Supplementary Appendix for "Simulating multiple equilibria in rational expectations models with occasionally-binding constraints: An algorithm and a policy application"[1]

This appendix provides further details of the numerical applications solved in Section 4 of the main paper, and there is an extra application with an RBC model which replicates some numerical results in Guerrieri and Iacoviello (2015, Section 4). In addition, we show how our algorithm can be applied to a model in which the constraint binds at steady state. The codes for the simulations are available at the author's GitHub page at: github.com/MCHatcher.

1 General framework

We consider models of the form

$$\begin{aligned} &B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, \quad \forall t \geq 1 \\ &\text{s.t. } x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \text{ for all } t, \text{ and } x^*_{1,t} > \underline{x}_1 \ \forall t > T \end{aligned} \tag{1}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i \ \forall i \in [5]$, $x_0 \in \mathbb{R}^n$ given, e_t is a vector of known shocks with $e_t = 0_{m\times 1}$ for all $t > T$, $\mathbb{1}_t \in \{0, 1\}$ is an indicator variable, and the 'shadow value' of the bounded variable is given by

$$x^*_{1,t} = F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + Ge_t + H \tag{2}$$

where $\underline{x}_1, H \in \mathbb{R}$, F is a $1 \times 3n$ vector with $f_{11} = 0$ and G is a $1 \times m$ vector.

The matrices $B_{i,t}$ are regime dependent. In the reference regime $B_{i,t} = \overline{B}_i$; in the alternative regime $B_{i,t} = \tilde{B}_i$. The indicator variable $\mathbb{1}_t$ determines which regime is realized at a given t. The assumption that $x^*_{1,t} > \underline{x}_1 \ \forall t > T$ is a terminal condition which states that the bounded variable, $x_{1,t}$, permanently escapes the bound after a finite number of periods T.

Solutions to the problem in (1) are found by trialling sequences for the indicator variable of the form $(\mathbb{1}_t)_{t=1}^T$ (with $\mathbb{1}_t$ specified for all t) and $\mathbb{1}_t = 1 \ \forall t > T$; this in turn implies sequences $\{B_{1,t}, B_{2,t}, B_{3,t}, B_{4,t}, B_{5,t}\}_{t\geq 1}$ which can be used to find a time path $(x_t)_{t\geq 1}$ using Proposition 1 of the main text. Only time paths consistent with the terminal condition and the occasionally-binding constraint are accepted as solutions (see Proposition 1).

Following Proposition 1, any solution(s) to problem (1) have the form

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for all } t > T \end{cases} \tag{3}$$

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where, for $t = 1, ..., T$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t} \tag{4}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}) \tag{5}$$

and $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ has eigenvalues in the unit circle, $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$, $\Psi_{T+1} = \overline{\Psi}$, $\Omega_{T+1} = \overline{\Omega}$, and $e_t = 0_{m\times 1}$ for all $t > T$.

2 Asset pricing model

The model has the form

$$\begin{aligned} q_t &= \beta(1-\rho)E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\ r_t &= \max\{\underline{r}, \phi q_t\} \\ u_t &= \rho_u u_{t-1} + e_t \end{aligned} \tag{6}$$

where $\beta, \rho \in (0,1)$, $\phi > 0$, $\underline{r} < 0$, $\rho_u \in (0,1)$ and all values of e_t are known.

Let $x_t = \begin{bmatrix} r_t & q_t & u_t \end{bmatrix}'$, such that the bounded variable is ordered first, and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -\phi & 0 \\ \sigma & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta(1-\rho) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho_u \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ \sigma & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i \text{ for } i \in \{2,3,4\}, \quad \overline{B}_5 = \begin{bmatrix} \underline{r} \\ 0 \\ 0 \end{bmatrix}.$$

Given $x_{1,t} = r_t$, the equation $r_t = \max\{\underline{r}, \phi q_t\}$ can be written in the form $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ by setting $\underline{x}_1 = \underline{r}$ and $x^*_{1,t} = \phi q_t$; note that the latter equation can be written in vector form as in (2) with $F = \begin{bmatrix} 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$.

2.1 Computing the M matrix

To compute the M matrix of impulse responses of the bounded variable, we solve the model in (6) ignoring the bound (i.e. with the max operator removed) and with a 'news shock' $v_{r,t} \in \{0,1\}$ added, such that: $r_t = \phi q_t + v_{r,t}$. The resulting model can be written as:

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \tilde{e}_t, \quad \forall t \geq 1 \tag{7}$$

where $\tilde{e}_t := \overline{B}_4 e_t + v_t$ and $v_t := \begin{bmatrix} v_{r,t} & 0 & 0 \end{bmatrix}'$, with $v_{r,t} = 0$ for all $t > T$, where T is the horizon at which the M matrix is being computed.

Recall that the 1st column of M lists the impulse response of r_t (at dates $t = 1, ..., T$) to the shock $v_{r,1} = 1$. In general, the jth column of M lists the impulse response of r_t (at dates $t = 1, ..., T$) to the shock $v_{r,j} = 1$, and this holds for columns $j = 1, ..., T$.

By Remark 2, the impulse responses and the M matrix are obtained as follows:

$$x_t = \overline{\Omega} x_{t-1} + \hat{\Gamma}\tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{8}$$

where $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_2(\Psi_{t+1} + \hat{\Gamma}\tilde{e}_{t+1})$ $\forall t \in [1, T]$, with $\Psi_t = 0_{n\times 1}$ for all $t \geq T$, and the M matrix is given by

$$M_{ij} = \frac{\partial x_{1,i}}{\partial v_{r,j}} = r_i|_{v_{r,j}=1} - r_i|_{v_{r,j}=0} \quad \text{for } i, j \in \{1, ..., T\}.$$

For the parameters $\beta = 0.99$, $\sigma = 5$, $\phi = 0.2$, $\rho = \rho_u = 0.5$, which were used for the numerical exercises in the main text, we checked whether the M matrix is a P-matrix for values of T up to 1,000; this turned out to be straightforward since we found that $M + M'$ is positive definite, which implies that M is a P-matrix (see e.g. Holden, 2022, Appendix: Lemma 1). Given these results, we can be confident that there is a unique solution to the model with bound imposed, and this squares with all our numerical perfect foresight simulations.[2]

2.2 Computing the policy function

To compute the policy function of the model in (6), we obtain the perfect foresight solution x_t as e_t varies. In particular, we fixed an $e_t \in [-0.2, 0.2]$, set a value for $x_{t-1} = \begin{bmatrix} 0 & q_{t-1} & u_{t-1} \end{bmatrix}'$ and specified values for $e_{t+1}, ...$; we then found the perfect foresight solution at those initial conditions using the method in Proposition 1. We repeated this process for different values of e_t linearly spaced in the interval $[-0.2, 0.2]$ while holding the other initial conditions fixed. Note that x_t corresponds to the first date on the perfect foresight path.

We consider $t = 1$ as the current period and we set $q_0 = u_0 = 0$, so $x_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}'$. We considered 60 different values of e_1 which are linearly-spaced on the interval $[-0.2, 0.2]$. To replicate the policy function in Guerrieri and Iacoviello (2015), we set $e_t = 0$ for all $t \geq 2$ as our baseline case (see main text). We also plot policy functions for the cases $e_t = 0.02$ for $t \in \{2, ..., 5\}$ (positive news shocks) and $e_t = -0.02$ for $t \in \{2, ..., 5\}$ (negative news shocks). In the latter two cases, everything is identical, except that e_t is non-zero for $t = 2, ..., 5$.

2.3 Perfect foresight solutions

The perfect foresight solutions plotted in Figure 4 are simply the perfect foresight paths for $t = 1, ..., 15$ when $e_1 = -0.1$. Note that r_1 and q_1 correspond to the points on the policy function in Figure 3 (see main text) when $e_1 = -0.1$, whereas the remaining values $r_2, ..., r_{15}$ and $q_2, ..., q_{15}$ are the paths of the interest rate and asset price in the subsequent periods.

[2]Recall that if M is a P-matrix then, for any initial conditions, there is a unique perfect foresight solution to the model with the bound at that T – so it is not necessary to search for multiple solutions at any $T' \leq T$ (since there cannot be more than one solution). Strictly speaking, multiple solutions cannot be ruled out for larger T, except by computing the M matrix; however, as a double check we ran our perfect foresight simulations for extremely large values of T (up to 5,000) and we found a unique solution in all cases.

3 New Keynesian model

3.1 Baseline model

The baseline model has the form:

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{9}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) \tag{10}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{11}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{12}$$

where $\theta_\pi > 1$, $\beta \in (0,1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0,1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

Let $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t \end{bmatrix}'$ and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i, \quad \text{for } i \in \{2,3,4\}, \quad \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Given $x_{1,t} = i_t$, the equation $i_t = \max\{\underline{i},\ i_t^*\}$ can be written in the form $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ by setting $\underline{x}_1 = \underline{i}$ and $x_{1,t}^* = i_t^*$; note that the latter equation can be written in vector form as in (2) with $F = \begin{bmatrix} 0 & 1 & 0_{1\times 10} \end{bmatrix}$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$. We set the parameters at the values given in the main text, whenever these parameters were not being varied as part of the analysis.

The M matrix is computed using the method in Section 2.1. We checked whether the M matrix is a P-matrix for pairs $(\theta_{\Delta y}, \theta_\pi)$ using a recursive test due to Tsatsomeros and Li (2000), for which Matlab code is available at Tsatsomeros' webpage.[3] We also computed perfect foresight paths for initial conditions $x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}'$, $e_1 = 0.01$, $e_t = 0\ \forall t \geq 2$.

[3]See https://www.math.wsu.edu/faculty/tsat/matlab.html.

3.2 Price-level targeting rule

With price-level targeting the model is amended to:

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{13}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)\left(\theta_p p_t + \theta_{\Delta y}(y_t - y_{t-1})\right) \tag{14}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t\pi_{t+1}) + e_t \tag{15}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{16}$$

$$p_t = p_{t-1} + \pi_t \tag{17}$$

where $\theta_p > 0$, $\beta \in (0,1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0,1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

Let $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t & p_t \end{bmatrix}'$ and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & 0 & -(1-\rho_i)\theta_p \\ \sigma^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\kappa & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},\ \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},\ \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & 0 & -(1-\rho_i)\theta_p \\ \sigma^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\kappa & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},\ \tilde{B}_i = \overline{B}_i,\ \text{ for } i \in \{2,3,4\},\ \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3.3 'Bad' solutions under price-level targeting

In the main text we note that when multiple solutions exist under a price-level targeting interest rate rule (14) (i.e, for small values of θ_p), the 'bad' solution has inflation and output cycle between positive and negative values before converging to the steady-state values. In this section we provide examples of this phenomenon (not all shown in the main paper).

In Figure 1 we plot a 'good' and 'bad' solution under the price-level targeting rule, when the response to the price level is sufficiently small that there are multiple solutions. Relative to zero or positive interest rate smoothing, there is a considerable improvement in the sense that the 'bad' solution is far less deflationary on impact and implies only a modest negative output gap. The bad solution cycles between positive and negative values in subsequent periods, before converging to steady state. Figure 2 demonstrates that the result in Figure 1 is not a 'one-off' for specific parameter values: in all cases, price-level targeting implies a 'bad' solution which is not so bad compared to the alternative policies we studied.

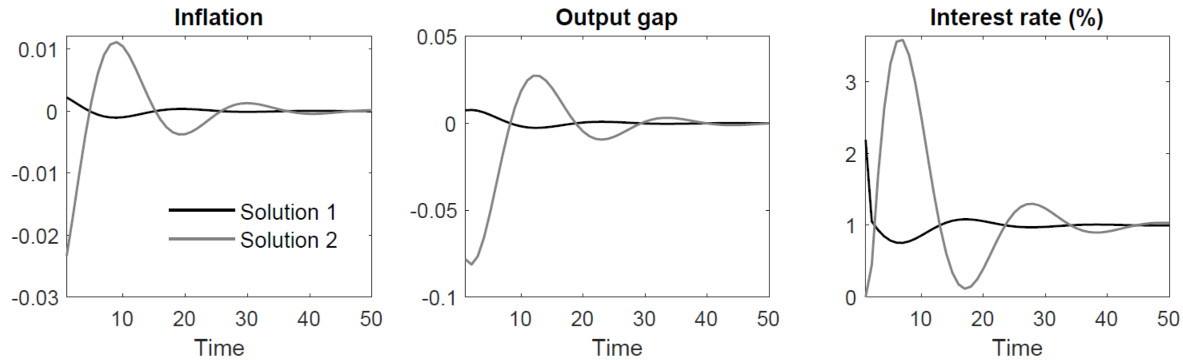

Figure 1: 'Good' and 'bad' solutions under price-level targeting when $\theta_p = 0.015$

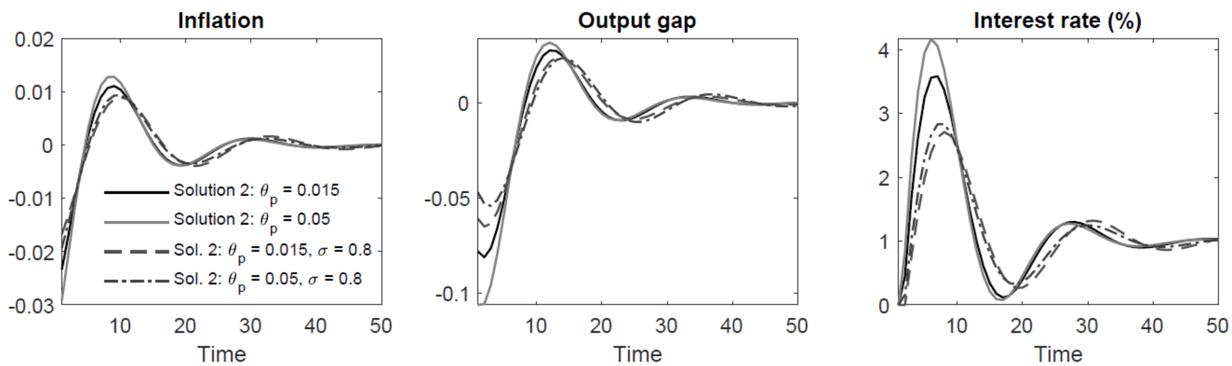

Figure 2: 'Bad' solutions under price-level targeting for various parameter values

4 RBC model with a constraint on investment

As an extra application, we considered a Real Business Cycle model with a lower bound on investment, as in Guerrieri and Iacoviello (2015, Section 4). This model requires us to log-linearize a non-linear model, in contrast to the other two applications,[4] and we also show how to specify the 'shadow value' $x^*_{1,t}$ in the equation $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$.

[4]In cases where log-linearization is tricky or laborious, the log-linearization can be done numerically in Dynare and the matrices $\overline{B}_j, \tilde{B}_j$, $j \in [5]$, can be extracted from the workspace; see Jones (2016).

A social planner chooses allocations $\{K_t, C_t\}_{t=0}^{\infty}$ to maximize utility $U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}-1}{1-\sigma}\right)$, subject to the following constraints:

$$C_t + I_t = A_t K_{t-1}^{\alpha} \tag{18}$$

$$K_t = (1-\delta)K_{t-1} + I_t \tag{19}$$

$$I_t \geq \phi I_{SS} \tag{20}$$

where $\sigma, I_{SS} > 0$, $\alpha, \phi \in (0,1)$, I_{SS} is the steady-state level of investment, and productivity is $A_t = A_{t-1}^{\rho} exp(\epsilon_t)$, where $\rho \in (0,1)$ and ϵ_t is a shock whose value is known at all dates.

Equations (18)–(20) are, respectively, the resource constraint, the capital accumulation equation, and a constraint that prevents investment from falling below a fraction ϕ of its steady-state value I_{SS}. The necessary conditions for a solution to the planner problem are (18)–(20) plus the consumption Euler equation and the complementary slackness condition:

$$C_t^{-\sigma} - \lambda_t = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta) - (1-\delta)\lambda_{t+1}) \tag{21}$$

$$\lambda_t(I_t - \phi I_{SS}) = 0 \tag{22}$$

where $\lambda_t \geq 0$ is the Lagrange multiplier on the investment constraint.

The investment constraint is slack when $\lambda_t = 0$ and binding when $\lambda_t > 0$. If $\lambda_t > 0$, then $I_t = \phi I_{SS}$ to ensure that the complementary slackness condition (22) holds. If $\lambda_t = 0$, then either $I_t = \phi I_{SS}$ or $I_t > \phi I_{SS}$ (but not $I_t < \phi I_{SS}$, since this would violate condition (20)). The two regimes are as follows. Under the reference regime (slack):

$$I_t = K_t - (1-\delta)K_{t-1}, \quad K_t = A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} - C_t$$

$$C_t^{-\sigma} = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta)), \quad \lambda_t = 0$$

and under the alternative regime (binding):

$$I_t = \phi I_{SS}, \quad K_t = I_t + (1-\delta)K_{t-1}, \quad C_t = A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} - K_t,$$

$$C_t^{-\sigma} - \lambda_t = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta) - (1-\delta)\lambda_{t+1}).$$

To put this non-linear model in the form of (1), we log-linearize the equations (under both regimes) around the steady state at which the investment constraint is slack.[5] To ease the process, we define the new variables $Y_t := A_t K_{t-1}^{\alpha}$ and $R_t := \alpha A_t K_{t-1}^{\alpha-1} + 1 - \delta$. The two regimes can then be written in terms of deviations from steady state as follows:

$$\hat{i}_t = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1}, \quad \hat{k}_t = (1-\delta)\hat{k}_{t-1} + (Y_{SS}/K_{SS})\hat{y}_t - (C_{SS}/K_{SS})\hat{c}_t$$

$$\hat{c}_t = E_t\hat{c}_{t+1} - (1/\sigma)E_t\hat{r}_{t+1}, \quad \lambda_t = 0, \quad \hat{y}_t = \hat{a}_t + \alpha\hat{k}_{t-1}$$

$$\hat{r}_t = \alpha R_{SS}^{-1}(Y_{SS}/K_{SS})\,\hat{a}_t - \alpha(1-\alpha)R_{SS}^{-1}(Y_{SS}/K_{SS})\,\hat{k}_{t-1}, \quad \hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t$$

[5] The steady state is $I_{SS} = \delta K_{SS}$, $C_{SS} = A_{SS}K_{SS}^{\alpha} - I_{SS}$, $K_{SS} = \left(\frac{\alpha\beta A_{SS}}{1-\beta(1-\delta)}\right)^{1/(1-\alpha)}$ and $A_{SS} = 1$.

under the reference regime, and

$$\hat{i}_t = \phi - 1, \quad \hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta \hat{i}_t, \quad C_{SS}\hat{c}_t = Y_{SS}\hat{y}_t + (1-\delta)K_{SS}\hat{k}_{t-1} - K_{SS}\hat{k}_t$$
$$C_{SS}^{\sigma}\lambda_t = -\sigma\hat{c}_t + \sigma E_t\hat{c}_{t+1} - E_t\hat{r}_{t+1} + (1-\delta)(C_{SS}^{\sigma}/R_{SS})E_t\lambda_{t+1}, \quad \hat{y}_t = \hat{a}_t + \alpha\hat{k}_{t-1}$$
$$\hat{r}_t = \alpha R_{SS}^{-1}\left(Y_{SS}/K_{SS}\right)\hat{a}_t - \alpha(1-\alpha)R_{SS}^{-1}\left(Y_{SS}/K_{SS}\right)\hat{k}_{t-1}, \quad \hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t$$

under the alternative regime.

We let $x_t := \begin{bmatrix}\hat{i}_t & \hat{k}_t & \hat{c}_t & \lambda_t & \hat{y}_t & \hat{r}_t & \hat{a}_t\end{bmatrix}'$ and $e_t := \begin{bmatrix}\epsilon_t\end{bmatrix}$, where 'hats' are log deviations from steady-state, i.e. $\hat{z}_t := \ln(Z_t/Z_{SS}) \approx (Z_t - Z_{SS})/Z_{SS}$. Note that $x_{1,t} = \hat{i}_t$. The constraint (20) is $\hat{i}_t \geq \phi - 1$ in deviations and we put this in the form $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ by setting $\underline{x}_1 = \phi - 1$ and $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t$. In the reference regime (slack), $\lambda_t = 0$, so $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1}$ and $\max\{\underline{x}_1, x_{1,t}^*\} = x_{1,t}^*$ if and only if $\delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} (= \hat{i}_t) \geq \phi - 1$. In the alternative regime (binding), $\lambda_t > 0$, so $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t = (\phi - 1) - \lambda_t < \phi - 1$, so $\max\{\underline{x}_1, x_{1,t}^*\} = \underline{x}_1 = \phi - 1$ as required.[6]

The shadow value $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t$ can be written as in (2) by setting $F = \begin{bmatrix}0 & \delta^{-1} & 0 & -1 & 0_{1\times 11} & -(1-\delta)\delta^{-1} & 0_{1\times 5}\end{bmatrix}$ and $G = H = \begin{bmatrix}0\end{bmatrix}$. The matrices $\overline{B}_j, \tilde{B}_j$, $j \in [5]$, under the two regimes have the form:

$$\overline{B}_1 = \begin{bmatrix} 1 & -\delta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{C_{SS}}{K_{SS}} & 0 & -\frac{Y_{SS}}{K_{SS}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\alpha Y_{SS}}{K_{SS}R_{SS}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_3 = \begin{bmatrix} 0 & -(1-\delta)\delta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1-\alpha)Y_{SS}}{K_{SS}R_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{K_{SS}}{C_{SS}} & 1 & 0 & -\frac{Y_{SS}}{C_{SS}} & 0 & 0 \\ 0 & 0 & \sigma & C_{SS}^{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\alpha Y_{SS}}{K_{SS}R_{SS}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & \frac{(1-\delta)C_{SS}^{\sigma}}{R_{SS}} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

[6] Note that if the computed value of λ_t is negative under the alternative regime, then we must reject the guess that the constraint binds in this period. In this case, $x_{1,t}^* > \underline{x}_1$ $(x_{1,t}^*|_{\mathbb{1}_t=0} = \hat{i}_t - \lambda_t = (\phi-1) - \lambda_t > \phi - 1)$, so $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\} = x_{1,t}^* > \phi - 1$, and the guess that $\mathbb{1}_t = 0$ is rejected as required.

$$
\tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\delta)K_{SS}}{C_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1-\alpha)Y_{SS}}{K_{SS}R_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \tilde{B}_4 = \overline{B}_4, \quad \tilde{B}_5 = \begin{bmatrix} \phi - 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
$$

We set $\beta = 0.96$, $\delta = 0.10$, $\rho = 0.90$, $\phi = 0.975$, $\sigma = 2$, $\alpha = 0.33$, as in Guerrieri and Iacoviello (2015). The policy functions (see Figure 3) and the impulse responses to a -4% productivity shock (see Figure 4) match those in Guerrieri and Iacoviello (2015, Fig. 2–3)

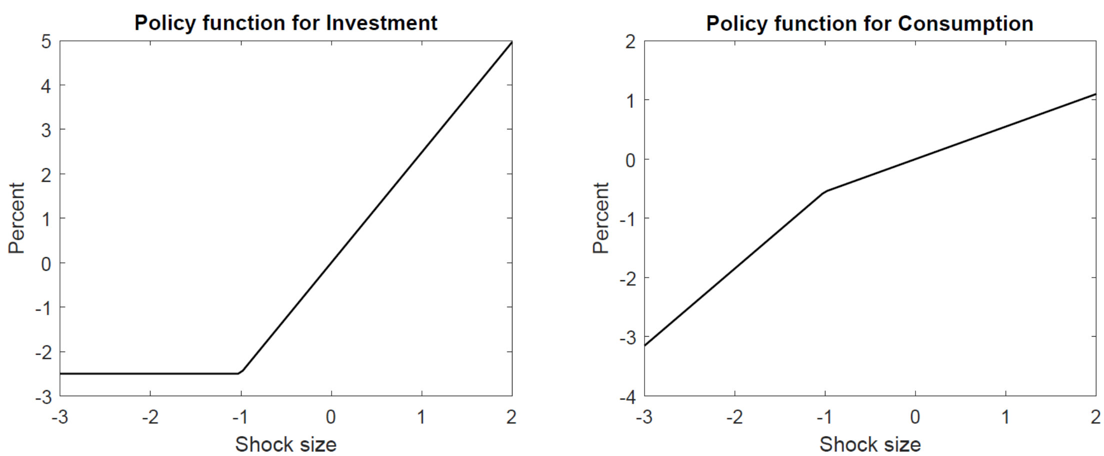

Figure 3: Policy functions for various shocks sizes e_t when $x_{t-1} = \overline{x}$

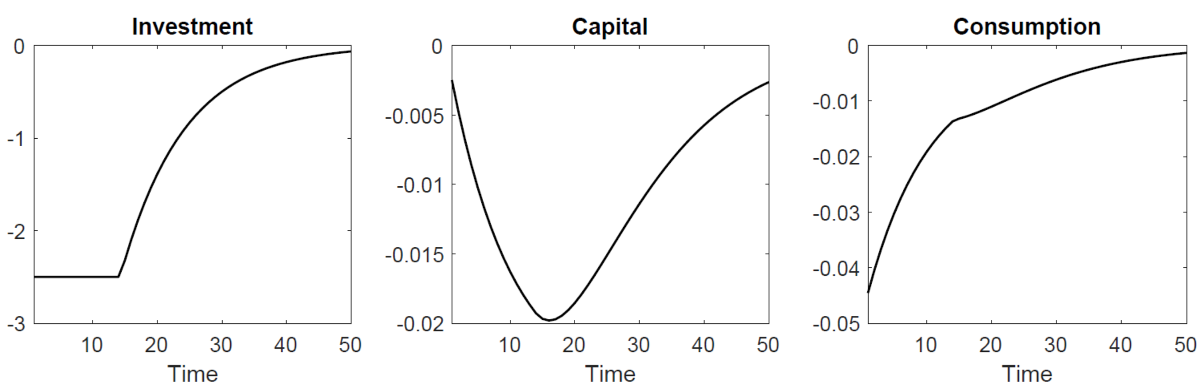

Figure 4: Perfect foresight solution for $e_1 = -0.04$ when $x_0 = \overline{x}$

5 Model with constraint binding at steady state

In this section we explain how our algorithm can be applied to models in which the constraint binds at steady state. An example of such a model is the borrowing constraint model that appears in Guerrieri and Iacoviello (2015, Online Appendix, Section C.1). Starting from the general model in (1)–(2), Assumptions 1-2 in the main paper must adapted for a model that converges to a steady state in which the constraint is binding. Assumptions 1 and 2 are now:

Assumption 1 *We assume that* $\det[\tilde{B}_1 - \tilde{B}_2 - \tilde{B}_3] \neq 0$*, such that there exists a unique steady state* $\overline{x} = (\tilde{B}_1 - \tilde{B}_2 - \tilde{B}_3)^{-1}\tilde{B}_5$ *at the alternative regime.*

Assumption 2 *Given a solution path* $(x_t)_{t=1}^{T}$ *that satisfies* (1)–(2) *and anticipated shocks* $(e_t)_{t=1}^{\infty}$ *with* $e_t = 0_{m\times 1}$ $\forall t > T$*, there exists is a unique stable terminal solution* $(x_t)_{t=T+1}^{\infty}$ *that satisfies* $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Psi}$ *and* $x_{1,t} = \underline{x}_1 > x_{1,t}^*$*, where* $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_3$ *has eigenvalues in the unit circle, and* $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}(\tilde{B}_2\tilde{\Psi} + \tilde{B}_5) = (I_n - \tilde{\Omega})\overline{x}$*.*

We can then restate Proposition 1 in the main paper as follows.

Proposition 1 *Consider model* (1)–(2) *for* $T \in \mathbb{N}_+$*, described by Assumptions 1–2 above. If one or more perfect foresight solutions exist, the solution(s) satisfy the following equations:*

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \tilde{\Omega}x_{t-1} + \tilde{\Psi} & \text{for } t > T \end{cases} \tag{23}$$

where, for $t = 1, ..., T$*,*

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t} \tag{24}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}) \tag{25}$$

provided that $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ $\forall t \in [1, T]$*; the terminal matrices* $\Omega_{T+1} = \tilde{\Omega}$*,* $\Psi_{T+1} = \tilde{\Psi}$*,* $\Gamma_{T+1} = 0_{n\times m}$ *and the terminal solution* $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Psi}$ $\forall t > T$ *satisfy Assumption 2; and the guessed structure* $(\mathbb{1}_t)_{t=1}^{T}$ *is verified by* $x_{1,t}^*$ *at all dates on the transition path, i.e.* $\forall t \in [1, T]$

$$B_{j,t} = \begin{cases} \overline{B}_j & \text{if and only if } \mathbb{1}_t = 1 \text{ and } x_{1,t}^*|_{\mathbb{1}_t=1} \geq \underline{x}_1 \\ \tilde{B}_j & \text{if and only if } \mathbb{1}_t = 0 \text{ and } x_{1,t}^*|_{\mathbb{1}_t=0} \leq \underline{x}_1 \end{cases}, \quad j = 1, .., 5. \tag{26}$$

Proof. The proof follows the same steps as the proof of Proposition 1 in the Appendix of the main paper, except that the terminal solution is $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Psi}$, $\forall t > T$, such that the matrices $\overline{\Omega}$, $\overline{\Psi}$ are replaced by $\tilde{\Omega}$, $\tilde{\Psi}$ in all expressions involving the terminal solution. ■

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