Groups of Order 9.

(1). When looking for groups of order 9, there is obviously the cyclic group of order 9, C9. List the generators of C9. Conclude that for all other groups of order 9, all non-identity elements must have order 3.

* We know that C9 is a cyclic group, and that the generators of a cyclic group are all of the elements that are coprime to the order of the group.
* The generators of C9 are {1, a, b, ab, b2, ab2}

(2). Pick a group G of order 9 so that G is not C9. Show that is must have an element, a, of order 3. Pick b in G - {1, a, a2}. Prove that {b, b2} and {1, a, a2} have no elements in common.

* Since we know that our group G is not equal to the cyclic group on 9 elements, we know that there is no element of order 9 in the group. Knowing that the order of an element must divide the order of the group, we know that the order of the elements in our group must be {1, 3, 9} and with 9 crossed off it must be {1,3}, and trivially a group cannot only contain elements of order 1, so the group must contain at least one element of order 3.

(3). Prove that {1, a, a2, b, ab, a2b, b2, ab2, a2b2} are all distinct and hence are the 9 members of the group.

* Trivially, the elements {1, b, a, b2, a2} cannot equal any other element in the group, and therefore have to be distinct.
* a2 = ab cannot be true or else a = b from multiplying on the left by a-1,
* ab = b2 cannot be true or else a = b from multiplying on the right by b-1.
* a2b = b2 cannot be true or else a2 = b from multiplying on the right by b-1.
* ab2 = a2b2 cannot be true or else a = a2 ⇒ 1 = a from multiplying on the right by b1 twice, and then a-1 once.
* a2b2 = a2 cannot be true or else b2 = 1 from multiplying on the left twice by a-1.

Therefore all the elements in the group are distinct.

(4). Clearly, ba must be in the group. Prove that ba is not in {1, a, a2, b, b2}, so it must be in the set {ab, a2b, ab2, a2b2}

* We know that ba cannot be {1, a, a2, b, b2} because the multiplication of any elements in the group must stay in the group. It is clear to see that multiplying any of these elements by ba will result in some pair of b’s and a’s, which does not exist in our set {1, a, a2, b, b2}, so ba must be in {ab, a2b, ab2, a2b2}.

(5). Prove that if G1 and G2 are groups, the Cartesian product G1 X G2 which consists of pairs (g1,g2) is a group. What is its identity?

* If we have an operation X on two groups G1 and G2 that maps each element of G1 to each element of G2, the resulting pairs form a group.
* Let an represent all the elements in G1, and bn represent all elements in G2 we must prove that the operation is associative. [(a1b1)(a2b2)]\*(a3b3) = [(a1b1)(a2b2)(a2b2)] = (a1b1) \* [(a2b2)(a3b3)]
* The identity element of the resulting group is the Identity element of G1, and the identity element of G2, namely (i1, i2).

(6). Show that if ba = ab, G is abelian. In this case, show that G can be expressed as the cartesian product of two simpler groups.

|  |  |  |  |
| --- | --- | --- | --- |
| G1 | 1 | a | a2 |
| 1 | 1 | a | a2 |
| a | a | a2 | 1 |
| a2 | a2 | 1 | a |

|  |  |  |  |
| --- | --- | --- | --- |
| G2 | 1 | b | b2 |
| 1 | 1 | b | b2 |
| b | b | b2 | 1 |
| b2 | b2 | 1 | b |

If we know the group is abelian, the cross product of these two simple groups will result in a group with order of the product of the orders of the 2 simpler groups. Only if the gcd of the orders of the simpler groups is 1. This is because if there is a divisor of the 2 simpler groups, and we know ba = ab, there will be overlapping elements that can be generated by either group, resulting in duplicate elements, and a group that does not divide the order of the resulting group.

(7). Construct a Cayley table under the assumption that ba = a2b. Show that the Cayley table is not associative by a counter example. Thus, ba cannot equal ab2.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ba=a2b | 1 | a | a2 | b | ab | a2b | b2 | ab2 | a2b2 |
| 1 | 1 | a | a2 | b | ab | a2b | b2 | ab2 | a2b2 |
| a | a | a2 | 1 | ab | a2b | b | ab2 | a2b2 | b2 |
| a2 | a2 | 1 | a | a2b | b | ab | a2b2 | b2 | ab2 |
| b | b | **a2b** | ab | b2 | a2b2 | ab2 | 1 | a2 | a |
| ab | ab | b | a2b | ab2 | b2 | a2b2 | a | 1 | a2 |
| a2b | a2b | ab | b | a2b2 | ab2 | b2 | a2 | a | 1 |
| b2 | b2 | ab2 | a2b2 | 1 | a | a2 | b | ab | a2b |
| ab2 | ab2 | a2b2 | b2 | a | a2 | 1 | ab | a2b | b |
| a2b2 | a2b2 | b2 | ab2 | a2 | 1 | a | a2b | b | a2b |

Counter Example:

* [(ab)(ab2)]\*(a) ⇒ a
* (ab)\*[(ab2)(a)] ⇒ 1
* Therefore with ba = a2b the table is not associative, and cannot be valid

(8). Construct a Cayley table under the assumption that ba = ab2. Show that the Cayley table is not associative by a counter example.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ba=ab2 | 1 | a | a2 | b | ab | a2b | b2 | ab2 | a2b2 |
| 1 | 1 | a | a2 | b | ab | a2b | b2 | ab2 | a2b2 |
| a | a | a2 | 1 | ab | a2b | b | ab2 | a2b2 | b2 |
| a2 | a2 | 1 | a | a2b | b | ab | a2b2 | b2 | ab2 |
| b | b | **ab2** | a2b | b2 | a | a2b2 | 1 | ab | a2 |
| ab | ab | a2b2 | b | ab2 | a2 | b2 | a | a2b | 1 |
| a2b | a2b | b | ab | a2b2 | b2 | ab2 | a2 | 1 | a |
| b2 | b2 | ab | a2b2 | 1 | ab2 | a2 | b | a | a2b |
| ab2 | ab2 | a2b | b2 | a | a2b2 | 1 | ab | a2 | b |
| a2b2 | a2b2 | b2 | ab2 | a2 | 1 | a | a2b | b | ab |

Counter Example:

* [(ab)(a)]\*(a2) ⇒ ab2
* (ab)\*[(a)(a2)] ⇒ ab

(9). Try to construct a Cayley table under the assumption that ba = a2b2. What problem do you run into in trying to construct this table? Put together an argument show that ba = a2b2 cannot be a rule in a group of order 9.

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| ba=a2b2 | 1 | a | a2 | b | ab | a2b | b2 | ab2 | a2b2 |
| 1 | 1 | a | a2 | b | ab | a2b | b2 | ab2 | a2b2 |
| a | a | a2 | 1 | ab | a2b | b | ab2 | a2b2 | b2 |
| a2 | a2 | 1 | a | a2b | b | ab | a2b2 | b2 | ab2 |
| b | b | **a2b2** |  | b2 |  |  | 1 |  |  |
| ab | ab | b2 |  | ab2 |  |  | a |  |  |
| a2b | a2b |  |  | a2b2 |  |  | a2 |  |  |
| b2 | b2 |  |  | 1 |  |  | b |  |  |
| ab2 | ab2 |  |  | a |  |  | ab |  |  |
| a2b2 | a2b2 |  |  | a2 |  |  | a2b |  |  |

It becomes quickly clear when attempting to construct this cayley table, that there are certain multiplications that result in infinitely generated rules, For example (ab)(a2) will continuously generate a new pair of (a2b2) that will have either a b on the left side (ba2b2) and will chain generate new pairs to the right resulting in an a on the right side, which generates the rule again until there is a b on the left side, and vice versa.

(10). Conclude that there are only two groups of order 9 and that they are both abelian. Give the groups in terms of cyclic groups Ck.

* After proving that we cannot construct a cayley table using ba = a2b2 but we can using ba = ab2 and ba = a2b, we know that the groups of order 9 are abelian. With the knowledge from earlier that the cross product of 2 groups forms a group of all the pairs of the originals, we can conclude that the non cyclic group of order 9 is the construction of C3 X C3.