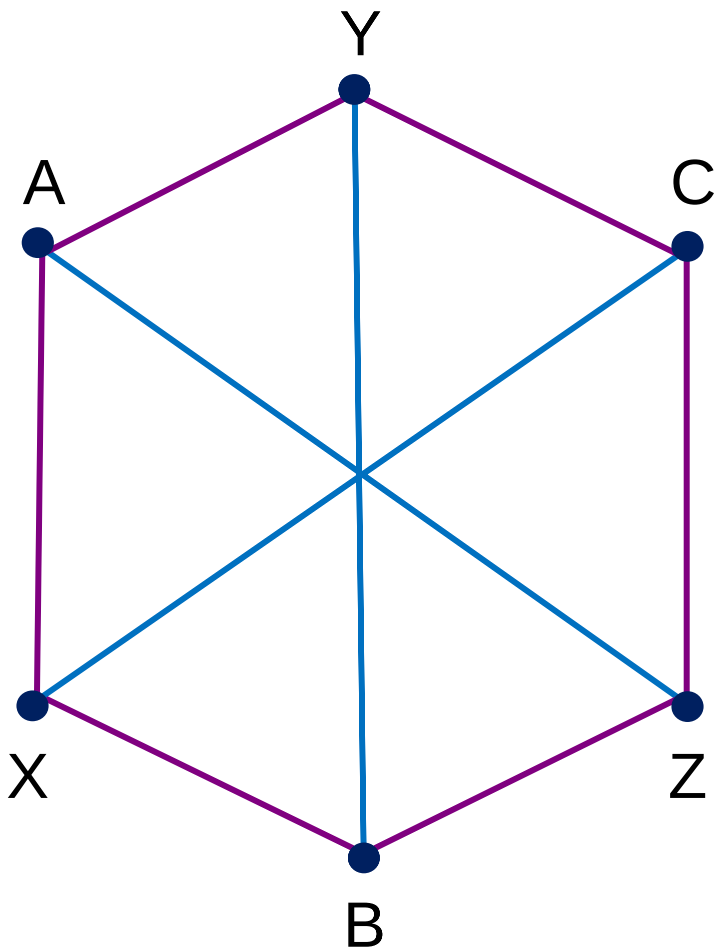


This is another form of $K_{3,3}$.

Remember this for later!

Example 2

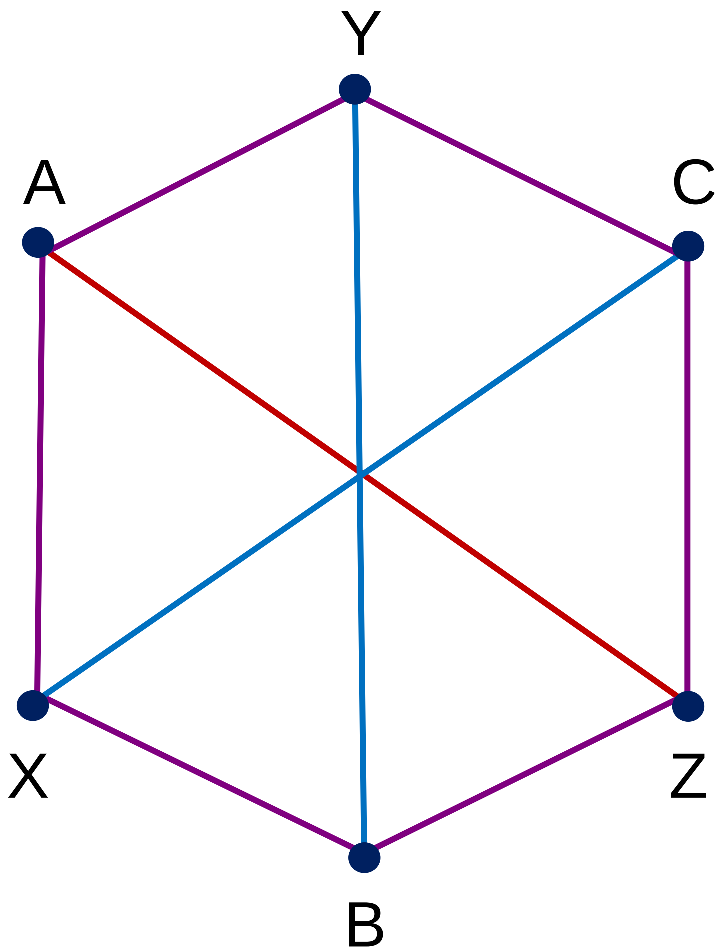
Select any inside edge and assign it to set P.



Set P	Set Q

Example 2

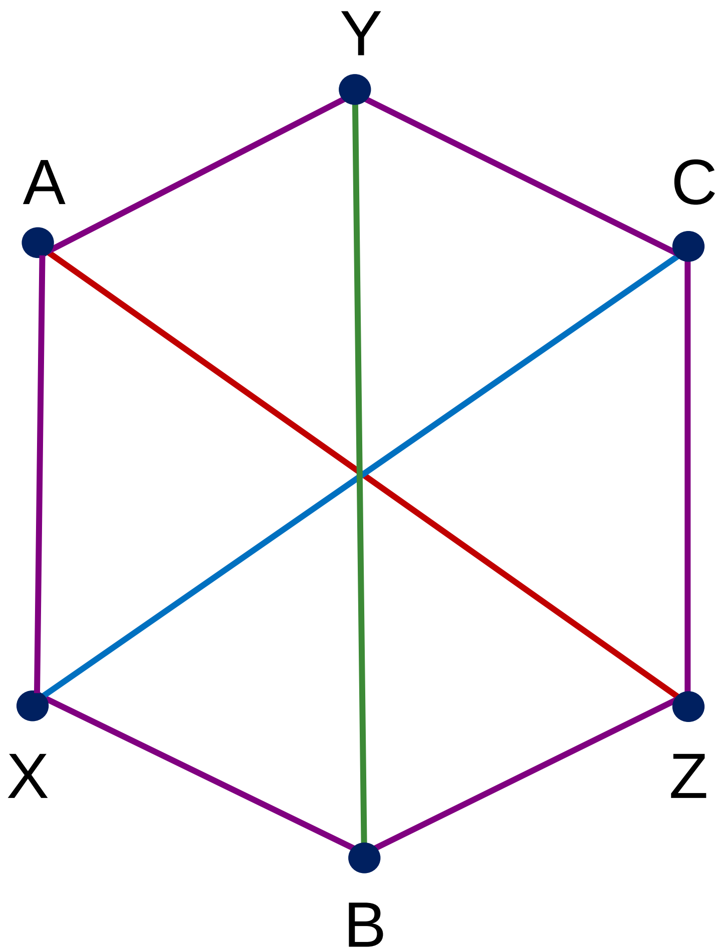
Assign AZ to set P.



Set P	Set Q
AZ	

Example 2

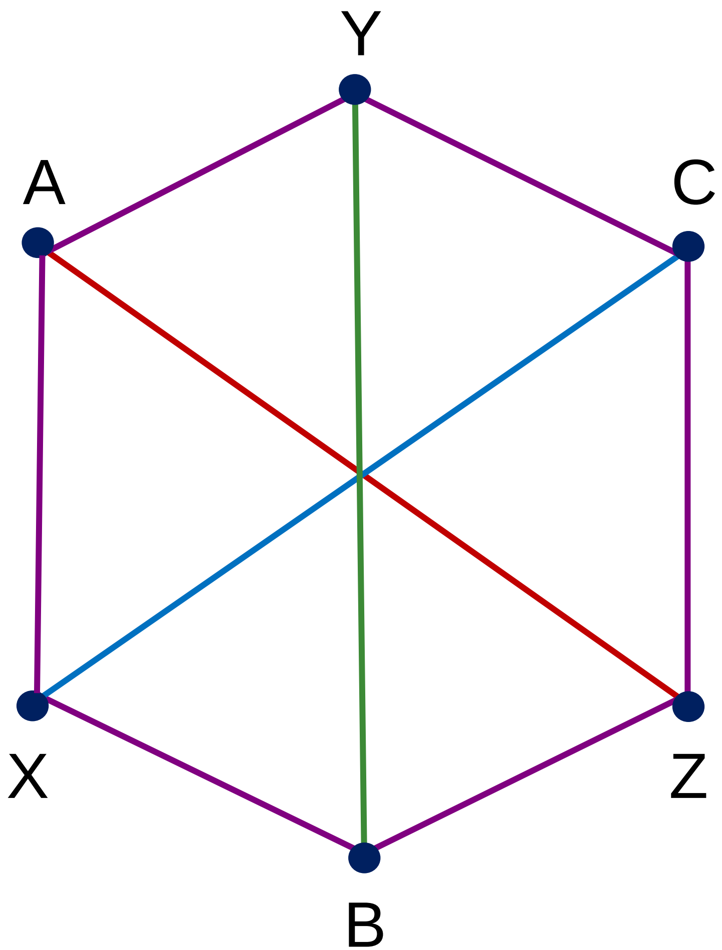
Since AZ crosses BY, assign BY to set Q.



Set P	Set Q
AZ	
	BY

Example 2

BY crosses CX, *BUT* AZ also crosses CX.



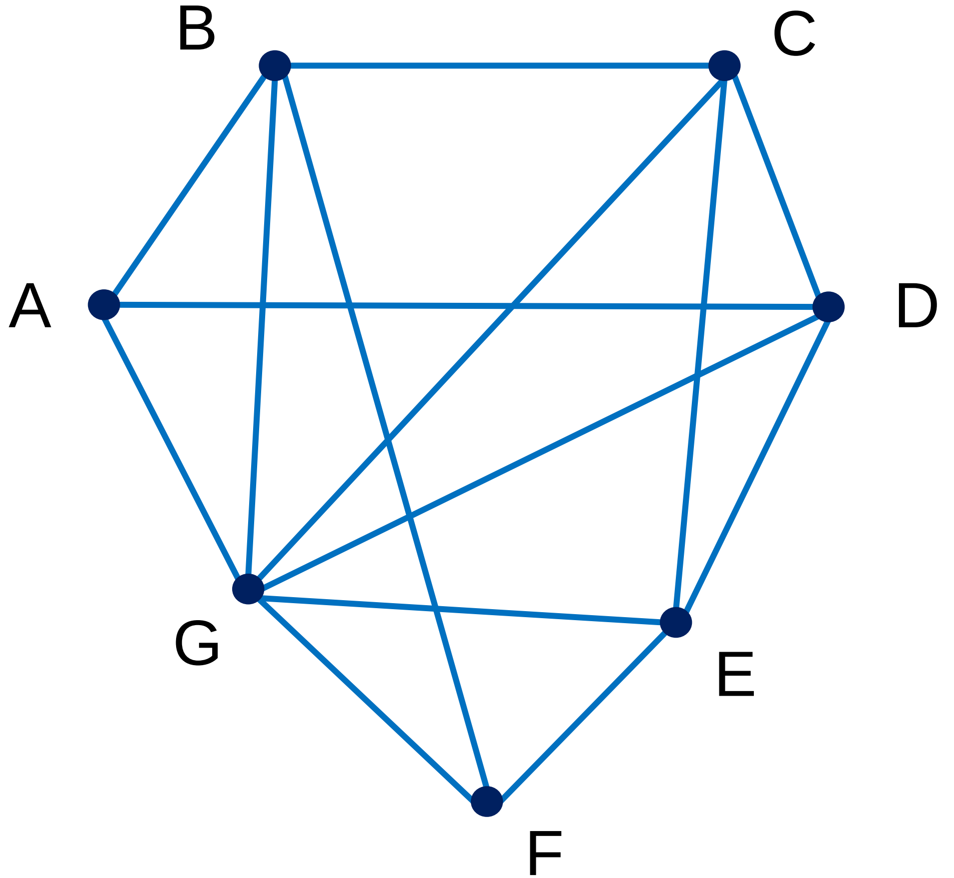
Set P	Set Q
AZ	
	BY

We cannot assign CX to either P or Q so have a contradiction.

Thus $K_{3,3}$ is non-planar

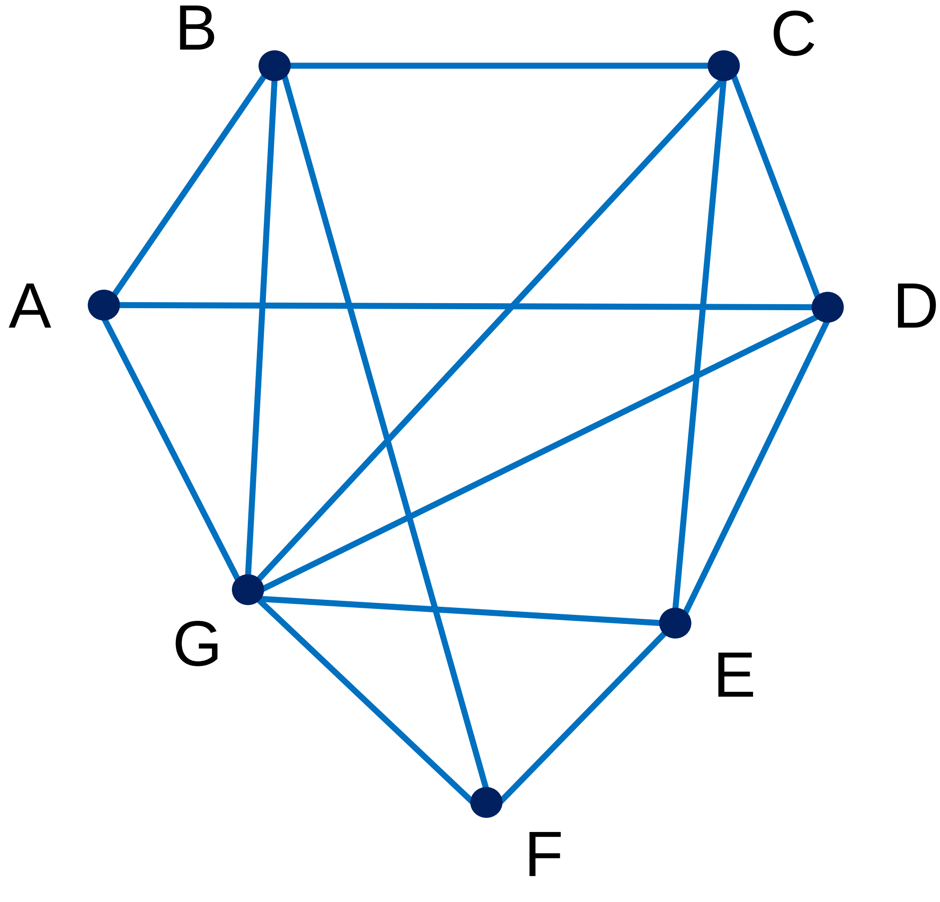
Example 3

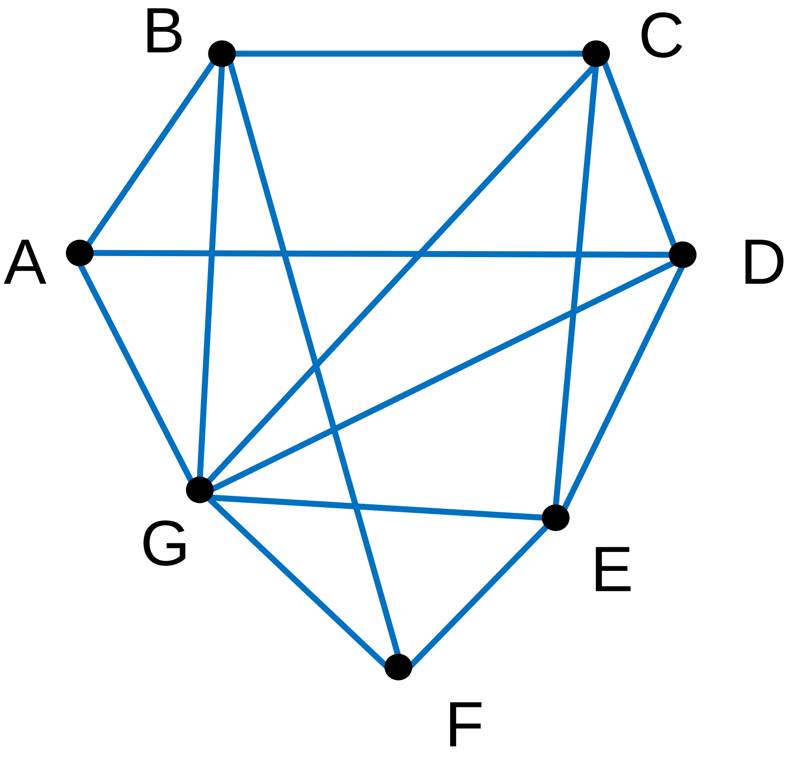
Prove that the graph *G* below is non-planar.



Example 3

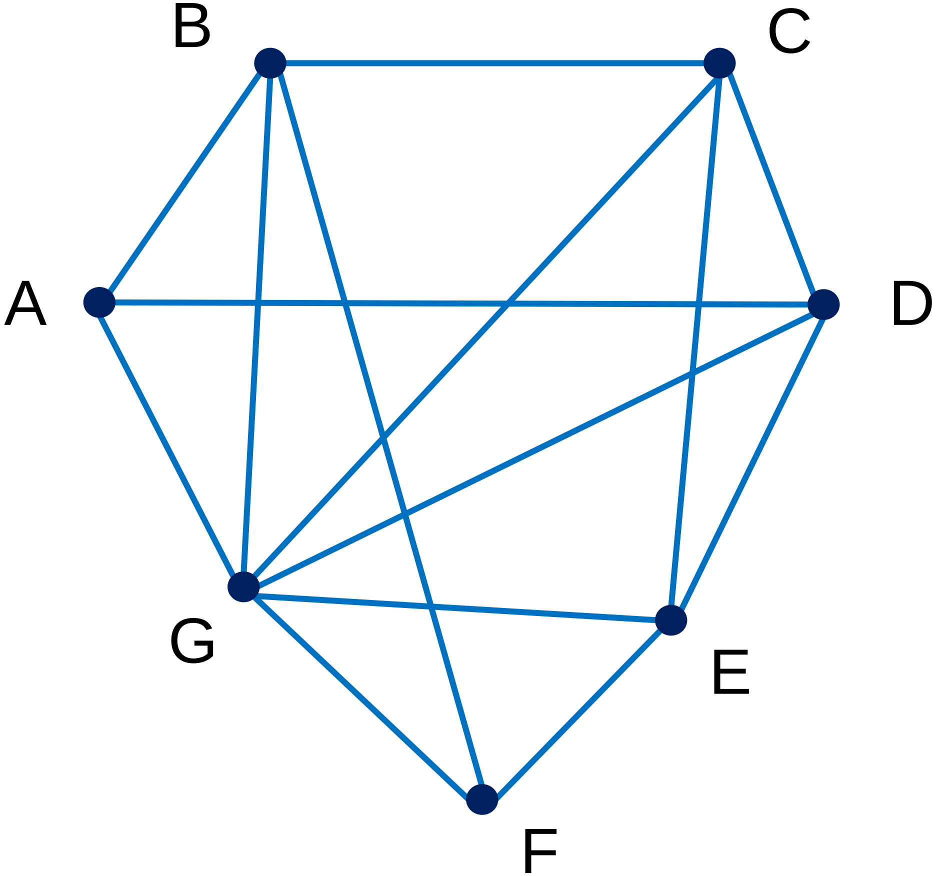
Delete edges BG, CE, DG, EG.

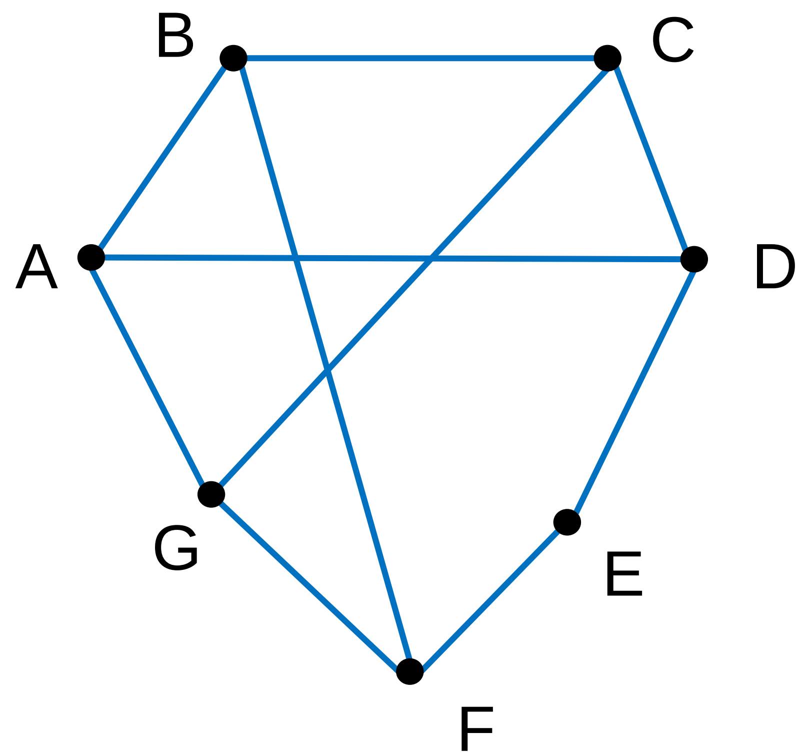




Example 3

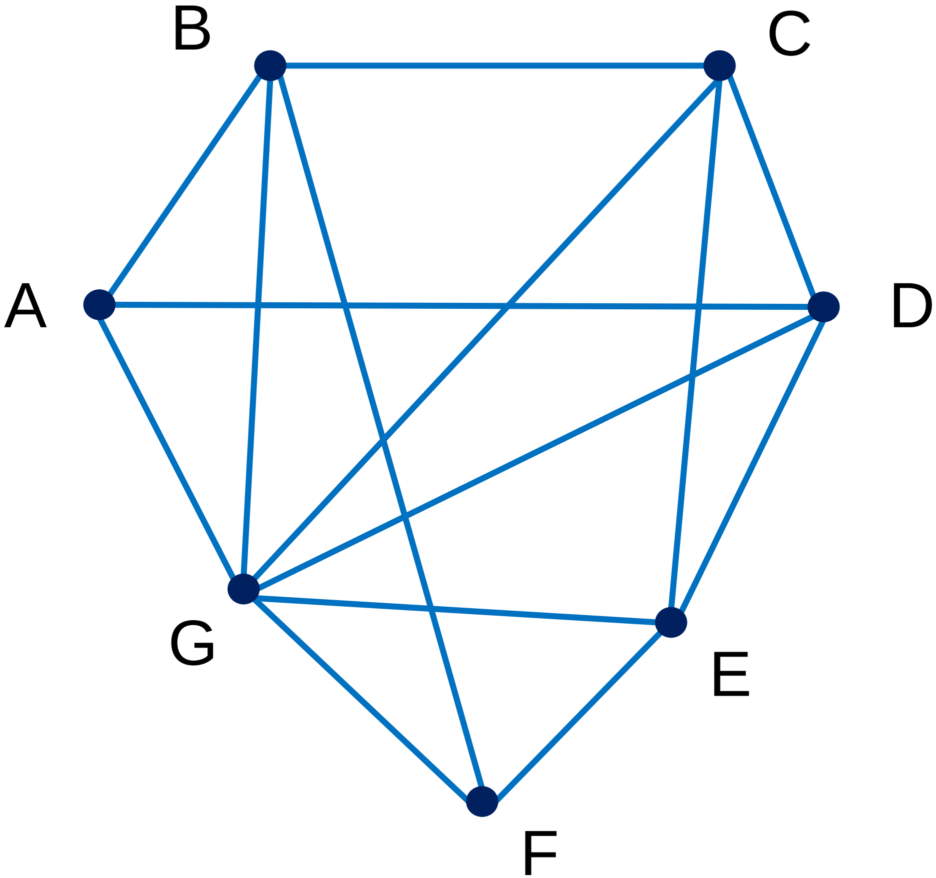
Delete edges BG, CE, DG, EG. Ignore vertex E.

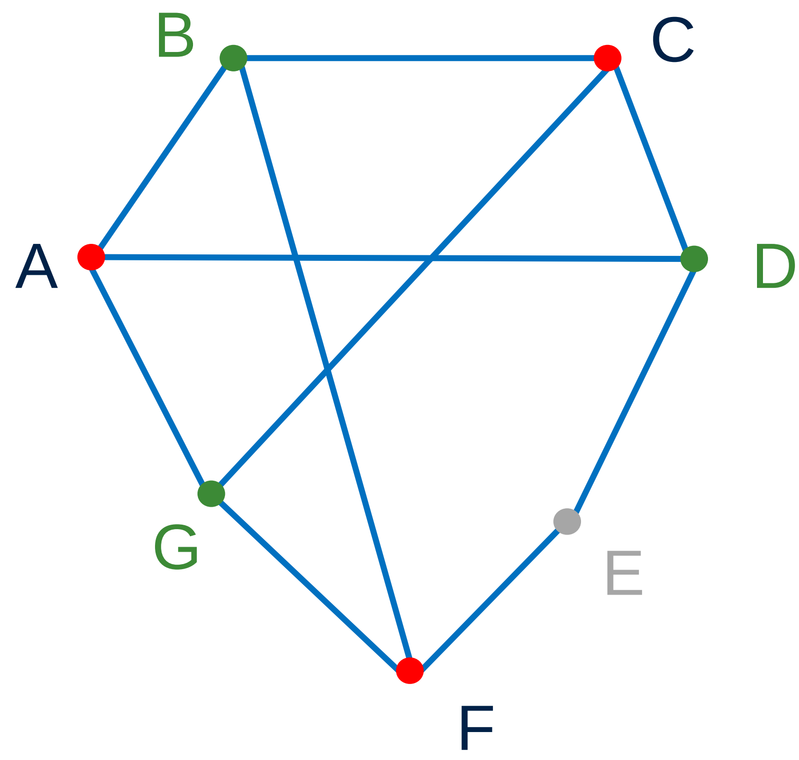




Example 3

Delete edges BG, CE, DG, EG. Ignore vertex E.



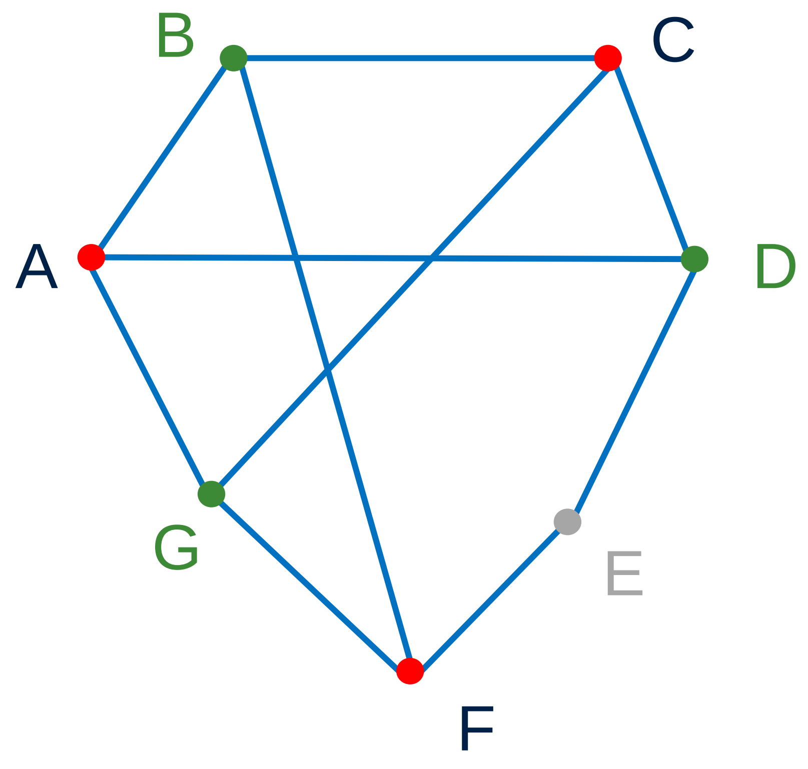


Example 3

Delete edges BG, CE, DG, EG. Ignore vertex E.

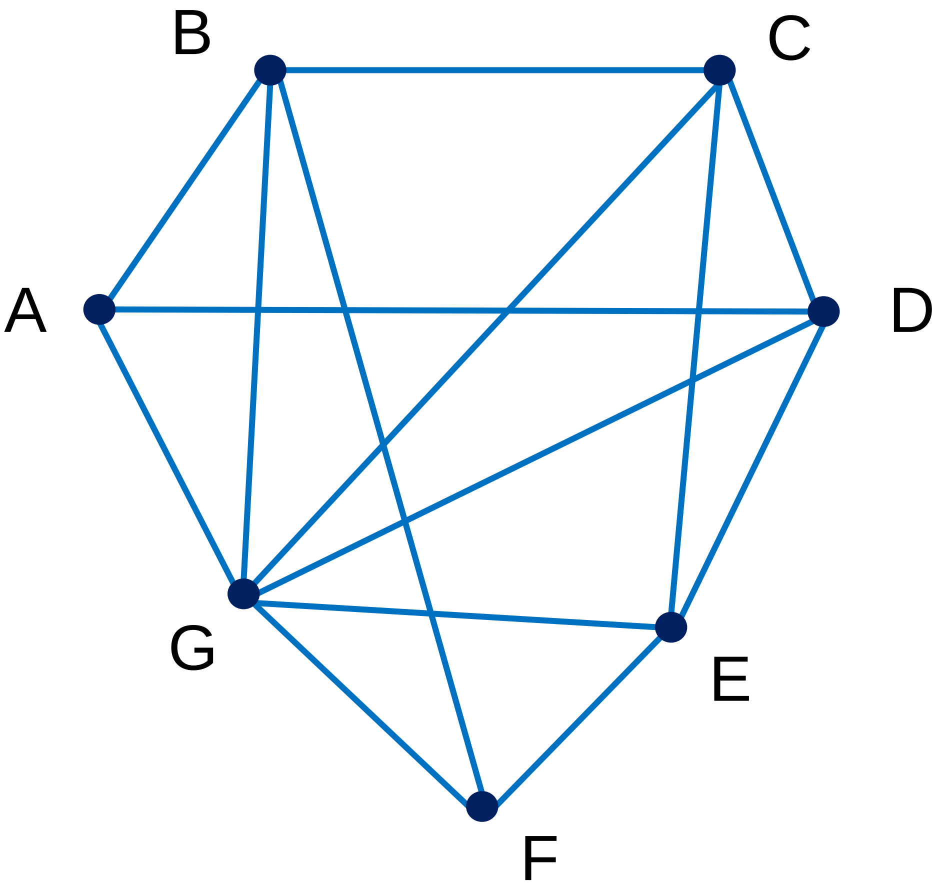
This subgraph of *G* contains a subdivision of $K_{3,3}$ on vertices {A, C, F} and {B, D, G}.

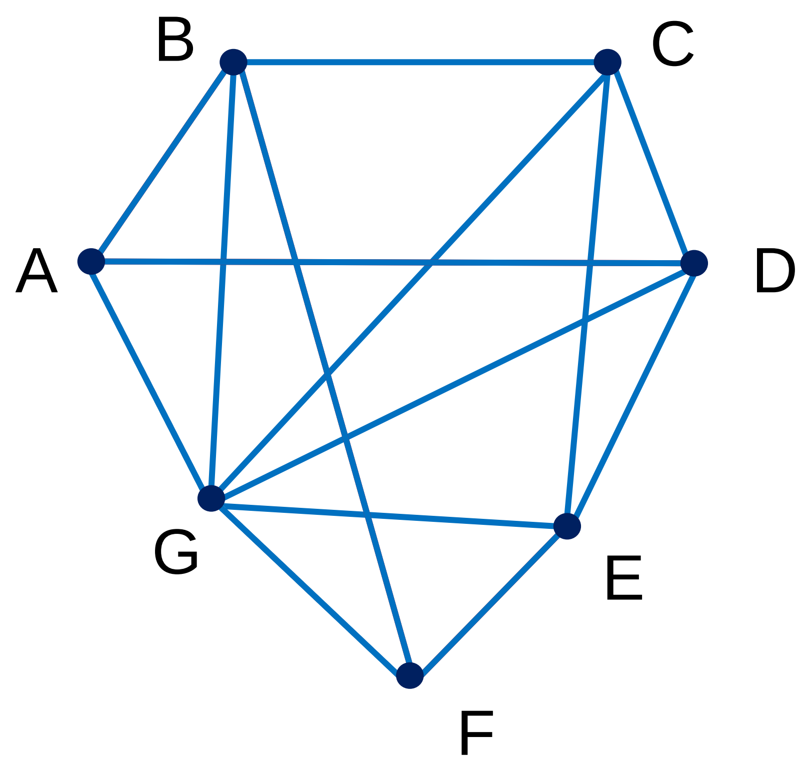
By Kuratowski's theorem, *G* is non-planar.



Example 3 – Alternative method

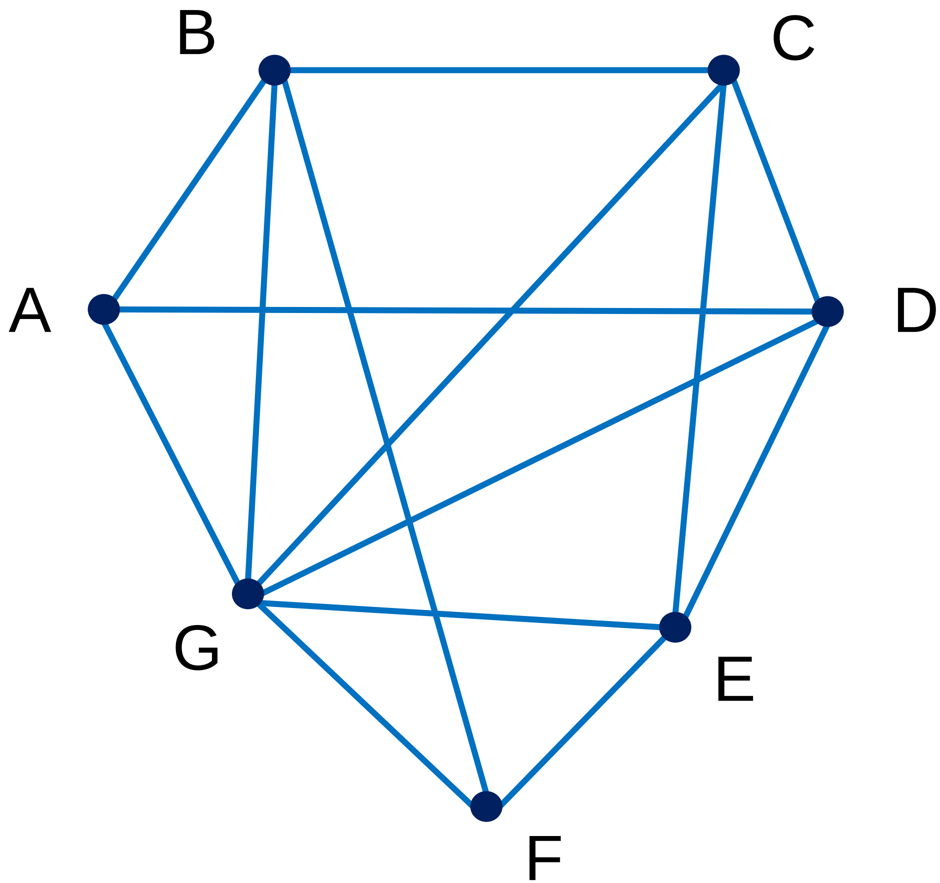
Delete edges AG and FG.

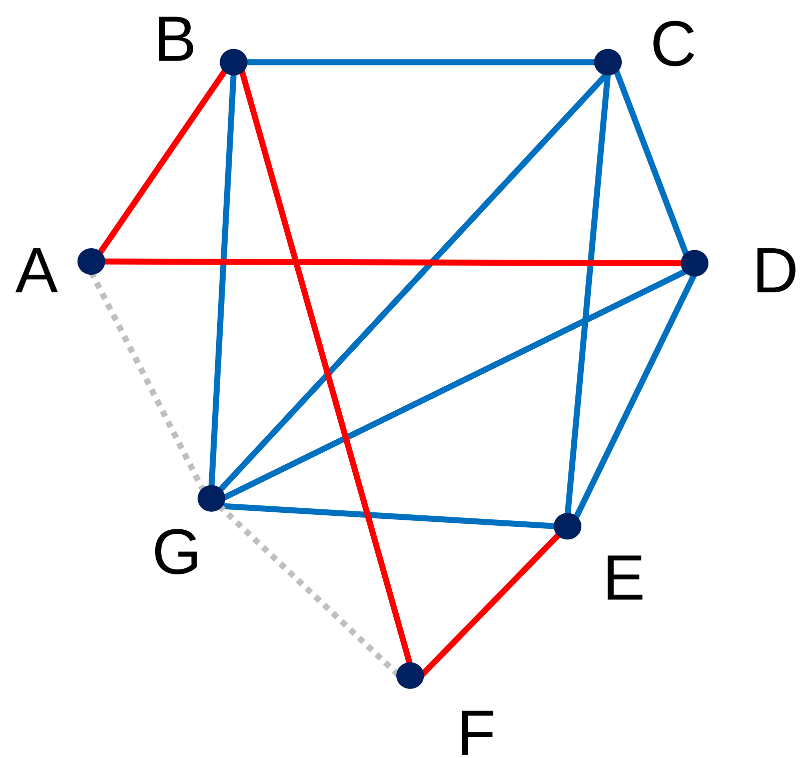




Example 3 – Alternative method

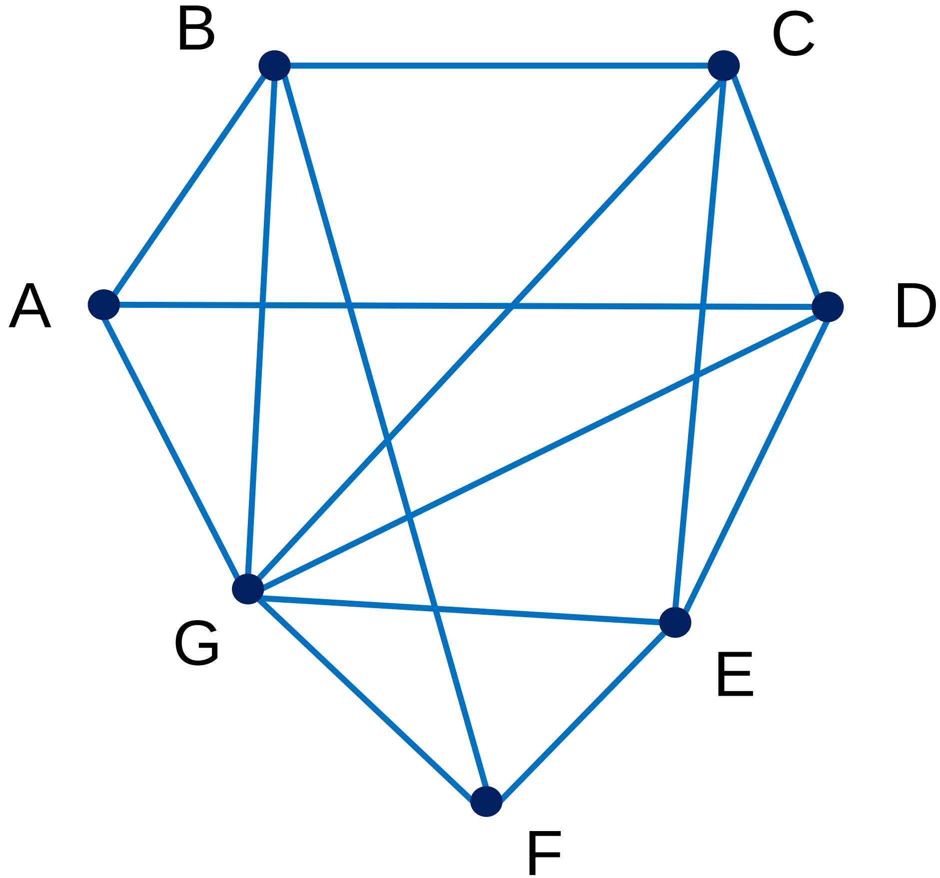
Delete edges AG and FG.

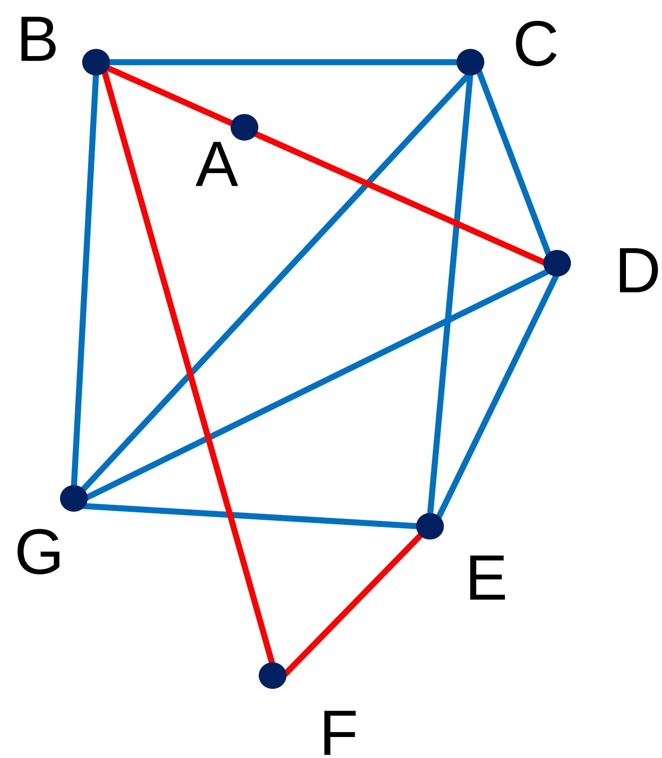




Example 3 – Alternative method

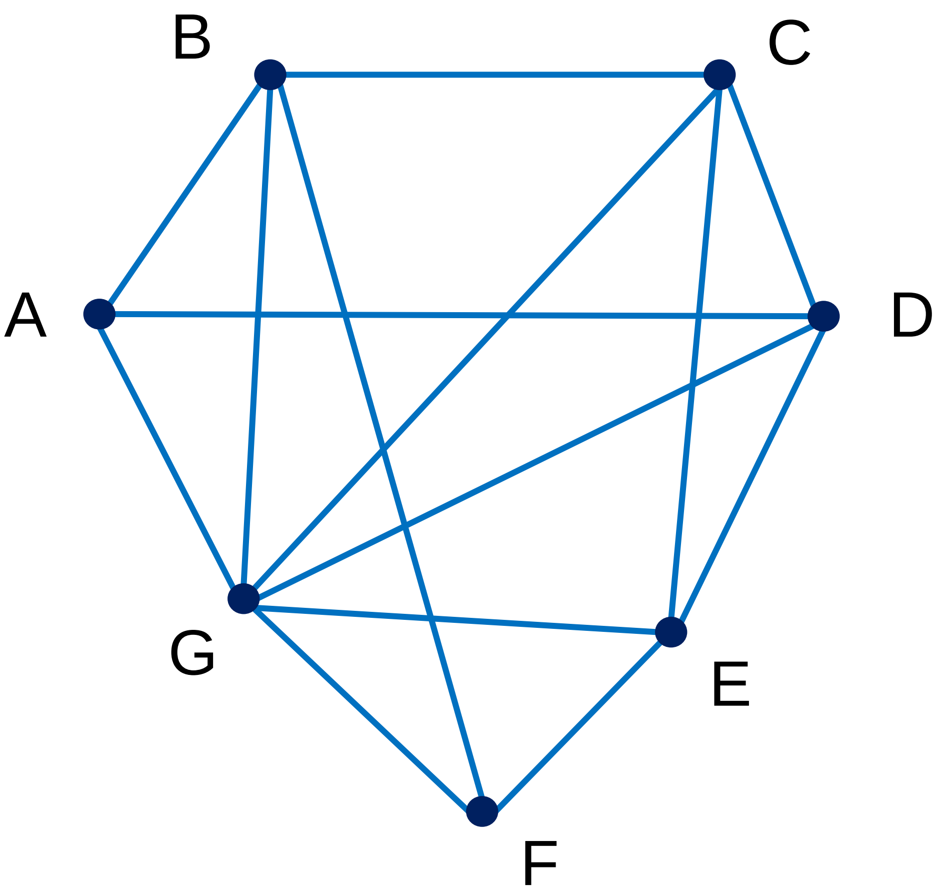
Delete edges AG and FG. Straighten BAD ...

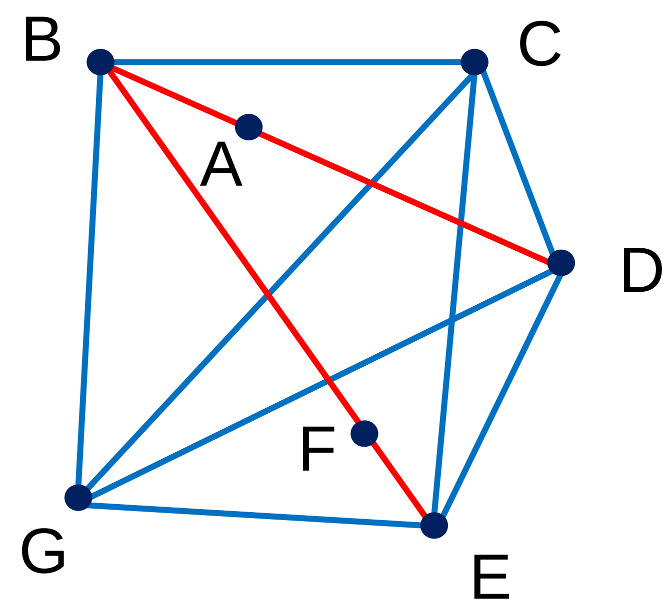




Example 3 – Alternative method

Delete edges AG and FG. Straighten BAD and BFE.



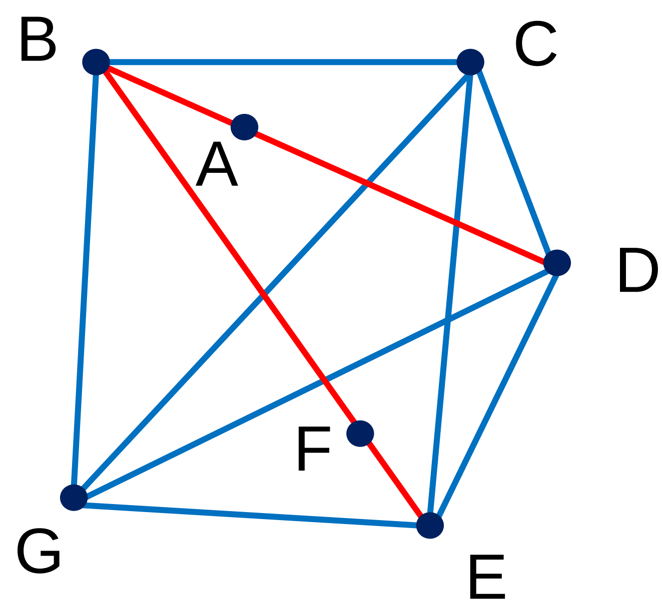


Example 3 – Alternative method

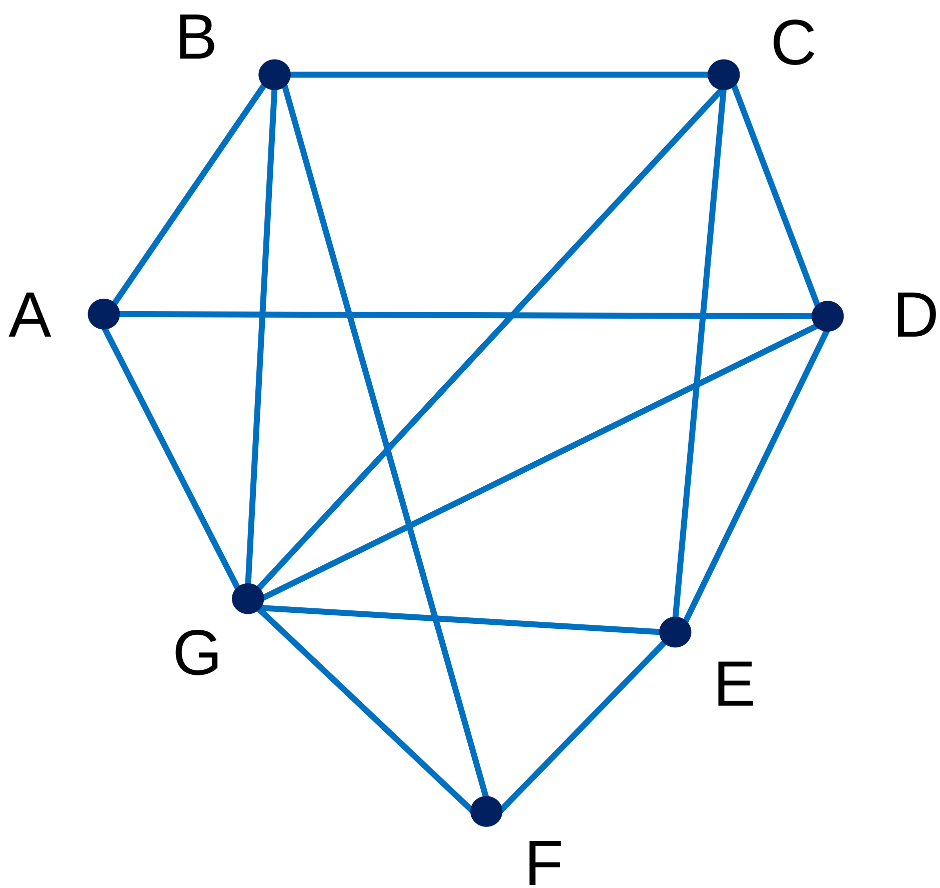
Delete edges AG and FG. Straighten BAD and BFE.

This subgraph of *G* also contains a subdivision of K_5.

By Kuratowski's theorem, *G* is non-planar.



Example 3: Thickness (OCR)

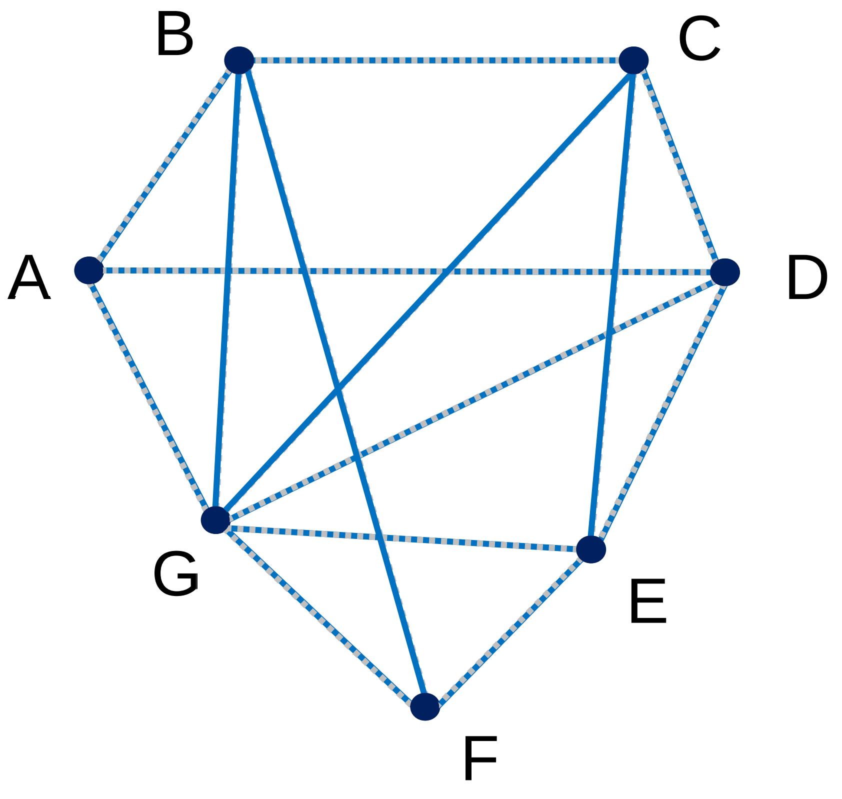


Thickness refers to the minimum number of planar subgraphs required to produce a given graph.

By definition, if a graph is planar it has a thickness of 1.

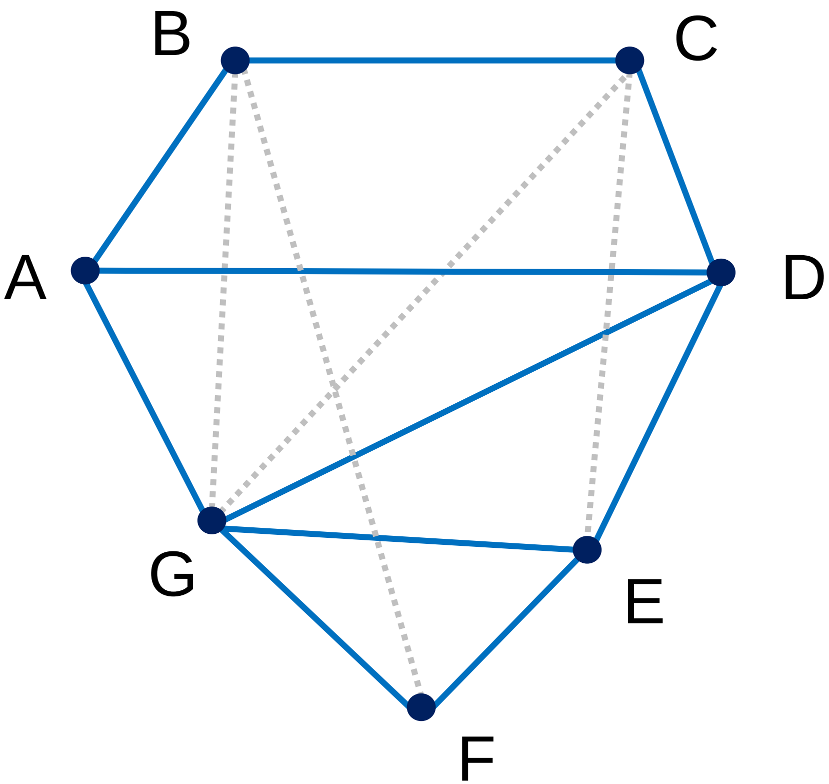
The graph *G*, shown left, is non-planar and so must have thickness > 1.

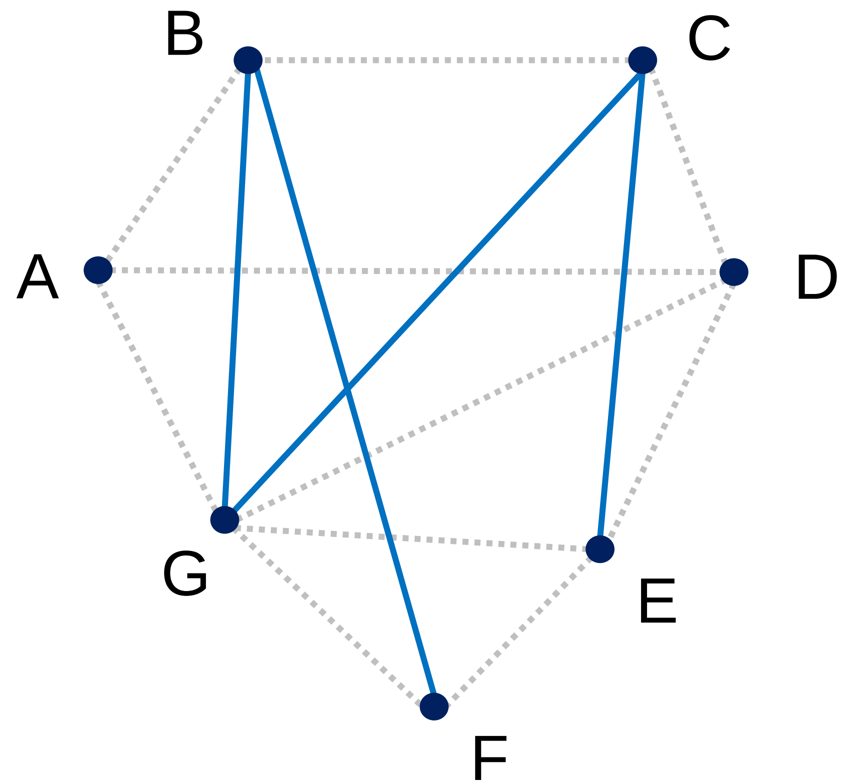
Example 3: Thickness (OCR)



Example 3: Thickness (OCR)

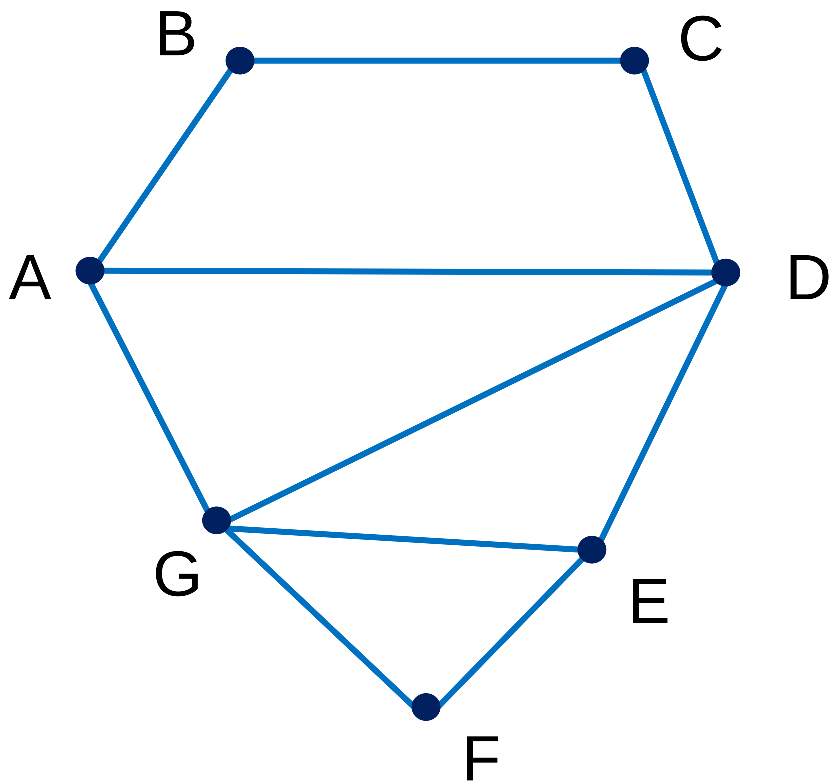
G is the union of the two planar graphs below.

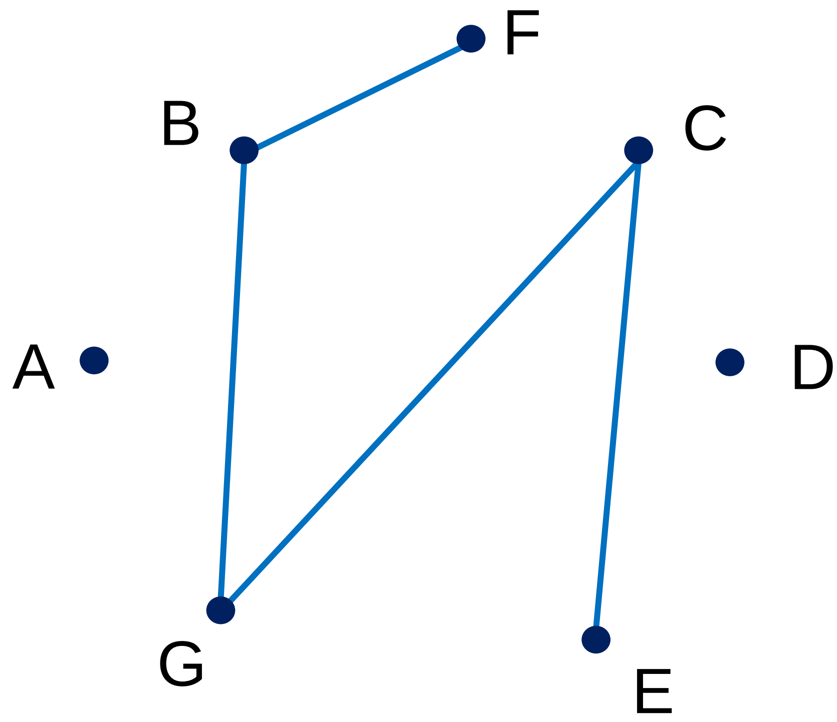




Example 3: Thickness (OCR)

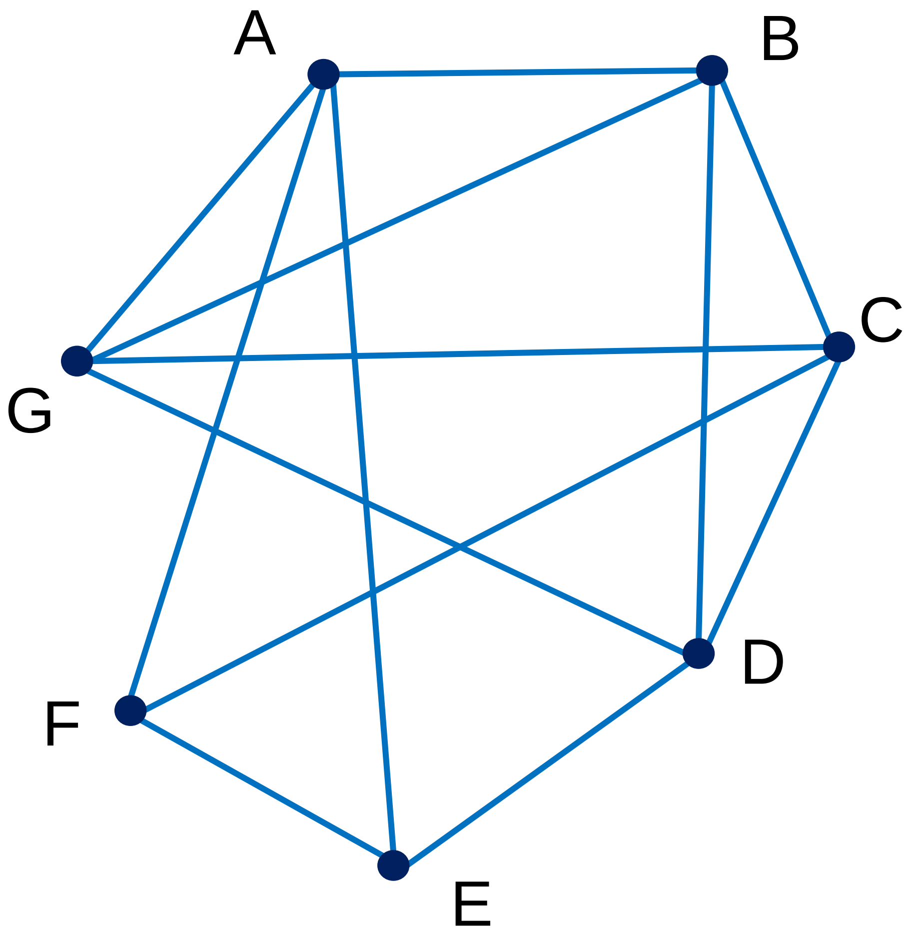
G therefore has thickness = 2.





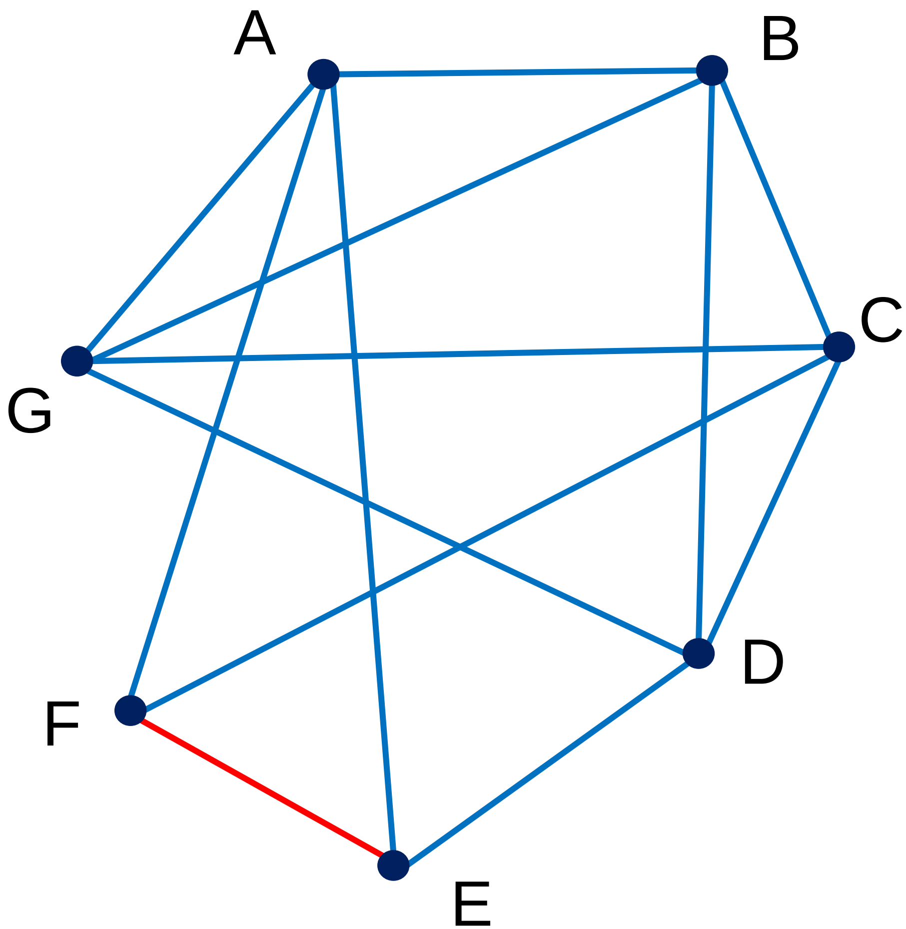
Example 4: Edge Contraction (OCR)

By using an edge contraction, show that *G* is non-planar.



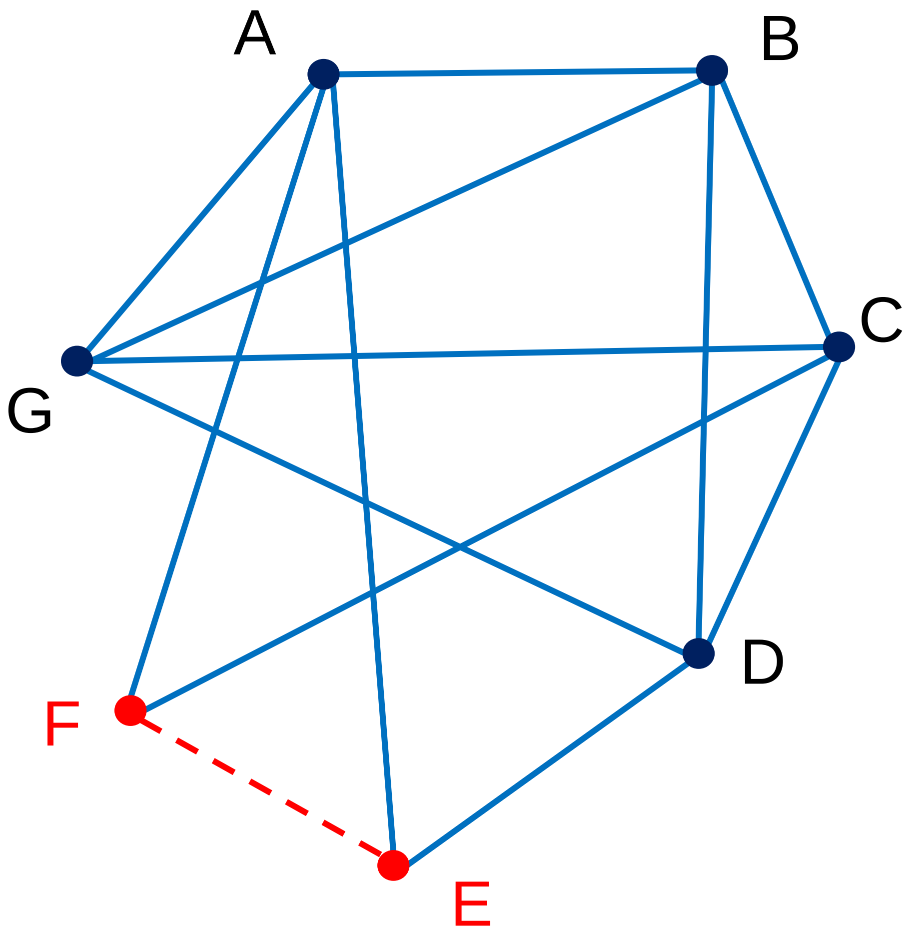
Example 4: Edge Contraction (OCR)

Contract edge EF (this is not the only correct possibility).



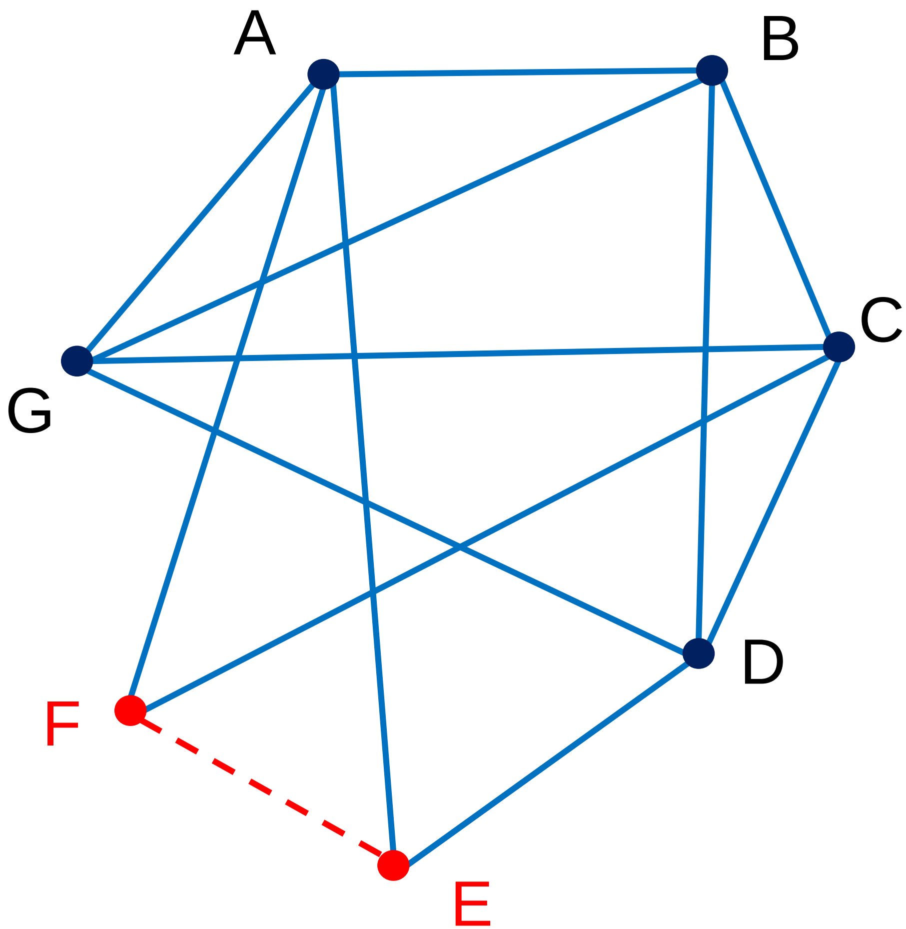
Example 4: Edge Contraction (OCR)

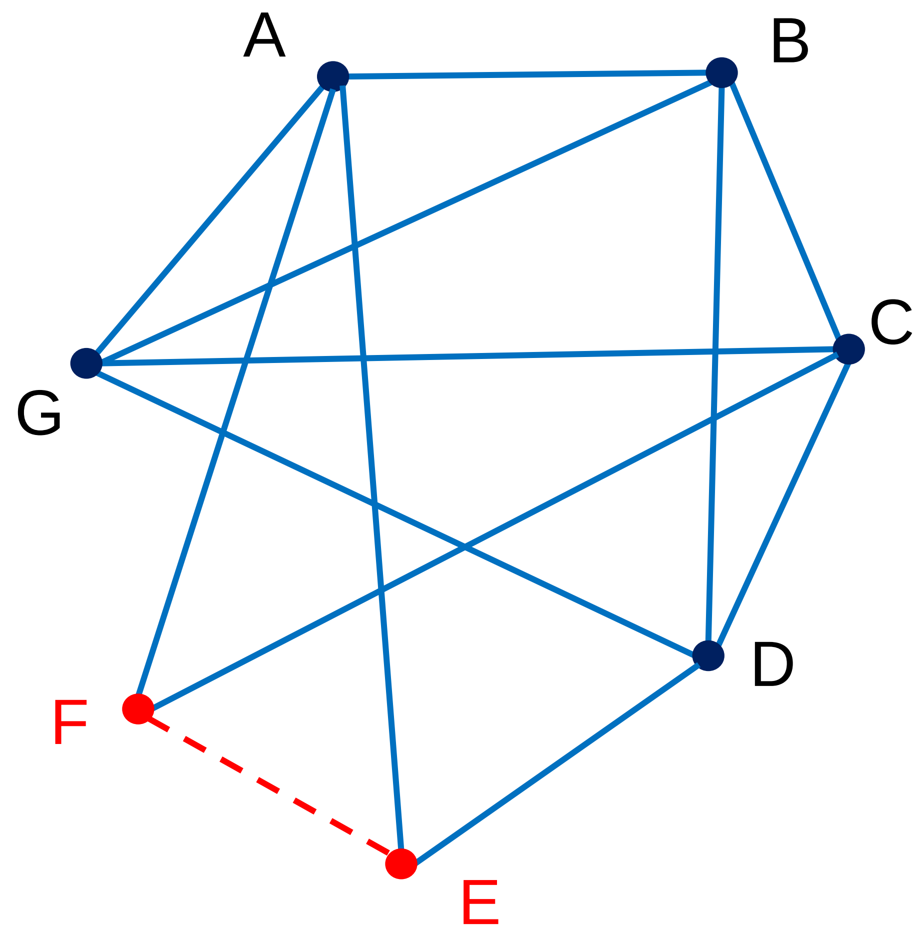
Delete the edge EF so that vertices E and F are concurrent.



Example 4: Edge Contraction (OCR)

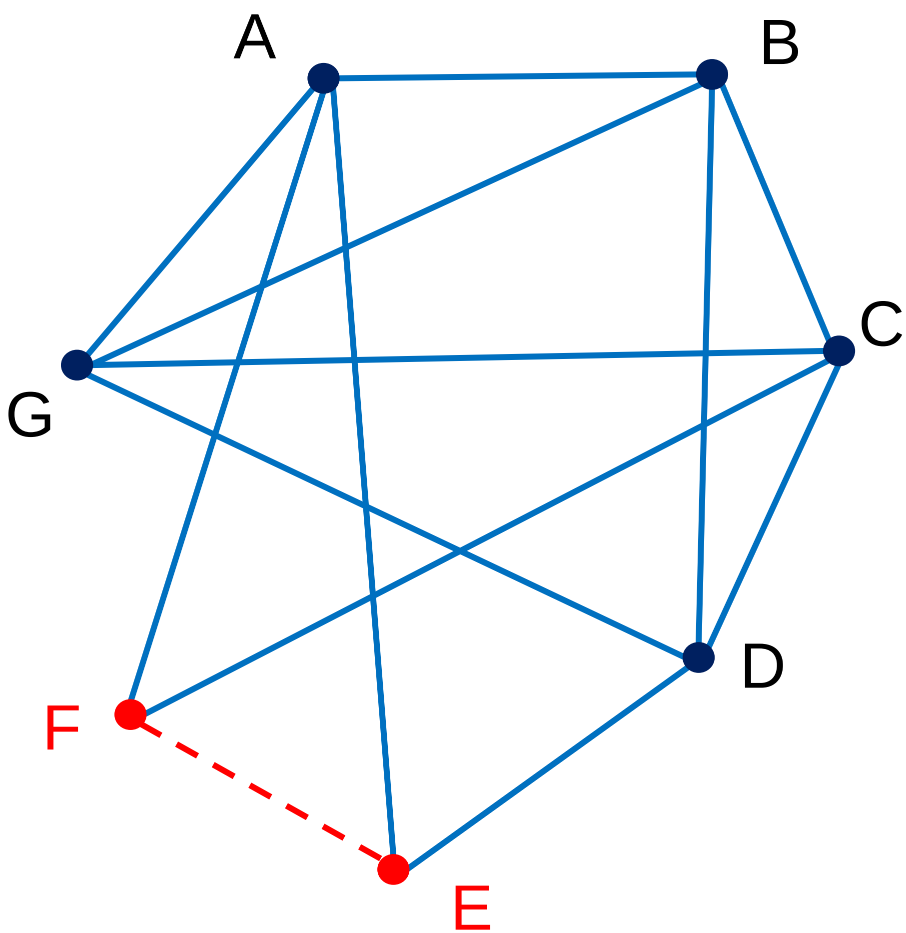
Delete the edge EF so that vertices E and F are concurrent.

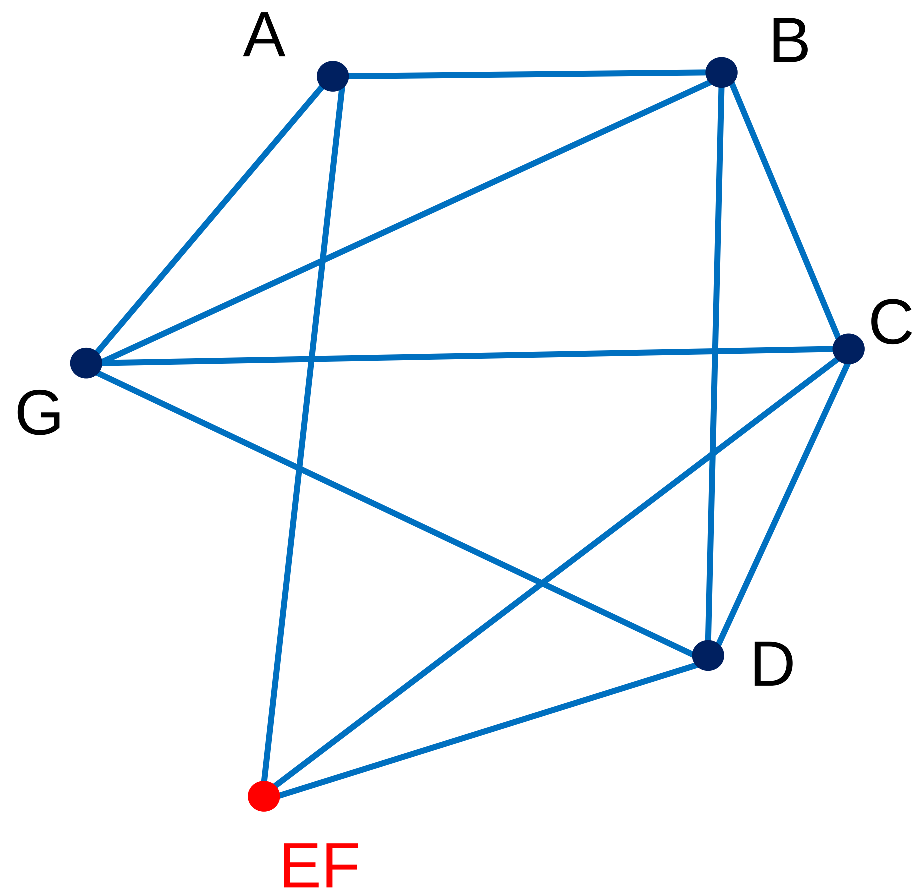




Example 4: Edge Contraction (OCR)

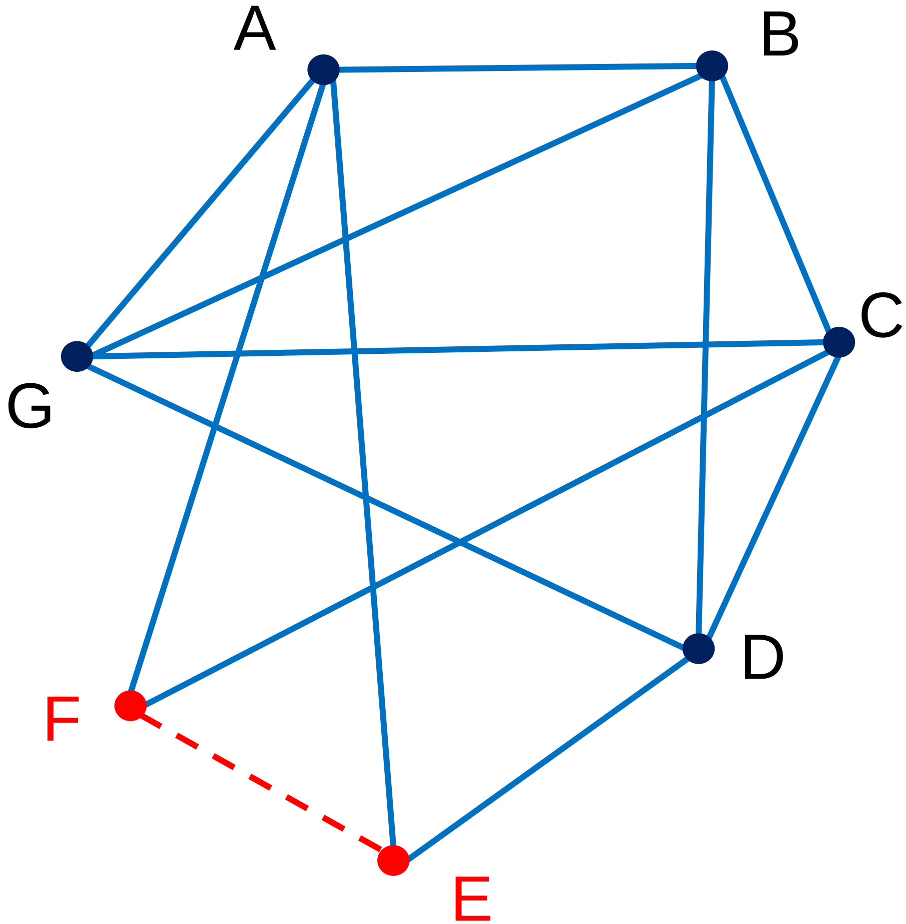
Delete the edge EF so that vertices E and F are concurrent.

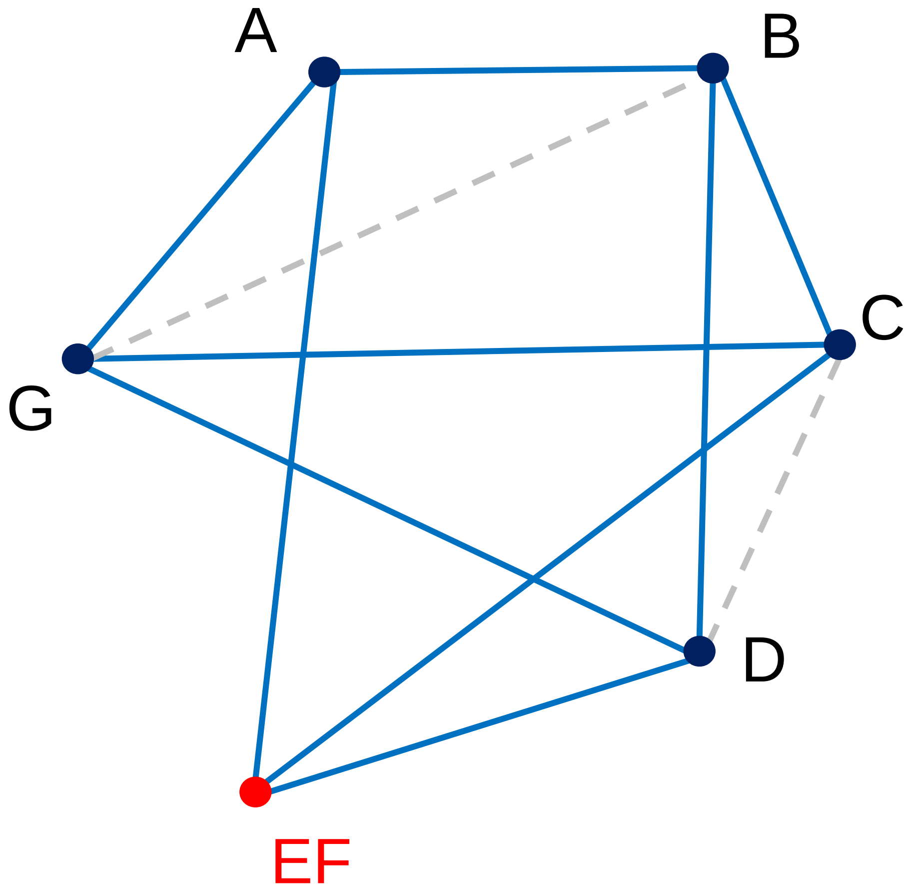




Example 4: Edge Contraction (OCR)

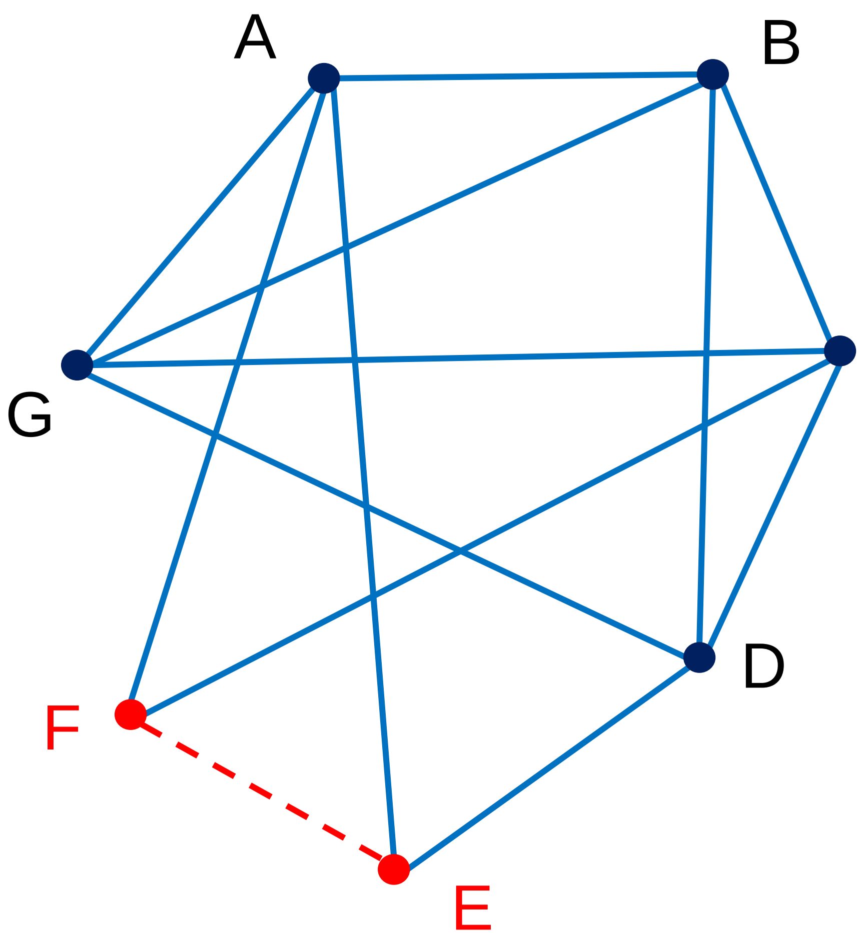
Now remove edges BG and CD to leave …

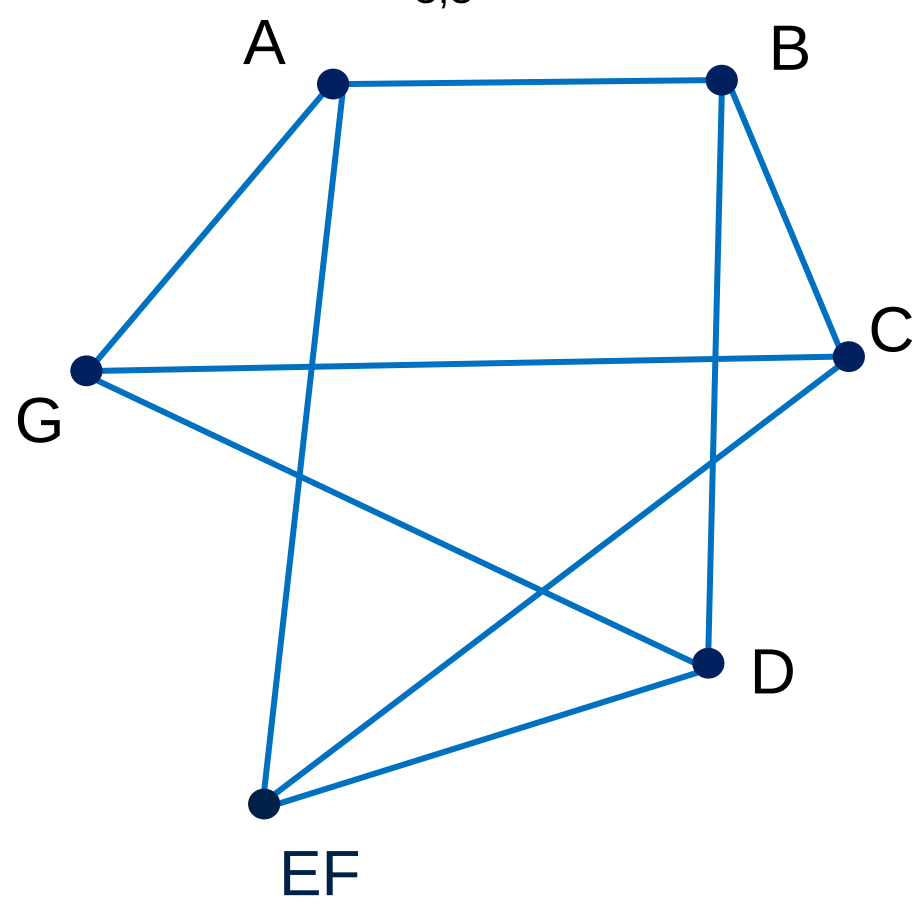




Example 4: Edge Contraction (OCR)

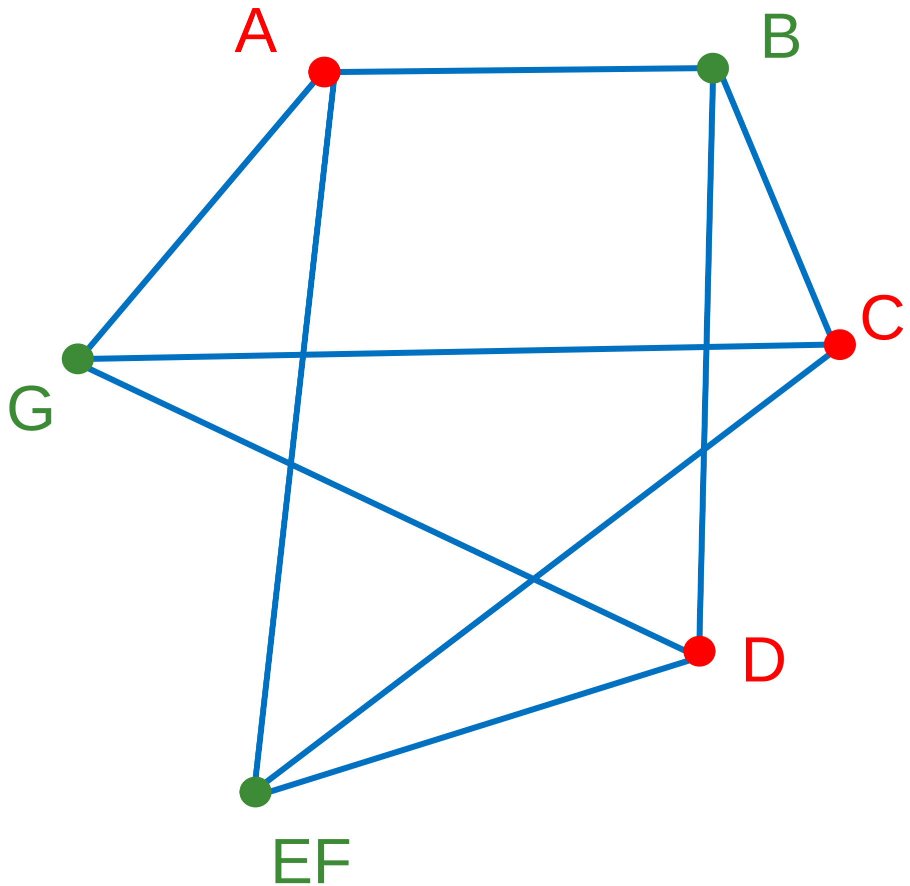
Now remove edges BG and CD to leave … $K_{3,3}$





Example 4: Edge Contraction (OCR)

This contraction of *G* contains $K_{3,3}$ on vertices {A, C, D} and {B, EF, G} and so by Kuratowski's Theorem *G* is non-planar.



Edexcel SAMS A Level Decision 1 Q2

2.

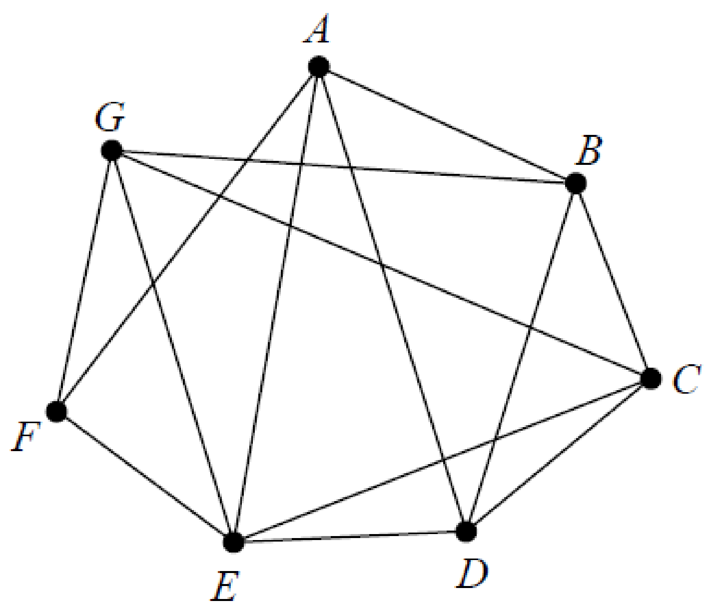


Figure 1

(a) Define what is meant by a **planar** graph.

(2)

(b) Starting at A, find a Hamiltonian cycle for the graph in Figure 1.

(1)

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Arc AG is added to Figure 1 to create the graph shown in Figure 2.

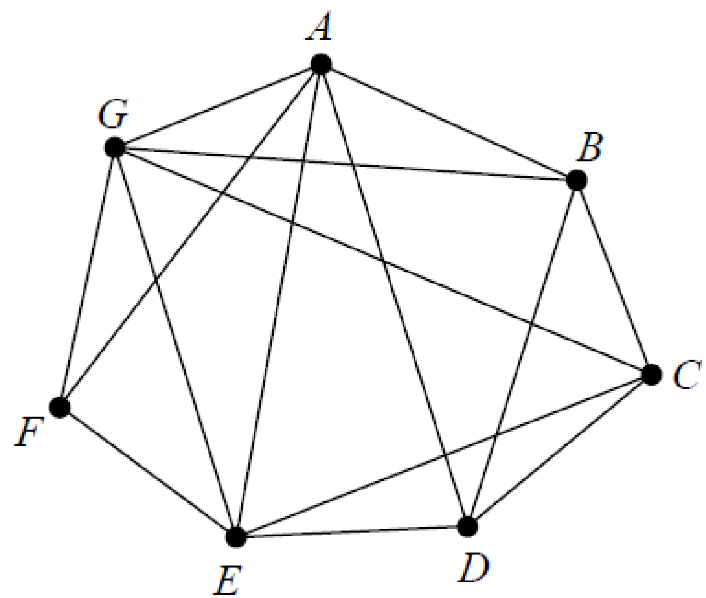


Figure 2

Taking ABCDEFGA as the Hamiltonian cycle,

(c) use the planarity algorithm to determine whether the graph shown in Figure 2 is planar. You must make your working clear and justify your answer.

(4)

Edexcel SAMS A Level Decision 1 Q2

Question	Scheme	Marks	AOs
2(a)	A planar graph is a graph that can be drawn on a plane so that …	B1	1.2
	… no arc meets another arc except at a vertex	B1	1.2
		(2)	
(b)	e.g. ABCDEGFA	B1	1.1b
		(1)	
(c)	Creates two lists of arcs	M1	2.1
	e.g. BG AD		
	CG BD	M1	1.1b
	EG AE		
	CE AF	A1	1.1b
	Since no arc appears in both lists, the graph is planar (**or** draws a planar version)	A1	2.4
		(4)	
			(7 marks)

About MEI

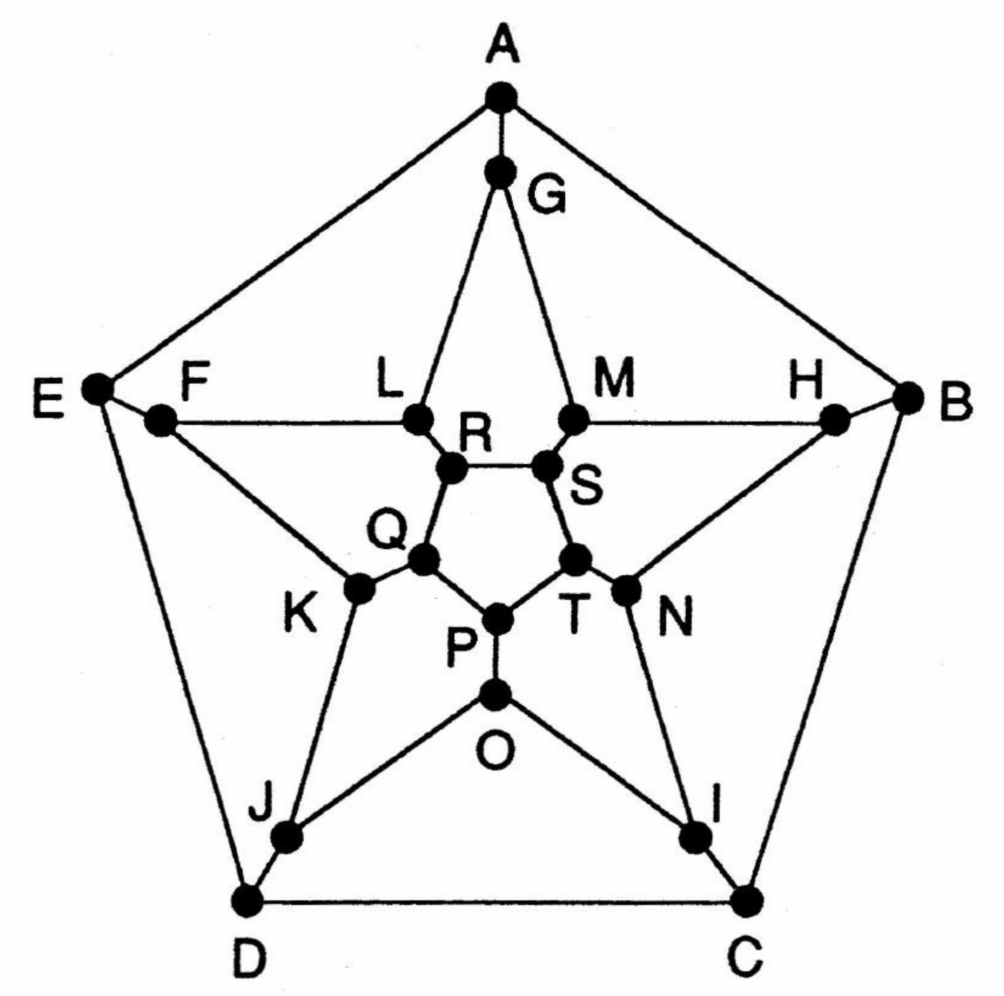
- Registered charity committed to improving mathematics education
- Independent UK curriculum development body
- We offer continuing professional development courses, provide specialist tuition for students and work with employers to enhance mathematical skills in the workplace
- We also pioneer the development of innovative teaching and learning resources

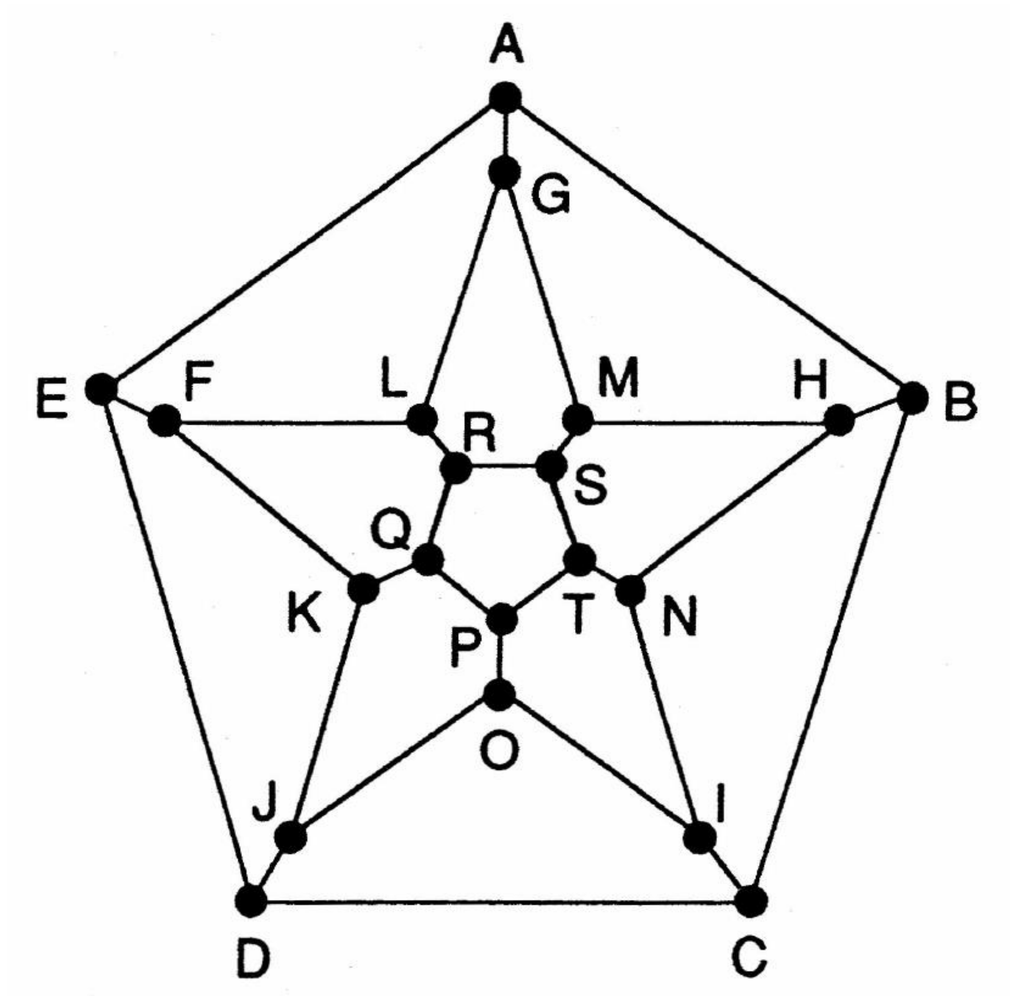
Teaching Discrete Mathematics

Activity: The Icosian Game

In 1857 William Rowan Hamilton invented **The Icosian Game** based on the twenty vertices of a dodecahedron, imagining these as landmarks on a grand World Tour. It was marketed by John Jacques & Sons under the name *"Around the World"*, with the aim of the game being to complete a cycle of twenty edges (an **'icosian'**) through all twenty vertices without repeats. The flat game board is a Schelegel diagram of a dodecahdron.

Can you find two different cycles through all 20 vertices?





Advanced Mathematics Support Programme

Topic mapping for Discrete/Decision Mathematics courses from 2017

The following tables give a side-by-side comparison of the content of the various Discrete/Decision Maths units. It is a summary only and more detail can be found in the Further Mathematics specification documents for each awarding organisation.

Topics shown with a green background are covered in TD1.
Topics shown with a purple background are covered in TD2.

Discrete/Decision Mathematics Topics		AQA	Edexcel	MEI	OCR A
Algorithms	Finite algorithms, definition		AS D1	MwA	AS
	Trace and Interpret and apply algorithms		AS D1	MwA	AS
	Repair, develop and adapt algorithms			MwA	AS
	Complexity/Order of an algorithm (Big O notation)		AS D1	MwA	AS
	Complexity and Efficiency				AS
	Proof, disproof, counter example			MwA	
	Heuristics			MwA	AS
	Sorting algorithms, comparisons and swaps		AS D1	MwA	AS
	- Bubble sort		AS D1		AS
	- Quick sort		AS D1	MwA	A Level
	- Shuttle sort				AS
	Packing algorithms, bin-packing		AS D1	MwA	AS
	Extend knowledge of packing methods, e.g. 2D/3D, Knapsack				A Level
Graph Theory	Nodes/vertices, degree/order	AS	AS D1	MwA	AS
	Arcs/edges, simple, connected, related vocabulary	AS	AS D1	MwA	AS
	Trees	AS		MwA	AS
	Euler's relation: V - E + F = 2	AS			A Level
	Bipartite graphs, $K_{m,n}$	AS		MwA	A Level
	Walk, trail, path, cycle	AS			AS
	Eulerian, semi-Eulerian graphs	AS	AS D1		AS
	Hamiltonian cycles	AS	A Level D1		A Level
	Complete graphs, K_n	AS	AS D1		AS
	Isomorphic Equivalence	A Level	AS D1		AS
	Planar graphs	AS	AS D1		A Level
	Subdivision and contraction	A Level			A Level
	Planarity Algorithm		A Level D1		
	Kuratowski's Theorem	A Level			A Level
	Thickness				A Level
	Complement of a graph	A Level			
	Ore's theorem				A Level

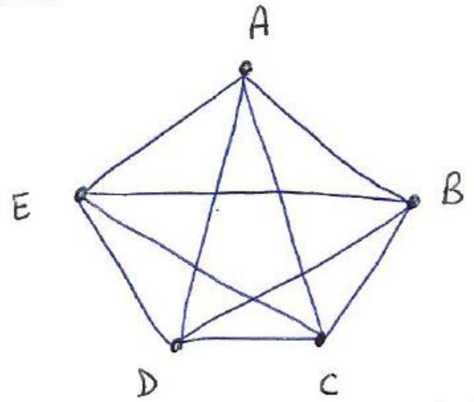
Discrete/Decision Mathematics Topics		AQA	Edexcel	MEI	OCR A
Networks	Modelling problems using weighted graphs	AS	AS D1	MwA	AS
	Adjacency/Incidence matrix	AS	AS D1	MwA	AS
	Minimum connector problem, Kruskal and Prim	AS	AS D1	MwA	AS
	Shortest path, Dijkstra		AS D1	MwA	AS
	Complexity of Kruskal's, Prim's and Dijkstra's algorithms			MwA	AS
	Floyd's algorithm		A Level D1		
	Route inspection (Chinese Postman)	AS	AS D1		A Level
	Travelling salesperson	AS	A Level D1		A Level
	Solving network problems using technology			MwA	
Network Flows	Digraph representation	AS	AS D2	MwA	AS
	Evaluating/interpreting a cut	AS	AS D2	MwA	
	Max flow – min cut theorem	AS	AS D2	MwA	
	Labelling procedure for flow augmentation	A Level	AS D2		
	Supersource, supersink	AS	A Level D2	MwA	
	Upper and lower capacities	A Level	A Level D2		
	Restriction at a vertex	A Level	A Level D2		
Linear Programming	Formulation of constrained problems into Linear Programs	AS	AS D1	MwA	AS
	Graphical solution using an objective function	AS	AS D1	MwA	AS
	Integer solution		AS D1	MwA	A Level
	Slack variables	A Level	A Level D1	MwA	A Level
	Simplex Method	A Level	A Level D1	MwA	A Level
	Interpretation of Simplex	A Level	A Level D1	MwA	A Level
	Big M method		A Level D1	MwA	
	Integer programming, branch-and-bound method				A Level
	Post-optimal analysis			MwA	A Level
	Formulate a range of network problems as LPs			MwA	
	Use of software and interpretation of output			MwA	
Critical Path Analysis	Precedence table for a set of sub-tasks	AS	AS D1	MwA	AS
	Construct an activity network (AQA: on node; others: on arc)	AS	AS D1	MwA	AS
	Forward/backward pass	AS	AS D1	MwA	AS
	Float times, critical activities, critical path	AS	AS D1	MwA	AS
	Refine model due to changes in context of problem	AS			
	Interfering float			MwA	A Level
	Gantt/Cascade chart	A Level	AS D1	MwA	A Level
	Resource histogram	A Level	A Level D1		
	Resource levelling	A Level	A Level D1		
	Scheduling	A Level	A Level D1	MwA	A Level
Game Theory	Two-player, zero-sum games	AS	AS D2		AS
	Pay-off matrix	AS	AS D2		AS
	Play-safe strategies	AS	AS D2		AS
	Stable solutions, value of the game	AS	AS D2		AS
	Reduction of Pay-off matrix using dominance	AS	A Level D2		AS
	Optimal mixed strategies	AS	AS D2		AS
	Graphical solution	AS	AS D2		AS
	Nash equilibrium solution				A Level
	Conversion of higher order games to LP problems	A Level	A Level D2		A Level

Discrete/Decision Mathematics Topics		AQA	Edexcel	MEI	OCR A
Allocation	Cost matrix reduction		**AS D2**		
	Hungarian algorithm		**AS D2**		
	Maximisation problems		**AS D2**		
	Formulation as a Linear Programming problem		A Level D2		
Transportation problems	North-west corner method for an initial feasible solution		A Level D2		
	Stepping-stone method for an improved solution		A Level D2		
	Improvement indices		A Level D2		
	Formulation as a Linear Programming problem		A Level D2		
Dynamic programming	Bellman's principle		A Level D2		
	Stage and State variables		A Level D2		
	Maximum, minimum, maximin, minimax problems		A Level D2		
	Network and table formats		A Level D2		
Recurrence relations	Modelling problems, e.g. population growth		**AS D2**		
	Solution of *first order* linear homogeneous and non-homogeneous relations		**AS D2**		
	Solution of *second order* linear homogeneous and non-homogeneous relations		A Level D2		
	Particular solution, complementary function, auxiliary equation		**AS D2**		
Decision Analysis	Construct and interpret decision trees		A Level D2		
	Decision nodes, chance nodes, end pay-offs		A Level D2		
	Expected monetary values (EMVs) and utility to compare courses of action		A Level D2		
Mathematical Preliminaries	Classifying Problems				**AS**
	Set notation, Venn diagrams				**AS**
	Pigeonhole principle				**AS**
	Arrangement and selection, Permutations & Combinations				**AS**
	Inclusion-exclusion principle for two sets				**AS**
	Inclusion-exclusion principle for more than two sets				A Level
	Derangements				A Level
Binary Operations	Modular Arithmetic and Matrix Multiplication	**AS**			
	Commutativity, associativity	**AS**			
	Cayley tables	**AS**			
	Identity, inverse	**AS**			
Group Theory	Language: order, period, subgroup, proper, trivial, non-trivial	A Level			
	Group axioms; closure, associativity, identity, inverse	A Level			
	Finite/infinite groups	A Level			
	Symmetry groups, abstract groups, modulo-n groups	A Level			
	Cyclic groups and abelian groups	A Level			
	Subgroups, order of group and element, Lagrange's theorem	A Level			
	Generators of groups	A Level			
	Isomorphism between groups of finite order	A Level			

Planarity consolidation exercise solutions

Solutions

1.



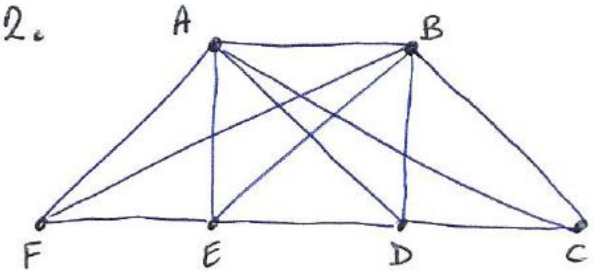
K_5 already has ABCDE(A) as an outer polygon.
Using the planarity algorithm we attempt to allocate edges to two distinct sets, P and Q.

	P	Q
Let AC be in set P:	AC	
AC crosses BE, ∴ BE is in Q:		BE
BE crosses AD, ∴ AD is in P:	AD	
AD crosses EC, ∴ EC is in Q:		EC
EC crosses DB, ∴ DB is in P.		
BUT AC crosses DB, ∴ DB is in Q		

} contradiction!

It is impossible to allot all edges to two distinct sets, so K_5 must be non-planar.

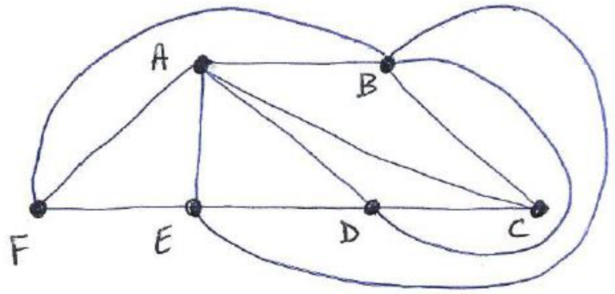
2.



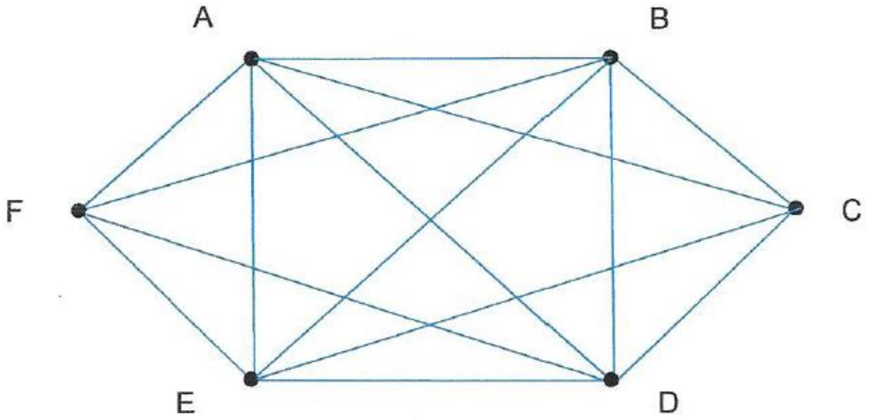
ABCDEF(A) is the Hamiltonian cycle forming the outer polygon.

	P (In)	Q (out)
Let AC be in set P:	AC	
AC crosses BF, BE, BD so they are in Q:		BF, BE, BD
BF (and BE) cross AE, AD so these are in P:	AE, AD	

As all edges have been added to these two distinct sets we can redraw the graph with edges in P inside the polygon and edges in Q outside the polygon.



3.

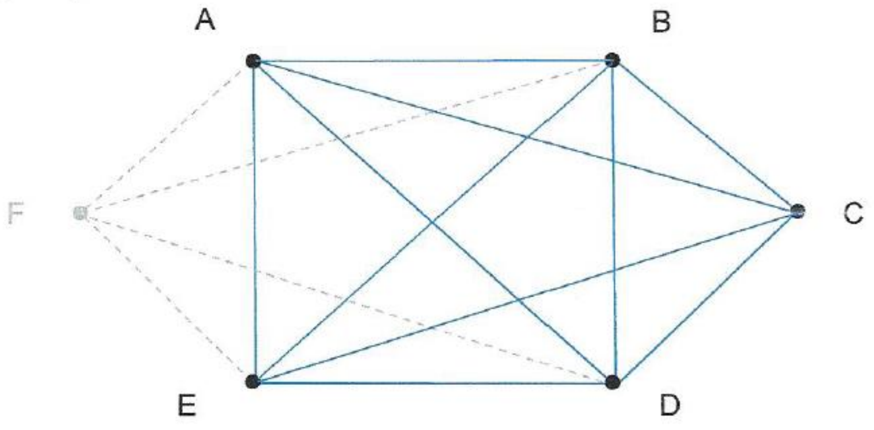


We could attempt the planarity algorithm using ABCDEF(A) as the obvious Hamiltonian cycle.

	P	Q
Let edge AC be in set P:	AC	
AC crosses BD, BE, BF so these must be in Q:		BD, BE, BF
BF crosses AD, AE so these must be in P:	AD, AE	

AD crosses CE, so CE must be in Q
BUT BD crosses CE so CE must be in P
Contradiction!

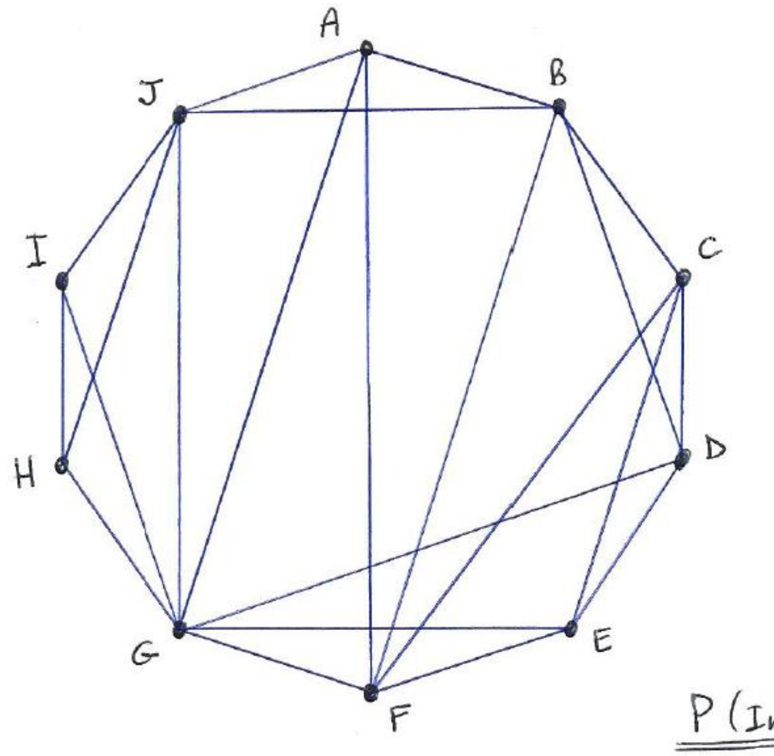
It is not possible to allocate all edges to two distinct sets so the graph is non-planar.



Or .... By deleting vertex F and its associated edges we can see that the graph contains K_5 as a subgraph and so by Kuratowski's theorem must be non-planar.

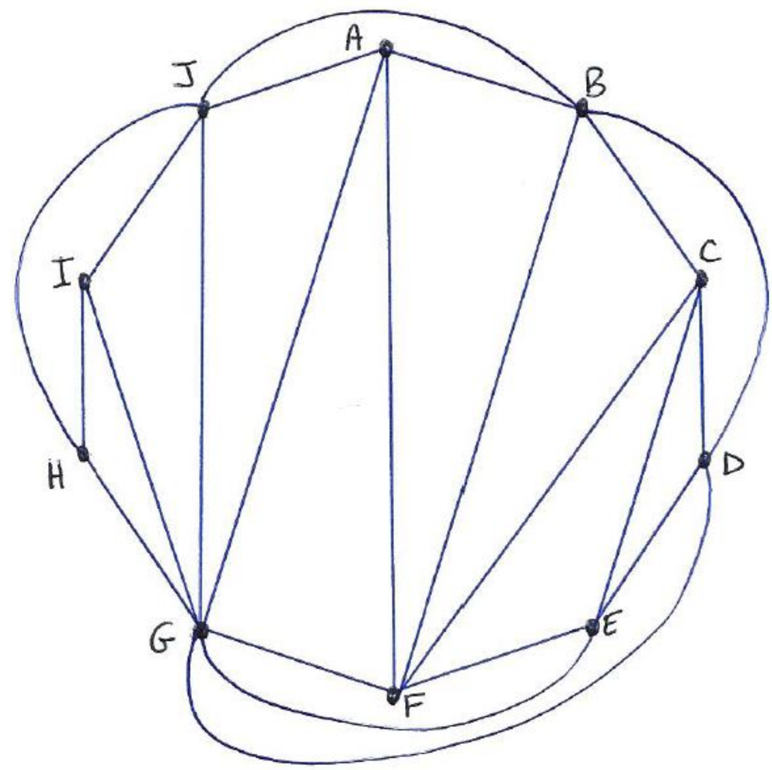
Solutions

4.



	P (In)	Q (Out)
Select edge AF to be in set P:	AF	
AF crosses BJ, DG, EG so these are in Q:		BJ, DG, EG
BJ crosses AG; DG crosses CE, CF, BF; add to P:	AG, CE, CF, BF	
CE (and CF) crosses BD; add BD to Q:		BD
Select GJ to be in set P:	GJ	
Select GI to be in set P; ⇒ HJ must be in Q:	GI	HJ

Now G can be redrawn:

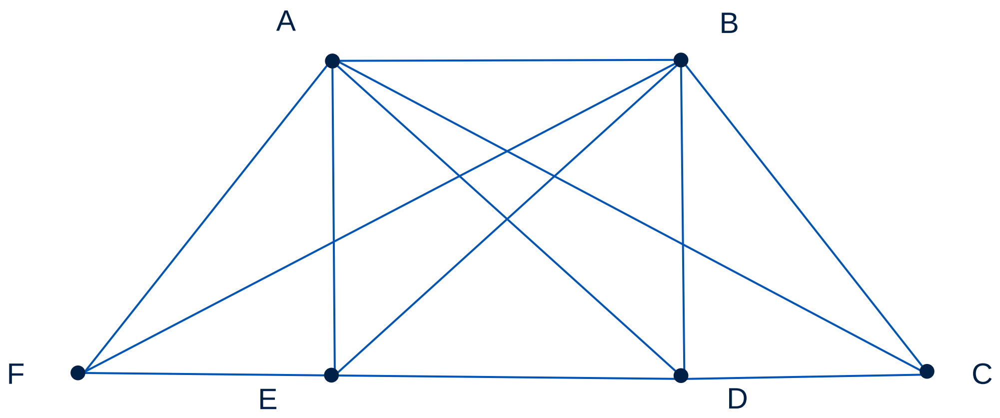


Teaching Discrete Mathematics

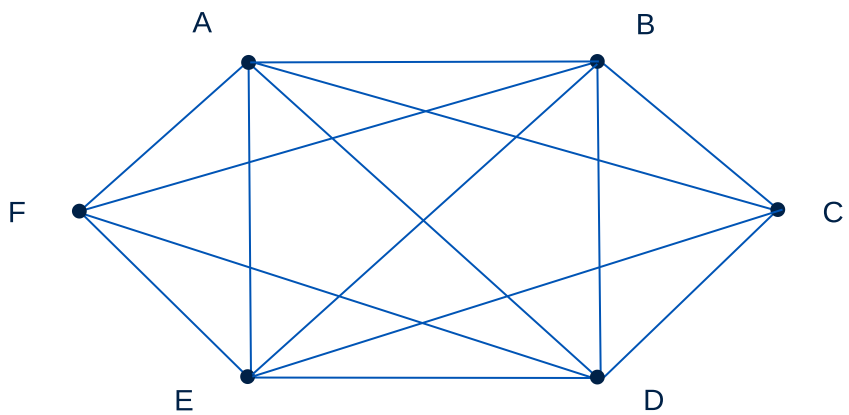
Planarity consolidation exercise

1. By attempting the planarity algorithm, show that K_5 is non-planar.

2. Using the planarity algorithm, redraw the graph below so that no edges are crossing.



3. Prove that the following graph is non-planar.



4. This is the incidence matrix for the graph *G*:

	A	B	C	D	E	F	G	H	I	J
A	0	1	0	0	0	1	1	0	0	1
B	1	0	1	1	0	1	0	0	0	1
C	0	1	0	1	1	1	0	0	0	0
D	0	1	1	0	1	0	1	0	0	0
E	0	0	1	1	0	1	1	0	0	0
F	1	1	1	0	1	0	1	0	0	0
G	1	0	0	1	1	1	0	1	1	1
H	0	0	0	0	0	0	1	0	1	1
I	0	0	0	0	0	0	1	1	0	1
J	1	1	0	0	0	0	1	1	1	0

a) Draw the graph *G* with the polygon A-B-C-D-E-F-G-H-I-J-(A) as the outside.

b) Either redraw the graph into planar form by using the planarity algorithm or use Kuratowski's theorem to prove non-planarity.

Teaching Discrete Mathematics

Planarity key points

Planarity

A **planar graph** can be drawn on a plane without any edges crossing. Any planar graph can also be drawn on a sphere without edges crossing but it is possible to draw more complex graphs on (say) the surface of a torus than can be drawn on a sphere.

The Planarity Algorithm

The following process can be used to re-draw a graph with crossing edges into an obviously planar graph (where it is possible to do so):

- Find a Hamiltonian cycle containing all vertices of the graph
- Re-draw the graph with this Hamiltonian cycle as an outer polygon with other edges inside
- Select any inside edge and assign this to set P
- Assign every edge that crosses the first edge to set Q
- Assign every edge that any edge in Q crosses to set P, etc.
- Finally, re-draw the graph keeping the Hamiltonian cycle outside as before but with edges in set P drawn inside the cycle and edges in set Q drawn outside the cycle (or vice versa).

If it is impossible to allocate all edges to two distinct sets, the graph must not be planar.

Kuratowski's Theorem

A **subdivision** of a graph can be formed by adding vertices of degree 2 along any of the edges. This will not affect the planarity of a graph.

A **contraction** of a graph is formed by the removal of an edge and the fusing of the original endpoints of that edge (e.g. A and B) to become a new single vertex (e.g. AB).

Although K_1, K_2, K_3 and K_4 are planar, K_5 is non-planar. Therefore when $n > 5$, K_n must also be non-planar as K_n must contain K_5.

The simplest bipartite graph which is non-planar is $K_{3,3}$.

For a given graph G, a **subgraph** will be any subset of the vertices and edges in G.

Kuratowski's Theorem states that a graph is planar if and only if it *does not* contain either K_5 or $K_{3,3}$ or a subdivision of either K_5 or $K_{3,3}$ as a subgraph.

Thickness

The **thickness of a graph** G is the minimum number of planar graphs into which the edges of G can be partitioned. Alternatively this can be expressed as: the thickness of a graph is the least number of planar subgraphs whose union is equal to the original graph.

Planar graphs have thickness 1.