Testing of Hypothesis

To reach decisions about populations on the basis of sample information, we make certain assumptions about the populations involved. Such assumptions, may or may not be true, are called statistical hypothesis.

There are two types of hypothesis. They are

Null Hypothesis : H_0 : Population statistic = sample statistic

Alternate Hypothesis : H_1 : Population statistic $\neq$ sample statistic

Note : By default, here hypothesis means null hypothesis.

In accepting or rejecting hypothesis, generally we commit two types of errors.

Type I error : Reject a hypothesis when it is to be accepted.

Type II error : Accept a hypothesis when it is to be rejected.

Note : In order for any test of hypothesis or rule of decision to be good, it must be designed so as to minimize the errors of decision.

Level of significance :

In testing a hypothesis, the maximum probability with which we would be willing to risk a type I error is called level of significance of the test.

For example, if 5% level of significance is chosen in designing a test of hypothesis, then there are 5 chances out of 100 that we would reject the hypothesis when it is to be accepted, that means we are about 95% confident that we made right decision.

Chi square test for goodness of fit :

We want to test whether the deviations of expected or theoretical data from the observed data are significant or not. That means we want to test the difference between observed data and expected or theoretical data is significant or not.

It is also used to test how a set of observations will fit a given distribution, which means it provides a test of goodness of fit and may be used to examine the validity of some hypothesis about an observed data.

As a test of goodness of fit, it can be used to study the correspondence between theoretical data and observed data.

Procedure to test goodness of fit :

(i) Set up a 'null hypothesis' and calculate $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$

where O_i's are observed values, E_i's are expected or theoretical values and

$\sum_{i=1}^{n} O_i = \sum_{i=1}^{n} E_i$

(ii) Find the number of degrees of freedom (d.f.) and take the corresponding value of χ^2 at a prescribed level of significance from the χ^2 test for goodness of fit table.

(iii) If calculated value is less than the table value, then accept the hypothesis otherwise reject the hypothesis.

To decide the number of d.f.

If any **theoretical or expected frequency** is less than 5, then we use the technique of 'pooling' which consists in adding the frequencies which are less than 5 with the preceding or succeeding frequency so that the resulting sum of theoretical (or expected) frequency is greater than or equal to 5.

Let 'n' be the number of theoretical (or expected) frequencies after pooling.

Now, the number of d.f. = $n - 1$ – number of relations used to get theoretical or expected frequencies.

Problems on chi-square test for goodness of fit :

1. 15,000 random numbers were taken for a data and the following frequencies of each digit were obtained.

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	1493	1441	1461	1552	1494	1454	1613	1491	1482	1519

Test the hypothesis that each digit had an equal chance of being chosen.

2. The theory predicts that the proportion of number of data values in four groups A, B, C, D should be in the ratio 9 : 3 : 3 : 1. In the survey of collection of 1600 data values, the numbers of data values in the four groups were found as 882, 313, 287, 118 respectively. Do the observed data values support the theory?

3. When fitting a binomial distribution, the following data was obtained.

X	0	1	2	3	4	5	6	7	8	9	10
Observed frequency	6	20	28	12	8	6	0	0	0	0	0
Theoretical frequency	7	19	24	18	8	3	1	0	0	0	0

Is the fitting correct?

Sampling distribution :

Consider a sample size of 'n' which can be drawn from a given population at random. Now we want to test the hypothesis with respect to a statistic.

If $n \geq 30$, then the sample is considered to be a large sample, otherwise it is considered to be a small sample.

Large samples :

Test for single mean :

Let 'μ' be the mean of the population.

Let 'n' be the size of the sample and $\overline{x}$ be the mean of the sample.

Let σ be the s.d. of the sample or population.

Here null hypothesis is

Sample has been drawn from the population

(or)

There is no significant difference between the sample and population means.

Calculate $z = \dfrac{\overline{x} - \mu}{\sigma / \sqrt{n}}$

Here z is a standard normal variable.

For two-tailed distribution :

If $|z| < 1.96$, then the hypothesis is accepted at 5% level of significance.

If $1.96 < |z| < 2.58$, then the hypothesis is accepted at 1% level of significance.

If $2.58 < |z| < 3$, then the hypothesis may be accepted at 1% level of significance.

If $|z| > 3$, then the hypothesis is rejected.

For one-tailed distribution :

For left tail : If $z > -1.65$, then the hypothesis is accepted at 5% level of significance.

For right tail : If $z < 1.65$, then the hypothesis is accepted at 5% level of significance.

Note :

95% confidence limits for μ are $\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$

99% confidence limits for μ are $\bar{x} \pm 2.58 \frac{\sigma}{\sqrt{n}}$

Note :

If population size is N, then

95% confidence limits for μ are $\bar{x} \pm 1.96 \sqrt{\frac{N-n}{N-1}} \frac{\sigma}{\sqrt{n}}$

99% confidence limits for μ are $\bar{x} \pm 2.58 \sqrt{\frac{N-n}{N-1}} \frac{\sigma}{\sqrt{n}}$

Problems on test for single mean :

1. A stenographer claims that she can take dictation at the rate of 120 words per minute. Test her claim on the basis of 100 trials in which she demonstrates a mean of 116 words per minute with a standard deviation of 15 words.

2. It is claimed that a random sample of 100 tyres with mean life of 15629 kms. is drawn from of population of tyres with mean life of 15200 kms. with standard deviation of 1248 kms. Test the validity of the claim.

3. An insurance agent claims that the average age of policy holders who insure through him is 30.5 years. A random sample of 100 policy holders who had insured through him gave the following distribution.

Age :	16-20	21-25	26-30	31-35	36-40
No. of Policy Holders	12	22	20	30	16

Test the claim at 5% level of significance.

4. A random sample of 400 items is found to have a mean of 82 and standard deviation of 18. Find the 95% confidence limits for the mean of population from which the sample is drawn.

(when the population size is known)

5. A random sample of 100 articles from a batch of 2000 articles shows that the average diameter of the articles is 0.354 inches with standard deviation of 0.048. Find the 95% confidence limits for the population mean.

Test for difference of means :

Here we consider two samples.

Sample 1 : Size = n_1, Mean = $\bar{x}_1$, s.d. = σ_1

Sample 2 : Size = n_2, Mean = $\bar{x}_2$, s.d. = σ_2

Here null hypothesis is

Both the samples are drawn from the same population

(or)

Samples are drawn from different populations which have the same mean

(or)

The samples are drawn from different populations which have insignificant difference as far as the means are concerned.

Calculate $$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^{\,2}}{n_1} + \frac{\sigma_2^{\,2}}{n_2}}}$$

Here z is a standard normal variable.

For two-tailed distribution :

If $|z| < 1.96$, then the hypothesis is accepted at 5% level of significance.

If $1.96 < |z| < 2.58$, then the hypothesis is accepted at 1% level of significance.

If $2.58 < |z| < 3$, then the hypothesis may be accepted at 1% level of significance.

If $|z| > 3$, then the hypothesis is rejected.

For one-tailed distribution :

For left tail : If $z > -1.65$, then the hypothesis is accepted at 5% level of significance.

For right tail : If $z < 1.65$, then the hypothesis is accepted at 5% level of significance.

Problems on test for difference of means :

1. 400 women shoppers are chosen at random in market A. Their average daily expenditure on food is found to be Rs.250 with standard deviation of Rs.40. The figures are Rs.220 and Rs.55 respectively in market B where also 400 women shoppers are chosen at random. Test the hypothesis that average daily food expenditure of the two samples of shoppers is equal.

2. An investigation of the relative merits of two kinds of flash light batteries showed that a random sample of 100 batteries of brand A lasted on the average 36.5 hours with a standard deviation of 1.8 hours, while a random sample of 80 batteries of brand B last on the average 36.8 hours with a standard deviation of 1.5 hours. Use 5% level of significance to test whether the observed difference between the average life times is significant.

Small samples :

Test for single mean :

Population : Mean = μ

Sample : Size = n, data values are $x_1, x_2, \dots, x_n$

Here null hypothesis is

The sample has been drawn from the population

(or)

There is no significant difference between population mean and sample mean

Calculate $t = \dfrac{\bar{x} - \mu}{S / \sqrt{n}}$ where $\bar{x} = \dfrac{\Sigma x_i}{n}$, $S = \sqrt{\dfrac{1}{n-1} \Sigma \left(x_i - \bar{x}\right)^2}$

Here S is the s.d. of the sample.

Take the corresponding value of 't' at a prescribed level of significance for (n-1) d.f. from the t- test table.

If the calculated value of | t | < table value of t, then the hypothesis is accepted otherwise it is rejected.

Confidence limits

95% confidence limits for population mean are $\bar{x} \pm (\text{table value of } t_{0.05}) \dfrac{S}{\sqrt{n}}$.

99% confidence limits for population mean are $\bar{x} \pm (\text{table value of } t_{0.01}) \dfrac{S}{\sqrt{n}}$.

Problems on test for single mean :

1. A machine is designed to produce insulating washers for electrical devices of average thickness of 0.025 cm. A random sample of 10 washers was found to have an average thickness of 0.024 cm. with a standard deviation of 0.002 cm. Test the significance of the deviation.

2. A soap manufacturing company was distributing a particular brand of soap through a large number of retail shops. Before a heavy advertisement campaign, the mean sales per week per shop was 140 dozens. After campaign, a sample of 26 shops was taken and mean sales per week per shop was found to be 147 dozens with standard deviation 16. Can you consider the advertisement effective?

3. Certain pesticide is packed into bags by a machine. A random sample of 10 bags is drawn and their contents are found to weigh (in Kgs.) as follows : 50, 49, 52, 44, 45, 48, 46, 45, 49 and 45. Test if the average packing can be taken to be 50 Kg.

Test for difference of means :

Sample 1 : Size = n_1, data values are $x_1, x_2, \ldots, x_{n_1}$

Sample 2 : Size = n_2, data values are $y_1, y_2, \ldots, y_{n_2}$

Here null hypothesis is

Both the samples are drawn from the same population

(or)

No significant difference between sample means

(or)

Both the samples are drawn from different populations which have the same mean

(or)

Both the samples are drawn from different populations for which difference of means is insignificant.

Calculate $t = \dfrac{\bar{x} - \bar{y}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

where $\bar{x} = \dfrac{\sum_{i=1}^{n_1} x_i}{n_1}$, $\bar{y} = \dfrac{\sum_{j=1}^{n_2} y_j}{n_2}$, $S = \sqrt{\dfrac{1}{n_1 + n_2 - 2}\left\{\sum_{i=1}^{n_1}(x_i - \bar{x})^2 + \sum_{j=1}^{n_2}(y_j - \bar{y})^2\right\}}$

Take the corresponding value of 't' at a prescribed level of significance for $(n_1+n_2 - 2)$ d.f. from the t- test table.

If the calculated value of $|\ t\ |$ < table value of t, then the hypothesis is accepted otherwise it is rejected.

Problems on test for difference of means :

1. The nicotine content in milligrams of two samples of tobacco were found to be as follows :

   Sample A : 24, 27, 26, 21, 25

   Sample B : 27, 30, 28, 31, 22, 36

   Can it be said that two samples came from populations having the same mean?

2. A group of 5 patients treated with medicine A weigh 42, 39, 48, 60 and 41 Kgs. Second group of 7 patients from the same hospital treated with medicine B weigh 38, 42, 56, 64, 68, 69 and 62 Kgs. Do you agree with the claim that medicine B increases the weight significantly?

3. A random sample of 20 daily workers of state A was found to have average daily earning of Rs.44 with sample variance 900. Another sample of 20 daily workers of state B was found to earn on average of Rs.30 per day with sample variance 400. Test whether the workers in state A are earning more than those in state B.

χ^2 test for the population variance :

Population : s.d. = σ or variance = σ^2

Sample : Size = n, data values are $x_1, x_2, \ldots, x_n$

Here null hypothesis is

The sample has been drawn from the population

(or)

There is no significant difference between population variance and sample variance.

Calculate $\chi^2 = \dfrac{\Sigma\left(x_i - \bar{x}\right)^2}{\sigma^2}$

Take the corresponding value of χ^2 at a prescribed level of significance for (n-1) d.f. from the χ^2 test table.

If the calculated value is less than the table value of χ^2, then the hypothesis is accepted otherwise it is rejected.

Problems on χ^2 test for single variance :

1. A sample of 20 observations gave a standard deviation 3.72. Is this compatible with the hypothesis that the sample is from a population with variance 4.35?

2. Weights (in Kgs.) of 10 students are given below:

   38, 40, 45, 53, 47, 43, 55, 48, 52 and 49.

   Can we say that variance of distribution of weights of all students, from which above sample of 10 students was drawn, is equal to 20 Sq.Kgs.?

F-test for equality of population variances :

Sample 1 : Size = n_1, data values are $x_1, x_2, \ldots, x_{n_1}$

Sample 2 : Size = n_2, data values are $y_1, y_2, \ldots, y_{n_2}$

Here null hypothesis is

Both the samples are drawn from the same population

(or)

No significant difference between sample variances

(or)

Both the samples are drawn from different populations which have the same variance

(or)

Both the samples are drawn from different populations for which difference of variances is insignificant.

Calculate $s_1^2 = \frac{1}{n_1 - 1} \sum_{i=1}^{n_1} (x_i - \bar{x})^2, \quad s_2^2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} (y_j - \bar{y})^2$

$$F = \begin{cases} \frac{s_1^2}{s_2^2}, & \text{if } s_1^2 > s_2^2 \\ \frac{s_2^2}{s_1^2}, & \text{if } s_1^2 < s_2^2 \end{cases}$$

Take the corresponding value of F at a prescribed level of significance for $(n_1\text{-}1, n_2\text{-}1)$ d.f. if $s_1^2 > s_2^2$ or $(n_2\text{-}1, n_1\text{-}1)$ d.f. if $s_1^2 < s_2^2$ from the F test table.

If the calculated value is less than the table value of F, then the hypothesis is accepted, otherwise it is rejected.

Problems on F-test for equality of variances :

1. It is known that the mean diameters of rivets produced by two firms A and B are practically the same, but the standard deviations may differ. For 22 rivets produced by firm A the standard deviation is 2.9 mm, while for 16 rivets produced by firm B the standard deviation is 3.8 mm. Compute the statistic you would use to test whether the products of firm A have the same variability as those of firm B.

2. In one sample of 8 observations the sum of the squares of deviations of the sample values from the mean was 84.4 and in another sample of 10 observations it was 102.6. Test whether this difference is significant at 5% level of significance.

Non-parametric tests :

In statistics, nonparametric tests are **methods of statistical analysis that do not require a distribution to meet the required assumptions to be analyzed** (especially if the data is not normally distributed). Due to this reason, they are sometimes referred to as distribution-free tests.

Example :

It may be noted that the χ^2-test for goodness of fit depends only on the set of observed and expected frequencies and on number of degrees of freedom. It does not make any assumptions regarding the parent population from which the observations are taken. Since χ^2-test for goodness of fit does not involve any population parameters, the test is known as non-parametric test or distribution test.

Estimation of parameters by the method of moments :

The method of moments involves equating sample moments with theoretical moments.

Definitions :

1. $E(X^k)$ is the k^{th} (theoretical) moment of the distribution (about the origin) for $k = 1,2,\ldots$
2. $E[(X-\mu)^k]$ is the k^{th} (theoretical) moment of the distribution (about the mean μ) for $k = 1,2,\ldots$
3. $M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^{\,k}$ is the k^{th} sample moment about origin for $k = 1,2,\ldots$
4. $M_k{}^* = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^k$ is the k^{th} sample moment about the mean $\overline{X}$ for $k = 1,2,\ldots$

Form of the method:

1. Equate the first sample moment about the origin $M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$ to the first theoretical moment $E(\mathrm{X})$.
2. Equate the second sample moment about the origin $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^{\,2}$ to the second theoretical moment $E(\mathrm{X}^2)$.
3. Continue equating the sample moments about the origin M_k with the corresponding theoretical moments, $k = 3,4,\ldots$ until you have as many equations as you have parameters.
4. Solve for the parameters.

The resulting values are called estimators by the method of moments.

Example :

Let X_1, X_2, …, X_n be the normal random variables with mean μ and variance σ^2. What are the estimators by method of moments for the mean μ and variance σ^2?

Solution :

The first and second theoretical moments about the origin are

$E(X_i) = \mu$ and $E(X_i^2) = \sigma^2 + \mu^2$

The first and second sample moments about the origin are

$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$ and $M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Equating the first sample moment about the origin to the first theoretical moment, we get $E(X) = \frac{1}{n}\sum_{i=1}^{n} X_i = \mu$

Equating the second sample moment about the origin to the second theoretical moment, we get $E(X^2) = \sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

The estimator for the mean μ by the method of moments is given by

$$\hat{\mu}_{MM} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

The estimator for the variance σ^2 by the method of moments is given by

$$\hat{\sigma}^2_{MM} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2$$

Maximum likelihood estimation of parameters :

Likelihood function :

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from a population density function $f(x,\theta)$. Then the likelihood function of the sample values $x_1, x_2, \ldots, x_n$ is their joint density function given by

$$L(\theta) = f(x_1,\theta)f(x_2,\theta)\ldots f(x_n,\theta) = \prod_{i=1}^{n} f(x_i,\theta) \quad \text{where } \theta \text{ is the parameter.}$$

The principle of maximum likelihood consists in finding an estimator for unknown parameter θ.

The parameter value $\theta = \hat{\theta}$ is the maximum likelihood estimator (MLE) if

$$\left.\frac{\partial L}{\partial \theta}\right|_{\theta=\hat{\theta}} = 0 \quad \text{and} \quad \left.\frac{\partial^2 L}{\partial \theta^2}\right|_{\theta=\hat{\theta}} < 0$$

Note : Since $L > 0$ and $\log L$ is a non-decreasing function of L, L and $\log L$ attain their extreme values (maxima and minima) at same value of $\theta = \hat{\theta}$.

Problems on maximum likelihood estimators :

1. Find the maximum likelihood estimate for the parameter λ of a Poisson distribution on the basis of a sample of size n.

Solution :

The probability function of the Poisson distribution with parameter λ is given by

$$P(X = x) = f(x,\lambda) = \frac{e^{-x}\lambda^x}{x!}, x = 0,1,2,\ldots$$

Likelihood function of random sample $x_1, x_2, \ldots, x_n$ of n observations is

$$L(\lambda) = \prod_{i=1}^{n} f(x_i,\lambda) = \frac{e^{-n\lambda}\,\lambda^{x_1+x_2+\ldots+x_n}}{x_1!x_2!\ldots x_n!}$$

$$\log L = -n\lambda + \left(\sum_{i=1}^{n} x_i\right)\log\lambda - \sum_{i=1}^{n}\log(x_i!) = -n\lambda + n\bar{x}\log\lambda - \sum_{i=1}^{n}\log(x_i!)$$

$$\frac{\partial}{\partial\lambda}\log L = 0 \quad \Rightarrow \quad -n + \frac{n\bar{x}}{\lambda} = 0 \quad \Rightarrow \quad \lambda = \bar{x}$$

$$\left.\frac{\partial^2}{\partial\lambda^2}\log L\right|_{\lambda=\bar{x}} = -\left.\frac{n\bar{x}}{\lambda^2}\right|_{\lambda=\bar{x}} < 0$$

Thus the MLE for λ is the sample mean $\bar{x}$

2. In random sampling from normal population with parameters μ and σ^2, find the maximum likelihood estimators for

(i) μ when σ^2 is known

(ii) σ^2 when μ is known

(iii) the simultaneous estimation of μ and σ^2.

Solution :

The probability function of the normal distribution with parameters μ and σ^2 is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Likelihood function of random sample $x_1, x_2, \ldots, x_n$ of n observations is

$$L = \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)\right] = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}\right)$$

$$\log L = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

(i) When σ^2 is known, the likelihood equation for estimating μ is given by

$$\frac{\partial}{\partial\mu}\log L = 0 \quad \Rightarrow \quad -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i-\mu)(-1) = 0$$

$$\Rightarrow \quad \sum_{i=1}^{n}(x_i-\mu) = 0 \quad \Rightarrow \quad \sum_{i=1}^{n} x_i - n\mu = 0 \quad \Rightarrow \quad \mu = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\frac{\partial^2}{\partial\mu^2}\log L = \frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(-1) = -\frac{n}{\sigma^2} < 0$$

Hence, for a normal distribution when σ^2 is known, the maximum likelihood estimation for μ is given by $\hat{\mu} = \bar{x}$

(ii) When μ is known, the likelihood equation for estimating σ^2 is given by

$$\frac{\partial}{\partial\sigma^2}\log L = 0 \quad \Rightarrow \quad -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\Rightarrow \quad n - \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \quad \Rightarrow \quad \sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial^2}{\partial(\sigma^2)^2}\log L = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\left.\frac{\partial^2}{\partial(\sigma^2)^2}\log L\right|_{\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}(n\sigma^2) = -\frac{n}{2\sigma^4} < 0$$

Hence, for a normal distribution when μ is known, the maximum likelihood estimation for σ^2 is given by $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

(iii) The likelihood equations for simultaneous estimation of μ and σ^2 are

$$\frac{\partial}{\partial\mu}\log L = 0 \quad \text{and} \quad \frac{\partial}{\partial\sigma^2}\log L = 0$$

Solving these, we get $\mu = \bar{x}$ and $\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2$

$$\frac{\partial^2}{\partial\mu^2}\log L\Big|_{\mu=\bar{x},\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = \frac{1}{2\sigma^2}\sum_{i=1}^{n}2(-1) = -\frac{n}{\sigma^2} < 0$$

$$\frac{\partial^2}{\partial(\sigma^2)^2}\log L\Big|_{\mu=\bar{x},\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}(n\sigma^2) = -\frac{n}{2\sigma^4} < 0$$

$$\frac{\partial^2}{\partial\mu\partial\sigma^2}\log L\Big|_{\mu=\bar{x},\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = \frac{1}{2\sigma^4}\sum_{i=1}^{n}2(x_i-\mu)\Big|_{\mu=\bar{x},\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2} = 0$$

$$\left[\left(\frac{\partial^2}{\partial\mu^2}\log L\right)\left(\frac{\partial^2}{\partial(\sigma^2)^2}\log L\right) - \left(\frac{\partial^2}{\partial\mu\partial\sigma^2}\log L\Big|\right)^2\right]_{\mu=\bar{x},\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= \left(-\frac{n}{\sigma^2}\right)\left(-\frac{n}{2\sigma^4}\right) - 0 = \frac{n^2}{2\sigma^6} > 0$$

Hence, for normal distribution the maximum likelihood estimations for μ and σ^2 are given by $\hat{\mu} = \bar{x}$ and $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = s^2$

where s^2 = sample variance