Supplementary Appendix for "Solving heterogeneous-belief asset pricing models with short selling constraints and many agents"

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This appendix provides (i) derivations of optimal demands with short-selling constraints, as in Sections 2.1 and 3.3 of the main text, and (ii) supporting material for the generalizations and extensions in Section 3.3, Section 5 (start) and Sections 5.1–5.3 of the main text.

1 Derivations

In this section we derive the optimal demand schedules for the cases of unconditional short-selling constraints (benchmark model) and conditional short-selling constraints (Section 3.3).

1.1 Derivation of demands in the benchmark model

Each type $h \in \mathcal{H}$ solves the following problem:[1]

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad z_{t,h} \geq 0 \tag{1}$$

where $w_{t+1,h} = (p_{t+1}+d_{t+1})z_{t,h} + (1+\tilde{r})(w_{t,h} - p_t z_{t,h})$ is future wealth, $w_{t,h} - p_t z_{t,h}$ is holdings of the risk-free asset, $a, \tilde{r} > 0$ are parameters and $\tilde{V}_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$, with $\sigma^2 > 0$.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2}\tilde{V}_{t,h}\left[w_{t+1,h}\right] + \lambda_{t,h} z_{t,h} \tag{2}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint, $z_{t,h} \geq 0$.

The first-order conditions are

$$z_{t,h}: \quad \tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h} = 0 \tag{3}$$

$$\lambda_{t,h}: \quad z_{t,h} \geq 0 \tag{4}$$

and the complementary slackness condition is:

$$\lambda_{t,h} z_{t,h} = 0. \tag{5}$$

By (3), $\lambda_{t,h} = -(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h})$. Note that $\lambda_{t,h} = 0$ if and only if $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t)$. This $z_{t,h}$ satisfies (5) (given $\lambda_{t,h} = 0$) and therefore it is an optimal demand iff $\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] \geq (1+\tilde{r})p_t$; see (4).

[1] We assume (as is standard) that these operators satisfy some basic properties of conditional expectation operators, namely, $\tilde{E}_{t,h}[y_t] = y_t$ and $\tilde{V}_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $\tilde{V}_{t,h}[x_t y_{t+1}] = x_t^2 \tilde{V}_{t,h}[y_{t+1}]$.

If $\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]<(1+\tilde{r})p_t$. we can reject $\lambda_{t,h}=0$, since the condition in (4) is not satisfied. It follows that $\lambda_{t,h}>0$ when $\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]<(1+\tilde{r})p_t$, and by the complementary slackness condition (5) it follows that $z_{t,h}=0$ (which satisfies (4)).

Therefore, the demand schedule of type $h\in\mathcal{H}$ is given by

$$z_{t,h}=\begin{cases}\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t\leq\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]}{1+\tilde{r}}\\ 0 & \text{if } p_t>\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]}{1+\tilde{r}}\end{cases} \tag{6}$$

as stated in Equation (2) of the main text.

1.2 Derivation of demands for conditional short-selling constraint

If $g(p_{t-1},...,p_{t-K})>0$, the short-selling constraint is absent at date t; if $g(p_{t-1},...,p_{t-K})\leq 0$ the short-selling constraint is present. We introduce an indicator variable $\mathbb{1}_t:=\mathbb{1}_{\{g(p_{t-1},...,p_{t-K})\leq 0\}}$ equal to 1 if the short-selling constraint is present at date t (i.e. if $g(p_{t-1},...,p_{t-K})\leq 0$), and equal to 0 otherwise. The problem of type $h\in\mathcal{H}$ at date t is thus amended from (1) to:

$$\max_{z_{t,h}}\ \tilde{E}_{t,h}[w_{t+1,h}]-\frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}]\qquad\text{s.t.}\qquad \mathbb{1}_t z_{t,h}\geq 0 \tag{7}$$

where $w_{t+1,h}=(p_{t+1}+d_{t+1})z_{t,h}+(1+\tilde{r})(w_{t,h}-p_t z_{t,h})$ as above.

Note that if $\mathbb{1}_t=0$, the portfolio choice of all types $h\in\mathcal{H}$ is unconstrained at date t, since $\mathbb{1}_t z_{t,h}=0\geq 0$ is satisfied for any $z_{t,h}\in\mathbb{R}$. On the other hand, if $\mathbb{1}_t=1$ then all types face the same maximization problem as in (1), i.e. short-selling is banned at date t.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}}\ \mathcal{L}_{t,h}=\tilde{E}_{t,h}\left[w_{t+1,h}\right]-\frac{a}{2}\tilde{V}_{t,h}\left[w_{t+1,h}\right]+\lambda_{t,h}\mathbb{1}_t z_{t,h} \tag{8}$$

where $\lambda_{t,h}\geq 0$ is the Lagrange multiplier on the constraint $\mathbb{1}_t z_{t,h}\geq 0$.

The first-order conditions are

$$z_{t,h}:\ \ \tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t-a\sigma^2 z_{t,h}+\lambda_{t,h}\mathbb{1}_t=0 \tag{9}$$

$$\lambda_{t,h}:\ \ \mathbb{1}_t z_{t,h}\geq 0 \tag{10}$$

and the complementary slackness condition is:

$$\lambda_{t,h}\mathbb{1}_t z_{t,h}=0. \tag{11}$$

If $\mathbb{1}_t=0$, then (10)–(11) are satisfied and $z_{t,h}=(a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t)\in\mathbb{R}$ by (9). Hence, demands are unconstrained if $\mathbb{1}_t=0$. If $\mathbb{1}_t=1$, the first-order conditions (9)–(11) are identical to (3)–(5), so the cases are the same as discussed below Eq. (5).

Therefore, demands are $z_{t,h}=(a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t)$ if $\mathbb{1}_t=0$ or $p_t\leq\frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}$; and $z_{t,h}=0$ otherwise (i.e. if $\mathbb{1}_t=1$ and $p_t>\frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}$).

2 Minor generalizations and nested cases

In this section we discuss some minor generalizations and nested cases briefly mentioned at the start of Section 5 in the main paper: individual investors or types who may be connected in a social network; housing as a physical investment asset subject to a short-selling ban; and short-selling constraints that permit negative positions up to some limit.

2.1 Individuals in a social network

First, note that if the types $h_1, h_2, ..., h_H$ are *individual investors* we may fix the population shares at $n_{t,h} = 1/H$. This interpretation is relevant for asset pricing models with many agents that differ in beliefs; for example, agent-based models as in LeBaron et al. (1999) or the *social network* model with individual-specific types in Hatcher and Hellmann (2022). Alternatively, some papers consider type adoption as in the Brock and Hommes (1998) model, but with local social networks. In the model of Panchenko et al. (2013), for example, type updating follows Brock and Hommes (1998) except that only the types (and performance) of investors in an agent's *social network* can be observed and adopted; see Panchenko et al. (2013, Eq. 10). Both the above cases are nested by the benchmark results because the demand schedules in these papers have the same functional form as in Equations (2) and (4) in the main paper, and beliefs and population shares satisfy Assumptions 1–2 (main paper).

2.2 Housing as the risky asset

In this section we demonstrate how our results can be applied when the risky asset is housing as in Bolt et al. (2019) and Hatcher (2021). In these models, housing is an investment asset that differs from shares due to the interpretation of 'dividends'. In Bolt et al. (2019) dividends are replaced by imputed rent based on an arbitrage condition between the user and rental costs, whereas Hatcher (2021) assumes linear housing utility scaled by a housing preference variable.[2] In both models, these additional variables are exogenous processes whose properties are known to the investors. We assume a fixed supply of housing $\overline{Z} > 0$.

With linear excess returns and an unconditional short-selling constraint, demands are:

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1+\hat{R}) - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1+\hat{R})}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1+\hat{R})}{1+\tilde{r}} \end{cases} \tag{BDDHL1}$$

where Q_t is the exogenous rental price and $\hat{R}$ is the fixed risk-free mortgage rate; and

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z}{1+\tilde{r}} \end{cases} \tag{Hatcher1}$$

[2] Using quadratic utility from housing in the framework of Hatcher (2021) is also possible (under certain conditions) as this mirrors the mean-variance assumption in the benchmark model.

where $\Theta_t > 0$ is an exogenous relative preference for housing utility versus financial wealth, and $\overline{U}_z > 0$ is a fixed marginal utility of housing (which does not depend on $z_{t,h}$).

Defining $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2\overline{Z}$ and $r := \tilde{r} - \overline{c}$, with $\tilde{E}_{t,h}[d_{t+1}] = Q_t(1+\hat{R})$ (or $\tilde{E}_{t,h}[d_{t+1}] = \Theta_t\overline{U}_z$), the above demands can be written as

$$z_{t,h} = \begin{cases} \frac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } p_t > \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \end{cases} \tag{12}$$

as in Equation (4) of the main text.

2.3 Short-selling constraints of the form: $z_{t,h} \geq L$

Suppose that negative positions are permitted up to some limit $L \leq 0$, such that $z_{t,h} \geq L$, $\forall t, h$. Formulating the maximization problem of type h as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] + \lambda_{t,h}(z_{t,h} - L) \tag{13}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint $z_{t,h} \geq L$.

The first-order conditions are

$$z_{t,h} : \tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h} = 0, \quad \lambda_{t,h} : z_{t,h} \geq L \tag{14}$$

and the complementary slackness condition is: $\lambda_{t,h}(z_{t,h} - L) = 0$.

Analogous to the discussion after (3)–(5), $\lambda_{t,h} = 0$ and $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t)$ is an optimal demand iff $\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t \geq a\sigma^2 L$, while if $\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t < a\sigma^2 L$, then $\lambda_{t,h} > 0$ and $z_{t,h} = L$.

Hence, the demand schedule of type $h \in \mathcal{H}$ is given by

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2 L}{1+\tilde{r}} \\ L & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2 L}{1+\tilde{r}}. \end{cases} \tag{15}$$

Defining $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2\overline{Z}$, $r := \tilde{r} - \overline{c}$ and $\tilde{z}_{t,h} := z_{t,h} - L$, the demands (15) are

$$\tilde{z}_{t,h} = \begin{cases} \frac{f_{t,h} + a\sigma^2\tilde{Z} - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h} + a\sigma^2\tilde{Z}}{1+r} \\ 0 & \text{if } p_t > \frac{f_{t,h} + a\sigma^2\tilde{Z}}{1+r} \end{cases}, \quad \text{where } \tilde{Z} := \overline{Z} - L. \tag{16}$$

Note that the demands in (31) have the same form as in Eq. (4) in the main paper, except that $z_{t,h}$ is replaced by $\tilde{z}_{t,h}$ and $\overline{Z}$ is replaced by $\tilde{Z}$. Similarly, market-clearing is:

$$\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z} \quad \Longrightarrow \quad \sum_{h\in\mathcal{H}} n_{t,h}\tilde{z}_{t,h} = \tilde{Z} \tag{17}$$

where $\sum_{h\in\mathcal{H}} n_{t,h} = 1$ is used. We can therefore state the following result.

Remark 1 *In the above model, demands and market-clearing are as in the benchmark model, except that $z_{t,h}$ is replaced by $\tilde{z}_{t,h}$ and $\overline{Z}$ is replaced by $\tilde{Z}$. Therefore, the market-clearing price and demands follow Proposition 1 with $\tilde{z}_{t,h}$ replacing $z_{t,h}$ and $\tilde{Z}$ replacing $\overline{Z}$.*

3 Supporting results for Remarks in the main paper

This section gives propositions supporting the Remarks in Section 3.3 and the 'Extensions' section #5 of the main paper. Any non-trivial proofs are in Section 6 of this appendix.

3.1 Conditional short-selling constraints

Recall from Section 1.2 that we introduced an indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K})\leq 0\}}$ that is equal to 1 if the short-selling constraint is in place at date t (i.e. if $g(p_{t-1},...,p_{t-K}) \leq 0$) and is equal to 0 otherwise (i.e. if $g(p_{t-1},...,p_{t-K}) > 0$).

The demands for types $h \in \mathcal{H}$ are given by:

$$z_{t,h} = \begin{cases} \frac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ and } \mathbb{1}_t = 1 \end{cases} \tag{18}$$

where $r := \tilde{r} - \overline{c}$ and $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$.

Proposition 1 (Proposition 1 (main) adapted to conditional constraint) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ or $\mathbb{1}_t = 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{19}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t)$ $\forall h \in \mathcal{H}$ with $z_{t,h} \in \mathbb{R}$ if $\mathbb{1}_t = 0$, and $z_{t,h} \geq 0$ otherwise (i.e. if $\mathbb{1}_t = 1$ and $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$).

(ii) If $\mathbb{1}_t = 1$ and $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$, then at least one type is short-selling constrained and there exist unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}$ and $\mathcal{S}_t^*$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\})$, and the market-clearing price and demands are

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{20}$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0$ $\forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0$ $\forall h \in \mathcal{S}_t^*$.

Proof. See Section 6.1 of this appendix. ■

3.2 Multiple markets and endogenous participation

In this section we adapt Proposition 1 for the case of multiple risky assets $m \in \{1, ..., M\}$ when participation shares $w_t^m \in (0,1)$ in each market are determined by attractiveness relative to other markets; see Westerhoff (2004) and Section 5.1 of the main paper.

We note in the main text that demand of type h in market m is given by

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1+r^m) p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m} \\ 0 & \text{if } p_t^m > \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m} \end{cases} \tag{21}$$

where $f_{t,h}^m := \tilde{f}_{t,h}^m + \overline{d}^m - a\sigma_m^2 \overline{Z}_m / w_t^m$ and $r^m := \tilde{r} - \overline{c}^m$.

Market-clearing in market m is given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m / w_t^m, \qquad \text{where } \tilde{z}_{t,h}^m := z_{t,h}^m / w_t^m. \tag{22}$$

With this change in variables, the market-clearing condition has the same form as in the benchmark model (aside from a scaling of supply by $1/w_t^m$). We therefore have the following.

Proposition 2 (Proposition 1 adapted to multiple markets) *Let p_t^m be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h}^m = \hat{n}_h(\mathbf{n}_{t-1}^m, \mathbf{u}_{t-1}^m)$ be the population share of type h in market m at date t, and let $\mathcal{B}_t^m \subseteq \mathcal{H}$ ($\mathcal{S}_t^m := \mathcal{H} \setminus \mathcal{B}_t^m$) be the set of unconstrained types (short-selling constrained types) in market m at date t. Then the following holds:*

(i) If $\sum_{h \in \mathcal{H}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{H}}\{f_{t,h}^m\}) \leq a\sigma_m^2 \overline{Z}_m / w_t^m$, then no type is short-selling constrained ($\mathcal{B}_t^{m*} = \mathcal{H}$, $S_t^{m*} = \emptyset$) and the market-clearing price in market m is

$$p_t^m = \frac{\sum_{h \in \mathcal{H}} n_{t,h}^m f_{t,h}^m}{1 + r^m} := p_t^{m*} \tag{23}$$

with demands $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1} (f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1 + r^m) p_t^m) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h \in \mathcal{H}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{H}}\{f_{t,h}^m\}) > a\sigma_m^2 \overline{Z}_m / w_t^m$, then at least one type is short-selling constrained at date t and there exist unique non-empty sets $\mathcal{B}_t^{m*} \subset \mathcal{H}$ and $\mathcal{S}_t^{m*}$ such that $\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{B}_t^{m*}}\{f_{t,h}^m\}) \leq a\sigma_m^2 \overline{Z}_m / w_t^m < \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (f_{t,h}^m - \max_{h \in \mathcal{S}_t^{m*}}\{f_{t,h}^m\})$, and the market-clearing price and demands are

$$p_t^m = \frac{\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m f_{t,h}^m - (1 - \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m) a\sigma_m^2 \overline{Z}_m / w_t^m}{(1 + r^m) \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m} > p_t^{m*} \tag{24}$$

and $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1} (f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1 + r^m) p_t^m) \geq 0 \ \forall h \in \mathcal{B}_t^{m*}$, $z_{t,h}^m = 0 \ \forall h \in \mathcal{S}_t^{m*}$.

Proof. It follows from the Proposition 1 Proof (main paper) when p_t, $f_{t,h}$, r and $\overline{Z}$ are replaced by p_t^m, $f_{t,h}^m$, r^m and $\overline{Z}_m / w_t^m$, and the demands $z_{t,h}$ are replaced by $z_{t,h}^m$ in (21). ■

3.3 Market maker: benchmark demand specification

As discussed in Section 5.3 of the main text, the demands in the case are the same as in Equations (2) and (4) of the paper, but the price is now given by

$$p_t = p_{t-1} + \mu[\lambda(Z_t - \overline{Z}) + (1-\lambda)(Z_{t-1} - \overline{Z})] \tag{25}$$

where $\mu > 0$, $\lambda \in (0,1]$ and $Z_t := \sum_{h\in\mathcal{H}} n_{t,h} z_{t,h}$ is aggregate demand per investor at date t, such that $Z_t - \overline{Z}$ can be interpreted as (average) excess demand per investor.

We now present amended versions of Proposition 1, Corollary 1 and the Computational Algorithm for the benchmark demand specification plus market maker price setting.

Proposition 3 *Let p_t be the price given by* (25) *at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained (short-selling constrained) types at date t. Then the following holds:*

1. If $p_{t-1} - \frac{1}{1+r}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$, $z_{t,h} \geq 0\ \forall h$) and price is given by

$$p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}}.$$

2. If $p_{t-1} - \frac{1}{1+r}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then one or more types are short-selling constrained with $z_{t,h} = 0$ and we have the following:

(i) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1} < \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, price is

$$p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

(ii) Else, $\exists \mathcal{B}_t^* = \emptyset, \mathcal{S}_t^* = \mathcal{H}$ such that $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h$), and price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See Section 6.2 of this Appendix. ■

Note that there are *three* distinct cases in Proposition 3, in contrast to Proposition 1 in the main paper, since *all types* may be short-selling constrained at the price set by the market maker. Corollary 1 and the Computational Algorithm are amended as follows.

Corollary 1 (amended) *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set such that types are ordered as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. Let $disp_{t,k} := \frac{\mu\lambda}{a\sigma^2}\sum_{h>k} n_{t,h}(f_{t,h} - f_{t,k}) - \frac{1}{1+r} f_{t,k}$ for $k \in \{1, ..., \tilde{H}_t - 1\}$, and $\tilde{g}_t := (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$. Then the price solution is:*

$$p_t = \begin{cases} \dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h=1}^{\tilde{H}_t} n_{t,h}f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}} := p_t^* & \text{if } disp_{t,1} \le \tilde{g}_t \\ \dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h=2}^{\tilde{H}_t} n_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h=2}^{\tilde{H}_t} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h=2}^{\tilde{H}_t} n_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \le \tilde{g}_t < disp_{t,1} \\ \dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h=3}^{\tilde{H}_t} n_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h=3}^{\tilde{H}_t} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h=3}^{\tilde{H}_t} n_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \le \tilde{g}_t < disp_{t,2} \\ \vdots & \vdots \\ \dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2} n_{t,\tilde{H}_t} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda n_{t,\tilde{H}_t})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1} n_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } disp_{t,\tilde{H}_t} \le \tilde{g}_t < disp_{t,\tilde{H}_t - 1} \\ p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}] := p_t^{(\tilde{H}_t)} & \text{if } disp_{t,\tilde{H}_t} > \tilde{g}_t \end{cases}$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \le k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } 2 \le k < k^*. \tag{26}$$

Proof. See Section 6.3 of this appendix. ■

Given Corollary 1, the computational algorithm needs to be amended as shown below. Note the algorithm uses our definition of $\tilde{g}_t := (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$ above.

3.3.1 Computational algorithm (Market maker and benchmark demands)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \frac{\mu\lambda}{a\sigma^2}\sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1}) - \frac{1}{1+r}f_{t,1}$. If $disp_{t,1} \le \tilde{g}_t$, accept $p_t = p_t^*$ as the price, compute demands and move to period $t+1$. Otherwise, move to Step 3.

3. Compute $disp_{t,\tilde{H}_t} = -\frac{1}{1+r}f_{t,\tilde{H}_t}$. If $disp_{t,\tilde{H}_t} > \tilde{g}_t$, accept $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ as the price, set all demands at zero and move to period $t+1$. Else, move to Step 4.

4. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k} + a\sigma^2\overline{Z} - (1+r)p_t^{guess}}{a\sigma^2} < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \le \tilde{g}_t < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a $k^* \in \{1, ..., \tilde{H}_t - 1\}$ is found such that $disp_{t,k^*+1} \le \tilde{g}_t < disp_{t,k^*}$ and go to step 5.

5. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}}$, compute demands at this price, and move to period $t+1$.

3.4 Market maker: alternative demand specification

Similar to Westerhoff (2004) we consider demands of the form $\tilde{a}_h(f_{t,h} - p_t)$, where $\tilde{a}_h > 0$ for all h. With a short-selling constraint $z_{t,h} \geq 0$, the demands are adjusted to

$$z_{t,h} = \begin{cases} \tilde{a}_h(\tilde{E}_{t,h}\left[p_{t+1}\right] - p_t) & \text{if } p_t \leq \tilde{E}_{t,h}\left[p_{t+1}\right] \\ 0 & \text{if } p_t > \tilde{E}_{t,h}\left[p_{t+1}\right]. \end{cases} \tag{27}$$

Given price beliefs in Assumption 1, we have $\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{c}p_t + \tilde{f}_{t,h}$, where we now assume $\overline{c} \in [0, 1)$ (since the interest rate is zero). We define $f_{t,h} := \tilde{f}_{t,h}$ and write the demands as

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} - (1-\overline{c})p_t) & \text{if } p_t \leq \frac{f_{t,h}}{1-\overline{c}} \\ 0 & \text{if } p_t > \frac{f_{t,h}}{1-\overline{c}}. \end{cases} \tag{28}$$

Since $\overline{c} \in [0, 1)$, the demands are decreasing in the current price. The price equation is $p_t = p_{t-1} + \mu[\lambda Z_t + (1-\lambda)Z_{t-1} - \overline{Z}]$, where $Z_t := \sum_{h\in\mathcal{H}} n_{t,h}z_{t,h}$ is aggregate demand.

Proposition 4 (Proposition 1 (main) adapted to new demand specification) *Let p_t be the price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Further, let $\tilde{n}_{t,h} := n_{t,h}\tilde{a}_h$ and $f_{t,h} := \tilde{f}_{t,h}$. Then the following holds:*

1. If $p_{t-1} - \frac{1}{1-\overline{c}}\min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + \mu\lambda\sum\limits_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq \mu\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}^*_t = \mathcal{H}$, $\mathcal{S}^*_t = \emptyset$, $z_{t,h} \geq 0\ \forall h \in \mathcal{H}$) and the price is

$$p_t = \frac{p_{t-1} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\overline{c})\sum\limits_{h\in\mathcal{H}}\tilde{n}_{t,h}}.$$

2. If $p_{t-1} - \frac{1}{1-\overline{c}}\min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + \mu\lambda\sum\limits_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > \mu\overline{Z}$, then one or more types $h \in \mathcal{H}$ are short-selling constrained with $z_{t,h} = 0$ and we have the following:

(i) If $\exists\ \mathcal{B}^*_t, \mathcal{S}^*_t \subset \mathcal{H}$ such that $\mu\lambda\sum\limits_{h\in\mathcal{B}^*_t}\tilde{n}_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{B}^*_t}\{f_{t,h}\}) - \frac{1}{1-\overline{c}}\min\limits_{h\in\mathcal{B}^*_t}\{f_{t,h}\} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < \mu\lambda\sum\limits_{h\in\mathcal{B}^*_t}\tilde{n}_{t,h}(f_{t,h} - \max\limits_{h\in\mathcal{S}^*_t}\{f_{t,h}\}) - \frac{1}{1-\overline{c}}\max\limits_{h\in\mathcal{S}^*_t}\{f_{t,h}\}$, price is given by

$$p_t = \frac{p_{t-1} + \mu\lambda\sum_{h\in\mathcal{B}^*_t}\tilde{n}_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\overline{c})\sum\limits_{h\in\mathcal{B}^*_t}\tilde{n}_{t,h}}$$

and demands are $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\overline{c})p_t) \geq 0\ \forall h \in \mathcal{B}^*_t$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}^*_t$.

(ii) Else, $\exists\mathcal{B}^*_t = \emptyset, \mathcal{S}^*_t = \mathcal{H}$ such that $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1-\overline{c}}\max_{h\in\mathcal{S}^*_t}\{f_{t,h}\} > \mu\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h \in \mathcal{H}$), and price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See Section 6.4 of this appendix. ∎

Corollary 2 (amended) *Let* $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ *be the set such that types are ordered as* $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. *Let* $disp_{t,k} := \mu\lambda \sum_{h=k+1}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - f_{t,k}) - \frac{f_{t,k}}{1-\bar{c}}$ *and* $g(p_{t-1}, Z_{t-1}) := \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, *for* $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. *The price solution is:*

$$p_t = \begin{cases} \dfrac{p_{t-1} + \mu\lambda \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq g(p_{t-1}, Z_{t-1}) \\ \dfrac{p_{t-1} + \mu\lambda \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,1} \\ \dfrac{p_{t-1} + \mu\lambda \sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,2} \\ \vdots & \vdots \\ \dfrac{p_{t-1} + \mu\lambda \tilde{n}_{t,\tilde{H}_t} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } disp_{t,\tilde{H}_t} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,\tilde{H}_t - 1} \\ p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}] := p_t^{(\tilde{H}_t)} & \text{if } disp_{t,\tilde{H}_t} > g(p_{t-1}, Z_{t-1}) \end{cases}$$

where $p_t^{(k^*)}$ *is the price if types* $1, ..., k^*$ *are short-selling constrained,* p_t^* *is the corresponding price if short-selling constraints were absent (which satisfies* $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$*), and*

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } 2 \leq k < k^*. \tag{29}$$

Proof. See Section 6.5 of this appendix. ■

Given Corollary 2, the computational algorithm needs to be amended as shown below.

3.4.1 Computational algorithm (Market maker and heterogeneous slopes)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \mu\lambda \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - f_{t,1}) - \frac{f_{t,1}}{1-\bar{c}}$. If $disp_{t,1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, accept $p_t = p_t^*$, compute demands and move to period $t+1$. Otherwise, go to Step 3.

3. Compute $disp_{t,\tilde{H}_t} = -\frac{f_{t,\tilde{H}_t}}{1-\bar{c}}$. If $disp_{t,\tilde{H}_t} > \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, accept $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ as the price, set all demands at zero, move to period $t+1$. Else, move to Step 4.

4. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \tilde{a}_k(f_{t,k} - (1-\bar{c})p_t^{guess}) < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a $k^* \in \{1, ..., \tilde{H}_t - 1\}$ is found such that $disp_{t,k^*+1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < disp_{t,k^*}$ and go to step 5.

5. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{p_{t-1} + \mu\lambda \sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, compute demands, move to $t+1$.

4 Additional heterogeneity

4.1 Heterogeneous responses to p_t

In the case of heterogeneous forecast coefficients on p_t, price beliefs are amended to

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \bar{c}_h p_t + \tilde{f}_{t,h} \tag{30}$$

where $\bar{c}_h \in [0, 1+\tilde{r})$ for all $h \in \mathcal{H}$.

As discussed below, this case requires an amendment to the computational algorithm because ranking types in terms of $f_{t,h}$ is no longer sufficient. We first provide an amended version of Proposition 1 before discussing the necessary changes to the algorithm.

Defining $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r_h := \tilde{r} - \bar{c}_h$, the demands are amended to:

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r_h)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}. \end{cases} \tag{31}$$

Using the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$, we have the following.

Proposition 5 (Heterogeneous p_t coefficients) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types). Let $f_{t,h}$, r_h be defined as above and $\tilde{n}_{t,h} := n_{t,h}(1+r_h)$. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}\sum_{h\in\mathcal{H}} \tilde{n}_{t,h} \leq 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}} \tag{32}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}\sum_{h\in\mathcal{H}} \tilde{n}_{t,h} > 0$, at least one type is short-selling constrained and $\exists$ unique $\mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \min_{h\in\mathcal{B}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \leq (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \max_{h\in\mathcal{S}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$, and the price is

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}} \tag{33}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. See Section 6.6 of this appendix. ■

Note that types with the lowest values of $(f_{t,h} + a\sigma^2\overline{Z})/(1+r_h)$ should be considered least optimistic, as they are more likely to be short-selling constrained at any given price. We thus define $\hat{f}_{t,h} := (f_{t,h} + a\sigma^2\overline{Z})/(1+r_h)$, which allows us to state the following result.

Corollary 3 *Let* $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ *be the set of types such that* $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, *where* $\hat{f}_{t,h} := \frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$. *Let* $disp_{t,k} := \sum_{h=k}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left(\frac{f_{t,k}+a\sigma^2\overline{Z}}{1+r_k}\right) \sum_{h=k}^{\tilde{H}_t} \tilde{n}_{t,h}$ *and* $\tilde{disp}_{t,k} := disp_{t,k} + n_{t,k} a\sigma^2\overline{Z}$ *for* $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := n_{t,h}(1 + r_h)$. *The market-clearing price is:*

$$p_t = \begin{cases} \frac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq (1 - \sum_{h=1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} \ (= 0) \\ \frac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2\overline{Z}}{\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq (1 - \sum_{h=2}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < \tilde{disp}_{t,1} \\ \frac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} - (n_{t,1} + n_{t,2}) a\sigma^2\overline{Z}}{\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq (1 - \sum_{h=3}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < \tilde{disp}_{t,2} \\ \vdots & \vdots \\ \frac{n_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t-1} n_{t,h}) a\sigma^2\overline{Z}}{\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t-1)} & \text{if } \tilde{disp}_{t,\tilde{H}_t-1} > (1 - \sum_{h=\tilde{H}_t-1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} \end{cases}$$

where $p_t^{(k^*)}$ *is the price if types* $1, ..., k^*$ *are short-selling constrained,* p_t^* *is the corresponding price if short-selling constraints were absent (which satisfies* $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$*), and*

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } 2 \leq k < k^*. \tag{34}$$

Proof. It follows from Proposition 5 and the proof of Corollary 1 (see main paper). ■

In light of the changes in Corollary 3, our algorithm needs to be amended as shown below. We stick with the above notation for which $\tilde{n}_{t,h} = n_{t,h}(1 + r_h)$ for all $h \in \{1, ..., \tilde{H}_t\}$.

4.1.1 Computational algorithm (Heterogeneous coefficients on p_t)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} = \frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left(\frac{f_{t,1}+a\sigma^2\overline{Z}}{1+r_1}\right) \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}$. If $disp_{t,1} \leq 0$, accept $p_t = p_t^*$ as the date t price, compute demands, move to period $t + 1$. Otherwise, go to Step 3.

3. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k}+a\sigma^2\overline{Z}-(1+r_k)p_t^{guess}}{a\sigma^2} < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq (1 - \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < \tilde{disp}_{t,k}$; if not, try $k = k_{prev}+1$ until a k^* is found such that $disp_{t,k^*+1} \leq (1 - \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < \tilde{disp}_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right] a\sigma^2\overline{Z}}{\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, compute demands, move to period $t + 1$.

4.1.2 Numerical example

We now turn to a numerical example. Consider $H = 3{,}000$ belief types with heterogeneity in the weights $\overline{c}_h$ on the current price (see Section 5.2.1 main paper and 4.1 above) and fixed population shares $n_{t,h} = 1/H$ for all t, h. There are three groups of investors consisting of 1,000 types each; within each group individuals use the same forecasting method, but their individual forecasts differ due to different weights $\overline{c}_h$ and idiosyncratic shocks in some periods (see below). Trend-followers expect the future change in price to be linked to past changes in price; contrarians believe the recent trend in prices will be reversed; and arbitrageurs base their expectations on the deviation of the current price from a fundamental price $\overline{p}$.

The first 1,000 types are trend-followers, types 1,001–2,000 are contrarians, and types 2,001–3,000 are arbitrageurs. All investors use the *current* price as a reference point, but we allow heterogeneity in the weights $\overline{c}_h$. In addition, each type has an idiosyncratic random component to beliefs $u_{t,h}$. Beliefs of trend-followers have the form $\tilde{E}_{t,h}[p_{t+1}] = \overline{c}_h p_t + g_h^1 \Delta p_{t-1} + g_h^2 \Delta p_{t-2} + u_{t,h}$, where $g_h^1, g_h^2 > 0$ and $\Delta p_t = p_t - p_{t-1}$. Contrarians have beliefs $\tilde{E}_{t,h}[p_{t+1}] = \overline{c}_h p_t + g_h^3 \Delta p_{t-1} + g_h^4 \Delta p_{t-2} + u_{t,h}$, where $g_h^3, g_h^4 < 0$. For arbitrageurs, $\tilde{E}_{t,h}[p_{t+1}] = \overline{c}_h p_t - g_h^5 (p_{t-1} - \overline{p}) + u_{t,h}$, where $g_h^5 > 0$. We set $d_t = \overline{d} = 1.1$, $\tilde{E}_{t,h}[d_{t+1}] = \overline{d}$ for all h, $\tilde{r} = 0.1$, $a = \sigma^2 = 1$ and $\overline{Z} = 0.1$. The fundamental price is therefore $\overline{p} = \frac{\overline{d} - a\sigma^2 \overline{Z}}{\tilde{r}} = 10$.

Prior to period 1, the parameters g_h^1 and g_h^2 are drawn from uniform distributions on (0,0.5) and (0,0.2), g_h^3, g_h^4 are drawn from a uniform distribution on (-0.1,0), and g_h^5 is drawn from a uniform distribution on (0.2,0.8). The $\overline{c}_h$ parameters are drawn from a uniform distribution on $(0.95, 1.05)$. The idiosyncratic shocks $u_{t,h}$ are set at zero in periods 1–10 and are drawn from a normal distribution $\mathcal{N}(0, 0.04^2)$ in all later periods. Initial prices are set at $\overline{p} + 0.6 = 10.6$. Figure 1 shows time series of the price and number of short-selling constrained types in a simulation of $T = 500$ periods, of which the first 40 periods are shown.

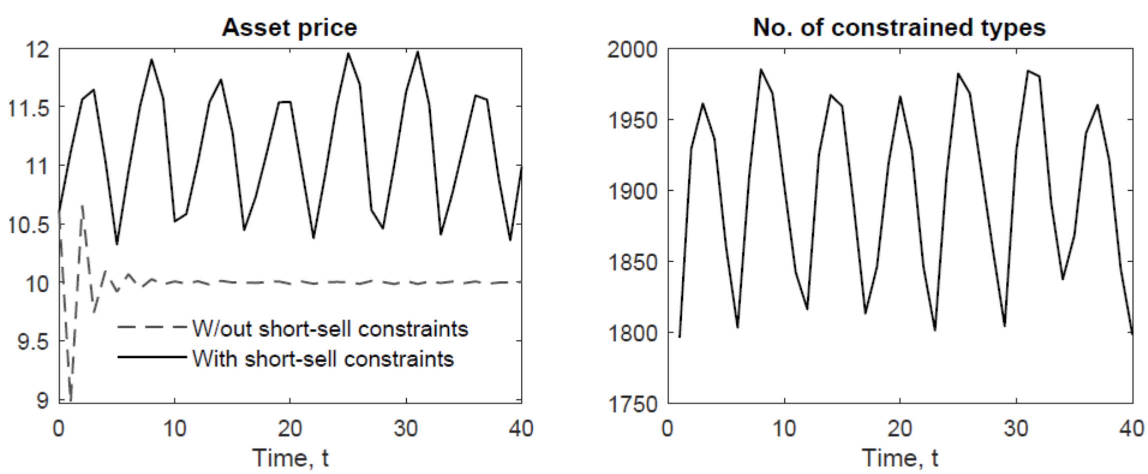

Figure 1: Simulation paths plotted over the first 40 periods ($H = 3,000$ types). The left panel shows the dynamics of the asset price p_t from periods $t = 1, .., 40$ and the right panel shows the number of short-selling constrained types, $|\mathcal{S}_t^*|$, in each period.

Figure 1 (left panel) shows that short-selling constraints have a substantial impact on the asset price dynamics. With short-selling constraints there are persistent price cycles (solid line); by contrast, when short-selling constraints are absent there are dampening price oscillations that rapidly converge toward the fundamental price (dashed line, left panel). The price is higher with short-selling constraints, and it oscillates depending the number of types which are short-selling constrained in a given period (see right panel).

Table 1: Computation times and accuracy: $H = 3,000$ types and $T = 500$ periods

Case	Short-sale constraints	Time (s)	Bind freq.	$\max(Error_t)$
No heterogeneity:	No	0.09	-	3.8e-16
$\overline{c}_h = 1$ for all h	Yes	0.16	497/500	4.3e-14
Heterogeneity 1:	**No**	**0.40**	**-**	**2.0e-15**
$\boldsymbol{\overline{c}_h \in (0.95, 1.05)}$	**Yes**	**0.66**	**500/500**	**8.9e-16**
Heterogeneity 2:	No	0.13	-	5.3e-16
$\overline{c}_h \in (0.995, 1.005)$	Yes	0.22	500/500	4.1e-15

Note: $\max(Error_t) = \max\{Error_1, ..., Error_T\}$, $Error_t := |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$. The middle row of the table (bold font) shows the results for the case in Figure 1 and described above. The other cases change the amount of heterogeneity in $\overline{c}_h$ with all other parameters and shocks fixed.

In Table 1 we report computation times and a measure of accuracy for the example in Figure 1, as well as two variations on this example. The first variant (bottom row) reduces the heterogeneity in $\overline{c}_h$, while the second case (top row) eliminates heterogeneity in $\overline{c}_h$ altogether. Our measure of accuracy (final column) is based on the maximum deviation from market clearing across all periods, i.e. $\max(Error_t) := \max_{1\leq t\leq T} |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$.

The results in Table 1 show that the solutions with short-selling constraints are computed quickly using our amended computational algorithm: computation times for a 500 period simulation with 3,000 types are below 1 second in all three cases. Further, both computation time and accuracy are comparable to the case where short-selling constraints are absent (see top rows in Table 1), which is based on the standard analytical solution p_t^* and does not require any search procedure.[3]

The price time series for the low heterogeneity and zero heterogeneity cases (not shown) are qualitatively similar to those in the case where short-selling constraints are absent (see Figure 1, dashed line). Intuitively, if heterogeneity is small its effects may largely 'wash out', but adding extra heterogeneity means constraints bind more often and on more types. Simulation codes are available at the author's GitHub page: https://github.com/MCHatcher.

[3]The simulations were run in Matlab 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

4.2 Heterogeneity in variances

In the case of heterogeneous subjective return variances, demands are given by

$$z_{t,h}=\begin{cases}\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\overline{d}-(1+\tilde{r})p_t}{a\sigma_h^2} & \text{if } p_t\leq\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\overline{d}}{1+\tilde{r}}\\ 0 & \text{if } p_t>\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\overline{d}}{1+\tilde{r}}\end{cases}\tag{35}$$

where $\sigma_h^2>0$ is the subjective return variance of type h and $\tilde{E}_{t,h}\left[p_{t+1}\right]=\overline{c}p_t+\tilde{f}_{t,h}$.

Defining $\tilde{a}_h:=(a\sigma_h^2)^{-1}$, $f_{t,h}:=\tilde{f}_{t,h}+\overline{d}-\overline{Z}/\tilde{a}_h$ and $r:=\tilde{r}-\overline{c}$, the demands in (35) are

$$z_{t,h}=\begin{cases}\tilde{a}_h(f_{t,h}+\overline{Z}/\tilde{a}_h-(1+r)p_t) & \text{if } p_t\leq\dfrac{f_{t,h}+\overline{Z}/\tilde{a}_h}{1+r}\\ 0 & \text{if } p_t>\dfrac{f_{t,h}+\overline{Z}/\tilde{a}_h}{1+r}.\end{cases}\tag{36}$$

In (35) and (36), the subjective variances are heterogeneous, but it should be clear that heterogeneity in risk aversion is also nested by this approach. Time variation is also straightforward: add time subscripts. By (36) and market-clearing, we have the following.

Proposition 6 (Heterogeneous subjective variances) *Let p_t be the market-clearing price at date $t\in\mathbb{N}_+$, let $n_{t,h}=\hat{n}_h(\mathbf{n}_{t-1},\mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t\subseteq\mathcal{H}$ ($\mathcal{S}_t:=\mathcal{H}\backslash\mathcal{B}_t$) be the set of unconstrained types (constrained types). Let $\tilde{a}_h=(a\sigma_h^2)^{-1}$, $\tilde{n}_{t,h}:=\tilde{a}_h n_{t,h}$ and $f_{t,h}$ as above. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h}-\min_{h\in\mathcal{H}}\{f_{t,h}+\overline{Z}/\tilde{a}_h\})\leq 0$, then no type is short-selling constrained ($\mathcal{B}_t^*=\mathcal{H}$, $\mathcal{S}_t^*=\emptyset$) and the market-clearing price is

$$p_t=\frac{\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h}}{(1+r)\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}}\tag{37}$$

with demands $z_{t,h}=\tilde{a}_h(f_{t,h}+\overline{Z}/\tilde{a}_h-(1+r)p_t)\geq 0\ \forall h\in\mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h}-\min_{h\in\mathcal{H}}\{f_{t,h}+\overline{Z}/\tilde{a}_h\})>0$, at least one type is short-selling constrained and $\exists$ unique $\mathcal{B}_t^*,\mathcal{S}_t^*\subset\mathcal{H}$ such that $\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h}-\min_{h\in\mathcal{B}_t^*}\{f_{t,h}+\overline{Z}/\tilde{a}_h\})\leq(1-\sum_{h\in\mathcal{B}_t^*}n_{t,h})\overline{Z}<\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h}-\max_{h\in\mathcal{S}_t^*}\{f_{t,h}+\overline{Z}/\tilde{a}_h\})$, and the price is

$$p_t=\frac{\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}f_{t,h}-(1-\sum_{h\in\mathcal{B}_t^*}n_{t,h})\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}}\tag{38}$$

with demands $z_{t,h}=\tilde{a}_h(f_{t,h}+\overline{Z}/\tilde{a}_h-(1+r)p_t)\geq 0\ \forall h\in\mathcal{B}_t^*$, $z_{t,h}=0\ \forall h\in\mathcal{S}_t^*$.

Proof. See Section 6.7 of this appendix. ∎

Note that types with the lowest values of $f_{t,h}+\overline{Z}/\tilde{a}_h$ should be considered least optimistic, as they are more likely to be short-selling constrained at any given price. We thus define $\hat{f}_{t,h}:=f_{t,h}+\overline{Z}/\tilde{a}_h$ to rank types by optimism, which allows us to state the following result.

Corollary 4 *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set such that types are ordered as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} := f_{t,h} + \overline{Z}/\tilde{a}_h$. Let $disp_{t,k} := \sum_{h=k}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - [f_{t,k} + \overline{Z}/\tilde{a}_k])$ and $\tilde{disp}_{t,k} := disp_{t,k} + n_{t,k}\overline{Z}$ for $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The market-clearing price is:*

$$p_t = \begin{cases} \dfrac{\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h}}{(1+r)\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq (1 - \sum_{h=1}^{\tilde{H}_t} n_{t,h})\overline{Z} \ (= 0) \\ \dfrac{\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - n_{t,1}\overline{Z}}{(1+r)\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq (1 - \sum_{h=2}^{\tilde{H}_t} n_{t,h})\overline{Z} < \tilde{disp}_{t,1} \\ \dfrac{\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - (n_{t,1} + n_{t,2})\overline{Z}}{(1+r)\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq (1 - \sum_{h=3}^{\tilde{H}_t} n_{t,h})\overline{Z} < \tilde{disp}_{t,2} \\ \vdots & \vdots \\ \dfrac{\tilde{n}_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t - 1} n_{t,h})\overline{Z}}{(1+r)\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } \tilde{disp}_{t,\tilde{H}_t - 1} > (1 - \sum_{h=\tilde{H}_t - 1}^{\tilde{H}_t} n_{t,h})\overline{Z} \end{cases}$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } 2 \leq k < k^*. \tag{39}$$

Proof. It follows from Proposition 6 and from the same steps used in the proof of Corollary 1 (see main paper) with the necessary alterations being made. ■

4.2.1 Computational algorithm (Heterogeneous subjective variances)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} = f_{t,h} + \overline{Z}/\tilde{a}_h$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - [f_{t,1} + \overline{Z}/\tilde{a}_1])$. If $disp_{t,1} \leq 0$, accept $p_t = p_t^*$ as the date t price, compute demands and move to period $t+1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \tilde{a}_k(f_{t,k} + \overline{Z}/\tilde{a}_k - (1+r)p_t^{guess}) < 0$, say $\underline{k}$. Starting at $k = \underline{k}$, check if $disp_{t,k+1} \leq (1 - \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < \tilde{disp}_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \leq (1 - \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < \tilde{disp}_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right]\overline{Z}}{(1+r)\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, compute demands, and move to period $t+1$.

4.2.2 Numerical example

We stick with the same numerical example as set out above, except we set $c_h = \overline{c} = 1$ for all h and draw the subjective variances of different types from a uniform distribution, such that $\sigma_h^2 \sim \mathcal{U}(\sigma_{min}^2, \sigma_{max}^2)$. The results in Table 2 show that the solutions with short-selling

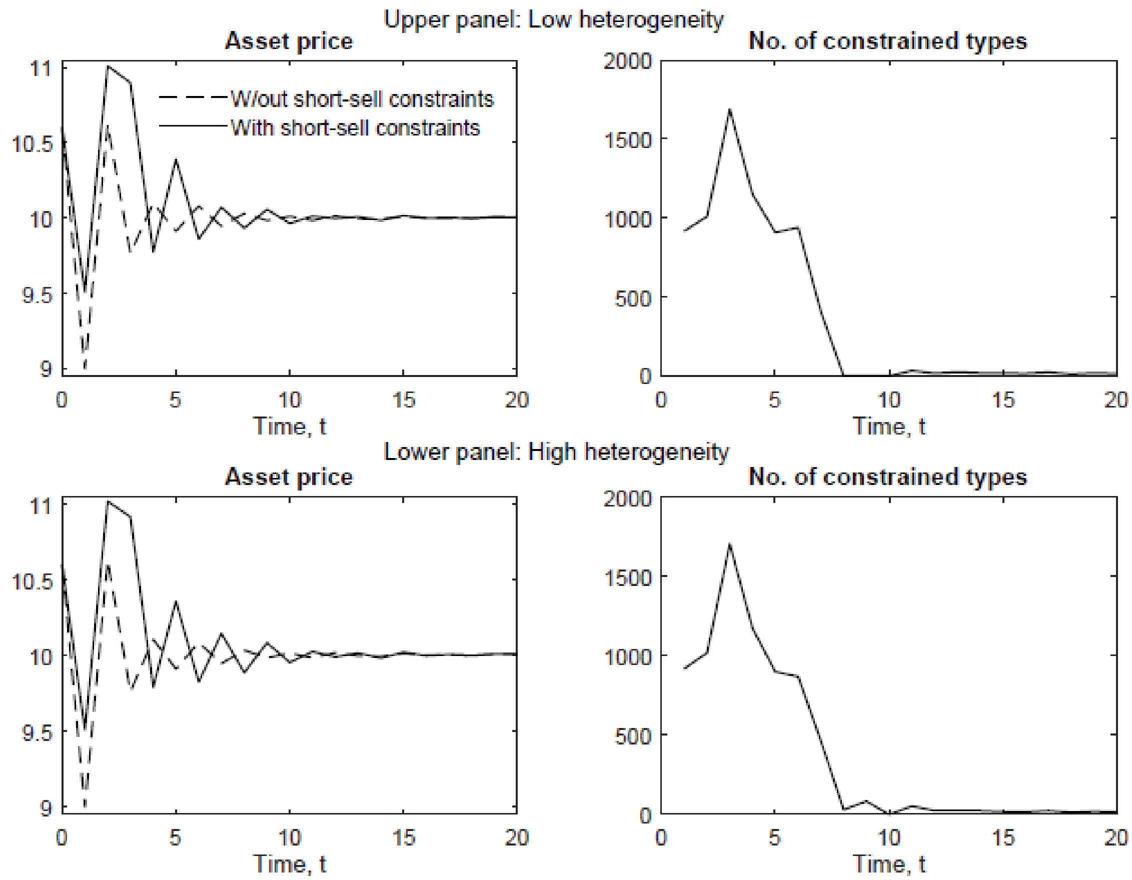

Figure 2: Simulation with low and high heterogeneity in σ_h^2 ($H = 3,000$ types)

Table 2: Computation times and accuracy: $T = 500$ periods, $H = 3,000$ types

Heterogeneity	Short-sale constraints	Time (s)	Bind freq.	max($Error_t$)
High case:	No	0.16	-	9.7e-16
$\sigma_h^2 \in (0.9, 1.1)$	Yes	0.23	499/500	3.4e-15
Low case:	No	0.14	-	8.1e-16
$\sigma_h^2 \in (0.99, 1.01)$	Yes	0.22	497/500	4.0e-15

Note: $\max(Error_t) = \max\{Error_1, ..., Error_T\}$, $Error_t := |\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$.

constraints are fast and accurate, being comparable to those when short-selling constraints are absent. The time series for the first 20 periods are plotted in Figure 2.

The codes used to generate the above results are available from the author's GitHub page at https://github.com/MCHatcher.

5 Generalization to population shares $n_{t,h} \in [0,1]$

In this section we relax the assumption $n_{t,h} \in (0,1)$ to allow $n_{t,h} \in [0,1]$ at dates $t \in \mathbb{N}_+$, subject to $\sum_{h\in\mathcal{H}} n_{t,h} = 1$. We show how our analytical results and algorithm can be adjusted for this weaker assumption. The cases $n_{t,h} = 1$ and $n_{t,h} = 0$ can be dealt with together because if $n_{t,h'} = 1$ for some h', then $n_{t,h} = 0$ for all $h \neq h'$;[4] and if $n_{t,h'} = 0$ for some types $h' \in \mathcal{H}$, then the sum of population shares across all other types is equal to 1. Let $\mathcal{H}_t^{sub} \subset \mathcal{H}$ be the subset containing all types with a zero population share at date t, such that $n_{t,h} = 0$ for all $h \in \mathcal{H}_t^{sub}$ and $n_{t,h} \in (0,1]$ for all $h \in \mathcal{H} \setminus \mathcal{H}_t^{sub}$, with $\sum_{\mathcal{H}\setminus\mathcal{H}_t^{sub}} n_{t,h} = 1$. Given that $n_{t,h} = 0$ for all $h \in \mathcal{H}_t^{sub}$, market-clearing at date t is given by

$$\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \sum_{h\in\mathcal{H}_t^*} n_{t,h} z_{t,h} = \overline{Z}, \quad \text{where } \mathcal{H}_t^* := \mathcal{H} \setminus \mathcal{H}_t^{sub} \neq \emptyset, \tag{40}$$

which is analogous to the problem we solve in the main text, except that the set of types under consideration ($\mathcal{H}_t^*$) is not restricted to be fixed over time.

Since $\mathcal{H}_t^*$ can be found without knowledge of the *current* (or future) market-clearing price p_t, nothing changes except $\mathcal{H}$ is replaced by $\mathcal{H}_t^*$ in Proposition 1. Formally, we have:

Proposition 7 (Proposition 1 adapted for $0 \leq n_{t,h} \leq 1$) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, let $\mathcal{H}_t^*$ be the set of types (with non-zero pop. shares) defined above, let $\mathcal{B}_t \subseteq \mathcal{H}_t^*$ ($\mathcal{S}_t := \mathcal{H}_t^* \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types) at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$, then no type is short-selling constrained at date t ($\mathcal{B}_t^* = \mathcal{H}_t^*$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}_t^*} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{41}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{H}_t^*$.

(ii) If $\sum_{h\in\mathcal{H}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}_t^*}\{f_{t,h}\}) > a\sigma^2\overline{Z}$, at least one type is short-selling constrained and $\exists$ unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}_t^*$, $\mathcal{S}_t^*$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\})$, and the price and demands are given by

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{42}$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. It follows from Proposition 1 in the main text when the set $\mathcal{H}$ is replaced by $\mathcal{H}_t^*$. ■

Corollary 1 does not require any amendments since we may re-define the function $\tilde{h}_t$ in the main text as a function from the set $\mathcal{H}_t^*$ to the set $\tilde{\mathcal{H}}_t$, i.e. $\tilde{h}_t : \mathcal{H}_t^* \to \tilde{\mathcal{H}}_t$. Finally, since we keep the set $\tilde{\mathcal{H}}_t$, the computational algorithm also does not require any changes.

[4] Recall that $n_{t,h} \in [0,1]$ and $\sum_{h\in\mathcal{H}} n_{t,h} = 1$. Hence, if $n_{t,h'} = 1$ for a type $h' \in \mathcal{H}$, then this implies that $\sum_{h\neq h'} n_{t,h} = 0$. The latter is possible only if $n_{t,h} = 0$ for all $h \neq h'$, since negative shares are ruled out.

6 Proofs

6.1 Proof of Proposition 1

By Section 3.1 the indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K})\leq 0\}}$ is equal to 1 if the short-selling constraint is in place at date t (i.e. if $g(p_{t-1},...,p_{t-K}) \leq 0$) and is 0 otherwise.

Case 1: $\mathbb{1}_t = 1$

If the short-selling constraint is in place at date t (i.e. if $\mathbb{1}_t = 1$), the cases for price and demands are equivalent to those for an unconditional short-selling constraint, as in Proposition 1 in the main paper and its proof. The short-selling constraint is slack for all types if $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ and binds for one or more types (but fewer than H) otherwise, i.e. if $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$.

Case 2: $\mathbb{1}_t = 0$

If the short-selling constraint is not in place at date t (i.e. if $\mathbb{1}_t = 0$) then demands are given by $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \in \mathbb{R}$ for all $h \in \mathcal{H}$, and thus the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ gives $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}}{1+r}$ (which is the same expression as for $\mathbb{1}_t = 1$, $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$). This conclusion holds regardless of whether $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ or $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$. ■

6.2 Proof of Proposition 3

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0\ \forall h \in \mathcal{H}$, which implies by the price equation that $p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} + \mu(1-\lambda)(Z_{t-1}-\overline{Z})}{1+\mu\lambda(1+r)(a\sigma^2)^{-1}} := p_t^*$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^* \geq 0\ \forall h \in \mathcal{H}$, which requires $\left(\frac{1}{1+r} + \frac{\mu\lambda}{a\sigma^2}\right)(a\sigma^2\overline{Z} + \min_{h\in\mathcal{H}}\{f_{t,h}\}) \geq p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})$, giving the inequality in Proposition 3 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$, so $p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1+\mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t \geq 0\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t < 0\ \forall h \in \mathcal{S}_t^*$, which requires $(\frac{1}{1+r} + \mu\lambda(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h})(a\sigma^2\overline{Z} + f_{t,h}) \geq (<)\ p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]\ \forall h \in \mathcal{B}_t^*$ ($\forall h \in \mathcal{S}_t^*$), giving the inequalities in Proposition Part 3(i).

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0\ \forall h \in \mathcal{H}$, which implies that $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t < 0\ \forall h \in \mathcal{H}$, i.e. iff $\max_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z} < (1+r)(p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}])$, which is the inequality in Proposition 3 Part 2(ii). ■

6.3 Proof of Corollary 1

The first (last) 'if' statement follows from Proposition 2 main paper as $disp_{t,1} \le \tilde{g}_t$ ($disp_{t,\tilde{H}_t} > \tilde{g}_t$) is equivalent to $p_{t-1} - \frac{1}{1+r}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \le (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$ (resp. $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > (\mu + \frac{a\sigma^2}{1+r})\overline{Z}$). The other cases follow as there are $\tilde{H}_t - 1$ other candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, i.e. $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, ..., \tilde{H}_t - 1\}$; $\mathcal{S}_t = \{1, 2\}, \mathcal{B}_t = \{3, ..., \tilde{H}_t - 1\}$;...$\mathcal{S}_t = \{1, ..., \tilde{H}_t\}, \mathcal{B}_t = \emptyset$. For arbitrary sets $\mathcal{S}_t = \{1, ..., k\}$, $B_t = \{k+1, ..., \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$, we have by Proposition 3 (above) that $p_t = p_t^{(k)}$ and the guess is verified if and only if $disp_{t,k+1} \le \tilde{g}_t < disp_{t,k}$.

It remains to show $p_t^* < p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}\ \forall k \in [2, k^*)$. Recall that the demands $z_{t,h}(p_t)$ are decreasing in the price for all h. Note that p_t^* satisfies $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^* - p_t^*$, where $\tilde{Z}_{t-1} := p_{t-1} + \mu(1-\lambda)Z_{t-1}$, $V_t^* := \mu\lambda\sum_{h=1}^{\tilde{H}_t} n_{t,h}z_{t,h}(p_t^*)$, and $z_{t,1} < 0$ since p_t^* is not consistent with short-selling constraints (or else $p_t = p_t^*$). Similarly, $p_t^{(1)}$ satisfies $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^{(1)} - p_t^{(1)}$, where $V_t^{(1)} := \mu\lambda\sum_{h=2}^{\tilde{H}_t} n_{t,h}z_{t,h}(p_t^{(1)})$. Suppose $p_t^{(1)} \le p_t^*$, which implies that $V_t^{(1)} > V_t^*$. This leads to the contradiction $V_t^{(1)} - p_t^{(1)} > \mu\overline{Z} - \tilde{Z}_{t-1}$; therefore $p_t^{(1)} > p_t^*$. For arbitrary k and $j = k-1, k$, note that $p_t^{(j)}$ satisfies $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^{(j)} - p_t^{(j)}$, where $V_t^{(j)} := \mu\lambda\sum_{h>j}^{\tilde{H}_t} n_{t,h}z_{t,h}(p_t^{(j)})$. Suppose $p_t^{(k)} \le p_t^{(k-1)}$. This leads to a contradiction since $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^{(k-1)} - p_t^{(k-1)} < V_t^{(k)} - p_t^{(k)}$; therefore $p_t^{(k)} > p_t^{(k-1)}$. Finally, $p_t^{(k^*)} > p_t^{(k)}$ $\forall k < k^*$ follows from applying the above argument for $j = k^* - 1, k^*$. ■

6.4 Proof of Proposition 4

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\overline{c})p_t) \ge 0\ \forall h \in \mathcal{H}$, which implies by the price equation that $p_t = \frac{p_{t-1} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}}{1 + \mu\lambda(1-\overline{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}} := p_t^*$, where $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The guess is verified if and only if $f_{t,h} \ge (1-\overline{c})p_t^*\ \forall h \in \mathcal{H}$, which requires $(1-\overline{c})^{-1}\left(1 + \mu\lambda(1-\overline{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}\right)f_{t,h} \ge p_{t-1} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}\ \forall h \in \mathcal{H}$. Note that this is equivalent to $(1-\overline{c})^{-1}\left(1 + \mu\lambda(1-\overline{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}\right)\min_{h\in\mathcal{H}}\{f_{t,h}\} \ge p_{t-1} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}$, which simplifies to the inequality in Proposition 4 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\bar{c})p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$, such that $p_t = \frac{p_{t-1}+\mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}}{1+\mu\lambda(1-\bar{c})\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}}$ by the price equation, with $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The guess is verified iff $f_{t,h} \geq (1-\bar{c})p_t\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} < (1-\bar{c})p_t\ \forall h \in \mathcal{S}_t^*$, i.e. $(1-\bar{c})^{-1}\left(1+\mu\lambda(1-\bar{c})\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}\right) f_{t,h} - (p_{t-1} + \mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}) \geq 0$ (< 0) $\forall h \in \mathcal{B}_t^*$ $(\forall h \in \mathcal{S}_t^*)$, which are equivalent to $\mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}}\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < \mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, as stated in Proposition 4 Part 2(i).

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0\ \forall h \in \mathcal{H}$, which implies $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ by the price equation. The guess is verified if and only if $f_{t,h} < (1-\bar{c})p_t\ \forall h \in \mathcal{H}$, i.e. if and only if $\frac{1}{1-\bar{c}}\max_{h\in\mathcal{H}}\{f_{t,h}\} < p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$, as stated in Proposition 4 Part 2(ii). ■

6.5 Proof of Corollary 2

The first (last) 'if' statement follows from Proposition 4 (above) as $disp_{t,1} \leq g(p_{t-1}, Z_{t-1})$ $(disp_{t,\tilde{H}_t} > g(p_{t-1}, Z_{t-1}))$ is equivalent to $p_{t-1}+\mu(1-\lambda)Z_{t-1}+\mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) - \frac{1}{1-\bar{c}}\min_{h\in\mathcal{H}}\{f_{t,h}\} \leq \mu\overline{Z}$ (resp. $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1-\bar{c}}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \mu\overline{Z}$). The other cases follow as there are $\tilde{H}_t - 1$ other candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, i.e. $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, ..., \tilde{H}_t - 1\}$; $\mathcal{S}_t = \{1,2\}, \mathcal{B}_t = \{3, ..., \tilde{H}_t - 1\}$;...$\mathcal{S}_t = \{1, ..., \tilde{H}_t\}, \mathcal{B}_t = \emptyset$. For arbitrary sets $\mathcal{S}_t = \{1, ..., k\}$, $B_t = \{k+1, ..., \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$ and by Proposition 4 (above) $p_t = p_t^{(k)}$ and the guess is verified iff $disp_{t,k+1} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,k}$. The proof that $p_t^* < p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}\ \forall k \in [2, k^*)$ follows the same steps as the proof of Corollary 1 in Section 6.3. ■

6.6 Proof of Proposition 5

Existence of a unique price follows from Anufriev and Tuinstra (2013, Proposition 2.1) when an appropriate relabelling of variables is used. We define $r_h := \tilde{r} - \bar{c}_h$ and $\tilde{n}_{t,h} := n_{t,h}(1+r_h)$.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0\ \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^* \geq 0\ \forall h \in \mathcal{H}$, i.e. $\left[\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\right]\sum_{h\in\mathcal{H}}\tilde{n}_{t,h} \geq \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}\ \forall h \in \mathcal{H}$, which simplifies to $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}\sum_{h\in\mathcal{H}}\tilde{n}_{t,h} \leq 0$ as in Proposition 5.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h} - (1+r_h)p_t\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h} - (1+r_h)p_t\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$, so market-clearing is $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$, giving $p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^{\mathcal{B}_t^*} \geq 0\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^{\mathcal{B}_t^*} < 0\ \forall h \in \mathcal{S}_t^*$, i.e. $\left[\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\right]\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h} \geq (<) \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}\ \forall h \in \mathcal{B}_t^*$ ($\forall h \in \mathcal{S}_t^*$), which simplify to $\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - \sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}\min_{h\in\mathcal{B}_t^*}\{\hat{f}_{t,h}\} \leq (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - \sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}\max_{h\in\mathcal{S}_t^*}\{\hat{f}_{t,h}\}$, where $\hat{f}_{t,h} := \frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$, which is the inequality given in Proposition 5. ■

6.7 Proof of Proposition 6

Existence of a unique price follows from Anufriev and Tuinstra (2013, Proposition 2.1) when an appropriate relabelling of variables is used. We define $\tilde{a}_h = (a\sigma_h^2)^{-1}$ and $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0\ \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h}}{(1+r)\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}}$. The guess is verified if and only if $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^* \geq 0\ \forall h \in \mathcal{H}$, i.e. $\left[f_{t,h} + \overline{Z}/\tilde{a}_h\right]\sum_{h\in\mathcal{H}}\tilde{n}_{t,h} \geq \sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h}$ $\forall h \in \mathcal{H}$, which simplifies to $\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq 0$ as in Proposition 6.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{H}\setminus\mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$, so market-clearing is $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$, giving $p_t = \frac{\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^{\mathcal{B}_t^*} \geq 0\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^{\mathcal{B}_t^*} < 0\ \forall h \in \mathcal{S}_t^*$, i.e. $\left[f_{t,h} + \overline{Z}/\tilde{a}_h\right]\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h} \geq (<) \sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}\ \forall h \in \mathcal{B}_t^*$ ($\forall h \in \mathcal{S}_t^*$), which simplify to $\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z} < \sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\})$, which is the inequality given in Proposition 6. ■

References

Anufriev, M. and Tuinstra, J. (2013). The impact of short-selling constraints on financial market stability in a heterogeneous agents model. *Journal of Economic Dynamics and Control*, 37(8):1523–1543.

Bolt, W., Demertzis, M., Diks, C., Hommes, C., and Van Der Leij, M. (2019). Identifying booms and busts in house prices under heterogeneous expectations. *Journal of Economic Dynamics and Control*, 103:234–259.

Brock, W. A. and Hommes, C. H. (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control*, 22(8-9):1235–1274.

Hatcher, M. (2021). Endogenous extrapolation and house price cycles. *Rebuilding Macroeconomics Working Paper No. 46*.

Hatcher, M. and Hellmann, T. (2022). Networks, beliefs, and asset prices. *SSRN Working Paper No. 4037357*.

LeBaron, B., Arthur, W. B., and Palmer, R. (1999). Time series properties of an artificial stock market. *Journal of Economic Dynamics and Control*, 23(9-10):1487–1516.

Panchenko, V., Gerasymchuk, S., and Pavlov, O. V. (2013). Asset price dynamics with heterogeneous beliefs and local network interactions. *Journal of Economic Dynamics and Control*, 37(12):2623–2642.

Westerhoff, F. H. (2004). Multiasset market dynamics. *Macroeconomic Dynamics*, 8(5):596–616.