

# Supplementary Appendix (For online publication only)

## “Solving heterogeneous-belief asset pricing models with short-selling constraints and many agents”

This appendix provides technical details of some generalizations and extensions of the benchmark model that are reported in the main text (Section 3.2 and Section 4.1.1).

### 1 Housing as the risky asset

In this section we show that if the risky asset is recast as housing as in the models of Uluc (2018), Bolt et al. (2019) or Hatcher (2021), then similar (or identical) analytical results to those in Proposition 1 are obtained.

#### 1.1 Models without tenure choice

Bolt et al. (2019) and Hatcher (2021) consider models in which housing is an investment asset that differs from shares due to the interpretation of ‘dividends’. In Bolt et al. (2019) dividends are replaced by imputed rent based on an arbitrage condition between the user and rental costs, whereas Hatcher (2021) assumes linear housing utility scaled by a housing preference variable which plays the same role as dividends in the baseline model.<sup>1</sup> In both models, these additional variables are assumed to be exogenous processes whose properties are known to the investors in the model. We assume a fixed supply of housing  $\bar{Z} > 0$ .

Assuming linear excess returns and short-selling constraints, demands for housing  $z_{t,h}$  are:<sup>2</sup>

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + Q_t(1+\hat{r}) - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + Q_t(1+\hat{r})}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + Q_t(1+\hat{r})}{1+r} \end{cases} \quad (\text{BDDHL1})$$

where  $Q_t$  is exogenous rental price and  $\hat{r}$  is the fixed risk-free mortgage rate; and

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \Theta_t \bar{U}' - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + \Theta_t \bar{U}'_z}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \Theta_t \bar{U}'_z}{1+r} \end{cases} \quad (\text{Hatcher1})$$

where  $\Theta_t > 0$  is relative preference for housing utility versus financial wealth (assumed exogenous), and  $\bar{U}' > 0$  is the fixed marginal utility of housing (which does not depend on  $z_{t,h}$ ).

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<sup>1</sup>Assuming quadratic utility in the framework of Hatcher (2021) does not pose problems if housing preference is fixed or deterministic. However, with stochastic preference shocks, there is no closed-form solution for the fundamental price, so an approximation would be needed to obtain analytical results.

<sup>2</sup>Note that Bolt et al. (2019) write the excess return on housing in non-linear form, in contrast to our simplifying assumption of a linear excess return here.

### 1.1.1 Derivation of the fundamental price

When short-selling constraints are slack, both the above demands are of the form:

$$z_{t,h} = \frac{\tilde{E}_{t,h}[p_{t+1}] + d_t - (1+r)p_t}{a\sigma^2} \quad (1)$$

where  $d_t$  has the interpretation of a ‘housing dividend’ paid at date  $t$ .

Using this expression, we can solve for the fundamental price of housing as follows:

$$p_t^* = \frac{\frac{r}{1+r} \sum_{i=0}^{\infty} \left(\frac{1}{1+r}\right)^i E_t d_{t+i} - a\sigma^2 \bar{Z}}{r} \quad (2)$$

where the expectations  $E_t d_{t+i}$  will depend on the assumptions we make about the evolution of the exogenous rental price  $Q_t$  and the exogenous housing preference  $\Theta_t$ .

We consider below two polar cases: IID disturbances (zero persistence) and permanent disturbances arising from a random walk. We will show that in both cases the demands (in terms of deviations from the fundamental price) have a common representation in which there is an additional ‘intercept term’ relative to the demands in the benchmark model.

- *Case of IID disturbances*

Suppose the process for  $d_t$  in (1) is of the form  $d_t = \bar{d} + \epsilon_t$ , where  $\bar{d} > 0$  and  $\epsilon_t$  is a zero-mean shock with constant variance. It follows that  $E_t d_{t+i} = \bar{d}$  if  $i \geq 1$  and  $E_t d_{t+i} = d_t$  otherwise (i.e. if  $i = 0$ ). Equation (2) therefore simplifies to:

$$p_t^* = \frac{d_t - \frac{1}{1+r}\epsilon_t - a\sigma^2 \bar{Z}}{r} \quad (3)$$

such that the fundamental price  $p_t^*$  differs relative to baseline model due to the shock  $\epsilon_t$ .

Note that (6) is the fundamental price when demands are given by (BDDHL1) if we let  $d_t = Q_t(1+\hat{r})$  with  $Q_t = \bar{Q} + v_t$ , where  $v_t$  is zero-mean and IID, such that  $d_t = \bar{d} + \epsilon_t$  provided  $\bar{d} := (1+\hat{r})\bar{Q}$  and  $\epsilon_t := (1+\hat{r})v_t$ . In this case, (6) gives  $d_t = rp_t^* + (1+r)^{-1}\epsilon_t + a\sigma^2 \bar{Z}$ , which implies that  $Q_t(1+\hat{r}) = rp_t^* + (1+r)^{-1}\epsilon_t + a\sigma^2 \bar{Z}$ .

Therefore, by (1) the demands in (BDDHL1) can be written in terms of deviations from the fundamental price,  $x_t := p_t - p_t^*$ , as

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \bar{Z} - (1+r)x_t}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \bar{Z}}{1+r} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \bar{Z}}{1+r} \end{cases} \quad (\text{BDDHL2})$$

where  $\tilde{E}_{t,h}[x_{t+1}] = \tilde{E}_{t,h}[p_{t+1}] - E_t[p_{t+1}^*]$ .

Note that the demands (BDDHL2) are *identical* to those in the benchmark model. Likewise, the demands in (Hatcher1), expressed in price deviations, are equivalent to (BDDHL2) by setting  $d_t = \Theta_t \bar{U}'$ , with  $\Theta_t = \bar{\Theta} + v_t$  and  $\epsilon_t := v_t \bar{U}'$ . We now consider permanent disturbances.

- *Case of permanent disturbances*

Suppose the process for  $d_t$  in (1) is a random walk without drift:  $d_t = d_{t-1} + \epsilon_t$ , where  $\bar{d} > 0$  and  $\epsilon_t$  is zero mean with constant variance. Then  $E_t d_{t+i} = d_t \forall i \geq 0$  and hence (2) becomes:

$$p_t^* = \frac{d_t - a\sigma^2 \bar{Z}}{r} \quad (4)$$

Note that (4) is the fundamental price given the demands (BDDHL1) if  $d_t = Q_t(1 + \hat{r})$  with  $Q_t = Q_{t-1} + v_t$ , where  $v_t$  is zero-mean and IID, such that  $d_t = d_{t-1} + \epsilon_t$  when  $\epsilon_t := (1 + \hat{r})v_t$ . Since (4) implies that  $d_t = rp_t^* + a\sigma^2 \bar{Z}$ , we have  $Q_t(1 + \hat{r}) = rp_t^* + a\sigma^2 \bar{Z}$  and the demands in (BDDHL1) can be written in terms of deviations from the fundamental price as:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \bar{Z} - (1+r)x_t}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \bar{Z}}{1+r} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2 \bar{Z}}{1+r}. \end{cases} \quad (\text{BDDHL3})$$

where  $x_t := p_t - p_t^*$ .

Note that the demands in (BDDHL3) are *identical* to those in the benchmark model. The same result applies to the demands in (Hatcher1), as this case simply requires a change in variables to  $d_t = \Theta_t \bar{U}'$ , with  $\Theta_t = \Theta_{t-1} + v_t$ , and  $\epsilon_t = v_t \bar{U}'$ .

### 1.1.2 Equivalence result

Analogous to the benchmark model, the housing market clears when  $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \bar{Z}$ . We thus have the following simple result based on the demands (BDDHL2), (BDDHL3).

**Remark 1** *In the above models, where demands are given by (BDDHL1) or (Hatcher1), the results in Proposition 1 and Corollary 1 remain intact, because the demands in terms of price deviations, (BDDHL2) and (BDDHL3), are identical to those in the benchmark model.*

## 1.2 A model with tenure choice

In Uluc (2018), housing is a risky investment asset, and there is also a housing tenure decision. There are three possible housing tenure choices: owner-occupiers who consume all services from housing investment; landlords who consume part of their housing investment position and rent out the rest; and tenants who rent consumed housing because their investment falls short of desired housing consumption. Maintenance costs differ for owner-occupiers, landlords and tenants; in the baseline model, housing investment is restricted to be non-negative by a short-selling constraint. Risk arises from differences in beliefs (which may be stochastic), such that the future return on housing investment is unknown at date  $t$ .

Since by definition all owner-occupiers are homeowners, short-selling constraints will be slack in an equilibrium in which all investors are owner-occupiers. We therefore restrict attention to equilibria (at date  $t$ ) with active rental markets, i.e. in which some investors choose to be

landlords and others choose to be tenants. We consider a version of the model with many behavioural investor types  $h \in \mathcal{H}$ , and we normalize supply of housing to 1 as in Uluc (2018). Demand for housing investment by type  $h \in \mathcal{H}$  is of the form:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + R_t - \delta_R - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + R_t - \delta_R}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + R_t - \delta_R}{1+r} \end{cases} \quad (5)$$

where  $p_t$  is the house price,  $R_t$  is the rental price, and  $\delta_R$  is the maintenance cost per unit of rented housing.

The fundamental price of housing given by

$$p_t^* = \bar{p} := \frac{a - b - \delta_O}{r} \quad (6)$$

where  $a, b > 0$  are parameters that enter due to quadratic utility from housing consumption, and  $\delta_O \in (0, \delta_R)$  is the maintenance of cost per unit of owner-occupied housing.

Note that the demands in (5) can be written in terms of deviations from the fundamental housing price,  $x_t := p_t - \bar{p}$ , as

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] + \hat{R}_t - (1+r)x_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + \hat{R}_t}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + \hat{R}_t}{1+r} \end{cases} \quad (7)$$

where  $\hat{R}_t := R_t + \delta_O - \delta_R - (a - b)$  and  $\tilde{E}_{t,h}[x_{t+1}] = \tilde{E}_{t,h}[p_{t+1}] - \bar{p}$ .

With a simple change in variables, the demands in (7) can be written in the same form in the benchmark model:

$$z_{t,h} = \begin{cases} \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\bar{Z} - (1+r)x_t}{a\sigma^2} & \text{if } p_t \leq \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\bar{Z}}{1+r} \\ 0 & \text{if } p_t > \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\bar{Z}}{1+r} \end{cases} \quad (8)$$

where  $\hat{E}_{t,h}[x_{t+1}] := \tilde{E}_{t,h}[x_{t+1}] + \hat{R}_t - a\sigma^2\bar{Z}$ .

The housing market clears when  $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \bar{Z} = 1$ , analogous to the benchmark model except that housing supply per investor is normalized to 1. We can thus state the following.

**Remark 2** *In the above model, the equilibrium price  $x_t$  and demands  $z_{t,h}$  follow Proposition 1, except that the beliefs  $\tilde{E}_{t,h}[x_{t+1}]$  are replaced by  $\hat{E}_{t,h}[x_{t+1}]$ . As a result, the ‘if’ conditions (relating to belief dispersion) are unchanged, whereas the expressions for the equilibrium price contain additional ‘intercept terms’ because  $\hat{E}_{t,h}[x_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \hat{R}_t - a\sigma^2\bar{Z}$ .*

Note that the equilibrium price depends on the rental price  $R_t$ , which is determined by rental market equilibrium  $\sum_{h \in \mathcal{H}} n_{t,h} c_{t,h} = 1$ , where  $c_{t,h}$  is the housing consumption demand of type  $h$  (which is a decreasing function of the rental price  $R_t$ ; see Uluc (2018), Eqs. (6) and (9)). Equilibrium prices  $x_t, R_t$  must also be consistent with optimal tenure choices by all types.

## 2 Endogenous participation in multiple asset markets

In this section we give a re-worked version of Proposition 1 for the case of multiple risky assets, where the extent of participation  $w_t^m \in (0, 1)$  in each market  $m \in \{1, 2, \dots, M\}$  is determined by its attractiveness relative to other markets, as in Westerhoff (2004).

We show in the main text (Section 4.1.1) that demand of type  $h$  in market  $m$  is given by

$$z_{t,h}^m = \begin{cases} w_t^m \left( \frac{\tilde{E}_{t,h}[x_{t+1}^m] + a\sigma_m^2 M \bar{Z}^m - (1+r)x_t^m}{a\sigma_m^2} \right) & \text{if } x_t^m \leq \frac{\tilde{E}_{t,h}[x_{t+1}^m] + a\sigma_m^2 M \bar{Z}^m}{1+r} \\ 0 & \text{if } x_t^m > \frac{\tilde{E}_{t,h}[x_{t+1}^m] + a\sigma_m^2 M \bar{Z}^m}{1+r} \end{cases} \quad (9)$$

where  $x_t^m := p_t^m - \bar{p}^m$  is the deviation from the fundamental price in market  $m$ .

As a result, the market-clearing condition in a given market  $m$  is:

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \bar{Z}^m / w_t^m \quad (10)$$

where  $\tilde{z}_{t,h}^m := (a\sigma_m^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}^m] + a\sigma_m^2 M \bar{Z}^m - (1+r)x_t^m)$  and  $\sum_{h \in \mathcal{H}} n_{t,h}^m = 1$ .

Notice that the only difference relative to the benchmark model (with one risky asset) is that the fixed supply of an  $\bar{Z}^m$  is scaled by the number of assets  $M$  in the adjusted demands  $\tilde{z}_{t,h}^m$  in (10) (cf. (9)) and by  $1/w_t^m$  on the right side of the market-clearing condition (10). We can thus state the following re-worked version of Proposition 1 that is adapted to this case.

**Proposition 1 (Adapted)** *Let  $x_t^m = p_t^m - \bar{p}^m$  be the equilibrium price at date  $t$  and  $\mathcal{B}_t^m \subseteq \mathcal{H}$  ( $\mathcal{S}_t^m := \mathcal{H} \setminus \mathcal{B}_t^m$ ) be the set of types that participate in market  $m$  (do not participate) at date  $t$ . Then the following holds:*

(i) *If  $\sum_{h \in \mathcal{H}} n_{t,h}^m (\tilde{E}_{t,h}[x_{t+1}^m] - \min_{h \in \mathcal{H}} \{\tilde{E}_{t,h}[x_{t+1}^m]\}) \leq a\sigma_m^2 \bar{Z}^m / w_t^m$ , then no type is short-selling constrained ( $\mathcal{B}_t^{m*} = \mathcal{H}$ ,  $\mathcal{S}_t^{m*} = \emptyset$ ) and the equilibrium price is*

$$x_t^m = \frac{\sum_{h \in \mathcal{H}} n_{t,h}^m f_h(x_{t-1}^m, \dots, x_{t-L}^m) + a\sigma_m^2 \bar{Z}^m \left( M - \frac{1}{w_t^m} \right)}{1+r} := x_t^{m*} \quad (11)$$

*with resulting demands  $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}^m] + a\sigma_m^2 \bar{Z}^m - (1+r)x_t^m) \geq 0 \forall h \in \mathcal{H}$ .*

(ii) *If  $\sum_{h \in \mathcal{H}} n_{t,h}^m (\tilde{E}_{t,h}[x_{t+1}^m] - \min_{h \in \mathcal{H}} \{\tilde{E}_{t,h}[x_{t+1}^m]\}) > a\sigma_m^2 \bar{Z}^m / w_t^m$ , at least one type is short-selling constrained and there exist unique non-empty sets  $\mathcal{B}_t^{m*} \subset \mathcal{H}$  and  $\mathcal{S}_t^{m*}$  such that*

$$\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (\tilde{E}_{t,h}[x_{t+1}^m] - \min_{h \in \mathcal{B}_t^{m*}} \{\tilde{E}_{t,h}[x_{t+1}^m]\}) \leq \frac{a\sigma_m^2 \bar{Z}^m}{w_t^m} < \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (\tilde{E}_{t,h}[x_{t+1}^m] - \max_{h \in \mathcal{S}_t^{m*}} \{\tilde{E}_{t,h}[x_{t+1}^m]\}),$$

$$x_t^m = \frac{\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m f_h(x_{t-1}^m, \dots, x_{t-L}^m) - \left( \frac{1}{w_t^m} - [\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m] M \right) a\sigma_m^2 \bar{Z}^m}{(1+r) \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m} > x_t^{m*} \quad (12)$$

*and  $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}^m] + a\sigma_m^2 \bar{Z}^m - (1+r)x_t^m) \geq 0 \forall h \in \mathcal{B}_t^{m*}$ ,  $z_{t,h}^m = 0 \forall h \in \mathcal{S}_t^{m*}$ .*

**Proof.** It follows from Proposition 1 Proof when variables are appended with  $m$  superscripts,  $a\sigma^2 \bar{Z}$  is replaced by  $a\sigma_m^2 M \bar{Z}^m$  in demands, and  $\bar{Z}$  is replaced by  $\bar{Z}^m / w_t^m$  in mkt-clearing. ■

## References

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