Simulating multiple equilibria in rational expectations models with occasionally-binding constraints: An algorithm and a policy application

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Abstract

This paper presents an algorithm for simulating otherwise-linear models with rational expectations and occasionally-binding constraints. We extend the guess-verify algorithm of Guerrieri and Iacoviello (2015) to detect and simulate multiple perfect foresight equilibria, and to allow arbitrary (finite) sequences of news shocks. We also show how to check whether multiple solutions can be ruled out in a given model, using results in Holden (2022). We apply our algorithm to a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria, including a 'bad' solution with strong deflation due to self-fulfilling pessimistic expectations. A price-level targeting rule does not always eliminate the bad solution, but it shrinks the indeterminacy region substantially and improves stabilization relative to conventional policies.

1 Introduction

Occasionally-binding constraints, such as borrowing limits and the lower bound on nominal interest rates, introduce a stark non-linearity in economic models. As a result, standard solution methods for linear rational expectations models (Blanchard and Kahn, 1980; Binder and Pesaran, 1997; Uhlig, 1999; Sims, 2002), which assume a time-invariant structure, must be adapted to cope with such constraints. An important contribution to the literature was made by Guerrieri and Iacoviello (2015). They show how to solve otherwise-linear rational expectations models with occasionally-binding constraints and a large number of state variables using a guess and verify method, and they also provide a toolkit (OccBin) that implements the solution algorithm in the popular software package Dynare. Their algorithm finds a single solution under perfect foresight and assuming zero anticipated future shocks.

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In this paper we extend their solution method so that multiple perfect foresight equilibria can be detected and simulated. Our algorithm allows non-zero anticipated future shocks, such that the case of zero 'news shocks' as in Guerrieri and Iacoviello (2015) is nested as a special case. We also show how researchers can use results in Holden (2022) to check whether multiple perfect foresight equilibria are ruled out in a given model. Hence, the paper is essentially a methodological contribution with a range of potential applications.

The modern literature on occasionally-binding constraints began with Eggertsson and Woodford (2003) and Jung et al. (2005); they both solve the benchmark New Keynesian model when there is a zero lower bound on nominal interest rates. Jung et al. (2005) solve a perfect foresight version of the model, whereas Eggertsson and Woodford (2003) solve the model for the case of a two-state Markov process with an absorbing state. The model is solved using global methods in Adam and Billi (2006) and Nakov (2008); however, these approaches are computationally intensive and hence less useful for models with many state variables. Eggertsson et al. (2021) provide a toolkit for solving otherwise-linear models with an occasionally-binding constraint when the shock process is a two-state Markov process with an absorbing state, whereas Boehl (2022) provides a fast method for solving large-scale models in which constraints bind for a single spell along a perfect foresight path.

There are also solution methods that do not require the above restrictions. Guerrieri and Iacoviello (2015) provide a general toolkit (OccBin) that can used to solve DSGE models with many state variables when the model is linear aside from occasionally-binding constraints; their approach is based on a recursive perfect foresight algorithm that uses a guess-verify procedure and the method of undetermined coefficients. The algorithm is user-friendly because it is built into the software package Dynare (see Adjemian et al., 2011). By comparison, Holden (2016) and Holden (2022) use a 'news shocks' approach alongside results from the literature on linear complementarity problems; this approach has the advantage that it will find all solutions and can also be applied to models that are not otherwise-linear.

The present paper contributes to the literature by extending the solution method in Guerrieri and Iacoviello (2015) to detect and simulate multiple equilibria. Their original algorithm assumes *zero* future shocks when agents form future expectations. We show how to simulate perfect foresight paths when future shocks can be non-zero in an arbitrarily large (but finite) number of periods, thus allowing a wider range of scenarios to be examined. Our algorithm first searches for multiple perfect foresight solutions using the guess-verify method of Guerrieri and Iacoviello (2015); all solutions are then stored and, in cases of multiple solutions, we resolve the indeterminacy by drawing a 'sunspot' that selects a particular solution (akin to the approach in Farmer et al., 2015). An advantage of our approach is its flexibility: researchers can choose freely the *probabilities* of selecting each solution.

We illustrate this approach using a Fisherian model with multiple equilibria: both a high-inflation and low-inflation solution exist for the same initial conditions. Since searching for multiple equilibria is inefficient if there is a unique solution, we also show how researchers can check whether multiple perfect foresight solutions are ruled out using results in Holden (2022). Our marginal contribution here is to adapt the results for our algorithm where, as in the Occbin Toolkit, models are in the Binder-Pesaran form (Binder and Pesaran, 1997).

We provide two applications. The first is an asset pricing example studied in Guerrieri and Iacoviello (2015), which has a unique solution. We compute the policy functions under perfect foresight in the cases of zero and non-zero 'news shocks'; here we show that there is a strong asymmetry to positive and negative sequences of news shocks due to the truncated feedback rule in this model. Our second application studies a New Keynesian model with a zero lower bound on nominal interest rates and *multiple equilibria* for some parameter values (see Brendon et al., 2013), due to a monetary policy response to the *change* in the output gap. Here we show that our algorithm replicates their finding of two equilibria: a 'good' solution for which the lower bound is not hit and a 'bad' solution for which inflation and the output gap are strongly negative due to self-fulfilling pessimistic expectations.

Multiplicity is widespread under a rule that includes an inflation target alongside the first-difference in the output gap (the 'speed limit'); however, switching the inflation target for a price-level target shrinks the indeterminacy region substantially, such that the 'bad solution' is eliminated for a wide range of parameter values. An important implication is that while price-level targeting improves stabilization relative to conventional policies, a response to the price level is not (in general) sufficient to ensure determinacy in New Keynesian models with a zero lower bound, thus shedding new light on a conclusion in Holden (2022).

The paper proceeds as follows. Section 2 outlines a benchmark solution that extends the approach in Guerrieri and Iacoviello (2015) to non-zero news shocks and multiple equilibria. Our main extensions related to detecting and simulating multiple equilibria are presented in Section 3, and Section 4 presents our applications. Finally, Section 5 concludes.

2 Model

Consider a multivariate rational expectations model with *perfect foresight*; the model is linear aside from multiple possible regimes due to occasionally-binding constraints. Time is discrete and starts at date $t = 1$; therefore, $t \in \mathbb{N}_+$. As in Guerrieri and Iacoviello (2015) we focus for exposition purposes on the case of a single occasionally-binding constraint, implying that there are two regimes: a "reference regime" and an "alternative regime". It does not matter whether the regime with the constraint slack is the reference regime or the alternative regime; in what follows we assume that the reference regime has the constraint slack.

The reference regime is described by (1), and the alternative regime is described by (2):

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5 \tag{1}$$

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \tag{2}$$

where x_t is an $n \times 1$ vector of endogenous state and jump variables, E_t is the conditional expectations operator, and e_t is an $m \times 1$ vector of 'shocks' whose values are *known.* Note that serially correlated exogenous processes can be included in the vector x_t.

Matrices $\overline{B}_i, \tilde{B}_i$, $i \in [5]$, contain the model parameters. The $\overline{B}_i, \tilde{B}_i$, $i \in \{1, 2, 3\}$, are $n \times n$ matrices, $\overline{B}_4, \tilde{B}_4$ are $n \times m$ matrices, and $\overline{B}_5, \tilde{B}_5$ are $n \times 1$ vectors of intercepts. As shown in Binder and Pesaran (1997), the above formulation is quite general as it can accommodate multiple leads and lags of the endogenous variables through an appropriate definition of x_t, as well as conditional expectations at different horizons and at earlier dates.

The first variable $x_{1,t}$ is subject to a lower bound constraint in all periods:

$$x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}, \qquad x^*_{1,t} := F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G e_t + H \tag{3}$$

where $\underline{x}_1, H \in \mathbb{R}$, F is a $1 \times 3n$ vector with $f_{11} = 0$ and G is a $1 \times m$ vector.

The specification in (3) allows the constrained variable to depend on the exogenous shocks and on contemporaneous, past or future values of endogenous variables; note that an upper bound constraint can easily be accommodated.[1] Variable $x^*_{1,t}$ is the 'shadow value' of the constrained variable. The vectors F, G, H are given by the equation that describes the bounded variable when the constraint is slack; for example, in a model with a lower bound on nominal interest rates, this equation is usually a Taylor(-type) rule.

In typical applications, one of the intercept matrices may be zero, as DSGE models are typically log-linearized around a non-stochastic steady state (see Uhlig, 1999). At any given date t, the economy is either in the reference regime or alternative regime. Given mutually exclusive regimes, we introduce an *indicator variable* $\mathbb{1}_t \in \{0, 1\}$ that is equal to 1 (0) if the reference regime (alternative regime) is in place in period t. Our model (1)–(3) is then:

$$\begin{aligned} & B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 1 \\ & \text{s.t. } x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \end{aligned} \tag{4}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i \; \forall i \in [5]$ and $x_0 \in \mathbb{R}^n$ is given.

The information set at time t includes all current, past and future values of the endogenous and exogenous variables; note that the indicator variable $\mathbb{1}_t$ is *endogenous.* As in Guerrieri and Iacoviello (2015) and Holden (2022), we assume the model returns to the reference regime forever after some finite date $T \geq 1$ (such that $\mathbb{1}_t = 1 \; \forall t > T$). Following Guerrieri and Iacoviello (2015), we find the sequence $(\mathbb{1}_t)_{t=1}^T$ using a guess-verify method.

[1] In a simple a borrowing-constraint model with $b_t \leq \bar{b}$ (see Guerrieri and Iacoviello, 2015, Appendix), we have $-b_t \geq -\bar{b}$, so we can replace b_t with $-x_{1,t}$, where $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$, $\underline{x}_1 = -\bar{b}$ and $x^*_{1,t}$ is determined by the budget constraint. Similarly, if $b_t \leq E_t q_{t+1} + h_t - r_t$ (see Iacoviello, 2005) we have $x_{1,t} = \max\{0, x^*_{1,t}\}$, where $x^*_{1,t} := E_t q_{t+1} + h_t - r_t - b_t$, and given $x_{1,t}$ we can recover b_t via $b_t = E_t q_{t+1} + h_t - r_t - x_{1,t}$.

That is, we guess a sequence of regimes $(\mathbb{1}_t)_{t=1}^T$ and date T, which agents take as known and given, and accept the resulting time path $(x_t)_{t=1}^\infty$ as a solution if and only if the guessed sequence of regimes is verified by the time path of the shadow variable $x_{1,t}^*$.[2]

2.1 The solution method

Definition 1. *A perfect foresight solution to model* (4) *is a function* $f : x_{t-1} \times e_t \to x_t$ *such that the system in* (4) *holds for all* $t \geq 1$*, given a sequence of known shocks* $(e_t)_{t=1}^\infty$.

An alternative way of characterizing a solution is in terms of a set of matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=1}^\infty$ that generalize the constant-coefficient decision rules of a linear rational expectations model:

$$x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t \tag{5}$$

where Ω_t is an $n \times n$ matrix, Γ_t is an $n \times m$ matrix, Ψ_t is an $n \times 1$ vector, and the t subscript indicates that the matrices are in general time-varying.

Following Guerrieri and Iacoviello (2015) and Kulish and Pagan (2017), the matrices $\Omega_t, \Gamma_t, \Psi_t$ can be determined recursively using simple formulas. Our perfect foresight assumption implies that the date t solution x_t will generally depend on both current shocks e_t and anticipated future shocks $e_{t+1}, \ldots, e_{t+T}$; such anticipated future shocks ('news shocks') will enter in (5) through the intercept matrix Ψ_t.

There are three key requirements for existence of a solution:

(i) Existence of a rational expectations solution at the reference regime. The corresponding solution matrices, $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$, are used to find a solution during the transition periods.

(ii) A series of regularity conditions $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ must be met for $t = 1, ..., T$, where $T + 1$ is the date at which the terminal solution is reached.

(iii) The resulting solution path x_t must satisfy the lower-bound constraint for all $t \geq 1$ and be away from the bound for all $t > T$ (terminal condition).

Requirement (i) is needed as the solution relies on 'backward induction' from a terminal solution. A terminal solution can be found using standard methods, such as Binder and Pesaran (1997), Sims (2002) or Dynare (Adjemian et al., 2011). Existence of a terminal solution is necessary but not sufficient for existence of a solution; the regularity conditions in (ii) must hold and the perfect foresight path must satisfy the occasionally-binding constraint and a terminal condition; see (iii). Note that existence of a perfect foresight solution is not guaranteed in well-specified models with occasionally-binding constraints (Holden, 2022).

[2] Because we use a guess-verify approach, our method can find a *finite* number of solutions (corresponding to all verified guesses), but it will not find infinitely many solutions if there is a continuum of solutions.

Uniqueness is discussed further once a benchmark solution is presented. For the moment we note that uniqueness of a terminal solution is necessary for a unique solution in (5). Cagliarini and Kulish (2013) provide rank conditions that can be used to check for existence and uniqueness of the path $(x_t)_{t=1}^{\infty}$ under an arbitrary finite sequence of exogenous structural changes $(\mathbb{1}_t)_{t=1}^{T}$, and Holden (2022) provides a sufficient condition for existence of a unique perfect foresight solution in otherwise-linear models with an occasionally-binding constraint, subject to a terminal condition (as here). We provide a more detailed discussion below.

A terminal solution is a time-invariant rational expectations solution; existence and uniqueness can therefore be checked using standard methods such as Blanchard and Kahn (1980) and Sims (2002). We assume that the Blanchard-Kahn conditions for uniqueness and stability are satisfied by the terminal solution and that the occasionally-binding constraint is satisfied. In other words, we assume the system in (4) returns permanently to the reference regime, and the unique and stable terminal solution $(x_t)_{t=T+1}^{\infty}$ is away from the lower bound, i.e. $x_{1.t} > \underline{x}_1$ for all $t > T$ (terminal condition). Formally we have:

Assumption 1. *We assume that* $\det[\overline{B}_1 - \overline{B}_2 - \overline{B}_3] \neq 0$*, such that there exists a unique steady state* $\overline{x} = (\overline{B}_1 - \overline{B}_2 - \overline{B}_3)^{-1}\overline{B}_5$ *at the reference regime.*

Assumption 2. *Given a solution path* $(x_t)_{t=1}^{T}$ *that satisfies* (3)–(4) *and anticipated shocks* $(e_t)_{t=1}^{\infty}$ *with* $e_t = 0_{m\times 1}$ $\forall t > T$*, there is a unique stable terminal solution* $(x_t)_{t=T+1}^{\infty}$ *that satisfies* $x_t = \overline{\Omega}x_{t-1} + \overline{\Psi}$*,* $x_{1,t} > \underline{x}_1$*, and* $\lim_{t\to\infty} x_t = \overline{x}$*, where* $\overline{x}_1 \geq \underline{x}_1$*,* $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ *has eigenvalues in the unit circle, and* $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5) = (I_n - \overline{\Omega})\overline{x}$*.*

Assumption 1 restricts attention to models with a unique steady state $\overline{x}$ under the reference regime. Assumption 2 says that the perfect foresight solution converges to this steady state and there is a finite date T such that the solution is away from the bound for all $t > T$. The final part of Assumption 2 rules out cases where, trivially, no perfect foresight solution exists because x_t converges on a steady state $\overline{x}$ that violates the lower bound constraint.[3]

In addition, Assumption 2 rules out many stable terminal solutions. It does not rule out multiple solutions in general, however, as there may be multiple solution paths, say $(x_t^1)_{t=1}^{T_1}$ and $(x_t^2)_{t=1}^{T_2}$, that satisfy the lower bound constraint (3) and have unique terminal solutions $(x_t^1)_{t=T_1+1}^{\infty}$ and $(x_t^2)_{t=T_2+1}^{\infty}$. We discuss multiple equilibria after presenting our algorithm.

2.2 Solving the model

We take date $t = 1$ as the current period (this permits us to study the solution at a given date by relabelling date 1 as date t). Let there be T^a vectors of shocks $e_1, ..., e_{T^a}$ that correspond to known shocks in periods 1 to T^a; we assume $e_t = 0_{m\times 1}$ for all $t > T^a$. Without loss of generality, we assume that $T^a = T$; note that there is no loss of generality since to study

[3]For models with the constraint is binding at steady-state, it is straightforward to amend Assumptions 1–2 and our solution algorithm as shown in Section 5 of the *Supplementary Appendix*.

anticipated shocks up a horizon smaller than T we can simply set all shocks beyond the desired horizon equal to zero. Given perfect foresight, expectations coincide with future values: $E_t[x_{t+1}] = x_{t+1}$. The system to be solved is therefore:

$$\begin{cases} B_{1,t}x_t = B_{2,t}x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, & 1 \leq t \leq T \\ \overline{B}_1 x_t = \overline{B}_2 x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_5, & \forall t > T \end{cases} \tag{6}$$

subject to $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ for $t = 1, ..., T$.

For all $t > T$, the reference regime is in place. By Assumption 2, there is a unique stable terminal solution $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$, $\forall t > T$, which is away from the lower bound. Therefore, the remaining system to be solved is:

$$\begin{aligned} &B_{1,1}x_1 = B_{2,1}x_2 + B_{3,1}x_0 + B_{4,1}e_1 + B_{5,1} \\ &\vdots \\ &B_{1,T}x_T = B_{2,T}x_{T+1} + B_{3,T}x_{T-1} + B_{4,T}e_T + B_{5,T} \end{aligned} \tag{7}$$

subject to $x_{T+1} = \overline{\Omega} x_T + \overline{\Psi}$ and $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ for $t = 1, ..., T$. We can thus state the following result which we build on later.

Proposition 1. *Consider the model (6)–(7) for $T \in \mathbb{N}_+$. If one or more perfect foresight solutions exist, then the solution(s) satisfy the following equations:*

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases} \tag{8}$$

where, for $t = 1, ..., T$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{4,t} \tag{9}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}) \tag{10}$$

provided that $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0 \; \forall t \in [1, T]$*; the terminal matrices $\Omega_{T+1} = \overline{\Omega}$, $\Psi_{T+1} = \overline{\Psi}$, $\Gamma_{T+1} = 0_{n \times m}$ and the terminal solution $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$ $\forall t > T$ satisfy Assumption 2; and the guessed structure $(\mathbb{1}_t)_{t=1}^T$ is verified by $x^*_{1,t}$ at all dates on the transition path, i.e. $\forall t \in [1, T]$*

$$B_{j,t} = \begin{cases} \overline{B}_j & \text{if and only if } \mathbb{1}_t = 1 \text{ and } x^*_{1,t}|_{\mathbb{1}_t = 1} \geq \underline{x}_1 \\ \tilde{B}_j & \text{if and only if } \mathbb{1}_t = 0 \text{ and } x^*_{1,t}|_{\mathbb{1}_t = 0} \leq \underline{x}_1 \end{cases}, \quad j = 1, .., 5. \tag{11}$$

Proof. See the Appendix. ■

Proposition 1 formalizes the guess-verify solution method. Note that the solution algorithm is recursive, like the original Occbin algorithm, and is therefore easy to program. The case of zero news shocks as in Guerrieri and Iacoviello (2015), and the OccBin Toolkit, is nested as the special case $e_t = 0_{m\times 1}$ for all $t > 1$. A second difference relative to Guerrieri and Iacoviello (2015) is that we allow for *multiple* perfect foresight solutions (see (11)) – i.e. the guess-verify search will not be automatically terminated once a solution has been found. A perfect foresight solution of the form in Proposition 1 will not exist if the invertibility conditions in (9)–(10) are not met or if no guessed structure $(\mathbb{1}_t)_{t=1}^T$ is verified.

The condition in (11) is the 'verify' part of the solution method. Given a guessed date T such that $x_{1,t} > \underline{x}_1$ for all $t > T$ and a guessed sequence of regimes $(\mathbb{1}_t)_{t=1}^T$, the guess is verified if and only if the condition in (11) involving the 'shadow values' $x^*_{1,t}$ is satisfied. Equation (11) makes clear that different sequences of regimes may be verified; e.g. there will be multiple equilibria if both $\mathbb{1}_{t'} = 1$ and $\mathbb{1}_{t'} = 0$ are verified for some t' $(\leq T)$ and $\mathbb{1}_t = 1$ for all $t \neq t'$.[4] Holden (2022) provides necessary and sufficient conditions for a *unique* perfect foresight solution in models with occasionally-binding constraints. We first provide an example with multiple equilibria before discussing when multiplicity can be ruled out.

Example 1. *Following Holden (2022, Example 2) suppose that for all $t \geq 1$ our model consists of a Taylor-type rule with a zero lower bound and the Fisher equation:*

$$i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + e_t\} \tag{12}$$

$$i_t = r + \pi_{t+1} \tag{13}$$

where $\phi - \psi > 1$, $\psi > 0$, $e_1, \pi_0 \in \mathbb{R}$, $e_t = 0$ $\forall t > 1$, and $r > 0$ is a fixed real interest rate. To simplify presentation, we set $\phi = 2$. The results are not specific to this case.

Away from the lower bound, the solution has the form $i_t = r + \omega^2\pi_{t-1}$ for all $t \geq 1$, $\pi_1 = \omega\pi_0 + \frac{1}{\phi-\omega}e_1$, and $\pi_t = \omega\pi_{t-1}$, for all $t > 1$, where $\omega = 1 - \sqrt{1-\psi} \in (0,1)$. This solution is stable (inflation converges to 0 and nominal rates to r), does not violate the lower bound in period 1 provided $r + \phi\pi_1 - \psi\pi_0 + e_1 \geq 0$, and is away from the bound for all $t > 1$ only if $r + \phi\pi_t - \psi\pi_{t-1} > 0$; hence this solution exists if and only if $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$.

As shown by Holden (2022), there is a second stable solution, for which the constraint binds only in period 1, i.e. $i_1 = 0$ and $\pi_t = \omega\pi_{t-1}$, $i_t = r + \omega^2\pi_{t-1}$ for all $t > 1$. Note that $i_1 = 0$ implies that $\pi_2 = -r$ by (13), so $\pi_1 = -r/\omega$ and $\pi_t = \omega\pi_{t-1} = \omega^{t-2}(-r)$ for all $t > 1$. Note that the bound binds in period 1 provided $r + \phi\pi_1 - \psi\pi_0 + e_1 \leq 0$ and is escaped thereafter only if $r + \phi\pi_t - \psi\pi_{t-1} > 0$ for all $t > 1$, so we require $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$.[5] Hence, for $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ both solutions exist, while for $\pi_0 < -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ there is no stable solution.

[4]This simple example is easy to explain and to understand. In general, however, sequences of verified regimes may differ in multiple periods and in the date T after which the terminal solution prevails.

[5]Note that for all $t > 1$, we have $i_t = r + \omega^2\pi_{t-1} = [1 - (\omega)^{t-1}]r > 0$.

We now show that our solution method finds the same solutions. Letting $x_t := [i_t \quad \pi_t]'$, *the matrices in the reference regime and the alternative regime are given by*

$$\overline{B}_1 = \begin{bmatrix} 1 & -\phi \\ 1 & 0 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{B}_3 = \begin{bmatrix} 0 & -\psi \\ 0 & 0 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} r \\ r \end{bmatrix}$$

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{B}_5 = \begin{bmatrix} 0 \\ r \end{bmatrix}.$$

Hence, analogous to (3)–(4)*, the model for all* $t \geq 1$ *is*

$$B_{1,t}x_t = B_{2,t}x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, \quad s.t.\ i_t = \max\{0, i_t^*\} \tag{14}$$

where $B_{j,t} := \mathbb{1}_t\overline{B}_j + (1 - \mathbb{1}_t)\tilde{B}_j \ \forall j \in \{1, ..., 5\}$, $e_t = 0$ *for all* $t > 1$, *and*

$$i_t^* = F\begin{bmatrix} x_t' & x_{t+1}' & x_{t-1}' \end{bmatrix}' + Ge_t + H, \quad \textit{with } F = \begin{bmatrix} 0 & \phi & 0 & 0 & 0 & -\psi \end{bmatrix},\ G = \begin{bmatrix} 1 \end{bmatrix},\ H = \begin{bmatrix} r \end{bmatrix}.$$

Consider first the solution away from the lower bound. This solution corresponds to the guess that $\mathbb{1}_t = 1$ *for all* $t \geq 1$, *such that* $B_{j,t} = \overline{B}_j \ \forall j$ *and* (14) *becomes*

$$\overline{B}_1x_t = \overline{B}_2x_{t+1} + \overline{B}_3x_{t-1} + \overline{B}_4e_t + \overline{B}_5, \quad \forall t \geq 1. \tag{15}$$

The guessed solution $x_t = [i_t \quad \pi_t]'$ *thus follows Proposition 1 with* $T = 1$ *and* $\mathbb{1}_t = 1$ *for all* t:

$$x_t = \begin{cases} \Omega_1 x_0 + \Gamma_1 e_1 + \Psi_1 & \textit{for } t = 1 \\ \overline{\Omega}x_{t-1} + \overline{\Psi} & \textit{for } t > 1 \end{cases} \tag{16}$$

where $\Omega_1 = \overline{\Omega}$, $\Psi_1 = \overline{\Psi}$, $\Gamma_1 = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_4 = \begin{bmatrix} -\frac{\omega}{\phi-\omega} & -\frac{1}{\phi-\omega} \end{bmatrix}'$, *and*

$$\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3 = \begin{bmatrix} 0 & \omega^2 \\ 0 & \omega \end{bmatrix}, \qquad \overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5) = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

with $\omega = 1 - \sqrt{1-\psi}$ *as above.*[6]

This guessed solution is verified provided $i_1^* = r + \phi\pi_1 - \psi\pi_0 + e_1 \geq 0$ *(see Proposition 1) and* $i_t^* = r + \phi\pi_t - \psi\pi_{t-1} > 0 \ \forall t > 1$, *which requires* $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ *as we saw above.*

Now consider the second solution. In this case we guess that the lower bound constraint binds only in period 1, such that $\mathbb{1}_1 = 0$ *and* $\mathbb{1}_t = 1 \ \forall t > 1$; *hence* $B_{j,t} = \tilde{B}_j$ *for* $t = 1$ *and*

[6] We can guess that the left column of $\overline{\Omega}$ is zero (because i_{t-1} does not enter the model), which makes it straightforward to solve the resulting quadratic equation in $\overline{\Omega}$ for the right-hand column. In general, $\overline{\Omega}, \overline{\Psi}$ can be found using numerical methods (e.g. Binder and Pesaran, 1997; Sims, 2002; Cho and Moreno, 2011).

$B_{j,t} = \overline{B}_j$ *for all* $t > 1$. *This guess implies that the terminal solution in* (16) *applies from period 2 onwards, such that* $T = 1$ *and* $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$ *for all* $t > 1$. *Hence, by Proposition 1 our guessed solution path is*

$$x_t = \begin{cases} \Psi_1 & \text{for } t = 1 \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > 1 \end{cases} \tag{17}$$

where $\Psi_1 = (\tilde{B}_1 - \tilde{B}_2 \overline{\Omega})^{-1} (\tilde{B}_2 \overline{\Psi} + \tilde{B}_5) = \begin{bmatrix} 0 & -\frac{r}{\omega} \end{bmatrix}'$.

To verify this solution, we require $i_1^* = r + \phi\pi_1 - \psi\pi_0 + e_1 \leq 0$ *and* $i_t^* = r + \phi\pi_t - \psi\pi_{t-1} > 0$ *for all* $t > 1$, *which requires* $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ *as already seen above.*

The two solutions are shown in Figure 1, along with the shadow interest rate i_t^* *under both solutions. Note that Solution 1 has a positive shadow rate in all periods that coincides with the actual interest rate; hence this solution is verified and is away from the bound in all periods. By comparison, Solution 2 hits the bound in period 1 and has a negative shadow rate in this period (so the constraint binds); hence this solution is also verified as argued above.*

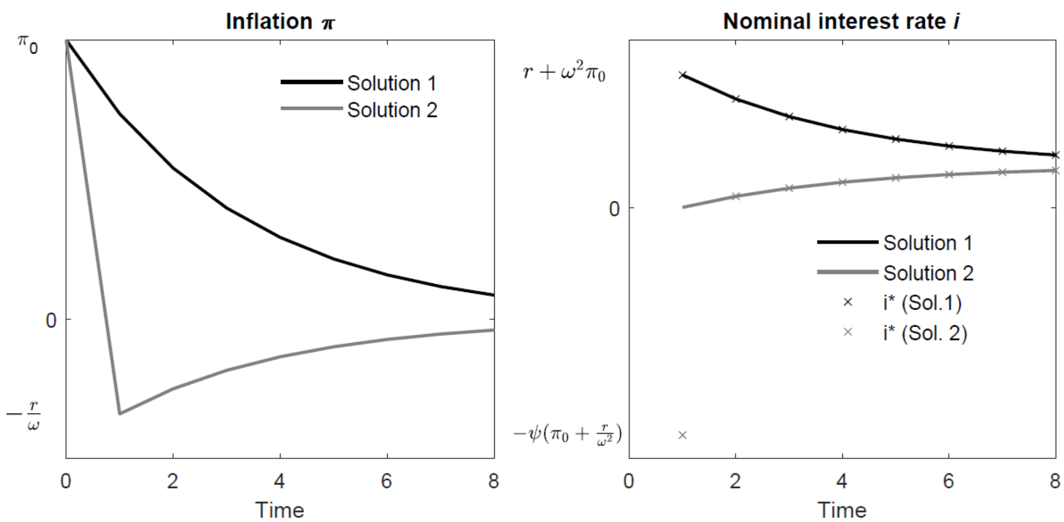

Figure 1: The two solutions when $\pi_0 > 0$ and $e_1 = 0$

Having studied a specific example, we now present a simple and general method for simulating multiple equilibria in ocassionally-binding constraint models. In particular, our approach allows users to attach *prior probabilities* to each perfect foresight solution; a realized solution is then selected by a random 'sunspot'. We also show how this approach can be extended to 'stochastic simulations' in which realized shocks can differ from those shocks on which expectations were based, such that the perfect foreight assumption is relaxed.

3 Simulating multiple equilibria

Example 1 is not special in having multiple solutions. As argued by Holden (2022), multiplicity is common in otherwise-linear models with occasionally-binding constraints; thus, it is important that solution algorithms can find multiple equilibria and construct simulation paths where multiplicity is not neglected. In this section we explain how this can be done, assuming there are multiple perfect foresight solutions that satisfy Proposition 1.

Suppose, therefore, there are a finite number of solutions $K \geq 2$ that have been found using the method in Proposition 1; for instance, as in Example 1 above. To pick between these solutions we suggest a simple procedure where the researcher chooses 'prior probabilities' $p_1, ..., p_K \in [0, 1]$ which determine the likelihood that a given solution k will be selected at date 1. We index the K different perfect foresight solutions by $(x_t^k)_{t=1}^{\infty}$ for $k = 1, ..., K$, and let $u_1 \sim \mathcal{U}_{(0,1)}$ be a 'sunspot' drawn from the uniform distribution on the interval $(0, 1)$. We can then summarize the choice of a solution as follows.

Remark 1. *Suppose there are $K \geq 2$ perfect foresight solutions that satisfy the conditions in Proposition 1. Given chosen probabilities $p_1, ..., p_K \geq 0$ such that $\sum_{k=1}^{K} p_k = 1$ and a random draw $u_1 \sim \mathcal{U}_{(0,1)}$, we can select a single perfect foresight solution as follows:*

$$(x_t)_{t=1}^{\infty} = \begin{cases} (x_t^1)_{t=1}^{\infty} & \text{if } u_1 \in (0, p_1] \\ (x_t^2)_{t=1}^{\infty} & \text{if } u_1 \in (p_1, p_1 + p_2] \\ \vdots & \\ (x_t^K)_{t=1}^{\infty} & \text{if } u_1 > p_1 + ... + p_{K-1} \end{cases} \tag{18}$$

i.e. for any $u_1 \in (\sum_{k=0}^{k^*-1} p_k, \sum_{k=0}^{k^*} p_k]$, where $k^* \in \{1, ..., K\}$ and $p_0 := 0$, the unique (selected) perfect foresight solution is $(x_t)_{t=1}^{\infty} = (x_t^{k^*})_{t=1}^{\infty}$.

Remark 1 gives a general method for choosing between multiple perfect foresight solutions; it can be applied to any set of perfect foresight solutions and is flexible due to the specification of prior beliefs. For instance, if the researcher thinks some solution(s) somewhat 'unrealistic' they may attach low (or zero) probability to those solution paths. On the other hand, a perfectly agnostic researcher would choose 'flat priors' of $p_k = 1/K$ for all k. A code in our algorithm first stores all (found) perfect foresight solutions to a particular model and then selects a single solution using Remark 1 and some specified probabilities.

This approach amounts to a one-off lottery among perfect foresight paths. For instance, given assigned probabilities p_1, p_2 in Example 1, Solution 1 will be chosen as the simulated path if we draw a $u_1 \in (0, p_1]$, and Solution 2 will be chosen otherwise (i.e. if $u_1 > p_1$). In practice, however, researchers will sometimes simulate not only the computed perfect foresight path but also 'stochastic simulations' in which the model is hit in each period with unanticipated shocks (e.g. this is what the 'stochastic sims' option in the OccBin toolkit does). Such simulation paths ignore risk effects in expectations but may be useful for studying robustness to relaxing the perfect foresight assumption. This simulation approach effectively 'restarts the clock' at each date; as a result, indeterminacy may arise for $t > 1$ if the inherited initial conditions in these periods differ from those along a perfect foresight path, i.e. if the realized shocks e_t differ from those on which expectations were based.[7]

We now discuss how the approach in Remark 1 may be used to construct such simulation paths. Suppose, therefore, that for $t > 1$ an unanticipated shock vector e_t is drawn from some distribution. Then at each date t we can find the solution(s) by using Proposition 1 and Remark 1, since it is a simple matter of relabelling to use these approaches in any period $t \in \mathbb{N}_+$. For example, given a selected solution x_1 and a new draw e_2, we may use x_1 as an initial condition to find the date 2 solutions, and then Remark 1 will gives us a single selected solution x_2 (and new initial condition), and so on. We now study a simple example.

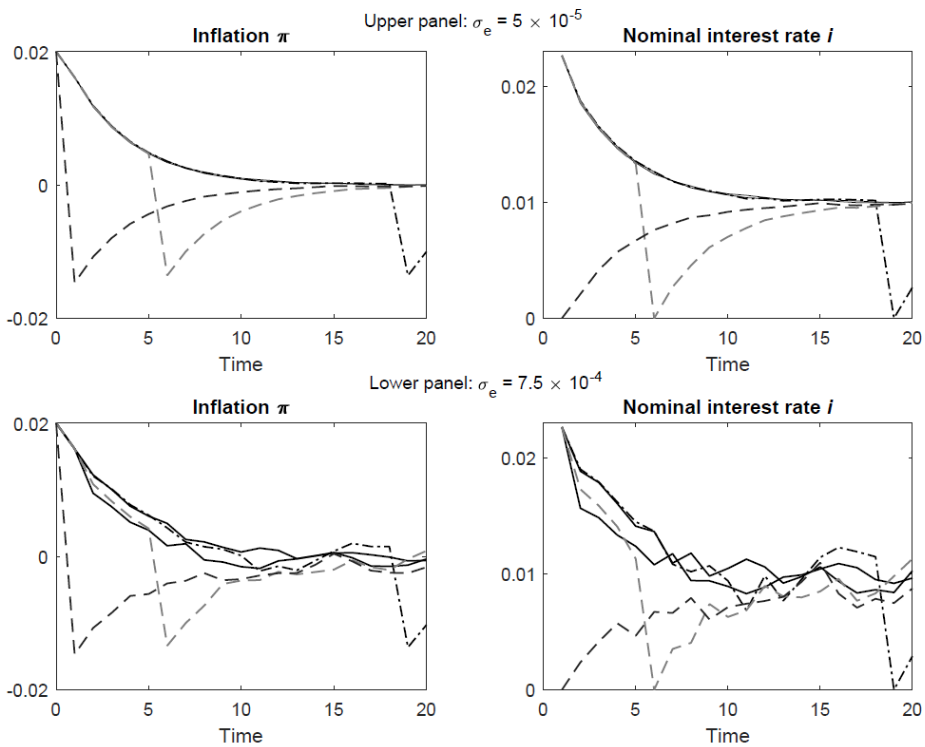

Figure 2: Five stochastic simulations: $p_1 = 0.95$ and initial values $\pi_0 = 0.02$, $e_1, e_2 = -0.001$

Example 2. *Continuing with Example 1, we set* $r = 0.01$*,* $\phi = 2$*,* $\psi = 0.93$*,* $\pi_0 = 0.02$*, along with date-*1 *anticipated shocks* $e_1 = e_2 = -0.001$ *and a probability of selecting solution 1 (away from the bound) of* $p_1 = 0.95$*. At dates* $t > 1$ *we solve for* i_t, π_t *conditional on the*

[7]Note that if the realized shocks are drawn from continuous distributions, then the probability they will coincide with those on which expectations were based is zero even if the support contains the original shocks.

inherited state π_{t-1} and fresh draws for the shocks e_t, e_{t+1}. We did this 5 times, thus giving 5 'stochastic' (i.e. non-deterministic) paths for interest rates and inflation; see Figure 2. At each date $t \geq 1$ we solve for both solutions and draw a sunspot $u_t \sim \mathcal{U}_{(0,1)}$ that selects either solution 1 (away from bound) or solution 2 (hits bound in current period) given the chosen probabilities of each solution; see Remark 1. Since we set the probability of solution 1 at 95%*, solution 1 (solution 2) is chosen at date t if and only if* $u_t \in (0, 0.95]$ *(*$u_t > 0.95$*).*

In the upper panel of Figure 2, the standard deviation of the policy shock is very small in order to isolate the impact of the sunspot, i.e. selection between the two equilibria. Of five simulations, three hit the zero lower bound in some period (see dashed lines); in these cases, we see strong deflation, in contrast to the positive inflation solutions away from the bound (cf. Figure 1). In the lower panel, the shock variance is large enough to make each individual simulation path discernible, but the main variations in inflation and interest rates arise from switching between multiple equilibria rather than disturbances to monetary policy.

3.1 The M matrix and uniqueness

We now consider a condition that is sufficient to rule out multiple perfect foresight solutions. Following Holden (2022), the 'M matrix' is a matrix $M \in \mathbb{R}^{T \times T}$ that lists the impulse responses of the bounded variable to news shocks $v_{1,t} = 1$ at dates $t = 1, ..., T$, when the bound is not imposed (i.e. when the max operator is removed). The M matrix is a key concept for uniqueness of perfect foresight solutions. For $i, j \in \{1, ..., T\}$ the ijth entry of the M matrix is the impulse response $x_{1,i}$ to an anticipated shock $v_{1,j}$ at date j. For example, the first column of M lists the T-period impulse response $x_{1,t}$ to a date 1 shock $v_{1,1} = 1$.

The impulse responses can be found by introducing an $n \times 1$ vector of known shocks, $v_t := [v_{1,t} \;\; 0_{1\times(n-1)}]'$, to the bounded variable, where $v_{1,t} \in \{0, 1\}$ for $t = 1, ..., T$ and $v_{1,t} = 0$ for all $t > T$. Letting $\hat{x}_t := x_t - \overline{x}$ (see Assumption 1), we can solve the following model:

$$\overline{B}_1 \hat{x}_t = \overline{B}_2 \hat{x}_{t+1} + \overline{B}_3 \hat{x}_{t-1} + \overline{B}_4 e_t + v_t, \quad \forall t \geq 1 \tag{19}$$

whose solution and M matrix are described in Remark 2.

Remark 2. *The solution to the perfect foresight model in* (19) *is given by*

$$\hat{x}_t = \overline{\Omega} \hat{x}_{t-1} + \hat{\Gamma} \tilde{e}_t + \Psi_t, \quad \forall t \geq 1 \tag{20}$$

where $\tilde{e}_t := \overline{B}_4 e_t + v_t$, $\hat{\Gamma} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1}$, $\Psi_t = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_2 (\Psi_{t+1} + \hat{\Gamma} \tilde{e}_{t+1})$ $\forall t \in [1, T]$, *with* $\Psi_t = 0_{n\times 1}$ *for all* $t \geq T$*, and the corresponding* M *matrix is*

$$M_{ij} = \frac{\partial \hat{x}_{1,i}}{\partial v_{1,j}} = \hat{x}_{1,i}|_{v_{1,j}=1} - \hat{x}_{1,i}|_{v_{1,j}=0} \quad \text{for } i, j \in \{1, ..., T\}.$$

Remark 2 gives a simple way to compute the M matrix using existing solution methods that can deal with anticipated shocks; see, for example, Cagliarini and Kulish (2013), Kulish and Pagan (2017) and Hatcher (2022). Note that uniqueness of the solution in Remark 2 follows from our assumption of a unique (stable) terminal solution; see Assumption 2.

Corollary 1 in Holden (2022) states that a generic otherwise-linear model with an occasionally-binding constraint and a terminal condition will have (i) a unique perfect foresight solution for all initial conditions if M is a P-matrix, and (ii) multiple solutions if M is not a P-matrix *and* a certain rank condition is met. A P-matrix is defined as follows.

Definition 2. *A matrix $M \in \mathbb{R}^{T\times T}$ is a P-matrix if and only if all of the principal sub-matrices of M have positive determinants.*

If M is a P-matrix, there cannot be multiple perfect foresight solutions. For instance, the model we studied in Example 1, which has either two solutions or no solution, does not satisfy the P-matrix requirement. Given this general result on uniqueness, our algorithm checks whether M is a P-matrix at a given value of T. If M is a P-matrix, then the guess-verify part of the algorithm can be terminated automatically after a solution is found, saving computation time. Further, we can state the following corollary for this case.

Corollary 1. *Suppose M is a P-matrix. Let a superscript (1) be the first row of a matrix and let $\hat{F}_t = \overline{\Omega}\hat{F}_{t-1}$, $\hat{G}_t = \overline{\Omega}\hat{G}_{t-1}$, $\hat{H}_t = \overline{\Omega}\hat{H}_{t-1} + \Psi_t$ $\forall t \in \{2, ..., T\}$, with $\hat{F}_1 = \overline{\Omega}$, $\hat{G}_1 = \overline{\Gamma} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_4$, $\hat{H}_1 = \Psi_1$ and $\Psi_t = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2(\Psi_{t+1} + \overline{\Gamma}e_{t+1}) + \overline{B}_5)$, $\Psi_{T+1} = \overline{\Psi}$. Then we can state the following:*

(i) If $\min_{t\in\{1,...,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\} > \underline{x}_1$, there is a unique solution which is away from the bound in all periods and given by

$$x_t = \begin{cases} \overline{\Omega}x_{t-1} + \overline{\Gamma}e_t + \Psi_t, & 1 \le t \le T \\ \overline{\Omega}x_{t-1} + \overline{\Psi}, & \forall t > T. \end{cases} \tag{21}$$

(ii) If $\min_{t\in\{1,...,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\} = \underline{x}_1$, there is a unique solution (21) *for which the constraint is slack for all $t \in \{1, ..., T\}$ but the bound is touched for some $t \in \{1, ..., T\}$;*

(iii) If $\min_{t\in\{1,...,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\} < \underline{x}_1$ and there exists $(\mathbb{1}_t)_{t=1}^T, (x_t)_{t=1}^T$ that satisfy the existence conditions in Proposition 1, there is a unique solution that differs from (21)*.*

Proof. It follows from Proposition 1. See the Appendix. ∎

Corollary 1 provides a condition to check whether there is a unique perfect foresight solution for which the lower bound constraint is slack in all periods, in models where the M matrix is P-matrix. If $\min_{t\in\{1,...,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\} > \underline{x}_1$ (part (i)) or

$\min_{t\in\{1,\ldots,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\} = \underline{x}_1$ (part (ii)), then the solution is given by x_t in (21). Note that finding $\min_{t\in\{1,\ldots,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\}$ is straightforward since the matrices $\hat{F}_t, \hat{G}_t, \hat{H}_t$ are determined recursively and depend only on the solution matrices $\overline{\Omega}, \overline{\Psi}$ of the terminal solution and the matrix $\overline{\Gamma} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_4$.[8] Clearly, if $\min_{t\in\{1,\ldots,T\}}\{\hat{F}_t^{(1)}x_0 + \hat{G}_t^{(1)}e_1 + \hat{H}_t^{(1)}\} \geq \underline{x}_1$, then computation time need not be wasted trialling guesses for which the constraint binds in one or more periods.

3.2 Existence

Proposition 1 and Remark 1 take the existence of one or more perfect foresight solutions to the model (6)–(7) as given . We now briefly discuss conditions for existence of a solution.

First, existence of a solution in Proposition 1 requires that the invertibility conditions $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0 \, \forall t \in [1, T]$ are met.[9] Our algorithm thus rejects guessed solutions immediately if an invertibility condition is violated, and moves on to the next guess. Second, even if the invertibility conditions are satisfied, existence of a solution is not guaranteed, as shown by Example 1. The problem in that example is that the M matrix is *not* a P-matrix, and hence there may be multiple solutions (if initial inflation is high enough) or no solution (if initial inflation is too low). Holden (2022) provides further discussion and examples.

4 Applications

We now present two applications: an asset pricing model with a truncated feedback rule as in Guerrieri and Iacoviello (2015) and a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria for some parameter values. Implementation details are provided in a *Supplementary Appendix*, along with some further examples.[10]

4.1 Asset pricing model

We first consider the simple asset pricing model in Guerrieri and Iacoviello (2015):

$$
\begin{aligned}
q_t &= \beta(1-\rho)E_t q_{t+1} + \rho q_{t-1} - \sigma r_t + u_t \\
r_t &= \max\{\underline{r}, \phi q_t\} \\
u_t &= \rho_u u_{t-1} + e_t
\end{aligned}
$$

where $\beta, \rho \in (0,1)$, $\phi, \sigma > 0$, $\underline{r} < 0$, $\rho_u \in (0,1)$ and all values of e_t are known.

[8] Note that matrix $\overline{\Gamma}$ does not appear in the terminal solution as future shocks are zero for all $t > T$.

[9] If the invertibility conditions are not satisfied, computing a pseudo-inverse, as in Chen et al. (2012), would arbitrarily select a solution path. We do not follow this approach here.

[10] See https://github.com/MCHatcher for the Supplementary Appendix and replication codes. One extra example provided is an RBC model with an investment constraint, as in Guerrieri and Iacoviello (2015).

The asset price q_t is determined by a difference equation with an expectation term and a control term. The control variable r_t can be interpreted as a net policy interest rate (in deviations from steady state) that operates subject to a zero lower bound, such that $\underline{r} = -(1/\beta - 1)$. Guerrieri and Iacoviello (2015) use this model to illustrate their perfect foresight solution; we therefore keep contact with a known example in the literature.

We pick the same parameter values as in Guerrieri and Iacoviello (2015): $\beta = 0.99$, $\sigma = 5$, $\phi = 0.2$, $\rho = \rho_u = 0.5$. We first computed the 'M matrix' of impulse responses to r_t (see Section 3.1); note that if the M matrix is a P-matrix then multiple solutions are ruled out for that value of T and for all smaller values. We set $T = 1,000$ and found that the M-matrix is a P-matrix.[11] Given this result, and our use of short sequences of non-zero news shocks (see below), we can be confident that the model has a unique perfect foresight solution.

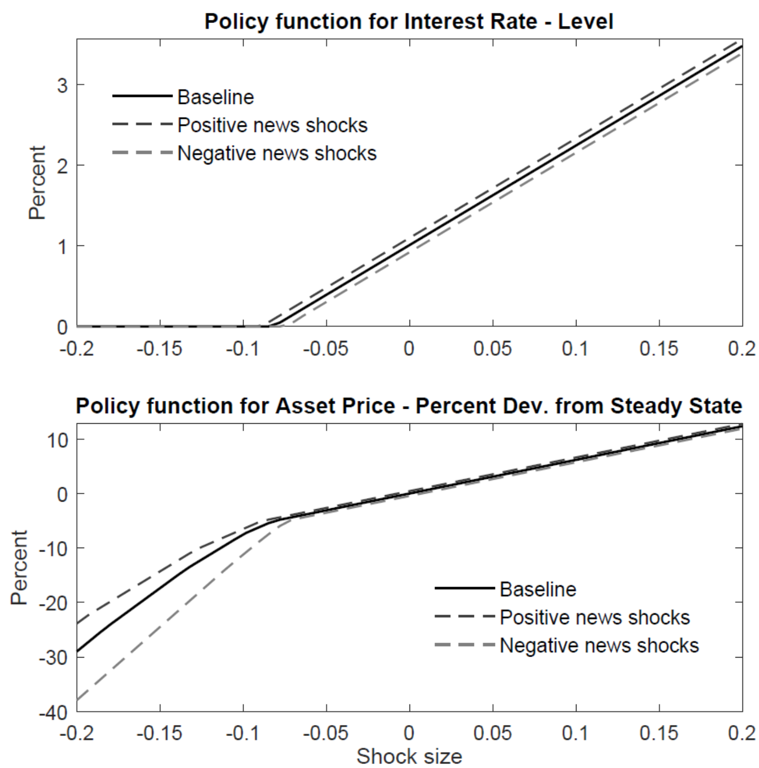

Figure 3: Policy functions for various shock sizes e_t when $q_{t-1} = u_{t-1} = 0$. In the baseline case all future (anticipated) shocks are set at 0. In the positive (negative) news case the news shocks are $e_s = 0.02$ ($e_s = -0.02$) for $s = t+1, ..., t+4$ and zero otherwise.

In Figure 3 we plot the policy functions of q_t and r_t for initial states $q_{t-1} = u_{t-1} = 0$, various values of e_t, and three different news shock sequences $(e_s)_{s=t+1}^{t+4}$, including the case of

[11]In this example it is not computationally intensive to verify that M is a P-matrix because M is a general positive definite matrix, which ensures that M is a P-matrix (see Holden, 2022, Appendix: Lemma 1). This positive-definiteness pre-check appears in our code such that unnecessary computation is avoided.

zero news shocks studied by Guerrieri and Iacoviello (i.e. $e_s = 0$ for all $s > t$). In the other two cases, we consider sequences of either all positive or all negative news shocks that last four periods; in particular, $e_s = \pm 0.02$ for $s = t+1, ..., t+4$ and $e_s = 0$ for all $s > t+4$. One motivation for this exercise is see how 'symmetric' the policy function is to positive versus negative sequences of news shocks of equal magnitude. Each policy function is computed at 60 different points, i.e. for 60 values of e_t in the interval $[-0.2, 0.2]$. The computation of each policy function takes Matlab around 0.7 seconds.[12]

The baseline policy function in Figure 3 (solid line) replicates of Guerrieri and Iacoviello (2015, Fig. 1). In the positive shocks scenario, agents receive 'good' news about future asset prices; as a result, the asset price increases through the expectations channel (lower panel). The impact of positive news is visible when the current shock is negative enough that the lower bound on the interest rate is binding (upper panel), as positive news ensures a faster escape from the bound, which raises the current asset price somewhat. In the case of negative news shocks, the expectational drag on asset prices increases the number of periods spent at the lower bound, such that asset prices fall compared to the zero news scenario.

Figure 4 plots the perfect foresight paths for $e_t = -0.1$ and all parameters unchanged.

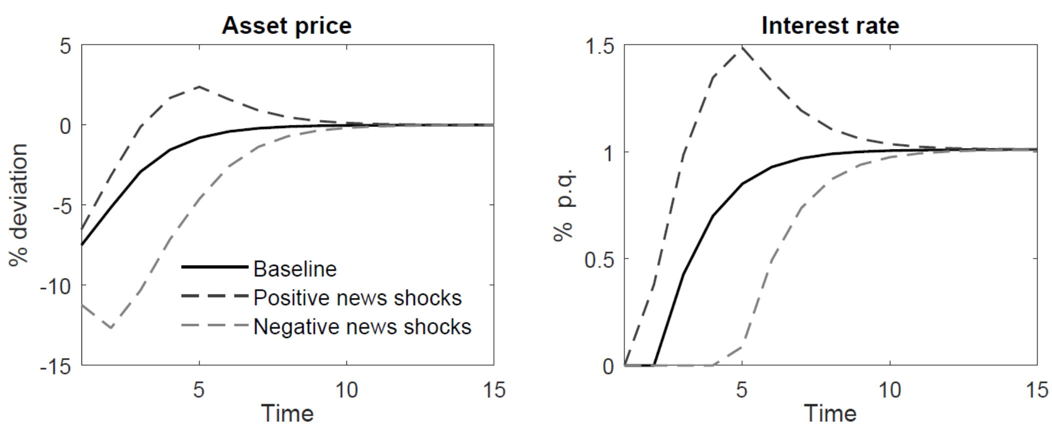

Figure 4: Perfect foresight solutions for different news shocks: $e_1 = -0.1$, $q_0 = u_0 = 0$. In the baseline case, all future (anticipated) shocks are set at 0. In the positive (negative) news case the news shocks are $e_t = 0.02$ ($e_t = -0.02$) for $t = 1, ..., 4$ and zero otherwise.

Time spent at the lower bound is clearly shorter the more positive news shocks are; for example, with positive news shocks, the interest rate is at the bound for only 1 period, compared to 2 periods in the case of zero news shocks, and 4 periods for positive news shocks (see right panel). Due to the difference in 'escape dates', the asset price responses on impact are quite different, as are the subsequent trajectories of the asset price; in fact, we see a hump-shaped response in the case of positive news shocks, whereas for negative news shocks

[12]The simulations were run in Matlab 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

we see U-shaped response of the asset price (left panel). The solutions for q and r in period 1 correspond to the policy functions in Figure 3 evaluated at the point $e_t = -0.1$.

4.2 A New Keynesian model

As a second application we consider a New Keynesian model which can exhibit multiple perfect foresight solutions, as in Brendon et al. (2013). The model has a zero lower bound on nominal interest rates, and the only other departure from the benchmark model is that monetary policy responds to the *change* in the output gap, similar to the 'speed limit' policies which Walsh (2003) considers on both theoretical and practical grounds:[13]

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{22}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) \tag{23}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{24}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{25}$$

where $\theta_\pi > 1$, $\beta \in (0,1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0,1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

4.2.1 Baseline analysis

We start by setting parameters at $\beta = 0.99$, $\sigma = 1$, $\rho_i = 0$ (no interest rate smoothing) and $\kappa = \frac{(1-0.85)(1-0.85\beta)}{0.85}(2+\sigma)$ as in Brendon et al. (2013); additionally, we set $\theta_\pi = 1.5$ and $\theta_{\Delta y} = 1.6$ to replicate the exercise in Holden (2022, Appendix E). Starting at steady state, we hit the economy with a 1% demand shock at date 1 (i.e. $e_1 = 0.01$) and search for perfect foresight solutions to the model (22)–(25) using our algorithm. We plot the solution paths in Figure 5: as expected, these solution paths replicate the results reported by Holden.

There are two perfect foresight solutions in Figure 5: one where the lower bound is never hit and both inflation and the output gap rise marginally above their steady-state values; and a second solution where interest rates are at the lower bound in the first two periods and there is strong and persistent deflation and negative output gaps. Intuitively, the latter 'bad' solution arises due to *self-fulfilling* expectations: if agents expect low inflation, then the rise in real rates lowers the output gap and inflation, validating the expectations. A strong response to the *change* in the output gap is important for this result since this means that the shadow interest rate is less expansionary over time than with a target output gap of zero. The solution that hits the bound for two periods is clearly inferior in terms of stabilization of

[13] Walsh (2003) shows that speed limit policies have some theoretical advantages in New Keynesian models, related to the introduction of history dependence, as well as a practical advantage for real-time policymaking: reduced mis-measurement of the first-difference of the output gap versus the output gap in levels.

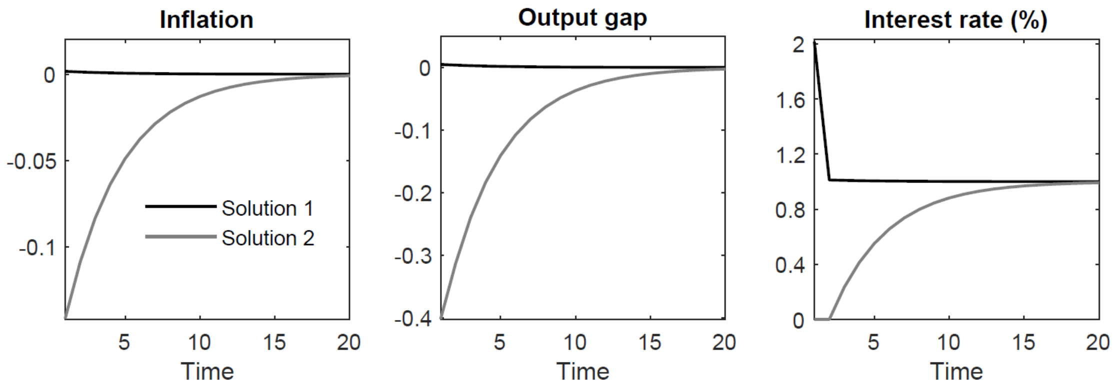

Figure 5: Multiple equilibria in the Brendon et al. model: $e_1 = 0.01$ and $i_0^* = y_0 = 0$

inflation and the output gap. We therefore study some alternative monetary policies below, to see if they can restore uniqueness by eliminating the 'bad' solution. Before doing so, we first confirm that multiplicity is a robust feature of this model.

We start by plotting some parameter regions for which the M matrix of impulse responses (see Remark 2) is a P matrix and is *not* a P matrix; see Figure 6.[14] Recall that there is a unique solution for all initial conditions if the M matrix is a P-matrix. We set $T = 2$ in Figure 6 and plot the regions for which the M matrix is a P-matrix (white), and is not a P-matrix (non-uniqueness region, black). We consider different combinations of the response coefficients θ_π, $\theta_{\Delta y}$ in the interest rate rule and we provide separate plots for three different values of the inverse elasticity of intertemporal substitution, σ.

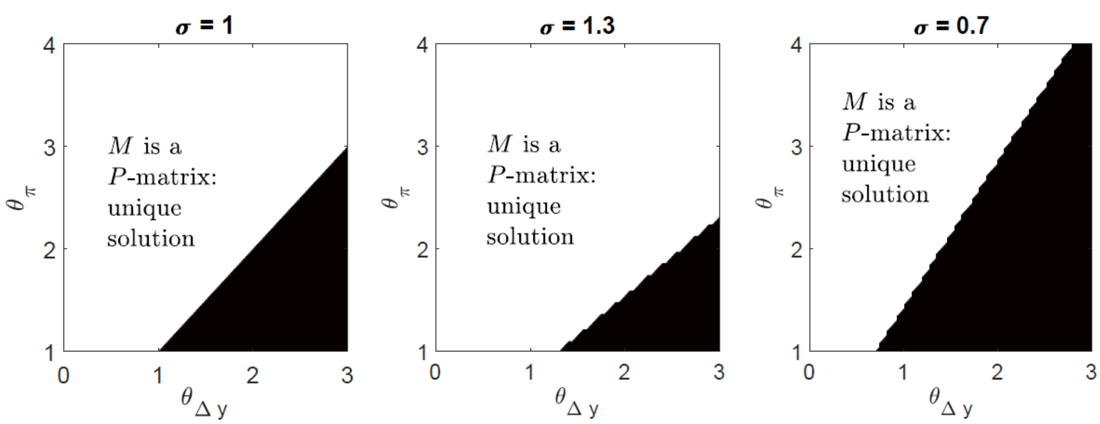

Figure 6: Regions in which M is not a P-matrix (black) when $T = 2$

Figure 6 shows that there is a unique solution if the response to the change in the output

[14]We check if M is a P-matrix using the recursive test in Tsatsomeros and Li (2000).

gap, $\theta_{\Delta y}$, is not too strong relative to the inflation response θ_π; see the white region to the left of the 'diagonal' in Figure 6. In the first panel, which uses the baseline value of $\sigma = 1$, we see that M is a P-matrix only if the response to the change in the output gap is *smaller* than the response coefficient on inflation. Note that the parameter values used in Figure 5 ($\theta_\pi = 1.5, \theta_{\Delta y} = 1.6$), where there are two solutions, lie in the 'non-uniqueness' region as expected. In fact, Brendon et al. (2013, Proposition 1) show that the model (22)–(25) has self-fulfilling equilibria where the lower bound is hit in the initial period if and only if $\theta_{\Delta y} > \sigma\theta_\pi$, while Holden (2022, Appendix E, Proposition 12) shows in a similar vein that the M matrix of this model is a P-matrix for $T = 1$ only if $\theta_{\Delta y} < \sigma\theta_\pi$. Figure 6 suggests this conclusion holds for $T = 2$, and we found the same conclusion for higher values of T.

In summary, the result of multiplicity seems quite robust in this model and this finding raises the question of whether alternative monetary policies could restore uniqueness by eliminating the bad solution. We investigate this question below while retaining the 'speed limit' aspect of the policy rule, which may have both theoretical and practical advantages as argued by Walsh (2003). We start by considering interest rate smoothing before turning to a price-level target pursued alongside the 'speed limit' target in the monetary policy rule.

4.2.2 Interest rate smoothing

Given the presence of 'good' and 'bad' solutions, we first ask whether policymakers could achieve a better outcome by smoothing the shadow interest rate in Equation (23). We start out by checking the regions where the M matrix is a P-matrix for $T = 2$, analogous to the exercise in Figure 6 except that interest rate smoothing is 'turned on'.

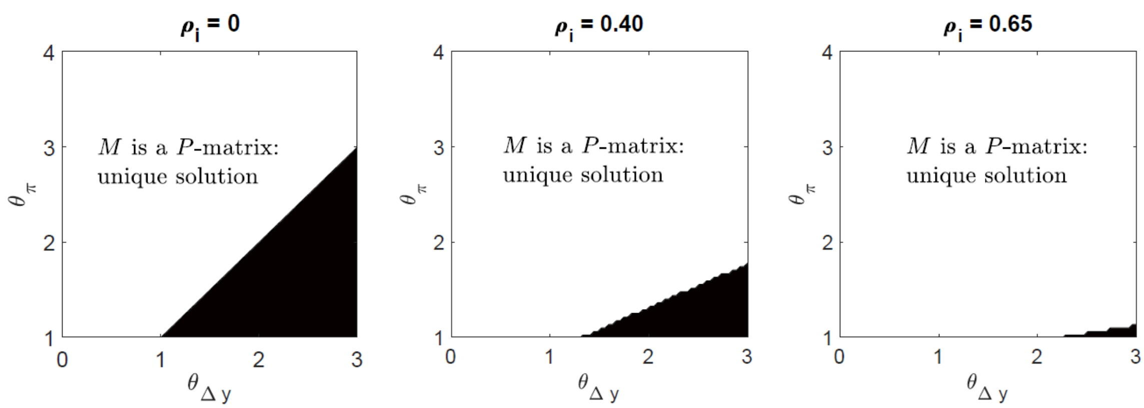

Figure 7: Regions in which M is not a P-matrix (black) for $T = 2$ and various ρ_i

The plots of the P-matrix regions in Figure 7 indicate that, *for a given* T, the determinacy region grows as the smoothing parameter ρ_i is increased. The intuition is quite simple: persistence in the shadow interest rate makes it *harder* to induce a lower bound episode over short horizons – i.e. for low values of T. However, we find the same result noted by Holden

(2022, Appendix E) *as T is increased*: the P-matrix regions under interest rate smoothing (see Figure 7) tend to those in the model *without* any smoothing, such that *multiplicity remains a widespread problem* and there are both 'good' and 'bad' equilibria.

The finding that multiplicity remains intact is quite robust, but multiplicity appears to be *absent for large enough values of ρ_i*. In other words, if the shadow interest rate is sufficiently inertial, then determinacy seems to be restored and the 'bad' equilibria are ruled out. For the parameter values that we consider, this result seems to be robust for smoothing parameters of around $\rho_i = 0.8$ or higher.[15] Some intuition for this result comes from scaling the response coefficients in (23) by $\frac{1}{1-\rho_i}$ and letting $\rho_i \to 1$. In this case, the shadow interest rate tends to $\Delta i_t^* = \theta_\pi \pi_t + \theta_{\Delta y} \Delta y_t$, which is consistent with any rule of the form $i_t^* = constant + \theta_\pi p_t + \theta_{\Delta y} y_t$, where $p_t = \pi_t + p_{t-1}$ is the log price level. The latter is a price-level targeting rule *without* any speed limit term. Since Holden (2022, Appendix E) concludes that such a rule restores determinacy in this model, it is intuitive that sufficiently high values of ρ_i in the original policy rule give the same conclusion.

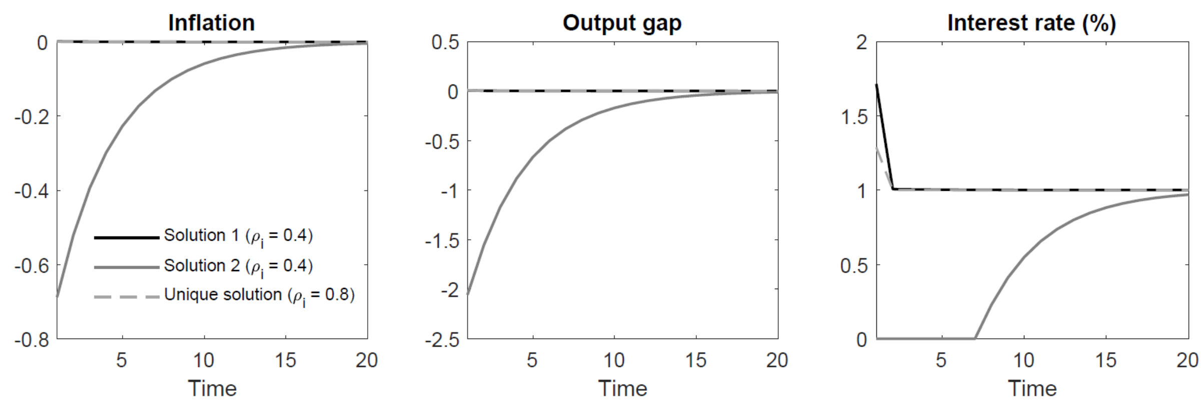

Figure 8: Perfect foresight solutions with interest rate smoothing: $e_1 = 0.01$ and $i_0^* = y_0 = 0$

In Figure 8 we consider a numerical simulation. We set $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$ and $e_1 = 0.01$ as in the baseline simulation in Figure 5; the only difference is that the interest rate smoothing parameter is set at either $\rho_i = 0.40$ (weak smoothing) or a high value $\rho_i = 0.80$ (strong smoothing). Recall that we found that $\rho_i = 0.40$ does not prevent multiplicity (once we consider appropriately large values of T), whereas for $\rho_i = 0.80$ we found a unique solution for all parameter values considered (i.e. the entire parameter region, not shown, was white).

With moderate interest rate smoothing ($\rho_i = 0.4$) there are two solutions, and the 'bad' solution is *exacerbated* relative to Figure 5: both inflation and the output gap fall by around *5 times* as much on impact because interest rates now spend 7 periods at the zero lower

[15] The computational burden of checking whether M is a P-matrix for very large T means that our results are strongly suggestive rather than conclusive. As a second check, we also studied some individual perfect foresight simulations for uniqueness (with affirmative results) for values of T up to 5,000.

bound, rather than 2 periods. Intuitively, pessimistic expectations can be self-fulfilling with interest rate smoothing since the shadow rate is not too expansionary after escaping the bound. Nevertheless, we see a unique solution for $\rho_i = 0.80$; the intuition seems to be that policy will remain expansionary for so long that pessimistic expectations cannot be justified via cuts in interest rates down to the zero lower bound.

In short, interest rate smoothing does not, in general, prevent the occurrence of multiple equilibria in the above model, and when the 'bad' equilibrium is present this policy rule can *worsen* destabilization of inflation and output due to pessimistic expectations. At the same time, we saw that highly inertial interest rate rules eliminate the bad solution.

4.2.3 Price-level targeting with a speed limit

We now consider a price-level targeting interest rate rule. We are motivated here by work showing that price-level targeting interest rate rules can mitigate or resolve indeterminacies in New Keynesian models (Giannoni, 2014; Holden, 2022). Our query is whether a response to the price level is *sufficient* to restore determinacy when maintaining the 'speed limit' term in the interest rate rule. We consider the latter to be important given the potential advantages of speed limit policies highlighted by Walsh (2003) and the necessity of an interest rate response to Δy_t for multiple solutions; see Figure 7 and Holden's analysis of the baseline New Keynesian model (with a zero lower bound) in which the shadow rate responds to y_t.

Accordingly, we assume that the shadow interest rate is given by

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)\left(\theta_p p_t + \theta_{\Delta y}(y_t - y_{t-1})\right) \tag{26}$$

where $\theta_p > 0$ and $p_t := \pi_t + p_{t-1}$ is the log price level.

Differently from the rule considered in Holden (2022, Appendix E), the shadow interest rate still responds to the *change* in output gap. We first consider the implications for uniqueness across a wide range of parameter values; see Figure 9.

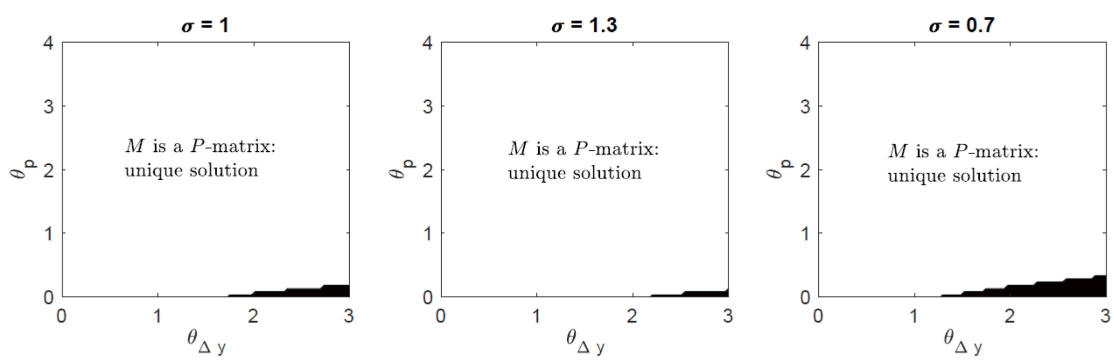

Figure 9: Regions in which M is not a P-matrix (black): price-level targeting rule when $T = 2$. In the figure, θ_p is the (long run) response coefficient on the log price level.

Figure 9 shows that indeterminacy arises only when a strong response to the speed limit term is combined with a relatively weak response to the price level (black region). Notably, for the *same* numerical reaction coefficients as in Figures 6 and 7, there is always a unique solution with the price-level targeting rule. Thus, our results suggest that *a non-trivial interest rate response to the price level is sufficient to restore determinacy.*

To illustrate the implications for inflation and output, in Figure 10 we plot some perfect foresight solutions for selected parameter values. Here, there is a relatively strong response to the price level and hence there is a unique solution in all cases, which corresponds to the 'good' solution for which the zero lower bound is not hit. Intuitively, equilibria associated with self-fulfilling pessimistic expectations are avoided because if the monetary policy rule is highly expansionary in response to pessimistic expectations, then such expectations cannot be validated as rational. Further, we see that the result of uniqueness is not overturned if we add modest or strong interest rate smoothing in the shadow interest rate.

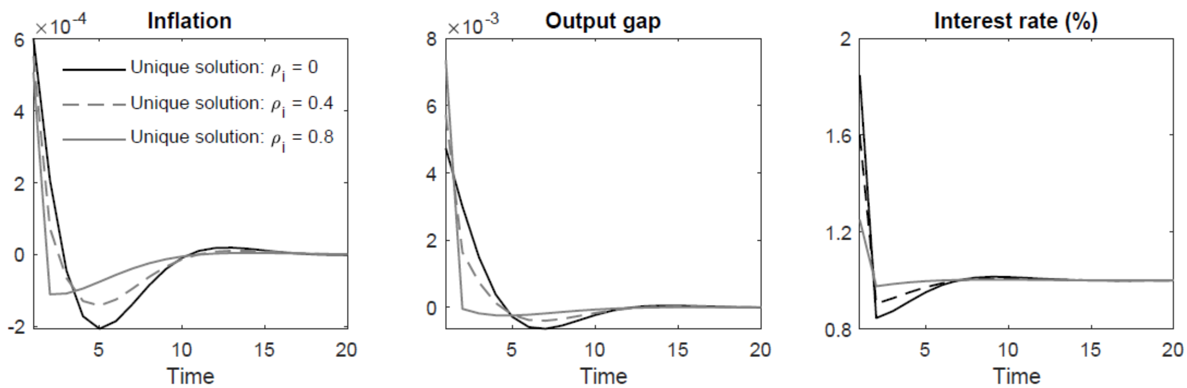

Figure 10: Unique perfect foresight paths under price-level targeting for $e_1 = 0.01$ and $i_0^* = y_0 = 0$ when $\sigma = 1$, $\theta_p = 1.5$ and $\theta_{\Delta y} = 1.6$: various values of ρ_i

Interestingly, when the reaction coefficient on the price level θ_p is small enough that both solutions exist, we see that the 'bad' solution under price-level targeting looks somewhat 'better' than under inflation targeting either with or without interest rate smoothing: the initial drops in inflation and output are much smaller than under the conventional policies. We give an example in Figure 11. Here we see that inflation, output and interest rates oscillate in a cyclical fashion around the 'good' solution, albeit within quite a narrow range. These two features – that the 'bad' solution is far less deflationary and displays cycles – seem to be quite robust, as shown in Section 3.3 of the *Supplementary Appendix*.

Notably, our conclusions on price-level targeting are less positive than in Holden (2022), where analytical and numerical results show that a price-level targeting rule ensures determinacy in a range of New Keynesian models. What our results highlight is that the assumption that the price-targeting rule responds to the *level* of the output gap, rather than the *change*

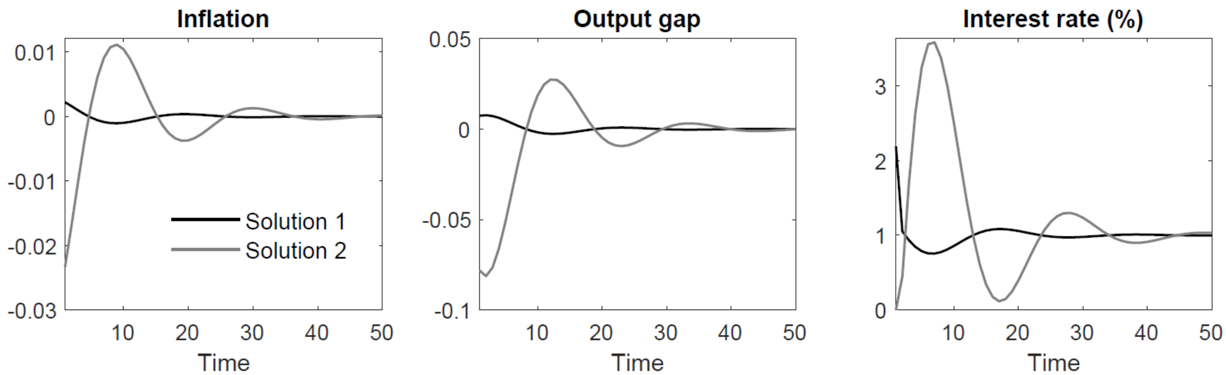

Figure 11: 'Good' and 'bad' solutions under price-level targeting for $e_1 = 0.01$, $i_0^* = y_0 = 0$ and a 'weak' response to the price level $\theta_p = 0.015$ when $\sigma = 1$ and $\theta_{\Delta y} = 1.6$

in the output gap, is crucial. In the present model, where the 'speed limit' $y_t - y_{t-1}$ enters the interest rate rule, determinacy requires a *sufficiently strong* response to the price level relative to the coefficient on the speed limit term, as shown in Figure 9.

In short, a price-level targeting rule is not sufficient to guarantee determinacy in New Keynesian models, *if* we allow such targets to be used alongside a response to the 'speed limit' when setting interest rates. Our results do, however, suggest that a *well-designed* price-level targeting rule – with a sufficiently strong response to the price level – will restore determinacy. Furthermore, price-level targeting has a degree of robustness in the sense that even when the 'bad solution' exists, the resulting destabilization of the output gap and inflation is somewhat less than under conventional rules with an inflation target and interest rate smoothing. These are the main policy conclusions that emerge from our analysis.

5 Conclusion

In this paper we presented an extension of the guess-verify algorithm in Guerrieri and Iacoviello (2015) for solving otherwise-linear *perfect foresight* models with occasionally-binding constraints . Our algorithm makes it easy for researchers to detect and simulate models with multiple equilibria and it also permits arbitrarily long (but finite) sequences of 'news shocks'. In cases of multiplicity, the selection of an equilibrium is based on a 'sunspot' and prior probabilities which are specified by the researcher. In addition, drawing on the results in Holden (2022) we also showed how, in the context of our algorithm, researchers can check whether multiple perfect foresight equilibria are ruled out in a given model.

We illustrated our algorithm through worked examples and two applications based on known models in the literature. The first application studied the policy function and impulse responses of an asset pricing model with a truncated feedback rule, as in Guerrieri and Iacoviello (2015). Here we highlighted the strong asymmetric effects of positive versus negative sequences of news shocks on the perfect foresight paths of the asset price and interest rates due to the different lengths of episodes at the bound in these cases.

Our second application studied a New Keynesian model with a zero lower bound and a (shadow) interest rate rule that responds to the *change* in the output gap – i.e. the 'speed limit'. We first confirmed that the model has two perfect foresight solutions for some parameter values, as shown by Brendon et al. (2013). One of these is a 'bad' solution for which self-fulfilling pessimistic expectations drive down inflation and the output gap. Whereas multiplicity arises for a wide range of parameter values with an inflation targeting rule that allows interest rate smoothing, replacing the inflation target with a price-level target substantially reduces the indeterminacy region. Our results suggest a simple rule-of-thumb: a strong enough interest rate response to the price level avoids indeterminacy by eliminating the bad solution. Further, price-level targeting is quite robust in the sense that when the 'bad solution' does exist, it is not very strongly deflationary, in contrast to the inflation targeting rule with or without interest rate smoothing.

The above results highlight some potential uses of our algorithm, such as policy analysis, and suggest some interesting avenues for future research.

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Appendix

Proof of Proposition 1

By assumption, a solution exists and the model returns permanently to the reference regime after some finite date $T \geq 1$ (see Assumption 2). Hence, the system to be solved $\forall t \geq 1$ is:

$$B_{1,t}x_t = B_{2,t}x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \quad \text{s.t.} \quad x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \text{ for } t = 1, ..., T \tag{A1}$$

where $B_{j,t} := \mathbb{1}_t \overline{B}_j + (1 - \mathbb{1}_t)\tilde{B}_j \ \forall j \in [5]$ and $\forall t > T$, $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$, $B_{j,t} = \overline{B}_j$, $e_t = 0_{m\times 1}$.

Consider first the periods $1 \leq t \leq T$. Suppose there exist a set of well-defined matrices $\{\Omega_t, \Gamma_t, \Psi_t\}$ such that $x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t$. Shifting this equation forward one period:

$$x_{t+1} = \Omega_{t+1}x_t + \Gamma_{t+1}e_{t+1} + \Psi_{t+1}, \quad 1 \leq t \leq T - 1. \tag{A2}$$

Substituting (A2) into (A1) and rearranging gives, for all $t \in \{1, ..., T - 1\}$,

$$(B_{1,t} - B_{2,t}\Omega_{t+1})x_t = B_{3,t}x_{t-1} + B_{4,t}e_t + B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}. \tag{A3}$$

Provided $\Omega_T, \Gamma_T, \Psi_T$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$, the set $\{\Omega_t, \Gamma_t, \Psi_t\}$ is well-defined for t where these matrices follow Proposition 1. Therefore, if $\Omega_T, \Gamma_T, \Psi_T$ well-defined and $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0 \ \forall t < T$, then $\Omega_t, \Gamma_t, \Psi_t$ are well-defined for $t = 1, ..., T$.

For $t > T$, we have by Assumption 2, $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$ where $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ and $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5)$. Hence, $x_{t+1} = \overline{\Omega} x_t + \overline{\Psi}$, $\forall t \geq T$. Matrices $\Omega_T, \Gamma_T, \Psi_T$ are determined by the first line of (A1) and the previous equation at date $t = T$:

$$B_{1,T}x_T = B_{2,T}x_{T+1} + B_{3,T}x_{T-1} + B_{4,T}e_T + B_{5,T}, \qquad x_{T+1} = \overline{\Omega} x_T + \overline{\Psi}$$

or $(B_{1,T} - B_{2,T}\overline{\Omega})x_T = B_{3,T}x_{T-1} + B_{4,T}e_T + B_{2,T}\overline{\Psi} + B_{5,T}$. Provided $\det[B_{1,T} - B_{2,T}\overline{\Omega}] \neq 0$, the matrices $\Omega_T, \Gamma_T, \Psi_T$ are given by the expressions in Proposition 1.

For the time path $(x_t)_{t=1}^T$ to satisfy the constraint $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \ \forall t \in \{1, ..., T\}$, the guessed structure $(\mathbb{1}_t)_{t=1}^T$ must be verified at all dates. Consider first date $t = 1$. If $\mathbb{1}_t = 1$, then $x_{1,t} = x^*_{1,t}$, which satisfies $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ if and only if $x^*_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$, which is equivalent to $x^*_{1,t}|_{\mathbb{1}_t=1} \geq \underline{x}_1$. On the other hand, if $\mathbb{1}_t = 0$, then $x_{1,t} = \underline{x}_1$, which satisfies $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ if and only if $\underline{x}_1 = \max\{\underline{x}_1, x^*_{1,t}\}$, which is equivalent to $x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1$.

Hence, the guess at $t = 1$ is verified only if either $\mathbb{1}_t = 1$ and $x^*_{1,t}|_{\mathbb{1}_t=1} \geq \underline{x}_1$ *or* $\mathbb{1}_t = 0$ and $x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1$, which is equivalent to:

$$B_{j,t} = \begin{cases} \overline{B}_j & \text{if and only if } \mathbb{1}_t = 1 \text{ and } x^*_{1,t}|_{\mathbb{1}_t=1} \geq \underline{x}_1 \\ \tilde{B}_j & \text{if and only if } \mathbb{1}_t = 0 \text{ and } x^*_{1,t}|_{\mathbb{1}_t=0} \leq \underline{x}_1 \end{cases}, \quad \text{for } j = 1, ..., 5. \tag{*}$$

By analogous arguments, the guessed structure for each subsequent t is verified if and only if condition (*) holds for this particular t. Hence, the guessed structure $(\mathbb{1}_t)_{t=1}^{T}$ is verified for $t = 1, ..., T$ if and only if (*) holds for all $t \in \{1, ..., T\}$. ■

Proof of Corollary 1

If the bound is not imposed, then by Proposition 1 we have $B_{j,t} = \overline{B}_j$ for all $j \in \{1, ..., 5\}$ and all $t \geq 1$; this implies by Assumption 2 that there is a unique solution of the form:

$$x_t = \overline{\Omega} x_{t-1} + \overline{\Gamma} e_t + \Psi_t, \quad \forall t \geq 1 \tag{A4}$$

where $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$, $\overline{\Gamma} := (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_4$ and

$$\Psi_t = \begin{cases} (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B_2}(\Psi_{t+1} + \overline{\Gamma} e_{t+1}) + \overline{B}_5) & \text{for } 1 \leq t \leq T-1 \\ \overline{\Psi} & \text{for } t \geq T \end{cases} \tag{A5}$$

with $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5) = (I_n - \overline{\Omega})\overline{x}$ (see Assumption 2).

Consider the initial period $t = 1$. By (A4), $x_1 = \hat{F}_1 x_0 + \hat{G}_1 e_1 + \hat{H}_1$ with $\hat{F}_1 = \overline{\Omega}$, $\hat{G}_1 = \overline{\Gamma}$, $\hat{H}_1 = \Psi_1$. For $t \geq 2$, suppose there exist matrices $(\hat{F}_t, \hat{G}_t, \hat{H}_t)_{t\geq 2}$ such that $x_t = \hat{F}_t x_0 + \hat{G}_t e_1 + \hat{H}_t$ $\forall t \geq 1$. Lagging one period gives $x_{t-1} = \hat{F}_{t-1} x_0 + \hat{G}_{t-1} e_1 + \hat{H}_{t-1}$ $\forall t \geq 2$. Inserting this expression into (A4) gives $x_t = \overline{\Omega}\hat{F}_{t-1} x_0 + \overline{\Omega}\hat{G}_{t-1} e_1 + \overline{\Omega}\hat{H}_{t-1} + \Psi_t$, $\forall t \geq 2$, which establishes the equivalences $\hat{F}_t = \overline{\Omega}\hat{F}_{t-1}$, $\hat{G}_t = \overline{\Omega}\hat{G}_{t-1}$, $\hat{H}_t = \overline{\Omega}\hat{H}_{t-1} + \Psi_t$ for all $t \geq 2$. These recursions are well-defined and unique since $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$ have these properties (see (A5)).

Part (i). Under our proposed solution $x_t = \hat{F}_t x_0 + \hat{G}_t e_1 + \hat{H}_t$, $\forall t \geq 1$, the path of the bounded variable is $x_{1,t} = \hat{F}_t^{(1)} x_0 + \hat{G}_t^{(1)} e_1 + \hat{H}_t^{(1)}$, where a superscript (1) denotes the first row of a matrix. This solution is away from the bound for all $t \in \{1, ..., T\}$ if and only if $x_{1,t} > \underline{x}_1$ $\forall t \in \{1, ..., T\}$, i.e. if and only if $\min_{t\in\{1,...T\}}\{\hat{F}_t^{(1)} x_0 + \hat{G}_t^{(1)} e_1 + \hat{H}_t^{(1)}\} > \underline{x}_1$. Finally, note that if the latter condition holds, then our guessed solution is also verified.

Part (ii). Our proposed solution in this case is the same as in Part (i) above, but it will be at the bound in some period(s) if and only if $x_{1,t} = \underline{x}_1$ for some $t \in \{1, ..., T\}$, implying that $\min_{t\in\{1,...T\}}\{\hat{F}_t^{(1)} x_0 + \hat{G}_t^{(1)} e_1 + \hat{H}_t^{(1)}\} = \underline{x}_1$, which satisfies the lower bound constraint.

Part (iii). If $\min_{t\in\{1,...T\}}\{\hat{F}_t^{(1)} x_0 + \hat{G}_t^{(1)} e_1 + \hat{H}_t^{(1)}\} < \underline{x}_1$, then the bound is violated in some period(s) $t \in \{1, ..., T\}$ and the proposed solution $\hat{F}_t x_0 + \hat{G}_t e_1 + \hat{H}_t$ is rejected. However, since we have assumed that M is a P-matrix, it follows that if the invertibility conditions in Proposition 1 hold, then there exists a unique solution x_t that differs from the above. Such a solution must hit the bound, with the constraint binding in one or more periods $t \in \{1, ..., T\}$, because we have just ruled out the path above with $\mathbb{1}_t = 1$ for all t. ■