SQT-1 (Probability)

Random Experiment → An activity repeated number of times under identical conditions and outcome of activity is not predictable is called Random Exp.
The set of possible outcomes of a random experiment is called sample space.

Exhaustive Case → The total number of possible outcomes in any trial is known as exhaustive events.

Mutually Exclusive Events → Two events are mutually exclusive or disjoint if they cannot both occur at the same time. A clear example is set of outcomes of a single coin, which can result in either heads or tails, but not both.

Independent Events → Two events are indepent if the incidence of one event does not affect the probability of other event.

Dependent Events → If the incidence of one event does affect the probability of other event, then events are dependent.

Probability of an event = $\dfrac{\text{number of favourable cases to the event}}{\text{total number of exhaustive events}}$

Axioms on probability

① If E is an event, then Probability of event E = P(E) has the bounds 0 and 1.

i.e $0 \le P(E) \le 1$

② If E is an impossible event, then $P(E)=0$

③ If E is a certain event, then $P(E)=1$

④ If E is an event, then $P(\bar{E}) = 1 - P(E)$

⑤ If $E_1, E_2, E_3, \ldots E_n$ are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup E_3 \ldots \cup E_n) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_n)$$

Addition theorem on probability

If A and B are any two events, then $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$

Conditional Probability

$P(A/B) =$ P(A given B)

P(the event A after the event B has already happened)

Multiplication theorem on probability

$$P(AB) = P(A\cap B) = \begin{cases} P(A)\, P(B/A) \\ P(B)\, P(A/B) \end{cases}$$

Note → ① If A and B are independent events then $P(A\cap B) = P(A)P(B)$

② If $E_1, E_2, \dots E_n$ are independent events, then

$P(E_1\cap E_2\cap E_3 \dots \cap E_n) = P(E_1)P(E_2) \dots P(E_n)$

Baye's theorem

If $E_1, E_2, \dots E_n$ are mutually exclusive events of sample space S and X is the subset of union of E_i's then

$$P(E_i/X) = \frac{P(X/E_i)\, P(E_i)}{\sum P(X/E_i)\, P(E_i)}$$

Q. Four cards are drawn from a pack of cards. Find the probability that

(i) all are diamonds

(ii) there is one card of each suit

(iii) there are two spades and two hearts

(iv) all cards having distinct numbers

Soln (i) $\frac{^{13}C_4}{^{52}C_4}$ (ii) $\frac{^{13}C_1 \times {^{13}C_1} \times {^{13}C_1} \times {^{13}C_1}}{^{52}C_4}$

(iii) $\frac{^{13}C_2 \times {^{13}C_2}}{^{52}C_4}$ (iv) $\frac{52}{52} \times \frac{48}{51} \times \frac{44}{50} \times \frac{40}{49}$ or $\frac{^{13}C_4 \times 4\times4\times4\times4}{^{52}C_4}$

Q A and B are throwing a pair of dice alternatively. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins the game of throwing dice then find the probability of A's chance of winning.

Soln Let E_1 = event that A throws 6

E_2 = event that B throws 7

6 = (1,5) (5,1) (2,4) (4,2) (3,3)

$P(E_1) = \frac{5}{36}$ $\quad P(E_2) = \frac{6}{36} = \frac{1}{6}$

7 = (1,6) (6,1) (3,4) (4,3) (2,5) (5,2)

P(A win) = ~~[illegible]~~

$P(E_1) + P(\bar{E_1} \cap \bar{E_2} \cap E_1) + P(\bar{E_1} \cap \bar{E_2} \cap \bar{E_1} \cap \bar{E_2} \cap E_1) + \cdots$

$= P(E_1) + P(\bar{E_1})P(\bar{E_2})P(E_1) + P(\bar{E_1})P(\bar{E_2})P(\bar{E_1})P(\bar{E_2})P(E_1) + \cdots$

$= \frac{5}{36} + \left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{5}{36}\right) + \left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{31}{36}\right)\left(\frac{5}{6}\right)\left(\frac{5}{36}\right) + \cdots$

$= \frac{5}{36} + \left(\frac{31\times5}{36\times6}\right)\times\left(\frac{5}{36}\right) + \left(\frac{31\times5}{36\times6}\right)^2 \times \frac{5}{36} + \cdots$

$= \dfrac{\frac{5}{36}}{1 - \frac{31\times5}{36\times6}} = \dfrac{\frac{5}{36}}{\frac{36\times6 - 31\times5}{36\times6}} = \frac{30}{61}$ Ans

Q A doctor has decided to prescribe two new drugs 'A' and 'B' to 200 heart patients as follow: 50 get drugs A, 50 get drug B, and 100 get both the drugs. The 200 patients were chosen so that each had an 80% chances of having a heart attack if given neither drug. Drug A reduces the probability of heart attack by 35%. Drug B reduces the probability of heart attack by 20% and the two drug, when given together, work independently. If a random selected patient has an heart attack, what is probability that the patient was given drug A, drug B and both drugs:

Sol^n $E_1 \rightarrow$ event that the patient was given drug A
$E_2 \rightarrow$ " " " " " " B
E_3 " " " " " " both the drugs

$P(E_1) = \frac{50}{200} = \frac{1}{4}$ $P(E_2) = \frac{50}{200} = \frac{1}{4}$ $P(E_3) = \frac{100}{200} = \frac{1}{2}$

$X \rightarrow$ event of getting heart attack by a patient

$P\left(\frac{X}{E_1}\right) = \frac{80}{100} \times 0.65 = 0.80 \times 0.65$

$P\left(\frac{X}{E_2}\right) = 0.80 \times 0.80$

$P\left(\frac{X}{E_3}\right) = 0.80 \times 0.80 \times 0.65$

Now have to find

$$P\left(\frac{E_1}{X}\right) = \frac{P\left(\frac{X}{E_1}\right) P(E_1)}{P\left(\frac{X}{E_1}\right) P(E_1) + P\left(\frac{X}{E_2}\right) P(E_2) + P\left(\frac{X}{E_3}\right) P(E_3)}$$

$P\left(\frac{E_2}{X}\right) =$

$P\left(\frac{E_3}{X}\right) =$

Q The completion of a construction project depends on whether the carpenters and plumbers working on the project will go on strike. The probabilities of delay are 100%, 80%, 40% and 5% if both go on strike, carpenters alone go on strike, plumbers alone go on strike and neither of them strikes respectively. Also there is 60% chance that plumbers strike if carpenters strike and if plumbers go on strike there is 30% chance that capenters would follow. It is known that the chance for the plumbers strike is 10%.

(a) Determine the probability of delay in completion of project.

(b) If there is a delay, in completion of project, determine

(i) probability that both carpenters and plumbers strike

(ii) probability that carpenters strike and plumbers do not

(iii) probability of carpenters strike.

Sol^n : P = event that plumbers strike (2)

C = event that carpenters strike

D = Delay in the completion of Project.

$P\left(\frac{D}{PC}\right) = 1$, $P\left(\frac{D}{\bar{P}C}\right) = 0.8$, $P\left(\frac{D}{P\bar{C}}\right) = 0.4$ $P\left(\frac{D}{\bar{P}\bar{C}}\right) = 0.05$

$P\left(\frac{P}{C}\right) = 0.6$, $P\left(\frac{C}{P}\right) = 0.3$, $P(P) = 0.1$

(a) $P(D) = P\left(\frac{D}{PC}\right) P(PC) + P\left(\frac{D}{\bar{P}C}\right) P(\bar{P}C) + P\left(\frac{D}{P\bar{C}}\right) P(P\bar{C}) + P\left(\frac{D}{\bar{P}\bar{C}}\right) P(\bar{P}\bar{C})$ —(1)

$P(PC) = \begin{cases} P\left(\frac{P}{C}\right) P(C) \\ \text{or} \\ P\left(\frac{C}{P}\right) P(P) \Rightarrow 0.3 \times 0.1 = 0.03 \end{cases}$

Also $P(C) = \frac{P(PC)}{P(P/C)} = \frac{0.03}{0.6} = 0.05$

$P(\bar{P}C) = \begin{cases} P\left(\frac{\bar{P}}{C}\right) P(C) \Rightarrow \left[1 - P\left(\frac{P}{C}\right)\right] P(C) \Rightarrow (1-0.6) \times 0.05 = 0.4 \times 0.05 = 0.02 \\ \text{or} \\ P\left(\frac{C}{\bar{P}}\right) P(\bar{P}) \end{cases}$

$P(P\bar{C}) = \begin{cases} P\left(\frac{P}{\bar{C}}\right) P(\bar{C}) \\ \text{or} \\ P\left(\frac{\bar{C}}{P}\right) P(P) \Rightarrow \left[1 - P\left(\frac{C}{P}\right)\right] P(P) \Rightarrow [1-0.3] \times 0.1 = 0.7 \times 0.1 = 0.07 \end{cases}$

$P(\bar{P}\bar{C}) = P(\bar{P} \cap \bar{C}) = P(\overline{P \cup C}) = 1 - P(P \cup C) = 1 - [P(P) + P(C) - P(P \cap C)]$

$\Rightarrow 1 - [0.1 + 0.05 - 0.03]$

$\Rightarrow 1 - [0.10 + 0.02] = 1 - 0.12 = 0.88$

$\boxed{P(PC) = 0.03}$ $\boxed{P(P\bar{C}) = 0.07}$ $\boxed{P(\bar{P}C) = 0.02}$ $\boxed{P(\bar{P}\bar{C}) = 0.88}$

Put all these value in (1)

(b) (i) $$P\left(\frac{PC}{D}\right) = \frac{P\left(\frac{D}{PC}\right) P(PC)}{P(D)}$$

(ii) $$P\left(\frac{\overline{PC}}{D}\right) = \frac{P\left(\frac{D}{\overline{PC}}\right) P(\overline{PC})}{P(D)}$$

(iii) $$P\left(\frac{C}{D}\right) = \frac{P\left(\frac{D}{\overline{PC}}\right) P(\overline{PC}) + P\left(\frac{D}{PC}\right) P(PC)}{P(D)}$$

Random Variables (R.V)

Types of R.V → ① Discrete R.V
② Continuous R.V.

Random Experiment
↓
possible outcomes are random
↓
Connect these possible outcomes to a variable
↓
random variable

Discrete R.V → If it takes prescribed set of values
eg throwing a dice → we get 1,2,3,4,5,6

Continuous R.V → All possible values in an interval.

Discrete Random Variable → If X is an event and its probable values are $x_1, x_2, \ldots, x_n$ that means X takes values $x_1, x_2 \ldots x_n$ randomly. Then the variable X is called a discrete random variable.

Continuous Random Variable → If X takes all possible values in the range a to b i.e $a \le x \le b$ then X takes the values randomly, then X is called a continuous random variable.

For any random variable we define

(i) Probability density function, or probability mass function (p.d.f)
(name used for continuous r.v. — density; name used for discrete r.v — mass)
(it is also known as probability distribution)

(ii) Probability distribution function (or cumulative probability distribution)

Discrete Probability density functions

The discrete random variable X takes the values $x_1, x_2, \ldots x_n$ and their respective probabilities are $p_1, p_2, \ldots p_n$ such that sum of p_i's is equal 1.

Then the set of values (x_i, p_i) $i = 1, 2, \ldots n$ represent the p.d.f for X.

$$p(x_i) = p_i \qquad i = 1, 2 \ldots n$$

$$\sum_{i=1}^{n} p_i = 1$$

$E(X)$ = Expectation of X

for discrete r.v X → If the set of values (x_i, p_i), $i = 1, 2, \dots n$ represent the p.d.f for X then $E(X)$ is defined as

$$\boxed{E(X) = \sum_{i=1}^{n} p_i x_i}$$

for continuous r.v. X → If $f(x)$, $a \le x \le b$ is the p.d.f of continuous r.v X, then $E(X)$ is defined as $\boxed{E(X) = \int_a^b x f(x)\, dx}$

Note → ① for a discrete r.v. X, if the set of values (x_i, p_i), $i = 1, 2, \dots n$ represent the p.d.f for X then $\boxed{E(\phi(x)) = \sum_{i=1}^{n} p_i \phi(x_i)}$

② for a continuous r.v. X if $f(x)$, $a \le x \le b$ is the p.d.f of continuous r.v. X then $\boxed{E(\phi(x)) = \int_a^b \phi(x) f(x)\, dx}$

Properties of Expectation

① If X, Y are two random variables then $E(X+Y) = E(X) + E(Y)$

② If X is random variable and c is a constant then $E(cX) = cE(X)$

③ If c is a constant, then $E(c) = c$ where c is a constant.

④ If X and Y are two independent random variables then $E(XY) = E(X)\, E(Y)$

⑤ The quantity of measure of relationship between X and Y is called as covariance between X and Y and is defined as

Covariance between X and Y = $Cov(X,Y) = E((X - E(X))(Y - E(Y)))$

$\Rightarrow E(XY) - E(X)E(Y)$

Proof $E[(X-E(X))(Y-E(Y))]$

$E[XY - XE(Y) - YE(X) + E(X)E(Y)]$

$E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)$

$E(XY) - E(X)E(Y)$

Note → ① If $Cov(X,Y)=0$ then the random variables X and Y are independent.

② If $Cov(X,Y)<0$ then X and Y are inversely propotional related random variables.

③ If $Cov(X,Y)>0$ then X and Y are directly probotional related random variables.

Calculation of measures of central tendency and measures of dispersion using p.d.f of a random variable X.

Let us suppose that the p.d.f of X is available where X is either a discrete or continuous r.v.

Mean of X = $E(X)$

C.I	freq	Mid value of C.I
l_1-u_1	f_1	$x_1 = \frac{l_1+u_1}{2}$
l_2-u_2	f_2	$x_2 = \frac{l_2+u_2}{2}$
⋮		
l_n-u_n	f_n	$x_n = \frac{l_n+u_n}{2}$

$$\text{Mean} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$$

$$= \frac{f_1x_1 + f_2x_2 + \cdots + f_nx_n}{N}$$

$$= \frac{f_1}{N}x_1 + \frac{f_2}{N}x_2 + \cdots + \frac{f_n}{N}x_n$$

$$= p_1x_1 + p_2x_2 + \cdots + p_nx_n$$

$$= \sum p_i x_i = E(X)$$

Q A random variable X has the following probability distribution (6)

X	0	1	2	3	4	5	6	7
P(x)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

① find K ② $P(X<6)$, $P(X\le 6)$, $P(0<x<5)$

③ Distribution function

④ if $P(x\le c) > \frac{1}{2}$ find min value of c

⑤ find $\left(\frac{1.5<x<4.5}{x>2}\right)$

Soln ① $\Sigma p_i = 1 \Rightarrow 10K^2+9K=1 \Rightarrow 10K^2+9K-1=0$

$10K^2+10K-K-1=0$

$10K(K+1)-1(K+1)=0$

$(K+1)(10K-1)=0$

$K=-1$ ✗ $\quad K=\frac{1}{10}$

②

X	0	1	2	3	4	5	6	7
p(x)	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

$P(X<6) = 1-P(X\ge 6) = 1-[P(6)+P(7)]$

$= 1-0.02-0.17 = 1-0.19 = 0.81$

$P(X\le 6) = 1-P(X>6) = 1-P(X=7)$

$= 1-0.17 = 0.83$

$P(0<X<5) = P(1)+P(2)+P(3)+P(4) = 0.1+0.2+0.2+0.3 = 0.8$

③ $F(x) = \begin{cases} 0 & x\le 0 \\ 0.1 & x\le 1 \\ 0.3 & x\le 2 \\ 0.5 & x\le 3 \\ 0.8 & x\le 4 \\ 0.81 & x\le 5 \\ 0.83 & x\le 6 \\ 1 & x\le 7 \end{cases}$

④ from ③ Ans : $c=4$ $\quad P(x\le 4) > \frac{1}{2}$

$0.8 > \frac{1}{2}$

⑤ $P\left(\frac{1.5<x<4.5}{x>2}\right) = \frac{P((1.5<x<4.5)\cap(x>2))}{P(x>2)} = \frac{P(x=3)+P(x=4)}{1-P(x\le 2)}$

$= \frac{0.2+0.3}{1-(0+0.1+0.2)} = \frac{0.5}{0.7} = \frac{5}{7}$

Q If X is a continuous random variable with the following p.d.f

$$f(x) = \begin{cases} \alpha(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

find ① α ② $P(X>1)$

Sol^n ① $\int_{-\infty}^{\infty} f(x)\,dx = 1$ since $f(x)$ is pdf

$$\int_0^2 \alpha(2x-x^2)\,dx + \underbrace{\int_{-\infty}^{0} 0\,dx}_{0} + \underbrace{\int_{2}^{\infty} 0\,dx}_{0} = 1$$

$$\alpha \int_0^2 (2x - x^2)\,dx = 1$$

$$\alpha \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1 \Rightarrow \alpha\left[4 - \frac{8}{3}\right] = 1 \Rightarrow \frac{\alpha \times 4}{3} = 1$$

$$\alpha = \frac{3}{4}$$

② $P(X>1) = \frac{3}{4}\int_1^2 (2x - x^2)\,dx = \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^2$

$$\Rightarrow \frac{3}{4}\left[4 - \frac{8}{3}\right] - \frac{3}{4}\left[1 - \frac{1}{3}\right] = \frac{3}{4}\left[\frac{4}{3}\right] - \frac{3}{4}\times\frac{2}{3}$$

$$= 1 - \frac{1}{2} = \frac{1}{2} \quad \text{Ans}$$

Q A random variable X has density function

$$f(x) = \begin{cases} kx^2 & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

find k, $P(1 \le x \le 2)$, $P(X \le 2)$, $P(X>1)$

Sol^n # $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{-3} 0\,dx + \int_{-3}^{3} kx^2\,dx + \int_{3}^{\infty} 0\,dx = 1$

$$\Rightarrow k\int_{-3}^{3} x^2\,dx = 1 \Rightarrow k\left[\frac{x^3}{3}\right]_{-3}^{3} = 1$$

$$\Rightarrow k[9+9] = 1 \Rightarrow \boxed{k = \frac{1}{18}}$$

$P(1 \le x \le 2) = \frac{1}{18}\int_1^2 x^2\,dx$

$P(X>1) = \frac{1}{18}\int_1^3 x^2\,dx$

$P(X \le 2) = \frac{1}{18}\int_{-3}^{2} x^2\,dx$

① The Probability density function of random variable X

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

① find $P(X \ge 1.5)$

② find cumulative distribution function

Sol^n ① $P(X \ge 1.5) = \int_{1.5}^{\infty} f(x)\,dx = \int_{1.5}^{2} (2-x)\,dx$

$$= \left[2x - \frac{x^2}{2}\right]_{1.5}^{2} = [4-2] - \left[3 - \frac{2.25}{2}\right] = 0.125$$

② Cumulative distribution function

$X \le 0$

$$P(X \le x) = \int_{-\infty}^{x} f(x)\,dx = 0$$

$0 < x < 1$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x\,dx = \frac{x^2}{2}$$

$x \le 2$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_0^1 + \int_1^x (2-x)\,dx$$

$$= \frac{1}{2} + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + 2x - \frac{x^2}{2} - \left(2 - \frac{1}{2}\right) = 2x - \frac{x^2}{2} - 1$$

$x > 2$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$$

$$F_X(x) = \begin{cases} 0 & -\infty < x < 0 \\ x^2/2 & 1 < x < 2 \\ -1 + 2x - x^2/2 & x \ge 2 \end{cases}$$

Q. Find the mean and variance of probability distribution, given by following table

X	1	2	3	4	5
$P(x)$	0.2	0.35	0.25	0.15	0.05

Soln

$$E(X) = \sum x\,p(x) = 1\times 0.2 + 2\times 0.35 + 3\times 0.25 + 4\times 0.15 + 5\times 0.05$$

$$\boxed{\bar{X} = E(X) = \text{mean} = 2.5}$$

$$E(x^2) = \sum x^2 p(x) = 1\times 0.2 + 4\times 0.35 + 9\times 0.25 + 16\times 0.15 + 25\times 0.05$$
$$= 7.5$$

$$\text{Variance} = E(x^2) - [E(x)]^2 = 7.5 - (2.5)^2 = 7.5 - 6.25 = 1.25$$

Q. A continuous random X has density function given by

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

find expected value and variance of X

$x^{n-1} e^{-ax}$

$$\int_0^\infty x e^{ntax} dx = \frac{\Gamma n}{a^n}$$

$\boxed{\Gamma 2 = 1}$
Gama

$\boxed{\Gamma 3 = 2\times 1}$

Soln

$$E(x) = \int_0^\infty x f(x)\,dx = \int_0^\infty x(2e^{-2x})$$

$$\Rightarrow 2\int_0^\infty x e^{-2x}$$

$$\Rightarrow 2\frac{\Gamma 2}{2^2} = \frac{1}{2}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) = \int_0^\infty x^2 f(x)\,dx$$

$$= 2\int_0^\infty x^2 e^{-2x} dx = \frac{2\Gamma 3}{2^3} = \frac{2\times 2\times 1}{2^3} = \frac{1}{2}$$

$$\text{Variance} = E(x^2) - [E(x)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Q1 A random variable X has the following probability density function.

X	1	2	3	4	5	6	7
P(X)	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

find mean, mode, median, and standard deviation of distribution

Soln

Total Prob = 1

$10K^2+9K=1 \Rightarrow 10K^2+9K-1=0$

$10K^2+10K-K-1=0$

$10K(K+1)-1(K+1)=0$

$K=\frac{1}{10}, \cancel{-1}$

Probability can never by -ve

So $K=\frac{1}{10}=0.1$

X	1	2	3	4	5	6	7
P(X)	0.1	0.2	0.2	0.3	0.01	0.02	0.17
cumulative F(X)	0.1	0.3	0.5	0.8	0.81	0.83	1.0

① Mean = $E(X) = \sum x(Px)$ = $1\times0.1+2\times0.2+3\times0.2+4\times0.3+5\times0.01+6\times0.02+7\times0.17$

② Mode = (For which probability is high in this case it is 0.3)

So $\boxed{\text{mode}=4}$

③ Median = 3 (for which it first satisfies $\sum_{i=1}^{K} p_i \geq 0.5$)

④ S.D. = $\sqrt{var}$

$var = E(X^2) - [E(X)]^2$

$E(X^2) = \sum x^2 P(X)$

Q2 find the mean and variance of sum of numbers appeared in throwing a pair of dice

Soln let X = sum of numbers appeared in throwing a pair of dice

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

mean = $E(X) = \sum x P(X)$

variance = $E(X^2) - [E(X)]^2$

$x^2 p(x) - [x p(x)]^2$

Q The frequency distribution of a measurable characteristic varying b/w 0 and 2 is given by

$$f(x) = \begin{cases} x^3 & 0 \le x \le 1 \\ (2-x)^3 & 1 \le x \le 2 \end{cases}$$

Find the corresponding probability density function.

Sol^n Total frequency $= \int_0^1 f(x)dx + \int_1^2 f(x)dx$

$$= \int_0^1 x^3 dx + \int_1^2 (2-x)^3 dx$$

$2-x = t$ $\quad$ $x=1$ $t=1$

$dx = dt$ $\quad$ $x=2$ $t=0$

$$= \left[\frac{x^4}{4}\right]_0^1 + \int_0^1 (t)^3 dt$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\text{P.d.f} = \frac{\text{frequency distribution of } X}{\text{total frequency of } X} = \begin{cases} \dfrac{x^3}{1/2} & 0 \le x \le 1 \\ \dfrac{(2-x^3)}{1/2} & 1 \le x \le 2 \end{cases}$$

Q In a continuous distribution, the probability density function is given by $f(x) = Kx(2-x)$, $0 \le x \le 2$

Find the mean, mode, median, mean deviation about mean and standard deviation of the distribution.

Sol^n

$$\int_0^2 Kx(2-x) = 1$$

$$K\int_0^2 (2x - x^2)dx = 1 \quad \Rightarrow \quad K\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$K\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1 \quad \Rightarrow \quad K\left[\left(4 - \frac{8}{3}\right) - 0\right] = 1$$

$$K\left[\frac{12-8}{3}\right] = 1$$

$$K\left(\frac{4}{3}\right) = 1$$

$$K = \frac{3}{4}$$

p.d.f $\rightarrow$ $f(x) = \frac{3}{4}(2x - x^2)$ $\quad 0 \le x \le 2$

① mean $= E(x) = \int_0^2 x f(x)\,dx = \frac{3}{4}\int_0^2 x(2x - x^2)\,dx$

$= \frac{3}{4}\left(\int_0^2 (2x^2 - x^3)\,dx\right)$

$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right]$

$= \frac{3}{4}\left[\frac{64-48}{12}\right] = \frac{3 \times 16}{4 \times 12} = 1$

② Mode $\Rightarrow$ (when p.d.f is max)

$f(x) = \frac{3}{4}(2x - x^2)$ $\quad 0 \le x \le 2$

$f'(x) = \frac{3}{4}(2 - 2x) = 0$

$2 - 2x = 0$

$x = 1$

when $x=1$ p.d.f is max

$f''(x) = \frac{3}{4}(-2) < 0$ it means max at $x=1$

$\boxed{x=1 \text{ is mode of distribution}}$

③ median $\Rightarrow$ let m is median

$\int_0^m f(x)\,dx = \frac{1}{2}$ or $\int_m^2 f(x)\,dx = \frac{1}{2}$

$\int_0^m \frac{3}{4}(2x - x^2)\,dx = \frac{1}{2}$

$\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^m = \frac{4}{2 \times 3}$

$m^2 - \frac{m^3}{3} - \frac{2}{3} = 0 \Rightarrow \frac{m^3}{3} - m^2 + \frac{2}{3} = 0$

$m^3 - 3m^2 + 2 = 0$

$(m-1)(m^2 - 2m - 2) = 0$

$$\begin{array}{r} m^2 - 2m - 2 \\ m-1 \,\overline{)\, m^3 - 3m^2 + 2} \\ m^3 - m^2 \\ \hline -2m^2 + 2 \\ -2m^2 + 2m \\ \hline -2m + 2 \\ -2m + 2 \\ \hline \times \end{array}$$

$\boxed{m=1}$ $\quad m = \frac{2 \pm \sqrt{4+8}}{2} = \frac{2 \pm 2\sqrt{3}}{2}$

$m = 1 \pm \sqrt{3}$

$m = 2.732, -0.732$

$\Rightarrow$ range $(0, 2)$

So only $m=1$ is in range

$\boxed{\text{So median} = 1}$

(4) mean deviation about mean

$$E(|x-\text{mean}|) = E(|X-1|) = \int_0^2 |x-1| f(x)\,dx$$

$$\Rightarrow -\int_0^1 (x-1) f(x)\,dx + \int_1^2 (x-1) f(x)\,dx$$

$$\Rightarrow -\int_0^1 (x-1) \times \frac{3}{4}(2x-x^2)\,dx + \int_1^2 (x-1)\frac{3}{4}(2x-x^2)\,dx$$

$(x-1)(2x-x^2)$

$2x^2-x^3-2x+x^2$

$3x^2-x^3-2x$

$$\Rightarrow \frac{3}{4}\left[-\int_0^1 (x-1)(2x-x^2)\,dx + \int_1^2 (x-1)(2x-x^2)\,dx\right.$$

(5) S.D $= \sqrt{\text{variance}}$ where var $= E(X^2) - [E(X)]^2$

$E(X)=1$

$$E(X^2) = \int_0^2 x^2 f(x)\,dx = \int_0^2 x^2 \times \frac{3}{4}(2x-x^2)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^3-x^4)\,dx = \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2$$

$$= \frac{3}{4}\left[8-\frac{32}{5}\right] = \frac{3}{4}\times\left[\frac{40-32}{5}\right] = \frac{3\times \cancel{8}^2}{\cancel{4}\times 5} = \frac{6}{5}$$

Var $= \frac{6}{5} - 1 = \frac{1}{5}$ $\boxed{SD = \sqrt{\frac{1}{5}}}$

(1) A continuous random variable X follows the probability law
$f(x) = kx^2 \quad 0 \le x \le 1$
find mean, median, S.D of distribution. Also find the prob that (i) X lies b/w 0.2 and 0.5 (ii) X less than 0.3 (iii) $X > \frac{3}{4}$ given that $X > \frac{1}{2}$

Soln $f(x) = kx^2$ Total Prob $= \int_0^1 kx^2\,dx = 1 = k\left[\frac{x^3}{3}\right]_0^1 = 1$

$k\left[\frac{1}{3}\right] = 1$

$k=3$

$\boxed{f(x) = \text{p.d.f} = 3x^2}$

(1) Mean . $E(x) = \int_0^1 x f(x)dx = \int_0^1 x \times 3x^2 dx = \int_0^1 3x^3 dx = \left[\frac{3x^4}{4}\right]_0^1 = \frac{3}{4}$

(2) Mode . (when p.d.f is max) ⇒ f(x) is max

$f(x) = 3x^2 \quad 0 < x < 1$

$f'(x) = 6x = 0$

$x = 0$

$f''(x) = 6 > 0$ it means strictly ~~decreasing~~ increasing function

at $x=0$ minvalue

max when $X=1$

$\boxed{\text{median} = 1}$

(3) Median (M)

$\int_0^m 3x^2 dx = \int_m^1 3x^2 dx = \frac{1}{2}$

$[x^3]_0^m = \frac{1}{2}$

$m^3 = \frac{1}{2} \Rightarrow \boxed{m = \frac{1}{\sqrt[3]{2}}}$ median $\rightarrow \in (0,1)$

(4) $SD = \sqrt{var}$ where $var = E(X^2) - (E(X))^2$

$E(X^2) = \int_0^1 x^2 f(x) = \int_0^1 x^2 \times 3x^2 dx = \int_0^1 3x^4 dx$

$= \left[\frac{3x^5}{5}\right]_0^1 = \frac{3}{5}$

$var = \frac{3}{5} - \left(\frac{3}{4}\right)^2$

$= \frac{3}{5} - \frac{9}{16} = \frac{48-45}{80}$

$= \frac{3}{80}$

$\boxed{SD = \sqrt{\frac{3}{80}}}$

Note: Given range [a,b]
$[c,d] \subseteq [a,b]$
$P(c \le X \le d) = \int_c^d f(x)dx$

Now (i) Prob that X lies b/w 0.2 and 0.5

$P(0.2 \le X \le 0.5) = \int_{0.2}^{0.5} 3x^2 dx = [x^3]_{0.2}^{0.5}$

$= 0.125 - 0.008$

$= 0.117$

(ii) Prob that X is less than 0.3

$P(X < 0.3) = \int_0^{0.3} 3x^2 dx = [x^3]_0^{0.3} = 0.027$

(iii) prob that $X > 3/4$ given that $X > 1/2$

Let $E_1 \to X > 3/4$ $\quad E_2 \to X > \frac{1}{2}$

$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(X > 3/4)}{P(X > \frac{1}{2})} = \frac{\int_{3/4}^{1} 3x^2 dx}{\int_{1/2}^{1} 3x^2 dx} = \frac{[x^3]_{3/4}^{1}}{[x^3]_{1/2}^{1}}$$

$$\Rightarrow \frac{\left[1 - \frac{27}{64}\right]}{\left[1 - \frac{1}{8}\right]} = \frac{(64-27) \times 8^1}{\frac{64}{8} \times 7} = \frac{37}{56} \text{ Ans}$$

Q. An engineering firm is faced with the task of preparing a proposal for a research project. The cost of preparing is Rs 5000/- and the probabilities for potential gross profit of Rs 50,000/-, Rs 30,000/-, Rs 10,000/- and Rs. 2,000/- are 0.2, 0.45, 0.3 and 0.05 respectively, provided the proposal is accepted. If the probability of firm's proposal will be accepted is 0.4, what is the expected net profit of the firm?

Ans Let X = Net profit of the firm

Net profit = Total profit − Expenditure
(Gross Profit)

this is when proposal is rejected

X	45000	25000	5000	−3000	−5000
p(x)	$0.4 \times 0.2 \Rightarrow 0.08$	$0.4 \times 0.45 \Rightarrow 0.180$	$0.4 \times 0.3 \Rightarrow 0.12$	$0.4 \times 0.05 \Rightarrow 0.02$	$1-(0.08+0.18+0.12+0.02) \Rightarrow 0.6$

where $\Sigma P(x) = 1$

$E(x) = \Sigma x P(x)$

0.08
0.02
0.18
0.12
0.40

Q The delay time of a construction project is described

Soln variable is here $g(x)$

$g(x) \rightarrow$	5lakh	6lakh	7lakh	7lakh
$P(g(x)) \rightarrow$	0.5	0.3	0.1	0.1

Since $0.5+0.3+0.1+0.1=1$
So we can calculate anything that we want

Probability of delay is same Probability of penalty in that days

Mean of penalty $E(g(x)) = \quad \Sigma g(x)\, p(g(x))$

$SD = \sqrt{\text{Variance of Penalty}}$

where variance of penalty $= \quad E((g(x))^2) - [E(g(x))]^2$

$E[\{g(x)\}^2] = \Sigma \{g(x)\}^2 p(g(x))$

$\boxed{\text{Median (M)}}$ → If X is discrete r.v., then M that value of x_i's which first satisfies

$$\sum_{i=1}^{k} p_i \geq 0.5 \quad , \quad k = 1, 2 \ldots$$

If X is continuous r.v, then we have $\int_a^M f(x)dx = \int_M^b f(x)dx = \frac{1}{2}$. By solving this eq^n we get the value of M. (M must lies b/w a to b)

$\boxed{\text{MODE}}$ → If X is discrete r.v., then mode of X is that value of x_i's for which highest p_i is available.

If X is continuous r.v then find the value of x for which f(x) is maximum in the range a to b. (mode may not be unique)

$\boxed{\text{Quartiles}}$ To find Q_1 (first quartile) (Used for measure of dispersion)

If X is a discrete r.v then Q_1 that value of x_i's which first satisfies

$$\sum_{i=1}^{k} p_i \geq 0.25 \, , \; k = 1, 2, \ldots$$

If X is continuous r.v then we have $\int_a^{Q_1} f(x)dx = \frac{1}{4}$ or $\int_{Q_1}^{b} f(x)dx = \frac{3}{4}$

By solving this equation, we get the value of Q_1.

To find Q_3 (third quartile)

If X is a discrete r.v then Q_3 that value of x_i's which first satisfies $\sum_{i=1}^{k} p_i \geq 0.75$, $k = 1, 2 \ldots$

If X is a continuous r.v. then we have $\int_a^{Q_3} f(x)dx = \frac{3}{4}$ or $\int_{Q_3}^{b} f(x)dx = \frac{1}{4}$ By solving this eq^n, we get the value of Q_3.

Quartile Deviation (Q.D):

$$\boxed{Q.D = \frac{Q_3 - Q_1}{2}}$$

Note → In a similar analogue, we can find deciles and percentiles

Mean deviation about A

Here A is the mean or median or mode

Mean deviation about the point A (in the case of a discrete r.v.)

$$= \sum_{i=1}^{n} p_i |x_i - A|$$

Mean deviation about point A (in the case of continuous r.v.)

$$= \int_a^b f(x)\, |x - A|\, dx$$

Variance

If X is a r.v., then variance of X is defined as $E[(x\text{-mean})^2] = E(X)^2 - (mean)^2$

If X is discrete r.v then

variance $= E(X^2) - [E(X)]^2$ where $E(X^2) = \sum_{i=1}^{n} p_i x_i^2$

If X is a continuous r.v then

variance $= E(X^2) - [E(X)]^2$ where $E(X^2) = \int_a^b x^2 f(x)\, dx$

Standard Deviation (S.D) = positive square root of variance.

Moment generating function (M.G.F.) → The moment generating function of the probability distribution of a r.v X about the point $x=a$ is defined as the expected value of $e^{t(x-a)}$ and is denoted by $M_a(t)$

M.G.F for a discrete r.v X about origin ($a=0$)

Moment generating function about origin $= \sum e^{tx_i} p_i$

$M_0(t)$ = Momement generating function

$$= \sum p_i \left(1 + \frac{tx_i}{1!} + \frac{t^2 x_i^2}{2!} + \frac{t^3 x_i^3}{3!} + \cdots \frac{t^r x_i^r}{r!} + \cdots\right)$$

$$= \sum p_i + t \sum p_i x_i + \frac{t^2}{2} \sum p_i x_i^2 + \cdots + \frac{t^r}{r!} \sum p_i x_i^r + \cdots$$

$$= 1 + t E(x) + \frac{t^2}{2} E(x^2) + \cdots + \frac{t^r}{r!} E(x^r) + \cdots$$

By MGF we can calculate means and variance also ⇒

$$\boxed{\left.\frac{d}{dt} M_0(t)\right|_{t=0} = E(X)} \qquad \boxed{\left.\frac{d^2}{dt^2} M_0(t)\right|_{t=0} = E(X^2)}$$

Chebyshev's inequality → If X is a random variable with mean μ and variance σ^2 then for any positive number k show that

$$\boxed{P(|X-\mu| \geq K\sigma) \leq \frac{1}{K^2}} \quad \text{or} \quad \boxed{P(|X-\mu| < K\sigma) \leq 1-\frac{1}{K^2}}$$

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$-\infty \quad \mu-K\sigma \quad \mu+K\sigma \quad \infty$

$$= \int_{-\infty}^{\mu-K\sigma} (x-\mu)^2 f(x)\,dx + \underbrace{\int_{\mu-K\sigma}^{\mu+K\sigma} \underset{\geq 0}{(x-\mu)^2}\, \underset{\geq 0}{f(x)}\,dx}_{\geq 0} + \int_{\mu+K\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

when integrand is ≥ 0
then its integration is also ≥ 0

$$\sigma^2 \geq \int_{-\infty}^{\mu-K\sigma} (x-\mu)^2 f(x)\,dx + \int_{\mu+K\sigma}^{\infty} (x-\mu)^2 f(x)\,dx$$

For the 1st interval $\quad x \leq \mu-K\sigma \quad \Rightarrow \mu - x \geq K\sigma \quad \Rightarrow (\mu-x)^2 \geq K^2\sigma^2$
or $(x-\mu)^2 \geq K^2\sigma^2$

for the 2nd interval $\quad x \geq \mu+K\sigma \quad \Rightarrow x-\mu \geq K\sigma \quad \Rightarrow (x-\mu)^2 \geq K^2\sigma^2$

$$\sigma^2 \geq K^2\sigma^2 \left[\int_{-\infty}^{\mu-K\sigma} f(x)\,dx + \int_{\mu+K\sigma}^{\infty} f(x)\,dx\right]$$

$$\sigma^2 \geq K^2\sigma^2 \left[P(-\infty < X \leq \mu-K\sigma) + P(\mu+K\sigma \leq X < \infty)\right]$$

$$\sigma^2 \geq K^2\sigma^2 \left[P(X-\mu \leq -K\sigma) + P(X-\mu \geq K\sigma)\right]$$

$$\sigma^2 \geq K^2\sigma^2 \left[P(|X-\mu| \geq K\sigma)\right]$$

$$P(|X-\mu| \geq K\sigma) \leq \frac{1}{K^2}$$

$$\boxed{\begin{array}{c} |x-a| \leq \varepsilon \\ a-\varepsilon \leq x \leq a+\varepsilon \end{array}}$$

$$\boxed{\begin{array}{c} |x-a| > \varepsilon \\ x < a-\varepsilon \text{ and} \\ x > a+\varepsilon \end{array}}$$

MGF for a continuous r.v X about origin

$$M_0(t) = \int_a^b e^{tx} f(x)\,dx$$

Here also

$$\boxed{\left.\frac{d\,M_0(t)}{dt}\right|_{t=0} = E(X)} \qquad \boxed{\frac{d^2(M_0 t)}{dt^2} = E(X^2)}$$

Remark

If $k\sigma = c$ then $P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$ or $P(|X-\mu| \leq c) \leq 1-\frac{\sigma^2}{c^2}$

Joint Probability distribution → Two random variables X and Y are said to be jointly distributed if they are defined on the same probability space.

Note → In joint probability distribution, both X and Y are the same type of random variables. (means either both are discrete or both are continuous)

Discrete Joint Probability Distribution → Let X and Y be two discrete random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2, \dots x_n\}$ and $Y(S) = \{y_1, y_2, \dots y_n\}$.

Let $P(X = x_i, Y = y_j) = p_{ij}$ for $i = 1, 2, \dots n$ and $j = 1, 2, \dots m$

If $\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij} = 1$, then the set of values $\{X(S), Y(S), \{p_{ij}\}\}$ represent joint probability distribution (or joint probability density function) for discrete random variables X and Y.

Discrete Marginal Probability Distribution → If $\{X(S), Y(S), \{p_{ij}\}\}$ represent joint probability distribution for discrete random variable X and Y then, the probability distribution of X is determined by

$P_X(x_i) = P(x_i, y_1) + P(x_i, y_2) + P(x_i, y_3) + \dots + P(x_i, y_m)$ for $i = 1, 2, 3, \dots n$

and the set of values $\{P_X(x_i), i = 1, 2, \dots n\}$ represent marginal prob distribution (or marginal prob density function) of X.

the probability distribution of Y is determined by

$P_Y(y_j) = P(x_1, y_j) + P(x_2, y_j) + P(x_3, y_j) + \dots + P(x_n, y_j)$. for $j = 1, 2, \dots m$

and the set of values $\{P_Y(y_j), j = 1, 2, \dots m\}$ represent marginal probability distribution (or marginal prob density function) of Y.

Q1: for the following joint probability X and Y, find
(i) $P(X \le 1, Y=2)$ (ii) $P(X \le 1)$ (iii) $P(Y \le 3)$ (iv) $P(X<2, Y \le 4)$
(v) marginal distributions of X and Y

Soln

X \ Y	1	2	3	4	5	6	$F_x(x)$ → marginal distribution of X
0	0	0	1/32	2/32	2/32	3/32	8/32
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16
2	1/32	1/32	1/64	1/64	0	2/64	8/64
$F_y(y)$	3/32	3/32	11/64	13/64	6/32	16/64	

↓ marginal distribution of Y

(i) $P(X \le 1, Y=2) = 0 + \frac{1}{16} = \frac{1}{16}$

(ii) $P(X \le 1) = \frac{8}{32} + \frac{10}{16}$

(iii) $P(Y \le 3) = \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(iv) $P(X<2, Y \le 4) = 0 + 0 + \frac{1}{32} + \frac{2}{32} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}$

Q2 for the following joint prob distribution, obtain the conditional distribution of X given Y = 2

Y \ X	-1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

$$P(X=x / Y=2) = \frac{P(x, Y=2)}{P(Y=2)}$$

Soln

$P(Y=2) = \frac{2}{15} + \frac{1}{15} + \frac{2}{15} = \frac{5}{15}$

~~X | -1 | 0 | 1; P(X/Y=2) | 1/15 + 3/15 + 2/15 ⇓ 6/15 ⇒ 2/5 | 2/15 + 2/15 + 1/15 ⇓ 5/15 | 1/3~~

X	-1	0	1
$P(X/Y=2)$	$\frac{2/15}{5/15} \Downarrow \frac{2}{5}$	$\frac{1/15}{5/15} \Downarrow \frac{1}{5}$	$\frac{2/15}{5/15} \Downarrow \frac{2}{5}$

Q3 If X and Y are r.v. having the joint prob density function $p(x,y) = \frac{x+2y}{27}$ where $x,y = 0,1,2$. Find the condition distribution of Y for a given X.

Soln

Y \ X	0	1	2
0	0	1/27	2/27
1	2/27	3/27	4/27
2	4/27	5/27	6/27
$P_X(x)$	6/27	9/27	12/27

$$P\left(\frac{Y}{X}\right) = \frac{P(x,y)}{P_X(x)}$$

Now $P\left(\frac{Y}{X}\right) \Rightarrow$

Y \ X	0	1	2
0	0	1/9	2/12
1	2/6	3/9	4/12
2	4/6	5/9	6/12

Q4 Two discrete r.v X and Y have the joint probability density function given by

$$p(x,y) = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!} \qquad \boxed{y = 0,1,2,\dots x}, \boxed{x = 0,1,2,\dots}$$

where $\lambda > 0$ and $0 < p < 1$

Find (i) the marginal probability density functions of X and Y → $F_X(x)$ and $F_Y(y)$

(ii) Conditional distribution of

(a) Y for a given X → $P(Y/X)$ (b) X for a given Y → $P(X/Y)$

Q5 The joint probability density function of two r.v. X and Y is given by $f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}$, $0<x<\infty$, $0<y<\infty$.

find the marginal distribution of X and Y, and the conditional distribution of Y for a given X.

Soln Marginal density function of X

$$f_X(x) = \int_{y=0}^{\infty} f(x,y)\,dy = \int_{y=0}^{\infty} \frac{9(1+x+y)\,dy}{2(1+x)^4(1+y)^4}$$

$$\Rightarrow \frac{9}{2(1+x)^4}\int_{y=0}^{\infty}\frac{(1+x+y)\,dy}{(1+y)^4} \Rightarrow \frac{9}{2(1+x)^4}\int_{t=1}^{\infty}\frac{(t+x)\,dt}{t^4}$$

$1+y=t$, $dy=dt$, $y=0 \rightarrow t=1$, $y=\infty \rightarrow t=\infty$

$$\Rightarrow \frac{9}{2(1+x)^4}\left[\int\frac{t}{t^4}dt + x\int\frac{1}{t^4}dt\right]$$

$$\Rightarrow \frac{9}{2(1+x)^4}\left[\frac{-1}{2t^2} + \frac{x\times -1}{3t^3}\right]_1^{\infty} \Rightarrow \frac{9}{2(1+x)^4}\left[(0+0)-\left(\frac{-1}{2}-\frac{x}{3}\right)\right]$$

$$\Rightarrow \frac{9}{2(1+x)^4}\left[\frac{1}{2}+\frac{x}{3}\right] = \frac{9\,3}{2(1+x)^4}\left[\frac{3+2x}{\cancel{6}\,2}\right] = \frac{3\,(3+2x)}{4(1+x)^4}$$

Similarly marginal density function of Y

$$f_Y(y) = \frac{9}{2(1+y)^4}\left[\frac{1}{2}+\frac{y}{3}\right]$$

Conditional distribution of Y for a given X

$$P(Y/X) = \frac{p(x,y)}{F_X(x)} = \frac{3\,\cancel{9}(1+x+y)\times \overset{2}{\cancel{4}}\cancel{(1+x)^4}}{\cancel{2}\cancel{(1+x)^4}(1+y)^4\times\cancel{3}(3+2x)} = \frac{6(1+x+y)}{(1+y)^4(3+2x)}$$

Q6 If X and Y are two random variables having joint probability density function $f(x,y) = \begin{cases}\frac{6-x-y}{8}, & 0<x<2,\ 2<y<4\\ 0, & \text{otherwise}\end{cases}$

find (i) $P(X<1, Y<3)$ (ii) $P(X+Y<3)$ (iii) $P(X<1/Y<3)$

Soln (i) $P(X<1, Y<3) = \int_{x=0}^{1}\int_{2}^{3} f(x,y)\,dy\,dx = \int_{x=0}^{1}\int_{y=2}^{3}\left(\frac{6-x-y}{8}\right)dy\,dx$

$$\Rightarrow \int_{x=0}^{1}\int_{y=2}^{3}\left(\frac{6-x-y}{8}\right)dy\,dx$$

$$\int_{x=0}^{1}\left(\frac{7-2x}{16}\right)dx$$

$$\left[\frac{7x}{16}-\frac{\cancel{2}x^2}{16\times\cancel{2}}\right]_0^1$$

$$\left[\frac{7}{16}-\frac{1}{16}\right]=\frac{6}{16}=\frac{3}{8}\text{ Ans}$$

$$\int\frac{6-x-y}{8}dy$$

$$\left(\frac{6-x}{8}\right)dy-\frac{y}{8}dy$$

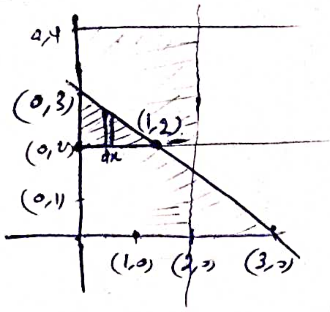
$$\left[\left(\frac{6-x}{8}\right)y-\frac{y^2}{2\times8}\right]_2^3$$

$$\left[\left(\frac{6-x}{8}\right)3-\frac{9}{2\times8}\right]-\left[\left(\frac{6-x}{8}\right)2-\frac{4}{2\times8}\right]$$

$$\left(\frac{6-x}{8}\right)3-\left(\frac{6-x}{8}\right)2-\frac{9}{16}+\frac{4}{16}$$

$$\frac{6-x}{8}-\frac{5}{16}=\frac{2(6-x)-5}{16}=\frac{7-2x}{16}$$

(ii) $P(X+Y<3)$



$$\Rightarrow\int_{x=0}^{1}\int_{y=2}^{3-x} f(x,y)\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\int_{y=2}^{3-x}\left(\frac{6-x-y}{8}\right)dy\,dx$$

$$\Rightarrow\int_{x=0}^{1}\frac18\left[6y-xy-\frac{y^2}{2}\right]_2^{3-x}dx \Rightarrow \int_{x=0}^{1}\left[\frac18\left[6(3-x)-x(3-x)-\frac{(3-x)^2}{2}\right]-\frac18\left[12-2x-\frac42\right]\right]dx$$

$$\Rightarrow\int_{x=0}^{1}\frac18\left[18-6x-3x+x^2-\left(\frac{9+x^2-6x}{2}\right)-12+2x+2\right]dx$$

$$\Rightarrow\int_{x=0}^{1}\frac18\left[18-9x+x^2-\left(\frac{9+x^2-6x}{2}\right)-10+2x\right]dx$$

$$\Rightarrow\frac18\left[18x-\frac{9x^2}{2}+\frac{x^3}{3}-\frac12\left(9x+\frac{x^3}{3}-\frac{6x^2}{2}\right)-10x+x^2\right]_0^1 = \frac18\left[18-\frac92+\frac13-\frac12\left(9+\frac13-3\right)-10+1\right]$$

(iii) $P(X<1 / Y<3) = \dfrac{P(X<1, Y<3)}{P(Y<3)} \xrightarrow{} \text{from ①} = \dfrac{3/8}{5/8} = \dfrac{3}{5}$

$$P(Y<3) = \int_{y=2}^{3} \int_{x=0}^{2} f(x,y)\,dx\,dy$$

$$\int \left(\frac{6-x-y}{8}\right) dx$$

$$\left[\frac{6x}{8} - \frac{x^2}{2\times 8} - \frac{yx}{8}\right]_0^2$$

$$\Rightarrow \left[\frac{6\times 2}{8} - \frac{4}{2\times 8} - \frac{2y}{8}\right]$$

$$\Rightarrow \left[\frac{3}{2} - \frac{1}{4} - \frac{y}{4}\right]$$

$$\Rightarrow \left[\frac{5}{4} - \frac{y}{4}\right]$$

$$\int_{y=2}^{3} \left(\frac{5}{4} - \frac{y}{4}\right) dy = \left[\frac{5}{4}y - \frac{y^2}{8}\right]_2^3$$

$$\Rightarrow \left[\frac{15}{4} - \frac{9}{8}\right] - \left[\frac{10}{4} - \frac{4}{8}\right]$$

$$\Rightarrow \frac{5}{4} - \frac{5}{8} = \frac{10-5}{8} = \frac{5}{8}$$

Distributions

Discrete Uniform Random Variable: A discrete random variable X, which assumes the values $x_1, x_2, \ldots, x_n$ each with probability $\frac{1}{n}$ is called discrete uniform random variable.

$$E(X) = \sum p_i x_i = \sum \frac{1}{n} x_i = \frac{1}{n}(x_1 + x_2 + x_3 + \cdots + x_n)$$

Continuous uniform random variable: A continuous random variable X is said to be a continuous uniform random variable on the interval (a,b) if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Binomial distribution ⇒ Let n be the number of trials

Let p be the probability of success in each trial

Let $q = 1-p$ be the probability of failure in each trial.

Let us define $\boxed{p(x) = {}^nC_x p^x q^{n-x} \quad x = 0,1,2,\ldots n}$ → (pdf)

Here $p(x)$ is the probability of x successes out of n trials.

$$\Rightarrow P(0) + P(1) + p(2) + \cdots + p(n)$$

$$\Rightarrow {}^nC_0 p^0 q^n + {}^nC_1 p^1 q^{n-1} + {}^nC_2 p^2 q^{n-2} + \cdots + {}^nC_n p^n q^0$$

$$\Rightarrow (q+p)^n = 1 \qquad \text{since } \boxed{p+q=1}$$

Therefore $p(x)$ represents the p.d.f of discrete r.v. X

Since $p(x), x = 0, 1, 2, \ldots n$ represent the terms of the B.E of $(q+p)^n$

the r.v X is called binomial variable.

Note To represent the p.d.f of binomial variable, we need two parameters n and p.

Moment Generating function of the binomial distribution about origin

MGF of BD about origin is given by

$$M_0(t) = E(e^{tx}) = \sum_{x=0}^{n} \overbrace{{}^nC_x p^x q^{n-x}}^{p(x)} \overbrace{e^{tx}}^{\phi(x)} = \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x} = (q+pe^t)^n$$

$$\boxed{E(\phi(x)) = \sum_{x} p(x)\phi(x)}$$

Mean and Variance of binomial distribution

Mean $= E(X) = \left.\frac{d\,M_0(t)}{dt}\right|_{t=0} = n(q+pe^t)^{n-1}pe^t\Big|_{t=0} = n(q+p)^{n-1}p = np$

So $\boxed{\text{mean} = np}$

$$E(X^2) = \left.\frac{d^2(M_0(t))}{dt^2}\right|_{t=0} = \left[\frac{d}{dt}\left[n(q+pe^t)^{n-1}pe^t\right]\right]_{t=0}$$

$$\Rightarrow n(n-1)(q+pe^t)^{n-2}(pe^t)^2 + n(q+pe^t)^{n-1}pe^t\Big]_{t=0}$$

$$\Rightarrow n(n-1)\times 1\times p^2 + n\times 1\times p$$

$$\Rightarrow n(n-1)p^2 + np = (n^2-n)p^2 + np = (np)^2 - np^2 + np$$

Variance $= E(X^2) - (E(X))^2 = n^2p^2 - np^2 + np - n^2p^2 = np - np^2$

$\Rightarrow np(1-p) = npq$

$\boxed{\text{Variance} = npq}$

Mode of the binomial distribution

$\Rightarrow$ Mode is that value of x for which $p(x)$ is maximum. Consider

$$\frac{p(x)}{p(x-1)} = \frac{{}^nC_x\, p^x q^{n-x}}{{}^nC_{x-1}\, p^{x-1} q^{n-x+1}} \Rightarrow \text{By solving} \Rightarrow 1 + \frac{(n+1)p - x}{xq} \quad x = 1, 2, \ldots, n$$

This can be >0, <0, $=0$

$$\Downarrow$$

$$\frac{\frac{n!}{x!(n-x)!}}{\frac{n!}{(x-1)!(n-x+1)!}}\left(\frac{p}{q}\right) \Rightarrow \frac{(n-x+1)p}{qx}$$

$$\Rightarrow \frac{(n+1)p - px}{qx} \Rightarrow \frac{(n+1)p - x(1-q)}{qx}$$

$$\Rightarrow \frac{(n+1)p - x + qx}{qx} \Rightarrow 1 + \frac{(n+1)p - x}{qx} \quad x = 1, 2, \ldots n$$

Mode of B.D = $\begin{cases} \text{integral part of } (n+1)p & , \text{ if } (n+1)p \text{ is not an integer} \\ (n+1)p \text{ and } (n+1)p-1 & , \text{ if } (n+1)p \text{ is an integer} \end{cases}$

Note for other measure of central tendency and measure of dispersion, we have to use the p.d.f as we have done in the case of a general r.v.

Q The prob that a pen manufactured by a company will be defective is $\frac{1}{10}$. If 12 such pens are manufactured, then find the probability that
(i) exactly two pens will be defective.
(ii) atleast three will be defective
(iii) no pen will be defective.

Soln $\boxed{n = 12}$

p = Prob that pen is defective $= \frac{1}{10} = 0.1$

$q = 1 - 0.1 = 0.9$

$\boxed{p = 0.1}$ $\boxed{q = 0.9}$

$p(x)$ = prob that x pens will be defective $= {}^nC_x p^x q^{n-x}$ $x = 0, 1, 2, \ldots$

(i) ${}^{12}C_2 (0.1)^2 (0.9)^{10}$

(ii) P(atleast three) = $P(x \geq 3) = p(x=3) + p(x=4) + \ldots + p(x=12)$

or

$P(x \geq 3) = 1 - (p(0) + p(1) + p(2))$

(iii) P(no pen will be defective) $= {}^{12}C_0 (0.1)^0 (0.9)^{12} = p(x=0)$

Q In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many samples would be expected to contain atleast 3 defectives.

Soln Sample size $= n = 20$

p = prob that a part is defective

mean $= np = 2$

$20 \times p = 2$

$p = \frac{2}{20} = \frac{1}{10}$ $\boxed{p = 0.1}$ $\boxed{q = 0.9}$

~~P(atleast 3 defective out of 1000) = ${}^{1000}C_3 (0.1)^3 (0.9)^{997}$~~

In a sample $p(x)$ = prob that x parts will be defective

p(atleast 3 defective parts in a sample) $= p(x \geq 3) = 1 - [p(0) + p(1) + p(2)]$

Ans = 0.323

The number of sample having atleast 3 defectives out of 1000 samples

$0.323 \times 1000 = 323$

Q3 An irregular six face die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 12,000 sets of 10 such throws would you expect it to give no even number. (irregular die)

Soln p = probability getting an even no. =

$p(x)$ = prob of getting x even number.

$\Rightarrow {}^nC_x p^n q^{n-n} = {}^{10}C_x p^x q^{10-x}$

By given condition $p(5) = 2p(4)$

$${}^{10}C_5 p^5 q^5 = 2 \times {}^{10}C_4 p^4 q^6$$

$$\frac{{}^{10}C_5}{2 \times {}^{10}C_4} = \frac{p^4 q^6}{p^5 q^5} = \frac{q}{p} = \frac{1}{2} \times \frac{10-5+1}{5} = \frac{3}{5}$$

$$\frac{1-p}{p} = \frac{3}{5} \Rightarrow 5 - 5p = 3p \Rightarrow 5 = 8p \quad \boxed{p = \frac{5}{8}}$$

$$\boxed{q = \frac{3}{8}}$$

$p(0)$ = Prob of getting no even number $= {}^{10}C_0 \left(\frac{5}{8}\right)^0 \left(\frac{3}{8}\right)^{10} = \left(\frac{3}{8}\right)^{10}$

The no. of sets in 1000 sets containing no even number $= \left(\frac{3}{8}\right)^{10} \times 1000$

$= 0.55 \simeq 1$

But no. of sets always is an integer so $\boxed{0.55 \simeq 1}$ $\boxed{\text{Ans} = 1}$

Q4 Comment on the following data

(i) The mean and variance of a binomial distribution are 3 and 4 respectively

$np = 3 \quad npq = 4$

$$\frac{npq}{np} = \frac{4}{3} \quad q = \frac{4}{3} > 1 \text{ not possible}$$

So given data is wrong

Q5 If a binomial distribution has mean and variance are 4 and 3 respectively. then find mode of the distribution.

Soln mean $= np = 4$ variance $= npq = 3$

$$\frac{npq}{np} = \frac{3}{4} \Rightarrow \boxed{q = \frac{3}{4}} \quad \boxed{p = \frac{1}{4}} \quad n \times \frac{1}{4} = 4$$

$n = 16$

Consider $(n+1)p = (16+1)\times\frac{1}{4} = \frac{17}{4} = 4.25$ → non integer

So mode is integral value of 4.25

$\boxed{\text{Mode} = 4}$

Q6 In a bombing action there is 50% chance that any bomb will strike the target. Two direct hits are required to destroy the target almost completely. How many bombs are required to be dropped to give a 99% chance or better of completely destroying the target?

Soln Let n be the number of bombs dropped.

p = Prob that a bomb will hit the target $= \frac{1}{2}$

$q = 1-\frac{1}{2} = \frac{1}{2}$

$p(n)$ = prob that n bombs will hit the target

$P(\text{atleast } 2) = p(x\geq 2) = 1-p(x<2) \geq \frac{99}{100}$

$= 1-[p(0)+p(1)] \geq \frac{99}{100}$

Atleast two bombs are necessary to destroy the target by 99% or more

$\Rightarrow 1-{}^nC_0\left(\frac{1}{2}\right)^n - {}^nC_1\left(\frac{1}{2}\right)^n \geq \frac{99}{100}$

$\Rightarrow 1-\left(\frac{1}{2}\right)^n(1+n) \geq \frac{99}{100}$

$\Rightarrow 1-\frac{99}{100} \geq \left(\frac{1}{2}\right)^n(1+n)$

$\Rightarrow \frac{1}{100} \geq \left(\frac{1}{2}\right)^n(1+n)$

$\Rightarrow \frac{2^n}{100} \geq (n+1)$

$\Rightarrow 2^n \geq 100(n+1)$

Let $n=10$

$2^{10} \geq 100(10+1)$ ✗

$2^{11} \geq 100(11+1)$ ✓

$n=11$

So $n=11$ Ans

Minimum number of bombs required to destroy the target 99% or more is $\underline{\underline{11}}$

Q7 The latest nationwide political poll indicates that the Americans who are randomly selected, the probability that they are conservative is 0.55, the probability that they are liberal is 0.3. and the prob that they are middle-of-the-road is 0.15. Assuming that these probabilities are accurate, answer the following questions pertaining to a randomly chosen group of 10 Americans.

(i) What is the prob that four are liberals?

(ii) What is the prob that none are conservative?

(iii) What is the prob that two are middle-of-the road?

(iv) What is the prob that atleast eight are liberal?

Soln middle of the road means the voter does not vote for anyone.

$n = 10$ $\quad p = \begin{cases} \text{if voter is conservative} & 0.55 \\ \text{if voter is liberal} & 0.3 \\ \text{if voter is middle-of-the-road} & 0.15 \end{cases}$

$$p(n) = {}^nC_x p^x q^{n-x} = {}^{10}C_x p^x q^{10-x}$$

(i) ${}^{10}C_4 (0.3)^4 (0.7)^6$

(ii) ${}^{10}C_0 (0.55)^0 (0.45)^{10}$

(iii) ${}^{10}C_2 (0.15)^2 (0.85)^8$

(iv) $p(x=8) + p(x=9) + p(x=10)$

${}^{10}C_8 (0.3)^8 (0.7)^2 + {}^{10}C_9 (0.3)^9 (0.7) + {}^{10}C_{10} (0.3)^{10}$