Online Appendix: "Simulating multiple equilibria in rational expectations models with occasionally-binding constraints: An algorithm and a policy application"[1]

This appendix provides further details of the numerical examples solved the main paper, and there are some sanity checks with numerical examples from Guerrieri and Iacoviello (2015). In addition, we show how our algorithm can be applied to a model in which the constraint binds at steady state and to the case of *multiple* occasionally-binding constraints. The codes for the simulations are available at the author's GitHub page at: github.com/MCHatcher.

1 General framework

We consider models of the form

$$\begin{aligned} &B_{1,t}x_t = B_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, \quad \forall t \geq 1 \\ &\text{s.t. } x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \text{ for all } t, \text{ and } x^*_{1,t} > \underline{x}_1 \; \forall t > T \end{aligned} \tag{1}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i \; \forall i \in [5]$, $x_0 \in \mathbb{R}^n$ given, e_t is a vector of known shocks with $e_t = 0_{m\times 1}$ for all $t > T$, $\mathbb{1}_t \in \{0, 1\}$ is an indicator variable, and the 'shadow value' of the bounded variable is given by

$$x^*_{1,t} = F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + Ge_t + H \tag{2}$$

where $\underline{x}_1, H \in \mathbb{R}$, F is a $1 \times 3n$ vector with $f_{11} = 0$ and G is a $1 \times m$ vector.

The matrices $B_{i,t}$ are regime dependent. In the reference regime $B_{i,t} = \overline{B}_i$; in the alternative regime $B_{i,t} = \tilde{B}_i$. The indicator variable $\mathbb{1}_t$ determines which regime is realized at a given t. The assumption that $x^*_{1,t} > \underline{x}_1 \; \forall t > T$ is a terminal condition which states that the bounded variable, $x_{1,t}$, permanently escapes the bound after a finite number of periods T.

1.1 Model solutions

Solutions to the problem in (1) are found by trialling sequences for the indicator variable of the form $(\mathbb{1}_t)_{t=1}^T$ (with $\mathbb{1}_t$ specified for all t) and $\mathbb{1}_t = 1 \; \forall t > T$; this in turn implies sequences $\{B_{1,t}, B_{2,t}, B_{3,t}, B_{4,t}, B_{5,t}\}_{t\geq 1}$ which can be used to find a time path $(x_t)_{t\geq 1}$ using the Algorithm in the main text. Only time paths consistent with the terminal condition and the occasionally-binding constraint are accepted as solutions.

As shown in our Algorithm, any solution(s) to problem (1) have the form

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for all } t > T \end{cases} \tag{3}$$

[1]Michael Hatcher, Department of Economics, University of Southampton, m.c.hatcher@soton.ac.uk.

where, for $t = 1, ..., T$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t} \tag{4}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}) \tag{5}$$

and $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ has eigenvalues in the unit circle, $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$, $\Psi_{T+1} = \overline{\Psi}$, $\Omega_{T+1} = \overline{\Omega}$, and $e_t = 0_{m\times 1}$ for all $t > T$.

1.2 Finding the M matrix

To compute the matrix $M \in \mathbb{R}^{T\times T}$ of impulse responses of the bounded variable to news shocks at dates $t = 1, ..., T$, we let $v_t := [v_{1,t} \ \ 0_{1\times(n-1)}]'$ be an $n \times 1$ vector of known shocks to the bounded variable, where $v_{1,t} \in \{0, 1\}$ for $t = 1, ..., T$ and $v_{1,t} = 0$ for all $t > T$. Letting $\hat{x}_t := x_t - \overline{x}$ (see Assumption 1, main paper), we can solve the following model:

$$\overline{B}_1\hat{x}_t = \overline{B}_2\hat{x}_{t+1} + \overline{B}_3\hat{x}_{t-1} + \overline{B}_4e_t + v_t, \quad \forall t \geq 1 \tag{6}$$

whose solution and M matrix are described in Remark 1.

Remark 1 *The solution to the perfect foresight model in* (6) *is given by*

$$\hat{x}_t = \overline{\Omega}\hat{x}_{t-1} + \hat{\Gamma}\tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{7}$$

where $\tilde{e}_t := \overline{B}_4e_t + v_t$*,* $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}$*,* $\Psi_t = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_2(\Psi_{t+1} + \hat{\Gamma}\tilde{e}_{t+1})$ $\forall t \in [1, T]$*, with* $\Psi_t = 0_{n\times 1}$ *for all* $t \geq T$*, and the corresponding* M *matrix is*

$$M_{ij} = \frac{\partial \hat{x}_{1,i}}{\partial v_{1,j}} = \hat{x}_{1,i}|_{v_{1,j}=1} - \hat{x}_{1,i}|_{v_{1,j}=0} \quad \textit{for } i, j \in \{1, ..., T\}.$$

2 Solution details: Fisherian example

Recall that for all $t \geq 1$ the model consists of a Taylor-type rule with a zero lower bound and the Fisher equation:

$$i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + e_t\} \tag{8}$$

$$i_t = r + E_t\pi_{t+1} \tag{9}$$

where $\phi - \psi > 1$, $\psi > 0$, $e_1, \pi_0 \in \mathbb{R}$, $e_t = 0$ $\forall t > 1$, and $r > 0$ is a fixed real interest rate. To simplify presentation, we set $\phi = 2$. The results are not specific to this case.

As discussed in the main paper, there are two solutions to the model (8)–(9): one is away from the bound in all periods, and the other has the constraint binding only in period 1. We now show that our Algorithm finds the same solutions. Letting $x_t := [i_t \ \ \pi_t]'$, the matrices in the reference regime and the alternative regime are given by

$$\overline{B}_1 = \begin{bmatrix} 1 & -\phi \\ 1 & 0 \end{bmatrix}, \ \overline{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \ \overline{B}_3 = \begin{bmatrix} 0 & -\psi \\ 0 & 0 \end{bmatrix}, \ \overline{B}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \overline{B}_5 = \begin{bmatrix} r \\ r \end{bmatrix}$$

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_5 = \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

Hence, analogous to (1)–(2), the model for all $t \geq 1$ is

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \text{s.t. } i_t = \max\{0, i_t^*\} \tag{10}$$

where $B_{j,t} := \mathbb{1}_t \overline{B}_j + (1 - \mathbb{1}_t)\tilde{B}_j \ \forall j \in \{1, ..., 5\}$, $e_t = 0$ for all $t > 1$, and

$$i_t^* = F \begin{bmatrix} x_t' & E_t x_{t+1}' & x_{t-1}' \end{bmatrix}' + G e_t + H, \quad \text{with } F = \begin{bmatrix} 0 & \phi & 0 & 0 & 0 & -\psi \end{bmatrix}, \ G = \begin{bmatrix} 1 \end{bmatrix}, \ H = \begin{bmatrix} r \end{bmatrix}.$$

Consider first the solution away from the lower bound. This solution corresponds to the guess that $\mathbb{1}_t = 1$ for all $t \geq 1$, such that $B_{j,t} = \overline{B}_j \ \forall j$ and (10) becomes

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5, \quad \forall t \geq 1. \tag{11}$$

The guessed solution $x_t = [i_t \ \ \pi_t]'$ thus follows the Algorithm with $T = 1$ and $\mathbb{1}_t = 1 \ \forall t$:

$$x_t = \begin{cases} \Omega_1 x_0 + \Gamma_1 e_1 + \Psi_1 & \text{for } t = 1 \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > 1 \end{cases} \tag{12}$$

where $\Omega_1 = \overline{\Omega}$, $\Psi_1 = \overline{\Psi}$, $\Gamma_1 = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_4 = \begin{bmatrix} -\frac{\omega}{\phi - \omega} & -\frac{1}{\phi - \omega} \end{bmatrix}'$, and

$$\overline{\Omega} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_3 = \begin{bmatrix} 0 & \omega^2 \\ 0 & \omega \end{bmatrix}, \qquad \overline{\Psi} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} (\overline{B}_2 \overline{\Psi} + \overline{B}_5) = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

with $\omega = 1 - \sqrt{1 - \psi}$ as above.[2]

This guessed solution is verified provided $i_1^* = r + \phi \pi_1 - \psi \pi_0 + e_1 \geq 0$ (see Algorithm) and $i_t^* = r + \phi \pi_t - \psi \pi_{t-1} > 0 \ \forall t > 1$, which requires $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ as we saw above.

Now consider the second solution. We guess that the lower bound constraint binds only in period 1, such that $\mathbb{1}_1 = 0$ and $\mathbb{1}_t = 1 \ \forall t > 1$; hence $B_{j,t} = \tilde{B}_j$ for $t = 1$ and $B_{j,t} = \overline{B}_j$ for all $t > 1$. This guess implies that $T = 1$ and $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$ for all $t > 1$. Hence, by the Algorithm our guessed solution path is

$$x_t = \begin{cases} \Psi_1 & \text{for } t = 1 \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > 1 \end{cases} \tag{13}$$

where $\Psi_1 = (\tilde{B}_1 - \tilde{B}_2 \overline{\Omega})^{-1} (\tilde{B}_2 \overline{\Psi} + \tilde{B}_5) = \begin{bmatrix} 0 & -\frac{r}{\omega} \end{bmatrix}'$.

To verify this solution, we require $i_1^* = r + \phi \pi_1 - \psi \pi_0 + e_1 \leq 0$ and $i_t^* = r + \phi \pi_t - \psi \pi_{t-1} > 0$ for all $t > 1$, which requires $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ as already seen. The two solutions are plotted in Figure 1 in the main paper, along with shadow nominal interest rates i_t^*.

[2] We can guess that the left column of $\overline{\Omega}$ is zero (because i_{t-1} does not enter the model), which makes it straightforward to solve the resulting quadratic equation in $\overline{\Omega}$ for the right-hand column. In general, $\overline{\Omega}, \overline{\Psi}$ can be found using numerical methods (e.g. Binder and Pesaran, 1997; Sims, 2002; Cho and Moreno, 2011).

3 New Keynesian model

3.1 Baseline model

The baseline model has the form:

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{14}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) \tag{15}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{16}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{17}$$

where $\theta_\pi > 1$, $\beta \in (0,1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0,1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

Let $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t \end{bmatrix}'$ and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix},\ \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},\ \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix},\ \tilde{B}_i = \overline{B}_i, \text{ for } i \in \{2,3,4\},\ \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Given $x_{1,t} = i_t$, $i_t = \max\{\underline{i},\ i_t^*\}$ can be written in the form $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ by setting $\underline{x}_1 = \underline{i}$ and $x_{1,t}^* = i_t^*$, or in vector form as in (2) with $F = \begin{bmatrix} 0 & 1 & 0_{1\times 10} \end{bmatrix}$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$. We set the parameters at the values given in the main text, whenever these parameters were not being varied as part of the analysis. Our computed perfect foresight paths correspond to initial conditions $x_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}'$, $e_1 = 0.01$, $e_t = 0\ \forall t \geq 2$, unless otherwise stated.

3.2 Computing the M matrix

To compute the M matrix of impulse responses of the bounded variable, we solve the model in (14)–(17) ignoring the bound (i.e. with the max operator removed) and with a 'news shock' $v_{r,t} \in \{0,1\}$ added, such that: $i_t = i_t^* + v_{i,t}$. The resulting model can be written as:

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \tilde{e}_t, \quad \forall t \geq 1 \tag{18}$$

where $\tilde{e}_t := \overline{B}_4 e_t + v_t$ and $v_t := \begin{bmatrix} v_{i,t} & 0 & 0 & 0 \end{bmatrix}'$, with $v_{i,t} = 0$ for all $t > T$, where T is the horizon at which the M matrix is being computed.

Recall that the 1st column of M lists the impulse response of i_t (at dates $t = 1, ..., T$) to the shock $v_{i,1} = 1$. In general, the jth column of M lists the impulse response of i_t (at dates $t = 1, ..., T$) to the shock $v_{i,j} = 1$, and this holds for columns $j = 1, ..., T$.

The impulse responses and the M matrix are obtained as follows:

$$x_t = \overline{\Omega} x_{t-1} + \hat{\Gamma} \tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{19}$$

where $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_2 (\Psi_{t+1} + \hat{\Gamma} \tilde{e}_{t+1})$ $\forall t \in [1, T]$, with $\Psi_t = 0_{n \times 1}$ for all $t \geq T$, and the M matrix is given by

$$M_{ij} = \frac{\partial x_{1,i}}{\partial v_{i,j}} = i_i|_{v_{i,j}=1} - i_i|_{v_{r,j}=0} \quad \text{for subscripts } i, j \in \{1, ..., T\}.$$

Given the parameters $\beta = 0.99$ and $\kappa = \frac{(1-0.85)(1-0.85\beta)}{0.85}(2+\sigma)$ (which we hold fixed), we assign values to ρ_i, σ, θ_π, $\theta_{\Delta y}$ and compute the M matrix using the method in Section 7.1 and check if it is a P-matrix using a recursive test due to Tsatsomeros and Li (2000).[3] Plots of the P-matrix regions under interest rate smoothing (not shown in the paper) are as follows.

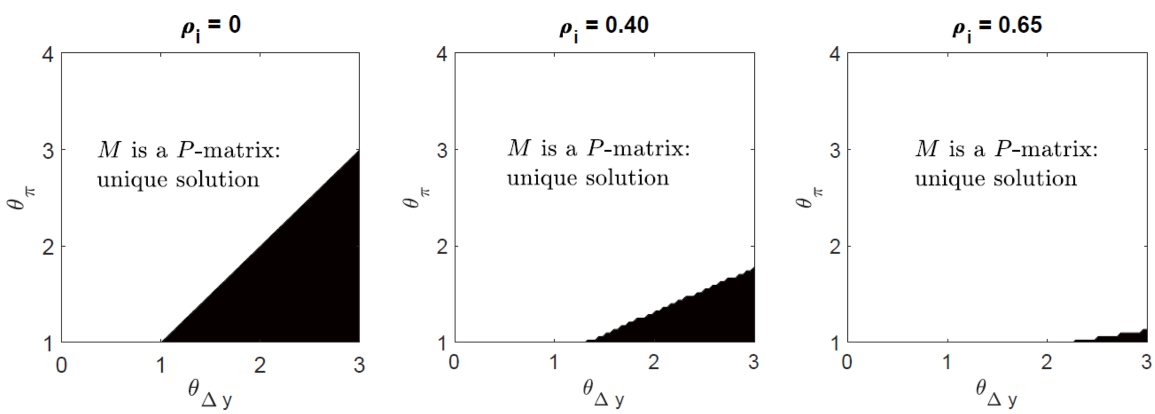

Figure 1: Regions in which M is not a P-matrix (black) for $T = 2$ and various ρ_i

[3] A MATLAB code is available on the author's webpage: https://www.math.wsu.edu/faculty/tsat/matlab.html.

3.3 Forward guidance

With forward guidance the shadow interest rate is amended to

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) + e_t^{FG} \tag{20}$$

where $e_t^{FG} < 0$ for all $t \in \mathcal{T}^{FG} \subset \mathbb{N}$ and $e_t^{FG} = 0$ otherwise.

Hence, letting $\tilde{i}_t^* := \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1}))$ the interest rate rule (20) is

$$i_t^* = \begin{cases} \tilde{i}_t^* - |e_t^{FG}| & \text{if } t \in \mathcal{T}^{FG} \\ \tilde{i}_t^* & \text{otherwise} \end{cases}. \tag{21}$$

We consider forward guidance horizons of the form $\mathcal{T}^{FG} = \{2, \ldots, t'\}$, where $t' \geq 2$. Hence, forward guidance occurs for consecutive periods $2, .., t'$ and the length of the forward guidance 'horizon' (or spell) is given by $t' - 1$; see also Table 1 in the paper.

Letting $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t \end{bmatrix}'$ as before and $\hat{e}_t = \begin{bmatrix} e_t & e_t^{FG} \end{bmatrix}'$, the reference regime (slack) is

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 \hat{e}_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} \\ 0 & 0 & 0 & \beta \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 \hat{e}_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & -(1-\rho_i)\theta_\pi \\ \sigma^{-1} & 0 & 1 & 0 \\ 0 & 0 & -\kappa & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i, \quad \text{for } i \in \{2,3,4\}, \; \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The M matrix under forward guidance is identical to the model with the baseline interest rate rule, (15). However, since forward guidance is modelled as negative shocks to the shadow interest rate, it does affect the initial conditions and hence may lead to determinacy when a rule absent news shocks would not (or vice versa). The simulation results in Table 1 of the paper check a large number of initial conditions (800) by varying the shocks according to

$$e_t^{FG} = -0.01 - \mathcal{U}_t, \quad \text{for } t \in \mathcal{T}^{FG} \text{ and } \mathcal{U}_t = \text{draw from uniform distribution on } (0, 0.01).$$

3.4 Price-level targeting rule

With price-level targeting the model is amended to

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{22}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)\left(\theta_p p_t + \theta_{\Delta y}(y_t - y_{t-1})\right) \tag{23}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{24}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{25}$$

$$p_t = p_{t-1} + \pi_t \tag{26}$$

where $\theta_p > 0$ is the reaction coefficient on the (log) price level.

Let $x_t = \begin{bmatrix} i_t & i_t^* & y_t & \pi_t & p_t \end{bmatrix}'$ and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & 0 & -(1-\rho_i)\theta_p \\ \sigma^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\kappa & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \sigma^{-1} & 0 \\ 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_i & -(1-\rho_i)\theta_{\Delta y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -(1-\rho_i)\theta_{\Delta y} & 0 & -(1-\rho_i)\theta_p \\ \sigma^{-1} & 0 & 1 & 0 & 0 \\ 0 & 0 & -\kappa & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i, \quad \text{for } i \in \{2,3,4\}, \ \tilde{B}_5 = \begin{bmatrix} \underline{i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

3.4.1 ‘Good’ solutions under price-level targeting

In Figure 2 we plot some perfect foresight solutions for parameter values corresponding to the determinacy region and the ‘good’ solution (for which the lower bound is not hit); we see the results are robust to modest or strong interest rate smoothing in the shadow rate.

3.4.2 ‘Bad’ solutions under price-level targeting

We note in the paper that when multiple solutions exist under a price-level targeting rule (23) (i.e. for very small θ_p), the ‘bad’ solution is not so bad in terms of stabilization of inflation and output. In this section we provide some extra examples of these ‘bad solutions’. In Figure 3 we plot a ‘good’ and ‘bad’ solution when the response to the price level is small enough to give multiple solutions. Figure 4 shows robustness to other parameter values.

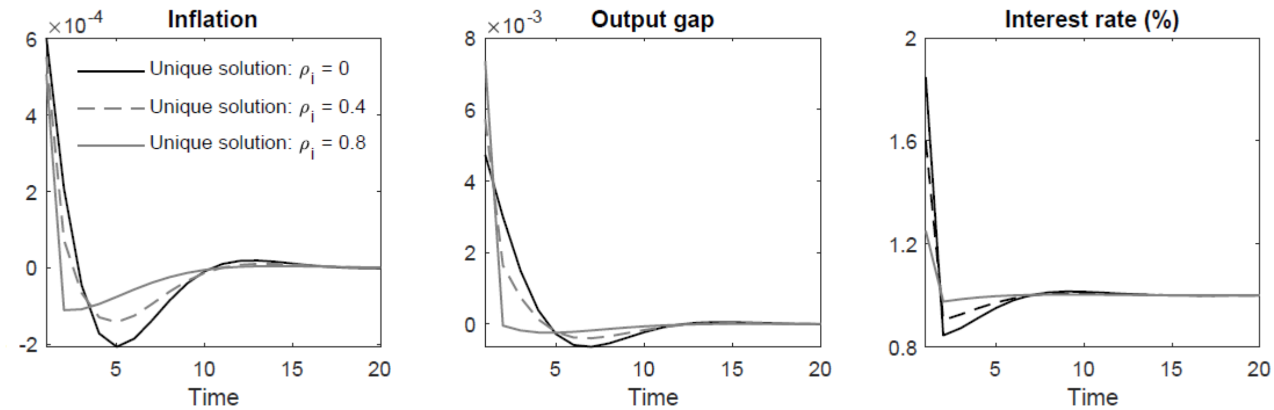

Figure 2: Unique perfect foresight paths under price-level targeting for $e_1 = 0.01$ and $i_0^* = y_0 = 0$ when $\sigma = 1$, $\theta_p = 1.5$ and $\theta_{\Delta y} = 1.6$: various values of ρ_i: 0, 0.4, 0.8.

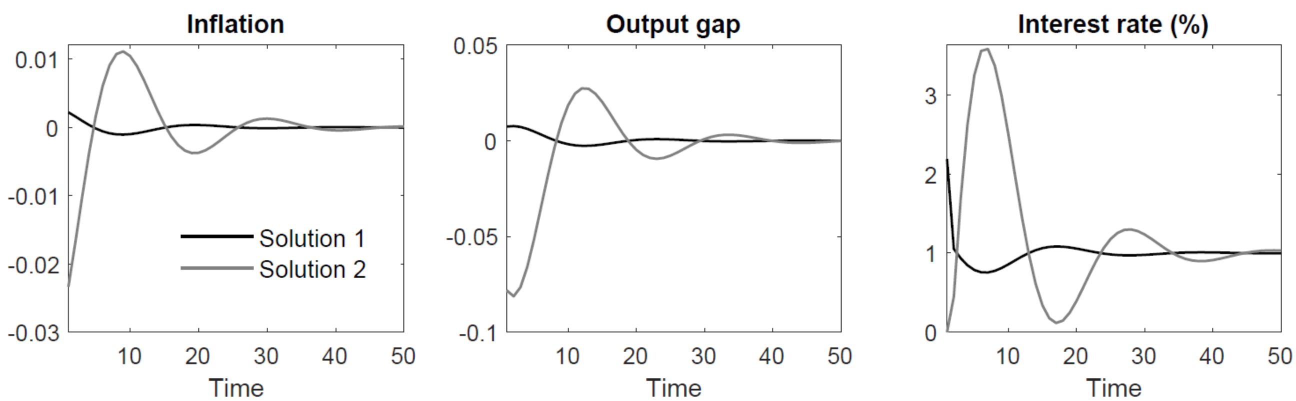

Figure 3: 'Good' and 'bad' solutions under price-level targeting when $\theta_p = 0.015$

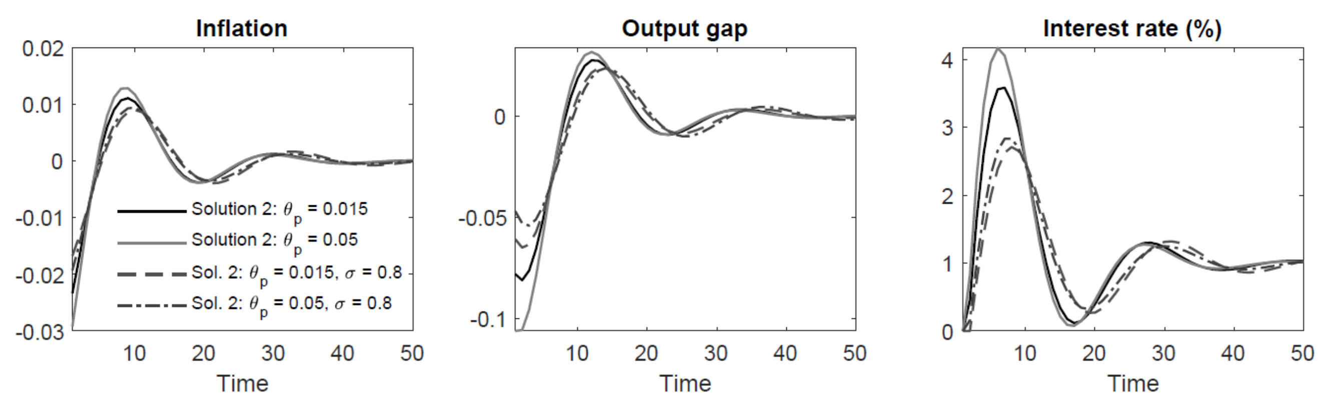

Figure 4: 'Bad' solutions under price-level targeting for various parameter values

4 Foundations of the Algorithm

By assumption, the model returns permanently to the reference regime after some date $T \geq 1$ and escapes the bound (see Assumption 2, main text). The system to be solved $\forall t \geq 1$ is

$$B_{1,t}x_t = B_{2,t}x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \quad \text{s.t.} \quad x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \text{ for } t = 1, ..., T \tag{27}$$

where $B_{j,t} := \mathbb{1}_t\overline{B}_j + (1 - \mathbb{1}_t)\tilde{B}_j$ $\forall j \in [5]$ and $\forall t > T$, $x_t = \overline{\Omega}x_{t-1} + \overline{\Psi}$, $B_{j,t} = \overline{B}_j$, $e_t = 0_{m\times 1}$.

Consider first the periods $1 \leq t \leq T$. Suppose there exist a set of well-defined matrices $\{\Omega_t, \Gamma_t, \Psi_t\}$ such that $x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t$. Shifting this equation forward one period:

$$x_{t+1} = \Omega_{t+1}x_t + \Gamma_{t+1}e_{t+1} + \Psi_{t+1}, \quad 1 \leq t \leq T - 1. \tag{28}$$

Substituting (28) into (27) and rearranging gives, for all $t \in \{1, ..., T - 1\}$,

$$(B_{1,t} - B_{2,t}\Omega_{t+1})x_t = B_{3,t}x_{t-1} + B_{4,t}e_t + B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}. \tag{29}$$

Provided $\Omega_T, \Gamma_T, \Psi_T$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$, the set $\{\Omega_t, \Gamma_t, \Psi_t\}$ is well-defined for t where these matrices follow the recursive formulas. Therefore, if $\Omega_T, \Gamma_T, \Psi_T$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ $\forall t < T$, $\Omega_t, \Gamma_t, \Psi_t$ are well-defined for $t = 1, ..., T$.

For $t > T$, we have by Assumption 2, $x_t = \overline{\Omega}x_{t-1} + \overline{\Psi}$ where $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ and $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$. Hence, $x_{t+1} = \overline{\Omega}x_t + \overline{\Psi}$, $\forall t \geq T$. Matrices $\Omega_T, \Gamma_T, \Psi_T$ are determined by the first line of (27) and the previous equation at date $t = T$:

$$B_{1,T}x_T = B_{2,T}x_{T+1} + B_{3,T}x_{T-1} + B_{4,T}e_T + B_{5,T}, \qquad x_{T+1} = \overline{\Omega}x_T + \overline{\Psi}$$

or $(B_{1,T} - B_{2,T}\overline{\Omega})x_T = B_{3,T}x_{T-1} + B_{4,T}e_T + B_{2,T}\overline{\Psi} + B_{5,T}$. Provided $\det[B_{1,T} - B_{2,T}\overline{\Omega}] \neq 0$, the matrices $\Omega_T, \Gamma_T, \Psi_T$ are given by the expressions in the Algorithm (see main text).

For the time path $(x_t)_{t=1}^T$ to satisfy the constraint $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ $\forall t \in \{1, ..., T\}$, the guessed structure $(\mathbb{1}_t)_{t=1}^T$ must be verified at all dates. Consider first date $t = 1$. If $\mathbb{1}_t = 1$, then $x_{1,t} = x^*_{1,t}$, which satisfies $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ if and only if $x^*_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$, which is equivalent to $x^*_{1,t}|_{\mathbb{1}_t=1} \geq \underline{x}_1$. On the other hand, if $\mathbb{1}_t = 0$, then $x_{1,t} = \underline{x}_1$, which satisfies $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ if and only if $\underline{x}_1 = \max\{\underline{x}_1, x^*_{1,t}\}$, which is equivalent to $x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1$.

Thus, the guess at $t = 1$ is verified iff $\mathbb{1}_t = 1$ and $x^*_{1,t}|_{\mathbb{1}_t=1} \geq \underline{x}_1$ *or* $\mathbb{1}_t = 0$ and $x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1$. By analogous arguments, the guessed structure for each subsequent t is verified if and only if the above condition holds for this particular t. Hence, the guessed structure $(\mathbb{1}_t)_{t=1}^T$ is verified when the following condition holds for all $t \in \{1, ..., T\}$ and $j = 1, ..., 5$:

$$\begin{cases} B_{j,t} = \overline{B}_j \text{ and } x^*_{1,t} \geq \underline{x}_1 \text{ for } t \text{ such that } \mathbb{1}_t = 1 \\ B_{j,t} = \tilde{B}_j \text{ and } x^*_{1,t} \leq \underline{x}_1 \text{ for } t \text{ such that } \mathbb{1}_t = 0. \end{cases} \tag{*}$$

Note that a guessed structure $(\mathbb{1}_t)_{t=1}^T$ is rejected if (*) does not hold for some $t \in [T]$.

5 Models with the constraint binding at steady state

In this section we explain how our algorithm can be applied to models in which the constraint binds at steady state. An example of such a model is the borrowing constraint model that appears in Guerrieri and Iacoviello (2015, Online Appendix, Section C.1).

Starting from the general model in (1)–(2), Assumptions 1-2 in the main paper must adapted for a model that converges to a steady state in which the constraint is binding.

Assumption 1 *We assume* $\det[\tilde{B}_1 - \tilde{B}_2 - \tilde{B}_3] \neq 0$, *such that there exists a unique steady state* $\overline{x} = (\tilde{B}_1 - \tilde{B}_2 - \tilde{B}_3)^{-1}\tilde{B}_5$ *at the alternative regime. This steady state satisfies* $\overline{x}_1 < \underline{x}_1$.

Assumption 2 *For any given initial value, there is a unique stable (terminal) solution at the alternative regime of the form* $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Psi}$, *where* $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}(\tilde{B}_2\tilde{\Psi} + \tilde{B}_5) = (I_n - \tilde{\Omega})\overline{x}$, $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_3$ *has eigenvalues in the unit circle, and* $x_t \to \overline{x}$ *as* $t \to \infty$.

Assumption 3 is unchanged relative to the main paper.

We can then restate the Algorithm in the main paper as follows.

1. Pick a $T \geq 1$ and a simulation length $T_s > T$. Guess a sequence $(\mathbb{1}_t)_{t=1}^T$ of 0s and 1s, starting with all 1s (slack in all periods) as an initial guess. Note: $\mathbb{1}_t = 1$ for $t > T$.

2. Find the structural matrices (or 'regimes') implied by the guess:

$$B_{i,t} = \mathbb{1}_t\overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i, \quad i \in [5]$$

   in periods $t = 1, \ldots, T_s$.

3. Compute $(x_t)_{t=1}^{T_s}$ and the shadow value of the bounded variable $(x^*_{1,t})_{t=1}^{T_s}$ via

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega}x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases}, \quad x^*_{1,t} = F\begin{bmatrix} x'_t & x'_{t+1} & x'_{t-1} \end{bmatrix}' + Ge_t + H$$

   where, for $t = 1, ..., T$ and initial matrices $\Omega_{T+1} = \tilde{\Omega}, \quad \Psi_{T+1} = \tilde{\Psi}, \quad \Gamma_{T+1} = 0_{n\times m}$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}B_{4,t}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}).$$

4. If $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ for $t = 1, \ldots, T$ and $x_{1,t} = \underline{x}_1$ $\forall t > T$, accept the guess and store the solution $(x_t)_{t=1}^{T_s}$; else reject. Return to Step 1 and repeat for a new guess.

Note that the only change is to the terminal matrices in Step 3, since backward induction now proceeds from the alternative regime rather than the reference regime.

6 Multiple occasionally-binding constraints

Thus far, we have dealt only with a single occasionally-binding constraint. We now consider *multiple* constraints and focus for simplicity on the case of *two* lower bound constraints (on variables 1 and 2) such that there are four regimes in total. Though we focus on this simple case, the guess-verify method extends straightforwardly to the case of an arbitrary number of constraints N, following an analogous approach to the algorithm presented below.[4]

With two constraints and hence four regimes, there are four different sets of structural matrices which correspond to the members of the set

$$\{(\overline{B}_1^s, \overline{B}_2^s, \overline{B}_3^s, \overline{B}_4^s, \overline{B}_5^s) : s = 1, \ldots, 4.\}$$

and the four indicator variables $\mathbb{1}_t^s \in \{0, 1\}$ for $s = 1, \ldots, 4$.

The indicator variables satisfy $\sum_{s=1}^4 \mathbb{1}_t^s = 1$ since regimes are mutually exclusive. Hence, if $\mathbb{1}_t^s = 1$, then $\mathbb{1}_t^k = 0$ for all $k \in \{1, 2, 3, 4\} \setminus \{s\}$. Regime 1 is the terminal structure (with both constraints slack), and we assume Assumptions 1 and 2 in the main paper hold.[5]

The system to be solved under perfect foresight is now:

$$\begin{aligned}
&B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 1 \\
\text{subject to}& \\
&x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\} \text{ for all } t, \text{ and } x_{1,t}^* > \underline{x}_1 \ \forall t > T \\
&x_{2,t} = \max\{\underline{x}_2, x_{2,t}^*\} \text{ for all } t, \text{ and } x_{2,t}^* > \underline{x}_2 \ \forall t > T
\end{aligned} \tag{30}$$

where $B_{i,t} := \sum_{s=1}^4 \mathbb{1}_t^s \overline{B}_i^s$ for $i = 1, \ldots, 5$, $x_0 \in \mathbb{R}^n$ given, and e_t is a vector of known shocks with $e_t = 0_{m \times 1}$ for all $t > T$, and the 'shadow values' of the bounded variables are given by

$$x_{1,t}^* = F_1 \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G_1 e_t + H_1 \tag{31}$$

$$x_{2,t}^* = F_2 \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G_2 e_t + H_2 \tag{32}$$

where $\underline{x}_1, \underline{x}_2, H_1, H_2 \in \mathbb{R}$, F_1, F_2 are $1 \times 3n$ vectors, with vector F_j having entry j equal to zero, and G_1, G_2 are $1 \times m$ vectors.

Variables 1 and 2 could be distinct economic variables or they could be used to impose multiple constraints on the same variable. For example, to put a lower bound $\underline{x}_1$ and an *upper bound* $\overline{x}_1^u$ on variable 1, we set $x_{2,t}^* = -x_{1,t}^*$ (so $F_2 = -F_1$, $G_2 = -G_1$, $H_2 = -H_1$).[6] We now show how the algorithm can deal with multiple constraints.

[4] For N constraints, there are 2^N (mutually-exclusive) regimes and hence 2^N indicator variables.

[5] Hence, we assume the steady state $\overline{x}$ at the reference regime is unique and satisfies $\underline{x}_1 \leq \overline{x}_1 \leq \overline{x}_1^u$, where $\underline{x}_1$ is the lower bound on variable 1 and $\overline{x}_1^u$ is the upper bound on variable 1 (if present; else $\overline{x}_1^u = \infty$).

[6] Note that if $z_t := \min\{\overline{x}_1, x_{1,t}^*\}$, then $-z_t = \max\{-\overline{x}_1, -x_{1,t}^*\}$, so we set $x_{2,t}^* = -x_{1,t}^*$ and $\overline{x}_2 = -\overline{x}_1^u$.

Given multiple occasionally-binding constraints, our Algorithm must be amended as follows:

1. Pick a $T \geq 1$ and a simulation length $T_s > T$. Guess on sequences $(\mathbb{1}_t^s)_{t=1}^T$ for $s = 1, 2, 3$ and compute $\mathbb{1}_t^4 = 1 - \sum_{s=1}^3 \mathbb{1}_t^s$ (implied) starting with $\mathbb{1}_t^1 = 1$ for all t, $\mathbb{1}_t^{s\neq 1} = 0$ for all t (constraints slack in all periods) as an initial guess. Note: $\mathbb{1}_t^1 = 1$ for all $t > T$.

2. Find the structural matrices (or 'regimes') implied by the guess:

$$B_{i,t} := \sum_{s=1}^{4} \mathbb{1}_t^s \overline{B}_i^s, \quad i \in [5]$$

   in periods $t = 1, \dots, T_s$.

3. Compute $(x_t)_{t=1}^{T_s}$ and the shadow values of the bounded variables $(x_{1,t}^*)_{t=1}^{T_s}, (x_{2,t}^*)_{t=1}^{T_s}$ via

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases}, \qquad \begin{aligned} x_{1,t}^* &= F_1 \begin{bmatrix} x_t' & x_{t+1}' & x_{t-1}' \end{bmatrix}' + G_1 e_t + H_1 \\ x_{2,t}^* &= F_2 \begin{bmatrix} x_t' & x_{t+1}' & x_{t-1}' \end{bmatrix}' + G_2 e_t + H_2 \end{aligned}$$

   where, for $t = 1, ..., T$ and initial matrices $\Omega_{T+1} = \overline{\Omega}$, $\Psi_{T+1} = \overline{\Psi}$, $\Gamma_{T+1} = 0_{n\times m}$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{4,t}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1} e_{t+1}) + B_{5,t}).$$

4. If $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$, $x_{2,t} = \max\{\underline{x}_2, x_{2,t}^*\}$ for $t = 1, ..., T$ and $x_{1,t} > \underline{x}_1$, $x_{2,t} > \underline{x}_2$ $\forall t > T$, accept the guess and store the solution $(x_t)_{t=1}^{T_s}$; else reject the guess in Step 1. Return to Step 1 and repeat for a new guess.

Case of N constraints

As noted above, the extension to an arbitrary *finite* number of constraints N is straightforward. In particular, the only non-trivial adjustments are as follows:

- There are $N \leq n$ occasionally-binding constraints and shadow values:[7]

$$x_{i,t} = \max\{\underline{x}_i, x_{i,t}^*\} \text{ for all } t, \quad x_{i,t}^* = F_i \begin{bmatrix} x_t' & x_{t+1}' & x_{t-1}' \end{bmatrix}' + G_i e_t + H_i, \quad i = 1, \dots, N.$$

- The time-varying structural matrices are $B_{i,t} := \sum_{s=1}^{2^N} \mathbb{1}_t^s \overline{B}_i^s$, $i \in [5]$, where $\mathbb{1}_t^s$ is an indicator variable equal to 1 if regime s applies at date t and zero otherwise.

- A guess on sequences $(\mathbb{1}_t^s)_{t=1}^T$ for $s = 1, \dots, 2^N$ (with $\mathbb{1}_t^1 = 1$, $\mathbb{1}_t^{\neq 1} = 0$ for all $t > T$) is verified if and only if $x_{i,t} = \max\{\underline{x}_i, x_{i,t}^*\}$ $\forall i \in [N]$ holds for all t and $x_{i,t} > \underline{x}_i$ $\forall t > T$.

[7] Note that the assumption $N \leq n$ is not restrictive because if, in the original model, one wanted to constrain say all n_{orig} variables from below and also constrain n^u of them from above, then one may define new variables $x_{n_{orig}+1}, ..., x_{n_{orig}+n^u}$ and let $n = n_{orign} + n^u$ such that x_t has n elements as required.

7 Example $1'$: Asset pricing model

The model in Guerrieri and Iacoviello (2015, Section 2.4) has the form

$$
\begin{aligned}
q_t &= \beta(1-\rho)E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\
r_t &= \max\{\underline{r}, \phi q_t\} \\
u_t &= \rho_u u_{t-1} + e_t
\end{aligned}
\tag{33}
$$

where $\beta, \rho \in (0,1)$, $\phi > 0$, $\underline{r} < 0$, $\rho_u \in (0,1)$ and all values of e_t are known.

Let $x_t = \begin{bmatrix} r_t & q_t & u_t \end{bmatrix}'$, such that the bounded variable is ordered first, and note that e_t (scalar) is the vector of known shocks. Then the reference regime (slack) is described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & -\phi & 0 \\ \sigma & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta(1-\rho) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho_u \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and the alternative regime (binding) is described by

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ \sigma & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_i = \overline{B}_i \text{ for } i \in \{2,3,4\}, \quad \overline{B}_5 = \begin{bmatrix} \underline{r} \\ 0 \\ 0 \end{bmatrix}.$$

Given $x_{1,t} = r_t$, the equation $r_t = \max\{\underline{r}, \phi q_t\}$ can be written in the form $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ by setting $\underline{x}_1 = \underline{r}$ and $x^*_{1,t} = \phi q_t$; note that the latter equation can be written in vector form as in (2) with $F = \begin{bmatrix} 0 & \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $G = H = \begin{bmatrix} 0 \end{bmatrix}$.

7.1 Computing the M matrix

To compute the M matrix of impulse responses of the bounded variable, we solve the model in (33) ignoring the bound (i.e. with the max operator removed) and with a 'news shock' $v_{r,t} \in \{0,1\}$ added, such that: $r_t = \phi q_t + v_{r,t}$. The resulting model can be written as:

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \tilde{e}_t, \quad \forall t \geq 1 \tag{34}$$

where $\tilde{e}_t := \overline{B}_4 e_t + v_t$ and $v_t := \begin{bmatrix} v_{r,t} & 0 & 0 \end{bmatrix}'$, with $v_{r,t} = 0$ for all $t > T$, where T is the horizon at which the M matrix is being computed.

The 1st column of M lists the impulse response of the policy rate r_t (at dates $t = 1, ..., T$) to the shock $v_{r,1} = 1$. In general, the jth column of M lists the impulse response of r_t (at dates $t = 1, ..., T$) to the shock $v_{r,j} = 1$, and this holds for columns $j = 1, ..., T$.

The impulse responses and the M matrix are obtained as follows:

$$x_t = \overline{\Omega} x_{t-1} + \hat{\Gamma}\tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{35}$$

where $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_2(\Psi_{t+1} + \hat{\Gamma}\tilde{e}_{t+1})\ \forall t \in [1, T]$, with $\Psi_t = 0_{n\times 1}$ for all $t \geq T$, and the M matrix is given by

$$M_{ij} = \frac{\partial x_{1,i}}{\partial v_{r,j}} = r_i|_{v_{r,j}=1} - r_i|_{v_{r,j}=0} \quad \text{for } i, j \in \{1, ..., T\}.$$

For the parameters $\beta = 0.99$, $\sigma = 5$, $\phi = 0.2$, $\rho = \rho_u = 0.5$, we checked whether the M matrix is a P-matrix for values of T up to 1,000; this turned out to be straightforward since we found that $M + M'$ is positive definite, which implies that M is a P-matrix (see e.g. Holden, 2022, Appendix: Lemma 1). Given these results, we can be confident that there is a unique solution to the model with bound imposed, and this squares with all our simulations.

7.2 Policy function and perfect foresight paths

To compute the policy function of model (33), we fixed an $e_t \in [-0.2, 0.2]$, set a value for $x_{t-1} = \begin{bmatrix} 0 & q_{t-1} & u_{t-1} \end{bmatrix}'$, specify values for $(e_{t+s})_{s\geq 1}$, and find a perfect foresight solution. We repeat this process for values of e_t linearly spaced in the interval $[-0.2, 0.2]$ while holding other initial conditions fixed. Each policy function is computed at 60 different points, which takes MATLAB around 0.7 seconds.[8] Some perfect foresight paths are plotted in Figure 5. The policy function in Figure 6 matches the one in Guerrieri and Iacoviello (2015, Fig. 1).

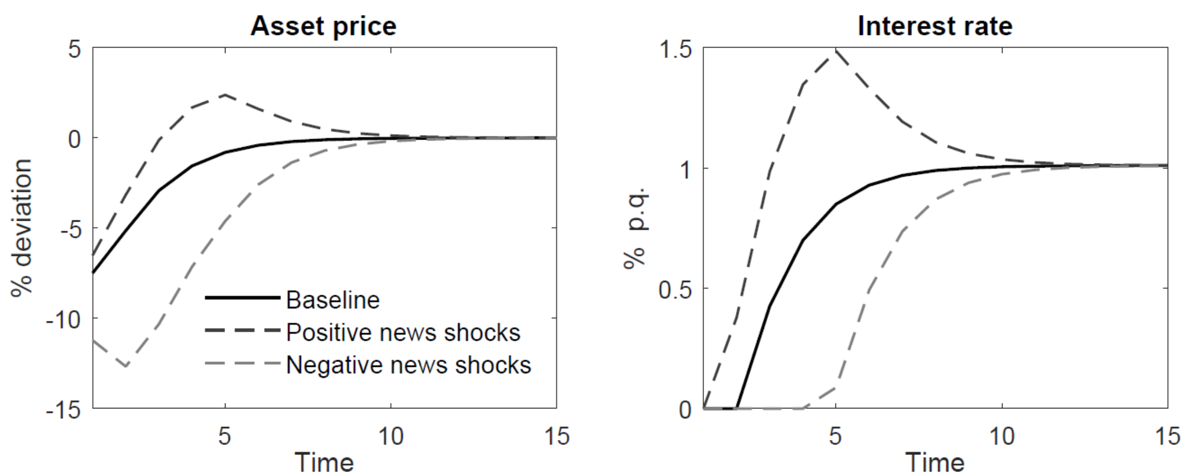

Figure 5: Perfect foresight solutions for different news shocks: $e_1 = -0.1$, $q_0 = u_0 = 0$. In the baseline case, all future (anticipated) shocks are set at 0. In the positive (negative) news case the news shocks are $e_t = 0.02$ ($e_t = -0.02$) for $t = 1, ..., 4$ and zero otherwise.

[8]The simulations were run in MATLAB 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

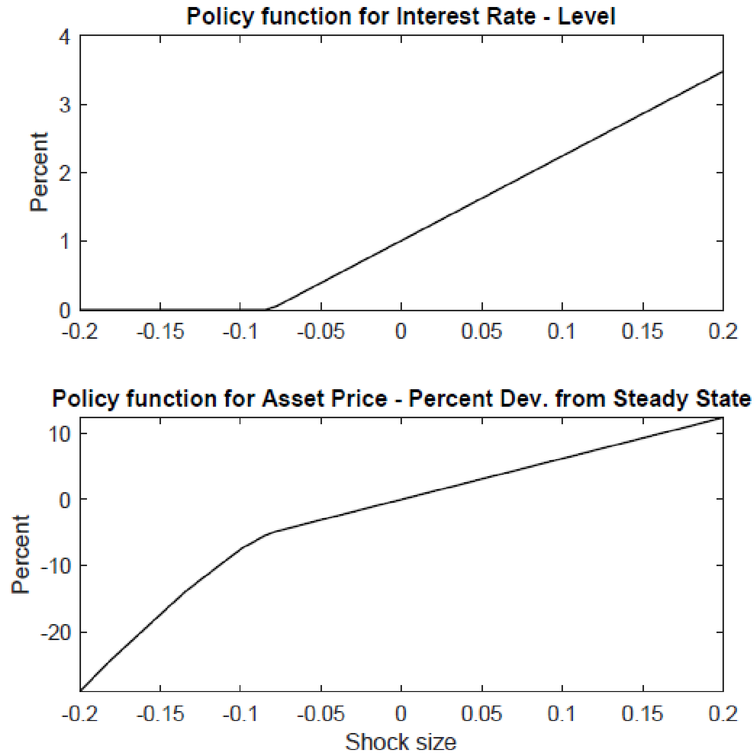

Figure 6: Policy functions for various e_t when $q_{t-1} = u_{t-1} = 0$ and news is zero

8 Example 2′: RBC model and investment constraint

We also considered a Real Business Cycle model with a lower bound on investment, as in Guerrieri and Iacoviello (2015, Section 4). This model requires us to log-linearize a non-linear model and to 'choose' a shadow value $x^*_{1,t}$ in the equation $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$.

A social planner chooses allocations $\{K_t, C_t\}_{t=0}^{\infty}$ to maximize utility $U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}-1}{1-\sigma}\right)$, subject to the following constraints:

$$C_t + I_t = A_t K_{t-1}^{\alpha} \tag{36}$$

$$K_t = (1-\delta)K_{t-1} + I_t \tag{37}$$

$$I_t \geq \phi I_{SS} \tag{38}$$

where $\sigma, I_{SS} > 0$, $\alpha, \phi \in (0,1)$, I_{SS} is the steady-state level of investment, and productivity is $A_t = A_{t-1}^{\rho} exp(\epsilon_t)$, where $\rho \in (0,1)$ and ϵ_t is a shock whose value is known at all dates.

Equations (36)–(38) are, respectively, the resource constraint, the capital accumulation equation, and a constraint that prevents investment from falling below a fraction ϕ of its steady-state value I_{SS}. The necessary conditions for a solution to the planner problem are (36)–(38)

plus the consumption Euler equation and the complementary slackness condition:

$$C_t^{-\sigma} - \lambda_t = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta) - (1-\delta)\lambda_{t+1}) \tag{39}$$

$$\lambda_t(I_t - \phi I_{SS}) = 0 \tag{40}$$

where $\lambda_t \geq 0$ is the Lagrange multiplier on the investment constraint.

The investment constraint is slack when $\lambda_t = 0$ and binding when $\lambda_t > 0$. If $\lambda_t > 0$, then $I_t = \phi I_{SS}$ to ensure that the complementary slackness condition (40) holds. If $\lambda_t = 0$, then either $I_t = \phi I_{SS}$ or $I_t > \phi I_{SS}$ (but not $I_t < \phi I_{SS}$, since this would violate condition (38)). The two regimes are as follows. Under the reference regime (slack):

$$I_t = K_t - (1-\delta)K_{t-1}, \quad K_t = A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} - C_t$$

$$C_t^{-\sigma} = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta)), \quad \lambda_t = 0$$

and under the alternative regime (binding):

$$I_t = \phi I_{SS}, \quad K_t = I_t + (1-\delta)K_{t-1}, \quad C_t = A_t K_{t-1}^{\alpha} + (1-\delta)K_{t-1} - K_t,$$

$$C_t^{-\sigma} - \lambda_t = \beta E_t(C_{t+1}^{-\sigma}(\alpha A_{t+1} K_t^{\alpha-1} + 1 - \delta) - (1-\delta)\lambda_{t+1}).$$

To put this non-linear model in the form of (1), we log-linearize the equations (under both regimes) around the steady state at which the investment constraint is slack.[9] To ease the process, we define the new variables $Y_t := A_t K_{t-1}^{\alpha}$ and $R_t := \alpha A_t K_{t-1}^{\alpha-1} + 1 - \delta$. The two regimes can then be written in terms of deviations from steady state as follows:

$$\hat{i}_t = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1}, \quad \hat{k}_t = (1-\delta)\hat{k}_{t-1} + (Y_{SS}/K_{SS})\hat{y}_t - (C_{SS}/K_{SS})\hat{c}_t$$

$$\hat{c}_t = E_t\hat{c}_{t+1} - (1/\sigma)E_t\hat{r}_{t+1}, \quad \lambda_t = 0, \quad \hat{y}_t = \hat{a}_t + \alpha\hat{k}_{t-1}$$

$$\hat{r}_t = \alpha R_{SS}^{-1}\left(Y_{SS}/K_{SS}\right)\hat{a}_t - \alpha(1-\alpha)R_{SS}^{-1}\left(Y_{SS}/K_{SS}\right)\hat{k}_{t-1}, \quad \hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t$$

under the reference regime, and

$$\hat{i}_t = \phi - 1, \quad \hat{k}_t = (1-\delta)\hat{k}_{t-1} + \delta\hat{i}_t, \quad C_{SS}\hat{c}_t = Y_{SS}\hat{y}_t + (1-\delta)K_{SS}\hat{k}_{t-1} - K_{SS}\hat{k}_t$$

$$C_{SS}^{\sigma}\lambda_t = -\sigma\hat{c}_t + \sigma E_t\hat{c}_{t+1} - E_t\hat{r}_{t+1} + (1-\delta)(C_{SS}^{\sigma}/R_{SS})E_t\lambda_{t+1}, \quad \hat{y}_t = \hat{a}_t + \alpha\hat{k}_{t-1}$$

$$\hat{r}_t = \alpha R_{SS}^{-1}\left(Y_{SS}/K_{SS}\right)\hat{a}_t - \alpha(1-\alpha)R_{SS}^{-1}\left(Y_{SS}/K_{SS}\right)\hat{k}_{t-1}, \quad \hat{a}_t = \rho\hat{a}_{t-1} + \epsilon_t$$

under the alternative regime.

We let $x_t := \begin{bmatrix}\hat{i}_t & \hat{k}_t & \hat{c}_t & \lambda_t & \hat{y}_t & \hat{r}_t & \hat{a}_t\end{bmatrix}'$ and $e_t := \begin{bmatrix}\epsilon_t\end{bmatrix}$, where 'hats' are log deviations from steady-state, i.e. $\hat{z}_t := \ln(Z_t/Z_{SS}) \approx (Z_t - Z_{SS})/Z_{SS}$. Note that $x_{1,t} = \hat{i}_t$. The constraint (38) is $\hat{i}_t \geq \phi - 1$ in deviations and we put this in the form $x_{1,t} = \max\{\underline{x}_1, x_{1,t}^*\}$ by setting $\underline{x}_1 = \phi - 1$ and $x_{1,t}^* = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t$. In the reference regime (slack), $\lambda_t = 0$, so

[9] The steady state is $I_{SS} = \delta K_{SS}$, $C_{SS} = A_{SS}K_{SS}^{\alpha} - I_{SS}$, $K_{SS} = \left(\frac{\alpha\beta A_{SS}}{1-\beta(1-\delta)}\right)^{1/(1-\alpha)}$ and $A_{SS} = 1$.

$x^*_{1,t} = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1}$ and $\max\{\underline{x}_1, x^*_{1,t}\} = x^*_{1,t}$ if and only if $\delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} (= \hat{i}_t) \geq \phi - 1$. In the alternative regime (binding), $\lambda_t > 0$, so $x^*_{1,t} = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t = (\phi - 1) - \lambda_t < \phi - 1$, so $\max\{\underline{x}_1, x^*_{1,t}\} = \underline{x}_1 = \phi - 1$ as required.[10]

The shadow value $x^*_{1,t} = \delta^{-1}\hat{k}_t - (1-\delta)\delta^{-1}\hat{k}_{t-1} - \lambda_t$ can be written as in (2) by setting $F = \begin{bmatrix} 0 & \delta^{-1} & 0 & -1 & 0_{1\times 11} & -(1-\delta)\delta^{-1} & 0_{1\times 5}\end{bmatrix}$ and $G = H = \begin{bmatrix}0\end{bmatrix}$. The matrices $\overline{B}_j, \tilde{B}_j$, $j \in [5]$, under the two regimes have the form:

$$\overline{B}_1 = \begin{bmatrix} 1 & -\delta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{C_{SS}}{K_{SS}} & 0 & -\frac{Y_{SS}}{K_{SS}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\alpha Y_{SS}}{K_{SS}R_{SS}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\sigma^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B}_3 = \begin{bmatrix} 0 & -(1-\delta)\delta^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1-\alpha)Y_{SS}}{K_{SS}R_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\delta & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{K_{SS}}{C_{SS}} & 1 & 0 & -\frac{Y_{SS}}{C_{SS}} & 0 & 0 \\ 0 & 0 & \sigma & C_{SS}^{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{\alpha Y_{SS}}{K_{SS}R_{SS}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & \frac{(1-\delta)C_{SS}^{\sigma}}{R_{SS}} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (1-\delta) & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{(1-\delta)K_{SS}}{C_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\alpha(1-\alpha)Y_{SS}}{K_{SS}R_{SS}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho \end{bmatrix}, \quad \tilde{B}_4 = \overline{B}_4, \quad \tilde{B}_5 = \begin{bmatrix} \phi - 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We set $\beta = 0.96$, $\delta = 0.10$, $\rho = 0.90$, $\phi = 0.975$, $\sigma = 2$, $\alpha = 0.33$, as in Guerrieri and Iacoviello (2015). The policy functions and sims. match those in Figs. 2-3 of their paper.

[10] Note that if the computed value of λ_t is negative under the alternative regime, then we must reject the guess that the constraint binds in this period. In this case, $x^*_{1,t} > \underline{x}_1$ ($x^*_{1,t}|_{\mathbb{1}_t=0} = \hat{i}_t - \lambda_t = (\phi-1) - \lambda_t > \phi - 1$), so $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} = x^*_{1,t} > \phi - 1$, and the guess that $\mathbb{1}_t = 0$ is rejected as required.

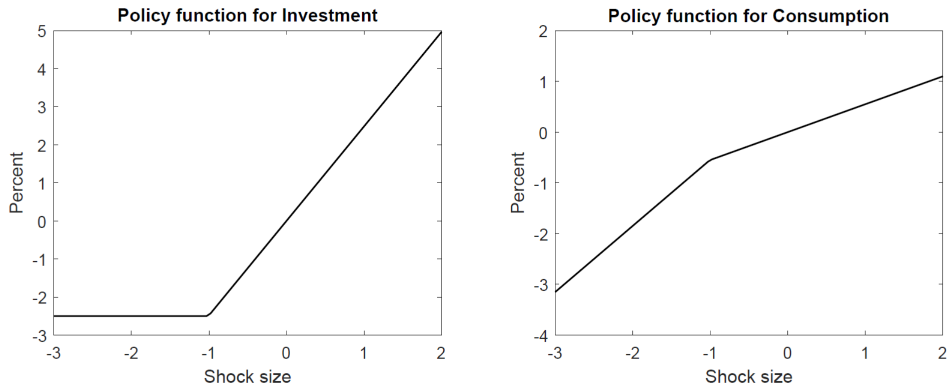

Figure 7: Policy functions for various shocks sizes e_t when $x_{t-1} = \overline{x}$

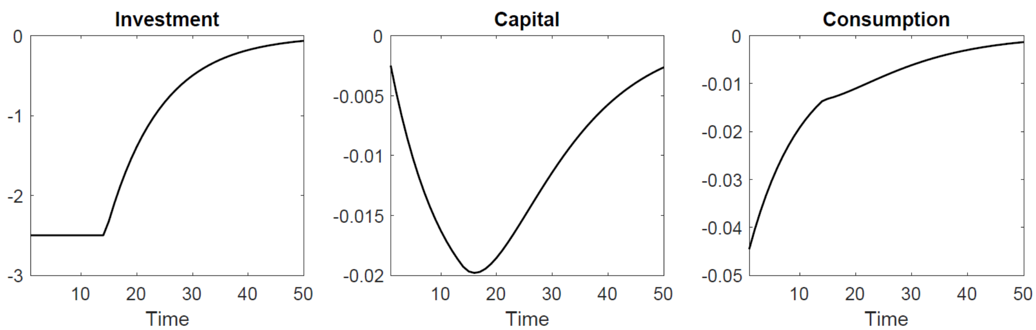

Figure 8: Perfect foresight solution for $e_1 = -0.04$ when $x_0 = \overline{x}$

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