

Supplementary Appendix

“Optimal pensions with endogenous labour supply”

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This appendix provides further details of the planner’s solution and the competitive equilibrium under the optimal pension policy, as discussed in the main text.

1 First-best allocation

We start by deriving the first-best allocation from the social planner problem; recall that lifetime utility of the young born at date t is $U_t = \ln(c_{t,y}) + \beta E_t[\ln(c_{t+1,o})] - \frac{\theta}{1+\chi} l_t^{1+\chi}$.

1.1 Planner problem

The social planner maximizes the welfare function

$$W_0 = E_0 \sum_{t=-1}^{\infty} \omega^t U_t = E_0 \sum_{t=0}^{\infty} \omega^t \left(\ln(c_{t,y}) + \frac{\beta}{\omega} \ln(c_{t,o}) - \frac{\theta}{1+\chi} l_t^{1+\chi} \right) + t.i.p. \quad (1)$$

subject to the resource constraint (where $y_t = A_t \tilde{k}_t^\alpha l_t^{1-\alpha}$, $\tilde{k}_t := k_t/(1+n_t)$, $k_t := K_t/N_{t-1}$)

$$y_t = c_{t,y} + \frac{c_{t,o}}{1+n_t} + k_{t+1} \quad \forall t \geq 0, \quad k_0 > 0 \quad (2)$$

and shocks from date 1 onwards

$$1+n_t = (1+n_{t-1})^{\rho_n} (1+\bar{n})^{(1-\rho_n)} \exp(\varepsilon_{n,t}), \quad A_t = A_{t-1}^{\rho_A} \exp(\varepsilon_{A,t}) \quad (3)$$

where $0 < \omega < 1$, the term $t.i.p. := \omega^{-1} \ln(c_{-1,y})$ is a given constant, the stochastic processes in (3) satisfy $\rho_j \in [0, 1)$, $\bar{n}, n_0 > -1$, $A_0 > 0$ and $\varepsilon_{j,t} \sim \mathcal{N}(0, \sigma_j^2)$, for $j = n, A$.

The planner’s maximization problem can be written recursively as

$$V(k_t, n_t, A_t) = \max_{c_{t,y}, l_t, k_{t+1}} \left\{ \ln(c_{t,y}) + \frac{\beta}{\omega} \ln(c_{t,o}) - \frac{\theta}{1+\chi} l_t^{1+\chi} + \omega E_t V(k_{t+1}, n_{t+1}, A_{t+1}) \right\} \quad (4)$$

subject to $c_{t,o} = (1+n_t)(A_t \tilde{k}_t^\alpha l_t^{1-\alpha} - c_{t,y} - k_{t+1})$, (3) and $k_0 = K_0/N_{-1} > 0$.

The first-order conditions are

$$\frac{1}{c_{t,y}} - \frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} = 0, \quad \frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} m p l_t - \theta l_t^\chi = 0 \quad (5)$$

$$-\frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} + \omega E_t V_k(k_{t+1}, n_{t+1}, A_{t+1}) = 0 \quad \implies \quad \frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} = \beta E_t \left[\frac{m p k_{t+1}}{c_{t+1,o}} \right] \quad (6)$$

where $mpl_t := (1 - \alpha)y_t/l_t$, $mpk_t := \alpha y_t/\tilde{k}_t$ and a version of the Benveniste-Schienkman condition, $V_k(k_t, n_t, A_t) = \frac{\beta}{\omega} \frac{(1+n_t)mpk_t}{c_{t,o}} \frac{\partial \tilde{k}_t}{\partial k_t} = \frac{\beta}{\omega} \frac{mpk_t}{c_{t,o}}$, has been used.

The first-order conditions (5)–(6) simplify to

$$c_{t,o} = \frac{\beta}{\omega}(1 + n_t)c_{t,y}, \quad \theta l_t^\chi = \frac{mpl_t}{c_{t,y}}, \quad \frac{1}{c_{t,y}} = \beta E_t \left[\frac{mpk_{t+1}}{c_{t+1,o}} \right] \quad (7)$$

as stated in the main text.

1.2 Planner solution

Let us guess that $k_{t+1} = \tilde{\Phi}y_t$, where $\tilde{\Phi}$ is an undetermined coefficient. Using this guess in (2) along with $c_{t,o} = \frac{\beta}{\omega}(1 + n_t)c_{t,y}$ from (7) yields

$$c_{t,y} = \frac{\omega}{\beta + \omega}(1 - \tilde{\Phi})y_t, \quad c_{t,o} = \frac{\beta}{\beta + \omega}(1 - \tilde{\Phi})(1 + n_t)y_t. \quad (8)$$

Multiplying both sides of the Euler equation in (7) by k_{t+1} and using $mpk_t k_t = \alpha(1 + n_t)y_t$:

$$\frac{k_{t+1}}{c_{t,y}} = \alpha \beta E_t \left[\frac{(1 + n_{t+1})y_{t+1}}{c_{t+1,o}} \right] \quad (9)$$

so, using the expressions in (8),

$$k_{t+1} = \frac{\alpha(\beta + \omega)}{1 - \tilde{\Phi}}c_{t,y} = \alpha\omega y_t \quad (10)$$

implying that $\tilde{\Phi} = \alpha\omega$, and hence

$$c_{t,y} = \frac{\omega}{\beta + \omega}(1 - \alpha\omega)y_t, \quad c_{t,o} = \frac{\beta}{\beta + \omega}(1 - \alpha\omega)(1 + n_t)y_t. \quad (11)$$

as stated in the main text, and recall that

$$y_t = A_t \tilde{k}_t^\alpha l_t^{1-\alpha} = \frac{A_t}{(1 + n_t)^\alpha} k_t^\alpha l_t^{1-\alpha}. \quad (12)$$

Finally, multiplying the middle equation in (7) by l_t and using $mpl_t l_t = (1 - \alpha)y_t$ and (11):

$$\theta l_t^{1+\chi} = \frac{(1 - \alpha)y_t}{c_{t,y}} = \frac{(1 + \frac{\beta}{\omega})(1 - \alpha)}{(1 - \alpha\omega)}$$

Therefore, optimal labour supply is given by

$$l_t = \left(\frac{(1 + \frac{\beta}{\omega})(1 - \alpha)}{\theta(1 - \alpha\omega)} \right)^{\frac{1}{1+\chi}}. \quad (13)$$

Equations (10)–(13) give the first-best allocation reported in the main text.

2 Decentralized economy

We split this section into three parts: a description of the environment; the competitive equilibrium; and implementation of the optimal (first-best) pension policy.

2.1 Environment

Households face an income tax (or subsidy) at rate $\tau \in \mathbb{R}$, a consumption tax (or subsidy) at rate $\tau_c \in \mathbb{R}$, and receive a paygo-type pension $P_t \in \mathbb{R}$.

The problem solved by a representative young born at date $t \geq 0$ is

$$\begin{aligned} \max_{s_t, l_t} U_t = \ln(c_{t,y}) + \beta E_t[\ln(c_{t+1,o})] - \theta \frac{l_t^{1+\chi}}{1+\chi} \quad \text{s.t.} \\ (1 + \tau_c)c_{t,y} = (1 - \tau)w_t l_t - s_t, \quad (1 + \tau_c)c_{t+1,o} = r_{t+1}s_t + P_{t+1} \end{aligned} \quad (14)$$

where the factor prices w_t , r_t and the pension P_{t+1} are taken as given.

The first-order conditions are

$$\frac{1}{c_{t,y}} = \beta E_t \left[\frac{r_{t+1}}{c_{t+1,o}} \right] \quad (15)$$

$$\theta l_t^\chi = \frac{(1 - \tau)w_t}{(1 + \tau_c)c_{t,y}} \quad (16)$$

where the consumption taxes τ_c ‘cancel out’ in (15).

Total tax contributions are given by

$$C_t = \tau N_t w_t l_t + \tau_c (N_t c_{t,y} + N_{t-1} c_{t,o}) \quad (17)$$

The government makes a pension transfer P_t to each old. We discuss the form that these transfers take in the next section.

A representative firm hires capital and labour and maximizes profit each period:

$$\max_{K_t, L_t} A_t K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t$$

which yields the factor prices

$$r_t = \alpha A_t \left(\frac{K_t}{L_t} \right)^{\alpha-1} = \alpha y_t / \tilde{k}_t \quad (= mpk_t) \quad (18)$$

$$w_t = (1 - \alpha) A_t \left(\frac{K_t}{L_t} \right)^\alpha = (1 - \alpha) y_t / l_t \quad (= mpl_t) \quad (19)$$

where mpk_t , mpl_t are defined in (6).

We now turn to a description of the competitive equilibrium given a pension policy τ, τ_c, P_t .

2.2 Competitive equilibrium

The competitive equilibrium of the decentralized economy is a set of allocations and prices such that the following conditions hold for all t :

- (i) s_t, l_t solve the maximization problem of the date t young given the shocks (3), taxes τ, τ_c , the pension transfer rule P_t , and the profit-maximizing factor prices (18).
- (ii) The pension system has a balanced budget: $P_t - C_t/N_{t-1} = 0$.
- (iii) Aggregate capital equals aggregate saving, $K_{t+1} = N_t s_t$, and the aggregate resource constraint and the per-person resource constraint, (2), hold.

The pension transfer rule is:

$$P_t = (1 + n_t)[\tau w_t l_t + \tau_c(1 - \Phi)y_t] \quad (20)$$

where $\Phi \in (0, 1)$ is a coefficient.¹

Given the rule (20), the old-age budget constraint reads as

$$\begin{aligned} (1 + \tau_c)c_{t+1,o} &= r_{t+1}s_t + P_{t+1} \\ &= r_{t+1}s_t + (1 + n_{t+1})[\tau w_{t+1}l_{t+1} + \tau_c(1 - \Phi)y_{t+1}] \end{aligned} \quad (21)$$

Note that $w_t l_t = (1 - \alpha)y_t$ (see (19)), so (21) amounts to

$$(1 + \tau_c)c_{t+1,o} = r_{t+1}s_t + (1 + n_{t+1})[(1 - \alpha)\tau + (1 - \Phi)\tau_c]y_{t+1} \quad (22)$$

By condition (iii), we have

$$s_t = k_{t+1} \implies r_{t+1}s_t = r_{t+1}k_{t+1} = \alpha(1 + n_{t+1})y_{t+1} \quad (23)$$

where $k_t := K_t/N_{t-1}$ (as above) and $r_t k_t = (1 + n_t)\alpha y_t$ is used (see (18)).

Therefore, (22) gives

$$\begin{aligned} (1 + \tau_c)c_{t+1,o} &= [\alpha + (1 - \alpha)\tau + (1 - \Phi)\tau_c](1 + n_{t+1})y_{t+1} \\ &= \frac{1}{\alpha}[\alpha + (1 - \alpha)\tau + (1 - \Phi)\tau_c]r_{t+1}k_{t+1}. \end{aligned} \quad (24)$$

which implies that

$$c_{t,o} = \left(\frac{\alpha + (1 - \alpha)\tau + (1 - \Phi)\tau_c}{1 + \tau_c} \right) (1 + n_t)y_t. \quad (25)$$

¹Our approach to determining Φ (see below) is mathematically equivalent to imposing budget balance $P_t = C_t/N_{t-1}$ from the start and using the method of undetermined coefficients. However, we prefer the ‘rule’ approach because it makes it easy to see how the optimal allocation is implemented; see Section 2.3.

Using the last line of (24) in the Euler equation (15):

$$k_{t+1} = \tilde{\beta}(1 + \tau_c)c_{t,y}, \quad \text{where } \tilde{\beta} := \frac{\beta\alpha}{\alpha + (1 - \alpha)\tau + (1 - \Phi)\tau_c}. \quad (26)$$

Using (26) in the young-age budget constraint in (14) gives

$$c_{t,y} = \frac{1}{(1 + \tilde{\beta})(1 + \tau_c)}(1 - \tau)w_t l_t = \frac{(1 - \alpha)(1 - \tau)}{(1 + \tilde{\beta})(1 + \tau_c)}y_t \quad (27)$$

where $w_t l_t = (1 - \alpha)y_t$ is used. Hence, by (26), we have

$$k_{t+1} = \frac{\tilde{\beta}(1 - \tau)}{1 + \tilde{\beta}}w_t l_t = \frac{\tilde{\beta}(1 - \tau)(1 - \alpha)}{1 + \tilde{\beta}}y_t. \quad (28)$$

By (16) and (27), labour supply satisfies

$$\theta l_t^{1+\chi} = \frac{(1 - \tau)(1 - \alpha)y_t}{(1 + \tau_c)c_{t,y}} = 1 + \tilde{\beta}. \quad (29)$$

such that

$$l_t = \left(\frac{1 + \tilde{\beta}}{\theta} \right)^{\frac{1}{1+\chi}}. \quad (30)$$

Finally, let us check which coefficients Φ satisfy the balanced-budget condition (ii):

$$P_t = C_t/N_{t-1}. \quad (31)$$

By (2), (17) and (20), equation (31) is satisfied if and only if

$$\begin{aligned} \tau(1 + n_t)w_t l_t + \tau_c(1 + n_t)(1 - \Phi)y_t &= \tau(1 + n_t)w_t l_t + \tau_c(1 + n_t)\left(c_{t,y} + \frac{c_{t,o}}{1 + n_t}\right) \\ &= \tau(1 + n_t)w_t l_t + \tau_c(1 + n_t)(y_t - k_{t+1}) \end{aligned} \quad (32)$$

which requires $k_{t+1} = \Phi y_t$, implying by (28) that $\Phi = \frac{\tilde{\beta}(1 - \tau)(1 - \alpha)}{1 + \tilde{\beta}}$. So by (26), Φ must solve

$$\tau_c \Phi^2 - [\alpha(1 + \beta) + (1 - \alpha)\tau + \tau_c]\Phi + \alpha\beta(1 - \alpha)(1 - \tau) = 0 \quad (33)$$

which has two real solutions

$$\Phi = \frac{[\alpha(1 + \beta) + (1 - \alpha)\tau + \tau_c] \pm \sqrt{[\alpha(1 + \beta) + (1 - \alpha)\tau + \tau_c]^2 - 4\alpha\beta(1 - \alpha)(1 - \tau)\tau_c}}{2\tau_c}$$

provided $[\alpha(1 + \beta) + (1 - \alpha)\tau + \tau_c]^2 \geq 4\alpha\beta(1 - \alpha)(1 - \tau)\tau_c$.² Thus, the coefficient Φ in the transfer rule (20) cannot be chosen independently of τ, τ_c .

We now show how τ, τ_c, Φ can be chosen to decentralize the first-best allocation as a competitive equilibrium given the balanced-budget condition.

²The two solutions are distinct if inequality is strict; otherwise Φ_+, Φ_- are repeated roots.

2.3 Implementing the first-best

In this section we show how τ, τ_c, Φ can be chosen to achieve the first-best allocation.

Under the first-best, $k_{t+1} = \alpha\omega y_t$, which requires $\Phi = \alpha\omega$ in the pension transfer rule (20):

$$P_t = (1 + n_t)[\tau w_t l_t + \tau_c(1 - \alpha\omega)y_t]. \quad (34)$$

Further, comparing eq. (29) to eq. (13), we see that labour supply cannot be optimal in the decentralized equilibrium unless $\tau_c = -\tau$, so we impose this relationship, which gives:

$$\begin{aligned} P_t &= (1 + n_t)\tau[w_t l_t - (1 - \alpha\omega)y_t] \\ &= -\tau\alpha(1 - \omega)(1 + n_t)y_t \end{aligned} \quad (35)$$

where $w_t l_t = (1 - \alpha)y_t$ is used.

With this rule, we have by (28):

$$k_{t+1} = \frac{\tilde{\beta}(1 - \tau)(1 - \alpha)}{1 + \tilde{\beta}}y_t, \quad \text{where } \tilde{\beta} = \frac{\beta}{1 - (1 - \omega)\tau}, \quad (36)$$

and the balanced-budget condition requires that

$$\alpha\omega = \Phi = \frac{\tilde{\beta}(1 - \tau)(1 - \alpha)}{1 + \tilde{\beta}} \quad (37)$$

such that

$$\tau = \tau^* := \frac{\beta(1 - \alpha) - \alpha\omega(1 + \beta)}{\beta(1 - \alpha) - \alpha\omega(1 - \omega)}, \quad \tau_c = -\tau \quad (38)$$

where τ^* can also be written as

$$\tau = \frac{\tau'}{1 - \frac{\omega(1 - \alpha\omega)}{(1 - \alpha)(\beta + \omega)}}, \quad \text{for } \tau' = \frac{\beta(1 - \alpha) - \alpha\omega(1 + \beta)}{(1 - \alpha)(\beta + \omega)}. \quad (39)$$

Thus, by (25)–(30) the equilibrium allocations are

$$k_{t+1} = \alpha\omega y_t \quad (40)$$

$$c_{t,y} = \left(\frac{1 - \alpha}{1 + \tilde{\beta}} \right) y_t = \frac{\omega}{\beta + \omega}(1 - \alpha\omega)y_t \quad (41)$$

$$c_{t,o} = \alpha \left(\frac{1 - (1 - \omega)\tau^*}{1 - \tau^*} \right) (1 + n_t)y_t = \frac{\beta}{\beta + \omega}(1 - \alpha\omega)(1 + n_t)y_t \quad (42)$$

$$l_t = \left(\frac{1 + \tilde{\beta}}{\theta} \right)^{\frac{1}{1+\chi}} = \left(\frac{(1 + \frac{\beta}{\omega})(1 - \alpha)}{\theta(1 - \alpha\omega)} \right)^{\frac{1}{1+\chi}} \quad (43)$$

which are the first-best allocations in (10)–(13).

Finally, note that $\alpha\omega$ is a root of (33) for $\tau = \tau^*, \tau_c = -\tau$, so if the government sets these taxes and sets $\Phi = \alpha\omega$ in its transfer rule, it ensures a first-best competitive equilibrium.