

Supplementary Appendix (For online publication only)

“Solving linear rational expectations models in the presence of structural change: Some refinements”

This appendix provides details of the numerical applications solved in Section 5 of the paper. The codes are available at the author’s github page at: github.com/MCHatcher.

1 New Keynesian model

The baseline model has the form

$$\pi_t = \frac{1}{(1 + \beta\alpha)} (\beta E_t \pi_{t+1} + (1 + \beta\alpha - \alpha - \beta)\pi^* + \alpha\pi_{t-1} + \psi\sigma y_t - \psi a_t - \mu_t) \quad (1)$$

$$y_t = E_t y_{t+1} - \sigma^{-1}(R_t - E_t \pi_{t+1}) + \frac{(1 - \rho_g)}{\sigma} g_t - \sigma^{-1} \ln(\beta) \quad (2)$$

$$R_t = (1 - \rho_R)\bar{R} + \rho_R R_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) \quad (3)$$

where the shocks to technology, demand and the mark-up follow AR(1) processes:

$$u_t = \rho_u u_{t-1} + \sigma_u \epsilon_{u,t}, \quad u \in \{a, g, \mu\}, \quad \epsilon_{u,t} \sim N(0, 1),$$

and $\bar{R} = \pi^* - \ln(\beta)$ is the steady-state nominal interest rate.

Let $x_t = [\pi_t \ y_t \ R_t \ a_t \ g_t \ \mu_t]'$ and $e_t = [\epsilon_{a,t} \ \epsilon_{g,t} \ \epsilon_{\mu,t}]'$. Then the (fixed-structure) version of the model can be written in the form

$$B_1 x_t = B_2 E_t x_{t+1} + B_3 x_{t-1} + B_4 e_t + B_5 \quad (4)$$

where

$$B_1 = \begin{bmatrix} 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 \\ -\theta_\pi & -(\theta_y + \theta_{dy}) & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{\beta}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 \\ \sigma^{-1} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{\alpha}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\theta_{dy} & \rho_R & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_a & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\mu \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sigma_a & 0 & 0 \\ 0 & \sigma_g & 0 \\ 0 & 0 & \sigma_\mu \end{bmatrix}, \quad B_5 = \begin{bmatrix} \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^* \\ -\sigma^{-1}\ln(\beta) \\ (1-\rho_R)\bar{R} - \theta_\pi\pi^* \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

1.1 Application 1 – Change in the inflation target

The inflation target is permanently lowered from π^* to π_{new}^* . The terminal structure is the one associated with π_{new}^* . Thus, the reference and alternative regimes are described by

$$\bar{B}_1 x_t = \bar{B}_2 E_t x_{t+1} + \bar{B}_3 x_{t-1} + \bar{B}_4 e_t + \bar{B}_5 \quad (5)$$

$$\tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5 \quad (6)$$

where

$$\bar{B}_1 = B_1, \quad \bar{B}_2 = B_2, \quad \bar{B}_3 = B_3, \quad \bar{B}_4 = B_4, \quad \bar{B}_5 = B_5$$

$$\tilde{B}_1 = B_1, \quad \tilde{B}_2 = B_2, \quad \tilde{B}_3 = B_3, \quad \tilde{B}_4 = B_4$$

with B_1, B_2, B_3, B_4, B_5 defined in Section 1 and

$$\tilde{B}_5 = \begin{bmatrix} \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha} \right) \pi_{new}^* \\ -\sigma^{-1} \ln(\beta) \\ (1-\rho_R) \bar{R}_{new} - \theta_\pi \pi_{new}^* \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{R}_{new} := \pi_{new}^* - \ln(\beta).$$

Given a change in the inflation target implemented in period $t = 8$ and announced in period $t = 4$, our model is

$$\bar{B}_1 x_t = \bar{B}_2 E_t x_{t+1} + \bar{B}_3 x_{t-1} + \bar{B}_4 e_t + \bar{B}_5, \quad \text{for } t \in [0, 3] \quad (7)$$

and

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 4 \quad (8)$$

where $B_{i,t} := \mathbb{1}_t \bar{B}_i + (1 - \mathbb{1}_t) \tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [4, 7] \\ 0 & \text{if } t \geq 8 \end{cases}.$$

Note that Eq. (7) defines the original fixed structure solution which is expected to prevail *forever* in periods 0 to 3.¹ Note that the above problem is nested by the baseline (Kulish and Pagan, 2017) solution in the main text when period 0 is relabelled as period 4.

1.2 Application 2 – Change in π^* (imperfect credibility)

1.2.1 Standard approach: expectations based on past inflation target

For convenience, let us define $\pi_{orig}^* := \pi^*$. The inflation expectations of the ‘doubting’ agents are given by $E_t^{IC} \pi_{t+1} = \pi_{orig}^*$. We assume their other expectations are formed rationally. Their population share is $1 - \lambda_{1,t}$, with $\lambda_{1,t} = \lambda \in (0, 1)$ for $t \in [4, 10]$ and $\lambda_{1,t} = 1$ for $t \geq 11$.

¹Given that agents’ expectations differ from reality, one may use notation such $\tilde{E}_t x_{t+1}$ to acknowledge that the beliefs are not rational (i.e. model-consistent) in these periods.

Thus our model, now including the economy-wide expectation $\tilde{E}_t x_{t+1}$, is given by

$$B_{1,t}x_t = B_{2,t}\tilde{E}_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \quad (9)$$

$$\tilde{E}_t x_{t+1} = \Lambda_t E_t x_{t+1} + (I_n - \Lambda_t) E_t^{IC} x_{t+1} \quad (10)$$

where $E_t^{IC} x_{t+1} = F_0 x_t + F_1 x_{t-1} + F_2 e_t + F_3$ (for user-specified matrices F_0, F_1, F_2, F_3) and Λ_t is an $n \times n$ diagonal matrix given by $\Lambda_t = \text{diag}(\lambda_{1,t}, 1, \dots, 1)$.

Since the inflation expectations of the doubters are constant, we have:

$$F_0 = F_1 = 0_{n \times n}, \quad F_2 = 0_{n \times m}, \quad F_3 = \begin{bmatrix} \pi_{orig}^* \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \implies E_t^{IC} x_{t+1} = F_3.$$

Substituting this result into (10) and substituting the result into (9), our model is given by

$$B_{1,t}x_t = \hat{B}_{2,t}E_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + \hat{B}_{5,t} \quad (11)$$

where $\hat{B}_{2,t} := \Lambda_t B_{2,t}$ and $\hat{B}_{5,t} := (I_n - \Lambda_t)F_3 + B_{5,t}$ and $\Lambda_t = I_n$ for all $t \geq 11$.

Note that Eq. (11) is in the same form as the system (8) and can, therefore, be solved by the same method.

1.2.2 Type 1 credibility

Under Type 1 credibility, our model is given by

$$B_{1,t}x_t = B_{2,t}\tilde{E}_t x_{t+1} + B_{3,t}x_{t-1} + B_{4,t}e_t + B_{5,t} \quad (12)$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t) E_t^{IC} x_{t+1} \quad (13)$$

where we assume that $\lambda_t = \lambda$ (given) for $t \in [4, 10]$ and $\lambda_t = 1$ for all $t \geq 11$, and the expectations of the doubting agents are given by $E_t^{IC} x_{t+1} = \bar{\Omega}x_t + \bar{\Gamma}E_t^{IC}[e_{t+1}] + \bar{\Psi} = \bar{\Omega}x_t + \bar{\Psi}$,² where

$$\Omega_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}\hat{B}_{3,t}, \quad \Gamma_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}\hat{B}_{4,t}, \quad (14)$$

$$\Psi_t^{IC} = (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1}(\hat{B}_{2,t}\Psi_{t+1}^{IC} + \hat{B}_{5,t}) \quad (15)$$

provided $\det[\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC}] \neq 0$ for all $t \in [0, \tilde{T}]$, where

$$\Omega_{T+1}^{IC} = \bar{\Omega} := (\bar{B}_1 - \bar{B}_2\bar{\Omega})^{-1}\bar{B}_3, \quad \Gamma_{T+1}^{IC} = \bar{\Gamma} := (\bar{B}_1 - \bar{B}_2\bar{\Omega})^{-1}\bar{B}_4,$$

$$\Psi_{T+1}^{IC} = \bar{\Psi} = (\bar{B}_1 - \bar{B}_2\bar{\Omega})^{-1}(\bar{B}_2\bar{\Psi} + \bar{B}_5)$$

²As noted in the main text (Section 4.2.2) we assume the doubting agents know the properties of the exogenous shocks e_t (e.g. zero mean) and have access to I_t (which does not include the *future* structure).

We assume that the doubting agents expect a permanent reversion to the original structure (with the higher inflation target) starting next period:

$$\hat{B}_{i,t} = \bar{B}_i, \quad \text{for } i \in [5] \implies \Omega_t^{IC} = \bar{\Omega}, \quad \Gamma_t^{IC} = \bar{\Gamma}, \quad \Psi_t^{IC} = \bar{\Psi}.$$

Substituting this expression for $E_t^{IC} x_{t+1}$ into (13) and substituting the result into (12) gives:

$$B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC} \quad (16)$$

where $B_{1,t}^{IC} := B_{1,t} - B_{2,t}(1 - \lambda_t)\bar{\Omega}$, $B_{2,t}^{IC} := B_{2,t}\lambda_t$ and $B_{5,t}^{IC} := B_{2,t}(1 - \lambda_t)\bar{\Psi} + B_{5,t}$.

Since Eq. (16) is in the same form as (8), it can be solved by the same method.

1.3 Application 3 – Forward guidance

1.3.1 ‘Plain vanillla’ forward guidance

The interest rate rule is now amended to

$$R_t = \begin{cases} \underline{R} := 0 & \text{for } t \in [2, 5] \\ (1 - \rho_R)\bar{R} + \rho_R R_{t-1} + \theta_\pi(\pi_t - \pi^*) + \theta_y y_t + \theta_{dy}(y_t - y_{t-1}) & \text{otherwise.} \end{cases}$$

Since the original structure is in place for $t \geq 6$, we let the original structure be the alternative (‘tilde’) regime, i.e. $\tilde{B}_i = B_i$ for $i \in [5]$, where the B_i matrices appear on p. 1. The reference structure is given by the ‘forward guidance regime’, i.e.

$$\bar{B}_1 = \begin{bmatrix} 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{B}_5 = \begin{bmatrix} \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right)\pi^* \\ -\sigma^{-1}\ln(\beta) \\ \underline{R} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $\bar{B}_i = B_i$ for $i = 2, 3, 4$.

Our model is given by

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 0 \quad (17)$$

where $B_{i,t} := \mathbb{1}_t \bar{B}_i + (1 - \mathbb{1}_t) \tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [2, 5] \\ 0 & \text{otherwise} \end{cases}.$$

1.3.2 Forward guidance with delayed announcement

We consider the case $K = 0$, which implies that a fraction $1 - \lambda$ of agents do not receive any prior announcement about the forward guidance policy. Denoting the economy-wide expectation $\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1 - \lambda) \hat{E}_t x_{t+1}$, Eq. (17) becomes

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \text{for } t \in [0, 1] \quad (18)$$

$$B_{1,t} x_t = B_{2,t} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}, \quad \forall t \geq 2 \quad (19)$$

where $B_{i,t} := \mathbb{1}_t \bar{B}_i + (1 - \mathbb{1}_t) \tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [2, 5] \\ 0 & \text{otherwise} \end{cases}$$

and $\tilde{E}_t x_{t+1} = \lambda E_t x_{t+1} + (1 - \lambda) \hat{E}_t x_{t+1}$. Since the uninformed agents expect the original structure to prevail (in all future periods) at dates 0 and 1, we have

$$\hat{E}_t x_{t+1} = \tilde{\Omega} x_t + \tilde{\Gamma} \hat{E}_t [e_{t+1}] + \tilde{\Psi} = \tilde{\Omega} x_t + \tilde{\Psi}, \quad \text{for } t = 0, 1$$

where $\tilde{\Omega} = (\tilde{B}_1 - \tilde{B}_2 \tilde{\Omega})^{-1} \tilde{B}_3$, $\tilde{\Gamma} = (\tilde{B}_1 - \tilde{B}_2 \tilde{\Omega})^{-1} \tilde{B}_4$, $\tilde{\Psi} = (\tilde{B}_1 - \tilde{B}_2 \tilde{\Omega})^{-1} (\tilde{B}_2 \tilde{\Psi} + \tilde{B}_5)$ (see Assumption 1 in the main text).

We can thus write the model compactly as

$$\hat{B}_{1,t} x_t = \hat{B}_{2,t} E_t x_{t+1} + \hat{B}_{3,t} x_{t-1} + \hat{B}_{4,t} e_t + \hat{B}_{5,t}, \quad \forall t \geq 0 \quad (20)$$

where, for $i \in [5]$, $\hat{B}_{i,t} = B_{i,t}$ for $t \geq 2$ and $\hat{B}_{i,t} = \tilde{B}_i$ for $t = 0, 1$, with

$$\hat{B}_1 := \tilde{B}_1 - \tilde{B}_2 (1 - \lambda) \tilde{\Omega}, \quad \hat{B}_2 := \lambda \tilde{B}_2, \quad \hat{B}_5 := \tilde{B}_5 + \tilde{B}_2 (1 - \lambda) \tilde{\Psi}$$

and $\hat{B}_i = B_i$ for $i = 2, 3$ (see p. 1).

1.3.3 Forward guidance (Type 2 credibility)

Under Type 2 credibility, our model is given by

$$B_{1,t} x_t = B_{2,t} \tilde{E}_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t} \quad (21)$$

$$\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t) E_t^{IC} x_{t+1} \quad (22)$$

$$E_t^{IC} x_{t+1} = p_t E_t^* x_{t+1} + (1 - p_t) \hat{E}_t x_{t+1} \quad (23)$$

where $\lambda_t = 0.7$ for $t \in [0, 5]$ and $\lambda_t = 1$ for all $t \geq 6$.

The expectations of the doubting agents are given by:

$$\hat{E}_t x_{t+1} = \Omega_{t+1}^{IC} x_t + \Gamma_{t+1}^{IC} \hat{E}_t [e_{t+1}] + \Psi_{t+1}^{IC} = \Omega_{t+1}^{IC} x_t + \Psi_{t+1}^{IC}$$

$$E_t^* x_{t+1} = \Omega_{t+1}^* x_t + \Gamma_{t+1}^* E_t^* [e_{t+1}] + \Psi_{t+1}^* = \Omega_{t+1}^* x_t + \Psi_{t+1}^*$$

where the matrices $\Omega_{t+1}^{IC}, \Psi_{t+1}^{IC}$ are given by the recursions (14)–(15), and the matrices $\Omega_{t+1}^*, \Gamma_{t+1}^*$ are given by the recursions in Proposition 1 with matrices $B_{1,t}, \dots, B_{5,t}$.

Substituting these expectations into (23) and substituting the result into (22) gives $\tilde{E}_t x_{t+1} = \lambda_t E_t x_{t+1} + (1 - \lambda_t)[\tilde{\Omega}_{t+1}^{IC} x_t + \tilde{\Psi}_{t+1}^{IC}]$, where $\tilde{\Omega}_{t+1}^{IC} := p_t \Omega_{t+1}^* + (1 - p_t) \Omega_{t+1}^{IC}$ and $\tilde{\Psi}_{t+1}^{IC} := p_t \Psi_{t+1}^* + (1 - p_t) \Psi_{t+1}^{IC}$. Hence, our model can be summarized as:

$$\begin{cases} B_{1,t}^{IC} x_t = B_{2,t}^{IC} E_t x_{t+1} + B_{3,t} x_{t-1} + B_{4,t} e_t + B_{5,t}^{IC}, & 0 \leq t \leq \tilde{T} \\ \tilde{B}_1 x_t = \tilde{B}_2 E_t x_{t+1} + \tilde{B}_3 x_{t-1} + \tilde{B}_4 e_t + \tilde{B}_5, & \forall t > \tilde{T}. \end{cases} \quad (24)$$

where $B_{1,t}^{IC} := B_{1,t} - B_{2,t}(1 - \lambda_t)\tilde{\Omega}_{t+1}^{IC}$, $B_{2,t}^{IC} := B_{2,t}\lambda_t$ and $B_{5,t}^{IC} := B_{2,t}(1 - \lambda_t)\tilde{\Psi}_{t+1}^{IC} + B_{5,t}$.

The computational approach can be summarized as follows:

1. Define the indicators $\{\mathbb{1}_t\}_{t=0}^\infty$ and $\{\mathbb{1}_t^{IC}\}_{t=0}^\infty$. In this example we have $\mathbb{1}_t = 1$ for $t \in [2, 5]$ and $\mathbb{1}_t = 0$ otherwise, and $\mathbb{1}_t^{IC} = 1$ for $t \in [2, 3]$ and $\mathbb{1}_t^{IC} = 0$ otherwise (because the doubting agents expect forward guidance to end 2 periods earlier than announced).
2. Compute the actual sequences of structures $B_{i,t} = \mathbb{1}_t \bar{B}_i + (1 - \mathbb{1}_t) \tilde{B}_i$ (for $i \in [5]$) and the sequence of structures to which the doubting agents attach positive (subjective) probability: $\hat{B}_{i,t} = \mathbb{1}_t^{IC} \bar{B}_i + (1 - \mathbb{1}_t^{IC}) \tilde{B}_i$.
3. Choose $\{\lambda_t\}_{t=0}^\infty$ and $\{p_t\}_{t=0}^\infty$. In this example, we set $\lambda_t = 0.7$ for $t \in [0, 5]$, $\lambda_t = 1$ for all $t \geq 6$ and $p_t = p = 0.5$.
4. For all $t \in [0, 5]$, compute the recursions

$$\begin{aligned} \Omega_t^* &= (B_{1,t} - B_{2,t}\Omega_{t+1}^*)^{-1} B_{3,t}, & \Gamma_t^* &= (B_{1,t} - B_{2,t}\Omega_{t+1}^*)^{-1} B_{4,t}, \\ \Psi_t^* &= (B_{1,t} - B_{2,t}\Omega_{t+1}^*)^{-1} (B_{2,t}\Psi_{t+1}^* + B_{5,t}), \\ \Omega_t^{IC} &= (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1} \hat{B}_{3,t}, & \Gamma_t^{IC} &= (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1} \hat{B}_{4,t}, \\ \Psi_t^{IC} &= (\hat{B}_{1,t} - \hat{B}_{2,t}\Omega_{t+1}^{IC})^{-1} (\hat{B}_{2,t}\Psi_{t+1}^{IC} + \hat{B}_{5,t}) \end{aligned}$$

subject to the terminal conditions $\Omega_{\tilde{T}+1}^* = \Omega_{\tilde{T}+1}^{IC} = \tilde{\Omega}$, $\Gamma_{\tilde{T}+1}^* = \Gamma_{\tilde{T}+1}^{IC} = \tilde{\Gamma}$.³

5. Find the matrices $B_{1,t}^{IC}, B_{2,t}^{IC}, B_{5,t}^{IC}$ and use these in Proposition 1 (in place of $B_{1,t}, B_{2,t}, B_{5,t}$) to find the solution matrices $\{\Omega_t, \Gamma_t, \Psi_t\}_{t=0}^{T_{sim}}$ and compute the solution $\{x_t\}_{t=0}^{T_{sim}}$.

³Note that in this example the perceived terminal structure of the doubting agents coincides with the actual one because they expect reversion to the Taylor-type rule to happen sooner than announced.

1.4 Forward guidance + indeterminacy

The interest rate rule is now

$$R_t = \begin{cases} \underline{R} := 0 & t \in [0, 7] \\ \rho_R R_{t-1} + (1 - \rho_R) \bar{R} + \theta_\pi (\pi_t - \pi^*) + \theta_y y_t + \delta_{dy} (y_t - y_{t-1}) & \text{otherwise} \end{cases}$$

where θ_π is such that the terminal solution is indeterminate, with degree 1 indeterminacy.

Following the approach in Farmer, Khramov and Nicolo (2015, JEDC), we assume one of the forward-looking variables (inflation or output) is hit by self-fulfilling expectations shocks, i.e. sunspots. We pick inflation and hence the terminal structure is given by (1)–(3) along with the additional inflation equation:

$$\pi_t = s_{t-1} + v_{\pi,t}, \quad \forall t > 7$$

where $s_t := E_t \pi_{t+1}$.

Substitute s_t in place $E_t \pi_{t+1}$ in (1)–(3) and let $\tilde{x}_t = [\pi_t \ y_t \ R_t \ a_t \ g_t \ \mu_t \ s_t]'$ and $\tilde{e}_t = [v_{\pi,t} \ \epsilon_{a,t} \ \epsilon_{g,t} \ \epsilon_{\mu,t}]'$. Then the system for $t > 7$ has the form:

$$\tilde{B}_1 \tilde{x}_t = \tilde{B}_2 E_t \tilde{x}_{t+1} + \tilde{B}_3 \tilde{x}_{t-1} + \tilde{B}_4 \tilde{e}_t + \tilde{B}_5, \quad \forall t > 7$$

where

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} & -\frac{\beta}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 & -\sigma^{-1} \\ -\theta_\pi & -(\theta_y + \theta_{dy}) & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\alpha}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\theta_{dy} & \rho_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\mu & 0 \end{bmatrix}, \quad \tilde{B}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_a & 0 & 0 \\ 0 & 0 & \sigma_g & 0 \\ 0 & 0 & 0 & \sigma_\mu \end{bmatrix}, \quad \tilde{B}_5 = \begin{bmatrix} 0 \\ \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right) \pi^* \\ -\sigma^{-1} \ln(\beta) \\ (1 - \rho_R) \bar{R} - \theta_\pi \pi^* \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

In the earlier periods $0 \leq t \leq 7$, the interest rate $R_t = \underline{R}$ is in place and inflation is determined as

$$\pi_t = s_{t-1}, \quad \text{for } t \in [0, 7]$$

since we assume coordination of expectations on the sunspot terminal solution.

Hence, the system for $t \in [0, 7]$ is:

$$\bar{B}_1 \tilde{x}_t = \bar{B}_2 E_t \tilde{x}_{t+1} + \bar{B}_3 \tilde{x}_{t-1} + \bar{B}_4 \tilde{e}_t + \bar{B}_5$$

where

$$\begin{aligned} \bar{B}_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\psi\sigma}{1+\beta\alpha} & 0 & \frac{\psi}{1+\beta\alpha} & 0 & \frac{1}{1+\beta\alpha} & -\frac{\beta}{1+\beta\alpha} \\ 0 & 1 & \sigma^{-1} & 0 & -\frac{(1-\rho_g)}{\sigma} & 0 & -\sigma^{-1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \bar{B}_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\alpha}{1+\beta\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_\mu & 0 \end{bmatrix}, \quad \bar{B}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sigma_a & 0 & 0 \\ 0 & 0 & \sigma_g & 0 \\ 0 & 0 & 0 & \sigma_\mu \end{bmatrix}, \quad \bar{B}_5 = \begin{bmatrix} 0 \\ \left(\frac{1+\beta\alpha-\alpha-\beta}{1+\beta\alpha}\right) \pi^* \\ -\sigma^{-1} \ln(\beta) \\ \underline{R} \\ 0 \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Hence, our model is given by

$$\mathcal{B}_{1,t} \tilde{x}_t = \mathcal{B}_{2,t} E_t \tilde{x}_{t+1} + \mathcal{B}_{3,t} \tilde{x}_{t-1} + \mathcal{B}_{4,t} \tilde{e}_t + \mathcal{B}_{5,t}, \quad \forall t \geq 0 \quad (25)$$

where $\mathcal{B}_{i,t} := \mathbb{1}_t \bar{B}_i + (1 - \mathbb{1}_t) \tilde{B}_i$ for $i \in [5]$ and

$$\mathbb{1}_t = \begin{cases} 1 & \text{if } t \in [0, 7] \\ 0 & \text{otherwise} \end{cases}.$$

2 Pension reform in the Diamond model

The baseline log-linearized model has the form:

$$\begin{aligned} \hat{c}_{t,y} &= -\frac{1}{\sigma} \hat{R}_{t+1} + \hat{c}_{t+1,o}, \quad \hat{R}_t = (\alpha - 1) \hat{k}_t, \quad \hat{w}_t = \alpha \hat{k}_t \\ \hat{c}_{t,y} &= \frac{(1-\tau)(1-\alpha)k^\alpha}{c_y} \hat{w}_t - \frac{(1+n)k}{c_y} \hat{k}_{t+1}, \quad \hat{c}_{t,o} = \alpha \hat{k}_t \end{aligned}$$

where $c_y = (1-\tau)(1-\alpha)k^\alpha - (1+n)k$, k and ‘hats’ are log deviations from steady state.

Steady state capital, k , is determined by the fixed point of the capital accumulation equation in the main text:

$$k^{1-\alpha} = \left[\frac{(1-\tau)(1-\alpha)}{1+n} \right] \frac{\beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}}}{\left(1 + \frac{(1-\alpha)\tau}{\alpha}\right) R + \beta^{\frac{1}{\sigma}} R^{\frac{1}{\sigma}}}, \quad \text{where } R = \alpha k^{\alpha-1}.$$

Some algebra shows that steady state capital must satisfy

$$(\alpha + (1 - \alpha)\tau)k^{\frac{1-\alpha}{\sigma}} + (\beta\alpha)^{\frac{1}{\sigma}}k^{1-\alpha} = \frac{(\beta\alpha)^{\frac{1}{\sigma}}(1 - \tau)(1 - \alpha)}{1 + n}. \quad (26)$$

Note that Eq. (26) does not hold for $k = 0$ since $LHS = 0 < RHS$, but since $\partial LHS / \partial k > 0$, there exists a unique $k^* \in (0, \infty)$ such that the equation is satisfied. We find k^* numerically.

Let $\hat{c}ap_t := \hat{k}_{t+1}$ and $\hat{x}_t = [\hat{c}_{t,y} \ \hat{R}_t \ \hat{w}_t \ \hat{c}ap_t \ \hat{c}_{t,o}]'$. Then the (fixed-structure) version of the model can be written in the form

$$B_1 \hat{x}_t = B_2 E_t \hat{x}_{t+1} + B_3 \hat{x}_{t-1} \quad (27)$$

where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{(\alpha-1)(1-\tau)\alpha k^\alpha}{c_y} & \frac{(1+n)k}{c_y} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & -\frac{1}{\sigma} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha - 1 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \end{bmatrix}$$

and $\hat{x}_t = x_t - x_{SS}$.

2.1 Pension reform: change in τ

The pension contribution rate is permanently lowered from τ to τ' in period $t = 1$. The terminal structure is associated with τ' . Thus, the reference and alternative regimes are

$$\bar{B}_1 \hat{x}_t = \bar{B}_2 E_t \hat{x}_{t+1} + \bar{B}_3 \hat{x}_{t-1} \quad (28)$$

$$\tilde{B}_1 \tilde{x}_t = \tilde{B}_2 E_t \tilde{x}_{t+1} + \tilde{B}_3 \tilde{x}_{t-1} \quad (29)$$

where $\bar{B}_1 = B_1$, $\bar{B}_2 = B_2$ and $\bar{B}_3 = B_3$ (see (27)) and

$$\tilde{B}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{(\alpha-1)(1-\tau')\alpha(k')^\alpha}{c'_y} & \frac{(1+n)k'}{c'_y} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} 0 & -\frac{1}{\sigma} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha - 1 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \end{bmatrix}.$$

where k' is post-reform steady-state capital and $c'_y = (1 - \tau')(1 - \alpha)(k')^\alpha - (1 + n)k'$.

Note that $\tilde{x}_t = x_t - x'_{SS}$ and $\hat{x}_t = x_t - x_{SS}$, so $\tilde{x}_t = \hat{x}_t + x_{SS} - x'_{SS}$. Using this substitution in (29), the terminal structure can be written in deviations from the original steady state as

$$\tilde{B}_1 \hat{x}_t = \tilde{B}_2 E_t \hat{x}_{t+1} + \tilde{B}_3 \hat{x}_{t-1} + \tilde{B}_5, \quad \text{where} \quad \tilde{B}_5 := (\tilde{B}_2 + \tilde{B}_3 - \tilde{B}_1)(x_{SS} - x'_{SS}).$$

Thus, our model is given by

$$B_{1,t} \hat{x}_t = B_{2,t} E_t \hat{x}_{t+1} + B_{3,t} \hat{x}_{t-1} + B_{5,t}, \quad \forall t \geq 0 \quad (30)$$

where $B_{i,t} := \mathbb{1}_t \bar{B}_i + (1 - \mathbb{1}_t) \tilde{B}_i$ for $i \in \{1, 2, 3, 5\}$ and $\mathbb{1}_t = 1$ if $t = 0$ and $\mathbb{1}_t = 0$ otherwise.

2.2 Welfare analysis

Lifetime utility is given by

$$U_t = \frac{c_{t,y}^{1-\sigma}}{1-\sigma} + \beta \frac{c_{t+1,o}^{1-\sigma}}{1-\sigma}.$$

Taking a Taylor series approximation of order 2 around the original steady state yields

$$\hat{U}_t = U_{SS} + c_y^{-\sigma}(c_{t,y} - c_y) + \beta c_o^{-\sigma}(c_{t+1,o} - c_o) - \frac{\sigma}{2} \left(c_y^{-(1+\sigma)}(c_{t,y} - c_y)^2 + \beta c_o^{-(1+\sigma)}(c_{t+1,o} - c_o)^2 \right) + \mathcal{O}(3).$$

where $U_{SS} := \frac{c_y^{1-\sigma}}{1-\sigma} + \frac{\beta c_o^{1-\sigma}}{1-\sigma}$.

For a given variable y_t in levels, let $y_t^{dev} := y_t - y$ and $\hat{y}_t := \ln(y_t/y)$. The two are related by

$$y_t^{dev} = y \left[\hat{y}_t + \frac{1}{2} \hat{y}_t^2 \right]$$

Hence, following e.g. Walsh (2017), our approximation of lifetime utility may be written:

$$\hat{U}_t = U_{SS} + c_y^{1-\sigma} \hat{c}_{t,y} + \beta c_o^{1-\sigma} \hat{c}_{t+1,o} - \frac{(\sigma-1)}{2} [c_y^{1-\sigma} \hat{c}_{t,y}^2 + \beta c_o^{1-\sigma} \hat{c}_{t+1,o}^2] \quad (31)$$

where terms of order 3 and above are omitted.

Analogously, an approximation of utility around the new (post-reform) steady state is

$$\tilde{U}_t = U'_{SS} + (c'_y)^{1-\sigma} \tilde{c}_{t,y} + \beta (c'_o)^{1-\sigma} \tilde{c}_{t+1,o} - \frac{(\sigma-1)}{2} [(c'_y)^{1-\sigma} \tilde{c}_{t,y}^2 + \beta (c'_o)^{1-\sigma} \tilde{c}_{t+1,o}^2] \quad (32)$$

where $\tilde{c}_{t,i} = c_{t,i} - c'_i$ for $i \in \{y, o\}$.

Equations (31)–(32) are used as our measures of lifetime utility. We assume there is a fixed social discount factor (per generation) of $\gamma \in (0, 1)$ so that social welfare is given by

$$W_0 = \sum_{t=-1}^{\infty} \gamma^t \dot{U}_t, \quad \text{where} \quad \dot{U}_t = \begin{cases} \hat{U}_t & \text{for } t = -1, 0 \\ \tilde{U}_t & \text{for } t \geq 1 \end{cases}. \quad (33)$$

Hence, we use the approximation of lifetime utility around the initial steady state for $t = -1, 0$ (while no reform has taken place) and the approximation of lifetime utility around the post-reform steady state in all periods $t \geq 1$ once the reform has been implemented.