Supplementary Appendix (For online publication only)

"Solving heterogeneous-belief asset pricing models with short-selling constraints and many agents"

This appendix provides technical details of some generalizations and extensions of the benchmark model that are reported in Section 3.2, Section 4.1 and Section 5 of the main text.

1 Housing as the risky asset

In this section we show that if the risky asset is housing as in Bolt et al. (2019) and Hatcher (2021), then similar (or identical) analytical results to those in Proposition 1 are obtained.

Bolt et al. (2019) and Hatcher (2021) consider models in which housing is an investment asset that differs from shares due to the interpretation of 'dividends'. In Bolt et al. (2019) dividends are replaced by imputed rent based on an arbitrage condition between the user and rental costs, whereas Hatcher (2021) assumes linear housing utility scaled by a housing preference variable.[1] In both models, these additional variables are exogenous processes whose properties are known to the investors. We assume a fixed supply of housing $\overline{Z} > 0$.

1.1 Derivation of demands and the fundamental price

Assuming linear excess returns and short-selling constraints, demands for housing $z_{t,h}$ are:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+Q_t(1+\hat{r})-(1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}]+Q_t(1+\hat{r})}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}]+Q_t(1+\hat{r})}{1+r} \end{cases} \tag{BDDHL1}$$

where Q_t is the exogenous rental price and $\hat{r}$ is the fixed risk-free mortgage rate; and

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+\Theta_t\overline{U}_z-(1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}]+\Theta_t\overline{U}_z}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}]+\Theta_t\overline{U}_z}{1+r} \end{cases} \tag{Hatcher1}$$

where $\Theta_t > 0$ is an exogenous preference for housing utility versus financial wealth, and $\overline{U}_z > 0$ is a fixed marginal utility of housing (which does not depend on $z_{t,h}$).

When short-selling constraints are slack, both the above demands are of the form:

$$z_{t,h} = \frac{\tilde{E}_{t,h}[p_{t+1}] + d_t - (1+r)p_t}{a\sigma^2} \tag{1}$$

where d_t has the interpretation of a 'housing dividend' paid at date t.

[1] Assuming quadratic utility from housing in the framework of Hatcher (2021) does not pose problems if housing preference is fixed or deterministic. However, with stochastic preference shocks, there is no closed-form solution for the fundamental price, so an approximation would be needed to obtain analytical results.

Using this expression, we can solve for the fundamental price of housing as follows:

$$p_t^* = \frac{\frac{r}{1+r}\sum_{i=0}^{\infty}\left(\frac{1}{1+r}\right)^i E_t d_{t+i} - a\sigma^2\overline{Z}}{r} \tag{2}$$

where the expectations $E_t d_{t+i}$ will depend on the assumptions we make about the evolution of the exogenous rental price Q_t and the exogenous housing preference Θ_t.

We consider below two polar cases: IID disturbances (as in the main text of the paper) and permanent disturbances (random walk, as in Bolt et al. (2019)). We will show that in both cases the demands, in terms of deviations from the fundamental price, have a common representation for which demands are identical to those in the benchmark model.

1.1.1 Case of IID disturbances

Suppose the process for d_t in (1) is of the form $d_t = \overline{d} + \epsilon_t$, where $\overline{d} > 0$ and ϵ_t is a zero-mean shock with constant variance. It follows that $E_t d_{t+i} = \overline{d}$ if $i \geq 1$ and $E_t d_{t+i} = d_t$ otherwise (i.e. if $i = 0$). Equation (2) therefore simplifies to:

$$p_t^* = \frac{d_t - \frac{1}{1+r}\epsilon_t - a\sigma^2\overline{Z}}{r} \tag{3}$$

such that the fundamental price p_t^* differs relative to baseline model due to the shock ϵ_t.

Note that (3) is the fundamental price when demands are given by (BDDHL1) if we let $d_t = Q_t(1+\hat{r})$ with $Q_t = \overline{Q} + v_t$, where v_t is zero-mean and IID, such that $d_t = \overline{d} + \epsilon_t$ provided $\overline{d} := (1+\hat{r})\overline{Q}$ and $\epsilon_t := (1+\hat{r})v_t$. In this case, (3) gives $d_t = rp_t^* + (1+r)^{-1}\epsilon_t + a\sigma^2\overline{Z}$, which implies that $Q_t(1+\hat{r}) = rp_t^* + (1+r)^{-1}\epsilon_t + a\sigma^2\overline{Z}$.

Therefore, by (1) the demands in (BDDHL1) can be written in terms of deviations from the fundamental price, $x_t := p_t - p_t^*$, as

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \end{cases} \tag{BDDHL2}$$

where $\tilde{E}_{t,h}\left[x_{t+1}\right] = \tilde{E}_{t,h}\left[p_{t+1}\right] - E_t[p_{t+1}^*]$.

Note the demands (BDDHL2) are *identical* to those in the benchmark model. Likewise, the demands in (Hatcher1), expressed in price deviations, are equivalent to (BDDHL2) by setting $d_t = \Theta_t\overline{U}_z$, with $\Theta_t = \overline{\Theta} + v_t$ and $\epsilon_t := \overline{U}_z v_t$. We now consider permanent disturbances.

1.1.2 Case of permanent disturbances

Suppose the process for d_t in (1) is a random walk without drift: $d_t = d_{t-1} + \epsilon_t$, where $\overline{d} > 0$ and ϵ_t is zero mean with constant variance. Then $E_t d_{t+i} = d_t \ \forall i \geq 0$ and hence (2) becomes:

$$p_t^* = \frac{d_t - a\sigma^2\overline{Z}}{r}. \tag{4}$$

Note that (4) is the fundamental price given the demands (BDDHL1) if $d_t = Q_t(1+\hat{r})$ with $Q_t = Q_{t-1} + v_t$, where v_t is zero-mean and IID, such that $d_t = d_{t-1} + \epsilon_t$ when $\epsilon_t := (1+\hat{r})v_t$. Since (4) gives $d_t = rp_t^* + a\sigma^2\overline{Z}$, we have $Q_t(1+\hat{r}) = rp_t^* + a\sigma^2\overline{Z}$ and the demands in (BDDHL1) in terms of deviations from the fundamental price $x_t := p_t - p_t^*$ are:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + + a\sigma^2\overline{Z}}{1+r} \end{cases} \tag{BDDHL3}$$

where $\tilde{E}_{t,h}\left[x_{t+1}\right] := \tilde{E}_{t,h}\left[p_{t+1}\right] - E_t[p_{t+1}^*]$.

Note that the demands in (BDDHL3) are *identical* to those in the benchmark model. The same result applies to the demands in (Hatcher1), as this case simply requires a change in variables to $d_t = \Theta_t\overline{U}_z$, with $\Theta_t = \Theta_{t-1} + v_t$, and $\epsilon_t = \overline{U}_z v_t$.

1.2 Equivalence result

Analogous to the benchmark model, the housing market clears when $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$. We thus have the following simple result based on the demands (BDDHL2), (BDDHL3).

Remark 1 *In the above models, where demands are given by* (BDDHL1) *or* (Hatcher1)*, the results in Proposition 1 and Corollary 1 remain intact, because the demands in terms of price deviations,* (BDDHL2) *and* (BDDHL3)*, are identical to those in the benchmark model.*

2 Multiple markets and endogenous participation

In this section we give a re-worked version of Proposition 1 for the case of multiple risky assets, where the extent of participation $w_t^m \in (0,1)$ in each market $m \in \{1,2,...,M\}$ is determined by its attractiveness relative to other markets, as in Westerhoff (2004).

We show in the main text (Section 4.1.1) that demand of type h in market m is given by

$$z_{t,h}^m = \begin{cases} w_t^m\left(\frac{\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m - (1+r)x_t^m}{a\sigma_m^2}\right) & \text{if } x_t^m \leq \frac{\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m}{1+r} \\ 0 & \text{if } x_t^m > \frac{\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m}{1+r} \end{cases} \tag{5}$$

where $x_t^m := p_t^m - \overline{p}^m$ is the deviation from the fundamental price in market m.

As a result, the market-clearing condition in a given market m is:

$$\sum_{h\in\mathcal{H}} n_{t,h}^m\left(\frac{\hat{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2\overline{Z}_m/w_t^m - (1+r)x_t^m}{a\sigma_m^2}\right) = \overline{Z}_m/w_t^m \tag{6}$$

where $\hat{E}_{t,h}\left[x_{t+1}^m\right] := \tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2\overline{Z}_m(M - 1/w_t^m)$.

Note that the only difference relative to the benchmark model (with one risky asset) is that the fixed supply of $\overline{Z}_m$ is scaled by $1/w_t^m$ and expectations are amended to $\hat{E}_{t,h}\left[x_{t+1}^m\right]$. We can thus state the following re-worked version of Proposition 1 that is adapted to this case.

Proposition 1 (Proposition 1 adapted to multiple markets) *Let $x_t^m = p_t^m - \overline{p}^m$ be the equilibrium price at date t and let $\mathcal{B}_t^m \subseteq \mathcal{H}$ ($\mathcal{S}_t^m := \mathcal{H} \setminus \mathcal{B}_t^m$) be the set of unconstrained types (short-selling constrained types) in market m at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}^m(\tilde{E}_{t,h}\left[x_{t+1}^m\right] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}\left[x_{t+1}^m\right]\}) \leq a\sigma_m^2\overline{Z}_m/w_t^m$, then no type is short-selling constrained ($\mathcal{B}_t^{m*} = \mathcal{H}$, $\mathcal{S}_t^{m*} = \emptyset$) and the equilibrium price is

$$x_t^m = \frac{\sum_{h\in\mathcal{H}} n_{t,h}^m f_h(x_{t-1}^m, ..., x_{t-L}^m) + a\sigma_m^2\overline{Z}_m\left(M - \frac{1}{w_t^m}\right)}{1+r} := x_t^{m*} \tag{7}$$

with demands $z_{t,h}^m = w_t^m(a\sigma_m^2)^{-1}(\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m - (1+r)x_t^m) \geq 0\ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h}^m(\tilde{E}_{t,h}\left[x_{t+1}^m\right] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}\left[x_{t+1}^m\right]\}) > a\sigma_m^2\overline{Z}_m/w_t^m$, at least one type is short-selling constrained and there exist unique non-empty sets $\mathcal{B}_t^{m*} \subset \mathcal{H}$ and $\mathcal{S}_t^{m*}$ such that $\sum_{h\in\mathcal{B}_t^{m*}} n_{t,h}^m(\tilde{E}_{t,h}\left[x_{t+1}^m\right] - \min\limits_{h\in\mathcal{B}_t^{m*}}\{\tilde{E}_{t,h}\left[x_{t+1}^m\right]\}) \leq \frac{a\sigma_m^2\overline{Z}_m}{w_t^m} < \sum_{h\in\mathcal{B}_t^{m*}} n_{t,h}^m(\tilde{E}_{t,h}\left[x_{t+1}^m\right] - \max\limits_{h\in\mathcal{S}_t^{m*}}\{\tilde{E}_{t,h}\left[x_{t+1}^m\right]\})$, and the associated equilibrium price and demands are

$$x_t^m = \frac{\sum_{h\in\mathcal{B}_t^{m*}} n_{t,h}^m f_h(x_{t-1}^m, ..., x_{t-L}^m) - \left(\frac{1}{w_t^m} - [\sum_{h\in\mathcal{B}_t^{m*}} n_{t,h}^m]M\right) a\sigma_m^2\overline{Z}_m}{(1+r)\sum_{h\in\mathcal{B}_t^{m*}} n_{t,h}^m} > x_t^{m*} \tag{8}$$

and $z_{t,h}^m = w_t^m(a\sigma_m^2)^{-1}(\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m - (1+r)x_t^m) \geq 0\ \forall h \in \mathcal{B}_t^{m*}$, $z_{t,h}^m = 0\ \forall h \in \mathcal{S}_t^{m*}$.

Proof. It follows from the Proposition 1 Proof when variables are appended with m superscripts, $\tilde{E}_{t,h}\left[x_{t+1}^m\right]$ is replaced by $\hat{E}_{t,h}\left[x_{t+1}^m\right]$ in (6), and $\overline{Z}$ is replaced by $\overline{Z}_m/w_t^m$. ∎

3 Beliefs with no dependence on the fundamental price

In this section we show that the results easily generalize to the case of one or more price predictors that do not depend on the (expected) fundamental price (for an example, see Gaunersdorfer and Hommes (2007)). In particular, consider a price predictor h of the form:

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = c_h + f_h(p_{t-1}, ..., p_{t-L}) \tag{9}$$

where $c_h \in \mathbb{R}$ is a constant and $f_h : \mathbb{R}^L \to \mathbb{R}$ is a deterministic function.

Note that since $x_t := p_t - \overline{p}$ and $\tilde{E}_{t,h}\left[x_{t+1}\right] := \tilde{E}_{t,h}\left[p_{t+1}\right] - E_t[p_{t+1}^*] = \tilde{E}_{t,h}\left[p_{t+1}\right] - \overline{p}$, the above predictor (9) can be written in terms of deviations from a fundamental price $\overline{p}$ as

$$\tilde{E}_{t,h}\left[x_{t+1}\right] = \tilde{c} + f_h(x_{t-1} + \overline{p}, ..., x_{t-L} + \overline{p}) \tag{10}$$

where $\tilde{c}_h := c_h - \overline{p}$.

Since we keep expectations in form $\tilde{E}_{t,h}\left[x_{t+1}\right]$ in Proposition 1 and Corollary 1, the 'if' conditions for slack and binding constraints are unchanged and the only difference is that when prices are "substituted out" in the expressions for the equilibrium price we must now substitute the right-hand-side of (10) rather than $f_h(x_{t-1}, ..., x_{t-L})$.

Alternatively, suppose that *all types* $h \in \mathcal{H}$ have price predictors of the form (9) (e.g. because the fundamental price is unknown to investors). Then there is no way to write demands in terms of deviations from the fundamental price, and we may instead work with

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+\overline{d}-(1+r)p_t}{a\sigma^2} = \frac{\hat{E}_{t,h}[p_{t+1}]+a\sigma^2\overline{Z}-(1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\hat{E}_{t,h}[p_{t+1}]+a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } p_t > \frac{\hat{E}_{t,h}[p_{t+1}]+a\sigma^2\overline{Z}}{1+r}. \end{cases} \tag{11}$$

where $\hat{E}_{t,h}\left[p_{t+1}\right] := \tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d} - a\sigma^2\overline{Z}$.

Note that the demands in (11) are of the same forms as those in the benchmark model, except that x_t is replaced by p_t and $\tilde{E}_{t,h}\left[x_{t+1}\right]$ is replaced by $\hat{E}_{t,h}\left[p_{t+1}\right]$. We can thus state the following simple result.

Remark 2 *In the above model, the equilibrium price p_t and demands $z_{t,h}$ follow Proposition 1, except that x_t is replaced by p_t and the beliefs $\tilde{E}_{t,h}\left[x_{t+1}\right]$ are replaced by $\hat{E}_{t,h}\left[p_{t+1}\right]$ in (11). As a result, the 'if' conditions (relating to belief dispersion) are as before with x_{t+1} replaced by p_{t+1}, whereas the expressions for the price p_t contain additional 'intercept terms' because $\hat{E}_{t,h}\left[p_{t+1}\right] = \tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d} - a\sigma^2\overline{Z} = \hat{c} + f_h(p_{t-1}, ..., p_{t-L})$, where $\hat{c} := \tilde{c} + \overline{d} - a\sigma^2\overline{Z}$.*

4 Alternative uptick rule

In this section we first derive expressions for investor demands under an alternative uptick rule (as studied in Section 5 of the paper) and then we provide a version of Proposition 1 that is adapted to the case of this rule. Under an alternative uptick rule, short-selling is banned in period t if and only if $p_{t-1} \leq (1-\kappa)p_{t-2}$ (or equivalently $x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p}$), where $\kappa \in [0, 1)$ represents the *threshold* reduction in prices in period $t-1$ (as a fraction of the past price) that will trigger the short-selling constraint to be enforced in period t.

4.1 Derivation of demands

The maximization problem solved by investor type $h \in \mathcal{H}$ is as follows:

$$\max_{z_{t,h}} \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2}v\tilde{a}r_{t,h}\left[w_{t+1,h}\right] \tag{12}$$

$$\text{s.t.} \quad z_{t,h} \geq 0 \text{ if } p_{t-1} \leq (1-\kappa)p_{t-2}, \quad z_{t,h} \in \mathbb{R} \text{ otherwise}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+r)(w_{t,h} - p_t z_{t,h})$ and

$$v\tilde{a}r_{t,h}\left[w_{t+1,h}\right] = var_t\left[w_{t+1,h}\right] = a\sigma^2 z_{t,h}$$

with $\sigma^2 := var_t[p_{t+1} + d_{t+1}]$ assumed constant.

Formulating the above problem as a Lagrangean we have

$$\max_{z_{t,h},\lambda_{t,h}} \mathcal{L}_t = \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2} v\tilde{a}r_{t,h}\left[w_{t+1,h}\right] + 1_t \lambda_{t,h} z_{t,h} \tag{13}$$

where 1_t is an indicator variable equal to 1 if $p_{t-1} \leq (1-\kappa)p_{t-2})$ and 0 otherwise and $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint $z_{t,h} \geq 0$.

The first-order conditions are

$$z_{t,h}: \quad \tilde{E}_{t,h}\left[p_{t+1} + d_{t+1}\right] - (1+r)p_t - a\sigma^2 z_{t,h} + 1_t \lambda_{t,h} = 0 \tag{14}$$

$$\lambda_{t,h}: \quad 1_t z_{t,h} \geq 0 \tag{15}$$

and the complementary slackness condition $1_t \lambda_{t,h} z_{t,h} = 0$.

Consider first Case 1: $p_{t-1} > (1-\kappa)p_{t-2}$ such that $1_t = 0$. Then by (13) we see that demand $z_{t,h}$ is unconstrained. It then follows from the first-order condition (14) that demand by investor type h is given by $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1} + d_{t+1}\right] - (1+r)p_t) := z^*_{t,h}$.[2] Now consider Case 2: $p_{t-1} \leq (1-\kappa)p_{t-2}$ such that $1_t = 1$. Then by (15), we have $z_{t,h} \geq 0$, which by (14) requires that either (i) $z_{t,h} = z^*_{t,h} \geq 0$ and $\lambda_{t,h} = 0$ (in which case the short-selling constraint is slack), or (ii) $z_{t,h} = 0$ and $\lambda_{t,h} = -(\tilde{E}_{t,h}\left[p_{t+1} + d_{t+1}\right] - (1+r)p_t) > 0$ by (14) and the complementary slackness condition (in which case $z^*_{t,h} < 0$ and the constraint binds).

The demand of type $h \in \mathcal{H}$ is thus given by

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d} - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d}}{1+r} \text{ or } p_{t-1} > (1-\kappa)p_{t-2} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d}}{1+r} \text{ and } p_{t-1} \leq (1-\kappa)p_{t-2} \end{cases} \tag{16}$$

where $\tilde{E}_{t,h}\left[d_{t+1}\right] = \overline{d}$ has been used.

Demand of type $h \in \mathcal{H}$ can therefore be written in terms of price deviations $x_t = p_t - \overline{p}$ as

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2 \overline{Z}}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \overline{Z}}{1+r} \text{ or } x_{t-1} > (1-\kappa)x_{t-2} - \kappa\overline{p} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \overline{Z}}{1+r} \text{ and } x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p} \end{cases} \tag{17}$$

as stated in Section 5 of the main text.

Note that an unconditional short-selling constraint $z_{t,h} \geq 0$ is the special case $1_t = 1$ $\forall t$ (see Case 2 above) such that the price condition is eliminated in equations (16)–(17).

[2] Note that all necessary conditions are satisfied since (15) collapses to $0 \geq 0$ and the complementary slackness condition becomes $0 = 0$. Both these conditions are satisfied regardless of the demand of type h.

4.2 Re-worked version of Proposition 1

Note that the only difference relative to the benchmark model with an unconditional short-selling constraint is that demands are unconstrained if the price condition is not met, i.e. if $x_{t-1} > (1-\kappa)x_{t-2} - \kappa\overline{p}$. We thus have the following adapted version of Proposition 1.

Proposition 2 (Proposition 1 adapted to alternative uptick rule) *Let $x_t = p_t - \overline{p}$ be the equilibrium price at date t and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\}) \leq a\sigma^2\overline{Z}$ or $x_{t-1} > (1-\kappa)x_{t-2} - \kappa\overline{p}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the equilibrium price is

$$x_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_h(x_{t-1}, ..., x_{t-L})}{1+r} := x_t^* \tag{18}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t) \in \mathbb{R}\ \forall h \in \mathcal{H}$.

(ii) If $x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p}$ and $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\}) > a\sigma^2\overline{Z}$, then at least one type is short-selling constrained and $\exists$ unique non-empty sets $\mathcal{B}_t^*, \mathcal{S}_t^*$ s.t. $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{B}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\})$, and the associated equilibrium price and demands are

$$x_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > x_t^* \tag{19}$$

and $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0\ \forall h \in \mathcal{B}_t^*,\ z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

Proof. It follows from the Proposition 1 Proof with the additional restriction that short positions are banned if and only if $x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p}$, such that $\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$, $x_t = x_t^*$ whenever $x_{t-1} > (1-\kappa)x_{t-2} - \kappa\overline{p}$. ■

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