Solving linear rational expectations models in the presence of structural change: Some refinements

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Abstract

Standard solution methods for linear rational expectations models assume a time-invariant structure. Recent work has gone beyond this by formulating solution methods for linear rational expectations models subject to structural changes, such as parameter shifts and policy reforms, that are announced in advance. This paper contributes to this literature by presenting a backward recursive solution method that is fast, general and easy to use: the only technical requirement is that the user 'write down' the structural matrices in each regime; the solution is then found recursively using these matrices. I provide a series of examples and show how the method can deal with some important structural changes – such as imperfectly credible policy reforms and shifts to multiple equilibria (sunspots) – that received little attention in recent literature.

1 Introduction

Standard methods for solving linear rational expectations models, such as Blanchard and Kahn (1980), Anderson and Moore (1985), Binder and Pesaran (1997), King and Watson (1998), Uhlig (1999), Klein (2000) and Sims (2002), assume a time-invariant structure – that is, the parameters of the system are taken to be constant. As a result, such methods cannot be used to study occasional changes in structure, such as parameter shifts or policy reforms, which may be partially or fully anticipated.[1]

There are many reasons the structure of economic models might shift. A short list would include implementation of new policies like forward guidance or quantitative easing; reforms to current policies, such as inflation targets, pensions or taxes, which may be phased in gradually; and shifts in technology, competitiveness of product markets, or the reaction functions

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[1] The cases we are thinking of are distinct from those studied in Markov-switching linear rational expectations models (see Davig and Leeper (2007) or Farmer et al. (2009)), which are appropriate for structural shifts that are unanticipated recurrent events whose probability distribution is known.

of policymakers. For example, Clarida et al. (2000) and Lubik and Schorfheide (2004) find evidence of structural change in the Federal Reserve's interest rate rule, in the form of a more aggressive response to inflation in the post-Volcker period. From a policymaker perspective, the consequences of policy reforms may differ substantially depending on whether the reform in question is announced or unanticipated (e.g. Mertens and Ravn (2012)). Hence, accurate policy evaluation requires solution methods that can deal with structural changes.

Recent work has moved in this direction by formulating solution methods for linear rational expectations models which are subject to structural change. This paper contributes to the literature by providing a general solution method that is fast, easy to use and does not rely on approximation of the true solution. It also presents explicit solutions for some important structural changes that received little attention in the recent literature.

The recent literature on structural change begins with Cagliarini and Kulish (2013). Building on the time-invariant solution of Sims (2002), they set out a general method for solving linear rational expectations models subject to anticipated structural changes and establish conditions on existence and uniqueness of solutions. Kulish and Pagan (2017) present an alternative solution based on the method of undetermined coeffficients, along the lines of Binder and Pesaran (1997). They show that since the model solution is a time-varying coefficient VAR, a likelihood function can be constructed to allow estimation of parameters and the dates of structural breaks. Moreover, they extend model solutions to the case of unanticipated structural changes, whereby expectations are not rational.

Applications of these methods include Jones and Kulish (2013), who study the impact of unconventional monetary policies that involve announcements about the future path of short-term and long-term interest rates (forward guidance); Kulish et al. (2017), who estimate expected durations at the zero lower bound using U.S. data; and Gibbs and Kulish (2017), who use a model with 'unanchored' expectations due to adaptive learning to study the output costs of disinflations. These works suggest that credible pre-announced policy changes may significantly improve economic outcomes relative to standard policies.

The present paper contributes to this literature in two ways. First, it presents a simple backward recursive method for solving linear rational expectations models subject to anticipated structural changes. The only technical requirement is that the user 'write down' the structural matrices in each regime; the solution is then calculated recursively using these matrices. As a result, explicit solutions can be stated for several cases of practical interest, including permanent and temporary structural changes, anticipated disturbances, policy reforms that are phased in gradually, and partial information for some fraction of agents.

This simplicity is a small but non-trivial improvement on current approaches. For example, the solution of Cagliarini and Kulish (2013) (which applies to fully anticipated structural changes) uses a time-stacking approach and thus involves complicated, high-dimensional matrices; moreover, in stochastic simulations a new system must be solved at each time step. By comparison, the solution method presented here avoids time-stacking and hence simplifies the task of 'writing down' and solving the model. Kulish and Pagan (2017) present a simpler solution based on the method of undetermined coefficients. The present paper is closely related as it also relies on this approach; however, whereas their method relies on a backward-forward algorithm, the solution I present is simpler because it is backward recursive and provides an explicit solution (if one exists). As a result, it should be of interest to researchers at policy institutions who would like to take methods 'off the shelf' and use them in a variety of applications, including solving medium or large-scale DSGE models.

Secondly, the paper considers some important structural changes that received little attention in the literature. In particular, I provide new solutions relating to imperfectly credible policy reforms and show how the solution method can deal with non-fundamental (sunspot) solutions, i.e. multiple equilibria. Of the main papers, only Kulish and Pagan (2017) consider imperfect credibility, in the form of beliefs that may differ from reality for a number of periods. Outside of this, there are many possible specifications of imperfect credibility, and this paper considers two new cases. I also show that the solution method extends straightforwardly to models with indeterminacy (sunspots) by building on results in Farmer et al. (2015). Hence, researchers may easily study anticipated structural changes that involve transitions from determinate model structures to indeterminate ones, and vice-versa.

The solution method is illustrated through several examples of structural change, including forward guidance, a change in the inflation target, and pension reform. Some examples have analytical solutions and are included to help build intuition; others are used to illustrate important concepts (e.g. optimal announcement dates) or to show that the solution method gives equivalent results to established algorithms in numerical applications. All the codes used in the examples will be made available on my webpage.

The paper proceeds as follows. Section 2 outlines rational expectations solutions in the absence of structural change. Section 3 sets out the model with structural change and several benchmark results are derived. Section 4 considers extensions, and Section 5 presents several numerical applications. Finally, Section 6 concludes.

2 Solutions in the absence of structural change

Our solution method is a variant of Binder and Pesaran (1997) and Cho and Moreno (2011), and is based on the method of undetermined coefficients. Following Binder and Pesaran (1997), a linear rational expectations model of n equations may be written in the form:

$$B_1 x_t = B_2 E_t x_{t+1} + B_3 x_{t-1} + B_4 z_t + B_5 \tag{1}$$

where x_t is an $n \times 1$ vector of endogenous state and jump variables, and z_t is an $m \times 1$ vector of exogenous shocks whose data generating process is known. All matrices conform to the specified dimensions. E_t is the expectations operator conditional on the information set I_t, which includes all current and past values of the endogenous and exogenous variables.

Time is discrete and runs from $t = 0$ onwards. As shown in Binder and Pesaran (1997), the formulation in Eq. (1) is quite general as it can accommodate multiple lags and leads of the endogenous variables and an arbitrary date at which expectations are formed. Note that the matrices B_2 and B_3 can be singular.

Provided that B_1 is non-singular, and assuming that the shocks follow a VAR(1) process, the model in (1) can be represented as

$$x_t = AE_t x_{t+1} + Bx_{t-1} + Cz_t + D \tag{2}$$

$$z_t = Rz_{t-1} + \epsilon_t, \quad E_t[\epsilon_{t+1}] = 0_{m\times 1} \tag{3}$$

where we assume the eigenvalues of the $m \times m$ matrix R are inside the unit circle.

The class of fundamental rational expectations solutions have the form:[2]

$$x_t = \Omega x_{t-1} + \Gamma z_t + \Psi \tag{4}$$

where Ω, Γ, Ψ are $n \times n$, $n \times m$ and $n \times 1$ matrices, respectively.

Given (3) and (4), we have $E_t x_{t+1} = \Omega x_t + \Gamma R z_t + \Psi$. Substituting this into Eq. (2) and rearranging gives the complete set of real-valued matrices consistent with (4):

$$\mathcal{S} = \{(\Omega, \Gamma, \Psi) | (\Omega, \Gamma, \Psi) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times 1}\}$$

where

$$\Omega = (I_n - A\Omega)^{-1}B, \tag{5}$$

$$\Gamma = (I_n - A\Omega)^{-1}(A\Gamma R + C), \quad \Psi = (I_n - A\Omega)^{-1}(A\Psi + D), \tag{6}$$

provided that $\det[I_n - A\Omega] \neq 0$.[3]

Note that (5) implies $A\Omega^2 - \Omega + B = 0$ and so determines Ω. Once Ω is found, Γ and Ψ can be determined, providing a solution to the model.[4] It is now standard to solve the model using methods developed by Blanchard and Kahn (1980), Uhlig (1999), King and Watson (1998), Klein (2000) and Sims (2002). These solution methods are by and large based on

[2] There is also a class of non-fundamental 'bubble' solutions described by $x_t = \Omega x_{t-1} + \Gamma z_t + \Psi + w_t$, where w_t is an arbitrary process satisfying $w_t = (I_n - A\Omega)^{-1} A E_t w_{t+1}$.

[3] Cho and Moreno (2011) state that the condition $\det[I_n - A\Omega] \neq 0$ in the above model is equivalent to the necessary condition for existence of solutions stated in King and Watson (1998).

[4] Eq. (6) implies that $\Psi = (I_n - A(\Omega + I_n))^{-1}D$ whereas the matrix results $\text{vec}(XYZ) = (Z^\intercal \otimes X)\text{vec}(Y)$ and $\text{vec}(XY) = (I \otimes X)\text{vec}(Y)$ (see e.g. Klein (2000), Appendix B) allow Γ to be determined from the equality $\text{vec}(\Gamma) = (I_n \otimes (I_n - A\Omega) - R^\intercal \otimes A)^{-1}\text{vec}(C)$ (provided that the relevant matrix inverses exist).

eigenvalue-eigenvector decomposition theory and can completely characterize the solutions in $\mathcal{S}$. For example, the QZ method, based on the Schur decomposition, uses the generalized eigenvalues of A and B to detect whether there is a unique or multiple number of real-valued stationary fundamental solutions (see Uhlig (1999) Theorem 3).

There are also recursive methods that solve directly for the matrices Ω, Γ, Ψ in (4), such as Binder and Pesaran (1997) and Cho and Moreno (2011). Since there are generally multiple (fundamental) solutions in $\mathcal{S}$, researchers have developed methods for selecting relevant solutions, like the minimum state variable (MSV) criterion (McCallum (1983)), E-stability (Evans and Honkapohja (2001)), or the MOD solution (McCallum (2007)). More recently, Cho and Moreno (2011) present a 'forward method' that finds the unique fundamental solution $\Omega^*, \Gamma^*, \Psi^*$ that satisfies the no-bubbles condition. Cho and McCallum (2015) present existence and determinacy conditions for solutions of the form (4).

3 Solutions in the presence of structural change

3.1 Model

We now consider a multivariate rational expectations model as above, except that the structural matrices of parameters may change over time. To incorporate structural change, I assume there are two possible regimes, denoted 1 and 2. I refer to regime 1 as the "reference regime" and regime 2 as the "alternative regime".

The reference reference is decribed by Eq. (7):

$$B_1 x_t = B_2 E_t x_{t+1} + B_3 x_{t-1} + B_4 z_t + B_5 \tag{7}$$

where x_t is an $n \times 1$ vector of endogenous variables, and E_t is the expectation operator.

The matrices B_i, $i \in \{1, 2, 3\}$, are $n \times n$ coefficient matrices of structural parameters. We assume that B_1 is non-singular, but B_2 and B_3 can be singular. z_t is an $m \times 1$ vector of exogenous variables with a known process, and B_4 is an $n \times m$ matrix of parameters. The $n \times 1$ column vector B_5 contains any intercepts.

The alternative regime is described by Eq. (8):

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 z_t + \tilde{B}_5 \tag{8}$$

Analogous to (7), the $\tilde{B}_i$, $i \in \{1, 2, 3\}$, are $n \times n$ matrices. $\tilde{B}_1$ is non-singular, whereras $\tilde{B}_2$, $\tilde{B}_3$ can be singular. The matrix $\tilde{B}_4$ is $n \times m$, and $\tilde{B}_5$ is $n \times 1$. In typical applications, one of the intercept matrices may be zero, as DSGE models are typically log-linearized around a non-stochastic steady state (see Uhlig (1999)).[5] At any given date t, either the reference regime applies or the alternative regime does. Hence, the regimes are mutually exclusive.

Given mutually exclusive regimes, we may introduce an *indicator variable* $\mathbb{1}_t$ that is equal to 1 if the reference regime applies in period t and 0 if the alternative regime is in place. Then we have the following system in any period $t \geq 0$:

$$B_1(\mathbb{1}_t)x_t = B_2(\mathbb{1}_t)E_t x_{t+1} + B_3(\mathbb{1}_t)x_{t-1} + B_4(\mathbb{1}_t)z_t + B_5(\mathbb{1}_t) \tag{9}$$

where $B_i(\mathbb{1}_t) \equiv \mathbb{1}_t B_i + (1 - \mathbb{1}_t)\tilde{B}_i$ for $i \in \{1, ..., 5\}$.

Let the structure under the reference regime be represented by the matrices $A \equiv B_1^{-1}B_2$, $B \equiv B_1^{-1}B_3$, $C \equiv B_1^{-1}B_4$ and $D \equiv B_1^{-1}B_5$. Analogously, the structure under the alternative regime is given by $\tilde{A} \equiv \tilde{B}_1^{-1}\tilde{B}_2$, $\tilde{B} \equiv \tilde{B}_1^{-1}\tilde{B}_3$, $\tilde{C} \equiv \tilde{B}_1^{-1}\tilde{B}_4$ and $\tilde{D} \equiv \tilde{B}_1^{-1}\tilde{B}_5$.

Pre-multiplying (9) by $B_1(\mathbb{1}_t)^{-1}$, and assuming z_t follows a VAR(1) process, we have:[6]

$$x_t = A(\mathbb{1}_t)E_t x_{t+1} + B(\mathbb{1}_t)x_{t-1} + C(\mathbb{1}_t)z_t + D(\mathbb{1}_t) \tag{10}$$

$$z_t = Rz_{t-1} + \epsilon_t, \quad E_t[\epsilon_{t+1}] = 0_{m \times 1} \tag{11}$$

where

$$\begin{aligned} A(\mathbb{1}_t) &\equiv B_1(\mathbb{1}_t)^{-1}B_2(\mathbb{1}_t) = \mathbb{1}_t A + (1 - \mathbb{1}_t)\tilde{A}, \quad B(\mathbb{1}_t) \equiv B_1(\mathbb{1}_t)^{-1}B_3(\mathbb{1}_t) = \mathbb{1}_t B + (1 - \mathbb{1}_t)\tilde{B} \\ C(\mathbb{1}_t) &\equiv B_1(\mathbb{1}_t)^{-1}B_4(\mathbb{1}_t) = \mathbb{1}_t C + (1 - \mathbb{1}_t)\tilde{C}, \quad D(\mathbb{1}_t) \equiv B_1(\mathbb{1}_t)^{-1}B_5(\mathbb{1}_t) = \mathbb{1}_t D + (1 - \mathbb{1}_t)\tilde{D} \end{aligned}$$

and we assume that the eigenvalues of the $m \times m$ matrix R are inside the unit circle. The initial vectors $x_{-1} \in \mathbb{R}^n$ and $z_{-1} \in \mathbb{R}^m$ are given.

The information set at time t includes all current and past values of the endogenous and exogenous variables, including the indicator variable. The indicator variable is an exogenous predetermined sequence $\{\mathbb{1}_t\}_{t=0}^{\infty}$; a change in its value indicates a structural change (i.e. regime shift). The information set also includes *future* values of the indicator variable. Definitions 1 and 2 explain how observability of future values of the indicator variable determines whether structural change is fully anticipated or partially anticipated. In what follows, I focus on fully anticipated structural change unless otherwise stated.

Definition 1 *Structural change is said to be anticipated if all future values of the indicator variable, $\{\mathbb{1}_{t+j}\}_{j=1}^{\infty}$, are part of agents' information set for all $t \geq 0$. Agents therefore have full information about future structural changes and are said to be informed.*

[5] If one of the regimes has a different steady state, intercepts will be present when the log-linearized model is written in terms of deviations x_t from a common steady state. See, e.g., Guerrieri and Iacoviello (2015).

[6] Given mutually exclusive regimes, a sufficient condition for existence of $B_1(\mathbb{1}_t)^{-1}$ is that B_1 and $\tilde{B}_1$ are non-singular. To see this, note that $B_1(\mathbb{1}_t)^{-1} = B_1^{-1}$ if $\mathbb{1}_t = 1$ and $B_1(\mathbb{1}_t)^{-1} = \tilde{B}_1^{-1}$ if $\mathbb{1}_t = 0$.

By Definition 1, expectations under anticipated structural change satisfy, $\forall j > 0$,

$$E_t [x_{t+j}] = E [x_{t+j} | I_t \cup \{\mathbb{1}_{t+1}, \mathbb{1}_{t+2}, ...\}] \quad \text{and} \quad E_t [z_{t+j}] = R^j z_t \tag{12}$$

where $I_t \equiv \{x_t, x_{t-1}, ..., z_t, z_{t-1}, ..., \mathbb{1}_t\}$ is the information set *excluding* future values of the indicator variable.

Definition 2 *Agents are said to uninformed if only K future values of the indicator variable, $\{\mathbb{1}_{t+j}\}_{j=1}^K$, are part of their information set and if, when forming expectations beyond this horizon, they assume the last known structure $\mathbb{1}_{t+K}$ will remain in place forever.*

By Definition 2, the expectations of uninformed agents satisfy, $\forall j > 0$,

$$\hat{E}_t [x_{t+j}] = E [x_{t+j} | I_t \cup \{\mathbb{1}_{t+1}, ..., \mathbb{1}_{t+K}\}] \quad \text{and} \quad \hat{E}_t [z_{t+j}] = R^j z_t \tag{13}$$

where I_t is defined in (12). Note that such agents have partial information: future structural changes are unknown to them until they are within K periods of the change in structure.

3.2 Solving the model

Definition 3 *A solution to the model with structural change is a function $f : x_{t-1} \times z_t \rightarrow x_t$ such that the system in* (10) *and* (11) *holds, given the current evaluation of the indicator variable $\mathbb{1}_t$ and agents' information about its future values.*

An alternative way of characterizing a solution is using matrix expressions that generalize the constant-coefficient decision rules of a linear rational expectations model. In particular, the function f can be expressed in terms of a set of matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=0}^{\infty}$, such that

$$x_t = \Omega_t x_{t-1} + \Gamma_t z_t + \Psi_t \tag{14}$$

where Ω_t is an $n \times n$ matrix, Γ_t is an $n \times m$ matrix and Ψ_t is an $n \times 1$ vector, and the t subscript indicates that the matrices are in general time-varying.

As Eq. (14) shows, the solution is in general non-linear, even though the regimes (7) and (8) are linear. In what follows, I show that the matrices Ω_t, Γ_t, Ψ_t can be determined recursively using simple formulas which are well-defined provided that a series of regularity conditions are met. The solution method takes as inputs information on changes in structure (reflected in $\mathbb{1}_t$) and the structural matrices in each regime – A, $\tilde{A}$, B, $\tilde{B}$, C, $\tilde{C}$, $\tilde{D}$ – and provides a recursive solution in terms of these matrices. The solution is thus easy to program and can be applied to many cases of practical interest.

There are two key requirements for existence of a solution:

(i) *Existence of a rational expectations solution must hold at the terminal regime.* Users may solve for such a solution using standard methods, such as Binder and Pesaran (1997) or Sims (2002). The corresponding solution matrices Ω^*, Γ^*, Ψ^* are used as inputs in the computation of the solution under structural change.

(ii) *A series of regularity conditions* $\det[I_n - A_t\Omega_{\tilde{T}-t}] \neq 0$ *must be met* for $t = 0, 1, .., \tilde{T}$, where $\tilde{T} + 1$ is the date at which the terminal structure is reached. If the regularity conditions are not met, then a solution will not exist even if requirement (i) is satisfied.

Requirement (i) is a necessary condition because expectations are forward-looking and hence the current solution relies on 'backward induction' from a terminal solution. I assume readers have access to standard solution methods to compute a terminal solution. Some of these solution methods, such as Blanchard and Kahn (1980), King and Watson (1998) and Klein (2000), partition the vector of endogenous variables into pre-determined and non-predetermined variables and write the model in first-order form. In these cases, the solution is not in the same form as (4), so further computation is needed to determine Ω, Γ, Ψ.

There are also methods that solve directly for the matrices Ω, Γ, Ψ, such as Binder and Pesaran (1997), Sims (2002) and Cho and Moreno (2011).[7] Alternatively, Jones (2016) shows that one may use Dynare (Adjemian et al. (2011)) to numerically log-linearize the equations of a calibrated non-linear model and then 'build' the corresponding structural-form matrices A, B, C, D. The 'forward method' of Cho and Moreno (2011) is convenient here because it uses the matrices A, B, C, D in a simple matrix recursion (that relies on certain invertibility conditions) to find the solution matrices Ω^*, Γ^*, Ψ^* of the *unique* fundamental solution that satisfies the *no-bubbles condition*. Hence, when there are multiple fundamental solutions due to the matrix quadratic in Ω (see (5)), the forward method selects an economically meaningful solution from this set. Given this desirable property, I use the forward method to compute fixed-structure solutions in all examples presented below.[8]

Note that existence of a solution in the terminal regime is necessary but not sufficient for existence of a solution under structural change. This is because the regularity conditions in (ii) are needed to ensure that a well-defined solution exists in periods where structural change influences the solution matrices – i.e. for all t until the terminal structure is reached.

[7]The gensys code of Sims (2002) is available online. To write the vector x_t in the same form as in Sims' method, conditional expectations of forward-looking variables should be included in x_t by defining auxiliary variables of the form $s_t = E_t x^p_{t+1}$, where x^p_t is a vector of the p forward-looking variables.

[8]Note that multiple fundamental solutions do not arise if the model is purely forward-looking ($B = 0_{n\times n}$), since the matrix quadratic becomes $\Omega = 0_{n\times n}$ (see (5)). In this case, $(I_n - A\Omega)$ is always invertible. The forward method of Cho and Moreno (2011) uses the Binder-Pesaran representation in the present paper; Cho and McCallum (2015) provide determinacy and stability conditions for such models.

3.3 General case

Consider first the general case in which for periods $t \in [0, \tilde{T}]$ there is an arbitrary sequence of structures involving the reference and alternative regime. The terminal structure is reached at date $t = \tilde{T} + 1$ and I assume, without loss of generality, that the terminal structure is the alternative regime.[9] The sequence of structures is reflected in the time path of the indicator variable, $\{\mathbb{1}_t\}_{t=0}^{\tilde{T}}$ and $\{\mathbb{1}_t = 0\}_{t=\tilde{T}+1}^{\infty}$, which is known at all dates (see Definition 1).

In the form of (10), the system to be solved for all $t \geq 0$ is:

$$x_t = \begin{cases} A_t E_t x_{t+1} + B_t x_{t-1} + C_t z_t + D_t, & 0 \leq t \leq \tilde{T} \\ \tilde{A} E_t x_{t+1} + \tilde{B} x_{t-1} + \tilde{C} z_t + \tilde{D}, & t > \tilde{T} \end{cases} \tag{15}$$

subject to Eqs. (11)–(12) and $M_t \equiv M(\mathbb{1}_t)$ for $M \in \{A, B, C, D\}$.

For all $t > \tilde{T}$, the alternative regime is in place. Therefore, the solution for $t > \tilde{T}$ is

$$x_t = \tilde{\Omega} x_{t-1} + \tilde{\Gamma} z_t + \tilde{\Psi} \tag{16}$$

where $\tilde{\Omega}$, $\tilde{\Gamma}$, $\tilde{\Psi}$ (defined below) are the standard time-invariant solution matrices.

Since (16) can be used to solve for x_t for $t > \tilde{T}$, the remaining system to be solved is:

$$\begin{aligned} x_0 &= A_0 E_0 x_1 + B_0 x_{-1} + C_0 z_0 + D_0 \\ &\vdots \\ x_{\tilde{T}} &= A_{\tilde{T}} E_{\tilde{T}} x_{\tilde{T}+1} + B_{\tilde{T}} x_{\tilde{T}-1} + C_{\tilde{T}} z_{\tilde{T}} + D_{\tilde{T}} \end{aligned} \tag{17}$$

where $E_{\tilde{T}} x_{\tilde{T}+1} = \tilde{\Omega} x_{\tilde{T}} + \tilde{\Gamma} R z_{\tilde{T}} + \tilde{\Psi}$ by (16).

Proposition 1 The solution to the system in (15) is given by

$$x_t = \begin{cases} \Omega_{\tilde{T}-t} x_{t-1} + \Gamma_{\tilde{T}-t} z_t + \Psi_{\tilde{T}-t} & 0 \leq t \leq \tilde{T} \\ \tilde{\Omega} x_{t-1} + \tilde{\Gamma} z_t + \tilde{\Psi} & t > \tilde{T} \end{cases} \tag{18}$$

where

$$\begin{aligned} \tilde{\Omega} &= (I_n - \tilde{A}\tilde{\Omega})^{-1} \tilde{B}, & \Omega_0 &= (I_n - A_{\tilde{T}} \tilde{\Omega})^{-1} B_{\tilde{T}} \\ \tilde{\Gamma} &= (I_n - \tilde{A}\tilde{\Omega})^{-1} (\tilde{A}\tilde{\Gamma} R + \tilde{C}), & \Gamma_0 &= (I_n - A_{\tilde{T}} \tilde{\Omega})^{-1} (A_{\tilde{T}} \tilde{\Gamma} R + C_{\tilde{T}}) \\ \tilde{\Psi} &= (I_n - \tilde{A}\tilde{\Omega})^{-1} (\tilde{A}\tilde{\Psi} + \tilde{D}), & \Psi_0 &= (I_n - A_{\tilde{T}} \tilde{\Omega})^{-1} (A_{\tilde{T}} \tilde{\Psi} + D_{\tilde{T}}) \end{aligned}$$

[9] Note that there is no loss of generality since either structure may be labelled as the alternative regime.

and for $t = 0, 1, ..., \tilde{T} - 1$,

$$\Omega_{\tilde{T}-t} = (I_n - A_t \Omega_{\tilde{T}-t-1})^{-1} B_t, \qquad \Gamma_{\tilde{T}-t} = (I_n - A_t \Omega_{\tilde{T}-t-1})^{-1} (A_t \Gamma_{\tilde{T}-t-1} R + C_t),$$

$$\Psi_{\tilde{T}-t} = (I_n - A_t \Omega_{\tilde{T}-t-1})^{-1} (A_t \Psi_{\tilde{T}-t-1} + D_t)$$

if $\det[I_n - A\tilde{\Omega}] \neq 0$, $\det[I_n - \tilde{A}\tilde{\Omega}] \neq 0$ and the following regularity condition is satisfied:

$$\det[I_n - A_t \Omega_{\tilde{T}-t-1}] \neq 0, \qquad \text{for } t = 0, 1, ..., \tilde{T} - 1.$$

Proof. See the Appendix. ■

Proposition 1 highlights some key features of the solution method set out in this paper: (i) the solution (if it exists) can be calculated using simple backward recursive matrix formulas which take the structural matrices as inputs and are well-defined provided that a series of invertibility conditions are met; (ii) the sequence of solution matrices during the 'transition period' ($\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}_{t=0}^{\tilde{T}}$) needs to be computed only once and then the deterministic and stochastic solutions follow immediately.[10]

These features are a small but non-trivial improvement over current approaches. For example, the solution method of Cagliarini and Kulish (2013) relies on time-stacking and hence the stochastic solution has to be recomputed at each date. The alternative solution of Kulish and Pagan (2017), while much closer to the present one, requires both backward and forward steps as part of the algorithm. By comparison, the method presented here is backward recursive and provides an explicit solution in the form of a difference equation, making it easy to implement for a variety of applications of practical interest.

3.4 Permanent anticipated structural change

I now consider some specific applications, starting with a *permanent* anticipated structural change at date $T+1$. The economy starts in the reference regime (i.e. $\mathbb{1}_0 = 1$) and in period $T+1$ there is a permanent shift from the reference regime to the alternative regime; all this is known to agents and incorporated in their expectations for all $t \geq 0$.

In the form of (10), the system to be solved for all $t \geq 0$ is:

$$x_t = A(\mathbb{1}_t) E_t x_{t+1} + B(\mathbb{1}_t) x_{t-1} + C(\mathbb{1}_t) z_t + D(\mathbb{1}_t) \tag{19}$$

subject to Eqs. (11)–(12) and $\mathbb{1}_t = 1$ if $t \leq T$ and 0 otherwise.

[10] Similar regularity conditions appear in the recursive algorithms of Binder and Pesaran (1997) and Cho and Moreno (2011) that compute solutions in fixed-structure models.

Hence, the system simplifies to

$$x_t = \begin{cases} AE_t x_{t+1} + Bx_{t-1} + Cz_t + D, & 0 \leq t \leq T \\ \tilde{A}E_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D}, & t > T. \end{cases} \tag{20}$$

For all $t > T$, the alternative regime is in place and expectations reflect this. Therefore, the solution for $t \geq T+1$ is $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Gamma}z_t + \tilde{\Psi}$, where $\tilde{\Omega}$, $\tilde{\Gamma}$, $\tilde{\Psi}$ are given in Proposition 1.

The remaining system to be solved is thus:

$$\begin{aligned} x_0 &= AE_0 x_1 + Bx_{-1} + Cz_0 + D \\ &\vdots \\ x_T &= AE_T x_{T+1} + Bx_{T-1} + Cz_T + D \end{aligned} \tag{21}$$

where $E_T x_{T+1} = \tilde{\Omega}x_T + \tilde{\Gamma}Rz_T + \tilde{\Psi}$. The solution is summarized in Corollary 1.

Corollary 1 (Permanent structural change) The solution to system (20) is given by Proposition 1 when $\tilde{T} = T$, except that

$$\Omega_0 = (I_n - A\tilde{\Omega})^{-1}B, \quad \Gamma_0 = (I_n - A\tilde{\Omega})^{-1}(A\tilde{\Gamma}R + C), \quad \Psi_0 = (I_n - A\tilde{\Omega})^{-1}(A\tilde{\Psi} + D),$$

and for $t = 0, 1, 2, ..., T-1$,

$$\Omega_{T-t} = (I_n - A\Omega_{T-t-1})^{-1}B, \qquad \Gamma_{T-t} = (I_n - A\Omega_{T-t-1})^{-1}(A\Gamma_{T-t-1}R + C),$$

$$\Psi_{T-t} = (I_n - A\Omega_{T-t-1})^{-1}(A\Psi_{T-t-1} + D)$$

if $\det[I_n - A\tilde{\Omega}] \neq 0$ and $\det[I_n - A\Omega_{T-t-1}] \neq 0$ for $t = 0, 1, ..., T-1$.

Proof. It follows from Proposition 1 – see the Online Appendix. ■

I now consider a simple example nested by Corollary 1.

Example 1 Rogoff and Obstfeld (1996) (Ch. 8) consider a Cagan model with a future shift in money supply that is anticipated from date 0:

$$p_t = \frac{1}{1+\eta}m_t + \frac{\eta}{1+\eta}E_t p_{t+1}, \qquad m_t = \begin{cases} \overline{m} & 0 \leq t \leq T \\ \overline{m}' & t > T \end{cases}$$

where $\eta > 0$ and p_t and m_t are the (log) price level and money supply.

The above model is already in the form $x_t = A_t E_t x_{t+1} + B_t x_{t-1} + C_t z_t + D_t$, with $x_t \equiv [p_t]$, $z_t \equiv [0]$ and $B_t = C_t = [0]$ for all t. The remaining matrices are given by

$$A_t = \begin{cases} [\frac{\eta}{1+\eta}] \equiv A & 0 \leq t \leq T \\ [\frac{\eta}{1+\eta}] \equiv \tilde{A} & t > T \end{cases}, \quad D_t = \begin{cases} [\frac{\overline{m}}{1+\eta}] \equiv D & 0 \leq t \leq T \\ [\frac{\overline{m}'}{1+\eta}] \equiv \tilde{D} & t > T. \end{cases}$$

Setting $\tilde{T} = T$ and using the formulas in Corollary 1 gives the same solution as in Rogoff and Obstfeld (1996) (p. 520):

$$p_t = \begin{cases} \Psi_{T-t} = \overline{m} + \left(\frac{\eta}{1+\eta}\right)^{T+1-t} (\overline{m}' - \overline{m}) & 0 \leq t \leq T \\ \tilde{\Psi} = \overline{m}' & t > T. \end{cases} \tag{22}$$

where $\Psi_0 = A\tilde{\Psi} + D$ and $\Psi_{T-t} = A\Psi_{T-t-1} + D = A^{T-t}\Psi_0 + \left(\frac{1-A^{T-t}}{1-A}\right) D$ for $t \in [0, T-1]$.

We see that the price level responds on the announcement date 0 and accelerates towards its new constant value $\overline{m}'$ as the implementation date $T+1$ approaches.

3.5 Temporary anticipated structural change

I now consider *temporary* anticipated structural changes, such as those triggered by pre-announced policies like forward guidance or austerity that are designed to last for a set amount of time. Suppose the alternative regime comes into force in period $T+1$ and remains in place until period T^*, where $T^* \geq T+1$. For all $t > T^*$ the reference regime applies. All this is known to agents from date 0 and incorporated in expectations.

In the form of Eq. (10), the system to be solved for all $t \geq 0$ is:

$$x_t = A(\mathbb{1}_t)E_t x_{t+1} + B(\mathbb{1}_t)x_{t-1} + C(\mathbb{1}_t)z_t + D(\mathbb{1}_t) \tag{23}$$

subject to Eqs. (11)–(12) and $\mathbb{1}_t = 0$ for $T+1 \leq t \leq T^*$ and $\mathbb{1}_t = 1$ otherwise.

Hence, the system to be solved forward from period $t = 0$ is:

$$\begin{aligned} x_t &= AE_t x_{t+1} + Bx_{t-1} + Cz_t + D, && 0 \leq t \leq T \\ x_t &= \tilde{A}E_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D}, && T+1 \leq t \leq T^* \\ x_t &= AE_t x_{t+1} + Bx_{t-1} + Cz_t + D, && t > T^*. \end{aligned} \tag{24}$$

Note that for $t > T^*$ the reference regime is in place. Therefore, the solution for $t > T^*$ is

$$x_t = \overline{\Omega}x_{t-1} + \overline{\Gamma}z_t + \overline{\Psi} \tag{25}$$

where $\overline{\Omega}$, $\overline{\Gamma}$, $\overline{\Psi}$ (defined below) are the standard time invariant solution matrices.

Since Eq. (25) can be used to solve for x_t at $t = T^* + 1$, we have at date $t = T^*$

$$x_{T^*} = \tilde{A} E_{T^*} x_{T^*+1} + \tilde{B} x_{T^*-1} + \tilde{C} z_{T^*} + \tilde{D}, \tag{26}$$

where $E_{T^*} x_{T^*+1} = \overline{\Omega} x_{T^*} + \overline{\Gamma} R z_{T^*} + \overline{\Psi}$ by (25).

Corollary 2 (Temporary structural change) The solution to system (24) is

$$x_t = \begin{cases} \Omega_{T^*-t} x_{t-1} + \Gamma_{T^*-t} z_t + \Psi_{T^*-t} & 0 \leq t \leq T^* \\ \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi} & t > T^* \end{cases}$$

where

$$\begin{aligned} \overline{\Omega} &= (I_n - A\overline{\Omega})^{-1} B, & \Omega_0 &= (I_n - \tilde{A}\overline{\Omega})^{-1} \tilde{B} \\ \overline{\Gamma} &= (I_n - A\overline{\Omega})^{-1} (A\overline{\Gamma} R + C), & \Gamma_0 &= (I_n - \tilde{A}\overline{\Omega})^{-1} (\tilde{A}\overline{\Gamma} R + \tilde{C}) \\ \overline{\Psi} &= (I_n - A\overline{\Omega})^{-1} (A\overline{\Psi} + D), & \Psi_0 &= (I_n - \tilde{A}\overline{\Omega})^{-1} (\tilde{A}\Psi + \tilde{D}) \end{aligned}$$

$$\Omega_{T^*-t} = \begin{cases} (I_n - A\Omega_{T^*-t-1})^{-1} B & t = 0, 1, ..., T \\ (I_n - \tilde{A}\Omega_{T^*-t-1})^{-1} \tilde{B} & T + 1 \leq t \leq T^* - 1 \end{cases}$$

$$\Gamma_{T^*-t} = \begin{cases} (I_n - A\Omega_{T^*-t-1})^{-1} (A\Gamma_{T^*-t-1} R + C) & t = 0, 1, ..., T \\ (I_n - \tilde{A}\Omega_{T^*-t-1})^{-1} (\tilde{A}\Gamma_{T^*-t-1} R + \tilde{C}) & T + 1 \leq t \leq T^* - 1 \end{cases}$$

$$\Psi_{T^*-t} = \begin{cases} (I_n - A\Omega_{T^*-t-1})^{-1} (A\Psi_{T^*-t-1} + D) & t = 0, 1, ..., T \\ (I_n - \tilde{A}\Omega_{T^*-t-1})^{-1} (\tilde{A}\Psi_{T^*-t-1} + \tilde{D}) & T + 1 \leq t \leq T^* - 1 \end{cases}$$

if $\det[I_n - A\overline{\Omega}] \neq 0$, $\det[I_n - \tilde{A}\overline{\Omega}] \neq 0$ and the below regularity conditions are satisfied:

$$\begin{cases} \det[I_n - A\Omega_{T^*-t-1}] \neq 0 & \text{for } t = 0, 1, ..., T - 1 \\ \det[I_n - \tilde{A}\Omega_{T^*-t-1}] \neq 0 & \text{for } t = T, ..., T^* - 1 \ . \end{cases}$$

Proof. It follows from Proposition 1 when $\tilde{T} = T^*$, $\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$ are relabelled as $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$, and the structure $\{A_t, B_t, C_t, D_t\}_{t=0}^{\tilde{T}}$ follows (24). See the Online Appendix. ■

I now consider a simple example nested by Corollary 2.

Example 2 Guerrieri and Iacoviello (2015) consider a simple asset pricing model with an interest rate rule. With forward guidance and no persistence the model is given by:

$$q_t = \beta E_t q_{t+1} - \sigma r_t + z_t, \qquad r_t = \begin{cases} r^* & \text{for } T + 1 \leq t \leq T^* \\ \phi q_t & \text{otherwise} \end{cases}$$

where $\sigma, \phi > 0$, $\beta, \rho \in (0, 1)$ and $z_t = \rho z_{t-1} + \epsilon_t$ with $E_t[\epsilon_{t+1}] = 0$.

With $x_t = [q_t \;\; r_t]^\intercal$ and $R = [\rho]$, the (non-zero) structural matrices in each regime are:

$$A = \begin{bmatrix} \frac{\beta}{1+\sigma\phi} & 0 \\ \frac{\beta\phi}{1+\sigma\phi} & 0 \end{bmatrix}, \; C = \begin{bmatrix} \frac{1}{1+\sigma\phi} \\ \frac{\phi}{1+\sigma\phi} \end{bmatrix}, \; \tilde{A} = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, \; \tilde{C} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \tilde{D} = \begin{bmatrix} -\sigma r^* \\ r^* \end{bmatrix}.$$

Suppose $T = 4$ and $T^* = 10$, i.e. forward guidance is implemented in periods 5-10. With forward guidance anticipated from date 0 (Corollary 2), the solution is

$$x_t = \begin{cases} \Gamma_{T^*-t} z_t + \Psi_{T^*-t} & \text{for } 0 \leq t \leq 10 \\ \Gamma z_t & \text{for } t > 10 \end{cases}$$

where $\Gamma = \begin{bmatrix} \frac{1}{1+\sigma\phi-\beta\rho} & \frac{\phi}{1+\sigma\phi-\beta\rho} \end{bmatrix}^\intercal$, $\Gamma_0 = \begin{bmatrix} \frac{1+\sigma\phi}{1+\sigma\phi-\beta\rho} & 0 \end{bmatrix}^\intercal$, $\Psi_0 = \begin{bmatrix} \frac{-\sigma\phi}{1-\beta} r^* & r^* \end{bmatrix}^\intercal$, and

$$\Gamma_{T^*-t} = \begin{cases} A\rho\Gamma_{T^*-t-1} + C & \text{for } t = 0, ..., 4 \\ \tilde{A}\rho\Gamma_{T^*-t-1} + \tilde{C} & \text{for } t = 5, .., 9, \end{cases} \qquad \Psi_{T^*-t} = \begin{cases} A\Psi_{T^*-t-1} & \text{for } t = 0, ..., 4 \\ \tilde{A}\Psi_{T^*-t-1} + \tilde{D} & \text{for } t = 5, .., 9. \end{cases}$$

3.6 Announcement effects

We are often interested in studying policies that are communicated to agents in advance, but which initially came as a surprise. In such cases, the policy announcement has a contemporaneous impact on endogenous variables, or 'announcement effect', that policymakers may wish to incorporate in policy evaluation. Example 1 can be used to illustrate this.

Example 1 revisited. Figure 1 plots the solution for the price level in the Cagan model when money supply is increased from $\overline{m}$ to $\overline{m}'$ in period $T + 1$ and this policy is announced in period 0. The announcement effect is the difference between the price level at $t = 0$, and the counterfactual price level at this date in the absence of any announcement (dashed line). Setting $t = 0$ in (22), the period 0 price level is

$$p_0 = \overline{m} + \left(\frac{\eta}{1+\eta}\right)^{T+1} (\overline{m}' - \overline{m})$$

where $\overline{m}$ is original (log) money supply.

Note that the counterfactual price level in the absence of an increase in money supply is $p_0^{init} = \overline{m}$. Hence, the announcement effect is

$$p_0 - p_0^{init} = \left(\frac{\eta}{1+\eta}\right)^{T+1} (\overline{m}' - \overline{m}) > 0.$$

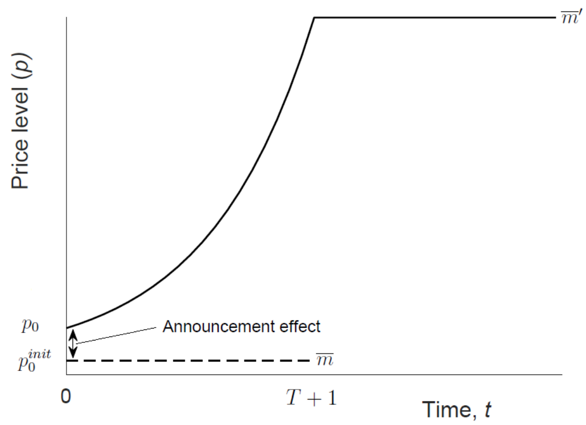

Figure 1: Anticipated increase in money supply at $T+1$, announced at $t = 0$

In general, there may be an 'announcement effect' upon any non-predetermined endogenous variable. Assuming (without loss of generality) that the reference regime prevalis in absence of structural change, we have for the general case in Proposition 1:

$$\text{Announcement effect} \equiv x_0 - x_0^{init} = (\Omega_{\tilde{T}} - \overline{\Omega})x_{-1} + (\Gamma_{\tilde{T}} - \overline{\Gamma})z_0 + \Psi_{\tilde{T}} - \overline{\Psi}$$

where $x_{-1} \in \mathbb{R}^n$ is given, and $\overline{\Omega}$, $\overline{\Gamma}$, $\overline{\Psi}$ are defined in Corollary 2.

Note that the announcement effect has three parts: (i) the change related to the response to the initial state of the system; (ii) the change brought about by the altered response to current shocks; and (iii) any shifts in the intercept terms. The above equations generalize straightforwardly to structural changes announced at an arbitrary date, t.

3.7 Anticipated shocks and parameter transitions

The above solutions can easily be modified to deal with anticipated disturbances. Suppose $z_t = Rz_{t-1} + \epsilon_t + \epsilon_t^a$, where ϵ_t are unanticipated shocks and ϵ_t^a is an $m \times 1$ vector of *news shocks* observed from period 0. We may then state the following result.

Corollary 3 Suppose $z_t = Rz_{t-1} + \epsilon_t + \epsilon_t^a$, with $E_t[\epsilon_{t+k}] = 0_{m\times 1}$ and anticipated 'news shocks' $E_t[\epsilon_{t+k}^a] = \epsilon_{t+k}^a$, $\forall k > 0$, where $\epsilon_t^a = 0_{m\times 1}$ for all $t \geq \tilde{T} + 1$. Then the solution to system (15) is given by Proposition 1, except that for $t = 0, 1, ..., \tilde{T} - 1$,

$$\Psi_{\tilde{T}-t} = (I_n - A_t\Omega_{\tilde{T}-t-1})^{-1}[A_t(\Psi_{\tilde{T}-t-1} + \Gamma_{\tilde{T}-t-1}\epsilon_{t+1}^a) + D_t].$$

Proof. It follows from Proposition 1 with minor changes. See Online Appendix. ■

Now suppose parameter values may evolve over time, as under a 'gradualist' policy. In this case there are several consecutive structural changes before the final structure is reached. To make this concrete, suppose the anticipated structural change in Proposition 1 were augmented with anticipated parameter changes in the intervening periods. There are thus $\tilde{T}$ different structures $\{A_t, B_t, C_t, D_t\}_{t=1}^{\tilde{T}}$ *in addition to* the initial and terminal structure. As a result, a necessary condition for existence is that an additional $\tilde{T}$ invertibility conditions be met to ensure that the structural matrices A_t, B_t, C_t, D_t are well-defined for all $t \in [1, \tilde{T}]$. In addition, each different regime i will have its own indicator variable, say $\mathbb{1}_t^i$.[11] The solution to this problem is covered by Proposition 1 but now the structural matrices A_t, B_t, C_t, D_t can (potentially) take on different values for all $t \in [0, \tilde{T}]$.

Remark 1 Suppose periods 0 to $\tilde{T}$ are characterized by consecutive anticipated structural changes, described by $\tilde{T}+1$ separate structures, $\{A_t, B_t, C_t, D_t\}_{t=0}^{\tilde{T}}$, in addition to the terminal structure $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$. Then the solution is given by Proposition 1.

Proof. It follows immediately from Proposition 1 since the notation in the latter allows an arbitrary sequence of structures in periods 0 to $\tilde{T}$. ■

An application would be a gradual adjustment of parameters towards the final structure; for example, a pension reform or tax amendment that is phased in gradually.

3.8 Coexistence of informed and uninformed agents

I now turn briefly to the case where there are some *uninformed* agents (see Definition 2). We may think of such agents as having a restricted information set: they do not have knowledge about future structural changes more than K periods ahead, and they expect the last known structure (defined by $\mathbb{1}_{t+K}$) to remain in place in later periods.

If some fraction $1-\lambda$ of agents are uninformed, the economy-wide expectation is given by

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)\hat{E}_t x_{t+1}.$$

This specification nests fully anticipated structural change when $\lambda = 1$ and all uninfomed agents when $\lambda = 0$. As $K \to \infty$ the uninformed receive full information (and hence $\hat{E}_t x_{t+1} = E_t x_{t+1}$ for all t); for $K = 1$, a fraction $1-\lambda$ of agents are very poorly informed: they do not learn about changes in structure until the previous period. Note that informed agents take

[11] Writing each regime as $B_{1,i}x_t = B_{2,i}E_t x_{t+1} + B_{3,i}x_{t-1} + B_{4,i}z_t + B_{5,i},\ i = 1, ..., \tilde{T}$ (see (7) and (8)), we reqiire $\det[B_{1,i}] \neq 0$ for all i for $A_i \equiv B_{1,i}^{-1}B_{2,i},\ B_i \equiv B_{1,i}^{-1}B_{3,i},\ C_i \equiv B_{1,i}^{-1}B_{4,i},\ D_i \equiv B_{1,i}^{-1}B_{5,i}$ to be defined. The associated indicator variables $\mathbb{1}_t^i$ and structural form are easy to define in terms of these matriices.

account of uninformed agents when forming their expectations, but the reverse is not true.[12]

There are many possible applications of this framework. For example, some agents may be slow to process information so that expectations are 'sticky' following the release of news. Alternatively, agents may lack clarity about the distant structure. The solution in this case is a simple extension of previous results and is summarized in the following corollary.

Corollary 4 Suppose that a fraction $1-\lambda$ of agents are uninformed and receive information about structural changes $K \geq 1$ periods in advance, where $1 \leq K \leq \tilde{T}$ and $\tilde{T}+1$ is the date the terminal structure first applies. Then we may state the following results for $\tilde{T}=T$ and $\tilde{T}=T^*$, respectively.

Permanent structural change. The solution is given by Corollary 1, except that for $0 \leq t \leq T-K$ the matrix recursions $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ are given by

$$\Omega_{T-t} = (I_n - A\Omega^*_{T-t-1})^{-1}B, \quad \Gamma_{T-t} = (I_n - A\Omega^*_{T-t-1})^{-1}(A\Gamma^*_{T-t-1}R + C)$$

$$\Psi_{T-t} = (I_n - A\Omega^*_{T-t-1})^{-1}(A\Psi^*_{T-t-1} + D)$$

provided $\det[I_n - A\Omega^*_{T-t-1}] \neq 0$ for all $t \in [0, T-K]$, where

$$M^*_{T-t-1} \equiv \lambda M_{T-t-1} + (1-\lambda)\overline{M}, \quad \text{for } M \in \{\Omega, \Gamma, \Psi\}.$$

Temporary structural change. The solution is given by Corollary 2, except that for $0 \leq t \leq T^* - K$ and $M \in \{\Omega, \Gamma, \Psi\}$, the lagged terms in the matrix expressions for $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ are given by:

$$\begin{array}{ll} \lambda M_{T-t-1} + (1-\lambda)\overline{M}, & t \in [0, T-K] \\ \lambda M_{T^*-t-1} + (1-\lambda)\tilde{M}_{T-t-1}, & t \in [T+1-K, T] \\ \lambda M_{T^*-t-1} + (1-\lambda)\tilde{M}, & t \in [T+1, T^*-K] \end{array}$$

provided the relevant matrix inverses exist for $t \in [0, T^*-K]$, and where the matrices $\tilde{\Omega}_{T-t-1}$, $\tilde{\Gamma}_{T-t-1}$, $\tilde{\Psi}_{T-t-1}$ follow the recursions in Corollary 1.

Proof. See the Online Appendix. ■

[12] I assume that the uninformed agents ignore the presence of informed agents. This makes sense because the information set of uninformed agents is a subset of the information set of the informed agents (see Definition 2), and hence $\hat{E}_t[\tilde{E}_t x_{t+1}] = \lambda \hat{E}_t[E_t x_{t+1}] + (1-\lambda)\hat{E}_t x_{t+1} = \hat{E}_t x_{t+1}$. Therefore, the uninformed agents solve a misspecified model in periods in which they are ignorant of structural change.

3.9 Optimal timing of announcements

One important application of the above results is *timing of announcements*. To make this concrete, suppose that the central bank plans to implement a new policy, and that these plans are unknown to members of the general public. If the policy is announced at date $T+1-K$, then at this point the public will learn about the future policy and their expectations will respond accordingly. Since different announcement dates will correspond to different paths for the endogenous variables, not all paths will be equally desirable. In other words, the central bank faces the question: *what is the optimal announcement date*?

We may think here of optimality being measured in terms of social welfare or some other target variables that matter to households. To study such problems, we can use the framework of uninformed agents who receive information about future structural changes K periods in advance (Definition 2). In particular, for a structural change that begins at date $t = T+1$, we can think of dates $0, 1, .., T$ as the possible announcement dates, which correspond to the cases $K = T+1$, $K = T$,...,$K = 1$, respectively. Note that the structural change in question may be temporary or permanent, but I assume that the *entire* future structure is known from the announcement date, $T+1-K$.

Remark 2 Suppose agents receive information about all future structural changes at date $T+1-K$, where $T+1$ is the date when structural change first occurs and $K \in [1, T+1]$ determines the announcement date. Then we may state the following.

Permanent structural change. The solution is given by Corollary I, except that

$$x_t = \begin{cases} \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi} & \text{if } 0 \leq t < T+1-K \\ \Omega_{T-t} x_{t-1} + \Gamma_{T-t} z_t + \Psi_{T-t} & \text{if } T+1-K \leq t \leq T. \end{cases}$$

Note that if $K = T+1$ we have the full information solution in Corollary 1.

Temporary structural change. The solution is given by Corollary 2, except that

$$x_t = \begin{cases} \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi} & \text{if } 0 \leq t < T+1-K \\ \Omega_{T^*-t} x_{t-1} + \Gamma_{T^*-t} z_t + \Psi_{T^*-t} & \text{if } T+1-K \leq t \leq T^* \end{cases}$$

where the matrices $\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}$ follow Corollary 2 for $T^*+1-K \leq t \leq T^*$.

Remark 2 says that, for delayed announcement, one may relabel date 0 as date $T+1-K$ and use 'truncated' versions of the recursions for the solution matrices. For an application of optimal announcement dates to pension reform, see the final section.

4 Extensions

This section considers two extensions. First, I consider the case of imperfect credibility of structural changes, whereby agents are sceptical about whether a structural change will be implemented as announced. Second, I show how the method presented above can deal with structural changes that involve multiple equilibria (i.e. sunspots).

4.1 Imperfect credibility

A reform is said to *imperfectly credible* if some agents doubt whether structural changes will be implemented as announced. Imperfect credibility concerns were an important consideration in the Bank of Canada's decision to renew its inflation target in 2011, rather than switch to a price-level target (see Ambler (2014)). I briefly review past approaches to imperfect credibility before providing two new solutions using the recursive method set out above.

4.1.1 Past approaches to imperfect credibility

One strand of literature has focused on settings where changes in structure are implemented as announced, but imperfect credibility is present in the form of doubts about whether the current structure (e.g. policy) will revert to the previous one. Any changes in structure are complete by date T, but imperfect credibility may persist beyond this to some date $\tilde{T} \geq T$.

This approach is typically modelled by having some fraction λ of agents form rational expectations, while the remaining fraction who doubt the announced structure base their expectations either on previous targets or realized outcomes. For example, Ball (1995), Goodfriend and King (2005) and Ascari and Ropele (2013) all model disinflations using inflation expectations of the form $\hat{E}_t\pi_{t+1} = \lambda E_t\pi_{t+1} + (1-\lambda)\pi^H$, where π^H is the old inflation target. Nicolae and Nolan (2006) also model disinflations under imperfect credibility, but in their model this enters through money supply expectations based partly on realized money supply rather than the announced path. In the above papers the degree of credibility λ is exogenous and, in some cases, time varying.

These expectations specifications can be easily be modelled using our method. For instance, if economy-wide expectations are given by $\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1-\lambda_t)E_t^{IC} x_{t+1}$, where $E_t^{IC} x_{t+1} = F_0 x_t + F_1 x_{t-1} + G z_t + H$ (for comformable matrices F_0, F_1, G and H) and $\lambda_t \in (0, 1]$ (given), then by (10) the model can be written the form

$$x_t = \hat{A}_t(\lambda_t E_t x_{t+1} + (1-\lambda_t)F_0 x_t) + \hat{B}_t x_{t-1} + \hat{C}_t z_t + \hat{D}_t \tag{27}$$

where $\hat{A}_t \equiv A(\mathbb{1}_t)$, $\hat{B}_t \equiv B(\mathbb{1}_t) + (1-\lambda_t)F_1$, $\hat{C}_t \equiv C(\mathbb{1}_t) + (1-\lambda_t)G$, $\hat{D}_t \equiv D(\mathbb{1}_t) + (1-\lambda_t)H$.

I assume that $\lambda_t = 1$ for all $t \geq \tilde{T} + 1$, as it seems reasonable to assume that doubts will eventually 'die out'. Structural change (if any occurs) happens prior to date $\tilde{T} + 1$, and I assume (without loss of generality) that the terminal structure is the alternative regime. To obtain a recursive solution, we may guess that $x_t = \Omega_{\tilde{T}-t} x_{t-1} + \Gamma_{\tilde{T}-t} z_t + \Psi_{\tilde{T}-t}$, which implies that $E_t x_{t+1} = \Omega_{\tilde{T}-t-1} x_t + \Gamma_{\tilde{T}-t-1} R z_t + \Psi_{\tilde{T}-t-1}$ in periods 0 to $\tilde{T} - 1$. Note that since $\lambda_t = 1$ for $t \geq \tilde{T} + 1$, the structure in (27) is fixed from date $\tilde{T} + 1$ onwards. It follows that $E_t x_{t+1} = \tilde{\Omega} x_t + \tilde{\Gamma} R z_t + \tilde{\Psi}$ for all $t \geq \tilde{T}$. The solution to this problem is as follows.

Corollary 5 Suppose a fraction λ_t of agents have rational expectations, and the remaining fraction $1 - \lambda_t$ have expectations $E_t^{IC} x_{t+1} = F_0 x_t + F_1 x_{t-1} + G z_t + H$, with $\lambda_t = 1$ for $t \geq \tilde{T} + 1$. Then the solution to (27) is given by Proposition 1, except that

$$\Omega_0 = [I_n - \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}} \tilde{\Omega} + (1 - \lambda_{\tilde{T}}) F_0)]^{-1} \hat{B}_{\tilde{T}}, \quad \Gamma_0 = [I_n - \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}} \tilde{\Omega} + (1 - \lambda_{\tilde{T}}) F_0)]^{-1} (\hat{A}_{\tilde{T}} \tilde{\Gamma} R + \hat{C}_{\tilde{T}}),$$

$$\Psi_0 = [I_n - \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}} \tilde{\Omega} + (1 - \lambda_{\tilde{T}}) F_0)]^{-1} (\hat{A}_{\tilde{T}} \tilde{\Psi} + \hat{D}_{\tilde{T}}),$$

and for $t = 0, 1, 2, ..., \tilde{T} - 1$,

$$\Omega_{\tilde{T}-t} = [I_n - \hat{A}_t(\lambda_t \Omega_{\tilde{T}-t-1} + (1 - \lambda_t) F_0)]^{-1} \hat{B}_t,$$

$$\Gamma_{\tilde{T}-t} = [I_n - \hat{A}_t(\lambda_t \Omega_{\tilde{T}-t-1} + (1 - \lambda_t) F_0)]^{-1} (\hat{A}_t \Gamma_{\tilde{T}-t-1} R + \hat{C}_t),$$
$$\Psi_{\tilde{T}-t} = [I_n - \hat{A}_t(\lambda_t \Omega_{\tilde{T}-t-1} + (1 - \lambda_t) F_0)]^{-1} (\hat{A}_t \Psi_{\tilde{T}-t-1} + \hat{D}_t)$$

provided that $\det[I_n - \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}} \tilde{\Omega} + (1 - \lambda_{\tilde{T}}) F_0)] \neq 0$ and

$$\det[I_n - \hat{A}_t(\lambda_t \Omega_{\tilde{T}-t-1} + (1 - \lambda_t) F_0)] \neq 0, \qquad \text{for all } t \in [0, \tilde{T} - 1].$$

Proof. It follows from Proposition 1 with minor changes. See Online Appendix. ■

The above approach is intuitive and easy to implement, but the expectations $E_t^{IC} x_{t+1}$ are clearly *ad hoc*. The imperfect credibility solutions I set out below are instead based on appropriately defined conditional expectations. The approach taken is similar to researchers at the Bank of Canada when assessing the performance of price-level targeting (PT) versus inflation targeting (IT) under imperfect credibility (Kryvtsov et al. (2008)), whereby inflation and output gap expectations were modelled as $\hat{E}_t q_{t+1} = p_t E[q_{t+1}|s_t, PT] + (1 - p_t) E[q_{t+1}|s_t, IT]$, where $q_t \in \{\pi_t, y_t\}$, p_t is the degree of credibility, and s_t is the state at the start of period t. However, these expectations are not rational because the agents who (correctly) believe in PT ignore those who do not when forming their expectations. By contrast, in the solutions presented below, a fraction λ of agents have model-consistent expectations *in all periods.*

Other approaches in the literature are less closely related because they either allow for recurrent unanticipated shifts in structure whose probability distribution is known (see

Schaumburg and Tambalotti (2007)) or depart from rational expectations altogether (Gibbs and Kulish (2017)). A recent paper that is more closely related is Haberis et al. (2019), in which the degree of credibility is endogenized under the assumption of certainty equivalence. I discuss a possible extension to endogenous credibility in more detail below.

4.1.2 Imperfect credibility: alternative solutions

Let $p_t \in [0,1]$ (given) be the degree of credibility of the current (fixed) structure, which may be interpreted as the sceptical agents' *perceived* probability that the current structure will remain in place in the next period. I assume that $p_t = 1$ for $t \geq T+1$ so that doubts eventually disappear. A fraction λ of agents have rational expectations and the remaining fraction $1-\lambda$ have doubts, in the form of imperfect credibility, in periods 0 to T. These agents have access to the information set I_t (see (12)) and hence they know the current structure and observe the current and past values of all variables. They also receive communication about the intended future structure (which they do not believe fully in periods $0 \leq t \leq T$). During periods 0 to T, the expectations $E_t^{IC} x_{t+1}$ are a credibility-weighted average of the expected solutions under full credibility and the previous regime. Where necessary, expectations are distinguished from rational expectations via the operator $\hat{E}_t$.

Type I Credibility

Suppose there is a *permanent* change in structure at the start of period $t = 0$. Agents observe the structural change and are told it is permanent, but some fraction $1-\lambda$ are sceptical and believe that will probability $1-p_t$ the structure will permanently revert to the previous one. These doubts last from periods 0 to T. The remaning fraction λ form rational expectations in all periods. Economy-wide expectations are thus given by

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda) E_t^{IC} x_{t+1}$$

where $E_t^{IC} x_{t+1}$ is the expectation of the sceptical agents.

We may conjecture that the solution has the form

$$x_t = \begin{cases} \Omega_{T-t} x_{t-1} + \Gamma_{T-t} z_t + \Psi_{T-t} & 0 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi} \equiv x_t^1 & t \geq T+1 \end{cases}$$

where $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$ are given in Corollary 2.

For future reference, the fixed-structure solution under the previous regime is

$$x_t^2 \equiv \tilde{\Omega} x_{t-1} + \tilde{\Gamma} z_t + \tilde{\Psi} \tag{28}$$

where $\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$ are given in Proposition 1.

The system to be solved for all $t \geq 0$ is

$$x_t = A\tilde{E}_t x_{t+1} + Bx_{t-1} + Cz_t + D \tag{29}$$

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)E_t^{IC} x_{t+1}, \qquad E_t^{IC} x_{t+1} = p_t \hat{E}_t x_{t+1}^1 + (1-p_t)\hat{E}_t x_{t+1}^2, \tag{30}$$

where $p_t = 1$ for $t \geq T+1$ and $\hat{E}_t$ is the expectation conditional on information I_t (see (12)).

Note that $E_t^{IC} x_{t+1}$ is specified as a credibility-weighted average of the (expected) fixed-structure solutions under each regime. This formulation has the intuitive intepretation that the sceptical agents think the economy will revert permanently to the previous structure with probability $1-p_t$ and stick with the current structure with probability p_t.

Proposition 2 The solution to model (29)–(30) is given by

$$x_t = \begin{cases} \Omega_{T-t} x_{t-1} + \Gamma_{T-t} z_t + \Psi_{T-t}, & 0 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi}, & t > T \end{cases}$$

where

$$\begin{aligned}
\Omega_0 &= [I_n - A_{p_T}(\lambda)]^{-1} B \\
\Gamma_0 &= [I_n - A_{p_T}(\lambda)]^{-1}[A([\lambda + (1-\lambda)p_T]\overline{\Gamma} + (1-\lambda)(1-p_T)\tilde{\Gamma})R + C] \\
\Psi_0 &= [I_n - A_{p_T}(\lambda)]^{-1}[A([\lambda + (1-\lambda)p_T]\overline{\Psi} + (1-\lambda)(1-p_T)\tilde{\Psi}) + D],
\end{aligned}$$

$\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$ are defined in Corollary 2 and, for $t = 0, 1, ..., T-1$,

$$\Omega_{T-t} = [(I_n - A_{p_t}(\lambda, \Omega_{T-t-1})]^{-1} B$$

$$\Gamma_{T-t} = [I_n - A_{p_t}(\lambda, \Omega_{T-t-1})]^{-1}[A(\lambda\Gamma_{T-t-1} + (1-\lambda)[p_t\overline{\Gamma} + (1-p_t)\tilde{\Gamma}])R + C]$$

$$\Psi_{T-t} = [I_n - A_{p_t}(\lambda, \Omega_{T-t-1})]^{-1}[A(\lambda\Psi_{T-t-1} + (1-\lambda)[p_t\overline{\Psi} + (1-p_t)\tilde{\Psi}]) + D]$$

provided $\det[I_n - A_{p_T}(\lambda)] \neq 0$ and the following regularity condition is satisfied:

$$\det[I_n - A_{p_t}(\lambda, \Omega_{T-t-1})] \neq 0, \qquad t = 0, 1, ..., T-1,$$

where

$$A_{p_T}(\lambda) \equiv A\left([\lambda + (1-\lambda)p_T]\overline{\Omega} + (1-\lambda)(1-p_T)\tilde{\Omega}\right)$$

$$A_{p_t}(\lambda, \Omega_{T-t-1}) \equiv A\left(\lambda\Omega_{T-t-1} + (1-\lambda)[p_t\overline{\Omega} + (1-p_t)\tilde{\Omega}]\right).$$

Proof. See the Appendix. ■

Note that since no changes in structure occur after date 0, agents' doubts about the new structure are ultimately unfounded. However, since $p_t = 1$ for all $t \geq T+1$, doubts eventually disappear and hence economy-wide expectations coincide with rational expectations from period $T+1$ onwards. The assumption that imperfect credibility 'dies out' is standard in the literature (see e.g. Nicolae and Nolan (2006)).

Type II Credibility

I now set out an alternative solution better suited for *temporary* changes in structure, such as forward guidance or fiscal stimulus. The key distinction is that sceptical agents form expectations that allow a future change in structure, while harbouring doubts that a permanent reversion will occur *sooner* than announced. I refer to this as Type II credibility.

Suppose a *temporary* structural change takes place at the start of period 0 and is permanently reversed in period $T+1$. Imperfect credibility is present during periods 0 to T, among fraction $1-\lambda$ of agents who fear that the reversion to the original structure will happen sooner than announced. The degree of credibility $p_t \in [0,1]$ is given. Any doubts are eliminated from period $T+1$ onwards, i.e. $p_t = 1$ once the final structure is in place.

We may guess that the solution has the form

$$x_t = \begin{cases} \Omega_{T-t}x_{t-1} + \Gamma_{T-t}z_t + \Psi_{T-t} & 0 \leq t \leq T \\ \tilde{\Omega}x_{t-1} + \tilde{\Gamma}z_t + \tilde{\Psi} \equiv x_t^2 & t \geq T+1 \end{cases} \tag{31}$$

where the solution matrices $\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$ are given in Proposition 1.

The system to be solved is

$$x_t = \begin{cases} A\tilde{E}_t x_{t+1} + Bx_{t-1} + Cz_t + D & 0 \leq t \leq T \\ \tilde{A}\tilde{E}_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & t \geq T+1 \end{cases} \tag{32}$$

subject to $\tilde{E}_t x_{t+1} = E_t x^1_{t+1} = \tilde{\Omega}x_t + \tilde{\Gamma}Rz_t + \tilde{\Psi}$ for all $t \geq T+1$ and, for $0 \leq t \leq T$,

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)E_t^{IC}x_{t+1}, \qquad E_t^{IC}x_{t+1} = p_t E_t^* x_{t+1} + (1-p_t)\hat{E}_t x^2_{t+1} \tag{33}$$

where $\hat{E}_t x^2_{t+1} = \tilde{\Omega}x_{t-1} + \tilde{\Gamma}Rz_t + \tilde{\Psi}$ by (31) and $E_t^* x_{t+1}$ is the (expected) solution under a fully credible permanent change in structure, as stated in Corollary 1.

Proposition 3 The solution to the model (32)–(33) is given by

$$x_t = \begin{cases} \Omega_{T-t}x_{t-1} + \Gamma_{T-t}z_t + \Psi_{T-t} & \text{for } 0 \leq t \leq T \\ \tilde{\Omega}x_{t-1} + \tilde{\Gamma}z_t + \tilde{\Psi} & \text{for } t \geq T+1 \end{cases}$$

where

$$
\begin{aligned}
\Omega_0 &= [I_n - A_{p_T}(\lambda, \Omega_0^*)]^{-1} B \\
\Gamma_0 &= [I_n - A_{p_T}(\lambda, \Omega_0^*)]^{-1} [A([\lambda + (1-\lambda)(1-p_T)]\tilde{\Gamma} + (1-\lambda)p_T\Gamma_0^*)R + C] \\
\Psi_0 &= [I_n - A_{p_T}(\lambda, \Omega_0^*)]^{-1}[A([\lambda + (1-\lambda)(1-p_T)]\tilde{\Psi} + (1-\lambda)p_T\Psi_0^* + D],
\end{aligned}
$$

$\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$ and $\Omega_0^*, \Gamma_0^*, \Psi_0^*$ are given by Corollary 1 and, for $t = 0, 1, ..., T-1$,

$$\Omega_{T-t} = [(I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega_{T-t-1}^*)]^{-1} B$$

$$\Gamma_{T-t} = [I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega_{T-t-1}^*)]^{-1}[A(\lambda\Gamma_{T-t-1} + (1-\lambda)[p_t\Gamma_{T-t-1}^* + (1-p_t)\tilde{\Gamma}])R + C]$$

$$\Psi_{T-t} = [I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega_{T-t-1}^*)]^{-1}[A(\lambda\Psi_{T-t-1} + (1-\lambda)[p_t\lambda\Psi_{T-t-1}^* + (1-p_t)\tilde{\Psi}]) + D]$$

provided $\det[I_n - A_{p_T}(\lambda, \Omega_0^*)] \neq 0$ and the following regularity condition is satisfied:

$$\det[I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega_{T-t-1}^*)] \neq 0, \qquad t = 0, 1, ..., T-1,$$

where

$$A_{p_T}(\lambda, \Omega_0^*) \equiv A\left([\lambda + (1-\lambda)(1-p_T)]\tilde{\Omega} + (1-\lambda)p_T\Omega_0^*\right)$$

$$A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega_{T-t-1}^*) \equiv A\left(\lambda\Omega_{T-t-1} + (1-\lambda)[p_t\Omega_{T-t-1}^* + (1-p_t)\tilde{\Omega}]\right)$$

and $\{\Omega_{T-t}^*, \Gamma_{T-t}^*, \Psi_{T-t}^*\}_{t=0}^{T-1}$ are given by the matrix recursion in Corollary 1.

Proof. See the Appendix. ■

The above solution pertains to imperfect credibility of *temporary* structural changes. Specifically, it applies when some fraction of agents suspect that a temporary policy will be reversed *sooner* than announced. There is a growing literature on reversals of pre-announced temporary policies, as it is well known that benevolent policymakers may have incentives to renege on such policies ex post. Doubts can also be motivated by political considerations, such as possible regime changes or the non-binding nature of manifesto promises.

Thus far, credibility p_t has been taken as exogenous and potentially time-varying. An extension would to be endogenize credibility in response to past economic outcomes $(x_{t-1}, , ...)$ or the temptation to renege as captured by social welfare. A recent paper that endogenizes credibility, under the assumption of certainty equivalence, is Haberis et al. (2019). It is non-trivial to endogenize credibility in the above solutions as future expectations over credibility matter for the current solution x_t; however, numerical methods may be used to solve the resulting problem, and the recursive solutions in Propositions 2 and 3 make this fairly straightforward to implement in practice.[13]

[13] In particular, under the assumption of certainty equivalence (i.e. zero future shocks), one may select

4.2 Indeterminacy

In this section I show how to deal with the case of non-fundamental (sunspot) solutions. Such solutions are of interest in their own right (see Farmer and Guo (1994)), but they are of particular relevance here because the propositions show that solutions under strucutural change are unique only if the terminal solution is determinate. If not, there are a multiplicity of stable terminal solutions and the user will need to pick one to resolve the indeterminacy. Standard methods may either fail in this case (e.g. Klein (2000)) or provide one solution along with a warning (e.g. Sims (2002)). Hence, it may be useful to outline an approach to pick a preferred solution from the set of indeterminate equilibria.

I apply the method of Farmer et al. (2015), which involves moving non-fundamental expectational errors (i.e. sunspots) to the vector of fundamental shocks. Since this reclassification resolves the indeterminacy problem, the resulting model can be solved using standard methods. They show how to apply this method to models written in the Sims (2002) form and to models solved in Dynare (Adjemian et al. (2011)). Here I show how this approach can be applied to models in the Binder-Pesaran canonical form used in the present paper.

Suppose k expectational variables are subject to self-fulfilling expectations shocks (i.e. sunspots).[14] The first step is to write the equations of these k variables: $x^k_{i,t} = s_{i,t-1} + \eta^k_{i,t}$, for $i = 1, .., k$, where $s_{i,t} \equiv E_t x^k_{i,t+1}$. Note that the expectational errors $\eta^k_{i,t}$ must satisfy $E_{t-1}[\eta^k_{i,t}] = 0$ for all i by the definition of rational expectations.

The augmented model thus consists of the original system plus k new equations:

$$x^k_t = s_{t-1} + \eta^k_t \tag{34}$$

$$B_1 x_t = B_2 E_t x_{t+1} + B_3 x_{t-1} + B_4 z_t + B_5 \tag{35}$$

where x^k_t, s_t and η^k_t are $k \times 1$ vectors.

Partitioning (35) into sunspot and non-sunspot variables leads to

$$\begin{bmatrix} B_{11} & B_{12} \end{bmatrix} \begin{bmatrix} x^k_t \\ x^{n-k}_t \end{bmatrix} = \begin{bmatrix} B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} E_t x^k_{t+1} \\ E_t x^{n-k}_{t+1} \end{bmatrix} + \begin{bmatrix} B_{31} & B_{32} \end{bmatrix} \begin{bmatrix} x^k_{t-1} \\ x^{n-k}_{t-1} \end{bmatrix} + B_4 z_t + B_5$$

solutions x_t for which the expected path $\{p^e_{t+j}\}_{j=1}^{T+1-t}$ is consistent with the endogenous path $\{p_{t+j}\}_{j=1}^{T+1-t}$. Note that this approach simply requires guessing on a path $\{p^e_{t+j}\}_{j=1}^{T+1-t}$ at each t and accepting as a solution the guesses for which the differences $p^e_{t+j} - p_{t+j}$ are approximately zero for all $j \in \{1, .., T+1-t\}$.

[14] Formally, if there are n_1 unstable generalized eigenvalues and p non-fundamental errors then (under some regularity assumptions) there are $k = n_1 - p$ degrees of indeterminacy (see Farmer et al. (2015)). The degree of indeterminacy can be determined using, for example, the Sims (2002) method. Note that if $p > k$ then the user must choose k of the expectations to be hit by sunspot fluctuations.

where B_{i1}, B_{i2}, $i \in \{1,2,3\}$, are $n \times k$ and $n \times n-k$ matrices, respectively.

Following Farmer et al. (2015), the identity $s_t = E_t x_{t+1}^k$ is used to substitute out $E_t x_{t+1}^k$. A partitioned version of the system in (34)–(35) in the Binder-Pesaran form is then

$$\tilde{B}_1 \begin{bmatrix} x_t^k \\ x_t^{n-k} \\ s_t \end{bmatrix} = \tilde{B}_2 \begin{bmatrix} E_t x_{t+1}^k \\ E_t x_{t+1}^{n-k} \\ E_t s_{t+1} \end{bmatrix} + \tilde{B}_3 \begin{bmatrix} x_{t-1}^k \\ x_{t-1}^{n-k} \\ s_{t-1} \end{bmatrix} + \tilde{B}_4 \begin{bmatrix} \eta_t^k \\ z_t \end{bmatrix} + B_5$$

or

$$\tilde{B}_1 \tilde{x}_t = \tilde{B}_2 E_t \tilde{x}_{t+1} + \tilde{B}_3 \tilde{x}_{t-1} + \tilde{B}_4 \tilde{z}_t + \tilde{B}_5 \tag{36}$$

$$\tilde{z}_t = \tilde{R}\tilde{z}_{t-1} + \tilde{\epsilon}_t, \qquad E_t[\tilde{\epsilon}_{t+1}] = 0_{k+m} \tag{37}$$

where

$$\tilde{B}_1 = \begin{bmatrix} I_k & 0_{k\times n-k} & 0_{k\times k} \\ B_{11} & B_{12} & -B_{21} \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0_{k\times k} & 0_{k\times n-k} & 0_{k\times k} \\ 0_{n\times k} & B_{22} & 0_{n\times k} \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} 0_{k\times k} & 0_{k\times n-k} & I_k \\ B_{31} & B_{32} & 0_{n\times k} \end{bmatrix}$$

$$\tilde{B}_4 = \begin{bmatrix} I_k & 0_{k\times m} \\ 0_{n\times k} & B_4 \end{bmatrix}, \quad \tilde{B}_5 = \begin{bmatrix} 0_{k\times 1} \\ B_5 \end{bmatrix}, \quad \tilde{R} = \begin{bmatrix} 0_{k\times k} & 0_{k\times m} \\ 0_{m\times k} & R \end{bmatrix}, \quad \tilde{\epsilon}_t = \begin{bmatrix} \epsilon_{\eta,t} \\ \epsilon_t \end{bmatrix},$$

and $\epsilon_{\eta,t}$ is a $k \times 1$ vector of sunspot shocks.

Provided that $\tilde{B}_1$ is invertible, our solution method can be used. Lastly, note that since the solution method requires that the vector of endogenous variables, $\tilde{x}_t$, be common across all time periods, the earlier structure(s) should be written in the same form as (36) and (37). This simply requires adding k equations of the form $s_t = E_t x_{t+1}^k$ to the model.[15] Since this requires standard manipulations, I spare readers the technical details.

Example 2 revisited. Consider the following version of the model in Example 2:

$$q_t = \beta E_t q_{t+1} - \sigma r_t + z_t, \qquad r_t = r^* \tag{38}$$

where I now assume $\beta \in (0,1]$.

If $\beta = 1$ the model is indeterminate: there are many stable solutions for q_t. To solve the model we can follow the approach set out above. Hence, we set $q_t = s_{t-1} + \eta_{q,t}$, where $s_t \equiv E_t q_{t+1}$ and $E_{t-1}[\eta_{q,t}] = 0$. After adding the equation for q_t to the system

[15] Note that in this case the sunspot shocks η_t^k do not enter the model and so are multiplied by zero matrices when the model is written in the amended Binder-Pesaran form in (36) –(37).

and setting $E_t q_{t+1} = s_t$ the system can be written in the form of (36) as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & \sigma & -1 \end{bmatrix} \begin{bmatrix} q_t \\ r_t \\ s_t \end{bmatrix} + 0_{3\times 3} \begin{bmatrix} E_t q_{t+1} \\ E_t r_{t+1} \\ E_t s_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_{t-1} \\ r_{t-1} \\ s_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_{q,t} \\ z_t \end{bmatrix} + \begin{bmatrix} 0 \\ r^* \\ 0 \end{bmatrix}. \quad (39)$$

If $\beta \in (0,1)$ the model is determinate and can be written in the same form as (39) by appending $s_t = E_t q_{t+1}$ to the original system, (38). One may then study, for example, a permanent anticipated shift from $\beta \in (0,1)$ to a sunspot solution where $\beta = 1$. Of course, the method is general and can easily be applied to more interesting cases.

5 Applications

I now consider some numerical applications. The first application uses a New Keynesian model to study monetary policy announcements. The second application is a pension reform in the Diamond (1965) model that is used to study optimal announcement dates.

5.1 A New Keynesian model

Following Cagliarini and Kulish (2013), we consider a simplified version of the New Keynesian model in Ireland (2007) in which the inflation target π^* is non-stochastic; the deviation of technology from its steady state, a_t, is stationary; and there are no habits in consumption. Under these assumptions, the model is given by the following set of log-linear equations:

$$y_t = E_t y_{t+1} - \sigma^{-1}(r_t - E_t \pi_{t+1}) + \frac{(1-\rho_g)}{\sigma} g_t - \sigma^{-1} \ln(\beta) \quad (40)$$

$$\pi_t = \frac{1}{(1+\beta\alpha)} \left(\beta E_t \pi_{t+1} + (1 + \beta\alpha - \alpha - \beta)\pi^* + \alpha \pi_{t-1} + \psi\sigma y_t - \psi a_t - e_t\right) \quad (41)$$

$$r_t = (1-\rho_r)\overline{r} + \rho_r r_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) \quad (42)$$

where the shocks to technology, demand and the mark-up follow AR(1) processes:

$$v_t = \rho_v v_{t-1} + \epsilon_{v,t}, \qquad v \in \{a, g, e\}, \qquad \epsilon_{v,t} \sim IID(0, \sigma_v^2).$$

Eqs. (40)–(42) are the IS curve, the Phillips curve and the interest rate rule. In these equations, y_t is output expressed in log deviations from steady state; π_t is the log inflation rate between periods t and $t-1$, and r_t is the log nominal interest rate. The steady-state nominal interest rate is $\overline{r} = \pi^* - \ln(\beta)$. I use the same parameter values as in Cagliarini and Kulish (2013): $\pi^* = 0.0125$ (quarterly), $\beta = 0.9925$, $\sigma = 1$, $\alpha = 0.25$, $\psi = 0.1$, $\rho_r = 0.65$, $\theta_\pi = 0.5$, $\theta_y = 0.10$, $\theta_{dy} = 0.2$; $\rho_g = \rho_a = \rho_e = 0.9$, and $\sigma_g = 0.02$, $\sigma_a = \sigma_e$.

5.1.1 A change in the inflation target

As a first exercise, suppose the central bank announces a lower inflation target. The announcement refers to the future value of π^*, which is to be reduced from 5% to 2.5% per annum. This exercise matches the one in Cagliarini and Kulish (2013). Starting from steady state, the economy is hit with an unanticipated one standard deviation demand shock, $\epsilon_g = 0.02$, in period 1. The resulting impulse responses are shown in Figure 2.

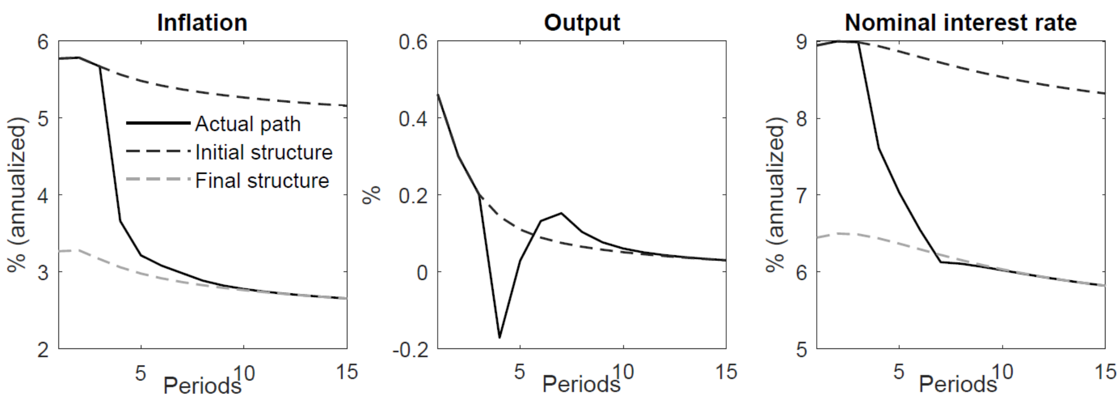

Figure 2: IRF with an anticipated change in π^*: announced period 4, implemented period 8

The impulse responses replicate those in Figure 3 of Cagliarini and Kulish (2013). Both inflation and output fall after the announcement is made in period 4. Because the monetary policy rule responds to the deviation of inflation from the current target rather than the announced target, a substantial negative inflation gap opens up and the nominal interest rate is cut; however, because expected inflation falls substantially, the real interest rate rises and hence a negative output gap results despite the cut in nominal interest rates.

5.1.2 Forward guidance

I now study forward guidance in a version of the above model. Given the presence of forward guidance, the interest rate rule (42) must be amended to

$$r_t = \begin{cases} \underline{r} \equiv 0 & t \in [T+1, T^*] \\ (1-\rho_r)\overline{r} + \rho_r r_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) & \text{otherwise.} \end{cases}$$

where $T^* - T$ is the number of periods of forward guidance.

The calibration is unchanged and technology and mark-up shocks are again set at zero. To motivate the use of forward guidance, I assume there is a large negative demand shock in period 0 ($\epsilon_g = -0.125$), similar to the exercise in Carlstrom et al. (2015).[16] The three cases of forward guidance are: a 2-period spell announced and implemented in period 0 (circles); a 4-period spell announced and implemented in period 0 (solid line); and a 2-period spell of forward guidance that is announced in period 0 but implemented in periods 2-3 (dash-dot). Note that the first two cases are nested by Corollary 1, whereas the case of pre-announced forward guidance is covered by Corollary 2.

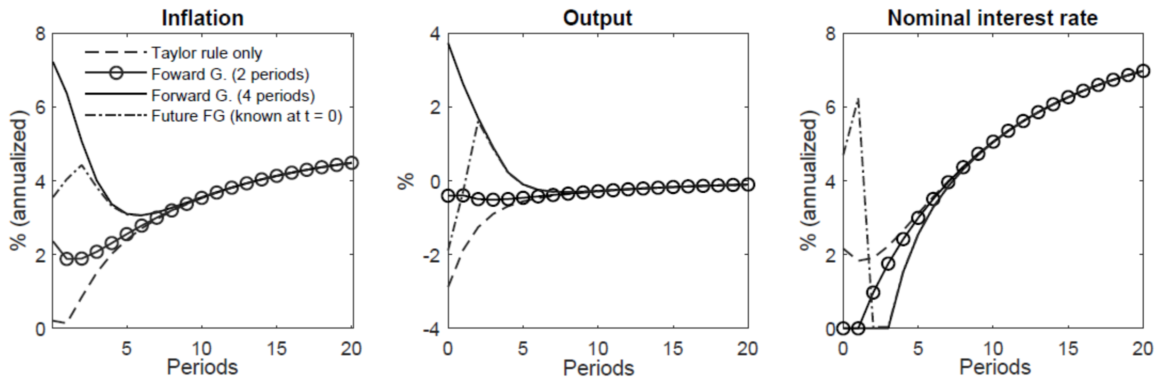

Figure 3: IRF to a large negative demand shock under forward guidance

Compared to a standard Taylor rule, forward guidance is more aggressive in cutting interest rates. The signs on impact depend on the forward guidance horizon, $T^* - T$. In the two period case, inflation falls around 3% below the target, and output falls by around 0.2%. In the 4-period case, the promise of extended zero rates generates strong expectations that raise inflation and output relative to steady state. Hence, at moderate horizons, longer forward guidance periods are more expansionary, as in Carlstrom et al. (2015). Finally, the pre-announced 2-period forward guidance policy shows that a promise today of zero future interest rates combined with a modest cut in rates from 9% to 5% (see dash-dot line, right panel) is enough to stabilize inflation and output somewhat relative to a Taylor rule.

5.2 Pension reform in the Diamond model

As a second application I consider a social security reform that reduces the pension benefit (and contribution) rate in the Diamond (1965) model. This example is useful because it illustrates the study of optimal announcement dates, as well as showing how the method set out above can be applied to log-linearized models derived from an underlying non-linear model whose steady state is affected by structural change. Since such pension reforms have been studied previously, we also keep contact with known results in the literature.

[16]Carlstrom et al. (2015) consider a smaller shock that persists for several periods without decay.

Pension reform in the log-utility case is studied by Fedotenkov (2016). I use the same perfect foresight model, except for the addition of CES utility with elasticity σ^{-1}. There are two generations alive at any given date t, the young (y) and the old (o). Households have two-period lives and care only about consumption; utility from future consumption is discounted by $\beta > 0$. The pension contribution rate is τ and the pension system is balanced budget. Population N_t grows at rate n and individual labour supply is 1. Output is produced by a representative firm with a Cobb-Douglas production function, $Y_t = K_t^{\alpha} N_t^{1-\alpha}$. Factor prices of labour and capital are w_t and R_t, respectively. Capital depreciates fully in a generation.

The central equation of the model is the capital accumulation equation:

$$k_{t+1} = \frac{\beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}}}{(1 + \frac{(1-\alpha)\tau}{\alpha}) R_{t+1} + \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1}{\sigma}}} \left(\frac{(1-\tau)(1-\alpha)}{1+n} \right) k_t^{\alpha}$$

where $k_{t+1} \equiv K_{t+1}/N_{t+1}$ and $R_t = \alpha k_t^{\alpha-1}$.

The above equation makes clear that capital accumulation depends on the pension contribution rate, τ. Except for the case of log utility ($\sigma = 1$), there is no analytical solution. Accordingly, I log-linearize the model and solve the resulting system of linear equations:

$$\begin{aligned} \hat{c}_{t,y} &= -\frac{1}{\sigma}\hat{R}_{t+1} + \hat{c}_{t+1,o}, \quad \hat{R}_t = (\alpha - 1)\hat{k}_t, \quad \hat{w}_t = \alpha \hat{k}_t \\ \hat{c}_{t,y} &= \frac{(1-\tau)(1-\alpha)k^{\alpha}}{c_y}\hat{w}_t - \frac{(1+n)k}{c_y}\hat{k}_{t+1}, \quad \hat{c}_{t,o} = \alpha \hat{k}_t \end{aligned}$$

where $c_y = (1-\tau)(1-\alpha)k^{\alpha} - (1+n)k$, k is steady state capital and 'hats' are log deviations from steady state.[17] These equations are, respectively, the Euler equation, the equilibrium factor prices, and the budget identities of young and old after market clearing is imposed.

As in Fedotenkov (2016), I consider a pension reform at date $t = 1$ that may be announced (in period 0) or unannounced. The pension contribution rate is reduced from $\tau = 0.20$ to $\tau' = 0.15$. The other parameters are $\alpha = 0.4$, $\beta = \frac{1}{(1+0.01)^{35}} = 0.760$, $n = 0$, and I initially set $\sigma \approx 1$ to keep contact with known results. The transition paths following the reform are shown in Figure 4 and match the results in Fedotenkov (2016) and the reply by Hatcher (2019). Note that whereas these authors solved the fully non-linear model under log utility, the results in Figure 4 are based on the linearized system above, but allowing for the change in steady state that occurs at date $t = 1$ when the reform is implemented.[18]

[17] There does not seem to be a general analytical solution for steady-state capital outside of the log utility case. However, the steady state capital is unique and I find it numerically.

[18] That is, the system is log-linearized around the original steady state up to $t = 0$ and around the new steady state from $t = 1$ onwards (in which τ is replaced with τ' and k and c_y are recomputed). The model is written in terms of deviations from the original steady state $x_t - x$, with any constants in matrix $\tilde{D}$.

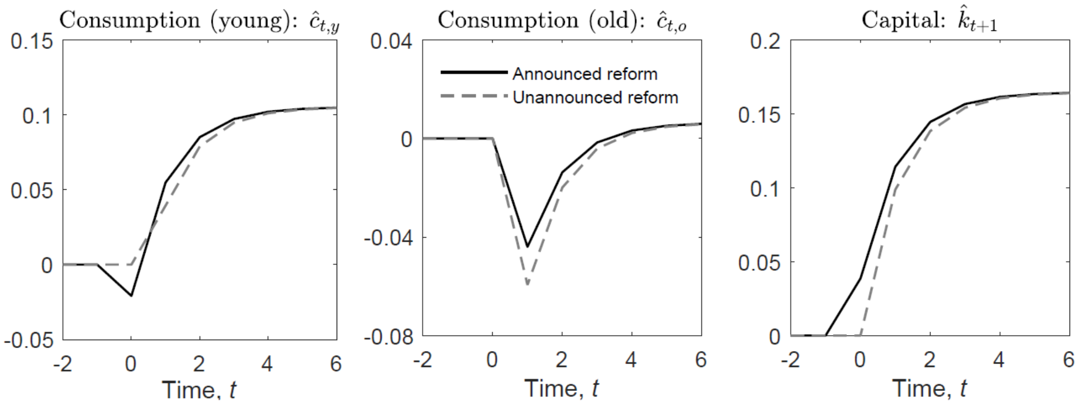

Figure 4: Impact of a pension reform at date 1. Announced vs unannounced reforms ($\sigma \approx 1$).

The announced reform initially lowers consumption by the young, as they respond to the future reduction in pension benefits by saving more so that capital accumulation rises (see left and right panels). Consumption of the old falls when the reform is implemented at $t = 1$, but the hit is less if the reform is announced, as pensioners can consume out of their extra savings when young. Since consumption of the young falls in the announcement period $t = 0$, lifetime utility goes down (Fedotenkov (2016)) and so may social welfare. Hatcher (2019) shows that period 0 social welfare is reduced for a wide range of social discount factors, again working with log utility. I now reconsider this result in the case of CES utility.

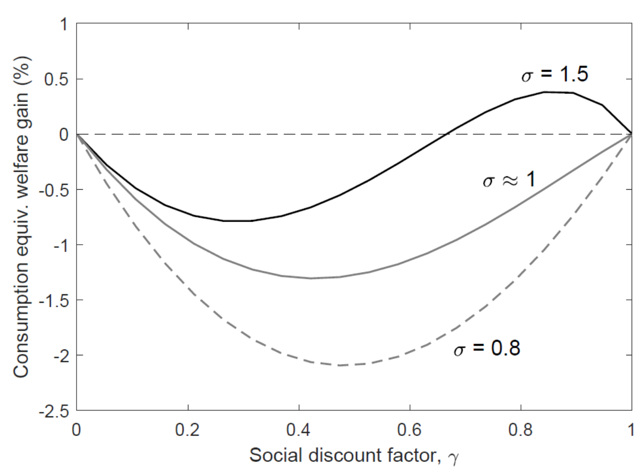

Figure 5: Is it optimal to announce pension reform? Welfare gain (loss) of announced reform.

The results in Figure 5 confirm the previous social welfare results under log utility ($\sigma = 1$), as well as for higher elasticities of intertemporal substitution ($0 < \sigma < 1$).[19] However, if the

[19] Social welfare W_0 is computed assuming a constant social discount factor $\gamma \in (0, 1)$ on future genrerations: $W_0 = \sum_{t=-1}^{\infty} \gamma^t \hat{U}_t$, where $\hat{U}_t$ is a second-order approximation of lifetime utility.

elasticity of substitution is low enough, social welfare may be higher under the announced reform for relatively high social discount factors (see solid line). This result is quite intuitive: if households are not as willing to substitute current for future consumption then the drop in young-age consumption is attenuated relative to Figure 4, and hence the utility losses of the young at $t = 0$ are smaller and can be more than offset by the (discounted) utility gains of future generations. The above result clearly speaks to the issue of optimal announcement dates, i.e. *when* to announce a future reform in order to maximize social welfare.

6 Conclusion

This paper has set out a recursive method for solving linear rational expectations models subject to structural change. It thus contributes to a recent literature that extends the solution of rational expectations models to cases of occasional anticipated changes in structure. There are many potential applications of such solutions, including announced policy reforms, the introduction of new policies like quantitative easing, and reforms to pensions or taxes that may be phased in gradually. The method was used to study several applications of practical interest. There are two main advantages relative to existing appoaches.

The first advantage is simplicity. Users specify the structural matrices in each regime, and the solution is calculated recursively using these matrices. Since the resulting recursions are backward-looking, all that is required is the matrix sequence be 'simulated' from initial values. The second advantage is flexibility. Explicit solutions were given for several cases of practical interest: permanent and temporary structural changes, anticipated disturbances, gradual policy reforms, and the case of partial information. In addition the baseline model was extended to deal with some cases that received little attention in the literature, namely, imperfect credibility of policy reforms and sunspot solutions due to multiple equilibria. The results should thus be useful to a wide audience, especially researchers at policy institutions who would like to take methods 'off the shelf' and use them in a variety of applications.

There are several promising avenues for future research. First, recursive methods like the one presented here may simplify the solution and estimation of otherwise-linear rational expectations models with an occasionally-binding constraint or state-contingent forward guidance. Second, since a linearized model may not provide a good approximation to an underlying nonlinear model, extending solutions to non-linear models would be of value. One method that could improve accuracy of linearized solutions is the dynamic perturbation approach of Mennuni and Stepanchuk (2016), which approximates a model at many points along the transition path. I leave a formal investigation of this issue for future research.

Appendix

Proof of Proposition 1

The model is given by

$$x_t = \begin{cases} A_t E_t x_{t+1} + B_t x_{t-1} + C_t z_t + D_t & 0 \leq t \leq \tilde{T} \\ \tilde{A} E_t x_{t+1} + \tilde{B} x_{t-1} + \tilde{C} z_t + \tilde{D} & t > \tilde{T} \end{cases} \tag{43}$$

where $z_t = R z_{t-1} + \epsilon_t$ and $E_t[\epsilon_{t+1}] = 0_{m\times 1}$.

Consider first the periods $0 \leq t \leq \tilde{T}$. Suppose there exist a set of well-defined matrices $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ (with known entries) such that for all $t \in [0, \tilde{T}]$

$$x_t = \Omega_{\tilde{T}-t} x_{t-1} + \Gamma_{\tilde{T}-t} z_t + \Psi_{\tilde{T}-t}.$$

Shifting this forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = \Omega_{\tilde{T}-t-1} x_t + \Gamma_{\tilde{T}-t-1} R z_t + \Psi_{\tilde{T}-t-1}, \qquad 0 \leq t \leq \tilde{T} - 1. \tag{44}$$

Substituting Eq. (44) into the first line of (43) gives

$$(I_n - A_t \Omega_{\tilde{T}-t-1}) x_t = B_t x_{t-1} + (A_t \Gamma_{\tilde{T}-t-1} R + C_t) z_t + A_t \Psi_{\tilde{T}-t-1} + D_t. \tag{45}$$

Provided $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A_t \Omega_{\tilde{T}-t-1}] \neq 0$, the set $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ is well defined for t where these matrices are given by Proposition 1. Therefore, if $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A_t \Omega_{\tilde{T}-t-1}] \neq 0$ for all $t \in [0, \tilde{T}-1]$, the sequences of $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ are well-defined for $t = 0, 1, ..., \tilde{T} - 1$.

For $t > \tilde{T}$, we may guess that $x_t = \tilde{\Omega} x_{t-1} + \tilde{\Gamma} z_t + \tilde{\Psi}$, and hence for all $t \geq \tilde{T}$,

$$E_t x_{t+1} = \tilde{\Omega} x_t + \tilde{\Gamma} R z_t + \tilde{\Psi}. \tag{46}$$

Using (46) in the second line of (43) gives:

$$(I_n - \tilde{A}\tilde{\Omega}) x_t = \tilde{B} x_{t-1} + (\tilde{A}\tilde{\Gamma} R + \tilde{C}) z_t + \tilde{A} \Psi_{T-t-1} + \tilde{D}, \qquad t > \tilde{T}.$$

Provided that $\det[I_n - \tilde{A}\tilde{\Omega}] \neq 0$, the matrices $\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$ are given by $\tilde{\Omega} = (I_n - \tilde{A}\tilde{\Omega})^{-1} \tilde{B}$, $\tilde{\Gamma} = (I_n - \tilde{A}\tilde{\Omega})^{-1} (\tilde{A}\tilde{\Gamma} R + \tilde{C})$, $\tilde{\Psi} = (I_n - \tilde{A}\tilde{\Omega})^{-1} (\tilde{A}\tilde{\Psi} + \tilde{D})$. Finally, $\Omega_0, \Gamma_0, \Psi_0$ are determined by Eqs. (43) and (46) at date $t = \tilde{T}$:

$$x_{\tilde{T}} = A_{\tilde{T}} E_{\tilde{T}} x_{\tilde{T}+1} + B_{\tilde{T}} x_{\tilde{T}-1} + C_{\tilde{T}} z_{\tilde{T}} + D_{\tilde{T}}, \qquad E_{\tilde{T}} x_{\tilde{T}+1} = \tilde{\Omega} x_{\tilde{T}} + \tilde{\Gamma} R z_{\tilde{T}} + \tilde{\Psi}$$

or $(I_n - A_{\tilde{T}} \tilde{\Omega}) x_{\tilde{T}} = B_{\tilde{T}} x_{\tilde{T}-1} + (A_{\tilde{T}} \tilde{\Gamma} R + C_{\tilde{T}}) z_{\tilde{T}} + A_{\tilde{T}} \tilde{\Psi} + D_{\tilde{T}}$. Provided $\det[I_n - A_{\tilde{T}} \tilde{\Omega}] \neq 0$, the expressions for Ω_0, Γ_0, Ψ_0 are well-defined and given by Proposition 1. ■

Proof of Proposition 2

For reference, the time-invariant solutions under each structure are

$$x_t^1 \equiv \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi}, \qquad x_t^2 \equiv \tilde{\Omega} x_{t-1} + \tilde{\Gamma} z_t + \tilde{\Psi} \tag{47}$$

where $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$ (resp. $\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}$) are given by the expressions in Corollary 2 (resp. Corollary 1), which are well defined provided that $\det[I_n - A\overline{\Omega}] \neq 0$ (resp. $\det[I_n - \tilde{A}\tilde{\Omega}] \neq 0$).

For all $t \geq 0$ the model is given by:

$$x_t = A\tilde{E}_t x_{t+1} + Bx_{t-1} + Cz_t + D \tag{48}$$

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda) E_t^{IC} x_{t+1}, \qquad E_t^{IC} x_{t+1} = p_t \hat{E}_t x_{t+1}^1 + (1-p_t)\hat{E}_t x_{t+1}^2, \tag{49}$$

where $z_t = Rz_{t-1} + \epsilon_t$, $\hat{E}_t[\epsilon_{t+1}] = E_t[\epsilon_{t+1}] = 0_{m\times 1}$ and $p_t = 1$ for $t \geq T+1$.

We may conjecture that the solution has the form:

$$x_t = \begin{cases} \Omega_{T-t} x_{t-1} + \Gamma_{T-t} z_t + \Psi_{T-t} & 0 \leq t \leq T \\ \overline{\Omega} x_{t-1} + \overline{\Gamma} z_t + \overline{\Psi} = x_t^1 & t \geq T+1 \end{cases} \tag{50}$$

Consider first the periods $0 \leq t \leq T$. Suppose there exist a set of well-defined matrices $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ (with entries known to rational agents) such that for all $t \in [0, T]$

$$x_t = \Omega_{T-t} x_{t-1} + \Gamma_{T-t} z_t + \Psi_{T-t}.$$

Shifting this forward one period and taking conditional expectations yields

$$E_t x_{t+1} = \Omega_{T-t-1} x_t + \Gamma_{T-t-1} R z_t + \Psi_{T-t-1}, \qquad 0 \leq t \leq T-1. \tag{51}$$

By (49) and (47) we have, for $0 \leq t \leq T-1$,

$$E_t^{IC} x_{t+1} = (p_t \overline{\Omega} + (1-p_t)\tilde{\Omega}) x_t + (p_t \overline{\Gamma} + (1-p_t)\tilde{\Gamma}) R z_t + p_t \overline{\Psi} + (1-p_t)\tilde{\Psi}. \tag{52}$$

Substituting (51)–(52) into (49) and substituting the result in (48) gives:

$$\begin{aligned} [I_n - A_{p_t}(\lambda, \Omega_{T-t-1})] x_t = & B x_{t-1} + [A(\lambda \Gamma_{T-t-1} + (1-\lambda)[p_t \overline{\Gamma} + (1-p_t)\tilde{\Gamma}])R + C] z_t \\ & + A(\lambda \Psi_{T-t-1} + (1-\lambda)[p_t \overline{\Psi} + (1-p_t)\tilde{\Psi}]) + D, \quad 0 \leq t \leq T-1 \end{aligned}$$

where $A_{p_t}(\lambda, \Omega_{T-t-1}) \equiv A\left(\lambda \Omega_{T-t-1} + (1-\lambda)[p_t \overline{\Omega} + (1-p_t)\tilde{\Omega}]\right)$.

Provided $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A_{p_t}(\lambda, \Omega_{T-t-1})] \neq 0$, the set $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ is well defined for t where these matrices are given by Proposition 2. Therefore, if $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A_{p_t}(\lambda, \Omega_{T-t-1})] \neq 0$ for $t = 0, 1, ..., T-1$, then the sequences of $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ are well-defined for $t = 0, 1, ..., T-1$.

In period $t = T$ we have by Eqs. (47)–(49):

$$x_T = A\tilde{E}_T x_{T+1} + Bx_{T-1} + Cz_T + D$$

$$\begin{aligned}\tilde{E}_T x_{T+1} &= (\lambda + (1-\lambda)p_T)E_T x_{T+1} + (1-\lambda)(1-p_T)\hat{E}_T x^2_{T+1} \\ &= (\lambda + (1-\lambda)p_T)(\overline{\Omega}x_T + \overline{\Gamma}Rz_T + \overline{\Psi}) + (1-\lambda)(1-p_T)(\tilde{\Omega}x_T + \tilde{\Gamma}Rz_T + \tilde{\Psi})\end{aligned}$$

where (50) and $p_t = 1$ for $t \geq T+1$ are used.

Combining these two results,

$$\begin{aligned}[I_n - A_{p_T}(\lambda)]x_T =& Bx_{T-1} + [A([\lambda + (1-\lambda)p_T]\overline{\Gamma} + (1-\lambda)(1-p_T)\tilde{\Gamma})R + C]z_T \\ & A([\lambda + (1-\lambda)p_T]\overline{\Psi} + (1-\lambda)(1-p_T)\tilde{\Psi}) + D\end{aligned}$$

where $A_{p_T}(\lambda) \equiv A\left([\lambda + (1-\lambda)p_T]\overline{\Omega} + (1-\lambda)(1-p_T)\tilde{\Omega}\right)$.

Provided $\det[I_n - A_{p_T}(\lambda)] \neq 0$, the expressions for Ω_0, Γ_0, Ψ_0 are well defined and given by Proposition 2. ■

Proof of Proposition 3

We conjecture that the solution has the form:

$$x_t = \begin{cases} \Omega_{T-t}x_{t-1} + \Gamma_{T-t}z_t + \Psi_{T-t} & \text{for } 0 \leq t \leq T \\ \tilde{\Omega}x_{t-1} + \tilde{\Gamma}z_t + \tilde{\Psi} \equiv x^2_t & \text{for } t \geq T+1. \end{cases} \tag{53}$$

The model is given by

$$x_t = \begin{cases} A\tilde{E}_t x_{t+1} + Bx_{t-1} + Cz_t + D & 0 \leq t \leq T \\ \tilde{A}\tilde{E}_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & t \geq T+1 \end{cases} \tag{54}$$

subject to $\tilde{E}_t x_{t+1} = E_t x^1_{t+1} = \tilde{\Omega}x_t + \tilde{\Gamma}Rz_t + \tilde{\Psi}$ and $p_t = 1$ for all $t \geq T+1$ and, for $0 \leq t \leq T$,

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)E^{IC}_t x_{t+1}, \qquad E^{IC}_t x_{t+1} = p_t E^*_t x_{t+1} + (1-p_t)\hat{E}_t x^2_{t+1} \tag{55}$$

where $z_t = Rz_{t-1} + \epsilon_t$ and $\hat{E}_t[\epsilon_{t+1}] = E_t[\epsilon_{t+1}] = 0_{m\times 1}$, $\hat{E}_t x^2_{t+1} = \tilde{\Omega}x_{t-1} + \tilde{\Gamma}Rz_t + \tilde{\Psi}$ by (53) and $E^*_t x_{t+1}$ denotes the (expected) solution under full credibility given in Corollary 1.

The solution for $t \geq T+1$ follows Proposition 1. Hence, we have:

$$(I_n - \tilde{A}\tilde{\Omega})x_t = \tilde{B}x_{t-1} + (\tilde{A}\tilde{\Gamma}R + \tilde{C})z_t + \tilde{A}\tilde{\Psi} + \tilde{D}, \qquad t \geq T+1.$$

Provided $\det[I_n - \tilde{A}\tilde{\Omega}] \neq 0$, the set $\{\tilde{\Omega}, \tilde{\Gamma}, \tilde{\Psi}\}$ is well-defined and given by Proposition 1.

Consider now the remaining periods $0 \leq t \leq T$. Suppose there exist a set of well-defined matrices $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ (with entries known to rational agents) such that for all $t \in [0, T]$

$$x_t = \Omega_{T-t}x_{t-1} + \Gamma_{T-t}z_t + \Psi_{T-t}.$$

Shifting this forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = \Omega_{T-t-1}x_t + \Gamma_{T-t-1}Rz_t + \Psi_{T-t-1}, \qquad 0 \le t \le T-1. \tag{56}$$

By (55) and (53) we have, for $0 \le t \le T-1$,

$$E_t^{IC} x_{t+1} = p_t(\Omega^*_{T-t-1}x_t + \Gamma^*_{T-t-1}Rz_t + \Psi^*_{T-t-1}) + (1-p_t)(\tilde{\Omega}x_t + \tilde{\Gamma}Rz_t\tilde{\Psi}), \tag{57}$$

where $\{\Omega^*_{T-t-1}, \Gamma^*_{T-t-1}\Psi^*_{T-t-1}\}_{t=0}^{T-1}$ follow the recursions in Corollary 1.

Using (56) and (57) in (54) gives, for $0 \le t \le T-1$,

$$\begin{aligned}[I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega^*_{T-t-1})]x_t =& Bx_{t-1} + A(\lambda\Psi_{T-t-1} + (1-\lambda)[p_t\lambda\Psi^*_{T-t-1} + (1-p_t)\tilde{\Psi}]) + D \\ &+ [A(\lambda\Gamma_{T-t-1} + (1-\lambda)[p_t\Gamma^*_{T-t-1} + (1-p_t)\tilde{\Gamma}])R + C]z_t,\end{aligned}$$

where $A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega^*_{T-t-1}) \equiv A\left(\lambda\Omega_{T-t-1} + (1-\lambda)[p_t\Omega^*_{T-t-1} + (1-p_t)\tilde{\Omega}]\right)$.

Provided that $\Omega_0, \Gamma_0, \Psi_0$ are well defined and $\det[I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega^*_{T-t-1})] \neq 0$, the set $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ is well defined for t where these matrices are given by Proposition 3. Therefore, if $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A_{p_t}(\lambda, \Omega_{T-t-1}, \Omega^*_{T-t-1})] \neq 0$ for all $t \in [0, T-1]$, then the sequences of $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ are well-defined for $t = 0, 1, ..., T-1$.

In period $t = T$ we have by Eqs. (54)–(55):

$$x_T = A\tilde{E}_T x_{T+1} + Bx_{T-1} + Cz_T + D$$

$$\begin{aligned}\tilde{E}_T x_{T+1} &= [\lambda + (1-\lambda)(1-p_T)]E_T x_{T+1} + (1-\lambda)p_T E^*_T x^2_{T+1} \\ &= [\lambda + (1-\lambda)(1-p_T)](\tilde{\Omega}x_T + \tilde{\Gamma}Rz_T + \tilde{\Psi}) + (1-\lambda)p_T(\Omega^*_0 x_T + \Gamma^*_0 Rz_T + \Psi^*_0)\end{aligned}$$

where (53) and $p_t = 1$ for $t \ge T+1$ are used.

Combining these two results,

$$\begin{aligned}[I_n - A_{p_T}(\lambda, \Omega^*_0)]x_T =& Bx_{T-1} + [A([\lambda + (1-\lambda)(1-p_T)]\tilde{\Gamma} + (1-\lambda)p_T\Gamma^*_0)R + C]z_T \\ &+ A([\lambda + (1-\lambda)(1-p_T)]\tilde{\Psi} + (1-\lambda)p_T\Psi^*_0) + D\end{aligned}$$

where $A_{p_T}(\lambda, \Omega^*_0) \equiv A\left([\lambda + (1-\lambda)(1-p_T)]\tilde{\Omega} + (1-\lambda)p_T\Omega^*_0\right)$.

Provided $\det[I_n - A\tilde{\Omega}] \neq 0$, the expressions for $\Omega^*_0, \Gamma^*_0, \Psi^*_0$ are well defined (see Cor. 1). Hence, if $\det[I_n - A\tilde{\Omega}] \neq 0$ and $\det[I_n - A_{p_T}(\lambda, \Omega^*_0)] \neq 0$, the expressions for $\Omega_0, \Gamma_0, \Psi_0$ are well defined and given by Proposition 3. ■

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Supplementary Appendix (Online Only)

Proof of Corollary 1

The model follows Proposition 1 except that $\tilde{T} = T$ and the structure is fixed for $t \in [0, T]$:

$$x_t = \begin{cases} AE_t x_{t+1} + Bx_{t-1} + Cz_t + D & 0 \leq t \leq T \\ \tilde{A}E_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & t > T \end{cases} \tag{58}$$

where $z_t = Rz_{t-1} + \epsilon_t$ and $E_t[\epsilon_{t+1}] = 0_{m\times 1}$.

Eq. (45) is now given by

$$(I_n - A\Omega_{T-t-1})x_t = Bx_{t-1} + (A\Gamma_{T-t-1}R + C)z_t + A\Psi_{T-t-1} + D, \quad 0 \leq t \leq T-1. \tag{59}$$

Provided $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A\Omega_{T-t-1}] \neq 0$, the set $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ is well defined for t where these matrices are given by Corollary 1. Therefore, if $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A\Omega_{T-t-1}] \neq 0$ for all $t \in [0, T-1]$, the sequences of $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ are well-defined for $t = 0, 1, ..., T-1$. At date $t = T$,

$$x_T = AE_T x_{T+1} + Bx_{T-1} + Cz_T + D, \qquad E_T x_{T+1} = \tilde{\Omega}x_T + \tilde{\Gamma}Rz_T + \tilde{\Psi}$$

or $(I_n - A\tilde{\Omega})x_T = Bx_{T-1} + (A\tilde{\Gamma}R + C)z_T + A\tilde{\Psi} + D$. Provided $\det[I_n - A\tilde{\Omega}] \neq 0$, the expressions for Ω_0, Γ_0, Ψ_0 are well defined and given by Corollary 1. The rest of the proof, for $t > T$, is identical to the proof of Proposition 1. ■

Proof of Corollary 2

The model is given by

$$x_t = \begin{cases} AE_t x_{t+1} + Bx_{t-1} + Cz_t + D & 0 \leq t \leq T \\ \tilde{A}E_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & T+1 \leq t \leq T^* \\ AE_t x_{t+1} + Bx_{t-1} + Cz_t + D & t > T^* \end{cases} \tag{60}$$

where $z_t = Rz_{t-1} + \epsilon_t$ and $E_t[\epsilon_{t+1}] = 0_{m\times 1}$.

The system in (60) is a special case of the general model in Proposition 1 where $\tilde{T} = T^*$, the terminal structure is the reference regime and, for $M \in \{A, B, C, D\}$, $M_t = \tilde{M}$ if $t \in [T+1, T^*]$ and $M_t = M$ otherwise. We may therefore guess that the solution for all $t \in [0, T^*]$ is of the form $x_t = \Omega_{T^*-t}x_{t-1} + \Gamma_{T^*-t}z_t + \Psi_{T^*-t}$. Shifting this equation forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = \Omega_{T^*-t-1}x_t + \Gamma_{T^*-t-1}Rz_t + \Psi_{T^*-t-1}, \qquad 0 \leq t \leq T^* - 1. \tag{61}$$

Using (61) in (60) gives:

$$(I_n - A\Omega_{T^*-t-1})x_t = Bx_{t-1} + (A\Gamma_{T^*-t-1}R + C)z_t + A\Psi_{T^*-t-1} + D, \qquad 0 \le t \le T$$

$$(I_n - \tilde{A}\Omega_{T^*-t-1})x_t = \tilde{B}x_{t-1} + (\tilde{A}\Gamma_{T^*-t-1}R + \tilde{C})z_t + \tilde{A}\Psi_{T^*-t-1} + \tilde{D}, \qquad T+1 \le t \le T^* - 1$$

Provided $\Omega_0, \Gamma_0, \Psi_0$ well defined and $\det[I_n - \tilde{A}\Omega_{T^*-t-1}] \neq 0$, the set $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ is well defined for $t \in [T+1, T^*\text{-}1]$ where these matrices are given in Corollary 2. Provided $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ well-defined for $t \in [T+1, T^*\text{-}1]$ and $\det[I_n - A\Omega_{T^*-t-1}] \neq 0$, the set $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ is well defined for $t \in [0, T]$. Hence, if $\Omega_0, \Gamma_0, \Psi_0$ well-defined, $\det[I_n - A\Omega_{T^*-t-1}] \neq 0$ for $t = 0, 1, ..., T$ and $\det[I_n - \tilde{A}\Omega_{T^*-t-1}] \neq 0$ for $t = T+1, ..., T^* - 1$, the sequences of $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ are well defined for $t = 0, 1, ..., T^* - 1$.

For $t > T^*$, the proof follows Proposition 1 except that the structural matrices $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ are replaced by (A, B, C, D) and the resulting solution matrices are labelled as $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$. Provided that $\det[I_n - A\overline{\Omega}] \neq 0$, matrices $\overline{\Omega}, \overline{\Gamma}, \overline{\Psi}$ follow the expressions in Corollary 2. Finally, at date $t = T^*$ we have:

$$x_{T^*} = \tilde{A}E_{T^*}x_{T^*+1} + \tilde{B}x_{T^*-1} + \tilde{C}z_{T^*} + \tilde{D}, \qquad E_T^* x_{T^*+1} = \overline{\Omega}x_{T^*} + \overline{\Gamma}Rz_{T^*} + \overline{\Psi}$$

or $(I_n - \tilde{A}\overline{\Omega})x_{T^*} = \tilde{B}x_{T^*-1} + (\tilde{A}\overline{\Gamma}R + \tilde{C})z_{T^*} + A\overline{\Psi} + \tilde{D}$. Provided $\det[I_n - \tilde{A}\overline{\Omega}] \neq 0$, the expressions for $\Omega_0, \Gamma_0, \Psi_0$ are well defined and given by Corollary 2. ■

Proof of Corollary 3

The model is given by

$$x_t = \begin{cases} A_t E_t x_{t+1} + B_t x_{t-1} + C_t z_t + D_t & 0 \le t \le \tilde{T} \\ \tilde{A}E_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & t > \tilde{T} \end{cases} \tag{62}$$

where $z_t = Rz_{t-1} + \epsilon_t + \epsilon_t^a$, $E_t[\epsilon_{t+1}] = 0_{m\times 1}$ and $E_t[\epsilon_{t+1}^a] = \begin{cases} \epsilon_{t+1}^a & 0 \le t \le T-1 \\ 0_{m\times 1} & t \ge T \end{cases}$.

We may guess that the solution for all $t \in [0, \tilde{T}]$ has the same form as in Proposition 1, $x_t = \Omega_{\tilde{T}-t}x_{t-1} + \Gamma_{\tilde{T}-t}z_t + \Psi_{\tilde{T}-t}$. Taking conditional expectations of this equation,

$$E_t x_{t+1} = \Omega_{\tilde{T}-t-1}x_t + \Gamma_{\tilde{T}-t-1}Rz_t + \Gamma_{\tilde{T}-t-1}\epsilon_{t+1}^a + \Psi_{\tilde{T}-t-1}, \qquad 0 \le t \le \tilde{T} - 1. \tag{63}$$

Substituting (63) into (62) we have, for $0 \le t \le \tilde{T} - 1$,

$$(I_n - A_t\Omega_{\tilde{T}-t-1})x_t = B_t x_{t-1} + (A_t\Gamma_{\tilde{T}-t-1}R + C_t)z_t + A_t(\Psi_{\tilde{T}-t-1} + \Gamma_{\tilde{T}-t-1}\epsilon_{t+1}^a) + D_t.$$

Provided $\Omega_0, \Gamma_0, \Psi_0$ well defined and $\det[I_n - A_t\Omega_{\tilde{T}-t-1}] \neq 0$, the set $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ is well-defined for t where these matrices are given in Corollary 3. So, if $\det[I_n - A_t\Omega_{\tilde{T}-t-1}] \neq 0$

for $t = 0, 1, ..., \tilde{T}-1$ and $\Omega_0, \Gamma_0, \Psi_0$ well defined, the sequences of $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ are well defined for $t = 0, 1..., \tilde{T}-1$. The remainder of the proof, for $t \geq \tilde{T}$, is identical to Proposition 1 since $E_t[\epsilon^a_{t+1}] = 0_{m\times 1}$ for all $t \geq \tilde{T}$. ■

Proof of Corollary 4

The proof is split into two parts covering (resp.) permanent and temporary structural change.

Permanent structural change

The model is given by

$$x_t = \begin{cases} A\tilde{E}_t x_{t+1} + Bx_{t-1} + Cz_t + D & 0 \leq t \leq T \\ \tilde{A}\tilde{E}_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & t > T \end{cases} \tag{64}$$

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)\hat{E}_t x_{t+1} \tag{65}$$

where $z_t = Rz_{t-1} + \epsilon_t$ and $E_t[\epsilon_{t+1}] = 0_{m\times 1}$.

Consider first periods $0 \leq t \leq T - K$, for which uniformed agents are ignorant of the future structural change. It follows that their expectations during these periods are:[20]

$$\hat{E}_t x_{t+1} = \overline{\Omega}x_t + \overline{\Gamma}Rz_t + \overline{\Psi}, \qquad 0 \leq t \leq T-K. \tag{66}$$

where $\overline{\Omega}$, $\overline{\Gamma}$, $\overline{\Psi}$ are given in Corollary 2.

During the periods $T + 1 - K \leq t \leq T$, the uninformed agents have the same expectations as the informed agents and hence,

$$\hat{E}_t x_{t+1} = E_t x_{t+1}, \qquad T+1-K \leq t \leq T. \tag{67}$$

Suppose there exist a set of well-defined matrices $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ (with entries known to informed agents) such that $x_t = \Omega_{T-t}x_{t-1} + \Gamma_{T-t}z_t + \Psi_{T-t}$ for all $t \in [0, T]$. Shifting this forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = \Omega_{T-t-1}x_t + \Gamma_{T-t-1}Rz_t + \Psi_{T-t-1}, \qquad 0 \leq t \leq T-1. \tag{68}$$

Substituting (68) and (66) into (65) and substituting the result into (64),

$$(I_n - A^*_{T-t-1})x_t = Bx_{t-1} + (A[\lambda\Gamma_{T-t-1} + (1-\lambda)\overline{\Gamma}]R + C)z_t + A(\lambda\Psi_{T-t-1} + (1-\lambda)\overline{\Psi}) + D.$$

[20] As stated in the main text, I assume that the uninformed agents ignore the presence of informed agents. This makes sense because the information set of uninformed agents is a subset of the information set of the informed agents, and hence $\hat{E}_t[\tilde{E}_t x_{t+1}] = \lambda\hat{E}_t[E_t x_{t+1}] + (1-\lambda)\hat{E}_t x_{t+1} = \hat{E}_t x_{t+1}$. Therefore, the uninformed agents solve a misspecified version of (64) in periods in which they are ignorant of structural change.

for $0 \leq t \leq T-K$, and

$$(I_n - A\Omega_{T-t-1})x_t = Bx_{t-1} + (A\Gamma_{T-t-1}R + C)z_t + A\Psi_{T-t-1} + D, \quad T+1-K \leq t \leq T-1,$$

where $A^*_{T-t-1} \equiv A[\lambda\Omega_{T-t-1} + (1-\lambda)\overline{\Omega}]$.

Provided $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - A\Omega_{T-t-1}] \neq 0$, the set $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ is well defined for $t \in [T+1-K, T-1]$. Provided $\det[I_n - A^*_{T-t-1}] \neq 0$ and $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ are well defined for $t = T+1-K, ..., T-1$, the set $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ is well defined for $t \in [0, T-K]$. Therefore, if $\Omega_0, \Gamma_0, \Psi_0$ are well defined, $\det[I_n - A\Omega_{T-t-1}] \neq 0$ for $t = T+1-K, .., T-1$ and $\det[I_n - A^*_{T-t-1}] \neq 0$ for $t = 0, 1, 2..., T-K$, the sequences of $\{\Omega_{T-t}, \Gamma_{T-t}, \Psi_{T-t}\}$ are well defined for $t = 0, 1..., T-1$. The remainder of the proof, relating to periods $t \geq T$, is identical to Corollary 1.

Temporary structural change

The model is given by

$$x_t = \begin{cases} A\tilde{E}_t x_{t+1} + Bx_{t-1} + Cz_t + D & 0 \leq t \leq T \\ \tilde{A}\tilde{E}_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & T+1 \leq t \leq T^* \\ AE_t x_{t+1} + Bx_{t-1} + Cz_t + D & t > T^* \end{cases} \tag{69}$$

$$\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)\hat{E}_t x_{t+1} \tag{70}$$

where $z_t = Rz_{t-1} + \epsilon_t$ and $E_t[\epsilon_{t+1}] = 0_{m\times 1}$.

Consider first the periods $0 \leq t \leq T-K$, for which uniformed agents are ignorant of any future structural change. It follows that their expectations during these periods are:

$$\hat{E}_t x_{t+1} = \overline{\Omega}x_t + \overline{\Gamma}Rz_t + \overline{\Psi}, \qquad 0 \leq t \leq T-K, \tag{71}$$

where $\overline{\Omega}$, $\overline{\Gamma}$, $\overline{\Psi}$ are given in Corollary 2.

Consider now periods $T+1 \leq t \leq T^*-K$, for which uninformed agents are ignorant of the final structural change. It follows that their expectations during these periods are:

$$\hat{E}_t x_{t+1} = \tilde{\Omega}x_t + \tilde{\Gamma}Rz_t + \tilde{\Psi}, \qquad T+1 \leq t \leq T^*-K, \tag{72}$$

where $\tilde{\Omega}$, $\tilde{\Gamma}$, $\tilde{\Psi}$ are given in Proposition 1.

During periods $T+1-K \leq t \leq T$ the expectations of uninformed agents incorporate the first structural change but not the second. It follows that the expectations are given by

$$\hat{E}_t x_{t+1} = \tilde{\Omega}_{T-t-1}x_t + \tilde{\Gamma}_{T-t-1}Rz_t + \tilde{\Psi}_{T-t-1}, \qquad T+1-K \leq t \leq T. \tag{73}$$

where matrices $\{\tilde{\Omega}_{T-t}, \tilde{\Gamma}_{T-t}, \tilde{\Psi}_{T-t}\}_{t=T+1-K}^{T}$ are given in Corollary 1.

Finally, during the period $T^* + 1 - K \leq t \leq T^*$, the uninformed agents have the same expectations as the informed agents and hence,

$$\hat{E}_t x_{t+1} = E_t x_{t+1}, \qquad T^* + 1 - K \leq t \leq T^*. \tag{74}$$

Suppose there exist a set of well-defined matrices $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ (with entries known to informed agents) such that $x_t = \Omega_{T^*-t} x_{t-1} + \Gamma_{T^*-t} z_t + \Psi_{T^*-t}$ for all $t \in [0, T^*]$. Shifting this forward one period and taking conditional expectations yields:

$$E_t x_{t+1} = \Omega_{T^*-t-1} x_t + \Gamma_{T^*-t-1} R z_t + \Psi_{T^*-t-1}, \qquad 0 \leq t \leq T^* - 1. \tag{75}$$

Substituting (71), (72), (74) and (75) into (70) and substituting the result into (69),

$$(I_n - A^*_{T^*-t-1})x_t = Bx_{t-1} + (A[\lambda\Gamma_{T^*-t-1} + (1-\lambda)\overline{\Gamma}]R + C)z_t + A(\lambda\Psi_{T^*-t-1} + (1-\lambda)\overline{\Psi}) + D$$

for $0 \leq t \leq T - K$,

$$(I_n - \hat{A}^*_{T^*-t-1})x_t = \tilde{B}x_{t-1} + (A[\lambda\Gamma_{T^*-t-1} + (1-\lambda)\tilde{\Gamma}_{T-t-1}]R + \tilde{C})z_t + A(\lambda\Psi_{T^*-t-1} + (1-\lambda)\tilde{\Psi}_{T-t-1}) + \tilde{D}$$

for $T + 1 - K \leq t \leq T$,

$$(I_n - \tilde{A}^*_{T^*-t-1})x_t = \tilde{B}x_{t-1} + (A[\lambda\Gamma_{T^*-t-1} + (1-\lambda)\tilde{\Gamma}]R + \tilde{C})z_t + A(\lambda\Psi_{T^*-t-1} + (1-\lambda)\tilde{\Psi}) + \tilde{D}$$

for $T + 1 \leq t \leq T^* - K$, and

$$(I_n - A\Omega_{T^*-t-1})x_t = Bx_{t-1} + (A\Gamma_{T^*-t-1}R + C)z_t + A\Psi_{T^*-t-1} + D, \qquad T^* + 1 - K \leq t \leq T^* - 1,$$

where $A^*_{T^*-t-1} \equiv A[\lambda\Omega_{T-t-1} + (1-\lambda)\overline{\Omega}]$, $\hat{A}^*_{T^*-t-1} \equiv \tilde{A}[\lambda\Omega_{T^*-t-1} + (1-\lambda)\tilde{\Omega}_{T-t-1}]$ and $\tilde{A}^*_{T^*-t-1} \equiv A[\lambda\Omega_{T^*-t-1} + (1-\lambda)\tilde{\Omega}]$.

If $\Omega_0, \Gamma_0, \Psi_0$ well-defined, $\det[I_n - A^*_{T^*-t-1}] \neq 0$ for $t = 0, ..., T-K$, $\det[I_n - \hat{A}^*_{T^*-t-1}] \neq 0$ for $t = T+1-K, ..., T$, $\det[I_n - \tilde{A}^*_{T^*-t-1}] \neq 0$ for $t = T+1, ..., T^*-K$, $\det[I_n - A\Omega_{T^*-t-1}] \neq 0$ for $t = T^*+1-K, ..., T^*-1$, the sequences of $\{\Omega_{T^*-t}, \Gamma_{T^*-t}, \Psi_{T^*-t}\}$ are well defined for $t = 0, ..., T^*-1$. The remainder of the proof, for $t \geq T^*$, is identical to Corollary 2. ■

Proof of Corollary 5

The model is given by

$$x_t = \begin{cases} A_t\tilde{E}_t x_{t+1} + B_t x_{t-1} + C_t z_t + D_t & 0 \leq t \leq \tilde{T} \\ \tilde{A}E_t x_{t+1} + \tilde{B}x_{t-1} + \tilde{C}z_t + \tilde{D} & t > \tilde{T} \end{cases} \tag{76}$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1-\lambda_t)E_t^{IC} x_{t+1}, \quad E_t^{IC} x_{t+1} = F_0 x_t + F_1 x_{t-1} + G z_t + H \tag{77}$$

where $z_t = Rz_{t-1}+\epsilon_t$, $E_t[\epsilon_{t+1}] = 0_{m\times 1}$, matrices F_0, F_1, G, H are conformable and real-valued, and $\lambda_t \in (0,1]$ with $\lambda_t = 1$ for $t \geq \tilde{T}+1$.

Using (77) in the top line of (76),

$$x_t = \hat{A}_t(\lambda_t E_t x_{t+1} + (1-\lambda_t)F_0 x_t) + \hat{B}_t x_{t-1} + \hat{C}_t z_t + \hat{D}_t, \qquad 0 \leq t \leq \tilde{T} \tag{78}$$

where $\hat{A}_t \equiv A(\mathbb{1}_t)$, $\hat{B}_t \equiv B(\mathbb{1}_t)+(1-\lambda_t)F_1$, $\hat{C}_t \equiv C(\mathbb{1}_t)+(1-\lambda_t)G$, $\hat{D}_t \equiv D(\mathbb{1}_t)+(1-\lambda_t)H$.

We may guess that the solution for all $t \in [0, \tilde{T}]$ has the same form as in Proposition 1, $x_t = \Omega_{\tilde{T}-t}x_{t-1} + \Gamma_{\tilde{T}-t}z_t + \Psi_{\tilde{T}-t}$. Taking conditional expectations of this equation,

$$E_t x_{t+1} = \Omega_{\tilde{T}-t-1}x_t + \Gamma_{\tilde{T}-t-1}Rz_t + \Psi_{\tilde{T}-t-1}, \qquad 0 \leq t \leq \tilde{T}-1. \tag{79}$$

Substituting Eq. (79) into (76) we have, for $0 \leq t \leq \tilde{T}-1$,

$$[I_n - \hat{A}_t(\lambda_t\Omega_{\tilde{T}-t-1} + (1-\lambda_t)F_0)]x_t = \hat{B}_t + (\hat{A}_t\Gamma_{\tilde{T}-t-1}R + \hat{C}_t)z_t + \hat{A}_t\Psi_{\tilde{T}-t-1} + \hat{D}_t$$

Provided $\Omega_0, \Gamma_0, \Psi_0$ well-defined and $\det[I_n - \hat{A}_t(\lambda_t\Omega_{\tilde{T}-t-1} + (1-\lambda_t)F_0)] \neq 0$, the set $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ is well-defined for t where these matrices are given in Corollary 5. Therefore, if $\det[I_n - \hat{A}_t(\lambda_t\Omega_{\tilde{T}-t-1} + (1-\lambda_t)F_0)] \neq 0$ for $t = 0, 1, 2..., \tilde{T}-1$ and $\Omega_0, \Gamma_0, \Psi_0$ well defined, the sequences of $\{\Omega_{\tilde{T}-t}, \Gamma_{\tilde{T}-t}, \Psi_{\tilde{T}-t}\}$ are well defined for $t = 0, 1, 2..., \tilde{T}-1$.

For $t > \tilde{T}$, we may guess that $x_t = \tilde{\Omega}x_{t-1} + \tilde{\Gamma}z_t + \tilde{\Psi}$ (see (76)), and hence, for all $t \geq \tilde{T}$,

$$E_t x_{t+1} = \tilde{\Omega}x_t + \tilde{\Gamma}Rz_t + \tilde{\Psi}. \tag{80}$$

At date $t = \tilde{T}$, we have by (76) and (78):

$$x_{\tilde{T}} = \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}}E_{\tilde{T}}x_{\tilde{T}+1} + (1-\lambda_{\tilde{T}})F_0x_{\tilde{T}}) + \hat{B}_{\tilde{T}}x_{\tilde{T}-1} + \hat{C}_{\tilde{T}}z_{\tilde{T}} + \hat{D}_{\tilde{T}} \quad E_{\tilde{T}}x_{\tilde{T}+1} = \tilde{\Omega}x_{\tilde{T}} + \tilde{\Gamma}Rz_{\tilde{T}} + \tilde{\Psi}$$

or $[I_n - \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}}\tilde{\Omega} + (1-\lambda_{\tilde{T}})F_0)]x_{\tilde{T}} = \hat{B}_{\tilde{T}}x_{\tilde{T}-1} + (\hat{A}_{\tilde{T}}\tilde{\Gamma}R + \hat{C}_{\tilde{T}})z_{\tilde{T}} + \hat{A}_{\tilde{T}}\tilde{\Psi} + \hat{D}_{\tilde{T}}$. Provided $\det[I_n - \hat{A}_{\tilde{T}}(\lambda_{\tilde{T}}\tilde{\Omega} + (1-\lambda_{\tilde{T}})F_0)] \neq 0$, the expressions for Ω_0, Γ_0, Ψ_0 are well defined and given by Corollary 5. The rest of the proof, for $t > \tilde{T}$, matches Proposition 1. ■