Simulating multiple equilibria in rational expectations models with occasionally-binding constraints: An algorithm and a policy application

Michael Hatcher,[*] December 2023

Abstract

This paper presents an algorithm for simulating multiple equilibria in otherwise-linear models with occasionally-binding constraints. Our algorithm extends the guess-verify approach of Guerrieri and Iacoviello (2015) to detect and simulate multiple perfect foresight equilibria, and allows arbitrary 'news shocks' up to a finite horizon. When there are multiple equilibria, we select one using a 'prior probabilities' approach and we show how to use this approach for *ex ante* analysis under the veil of ignorance and for stochastic simulations with switching between equilibria on the simulated path. We apply our algorithm to a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria, including a 'bad' solution based on self-fulfilling pessimistic expectations. A price-level targeting rule does not always eliminate the bad solution, but it expands the determinacy region substantially and improves stabilization relative to conventional interest rate policies or forward guidance.

1 Introduction

Occasionally-binding constraints, such as borrowing limits and the lower bound on nominal interest rates, introduce a stark non-linearity in economic models. As a result, standard solution methods for linear rational expectations models (Blanchard and Kahn, 1980; Binder and Pesaran, 1997; Uhlig, 1999; Sims, 2002), which assume a time-invariant structure, must be adapted to cope with such constraints. An important contribution to the literature was made by Guerrieri and Iacoviello (2015). They show how otherwise-linear rational expectations models with occasionally-binding constraints and many state variables can be solved

[*]Department of Economics, University of Southampton, SO17 1BJ, m.c.hatcher@soton.ac.uk.
For useful comments I thank Alessandro Mennuni, Serhiy Stepanchuk, Tom Holden, Michael Reiter, Andrew Gimber, Federico Di Pace, conference participants at the CEF 2023 (Nice) and MMF 2023 (Portsmouth) and seminar participants at the Bank of England and the Central Bank of Ireland.

using a guess and verify method, and they also provide a toolkit (OccBin) that implements the solution algorithm in the popular software package Dynare. When a solution exists, their algorithm finds one solution under perfect foresight and assuming zero anticipated future shocks. However, it is known that models with occasionally-binding constraints may have *multiple* perfect foresight equilibria (see Holden, 2022), and neglecting these additional solutions may have non-trivial quantitative or policy implications.

In this paper we therefore extend the Guerrieri and Iacoviello (2015) solution method so that *multiple* perfect foresight equilibria can be detected and simulated. Our algorithm also differs in allowing non-zero 'news shocks' up to a finite horizon. We show how researchers can select a perfect foresight solution (when there are many) using a 'prior probabilities' approach and we extend this approach for *ex ante* analysis and stochastic simulation using an extended path method. In the latter case, we show that switching between different equilibria can occur along a simulated path and have non-trivial implications for macroeconomic fluctuations.

The modern literature on occasionally-binding constraints began with Eggertsson and Woodford (2003) and Jung et al. (2005); they solve the benchmark New Keynesian model with a zero lower bound on nominal interest rates, the former in a version of the model with a two-state Markov process, and the latter under perfect foresight. The papers most relevant to the current paper are those which have studied *perfect foresight* solutions to generic models with occasionally-binding constraints, including the computational papers of Guerrieri and Iacoviello (2015), Holden (2016), Boehl (2022), and the theory paper by Holden (2022). Of these papers, only Holden (2016, 2022) consider multiple equilibria.

The present paper contributes to the literature in two ways. First, relative to Guerrieri and Iacoviello (2015), we extend their solution method to detect and simulate multiple equilibria, and we allow non-zero anticipated news, such as 'forward guidance' shocks. Hence, our algorithm allows a much wider range of economic scenarios to be examined. As argued by Farmer et al. (2015, p. 17), "a model with an indeterminate set of equilibria is an incomplete model." Therefore, we also show, second, how researchers can select a perfect foresight solution (from a set of multiple solutions) using a novel 'prior probabilities' approach. Our algorithm first searches for multiple perfect foresight solutions using the guess-verify approach; all solutions are then stored and, in cases of multiple equilibria, we resolve the indeterminacy by drawing a 'sunspot' that selects a particular solution at the *start of time*.

A key advantage of our approach is flexibility: researchers can specify the *probabilities* of each solution based on their 'prior beliefs', and this also enables *ex ante* analysis of scenarios or policies under the veil of ignorance. We also show how the 'prior probabilities' approach can be used – in conjunction with an extended path method – to construct stochastic simulation paths in which there can be switching between equilibria on the simulated path, and the resulting macroeconomic fluctuations can be large even if the structural shocks are small.

We provide two applications. The first is a Fisherian model with multiple equilibria: both a high-inflation and low-inflation solution exist for the same initial conditions. We show how our 'prior probabilities' approach can be used in this model to allow policymakers to evaluate expected economic outcomes under the 'veil of ignorance' (given occasionally-binding constraints), and we also report some stochastic simulations paths for this model.

Our main application studies a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria for some parameter values (see Brendon et al., 2013). Here we show that our algorithm replicates their finding of two perfect foresight equilibria: a 'good' solution for which the lower bound is not hit and a 'bad' solution for which inflation and the output gap are strongly negative due to *self-fulfilling* pessimistic expectations.

Multiplicity is widespread under a rule that includes an inflation target and the first-difference in the output gap (the 'speed limit'), and neither interest rate smoothing nor forward guidance solve the problem. However, switching the inflation target for a price-level target expands the determinacy region substantially – i.e. the 'bad solution' is eliminated for a wide range of parameter values. A policy implication is that while price-level targeting improves stabilization relative to conventional policies or forward guidance, a response to the price level is *not* (in general) sufficient to ensure determinacy in New Keynesian models with a zero lower bound, thus shedding new light on a conclusion in Holden (2022).

The paper proceeds as follows. Section 2 outlines the solution method and describes how we extend the benchmark solution method to study multiple equilibria. Section 3 provides formal details of our 'prior probabilities' approach to simulating multiple equilibria, including ex ante analysis and the construction of stochastic simulations. Section 4 presents our application to a New Keynesian model. Finally, Section 5 concludes.

2 Methodology

Consider a multivariate rational expectations model with *perfect foresight*. The model is linear aside from multiple possible regimes due to occasionally-binding constraints, and time is discrete: $t \in \mathbb{N}_+$. As in Guerrieri and Iacoviello (2015), we focus for exposition on the case of a single occasionally-binding constraint, implying that there are two regimes.[1] The reference regime (slack) is described by (1), and the alternative regime (bind) by (2):

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5 \tag{1}$$

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \tag{2}$$

where x_t is an $n \times 1$ vector of endogenous state and jump variables, E_t is the conditional expectations operator, and e_t is an $m \times 1$ vector of 'shocks' whose values are *known*. Note that serially correlated exogenous processes can be included in the vector x_t.

Matrices $\overline{B}_i, \tilde{B}_i$, $i \in [5]$, contain the model parameters. The $\overline{B}_i, \tilde{B}_i$, $i \in \{1, 2, 3\}$, are $n \times n$

[1]We discuss the case of multiple constraints in Section 6 of the Supplementary Appendix.

matrices, $\overline{B}_4, \tilde{B}_4$ are $n \times m$ matrices, and $\overline{B}_5, \tilde{B}_5$ are $n \times 1$ vectors of intercepts. As shown in Binder and Pesaran (1997), the above formulation is quite general as it can accommodate multiple leads and lags of the endogenous variables through an appropriate definition of x_t, as well as conditional expectations formed at earlier dates.

The first variable $x_{1,t}$ is subject to a lower bound constraint in all periods:

$$x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}, \qquad x^*_{1,t} := F \begin{bmatrix} x_t \\ E_t x_{t+1} \\ x_{t-1} \end{bmatrix} + G e_t + H \tag{3}$$

where $\underline{x}_1, H \in \mathbb{R}$, F is a $1 \times 3n$ vector with $f_{11} = 0$ and G is a $1 \times m$ vector.

The specification in (3) allows the constrained variable to depend on the exogenous shocks and on contemporaneous, past or future values of endogenous variables; note that an upper bound constraint can easily be accommodated.[2] Variable $x^*_{1,t}$ is the 'shadow value' of the constrained variable. The vectors F, G, H are given by the equation that describes the bounded variable when the constraint is slack; for example, in a model with a lower bound on nominal interest rates, this equation is typically a Taylor(-type) rule.

In typical applications, one of the intercept matrices may be zero, as DSGE models are usually log-linearized around a non-stochastic steady state (see Uhlig, 1999). At any given date t, the economy is either in the reference regime or alternative regime. Given mutually exclusive regimes, we introduce an *indicator variable* $\mathbb{1}_t \in \{0, 1\}$ that is equal to 1 (0) if the reference regime (alternative regime) is in place in period t. Our model (1)–(3) is then:

$$\begin{aligned} &B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 1 \\ &\text{s.t. } x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\} \end{aligned} \tag{4}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i \ \forall i \in [5]$ and $x_0 \in \mathbb{R}^n$ is given.

The information set at time t includes all current, past and future values of the endogenous and exogenous variables; note that the indicator variable $\mathbb{1}_t$ is *endogenous*. As in Guerrieri and Iacoviello (2015) and Holden (2022), we assume the model returns to the reference regime forever after some finite date $T \geq 1$ (i.e. $\mathbb{1}_t = 1 \ \forall t > T$).[3] Following Guerrieri and Iacoviello (2015), we find the sequence $(\mathbb{1}_t)_{t=1}^T$ using a guess-verify method. That is, we guess a sequence of regimes $(\mathbb{1}_t)_{t=1}^T$ and date T, and accept the resulting time path $(x_t)_{t=1}^\infty$ as a solution only if the guessed sequence of regimes is verified by the shadow variable $x^*_{1,t}$.

[2] For example, in a simple a borrowing-constraint model with $b_t \leq \bar{b}$, we have $-b_t \geq -\bar{b}$, so we can replace b_t with $-x_{1,t}$, where $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$, $\underline{x}_1 = -\bar{b}$ and $x^*_{1,t}$ is determined by the budget constraint.

[3] For models with the constraint binding at steady-state (in which case the terminal solution is at the alternative regime), it is straightforward to amend our solution algorithm by switching the role of the two regimes; for technical details, see Section 5 of the *Supplementary Appendix*.

2.1 Preliminaries

Definition 1. *A perfect foresight solution to model* (4) *is a function* $f : x_{t-1} \times e_t \rightarrow x_t$ *such that the system in* (4) *holds for all* $t \geq 1$*, given a sequence of known 'news' shocks* $(e_t)_{t=1}^{\infty}$*.*

An alternative way of characterizing a solution is in terms of a set of matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=1}^{\infty}$ that generalize the constant-coefficient decision rules of a linear rational expectations model:

$$x_t = \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t \tag{5}$$

where Ω_t is an $n \times n$ matrix, Γ_t is an $n \times m$ matrix, Ψ_t is an $n \times 1$ vector, and the t subscript indicates that the matrices are in general time-varying.

Following Guerrieri and Iacoviello (2015) and Kulish and Pagan (2017), the matrices $\Omega_t, \Gamma_t, \Psi_t$ are determined *recursively* using simple formulas. Our perfect foresight assumption implies that the solution x_t will generally depend on both current shocks e_t and anticipated future shocks $e_{t+1}, e_{t+2}, \ldots$, which enter the solution via the 'intercept' matrix Ψ_t.

There are three key requirements for the application of our algorithm:

(i) Existence of a rational expectations solution at the reference regime (terminal solution).

(ii) A series of regularity conditions $\det[B_{1,t} - B_{2,t}\Omega_{t+1}] \neq 0$ must be met for $t = 1, ..., T$, where $T + 1$ is the date at which the terminal solution is reached.

(iii) The resulting solution path x_t must satisfy the lower-bound constraint for all $t \geq 1$ and be away from the bound for all $t > T$ (terminal condition).

Requirements (i)–(iii) also apply in Guerrieri and Iacoviello (2015). The terminal solution in (i) can be found using standard methods, such as Blanchard and Kahn (1980), Binder and Pesaran (1997), Sims (2002) or Dynare (Adjemian et al., 2011), which can check if the solution is unique. Requirement (i) is necessary but not sufficient for existence of a solution; the regularity conditions in (ii) must hold and the perfect foresight path must satisfy the occasionally-binding constraint and the terminal condition in (iii); see Holden (2022).

We assume the Blanchard-Kahn conditions for uniqueness and stability are satisfied by the terminal solution, and it is away from the lower bound for all $t > T$. Formally, we have:

Assumption 1. *We assume* $\det[\overline{B}_1 - \overline{B}_2 - \overline{B}_3] \neq 0$*, such that there exists a unique steady state* $\overline{x} = (\overline{B}_1 - \overline{B}_2 - \overline{B}_3)^{-1}\overline{B}_5$ *at the reference regime. This steady state satisfies* $\overline{x}_1 \geq \underline{x}_1$*.*

Assumption 2. *For any given initial value, there is a unique stable (terminal) solution at the reference regime of the form* $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$*, where* $\overline{\Psi} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}(\overline{B}_2\overline{\Psi} + \overline{B}_5) = (I_n - \overline{\Omega})\overline{x}$*, and* $\overline{\Omega} = (\overline{B}_1 - \overline{B}_2\overline{\Omega})^{-1}\overline{B}_3$ *has eigenvalues in the unit circle, so* $x_t \rightarrow \overline{x}$ *as* $t \rightarrow \infty$*.*

Assumption 3. *Agents know all future shocks* $(e_t)_{t=1}^{\infty}$*, and* $e_t = 0_{m\times 1}$ *for all* $t > T$*.*

Assumptions 1–3 are analogous to the assumptions in Holden (2022, Supp. Appendix). Assumption 1 restricts attention to models with a unique steady state $\overline{x}$ at the reference regime that does not violate the lower bound constraint. Assumption 2 ensures that the terminal solution at the reference regime converges to this steady state. Lastly, Assumption 3 states that agents have *perfect foresight* and that news shocks 'die out' after date T.

2.2 Solution algorithm

We take date $t = 1$ as the current period. Given perfect foresight, expectations coincide with future values: $E_t[x_{t+1}] = x_{t+1}$ for all $t \geq 1$. The system to be solved is therefore:

$$\begin{cases} B_{1,t}x_t = B_{2,t}x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t}, & 1 \leq t \leq T \\ \overline{B}_1 x_t = \overline{B}_2 x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_5, & \forall t > T \end{cases} \tag{6}$$

subject to $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ for $t = 1, ..., T$.

By assumption, the reference regime holds for all $t > T$, and the terminal solution $x_t = \overline{\Omega} x_{t-1} + \overline{\Psi}$ is away from the bound. Thus, agents can use backward induction from the terminal solution $x_{T+1} = \overline{\Omega} x_T + \overline{\Psi}$ in period T, giving the following solution algorithm which uses a 'guess and verify' approach.

1. Pick a $T \geq 1$ and a simulation length $T_s > T$. Guess a sequence $(\mathbb{1}_t)_{t=1}^T$ of 0s and 1s, starting with all 1s (slack in all periods) as an initial guess. Note: $\mathbb{1}_t = 1$ for $t > T$.

2. Find the structural matrices (or 'regimes') implied by the guess:

$$B_{i,t} = \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i, \quad i \in [5]$$

   in periods $t = 1, \ldots, T_s$.

3. Compute $(x_t)_{t=1}^{T_s}$ and the shadow value of the bounded variable $(x^*_{1,t})_{t=1}^{T_s}$ via

$$x_t = \begin{cases} \Omega_t x_{t-1} + \Gamma_t e_t + \Psi_t & \text{for } 1 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Psi} & \text{for } t > T \end{cases}, \quad x^*_{1,t} = F \begin{bmatrix} x'_t & x'_{t+1} & x'_{t-1} \end{bmatrix}' + Ge_t + H$$

   where, for $t = 1, ..., T$ and initial matrices $\Omega_{T+1} = \overline{\Omega}$, $\Psi_{T+1} = \overline{\Psi}$, $\Gamma_{T+1} = 0_{n \times m}$,

$$\Omega_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{3,t}, \qquad \Gamma_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1} B_{4,t}$$

$$\Psi_t = (B_{1,t} - B_{2,t}\Omega_{t+1})^{-1}(B_{2,t}(\Psi_{t+1} + \Gamma_{t+1}e_{t+1}) + B_{5,t}).$$

4. If $x_{1,t} = \max\{\underline{x}_1, x^*_{1,t}\}$ for $t = 1, \ldots, T$ and $x_{1,t} > \underline{x}_1$ $\forall t > T$, accept the guess and store the solution $(x_t)_{t=1}^{T_s}$; else reject. Return to Step 1 and repeat for a new guess.

The above algorithm has two additions relative to Guerrieri and Iacoviello (2015). First, it allows for the possibility of *multiple* perfect foresight solutions. Hence, if a guessed structure is verified (see Step 4), then the resulting path for x_t is accepted as a perfect foresight solution and stored for later use, along with any other solutions found by repeating Steps 1–4. Second, the algorithm permits the inclusion of 'news shocks' up to a finite horizon, whereas the original algorithm sets all future shocks at *zero*. Conveniently, our solution still has the same general form as in Guerrieri and Iacoviello (2015), since the intercept matrix Ψ_t includes the next period shocks, and is determined by a recursive formula (it depends on its future value). Hence, this generalization – which can be used to model news about technology or policy announcements – comes at essentially zero computational cost.

We also present in Section 3 a method for performing model simulations when there are multiple equilibria. As argued by Farmer et al. (2015, p.17), "a model with an indeterminate set of equilibria is an incomplete model" and will need to be *closed* by some means in order to simulate or estimate the model.[4] For our case of multiple *perfect foresight* solutions, we propose to close the model by drawing a one-off 'sunspot' at the *beginning of time* which selects one of the equilibria, implying that all agents coordinate their expectations on it. Note that this approach does not contradict the perfect foresight assumption (which implies that risk is absent) because the exogenous sunspot is *not* part of the solution path itself from date 1 onwards, but only as a means of *initially selecting* one of the equilibria, such that there is a determinate (i.e. unique) perfect foresight path which can be simulated.

Our novel approach lets researchers to assign to each equilibrium a 'prior probability' which represents the probability that agents will coordinate beliefs on it (see Remark 1). Hence, if there were a bank run equilibrium and a no-run equilibrium, then a researcher may assign a low (or zero) probability to the 'run equilibrium' if they view it as somewhat implausible. At the other extreme, a researcher may take an agnostic approach by assigning equal probability to each equilibrium – i.e. 'flat priors'. Given these prior beliefs, we show how researchers or policymakers can compute *expected* outcomes, such as means or welfare losses under the 'veil of ignorance'.[5] Hence, our approach does not just select a perfect foresight equilibrium, but also allows welfare evaluation or robustness analysis.

We also extend our 'prior probabilities' approach for stochastic simulation in Section 3.4. In this case, we relax the perfect foresight assumption by allowing *unanticipated shocks* after period 1, and solve the model via an extended path method with risk ignored by agents when they form expectations. Along such simulation paths, switching between different perfect foresight solutions can occur and contribute substantially to macroeconomic fluctuations.

[4]Farmer et al. (2015) attribute the incompleteness argument to McCallum (1983), and a recent restatement of the point is given in Ascari and Mavroeidis (2022, p. 1): "structural models with no solution are incoherent, and those with multiple solutions are incomplete."

[5]Our interpretation of 'veil of ignorance' here is consistent with Rawls' 'original position' in the sense that the outcomes are viewed by policymakers (or researchers) as of date 0, i.e. before the start of time.

2.3 Implementation of the algorithm

Our underlying algorithm is built on the solution method of Guerrieri and Iacoviello (2015), i.e. a 'guess and verify' approach based on the method of undetermined coefficients. Our extension for multiple equilibria is based on the simple idea that continuing the guess-verify procedure after a solution has been found may yield additional solutions to the linear complementarity problem.[6] As in Guerrieri and Iacoviello (2015), computation of a solution requires a series of inversions of the matrix $(B_{1,t} - B_{2,t}\Omega_{t+1})$ for $t = 1, \ldots, T$, as seen in Step 3 of the Algorithm above. In case of non-invertibility, our algorithm automatically abandons the current guess and starts a new one so that computation time is not wasted.[7] Our algorithm is programmed in MATLAB and the codes are available online.

An important issue when using guess-verify is how the guessed sequences of regimes, given by $(\mathbb{1}_t)_{t=1}^{T}$, are determined. We take a pragmatic approach by trying as an initial guess the case for which the constraint is slack in all periods (see Step 1) and then trying guesses with a single spell at the lower bound, with each guess differing in the duration of the spell. We initially start these spells from date 1 (the first period), though it is straightforward to allow additional guesses with spells starting at a later date, or the case of multiple spells at the bound (as is possible, for example, in models with cyclical dynamics). Our default algorithm considers the single spell case as a baseline so that computation time is not high. An advantage of our algorithm is that it is straightforward for users to amend the guessed sequences of regimes in order to suit the application at hand.

There is also the question of when to *terminate* the guess-verify procedure. As shown by Holden (2022), if a matrix M of impulse responses of the bounded variable to the news shocks has the property of being a P-matrix, then there is a *unique solution* to the linear-complementarity problem for all initial conditions. In this case, it makes sense from a computational point of view to terminate the guess-verify procedure as soon as a solution is found (as there can be only one). We therefore build into our algorithm a pre-check as to whether, for given parameter values, a particular model has the P-matrix property, and if so, we can automatically terminate the search procedure when a solution has been found. Note that a matrix M is a P-matrix if and only if all its principal minors are positive.

It is generally computationally-intensive to check if a moderately-sized matrix M is a P-matrix, and computation time increases exponentially in the worst case (see Holden, 2016). In our algorithm, we rely on the recursive test for P-matrices in Tsatsomeros and Li (2000), which is exponential with base 2; however, to prevent running the test at times when this is not necessary, we have also built some pre-checks into our algorithm.[8]

[6]Because we use a guess-verify approach, our method can find a *finite* number of solutions (corresponding to all verified guesses), but it will not find infinitely many solutions if there is a continuum of solutions.

[7]If the invertibility conditions are not satisfied, computing a pseudo-inverse, as in Chen et al. (2012), would arbitrarily select a solution path. We do not follow this approach here.

[8]For example, one such pre-check is: if $M + M'$ is positive definite, then M is a P-matrix.

3 Simulating multiple equilibria

As argued by Holden (2022), multiplicity of equilibria is a robust feature of otherwise-linear models with occasionally-binding constraints. Therefore, it is important that solution algorithms can find multiple *perfect foresight* equilibria and construct simulation paths where multiplicity is not neglected. In this section we provide our approach to simulating multiple equilibria; we therefore assume throughout this section that there are multiple solutions.

3.1 Selection of a perfect foresight solution

Suppose that a finite number of solutions $K \geq 2$ are found using our Algorithm. To resolve the indeterminacy we suggest a simple procedure whereby the researcher specifies 'prior probabilities' $p_1, ..., p_K \in [0, 1]$, $\sum_{k=1}^{K} p_k = 1$, where p_k is the probability that agents will coordinate their expectations on equilibrium k. A realized solution is then selected by a random 'sunspot' – i.e. an exogenous process from outside the model – at the *start of time*. We index the K perfect foresight solutions by $(x_t^k)_{t=1}^{\infty}$ for $k = 1, ..., K$, and let $u_1 \sim \mathcal{U}_{(0,1)}$ be a sunspot drawn from the uniform distribution on $(0, 1)$ at date $t = 1$.

Remark 1. *Suppose there are $K \geq 2$ perfect foresight solutions. Given probabilities $p_1, \ldots, p_K$ and a random draw $u_1 \sim \mathcal{U}_{(0,1)}$, we can select a single solution at date 1 as follows:*

$$(x_t)_{t=1}^{\infty} = \begin{cases} (x_t^1)_{t=1}^{\infty} & \text{if } u_1 \in (0, p_1] \\ (x_t^2)_{t=1}^{\infty} & \text{if } u_1 \in (p_1, p_1 + p_2] \\ \vdots & \\ (x_t^K)_{t=1}^{\infty} & \text{if } u_1 > p_1 + ... + p_{K-1} \end{cases} \tag{7}$$

i.e. for any $u_1 \in (\sum_{k=0}^{k^*-1} p_k, \sum_{k=0}^{k^*} p_k]$, where $k^* \in \{1, ..., K\}$ and $p_0 := 0$, the unique (selected) perfect foresight solution is $(x_t)_{t=1}^{\infty} = (x_t^{k^*})_{t=1}^{\infty}$.

Remark 1 gives a general method for choosing a perfect foresight solution when there are many. It can be applied to any finite set of perfect foresight solutions and is flexible due to the free specification of probabilities. For instance, if the researcher thinks some solution(s) somewhat 'unrealistic' they may attach low (or zero) probability to those solution paths. On the other hand, a perfectly agnostic researcher would use 'flat priors' $p_k = 1/K$ for all k.

A code in our algorithm first stores all (found) perfect foresight solutions to a particular model and then selects a single solution using Remark 1 and some specified probabilities, akin to a lottery among perfect foresight paths, as we show in an example below. Our algorithm can therefore be used to automate the procedure of selecting an equilibrium (to some extent) while also taking prior beliefs, and their economic implications, seriously.[9]

[9] We say "automate...(to some extent)" as researchers may find it beneficial to first view the solution

3.2 Ex ante analysis

As well as resolving the indeterminacy problem (i.e. 'incompleteness'), the approach in Remark 1 allows researchers to study the *ex ante* implications of constraints that give rise to multiple perfect foresight equilibria, by computing an expectation under the *veil of ignorance*. As an example, one may evaluate the expected cost of a particular policy or friction that gives rise to multiple equilibria, as we show later in this section, by using the probabilities p_k in in Remark 1 in conjunction with the associated perfect foresight paths $(x_t^k)_{t\geq 1}$.

To make this concrete, note that given K perfect foresight solutions with assigned probabilities $p_1, \ldots, p_K$, the expected value of variable j at date t may be computed as

$$E_0[x_{j,t}] = \sum_{k=1}^{K} p_k x_{j,t}^k \tag{8}$$

where $x_{j,t}^k$ is the value of variable j at date t conditional on solution k being chosen.[10]

In a similar vein, one may compute expected welfare based, for example, on a quadratic approximation. If we let $W = g(\{x_t\})$ denote the corresponding welfare measure, then its expectation as of date 0 will have the form

$$E_0[W] = \sum_{k=1}^{K} p_k g(\{x_t^k\}) \tag{9}$$

where $\{x_t^k\}$ is the solution path from date 1 onwards conditional on solution k being chosen.

It should emphasised that expressions such as (8) and (9) also provide a basis for robustness-type analysis in the sense that the impact of different 'prior beliefs' can easily be studied and best and worst-case scenarios identified. We now give an example of a model with multiple solutions and illustrate the use of *ex ante* analysis as outlined above.

3.3 Fisherian example

Following Holden (2022, Example 2) suppose that for all $t \geq 1$ our model consists of a Taylor-type rule subject to a zero lower bound and the Fisher equation:

$$i_t = \max\{0, r + \phi\pi_t - \psi\pi_{t-1} + e_t\} \tag{10}$$

$$i_t = r + E_t\pi_{t+1} \tag{11}$$

where $\phi - \psi > 1$, $\psi > 0$, $e_1, \pi_0 \in \mathbb{R}$, $e_t = 0\ \forall t > 1$, and $r > 0$ is a fixed real interest rate. To simplify presentation, we set $\phi = 2$. The results are not specific to this case.

paths and then assign probabilities. We return to this point when discussing stochastic simulation below.

[10]Clearly, this approach generalizes straightforwardly to the vector x_t via $E_0[x_t] = \sum_{k=1}^{K} p_k x_t^k$.

There are two solutions to the model (10)–(11). The first solution is away from the bound in all periods and has the form: $\pi_1 = \omega\pi_0 - \frac{1}{\phi-\omega}e_1$, $i_1 = r + \omega\pi_1$ at $t = 1$ and $\pi_t = \omega\pi_{t-1}$, $i_t = r + \omega^2\pi_{t-1}$ for all $t > 1$, where $\omega = 1 - \sqrt{1-\psi} \in (0, 1)$. This solution is stable (inflation converges to 0 and nominal rates to r), does not violate the lower bound in period 1 provided $r + \phi\pi_1 - \psi\pi_0 + e_1 \geq 0$, and is away from the bound for all $t > 1$ if $r + \phi\pi_t - \psi\pi_{t-1} > 0$; hence this solution exists if $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$.

The second solution has the constraint binding only in period 1, i.e. $i_1 = 0$ and $\pi_t = \omega\pi_{t-1}$, $i_t = r + \omega^2\pi_{t-1}$ for all $t > 1$. Note that $i_1 = 0$ implies that $\pi_2 = -r$ by (11), so $\pi_1 = -r/\omega$ and $\pi_t = \omega\pi_{t-1} = \omega^{t-2}(-r)$ for all $t > 1$. Note that the bound binds in period 1 provided $r + \phi\pi_1 - \psi\pi_0 + e_1 \leq 0$ and is escaped thereafter if $r + \phi\pi_t - \psi\pi_{t-1} > 0$ for all $t > 1$, so we require $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$.[11] Hence, for $\pi_0 \geq -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ both solutions exist, and for $\pi_0 < -\frac{r}{\omega^2} + \frac{e_1}{\psi}$ there is no stable solution that satisfies the lower bound constraint.

Our Algorithm finds both these solutions; for details see the Supplementary Appendix. The two solutions are plotted in Figure 1, along with the shadow interest rate i_t^* in both cases. Note that Solution 1 has a positive shadow rate in all periods that coincides with the actual interest rate; hence this solution is verified and is away from the bound in all periods. By comparison, Solution 2 hits the bound in period 1 and has a negative shadow rate in this period (so the constraint binds); hence this solution is also verified as argued above.

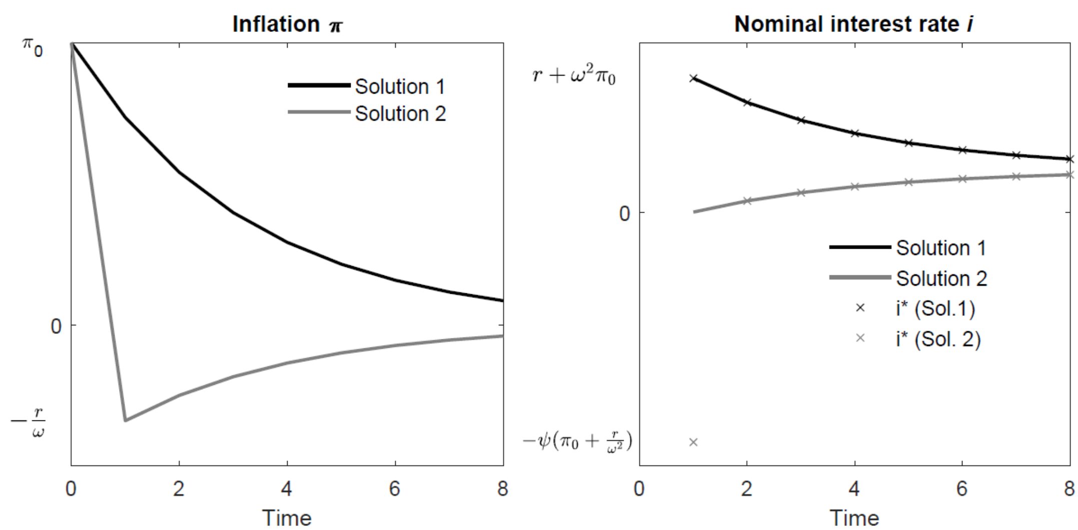

Figure 1: The two solutions when $\pi_0 > 0$ and $e_1 = 0$

We now consider *ex ante* analysis in this example. We stick for simplicity with case of *zero* monetary policy shocks as in Figure 1 ($e_t = 0$ for all t) and positive initial inflation $\pi_0 > 0$. Suppose a policymaker puts probability p_1 on Solution 1 and probability $p_2 = 1 - p_1$

[11] Note that when $i_1 = 0$ (Solution 2) we have $i_t = r + \omega^2\pi_{t-1} = [1 - (\omega)^{t-1}]r > 0$ for all $t > 1$.

on Solution 2, and has ad hoc loss function $L = \sum_{t=1}^{\infty} \beta^{t-1}\pi_t^2$ that penalizes equally positive and negative deviations of inflation from a zero target. Given the probabilities p_1, p_2, the expected loss at period 0 is $E_0[L] = p_1 L_1 + p_2 L_2 = \sum_{t=1}^{\infty} \beta^{t-1}[p_1(\pi_t^1)^2 + p_2(\pi_t^2)^2]$, where π_t^1 (π_t^2) is inflation at date t under Solution 1 (Solution 2). Given our assumption of zero shocks, $L_1 = \sum_{t=1}^{\infty} \beta^{t-1}(\pi_t^1)^2 = \frac{\omega^2}{1-\beta\omega^2}\pi_0^2$ and $L_2 = \sum_{t=1}^{\infty} \beta^{t-1}(\pi_t^2)^2 = \frac{(r/\omega)^2}{1-\beta\omega^2}$, so the expected loss

$$E_0[L] = \frac{p_1\omega^2\pi_0^2 + (1-p_1)(r/\omega)^2}{1-\beta\omega^2}$$

is linear in the probability of Solution 1, p_1, and quadratic in π_0.

In Figure 2 (left panel) we plot the expected loss as the probability of Solution 1, p_1, is increased from 0 to 1. We plot this relationship for several different values of π_0, including the initial inflation rate $\pi_0^* = r/\omega^2$ such that the losses are equal, i.e. $L_1 = L_2$.[12] For inflation rates above π_0^*, the expected loss increases with p_1; this is because relatively high initial inflation raises the loss under Solution 1 by increasing subsequent inflation, which declines geometrically from the initial value π_0 (see Figure 1). Going in the other direction, reducing initial inflation below π_0^* makes inflation under Solution 1 'closer' to the zero inflation target, such that the loss L_1 falls below L_2; note that Solution 2, for which lower bound binds in the period 1, is *independent* of the initial inflation rate, as shown in Figure 1.

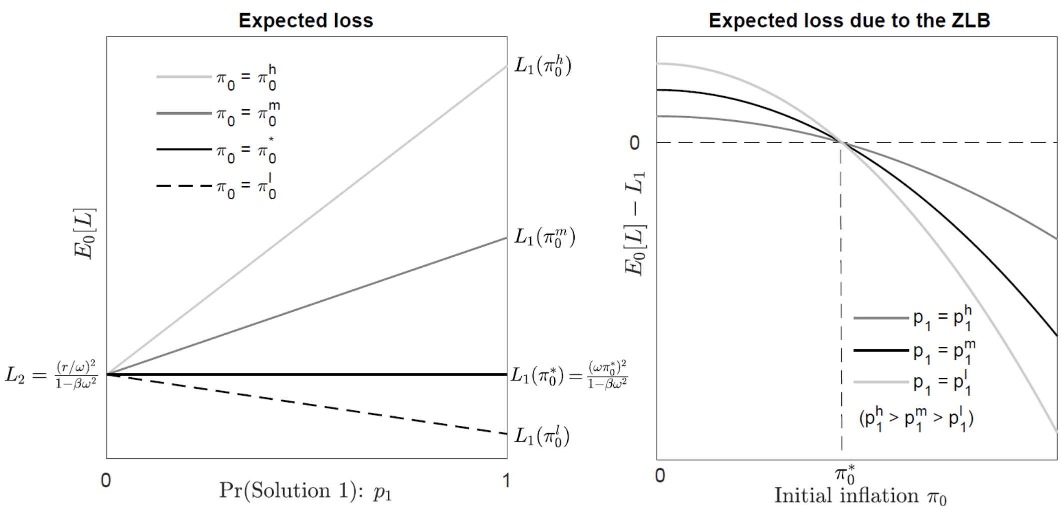

Figure 2: Expected loss $E_0[L]$ as p_1 is increased: various π_0 (left panel) and the expected loss due to the zero lower bound $E_0[L] - L_1$ as π_0 is increased, at three values of p_1 (right panel). The initial values in the left panel satisfy $\pi_0^h > \pi_0^m > \pi_0^* > \pi_0^l$, where π_0^* is the positive initial inflation that makes the loss under solution 1, $L_1(\pi_0)$, equal the loss L_2 under solution 2.

[12]In general, there are two initial inflation rates that make the losses equal, $\pi_0^* = \pm\frac{r}{\omega^2}$. However, we confine our attention to positive initial inflation in this example.

In Figure 2 (right panel) we plot the expected loss due to the zero lower bound, $E_0[L]-L_1$, which can be interpreted as a measure of the extra cost of the lower bound friction (which makes Solution 2 an equilibrium).[13] The expected loss due to the lower bound falls as initial inflation is increased, since this raises the Solution 1 loss L_1 by more than it raises the expected loss $E_0[L]$. Increasing the probability of Solution 1 'flattens' the relationship between $E_0[L]-L_1$ since the expected loss given the lower bound, $E_0[L_1]$, becomes closer to L_1 the higher the probability attached to Solution 1. While we are working with a simple model here, the results are suggestive of the type of policy and robustness analysis possible by using Remark 1 plus some welfare measure (such as a loss function or lifetime utility).

In the general case with news shocks, expected welfare will usually depend on these anticipated shocks, as well as initial values of state variables. This raises the point that under the veil of ignorance (i.e. *period 0*), the news shocks $(e_t)_{t=1}^{\infty}$ may be unknown, such that *ex ante* analysis could also be used to compute an expectation over different sequences of news shocks given by draws from some probability distribution (Monte Carlo approach). We do not pursue this point any further here, but we simply note that Remark 1 has uses that extend well beyond simple selection of a perfect foresight solution at start of time $t = 1$.

3.4 Extension: stochastic simulations

Finally, we consider *stochastic* simulations where equilibrium selection can occur each period. In this case, we assume that agents mistakenly think that they know all future shocks; as result, they have *im*perfect foresight and ignore risk, such that their expectations are not strictly rational, in contrast to the perfect foresight paths studied thus far.

Using this approach, researchers can construct stochastic simulations in which agents form expectations under certainty equivalence. Both Dynare (Adjemian et al., 2011) and OccBin (Guerrieri and Iacoviello, 2015) have built-in options for such simulations using an extended path method,[14] and an example that uses this approach (in a non-linear setting) is Christiano et al. (2015, see Section IV.D). Whereas these algorithms are typically applied to occasionally-binding constraint models with a unique solution, we present an approach for simulating stochastic paths in otherwise-linear models with *multiple equilibria*.

Our approach draws on Remark 1, but it uses that approach to select an equilibrium in every period t in which the model is simulated; note that this is necessary because while the entire solution path is known as of date 1 under perfect foresight, this is *not* true if expectations are not model-consistent, since the actual (realized) path will generally differ

[13] To motivate this exercise, we may think of an asset like central bank digital currency that could potentially remove the lower bound constraint. From a policy perspective, we would like to know *ex ante* whether it is worth providing such a currency (at some cost); this will depend at least partly on the macroeconomic benefits that come from eliminating the lower bound on nominal interest rates (as studied in our exercise).

[14] Occbin allows users to find a perfect foresight solution from date 1 onwards under the assumption of zero future shocks, which gives an initial solution x_1. The process is then repeated in period 2 (and later periods) but with a shock vector e_t drawn at each t; this is implemented via the 'stoch simul' option in Dynare.

from the one that agents expected. Hence, equilibrium selection arises every period in response to the new initial conditions x_{t-1}, e_t which agents are confronted with.

To make this concrete, suppose that at date t the shock vector e_t is drawn from some distribution. Then given e_t, x_{t-1} and agents' beliefs about future shocks, we can find the solution(s) by using our Algorithm, and one of these solution(s) can be selected by a sunspot (akin to Remark 1). The same procedure is then repeated in period $t+1$, given x_t, e_{t+1} and the agents' expected future shocks. Provided a solution exists for all simulated t, we can construct a stochastic path of desired length using this method:

1. Choose some systematic rule for assigning probabilities to different equilibria at each date, e.g. the 'flat priors' approach, $p_k = \frac{1}{\text{no. of equilibria}}$ for each solution k.[15]

2. Given x_0, e_1 and expected shocks $e_2^a, ..., e_T^a$, use the Algorithm to find the solution paths $(x_t^k)_{t=1}^T$ for $k = 1, ..., K_1$, where K_1 is the number of solutions at date 1. Use the approach in Remark 1 to select one of these solutions, $(x_t^{k_1^*})_{t=1}^T$, where $k_1^* \in \{1, ..., K_1\}$. Set $x_1 = x_1^{k_1^*}$ (our first simulated point) and move to period 2.

3. Draw vector e_2 from some distribution. Given x_1, e_2 and expected shocks $e_3^a, ..., e_{T+1}^a$, use the Algorithm to find the solution paths $(x_t^k)_{t=2}^{T+1}$ for $k = 1, ..., K_2$, where K_2 is the number of solutions in period 2. Use the approach in Remark 1 to select one of these solutions, $(x_t^{k_2^*})_{t=2}^{T+1}$, where $k_2^* \in \{1, ..., K_2\}$. Set $x_2 = x_2^{k_2^*}$ (second simulated point).

4. Repeat in periods 3,4 etc. to get a simulation path of the desired length, say $(x_t)_{t=1}^{T^*}$.

We now illustrate stochastic simulation using the Fisherian model in Section 3.3. We set $r = 0.01$, $\phi = 2$, $\psi = 0.93$, $\pi_0 = 0.02$, along with date-1 anticipated shocks $e_1 = e_2^a = -0.001$. We set the probability of selecting solution 1 (away from the bound) at $p_1 = 0.95$, so there is a 5% probability of selecting the solution at the bound. At dates $t > 1$ we solve for i_t, π_t conditional on the inherited state π_{t-1} and fresh draws for the monetary policy shocks e_t, e_{t+1} from a normal distribution with mean zero and standard deviation σ_e (see Figure 3).

We found two solutions in all simulated periods, which are variations of the 'high' and 'low' inflation solutions in Figure 1. To resolve the indeterminacy, at each date t we drew a sunspot $u_t \sim \mathcal{U}_{(0,1)}$ that selects either solution 1 (away from bound) or solution 2 (hits bound) in period t. Given our assumption that the probability of solution 1 is 95%, solution 1 (solution 2) is selected at date t if and only if $u_t \in (0, 0.95]$ ($u_t > 0.95$).

In the upper panel of Figure 3, the standard deviation of the policy shock is very small to isolate the impact of the sunspot, i.e. selection between the two equilibria. Of five simulations,

[15] In general, assigning probabilities in stochastic simulations is not straightforward since the number (and nature) of equilibria is not usually known *a priori*. Thus, researchers may either follow simple rules such as 'use flat priors' (as suggested in Step 1); 'assign low (or high) probability to solutions where the bound is hit'; or 'run a simulation first, peek at the equilibria at each t and then assign probabilities to them'.

three hit the zero lower bound in some period (see dashed lines); in these cases, we see strong deflation, in contrast to the solutions path away from the bound (cf. Figure 1). In the lower panel, the shock variance is large enough to make each stochastic simulation path discernible, but the main variations in inflation and interest rates arise from switching between multiple equilibria rather than from disturbances to interest rates set by monetary policymakers. One implication is that the average simulated value of variables, in a long simulation, may differ non-trivially from the values at the reference regime steady-state.[16]

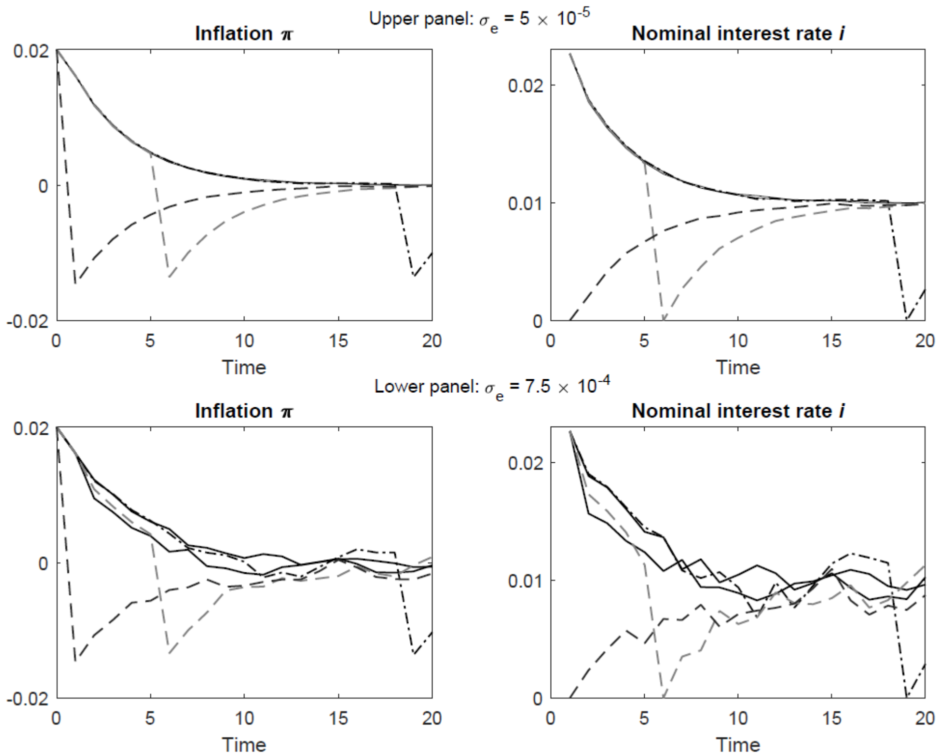

Figure 3: Five stochastic simulations: $p_1 = 0.95$ and initial values $\pi_0 = 0.02$, $e_1, e_2 = -0.001$

3.5 Existence and uniqueness

In the Fisherian example above, there are two solutions for large enough initial inflation. However, as remarked in Section 2.3, some models may have a unique solution for all initial conditions, and others no perfect foresight solution. Recall that a unique solution exists for all initial conditions if and only if the M matrix of impulse responses is a P-matrix (see Holden, 2022); otherwise, there may be multiple solutions or no solution. In the next section, which contains our main policy application, we make considerable use of this test.

[16]E.g. simulations of the above example shows that average inflation is depressed substantially by the occasional occurrences of the deflationary solution and is negative for large enough t (or p_1 small enough).

4 Policy Application

We now apply our algorithm to a New Keynesian model with a zero lower bound on nominal interest rates and multiple equilibria for some parameter values. Implementation details are provided in a *Supplementary Appendix*; we restrict attention to perfect foresight equilibria throughout, so we do not draw on Section 3.4 (stochastic simulation) at all here.[17]

4.1 A New Keynesian model

We consider the New Keynesian model studied in Brendon et al. (2013). Besidess the zero lower bound, the only other departure from the benchmark model is a policy response to the *change* in the output gap, similar to the 'speed limit' policies studied by Walsh (2003):

$$i_t = \max\{\underline{i},\ i_t^*\} \tag{12}$$

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) \tag{13}$$

$$y_t = E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + e_t \tag{14}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{15}$$

where $\theta_\pi > 1$, $\beta \in (0,1)$, $\theta_{\Delta y}, \kappa, \sigma > 0$, $\rho_i \in [0,1)$, $\underline{i} = \beta - 1$ and all values of e_t are known.

We start by setting parameters at $\beta = 0.99$, $\sigma = 1$, $\rho_i = 0$ (no interest rate smoothing) and $\kappa = \frac{(1-0.85)(1-0.85\beta)}{0.85}(2+\sigma)$ as in Brendon et al. (2013); additionally, we set $\theta_\pi = 1.5$ and $\theta_{\Delta y} = 1.6$ to replicate the exercise in Holden (2022, Appendix E). Starting at steady state, we hit the economy with a 1% demand shock at date 1 (i.e. $e_1 = 0.01$) and search for perfect foresight solutions to the model (12)–(15) using our algorithm. We plot the solution paths in Figure 4: as expected, these solution paths replicate the results reported by Holden.

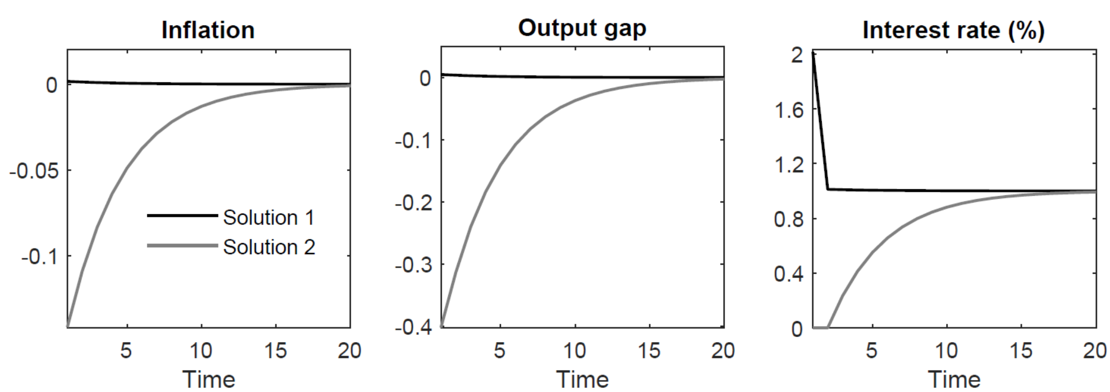

Figure 4: Multiple equilibria in the Brendon et al. model: $e_1 = 0.01$ and $i_0^* = y_0 = 0$

[17] See https://github.com/MCHatcher for the Supplementary Appendix and replication codes.

There are two perfect foresight solutions: one where the lower bound is never hit and both inflation and the output gap rise marginally above their steady-state values; and a second solution where interest rates are at the lower bound in the first two periods and there is strong and persistent deflation and large negative output gaps (Figure 4, all panels). The 'bad' solution arises due to *self-fulfilling* expectations: if agents expect low inflation, then the rise in real rates lowers the output gap and inflation, validating the expectations. A strong response to the *change* in the output gap is important for this result since this means that the shadow interest rate – responding to growth – is less expansionary than a level target at a zero output gap. The solution that hits the bound for two periods is clearly inferior in terms of stabilization of inflation and output gaps. We therefore study some alternative monetary policies below, to see if they can restore uniqueness by eliminating the 'bad' solution. Before doing so, we first confirm that multiplicity is a robust feature of this model.

We start by plotting some parameter regions for which the M matrix of impulse responses is a P matrix and is *not* a P matrix; see Figure 5.[18] Recall that there is a unique solution for all initial conditions if the M matrix is a P-matrix. We set $T = 2$ in Figure 5 and plot the regions for which the M matrix is a P-matrix (determinacy region, white), and is not a P-matrix (black). We consider different combinations of the response coefficients θ_π, $\theta_{\Delta y}$ in the interest rate rule and also vary the inverse elasticity of intertemporal substitution, σ.

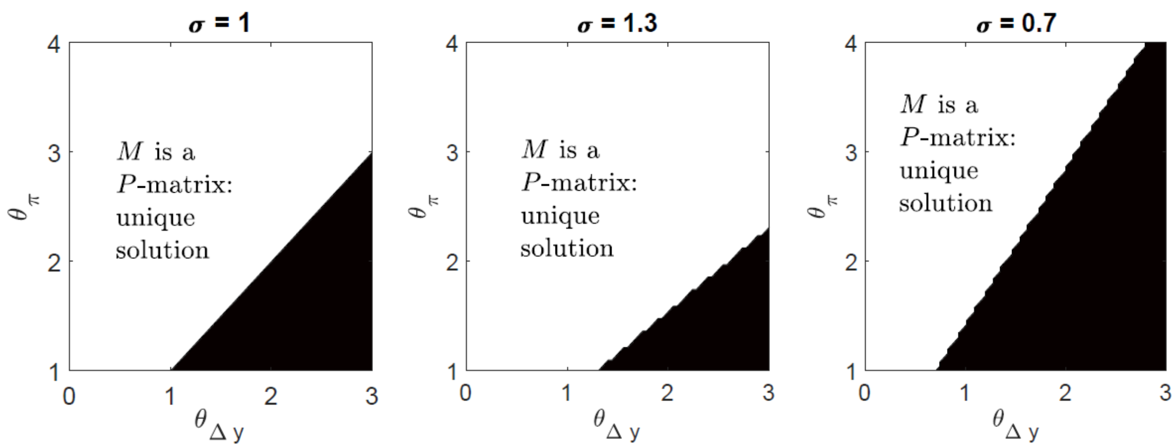

Figure 5: Regions in which M is not a P-matrix (black) when $T = 2$

Figure 5 indicates that there is a unique solution if the response to the change in the output gap, $\theta_{\Delta y}$, is not too strong relative to the inflation response θ_π; see the white region. In the first panel, which uses the baseline value of $\sigma = 1$, we see that M is a P-matrix only if the response to the change in the output gap is *smaller* than the response coefficient on inflation. Note that the parameter values used in Figure 4 ($\theta_\pi = 1.5, \theta_{\Delta y} = 1.6$), where

[18]We check if M is a P-matrix using the recursive test in Tsatsomeros and Li (2000).

there are two solutions, lie in the black region as expected. In fact, Brendon et al. (2013, Proposition 1) show that the model (12)–(15) has self-fulfilling equilibria where the lower bound is hit in the initial period if and only if $\theta_{\Delta y} > \sigma\theta_\pi$. Figure 5 is consistent with the conclusion for the case $T = 2$, and we found the same for higher values of T.

In summary, multiplicity is a robust finding in this model and this raises the question of whether alternative monetary policies could restore uniqueness by eliminating the bad solution. We investigate this below while retaining the 'speed limit' aspect of the policy rule, which may have theoretical and practical advantages as argued by Walsh (2003). We start with interest rate smoothing before turning to unconventional monetary policy rules.

4.2 Interest rate smoothing

We first ask whether policymakers could achieve a better outcome through smoothing the shadow interest rate in Equation (13) by setting $\rho_i \in (0, 1)$. We started out by checking the regions where the M matrix is a P-matrix for $T = 2$, analogous to the exercise in Figure 5'.

Plots of the P-matrix regions (see Supplementary Appendix) indicate that, *for a given* T, the determinacy region grows as the smoothing parameter ρ_i is increased. The intuition is quite simple: persistence in the shadow interest rate makes it *harder* to induce a lower bound episode over short horizons – i.e. for low values of T. However, we find the same result noted by Holden (2022, Appendix E) *as* T *is increased*: the P-matrix regions under interest rate smoothing tend to those in the model *without* any smoothing, such that *multiplicity remains a widespread problem* and there are both 'good' and 'bad' equilibria.

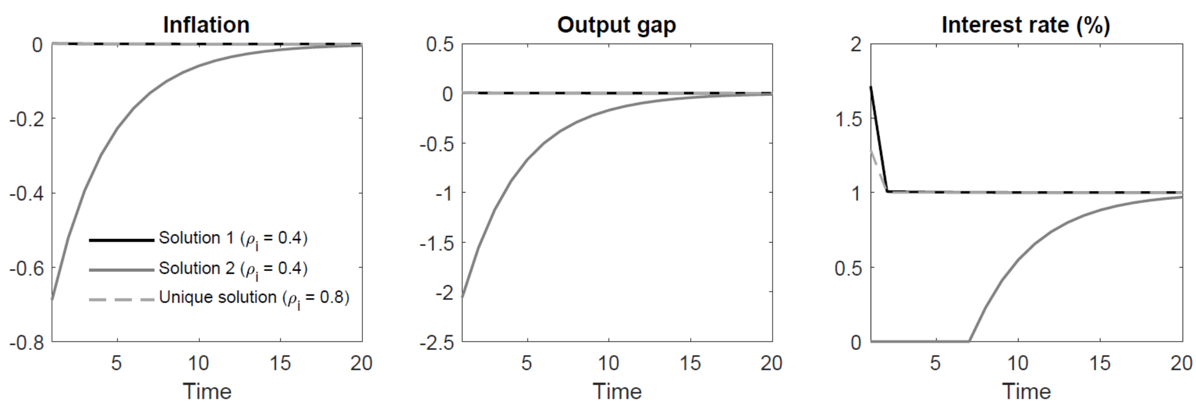

Figure 6: Perfect foresight solutions with interest rate smoothing: $e_1 = 0.01$ and $i_0^* = y_0 = 0$

In Figure 6 we present a numerical simulation. We set $\theta_\pi = 1.5$, $\theta_{\Delta y} = 1.6$ and $e_1 = 0.01$ as in the baseline simulation in Figure 4; the only difference is that the interest rate smoothing parameter is set at either $\rho_i = 0.40$ (weak smoothing) or a high value $\rho_i = 0.80$ (strong smoothing). With moderate interest rate smoothing ($\rho_i = 0.40$) there are two solutions and the 'bad' solution is *exacerbated* relative to Figure 4: inflation and the output gap fall by around *5 times* as much on impact as interest rates now spend 7 periods at the lower bound, rather than 2. Intuitively, it is easier to induce a lengthy spell at the lower bound when the policy rate is persistent, provided ρ_i is not large enough to eliminate the bad solution.

The finding of multiplicity is quite robust, but determinacy is restored *for large enough* ρ_i. Thus, if the shadow interest rate is sufficiently inertial, the 'bad' equilibrium is eliminated. For our parameters, this result seems to be robust for smoothing around $\rho_i = 0.8$ or higher.[19] Some intuition can be gained by scaling the response coefficients in (13) by $\frac{1}{1-\rho_i}$ and letting $\rho_i \to 1$. In this case, the shadow interest rate tends to $\Delta i_t^* = \theta_\pi \pi_t + \theta_{\Delta y} \Delta y_t$, which is consistent with any rule of the form $i_t^* = constant + \theta_\pi p_t + \theta_{\Delta y} y_t$, where $p_t = \pi_t + p_{t-1}$ is the log price level. The latter is a price-level targeting rule *without* any speed limit term. Since Holden (2022, Appendix E) finds such a rule restores determinacy in this model, it is intuitive that sufficiently high values of ρ_i in the original policy rule give the same conclusion.

In short, interest rate smoothing does not, in general, prevent the occurrence of multiple equilibria in the above model, and when the 'bad' equilibrium is present such a policy rule *worsens* destabilization of inflation and the output gap. At the same time, however, we saw that highly inertial interest rate rules eliminate the bad solution.

4.3 Forward guidance

Since neither inflation targeting or interest rate smoothing are robust solutions to indeterminacy, we now consider unconventional monetary policy in the form of *forward guidance*. We are motivated here by the observation that forward guidance promises an extended period of expansionary future monetary policy, such that pessimistic self-fulfilling expectations of inflation and the output gap might no longer be rational.

To model forward guidance, we consider 'news' to the shadow interest rate, such that

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)(\theta_\pi \pi_t + \theta_{\Delta y}(y_t - y_{t-1})) + e_t^{FG} \tag{16}$$

where $e_t^{FG} \leq 0$ is a forward guidance 'news shock'.

We assume that $e_t^{FG} < 0$ when forward guidance is in place. Our question is whether such a policy avoids the 'bad' solution or at least mitigates destabilization in inflation and output. Table 1 records the fractions of perfect foresight solutions in which both a 'good' and 'bad' solution exist as the forward guidance period is increased; all parameters and initial conditions (except e_t^{FG} shocks) are kept fixed at the baseline values used in Figure 4.

The results in Table 1 suggest that 'short doses' of forward guidance are *not* effective in

[19] The computational burden of checking whether M is a P-matrix for very large T means that our results are strongly suggestive rather than conclusive. As a second check, we also studied some individual perfect foresight simulations for uniqueness (with affirmative results) for values of T up to 5,000.

Table 1: Determinacy of perfect foresight paths: forward guidance (FG)

FG Horizon	Unique	Indeterminacy	Time at Bound (Max, Min)
Period 2 only	0	100%	Mean: 2 periods (2,2)
Periods 2–3	0	100%	Mean: 3 periods (3,3)
Periods 2–4	**100%**	**0%**	**Bound not hit**
Periods 2–5	26.1%	73.9%	Mean: 5 periods (5,5)
Periods 2–6	62.8%	37.2%	Mean: 2.9 periods (6,1)
Periods 2–8	0%	100%	Mean: 1.4 periods (2,1)

Note: Forward guidance is modelled via news shocks $e_t^{FG} = -0.01 - Uniform(0, 0.01)$ which start in period 2 and last until the stated date. Means and percentages based on 800 simulations.

ensuring determinacy (top rows). There is a big change at the 3-period horizon, however, for which *all* 800 perfect foresight simulations have a unique solution and no 'bad solution' exists. Unfortunately, increasing the forward guidance horizon further does *not* reinforce this result, but instead restores multiplicity (see bottom rows). In short, while the results in Table 1 suggest that a specific form of forward guidance is effective against indeterminacy in this model, robustness is lacking. Making forward guidance more aggressive (larger negative shocks) or allowing interest rate smoothing does not seem to overturn this conclusion.

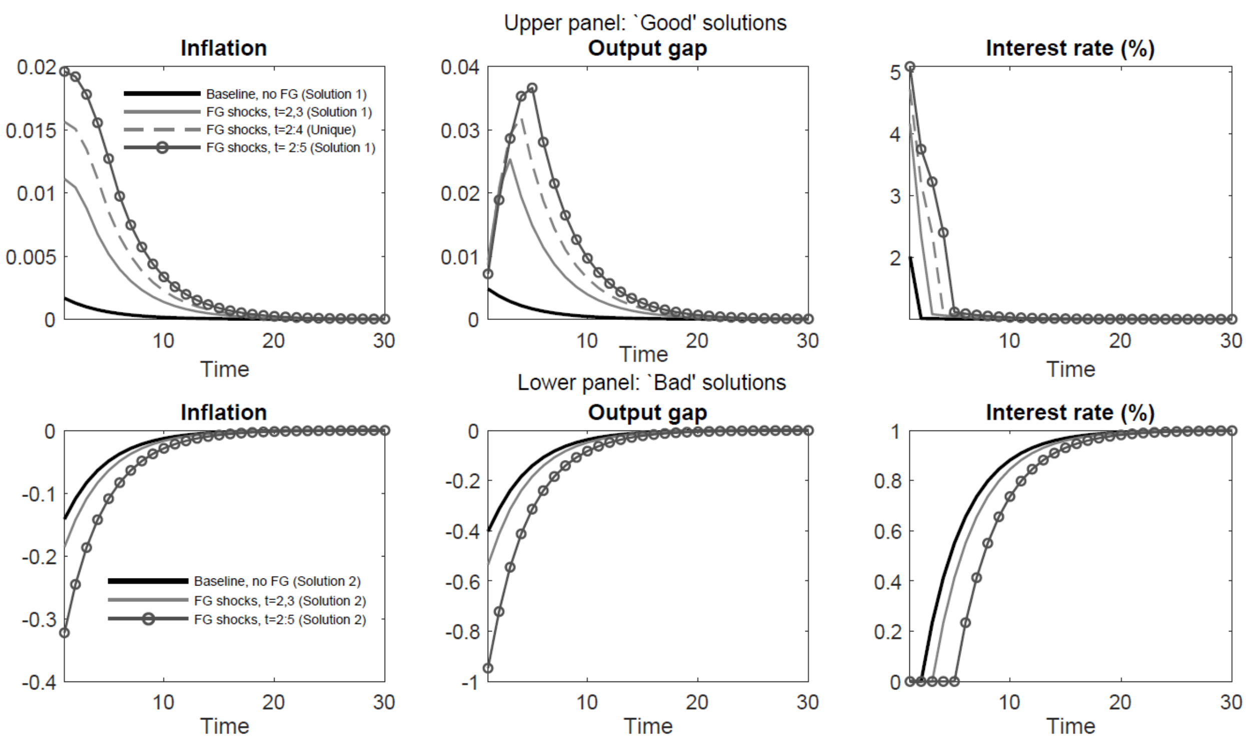

Figure 7: 'Good' and 'bad' solutions with forward guidance: $e_1 = 0.01$, $i_0^* = y_0 = 0$, $\theta_\pi = 1.5$, $\sigma = 1$ and $\theta_{\Delta y} = 1.6$. Forward guidance news shocks: $e_t^{FG} = -0.02$ in the stated periods.

Further to this, Figure 7 shows that the 'good' solution under forward guidance has inflation and output 'overstimulated' (as expected), while the 'bad' solution has very poor stabilization relative to the original policy rule (solid black line). While the problem of overstimulation could be resolved if forward guidance is made *conditional* on hitting the bound, such a policy could be counterproductive if it induces coordination on a bad equilibrium.[20]

4.4 Price-level targeting

Given that indeterminacy is not resolved by the policies considered thus far, we now consider price-level targeting. A key motivation is work showing that price-level targeting interest rate rules can mitigate or resolve indeterminacy in New Keynesian models (Giannoni, 2014; Holden, 2022). Our query is whether a response to the price level is *sufficient* to restore determinacy when retaining the 'speed limit' term in the interest rate rule. We retain the speed limit because policymakers may find such policies attractive (see Walsh, 2003) but inadvertently bring about multiplicity (Figures 4–6); this is a problem we want to solve.

Accordingly, we assume the shadow interest rate under price-level targeting is given by

$$i_t^* = \rho_i i_{t-1}^* + (1-\rho_i)\left(\theta_p p_t + \theta_{\Delta y}(y_t - y_{t-1})\right) \tag{17}$$

where $\theta_p > 0$ and $p_t := \pi_t + p_{t-1}$ is the log price level.

Differently from the rule in Holden (2022, Appendix E), the shadow interest rate still responds to the *change* in output gap. We now consider implications for determinacy.

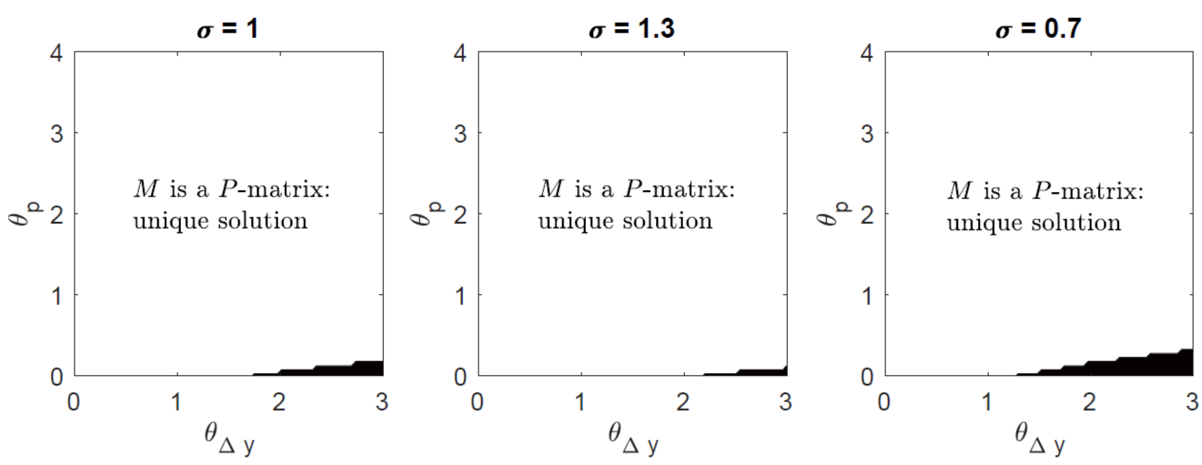

Figure 8: Regions in which M is not a P-matrix (black): price-level targeting rule when $T = 2$. In the figure, θ_p is the (long run) response coefficient on the log price level.

[20]For example, announcing that forward guidance is conditional on hitting the bound might be interpreted, by agents, as a signal that this outcome is what the central bank expects to happen.

Figure 8 shows that determinacy is not guaranteed if a strong response to the speed limit term is combined with a very weak response to the price level (black region) – though for the *same* numerical reaction coefficients as in Figure 5, there is always a unique solution. Thus, a non-trivial interest rate response to the price level is sufficient to ensure determinacy; intuitively, equilibria based on self-fulfilling pessimistic expectations are avoided if monetary policy rule is highly expansionary with a level target. Perfect foresight simulations suggest that the 'good solution' under price-level targeting (with nominal rates away from the bound) is comparable to those under the other rules, as shown in the *Supplementary Appendix*.

Interestingly, if the reaction coefficient on the price level θ_p is small enough that *both* solutions exist, we see that the 'bad' solution under price-level targeting looks somewhat 'better' than under inflation targeting, interest rate smoothing or forward guidance: the initial drops in inflation and output are much smaller than under these other policies. We give an example in Figure 9. Here we see that inflation, output and interest rates oscillate in a cyclical fashion around the 'good' solution, but within quite a narrow range.[21]

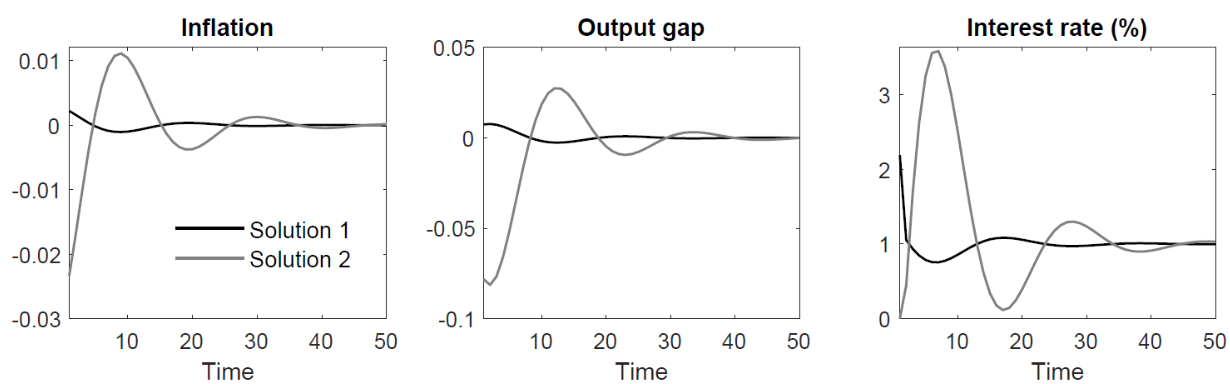

Figure 9: 'Good' and 'bad' solutions under price-level targeting for $e_1 = 0.01$, $i_0^* = y_0 = 0$ and a 'weak' response to the price level $\theta_p = 0.015$ when $\sigma = 1$ and $\theta_{\Delta y} = 1.6$

Notably, our conclusions on price-level targeting are less positive than in Holden (2022), where analytical and numerical results are used to show that a price-level targeting rule ensures determinacy in a range of New Keynesian models. What our results highlight is that the assumption that the price-targeting rule responds to the *level* of the output gap, rather than the *change* in the output gap, is crucial. In the present model, where the 'speed limit' $y_t - y_{t-1}$ enters the interest rate rule, determinacy requires a *sufficiently strong* response to the price level *relative to* the coefficient on the speed limit term, as shown in Figure 8.

[21] For additional examples, with similar properties, see Section 3.4 of the *Supplementary Appendix*.

4.5 Discussion

In short, price-level targeting does *not* guarantee determinacy in New Keynesian models *if* we allow such targets to be used alongside a 'speed limit' when setting interest rates. Our results do suggest, however, that a *well-designed* price-level targeting rule – with a strong enough response to the price level – will restore determinacy. Furthermore, outcomes under price-level targeting are robust in the sense that when the 'bad solution' exists, the resulting destabilization of the output gap and inflation is less than with an inflation target, interest rate smoothing or forward guidance; and this latter conclusion also extends to welfare based on a microfounded loss function.[22] These are the main policy implications of our analysis.

5 Conclusion

In this paper we extended the guess-verify algorithm in Guerrieri and Iacoviello (2015) to detect and simulate multiple equilibria, including simulations with 'news shocks' or stochastic simulations in which regime switches may occur on the simulated path. In cases of multiplicity, the selection of an equilibrium is based on a sunspot plus 'prior probabilities' specified by the researcher, and we showed how *ex ante* analysis can be done using these 'priors'.

We illustrated our algorithm using a simple Fisherian model with two solutions for a wide range of initial values. This example was used to show how our 'prior probabilities' approach can be used to compute expected outcomes, welfare effects and to assess robustness to different prior beliefs, under the veil of ignorance. We also presented some stochastic simulations of this model (using an extended path method) which demonstrate how unanticipated shocks produce switching between high and low inflation solutions along the simulated path and can be an independent source of non-trivial macroeconomic fluctuations.

Our main application studied a New Keynesian model with a zero lower bound, a policy response to the *change* in the output gap (or 'speed limit'), and multiple equlibria for some parameter values. One of these is a 'bad' solution for which self-fulfilling pessimistic expectations drive down inflation and the output gap, and interest rates spend some time at the lower bound. Multiplicity arises for a wide range of parameter values with an inflation targeting rule that allows interest rate smoothing or forward guidance, but replacing the inflation target with a price-level target expands the determinacy region much. Our results suggest a simple rule-of-thumb: a strong enough response of rates to the price level avoids indeterminacy by eliminating the bad solution. Further, price-level targeting is quite robust: when the 'bad solution' does exist, it is not strongly deflationary as under the other policies.

The above results highlight some uses of our algorithm, such as policy analysis and stochastic simulation, and suggest some interesting avenues for future research.

[22]The loss function is $L = \sum_{t=1}^{\infty} \beta^{t-1}[\pi_t^2 + \lambda y_t^2]$, where $\lambda = \frac{\kappa}{(1+2\theta)\theta}$ and θ is the elasticity of substitution. The target output gap will be zero given a fiscal policy that offsets steady-state distortions; see Walsh (2017).

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