# Supplementary Appendix
# "Optimal pensions with endogenous labour supply"

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This appendix provides further details of the planner's solution and the competitive equilibrium under the optimal pension policy, as discussed in the main text.

# 1 First-best allocation

We start by deriving the first-best allocation from the social planner problem; recall that lifetime utility of the young born at date $t$ is $U_t = \ln(c_{t,y}) + \beta E_t[\ln(c_{t+1,o})] - \frac{\theta}{1+\chi} l_t^{1+\chi}$.

## 1.1 Planner problem

The social planner maximizes the welfare function

$$W_0 = E_0 \sum_{t=-1}^{\infty} \omega^t U_t = E_0 \sum_{t=0}^{\infty} \omega^t \left( \ln(c_{t,y}) + \frac{\beta}{\omega} \ln(c_{t,o}) - \frac{\theta}{1+\chi} l_t^{1+\chi} \right) + t.i.p. \tag{1}$$

subject to the resource constraint (where $y_t = A_t \tilde{k}_t^{\alpha} l_t^{1-\alpha}$, $\tilde{k}_t := k_t/(1+n_t)$, $k_t := K_t/N_{t-1}$)

$$y_t = c_{t,y} + \frac{c_{t,o}}{1+n_t} + k_{t+1} \quad \forall t \geq 0, \qquad k_0 > 0 \tag{2}$$

and shocks from date 1 onwards

$$1 + n_t = (1 + n_{t-1})^{\rho_n} (1 + \overline{n})^{(1-\rho_n)} \exp(\varepsilon_{n,t}), \qquad A_t = A_{t-1}^{\rho_A} \exp(\varepsilon_{A,t}) \tag{3}$$

where $0 < \omega < 1$, the term $t.i.p. := \omega^{-1} \ln(c_{-1,y})$ is a given constant, the stochastic processes in (3) satisfy $\rho_j \in [0,1)$, $\overline{n}, n_0 > -1, A_0 > 0$ and $\varepsilon_{j,t} \sim \mathcal{N}(0, \sigma_j^2)$, for $j = n, A$.

The planner's maximization problem can be written recursively as

$$V(k_t, n_t, A_t) = \max_{c_{t,y},\, l_t,\, k_{t+1}} \{ \ln(c_{t,y}) + \frac{\beta}{\omega} \ln(c_{t,o}) - \frac{\theta}{1+\chi} l_t^{1+\chi} + \omega E_t V(k_{t+1}, n_{t+1}, A_{t+1}) \} \tag{4}$$

subject to $c_{t,o} = (1+n_t)(A_t \tilde{k}_t^{\alpha} l_t^{1-\alpha} - c_{t,y} - k_{t+1})$, (3) and $k_0 = K_0/N_{-1} > 0$.

The first-order conditions are

$$\frac{1}{c_{t,y}} - \frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} = 0, \qquad \frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} mpl_t - \theta l_t^{\chi} = 0 \tag{5}$$

$$-\frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} + \omega E_t V_k(k_{t+1}, n_{t+1}, A_{t+1}) = 0 \quad \Longrightarrow \quad \frac{\beta}{\omega} \frac{(1+n_t)}{c_{t,o}} = \beta E_t \left[ \frac{mpk_{t+1}}{c_{t+1,o}} \right] \tag{6}$$

where $mpl_t := (1-\alpha)y_t/l_t$, $mpk_t := \alpha y_t/\tilde{k}_t$ and a version of the Benveniste-Schienkman condition, $V_k(k_t, n_t, A_t) = \frac{\beta}{\omega}\frac{(1+n_t)mpk_t}{c_{t,o}}\frac{\partial \tilde{k}_t}{\partial k_t} = \frac{\beta}{\omega}\frac{mpk_t}{c_{t,o}}$, has been used.

The first-order conditions (5)–(6) simplify to

$$c_{t,o} = \frac{\beta}{\omega}(1+n_t)c_{t,y}, \qquad \theta l_t^{\chi} = \frac{mpl_t}{c_{t,y}}, \qquad \frac{1}{c_{t,y}} = \beta E_t\left[\frac{mpk_{t+1}}{c_{t+1,o}}\right] \tag{7}$$

as stated in the main text.

## 1.2 Planner solution

Let us guess that $k_{t+1} = \tilde{\Phi} y_t$, where $\tilde{\Phi}$ is an undetermined coefficient. Using this guess in (2) along with $c_{t,o} = \frac{\beta}{\omega}(1+n_t)c_{t,y}$ from (7) yields

$$c_{t,y} = \frac{\omega}{\beta+\omega}(1-\tilde{\Phi})y_t, \qquad c_{t,o} = \frac{\beta}{\beta+\omega}(1-\tilde{\Phi})(1+n_t)y_t. \tag{8}$$

Multiplying both sides of the Euler equation in (7) by $k_{t+1}$ and using $mpk_t k_t = \alpha(1+n_t)y_t$:

$$\frac{k_{t+1}}{c_{t,y}} = \alpha\beta E_t\left[\frac{(1+n_{t+1})y_{t+1}}{c_{t+1,o}}\right] \tag{9}$$

so, using the expressions in (8),

$$k_{t+1} = \frac{\alpha(\beta+\omega)}{1-\tilde{\Phi}}c_{t,y} = \alpha\omega y_t \tag{10}$$

implying that $\tilde{\Phi} = \alpha\omega$, and hence

$$c_{t,y} = \frac{\omega}{\beta+\omega}(1-\alpha\omega)y_t, \qquad c_{t,o} = \frac{\beta}{\beta+\omega}(1-\alpha\omega)(1+n_t)y_t. \tag{11}$$

as stated in the main text, and recall that

$$y_t = A_t\tilde{k}_t^{\alpha}l_t^{1-\alpha} = \frac{A_t}{(1+n_t)^{\alpha}}k_t^{\alpha}l_t^{1-\alpha}. \tag{12}$$

Finally, multiplying the middle equation in (7) by $l_t$ and using $mpl_t l_t = (1-\alpha)y_t$ and (11):

$$\theta l_t^{1+\chi} = \frac{(1-\alpha)y_t}{c_{t,y}} = \frac{(1+\frac{\beta}{\omega})(1-\alpha)}{(1-\alpha\omega)}$$

Therefore, optimal labour supply is given by

$$l_t = \left(\frac{(1+\frac{\beta}{\omega})(1-\alpha)}{\theta(1-\alpha\omega)}\right)^{\frac{1}{1+\chi}}. \tag{13}$$

Equations (10)–(13) give the first-best allocation reported in the main text.

## 2 Decentralized economy

We split this section into three parts: a description of the environment; the competitive equilibrium; and implementation of the optimal (first-best) pension policy.

### 2.1 Environment

Households face an income tax (or subsidy) at rate $\tau \in \mathbb{R}$, a consumption tax (or subsidy) at rate $\tau_c \in \mathbb{R}$, and receive a paygo-type pension $P_t \in \mathbb{R}$.

The problem solved by a representative young born at date $t \geq 0$ is

$$\begin{aligned} \max_{s_t, l_t} U_t = \ln(c_{t,y}) + \beta E_t[\ln(c_{t+1,o})] - \theta \frac{l_t^{1+\chi}}{1+\chi} \qquad \text{s.t.} \\ (1+\tau_c)c_{t,y} = (1-\tau)w_t l_t - s_t, \qquad (1+\tau_c)c_{t+1,o} = r_{t+1}s_t + P_{t+1} \end{aligned} \tag{14}$$

where the factor prices $w_t$, $r_t$ and the pension $P_{t+1}$ are taken as given.

The first-order conditions are

$$\frac{1}{c_{t,y}} = \beta E_t \left[ \frac{r_{t+1}}{c_{t+1,o}} \right] \tag{15}$$

$$\theta l_t^{\chi} = \frac{(1-\tau)w_t}{(1+\tau_c)c_{t,y}} \tag{16}$$

where the consumption taxes $\tau_c$ 'cancel out' in (15).

Total tax contributions are given by

$$C_t = \tau N_t w_t l_t + \tau_c (N_t c_{t,y} + N_{t-1} c_{t,o}) \tag{17}$$

The government makes a pension transfer $P_t$ to each old. We discuss the form that these transfers take in the next section.

A representative firm hires capital and labour and maximizes profit each period:

$$\max_{K_t, L_t} \; A_t K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

which yields the factor prices

$$r_t = \alpha A_t \left( \frac{K_t}{L_t} \right)^{\alpha - 1} = \alpha y_t / \tilde{k}_t \qquad (= mpk_t) \tag{18}$$

$$w_t = (1-\alpha) A_t \left( \frac{K_t}{L_t} \right)^{\alpha} = (1-\alpha) y_t / l_t \quad (= mpl_t) \tag{19}$$

where $mpk_t$, $mpl_t$ are defined in (6).

We now turn to a description of the competitive equilibrium given a pension policy $\tau, \tau_c, P_t$.

## 2.2 Competitive equilibrium

The competitive equilibrium of the decentralized economy is a set of allocations and prices such that the following conditions hold for all $t$:

(i) $s_t$, $l_t$ solve the maximization problem of the date $t$ young given the shocks (3), taxes $\tau, \tau_c$, the pension transfer rule $P_t$, and the profit-maximizing factor prices (18).

(ii) The pension system has a balanced budget: $P_t - C_t/N_{t-1} = 0$.

(iii) Aggregate capital equals aggregate saving, $K_{t+1} = N_t s_t$, and the aggregate resource constraint and the per-worker resource constraint, (2), hold.

The pension transfer rule is:

$$P_t = (1+n_t)[\tau w_t l_t + \tau_c(1-\Phi)y_t] \tag{20}$$

where $\Phi \in (0,1)$ is a coefficient.[1]

Given the rule (20), the old-age budget constraint reads as

$$\begin{aligned}(1+\tau_c)c_{t+1,o} &= r_{t+1}s_t + P_{t+1} \\ &= r_{1+1}s_t + (1+n_{t+1})[\tau w_{t+1}l_{t+1} + \tau_c(1-\Phi)y_{t+1}]\end{aligned} \tag{21}$$

Note that $w_t l_t = (1-\alpha)y_t$ (see (19)), so (21) amounts to

$$(1+\tau_c)c_{t+1,o} = r_{1+1}s_t + (1+n_{t+1})\left[(1-\alpha)\tau + (1-\Phi)\tau_c\right]y_{t+1} \tag{22}$$

By condition (iii), we have

$$s_t = k_{t+1} \quad\implies\quad r_{t+1}s_t = r_{t+1}k_{t+1} = \alpha(1+n_{t+1})y_{t+1} \tag{23}$$

where $k_t := K_t/N_{t-1}$ (as above) and $r_t k_t = (1+n_t)\alpha y_t$ is used (see (18)).

Therefore, (22) gives

$$\begin{aligned}(1+\tau_c)c_{t+1,o} &= \left[\alpha + (1-\alpha)\tau + (1-\Phi)\tau_c\right](1+n_{t+1})y_{t+1} \\ &= \frac{1}{\alpha}\left[\alpha + (1-\alpha)\tau + (1-\Phi)\tau_c\right]r_{t+1}k_{t+1}.\end{aligned} \tag{24}$$

which implies that

$$c_{t,o} = \left(\frac{\alpha + (1-\alpha)\tau + (1-\Phi)\tau_c}{1+\tau_c}\right)(1+n_t)y_t. \tag{25}$$

[1] Our approach to determining $\Phi$ (see below) is mathematically equivalent to imposing budget balance $P_t = C_t/N_{t-1}$ from the start and using the method of undetermined coefficients. However, we prefer the 'rule' approach because it makes it easy to see how the optimal allocation is implemented; see Section 2.3.

Using the last line of (24) in the Euler equation (15):

$$k_{t+1} = \tilde{\beta}(1+\tau_c)c_{t,y}, \qquad \text{where} \quad \tilde{\beta} := \frac{\beta\alpha}{\alpha + (1-\alpha)\tau + (1-\Phi)\tau_c}. \tag{26}$$

Using (26) in the young-age budget constraint in (14) gives

$$c_{t,y} = \frac{1}{(1+\tilde{\beta})(1+\tau_c)}(1-\tau)w_t l_t = \frac{(1-\alpha)(1-\tau)}{(1+\tilde{\beta})(1+\tau_c)}y_t \tag{27}$$

where $w_t l_t = (1-\alpha)y_t$ is used. Hence, by (26), we have

$$k_{t+1} = \frac{\tilde{\beta}(1-\tau)}{1+\tilde{\beta}}w_t l_t = \frac{\tilde{\beta}(1-\tau)(1-\alpha)}{1+\tilde{\beta}}y_t. \tag{28}$$

By (16) and (27), labour supply satisfies

$$\theta l_t^{1+\chi} = \frac{(1-\tau)(1-\alpha)y_t}{(1+\tau_c)c_{t,y}} = 1+\tilde{\beta}. \tag{29}$$

such that

$$l_t = \left(\frac{1+\tilde{\beta}}{\theta}\right)^{\frac{1}{1+\chi}}. \tag{30}$$

Finally, let us check which coefficients $\Phi$ satisfy the balanced-budget condition (ii):

$$P_t = C_t/N_{t-1}. \tag{31}$$

By (2), (17) and (20), equation (31) is satisfied if and only if

$$\begin{aligned} \tau(1+n_t)w_t l_t + \tau_c(1+n_t)(1-\Phi)y_t &= \tau(1+n_t)w_t l_t + \tau_c(1+n_t)(c_{t,y} + \frac{c_{t,o}}{1+n_t}) \\ &= \tau(1+n_t)w_t l_t + \tau_c(1+n_t)(y_t - k_{t+1}) \end{aligned} \tag{32}$$

which requires $k_{t+1} = \Phi y_t$, implying by (28) that $\Phi = \frac{\tilde{\beta}(1-\tau)(1-\alpha)}{1+\tilde{\beta}}$. So by (26), $\Phi$ must solve

$$\tau_c\Phi^2 - [\alpha(1+\beta) + (1-\alpha)\tau + \tau_c]\Phi + \alpha\beta(1-\alpha)(1-\tau) = 0 \tag{33}$$

which has two real solutions

$$\Phi = \frac{[\alpha(1+\beta) + (1-\alpha)\tau + \tau_c] \pm \sqrt{[\alpha(1+\beta) + (1-\alpha)\tau + \tau_c]^2 - 4\alpha\beta(1-\alpha)(1-\tau)\tau_c}}{2\tau_c}$$

provided $[\alpha(1+\beta) + (1-\alpha)\tau + \tau_c]^2 \geq 4\alpha\beta(1-\alpha)(1-\tau)\tau_c$.[2] Thus, the coefficient $\Phi$ in the transfer rule (20) cannot be chosen independently of $\tau, \tau_c$.

We now show how $\tau, \tau_c, \Phi$ can be chosen to decentralize the first-best allocation as a competitive equilibrium given the balanced-budget condition.

[2] The two solutions are distinct if inequality is strict; otherwise $\Phi_+, \Phi_-$ are repeated roots.

## 2.3 Implementing the first-best

In this section we show how $\tau, \tau_c, \Phi$ can be chosen to achieve the first-best allocation.

Under the first-best, $k_{t+1} = \alpha\omega y_t$, which requires $\Phi = \alpha\omega$ in the pension transfer rule (20):

$$P_t = (1+n_t)[\tau w_t l_t + \tau_c(1-\alpha\omega)y_t]. \tag{34}$$

Further, comparing eq. (29) to eq. (13), we see that labour supply cannot be optimal in the decentralized equilibrium unless $\tau_c = -\tau$, so we impose this relationship, which gives:

$$\begin{aligned} P_t &= (1+n_t)\tau[w_t l_t - (1-\alpha\omega)y_t] \\ &= -\tau\alpha(1-\omega)(1+n_t)y_t \end{aligned} \tag{35}$$

where $w_t l_t = (1-\alpha)y_t$ is used.

With this rule, we have by (28):

$$k_{t+1} = \frac{\tilde{\beta}(1-\tau)(1-\alpha)}{1+\tilde{\beta}} y_t, \qquad \text{where} \;\; \tilde{\beta} = \frac{\beta}{1-(1-\omega)\tau}, \tag{36}$$

and the balanced-budget condition requires that

$$\alpha\omega = \Phi = \frac{\tilde{\beta}(1-\tau)(1-\alpha)}{1+\tilde{\beta}} \tag{37}$$

such that

$$\tau = \tau^* := \frac{\beta(1-\alpha) - \alpha\omega(1+\beta)}{\beta(1-\alpha) - \alpha\omega(1-\omega)}, \quad \tau_c = -\tau \tag{38}$$

where $\tau^*$ can also be written as

$$\tau = \frac{\tau'}{1 - \frac{\omega(1-\alpha\omega)}{(1-\alpha)(\beta+\omega)}}, \qquad \text{for } \tau' = \frac{\beta(1-\alpha) - \alpha\omega(1+\beta)}{(1-\alpha)(\beta+\omega)}. \tag{39}$$

Thus, by (25)–(30) the equilibrium allocations are

$$k_{t+1} = \alpha\omega y_t \tag{40}$$

$$c_{t,y} = \left(\frac{1-\alpha}{1+\tilde{\beta}}\right) y_t = \frac{\omega}{\beta+\omega}(1-\alpha\omega)y_t \tag{41}$$

$$c_{t,o} = \alpha\left(\frac{1-(1-\omega)\tau^*}{1-\tau^*}\right)(1+n_t)y_t = \frac{\beta}{\beta+\omega}(1-\alpha\omega)(1+n_t)y_t \tag{42}$$

$$l_t = \left(\frac{1+\tilde{\beta}}{\theta}\right)^{\frac{1}{1+\chi}} = \left(\frac{(1+\frac{\beta}{\omega})(1-\alpha)}{\theta(1-\alpha\omega)}\right)^{\frac{1}{1+\chi}} \tag{43}$$

which are the first-best allocations in (10)–(13).

Finally, note that $\alpha\omega$ is a root of (33) for $\tau = \tau^*$, $\tau_c = -\tau$, so if the government sets these taxes and sets $\Phi = \alpha\omega$ in its transfer rule, it ensures a first-best competitive equilibrium.