Solving linear forward-looking equations

This note shows the steps necessary to solve linear forward-looking equations when ‘bubbles’ are ruled out by a terminal condition. The two examples below are taken from the lecture notes, and there are two exercises at the end to test your understanding of this material.

Example 1: Perfect foresight Cagan model

The lecture notes derive the following expressions for the (log) price level:

$$p_t = \frac{1}{1+\eta} m_t + \frac{\eta}{1+\eta} p_{t+1} \tag{1}$$

$$p_t = \frac{1}{1+\eta} m_t + \frac{\eta}{(1+\eta)^2} m_{t+1} + \frac{\eta^2}{(1+\eta)^2} p_{t+2} \tag{2}$$

$$p_t = \frac{1}{1+\eta} m_t + \frac{\eta}{(1+\eta)^2} m_{t+1} + \frac{\eta^2}{(1+\eta)^3} m_{t+2} + \frac{\eta^3}{(1+\eta)^3} p_{t+3} \tag{3}$$

The general pattern that emerges is that the price level is equal to its discounted value in some future period, plus the contribution of money supplies in the intervening periods. Mathematically, we can represent this pattern as follows:

$$p_t = \frac{1}{1+\eta} \sum_{s=t}^{t+T-1} \left(\frac{\eta}{1+\eta} \right)^{s-t} m_s + \left(\frac{\eta}{1+\eta} \right)^{T} p_{t+T} \tag{4}$$

This equation means we can write the current price level as function of the price level T periods from now (where T is a positive integer), without having to write a really long equation. Note that Equation (1) arises as a special case of this equation when we set $T = 1$, Equation (2) arises when $T = 2$, and Equation (3) when $T = 3$. Equation (4) is not a full solution for the price level as it has p_{t+T} on the RHS. However, using Equation (4), we can work out what the price level would be if we kept on substituting for the future price level.

The answer is given by the limit of Equation (4) as $T \rightarrow \infty$, or

$$p_t = \frac{1}{1+\eta} \sum_{s=t}^{\infty} \left(\frac{\eta}{1+\eta} \right)^{s-t} m_s + \lim_{T\to\infty} \left(\frac{\eta}{1+\eta} \right)^{T} p_{t+T} \tag{5}$$

We can rule out bubbles in the price level with the terminal condition: $\lim_{T\to\infty} \left(\frac{\eta}{1+\eta} \right)^{T} p_{t+T} = 0$.

Therefore, under the assumption of no bubbles, our final solution for the price level is

$$p_t = \frac{1}{1+\eta} \sum_{s=t}^{\infty} \left(\frac{\eta}{1+\eta} \right)^{s-t} m_s \qquad \text{(O\&R, p. 518)} \tag{6}$$

Example 2: Stochastic flex-price model of the exchange rate

The lecture notes derive the following expressions for the (log) nominal exchange rate:

$$e_t = \frac{1}{1+\eta} x_t + \frac{\eta}{1+\eta} E_t e_{t+1} \tag{7}$$

$$e_t = \frac{1}{1+\eta} x_t + \frac{\eta}{(1+\eta)^2} E_t x_{t+1} + \frac{\eta^2}{(1+\eta)^2} E_t e_{t+2} \tag{8}$$

$$e_t = \frac{1}{1+\eta} x_t + \frac{\eta}{(1+\eta)^2} E_t x_{t+1} + \frac{\eta^2}{(1+\eta)^3} E_t x_{t+2} + \frac{\eta^3}{(1+\eta)^3} E_t e_{t+3} \tag{9}$$

where $x_t = m_t - \phi y_t + \eta i^*_{t+1} - p^*_t$.

The general pattern that emerges is that the nominal exchange rate is equal to its discounted expected value in some future period, plus the contribution of current and future expected x's (i.e. fundamentals) in the intervening periods.

Mathematically, we can represent this pattern as follows:

$$e_t = \frac{1}{1+\eta} \sum_{s=t}^{t+T-1} \left(\frac{\eta}{1+\eta} \right)^{s-t} E_t x_s + \left(\frac{\eta}{1+\eta} \right)^T E_t e_{t+T} \tag{10}$$

where $x_s = m_s - \phi y_s + \eta i^*_{s+1} - p^*_s$.

Note that Equation (7) arises as a special case of Equation (10) when we set $T = 1$, Equation (8) arises when $T = 2$, and Equation (9) arises when $T = 3$. To reach the final exchange rate solution, we need to take the limit of Equation (10) as $T \to \infty$, or

$$e_t = \frac{1}{1+\eta} \sum_{s=t}^{\infty} \left(\frac{\eta}{1+\eta} \right)^{s-t} E_t x_s + \lim_{T\to\infty} \left(\frac{\eta}{1+\eta} \right)^T E_t e_{t+T} \tag{11}$$

We rule out bubbles in the exchange rate with the terminal condition: $\lim_{T\to\infty} \left(\frac{\eta}{1+\eta} \right)^T E_t e_{t+T} = 0$.

Hence, under the assumption that there are no (expected) 'bubbles' in the exchange rate, the final solution for the nominal exchange rate is given by

$$e_t = \frac{1}{1+\eta} \sum_{s=t}^{\infty} \left(\frac{\eta}{1+\eta} \right)^{s-t} E_t x_s \qquad \text{(O\&R, p. 528)} \tag{12}$$

Exercise 1:

Consider the stochastic Cagan model.

(A) Show that the terminal condition in this model is given by

$$\lim_{T\to\infty}\left(\frac{\eta}{1+\eta}\right)^{T} E_t\, p_{t+T} = 0$$

(B) What does this terminal condition say in words?

Exercise 2:

Consider the analytical solution of the Dornbusch overshooting model (O&R, Chapter 9.2.3). Obstfeld and Rogoff argue that this solution requires the assumption that

$$\lim_{T\to\infty}\left(\frac{\eta}{1+\eta}\right)^{T} e_{t+T} = 0$$

With the aid of the solutions for Example 1 and Example 2 above, show that this assumption is necessary in order to derive Equation (14) in the main text (see p. 616).