Supplementary Appendix (For online publication only)

"Solving linear rational expectations models in the presence of structural change: Some refinements"

This appendix provides details of the numerical applications solved in Section 5 of the paper. The codes are available at the author's github page at: github.com/MCHatcher.

1 New Keynesian model

The baseline model has the form

$$\pi_t = \frac{1}{(1+\beta\alpha)}\left(\beta E_t\pi_{t+1} + (1+\beta\alpha-\alpha-\beta)\pi^* + \alpha\pi_{t-1} + \psi\sigma y_t - \psi a_t - \mu_t\right) \tag{1}$$

$$y_t = E_t y_{t+1} - \sigma^{-1}(R_t - E_t\pi_{t+1}) + \frac{(1-\rho_g)}{\sigma}g_t - \sigma^{-1}\ln(\beta) \tag{2}$$

$$R_t = (1-\rho_R)\overline{R} + \rho_R R_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) \tag{3}$$

where the shocks to technology, demand and the mark-up follow AR(1) processes:

$$u_t = \rho_u u_{t-1} + \sigma_u \epsilon_{u,t}, \qquad u \in \{a, g, \mu\}, \qquad \epsilon_{u,t} \sim N(0,1),$$

and $\overline{R} = \pi^* - \ln(\beta)$ is the steady-state nominal interest rate.

Let $x_t = \begin{bmatrix}\pi_t & y_t & R_t & a_t & g_t & \mu_t\end{bmatrix}'$ and $e_t = \begin{bmatrix}\epsilon_{a,t} & \epsilon_{g,t} & \epsilon_{\mu,t}\end{bmatrix}'$. Then the (fixed-structure) version of the model can be written in the form

$$B_1 x_t = B_2 E_t x_{t+1} + B_3 x_{t-1} + B_4 e_t + B_5 \tag{4}$$

where

$$B_1 = \begin{bmatrix} 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 \\ -\theta_\pi & -(\theta_y+\theta_{dy}) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{\beta}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 \\ \sigma^{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{\alpha}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\theta_{dy} & \rho_R & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_a & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\mu \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_a & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & \sigma_\mu \end{bmatrix}, \quad B_5 = \begin{bmatrix} \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^* \\ -\sigma^{-1}\ln(\beta) \\ (1-\rho_R)\overline{R} - \theta_\pi\pi^* \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

1.1 Application 1 – Change in the inflation target

The inflation target is permanently lowered from π^* to π^*_{new}. The terminal structure is the one associated with π^*_{new}. Thus, the reference and alternative regimes are described by

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5 \tag{5}$$

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \tag{6}$$

where

$$\overline{B}_1 = B_1, \quad \overline{B}_2 = B_2, \quad \overline{B}_3 = B_3, \quad \overline{B}_4 = B_4, \quad \overline{B}_5 = B_5$$
$$\tilde{B}_1 = B_1, \quad \tilde{B}_2 = B_2, \quad \tilde{B}_3 = B_3, \quad \tilde{B}_4 = B_4$$

with B_1, B_2, B_3, B_4, B_5 defined in Section 1 and

$$\tilde{B}_5 = \begin{bmatrix} \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^*_{new} \\ -\sigma^{-1}\ln(\beta) \\ (1-\rho_R)\overline{R}_{new} - \theta_\pi \pi^*_{new} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \overline{R}_{new} := \pi^*_{new} - \ln(\beta).$$

Given a change in the inflation target implemented in period $t = 8$ and announced in period $t = 4$, our model is

$$\overline{B}_1 x_t = \overline{B}_2 E_t x_{t+1} + \overline{B}_3 x_{t-1} + \overline{B}_4 e_t + \overline{B}_5, \quad \text{for } t \in [0,3] \tag{7}$$

and

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 4 \tag{8}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1-\mathbb{1}_t)\tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [4,7] \\ 0 & \text{if } t \geq 8 \end{cases}.$$

Note that Eq. (7) defines the original fixed structure solution which is expected to prevail *forever* in periods 0 to 3.[1] Note that the above problem is nested by the baseline (Kulish and Pagan, 2017) solution in the main text when period 0 is relabelled as period 4.

1.2 Application 2 – Change in π^* (imperfect credibility)

1.2.1 Standard approach: expectations based on past inflation target

For convenience, let us define $\pi^*_{orig} := \pi^*$. The inflation expectations of the 'doubting' agents are given by $E_t^{IC}\pi_{t+1} = \pi^*_{orig}$. We assume their other expectations are formed rationally. Their population share is $1-\lambda_{1,t}$, with $\lambda_{1,t} = \lambda \in (0,1)$ for $t \in [4,10]$ and $\lambda_{1,t} = 1$ for $t \geq 11$.

[1] Given that agents' expectations differ from reality, one may use notation such $\tilde{E}_t x_{t+1}$ to acknowledge that the beliefs are not rational (i.e. model-consistent) in these periods.

Thus our model, now including the economy-wide expectation $\tilde{E}_t x_{t+1}$, is given by

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t} \tag{9}$$

$$\tilde{E}_t x_{t+1} = \Lambda_t E_t x_{t+1} + (I_n - \Lambda_t) E_t^{IC} x_{t+1} \tag{10}$$

where $E_t^{IC} x_{t+1} = F_0 x_t + F_1 x_{t-1} + F_2 e_t + F_3$ (for user-specified matrices F_0, F_1, F_2, F_3) and Λ_t is an $n \times n$ diagonal matrix given by $\Lambda_t = diag(\lambda_{1,t}, 1, ..., 1)$.

Since the inflation expectations of the doubters are constant, we have:

$$F_0 = F_1 = 0_{n\times n}, \quad F_2 = 0_{n \times m}, \quad F_3 = \begin{bmatrix} \pi^*_{orig} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies E_t^{IC} x_{t+1} = F_3.$$

Substituting this result into (10) and substituting the result into (9), our model is given by

$$B_{1,t} x_t = \hat{B}_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + \hat{B}_{5,t} \tag{11}$$

where $\hat{B}_{2,t} := \Lambda_t B_{2,t}$ and $\hat{B}_{5,t} := (I_n - \Lambda_t) F_3 + B_{5,t}$ and $\Lambda_t = I_n$ for all $t \geq 11$.

Note that Eq. (11) is in the same form as the system (8) and can, therefore, be solved by the same method.

1.2.2 Type 1 credibility

Under Type 1 credibility, our model is given by

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t} \tag{12}$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t) E_t^{IC} x_{t+1} \tag{13}$$

where we assume that $\lambda_t = \lambda$ (given) for $t \in [4, 10]$ and $\lambda_t = 1$ for all $t \geq 11$, and the expectations of the doubting agents are given by $E_t^{IC} x_{t+1} = \overline{\Omega} x_t + \overline{\Gamma} E_t^{IC}[e_{t+1}] + \overline{\Psi} = \overline{\Omega} x_t + \overline{\Psi}$,[2] where

$$\Omega_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t} \Omega_{t+1}^{IC})^{-1} \hat{B}_{3,t}, \qquad \Gamma_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t} \Omega_{t+1}^{IC})^{-1} \hat{B}_{4,t}, \tag{14}$$

$$\Psi_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t} \Omega_{t+1}^{IC})^{-1} (\hat{B}_{2,t} \Psi_{t+1}^{IC} + \hat{B}_{5,t}) \tag{15}$$

provided $\det[\hat{B}_{1,t} - \hat{B}_{2,t} \Omega_{t+1}^{IC}] \neq 0$ for all $t \in [0, \tilde{T}]$, where

$$\Omega_{T+1}^{IC} = \overline{\Omega} := (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_3, \qquad \Gamma_{T+1}^{IC} = \overline{\Gamma} := (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} \overline{B}_4,$$

$$\Psi_{T+1}^{IC} = \overline{\Psi} = (\overline{B}_1 - \overline{B}_2 \overline{\Omega})^{-1} (\overline{B}_2 \overline{\Psi} + \overline{B}_5)$$

[2] As noted in the main text (Section 4.2.2) we assume the doubting agents know the properties of the exogenous shocks e_t (e.g. zero mean) and have access to I_t (which does not include the *future* structure).

We assume that the doubting agents expect a permanent reversion to the original structure (with the higher inflation target) starting next period:

$$\hat{B}_{i,t} = \overline{B}_i, \quad \text{for } i \in [5] \quad \Longrightarrow \quad \Omega_t^{IC} = \overline{\Omega},\ \Gamma_t^{IC} = \overline{\Gamma},\ \Psi_t^{IC} = \overline{\Psi}.$$

Substituting this expression for $E_t^{IC} x_{t+1}$ into (13) and substituting the result into (12) gives:

$$B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC} \tag{16}$$

where $B_{1,t}^{IC} := B_{1,t} - B_{2,t}(1-\lambda_t)\overline{\Omega}$, $B_{2,t}^{IC} := B_{2,t}\lambda_t$ and $B_{5,t}^{IC} := B_{2,t}(1-\lambda_t)\overline{\Psi} + B_{5,t}$.

Since Eq. (16) is in the same form as (8), it can be solved by the same method.

1.3 Application 3 – Forward guidance

1.3.1 'Plain vanillla' forward guidance

The interest rate rule is now amended to

$$R_t = \begin{cases} \underline{R} := 0 & \text{for } t \in [2,5] \\ (1-\rho_R)\overline{R} + \rho_R R_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) & \text{otherwise.} \end{cases}$$

Since the original structure is in place for $t \geq 6$, we let the original structure be the alternative ('tilde') regime, i.e. $\tilde{B}_i = B_i$ for $i \in [5]$, where the B_i matrices appear on p. 1. The reference structure is given by the 'forward guidance regime', i.e.

$$\overline{B}_1 = \begin{bmatrix} 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^* \\ -\sigma^{-1}\ln(\beta) \\ \underline{R} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\overline{B}_i = B_i$ for $i = 2, 3, 4$.

Our model is given by

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 0 \tag{17}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1-\mathbb{1}_t)\tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [2,5] \\ 0 & \text{otherwise} \end{cases}.$$

1.3.2 Forward guidance with delayed announcement

We consider the case $K = 0$, which implies that a fraction $1-\lambda$ of agents do not receive any prior announcement about the forward guidance policy. Denoting the economy-wide expectation $\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)\hat{E}_t x_{t+1}$, Eq. (17) becomes

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \text{for } t \in [0,1] \tag{18}$$

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 2 \tag{19}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1-\mathbb{1}_t)\tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [2,5] \\ 0 & \text{otherwise} \end{cases}$$

and $\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1-\lambda)\hat{E}_t x_{t+1}$. Since the uninformed agents expect the original structure to prevail (in all future periods) at dates 0 and 1, we have

$$\hat{E}_t x_{t+1} = \tilde{\Omega} x_t + \tilde{\Gamma}\hat{E}_t[e_{t+1}] + \tilde{\Psi} = \tilde{\Omega} x_t + \tilde{\Psi}, \quad \text{for } t = 0,1$$

where $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_3$, $\tilde{\Gamma} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}\tilde{B}_4$, $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2\tilde{\Omega})^{-1}(\tilde{B}_2\tilde{\Psi} + \tilde{B}_5)$ (see Assumption 1 in the main text).

We can thus write the model compactly as

$$\hat{B}_{1,t} x_t = \hat{B}_{2,t} E_t x_{t+1} + \hat{B}_{3,t} x_{t-1} + \hat{B}_{4,t} e_t + \hat{B}_{5,t}, \quad \forall t \geq 0 \tag{20}$$

where, for $i \in [5]$, $\hat{B}_{i,t} = B_{i,t}$ for $t \geq 2$ and $\hat{B}_{i,t} = \hat{B}_i$ for $t = 0,1$, with

$$\hat{B}_1 := \tilde{B}_1 - \tilde{B}_2(1-\lambda)\tilde{\Omega}, \qquad \hat{B}_2 := \lambda\tilde{B}_2, \qquad \hat{B}_5 := \tilde{B}_5 + \tilde{B}_2(1-\lambda)\tilde{\Psi}$$

and $\hat{B}_i = B_i$ for $i = 2,3$ (see p. 1).

1.3.3 Forward guidance (Type 2 credibility)

Under Type 2 credibility, our model is given by

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t} \tag{21}$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1-\lambda_t) E_t^{IC} x_{t+1} \tag{22}$$

$$E_t^{IC} x_{t+1} = p_t E_t^* x_{t+1} + (1-p_t)\hat{E}_t x_{t+1} \tag{23}$$

where $\lambda_t = 0.7$ for $t \in [0,5]$ and $\lambda_t = 1$ for all $t \geq 6$.

The expectations of the doubting agents are given by:

$$\hat{E}_t x_{t+1} = \Omega_{t+1}^{IC} x_t + \Gamma_{t+1}^{IC}\hat{E}_t[e_{t+1}] + \Psi_{t+1}^{IC} = \Omega_{t+1}^{IC} x_t + \Psi_{t+1}^{IC}$$

$$E_t^* x_{t+1} = \Omega_{t+1}^* x_t + \Gamma_{t+1}^* E_t^*[e_{t+1}] + \Psi_{t+1}^* = \Omega_{t+1}^* x_t + \Psi_{t+1}^*$$

where the matrices $\Omega_{t+1}^{IC}, \Psi_{t+1}^{IC}$ are given by the recursions (14)–(15), and the matrices $\Omega_{t+1}^*, \Gamma_{t+1}^*$ are given by the recursions in Proposition 1 with matrices $B_{1,t}, ..., B_{5,t}$.

Substituting these expectations into (23) and substituting the result into (22) gives $\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1-\lambda_t)[\tilde{\Omega}_{t+1}^{IC} x_t + \tilde{\Psi}_{t+1}^{IC}]$, where $\tilde{\Omega}_{t+1}^{IC} := p_t \Omega_{t+1}^* + (1-p_t)\Omega_{t+1}^{IC}$ and $\tilde{\Psi}_{t+1}^{IC} := p_t \Psi_{t+1}^* + (1-p_t)\Psi_{t+1}^{IC}$. Hence, our model can be summarized as:

$$\begin{cases} B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC}, & 0 \le t \le \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T}. \end{cases} \tag{24}$$

where $B_{1,t}^{IC} := B_{1,t} - B_{2,t}(1-\lambda_t)\tilde{\Omega}_{t+1}^{IC}$, $B_{2,t}^{IC} := B_{2,t}\lambda_t$ and $B_{5,t}^{IC} := B_{2,t}(1-\lambda_t)\tilde{\Psi}_{t+1}^{IC} + B_{5,t}$.

The computational approach can be summarized as follows:

1. Define the indicators $\{\mathbb{1}_t\}_{t=0}^{\infty}$ and $\{\mathbb{1}_t^{IC}\}_{t=0}^{\infty}$. In this example we have $\mathbb{1}_t = 1$ for $t \in [2,5]$ and $\mathbb{1}_t = 0$ otherwise, and $\mathbb{1}_t^{IC} = 1$ for $t \in [2,3]$ and $\mathbb{1}_t^{IC} = 0$ otherwise (because the doubting agents expect forward guidance to end 2 periods earlier than announced).

2. Compute the actual sequences of structures $B_{i,t} = \mathbb{1}_t \overline{B}_i + (1-\mathbb{1}_t)\tilde{B}_i$ (for $i \in [5]$) and the sequence of structures to which the doubting agents attach positive (subjective) probability: $\hat{B}_{i,t} = \mathbb{1}_t^{IC}\overline{B}_i + (1-\mathbb{1}_t^{IC})\tilde{B}_i$.

3. Choose $\{\lambda_t\}_{t=0}^{\infty}$ and $\{p_t\}_{t=0}^{\infty}$. In this example, we set $\lambda_t = 0.7$ for $t \in [0,5]$, $\lambda_t = 1$ for all $t \ge 6$ and $p_t = p = 0.5$.

4. For all $t \in [0,5]$, compute the recursions

$$\Omega_t^* = (B_{1,t} - B_{2,t}\Omega_{t+1}^*)^{-1} B_{3,t}, \qquad \Gamma_t^* = (B_{1,t} - B_{2,t}\Omega_{t+1}^*)^{-1} B_{4,t},$$

$$\Psi_t^* = (B_{1,t} - B_{2,t}\Omega_{t+1}^*)^{-1}(B_{2,t}\Psi_{t+1}^* + B_{5,t}),$$

$$\Omega_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}\hat{B}_{3,t}, \quad \Gamma_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}\hat{B}_{4,t},$$

$$\Psi_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}(\hat{B}_{2,t}\Psi_{t+1}^{IC} + \hat{B}_{5,t})$$

   subject to the terminal conditions $\Omega_{\tilde{T}+1}^* = \Omega_{\tilde{T}+1}^{IC} = \tilde{\Omega}$, $\Gamma_{\tilde{T}+1}^* = \Gamma_{\tilde{T}+1}^{IC} = \tilde{\Gamma}$.[3]

5. Find the matrices $B_{1,t}^{IC}, B_{2,t}^{IC}, B_{5,t}^{IC}$ and use these in Proposition 1 (in place of $B_{1,t}, B_{2,t}, B_{5,t}$) to find the solution matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=0}^{T_{sim}}$ and compute the solution $\{x_t\}_{t=0}^{T_{sim}}$.

[3]Note that in this example the perceived terminal structure of the doubting agents coincides with the actual one because they expect reversion to the Taylor-type rule to happen sooner than announced.

1.4 Forward guidance + indeterminacy

The interest rate rule is now

$$R_t = \begin{cases} \underline{R} := 0 & t \in [0,7] \\ \rho_R R_{t-1} + (1-\rho_R)\overline{R} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \delta_{dy}(y_t - y_{t-1}) & \text{otherwise} \end{cases}$$

where θ_π is such that the terminal solution is indeterminate, with degree 1 indeterminacy.

Following the approach in Farmer, Khramov and Nicolo (2015, JEDC), we assume one of the forward-looking variables (inflation or output) is hit by self-fulfilling expectations shocks, i.e. sunspots. We pick inflation and hence the terminal structure is given by (1)–(3) along with the additional inflation equation:

$$\pi_t = s_{t-1} + v_{\pi,t}, \quad \forall t > 7$$

where $s_t := E_t \pi_{t+1}$.

Substitute s_t in place $E_t\pi_{t+1}$ in (1)–(3) and let $\tilde{x}_t = \begin{bmatrix} \pi_t & y_t & R_t & a_t & g_t & \mu_t & s_t \end{bmatrix}'$ and $\tilde{e}_t = \begin{bmatrix} v_{\pi,t} & \epsilon_{a,t} & \epsilon_{g,t} & \epsilon_{\mu,t} \end{bmatrix}'$. Then the system for $t > 7$ has the form:

$$\tilde{B}_1 \tilde{x}_t = \tilde{B}_2 E_t \tilde{x}_{t+1} + \tilde{B}_3 \tilde{x}_{t-1} + \tilde{B}_4 \tilde{e}_t + \tilde{B}_5, \quad \forall t > 7$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} & -\frac{\beta}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 & -\sigma^{-1} \\ -\theta_\pi & -(\theta_y + \theta_{dy}) & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\alpha}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\theta_{dy} & \rho_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\mu & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_a & 0 & 0 \\ 0 & 0 & \sigma_g & 0 \\ 0 & 0 & 0 & \sigma_\mu \end{bmatrix}, \quad \tilde{B}_5 = \begin{bmatrix} 0 \\ \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^* \\ -\sigma^{-1}\ln(\beta) \\ (1-\rho_R)\overline{R} - \theta_\pi\pi^* \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In the earlier periods $0 \leq t \leq 7$, the interest rate $R_t = \underline{R}$ is in place and inflation is determined as

$$\pi_t = s_{t-1}, \quad \text{for } t \in [0,7]$$

since we assume coordination of expectations on the sunspot terminal solution.

Hence, the system for $t \in [0, 7]$ is:

$$\overline{B}_1 \tilde{x}_t = \overline{B}_2 E_t \tilde{x}_{t+1} + \overline{B}_3 \tilde{x}_{t-1} + \overline{B}_4 \tilde{e}_t + \overline{B}_5$$

where

$$\overline{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} & -\frac{\beta}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 & -\sigma^{-1} \\ 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \overline{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overline{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\alpha}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\mu & 0 \end{bmatrix}, \quad \overline{B}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_a & 0 & 0 \\ 0 & 0 & \sigma_g & 0 \\ 0 & 0 & 0 & \sigma_\mu \end{bmatrix}, \quad \overline{B}_5 = \begin{bmatrix} 0 \\ \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^* \\ -\sigma^{-1}\ln(\beta) \\ \underline{R} \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, our model is given by

$$\mathcal{B}_{1,t}\tilde{x}_t = \mathcal{B}_{2,t} E_t \tilde{x}_{t+1} + \mathcal{B}_{3,t}\tilde{x}_{t-1} + \mathcal{B}_{4,t}\tilde{e}_t + \mathcal{B}_{5,t}, \quad \forall t \geq 0 \tag{25}$$

where $\mathcal{B}_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [0, 7] \\ 0 & \text{otherwise} \end{cases}.$$

2 Pension reform in the Diamond model

The baseline log-linearized model has the form:

$$\hat{c}_{t,y} = -\frac{1}{\sigma}\hat{R}_{t+1} + \hat{c}_{t+1,o}, \quad \hat{R}_t = (\alpha - 1)\hat{k}_t, \quad \hat{w}_t = \alpha\hat{k}_t$$
$$\hat{c}_{t,y} = \frac{(1-\tau)(1-\alpha)k^\alpha}{c_y}\hat{w}_t - \frac{(1+n)k}{c_y}\hat{k}_{t+1}, \quad \hat{c}_{t,o} = \alpha\hat{k}_t$$

where $c_y = (1-\tau)(1-\alpha)k^\alpha - (1+n)k$, k and ‘hats’ are log deviations from steady state.

Steady state capital, k, is determined by the fixed point of the capital accumulation equation in the main text:

$$k^{1-\alpha} = \left[\frac{(1-\tau)(1-\alpha)}{1+n}\right] \frac{\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}}}{\left(1 + \frac{(1-\alpha)\tau}{\alpha}\right) R + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}}}, \quad \text{where } R = \alpha k^{\alpha-1}.$$

Some algebra shows that steady state capital must satisfy

$$(\alpha + (1-\alpha)\tau)k^{\frac{1-\alpha}{\sigma}} + (\beta\alpha)^{\frac{1}{\sigma}} k^{1-\alpha} = \frac{(\beta\alpha)^{\frac{1}{\sigma}}(1-\tau)(1-\alpha)}{1+n}. \tag{26}$$

Note that Eq. (26) does not hold for $k = 0$ since $LHS = 0 < RHS$, but since $\partial LHS/\partial k > 0$, there exists a unique $k^* \in (0, \infty)$ such that the equation is satisfied. We find k^* numerically.

Let $\hat{cap}_t := \hat{k}_{t+1}$ and $\hat{x}_t = \begin{bmatrix} \hat{c}_{t,y} & \hat{R}_t & \hat{w}_t & \hat{cap}_t & \hat{c}_{t,o} \end{bmatrix}'$. Then the (fixed-structure) version of the model can be written in the form

$$B_1 \hat{x}_t = B_2 E_t \hat{x}_{t+1} + B_3 \hat{x}_{t-1} \tag{27}$$

where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{(\alpha-1)(1-\tau)\alpha k^\alpha}{c_y} & \frac{(1+n)k}{c_y} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & -\frac{1}{\sigma} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha-1 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \end{bmatrix}$$

and $\hat{x}_t = x_t - x_{SS}$.

2.1 Pension reform: change in τ

The pension contribution rate is permanently lowered from τ to τ' in period $t = 1$. The terminal structure is associated with τ'. Thus, the reference and alternative regimes are

$$\overline{B}_1 \hat{x}_t = \overline{B}_2 E_t \hat{x}_{t+1} + \overline{B}_3 \hat{x}_{t-1} \tag{28}$$

$$\tilde{B}_1 \tilde{x}_t = \tilde{B}_2 E_t \tilde{x}_{t+1} + \tilde{B}_3 \tilde{x}_{t-1} \tag{29}$$

where $\overline{B}_1 = B_1$, $\overline{B}_2 = B_2$ and $\overline{B}_3 = B_3$ (see (27)) and

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{(\alpha-1)(1-\tau')\alpha(k')^\alpha}{c'_y} & \frac{(1+n)k'}{c'_y} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} 0 & -\frac{1}{\sigma} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha-1 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \end{bmatrix}.$$

where k' is post-reform steady-state capital and $c'_y = (1-\tau')(1-\alpha)(k')^\alpha - (1+n)k'$.

Note that $\tilde{x}_t = x_t - x'_{SS}$ and $\hat{x}_t = x_t - x_{SS}$, so $\tilde{x}_t = \hat{x}_t + x_{SS} - x'_{SS}$. Using this substitution in (29), the terminal structure can be written in deviations from the original steady state as

$$\tilde{B}_1 \hat{x}_t = \tilde{B}_2 E_t \hat{x}_{t+1} + \tilde{B}_3 \hat{x}_{t-1} + \tilde{B}_5, \quad \text{where} \quad \tilde{B}_5 := (\tilde{B}_2 + \tilde{B}_3 - \tilde{B}_1)(x_{SS} - x'_{SS}).$$

Thus, our model is given by

$$B_{1,t} \hat{x}_t = B_{2,t} E_t \hat{x}_{t+1} + B_{3,t} \hat{x}_{t-1} + B_{5,t}, \quad \forall t \geq 0 \tag{30}$$

where $B_{i,t} := \mathbb{1}_t \overline{B}_i + (1 - \mathbb{1}_t)\tilde{B}_i$ for $i \in \{1, 2, 3, 5\}$ and $\mathbb{1}_t = 1$ if $t = 0$ and $\mathbb{1}_t = 0$ otherwise.

2.2 Welfare analysis

Lifetime utility is given by

$$U_t = \frac{c_{t,y}^{1-\sigma}}{1-\sigma} + \beta\frac{c_{t+1,o}^{1-\sigma}}{1-\sigma}.$$

Taking a Taylor series approximation of order 2 around the original steady state yields

$$\hat{U}_t = U_{SS}+c_y^{1-\sigma}(c_{t,y}-c_y)+\beta c_o^{1-\sigma}(c_{t+1,o}-c_o)-\frac{\sigma}{2}\left(c_y^{-(1+\sigma)}(c_{t,y}-c_y)^2+\beta c_o^{-(1+\sigma)}(c_{t+1,o}-c_o)^2\right)+\mathcal{O}(3).$$

where $U_{SS} := \frac{c_y^{1-\sigma}}{1-\sigma} + \frac{\beta c_o^{1-\sigma}}{1-\sigma}$.

For a given variable y_t in levels, let $y_t^{dev} := y_t - y$ and $\hat{y}_t := \ln(y_t/y)$. The two are related by

$$y_t^{dev} = y\left[\hat{y}_t + \frac{1}{2}\hat{y}_t^2\right]$$

Hence, following e.g. Walsh (2017), our approximation of liftetime utility may be written:

$$\hat{U}_t = U_{SS} + c_y^{1-\sigma}\hat{c}_{t,y} + \beta c_o^{1-\sigma}\hat{c}_{t+1,o} - \frac{(\sigma-1)}{2}\left[c_y^{1-\sigma}\hat{c}_{t,y}^2 + \beta c_o^{1-\sigma}\hat{c}_{t+1,o}^2\right] \tag{31}$$

where terms of order 3 and above are omitted.

Analogously, an approximation of utility around the new (post-reform) steady state is

$$\tilde{U}_t = U'_{SS} + (c'_y)^{1-\sigma}\tilde{c}_{t,y} + \beta(c'_o)^{1-\sigma}\tilde{c}_{t+1,o} - \frac{(\sigma-1)}{2}\left[(c'_y)^{1-\sigma}\tilde{c}_{t,y}^2 + \beta(c'_o)^{1-\sigma}\tilde{c}_{t+1,o}^2\right] \tag{32}$$

where $\tilde{c}_{t,i} = c_{t,i} - c'_i$ for $i \in \{y, o\}$.

Equations (31)–(32) are used as our measures of lifetime utility. We assume there is a fixed social discount factor (per generation) of $\gamma \in (0,1)$ so that social welfare is given by

$$W_0 = \sum_{t=-1}^{\infty}\gamma^t\dot{U}_t, \qquad \text{where} \quad \dot{U}_t = \begin{cases}\hat{U}_t & \text{for } t = -1, 0 \\ \tilde{U}_t & \text{for } t \geq 1\end{cases}. \tag{33}$$

Hence, we use the approximation of lifetime utility around the initial steady state for $t = -1, 0$ (while no reform has taken place) and the approximation of lifetime utility around the post-reform steady state in all periods $t \geq 1$ once the reform has been implemented.