Supplementary Appendix for "Solving heterogeneous-belief asset pricing models with short selling constraints and many agents"

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This appendix provides (i) derivations of optimal demands with short-selling constraints, as in Sections 2.1 and 4.1 of the main text, and (ii) supporting material for the generalizations and extensions in Sections 2.3, 3.3–3.4, 4.1–4.2 and 4.3.2 of the main text.

1 Derivations

In this section we derive the optimal demand schedules for the cases of unconditional short-selling constraints (benchmark model) and conditional short-selling constraints (Section 4.1).

1.1 Derivation of demands in the benchmark model

Each type $h \in \mathcal{H}$ solves the following problem:[1]

$$\max_{z_{t,h}} \ \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad z_{t,h} \geq 0 \qquad (1)$$

where $w_{t+1,h} = (p_{t+1}+d_{t+1})z_{t,h}+(1+\tilde{r})(w_{t,h}-p_t z_{t,h})$ is future wealth, $w_{t,h}-p_t z_{t,h}$ is holdings of the risk-free asset, $a, \tilde{r} > 0$ are parameters and $\tilde{V}_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$, with $\sigma^2 > 0$.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}} \ \mathcal{L}_{t,h} = \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2}\tilde{V}_{t,h}\left[w_{t+1,h}\right] + \lambda_{t,h}z_{t,h} \qquad (2)$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint, $z_{t,h} \geq 0$.

The first-order conditions are

$$z_{t,h} : \quad \tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h} = 0 \qquad (3)$$

$$\lambda_{t,h} : \quad z_{t,h} \geq 0 \qquad (4)$$

and the complementary slackness condition is:

$$\lambda_{t,h}z_{t,h} = 0. \qquad (5)$$

By (3), $\lambda_{t,h} = -(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h})$. Note that $\lambda_{t,h} = 0$ if and only if $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t)$. This $z_{t,h}$ satisfies (5) (given $\lambda_{t,h} = 0$) and therefore it is an optimal demand iff $\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] \geq (1+\tilde{r})p_t$; see (4).

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[1]We assume (as is standard) that these operators satisfy some basic properties of conditional expectation operators, namely, $\tilde{E}_{t,h}[y_t] = y_t$ and $\tilde{V}_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1}+y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $\tilde{V}_{t,h}[x_t y_{t+1}] = x_t^2 \tilde{V}_{t,h}[y_{t+1}]$.

If $\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] < (1+\tilde{r})p_t$. we can reject $\lambda_{t,h} = 0$, since the condition in (4) is not satisfied. It follows that $\lambda_{t,h} > 0$ when $\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] < (1+\tilde{r})p_t$, and by the complementary slackness condition (5) it follows that $z_{t,h} = 0$ (which satisfies (4)).

Therefore, the demand schedule of type $h \in \mathcal{H}$ is given by

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}} \end{cases} \tag{6}$$

as stated in Equation (2) of the main text.

1.2 Derivation of demands for conditional short-selling constraint

If $g(p_{t-1}, ..., p_{t-K}) > 0$, the short-selling constraint is absent at date t; if $g(p_{t-1}, ..., p_{t-K}) \leq 0$ the short-selling constraint is present. We introduce an indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K}) \leq 0\}}$ equal to 1 if the short-selling constraint is present at date t (i.e. if $g(p_{t-1}, ..., p_{t-K}) \leq 0$), and equal to 0 otherwise. The problem of type $h \in \mathcal{H}$ at date t is thus amended from (1) to:

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad \mathbb{1}_t z_{t,h} \geq 0 \tag{7}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+\tilde{r})(w_{t,h} - p_t z_{t,h})$ as above.

Note that if $\mathbb{1}_t = 0$, the portfolio choice of all types $h \in \mathcal{H}$ is unconstrained at date t, since $\mathbb{1}_t z_{t,h} = 0 \geq 0$ is satisfied for any $z_{t,h} \in \mathbb{R}$. On the other hand, if $\mathbb{1}_t = 1$ then all types face the same maximization problem as in (1), i.e. short-selling is ruled out at date t.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] + \lambda_{t,h}\mathbb{1}_t z_{t,h} \tag{8}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the constraint $\mathbb{1}_t z_{t,h} \geq 0$.

The first-order conditions are

$$z_{t,h} : \quad \tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h}\mathbb{1}_t = 0 \tag{9}$$

$$\lambda_{t,h} : \quad \mathbb{1}_t z_{t,h} \geq 0 \tag{10}$$

and the complementary slackness condition is:

$$\lambda_{t,h}\mathbb{1}_t z_{t,h} = 0. \tag{11}$$

If $\mathbb{1}_t = 0$, then (10)–(11) are satisfied and $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t) \in \mathbb{R}$ by (9). Hence, demands are unconstrained if $\mathbb{1}_t = 0$. If $\mathbb{1}_t = 1$, the first-order conditions (9)–(11) are identical to (3)–(5), so the cases are the same as discussed below (5).

Therefore, demands are $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t)$ if $\mathbb{1}_t = 0$ or $p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}$; and $z_{t,h} = 0$ otherwise (i.e. if $\mathbb{1}_t = 1$ and $p_t > \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}$).

2 Nested cases

In this section we show some cases (housing as the risky asset; constraints $z_{t,h} \geq L$) which are essentially nested by the benchmark model as discussed in Section 3.2 of the paper.

2.1 Housing as the risky asset

In this section we demonstrate how our results can be applied when the risky asset is housing as in Bolt et al. (2019) and Hatcher (2021). In these models, housing is an investment asset that differs from shares due to the interpretation of 'dividends'. In Bolt et al. (2019) dividends are replaced by imputed rent based on an arbitrage condition between the user and rental costs, whereas Hatcher (2021) assumes linear housing utility scaled by a housing preference variable.[2] In both models, these additional variables are exogenous processes whose properties are known to the investors. We assume a fixed supply of housing $\overline{Z} > 0$.

With linear excess returns and an unconditional short-selling constraint, demands are:

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1 + \hat{R}) - (1 + \tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1 + \hat{R})}{1 + \tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1 + \hat{R})}{1 + \tilde{r}} \end{cases} \tag{BDDHL1}$$

where Q_t is the exogenous rental price and $\hat{R}$ is the fixed risk-free mortgage rate; and

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z - (1 + \tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z}{1 + \tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z}{1 + \tilde{r}} \end{cases} \tag{Hatcher1}$$

where $\Theta_t > 0$ is an exogenous relative preference for housing utility versus financial wealth, and $\overline{U}_z > 0$ is a fixed marginal utility of housing (which does not depend on $z_{t,h}$).

Defining $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2\overline{Z}$ and $r := \tilde{r} - \overline{c}$, with $\tilde{E}_{t,h}\left[d_{t+1}\right] = Q_t(1 + \hat{R})$ (or $\tilde{E}_{t,h}\left[d_{t+1}\right] = \Theta_t\overline{U}_z$), the above demands can be written as

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1 + r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1 + r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1 + r} \end{cases} \tag{12}$$

as in Equation (4) of the main text.

Hence, our approach can easily accommodate these cases. Note that the above specifications will imply a change in the fundamental price even if shocks are IID, since the *current* 'dividend' rather than the expected future value enters into demand. Nevertheless, it remains straightforward to write demands in deviations from the fundamental price.

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[2]Using quadratic utility from housing in the framework of Hatcher (2021) is also possible (under certain conditions) as this mirrors the mean-variance assumption in the benchmark model.

2.2 Short-selling constraints of the form: $z_{t,h} \geq L$

Suppose that negative positions beyond some limit $L \leq 0$ are not permitted in any period. We can formulate this assumption as an unconditional short-selling limit $z_{t,h} \geq L, \forall t, h$.[3]

Formulating the maximization problem of type h as a Lagrangean:

$$\max_{z_{t,h}, \lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] + \lambda_{t,h}(z_{t,h} - L) \tag{13}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint $z_{t,h} \geq L$.

The first-order conditions are

$$z_{t,h}: \tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h} = 0, \quad \lambda_{t,h}: z_{t,h} \geq L \tag{14}$$

and the complementary slackness condition is: $\lambda_{t,h}(z_{t,h} - L) = 0$.

Analogous to the discussion after (3)–(5), $\lambda_{t,h} = 0$ and $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t)$ is an optimal demand iff $\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t \geq a\sigma^2 L$, while if $\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t < a\sigma^2 L$, then $\lambda_{t,h} > 0$ and $z_{t,h} = L$.

Hence, the demand schedule of type $h \in \mathcal{H}$ is given by

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2 L}{1+\tilde{r}} \\ L & \text{if } p_t > \dfrac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2 L}{1+\tilde{r}}. \end{cases} \tag{15}$$

Defining $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2\overline{Z}$, $r := \tilde{r} - \bar{c}$ and $\tilde{z}_{t,h} := z_{t,h} - L$, the demands in (15) can be written in the form:

$$\tilde{z}_{t,h} = \begin{cases} \dfrac{f_{t,h} + a\sigma^2\tilde{Z} - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\tilde{Z}}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\tilde{Z}}{1+r} \end{cases} \tag{16}$$

where $\tilde{Z} := \overline{Z} - L$.

Note that the demands in (30) have the same form as in Eq. (4) in the main paper, except that $z_{t,h}$ is replaced by $\tilde{z}_{t,h}$ and $\overline{Z}$ is replaced by $\tilde{Z}$. Similarly, market-clearing is:

$$\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \overline{Z} \implies \sum_{h \in \mathcal{H}} n_{t,h} \tilde{z}_{t,h} = \tilde{Z} \tag{17}$$

where $\sum_{h \in \mathcal{H}} n_{t,h} = 1$ is used.[4]

We can therefore state the following result.

Remark 1 *In the above model, demands and market-clearing are as in the benchmark model, except that $z_{t,h}$ is replaced by $\tilde{z}_{t,h}$ and $\overline{Z}$ is replaced by $\tilde{Z}$. Therefore, the market-clearing price and demands follow Proposition 1 with $\tilde{z}_{t,h}$ replacing $z_{t,h}$ and $\tilde{Z}$ replacing $\overline{Z}$.*

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[3]The case of a conditional short-selling limit is also straightforward; see Section 1.2.

[4]The 'deviations' representation of market-clearing in (17) holds regardless of whether some types (those in the set $\mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$) are short-selling constrained with $z_{t,h} = L$ while the others (in $\mathcal{B}_t^*$) are unconstrained.

3 Supporting results for Remarks in the main paper

This section provides propositions supporting the Remarks given in the 'Extensions' section of the main paper. Any non-trivial proofs appear in Section 6.

3.1 Conditional short-selling constraints

Recall from Section 1.2 that we introduced an indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K}) \leq 0\}}$ that is equal to 1 if the short-selling constraint is in place at date t (i.e. if $g(p_{t-1},...,p_{t-K}) \leq 0$) and is equal to 0 otherwise (i.e. if $g(p_{t-1},...,p_{t-K}) > 0$).

The demands for types $h \in \mathcal{H}$ are given by:

$$
z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ and } \mathbb{1}_t = 1 \end{cases} \tag{18}
$$

where $r := \tilde{r} - \overline{c}$ and $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$.

Proposition 1 (Proposition 1 (main) adapted to conditional constraint) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Then the following holds:*

(i) If $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ or $\mathbb{1}_t = 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$
p_t = \frac{\sum_{h \in \mathcal{H}} n_{t,h}f_{t,h}}{1+r} := p_t^* \tag{19}
$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \; \forall h \in \mathcal{H}$ with $z_{t,h} \in \mathbb{R}$ if $\mathbb{1}_t = 0$, and $z_{t,h} \geq 0$ otherwise (i.e. if $\mathbb{1}_t = 1$ and $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$).

(ii) If $\mathbb{1}_t = 1$ and $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$, then at least one type is short-selling constrained and there exist unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}$ and $\mathcal{S}_t^*$ such that $\sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_t^*}\{f_{t,h}\})$, and the market-clearing price and demands are

$$
p_t = \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h}f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h \in \mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{20}
$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \; \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \; \forall h \in \mathcal{S}_t^*$.

Proof. See Section 6.1 of this appendix. ∎

5

3.2 Multiple markets and endogenous participation

In this section we adapt Proposition 1 for the case of multiple risky assets $m \in \{1, ..., M\}$ when participation shares $w_t^m \in (0,1)$ in each market are determined by attractiveness relative to other markets; see Westerhoff (2004) and Section 4.2 of the paper.

We note in the main text (Section 4.2) that demand of type h in market m is given by

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m/w_t^m - (1+r^m)p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m/w_t^m}{1+r^m} \\ 0 & \text{if } p_t^m > \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m/w_t^m}{1+r^m} \end{cases} \tag{21}$$

where $f_{t,h}^m := \tilde{f}_{t,h}^m + \overline{d}^m - a\sigma_m^2 \overline{Z}_m/w_t^m$ and $r^m := \tilde{r} - \overline{c}^m$.

Market-clearing in market m is given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m/w_t^m, \qquad \text{where } \tilde{z}_{t,h}^m := z_{t,h}^m/w_t^m. \tag{22}$$

With this change in variables, the market-clearing condition has the same form as in the benchmark model (aside from a scaling of supply by $1/w_t^m$). We therefore have the following.

Proposition 2 (Proposition 1 adapted to multiple markets) *Let p_t^m be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h}^m = \hat{n}_h(\mathbf{n}_{t-1}^m, \mathbf{u}_{t-1}^m)$ be the population share of type h in market m at date t, and let $\mathcal{B}_t^m \subseteq \mathcal{H}$ ($\mathcal{S}_t^m := \mathcal{H} \setminus \mathcal{B}_t^m$) be the set of unconstrained types (short-selling constrained types) in market m at date t. Then the following holds:*

(i) If $\sum_{h \in \mathcal{H}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{H}}\{f_{t,h}^m\}) \leq a\sigma_m^2 \overline{Z}_m/w_t^m$, then no type is short-selling constrained ($\mathcal{B}_t^{m*} = \mathcal{H}$, $S_t^{m*} = \emptyset$) and the market-clearing price in market m is

$$p_t^m = \frac{\sum_{h \in \mathcal{H}} n_{t,h}^m f_{t,h}^m}{1 + r^m} := p_t^{m*} \tag{23}$$

with demands $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1}(f_{t,h}^m + a\sigma_m^2 \overline{Z}_m/w_t^m - (1+r^m)p_t^m) \geq 0 \; \forall h \in \mathcal{H}$.

(ii) If $\sum_{h \in \mathcal{H}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{H}}\{f_{t,h}^m\}) > a\sigma_m^2 \overline{Z}_m/w_t^m$, then at least one type is short-selling constrained at date t and there exist unique non-empty sets $\mathcal{B}_t^{m*} \subset \mathcal{H}$ and $\mathcal{S}_t^{m*}$ such that $\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{B}_t^{m*}}\{f_{t,h}^m\}) \leq a\sigma_m^2 \overline{Z}_m/w_t^m < \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (f_{t,h}^m - \max_{h \in \mathcal{S}_t^{m*}}\{f_{t,h}^m\})$, and the market-clearing price and demands are

$$p_t^m = \frac{\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m f_{t,h}^m - (1 - \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m)a\sigma_m^2 \overline{Z}_m/w_t^m}{(1+r^m)\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m} > p_t^{m*} \tag{24}$$

and $z_{t,h}^m = w_t^m(a\sigma_m^2)^{-1}(f_{t,h}^m + a\sigma_m^2 \overline{Z}_m/w_t^m - (1+r^m)p_t^m) \geq 0 \; \forall h \in \mathcal{B}_t^{m*}$, $z_{t,h}^m = 0 \; \forall h \in \mathcal{S}_t^{m*}$.

Proof. It follows from the Proposition 1 Proof (main paper) when p_t, $f_{t,h}$, r and $\overline{Z}$ are replaced by p_t^m, $f_{t,h}^m$, r^m and $\overline{Z}_m/w_t^m$, and the demands $z_{t,h}$ are replaced by $z_{t,h}^m$ in (21). ∎

3.3 Market maker: benchmark demand specification

Following Proposition 2 in the main paper (benchmark demands, Section 4.3.1) we now present amended versions of Corollary 1 and Computational Algorithm in this section.

Corollary 1 (amended) *Let* $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ *be the set such that types are ordered as* $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. *Let* $disp_{t,k} := \frac{\mu\lambda}{a\sigma^2} \sum_{h>k} n_{t,h}(f_{t,h} - f_{t,k}) - \frac{1}{1+r} f_{t,k}$ *for* $k \in \{1, ..., \tilde{H}_t - 1\}$, *and* $\tilde{g}_t := (\mu + (1+r)^{-1} a\sigma^2) \overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$. *The price solution is:*

$$
p_t = \begin{cases}
\dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2} \sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}} := p_t^* & \text{if } disp_{t,1} \leq \tilde{g}_t \\[2ex]
\dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2} \sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h=2}^{\tilde{H}_t} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1} \sum_{h=2}^{\tilde{H}_t} n_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq \tilde{g}_t < disp_{t,1} \\[2ex]
\dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2} \sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h=3}^{\tilde{H}_t} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1} \sum_{h=3}^{\tilde{H}_t} n_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq \tilde{g}_t < disp_{t,2} \\[1ex]
\quad\vdots & \quad\vdots \\[1ex]
\dfrac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2} n_{t,\tilde{H}_t} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda n_{t,\tilde{H}_t})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1} n_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } disp_{t,\tilde{H}_t} \leq \tilde{g}_t < disp_{t,\tilde{H}_t - 1} \\[2ex]
p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}] := p_t^{(\tilde{H}_t)} & \text{if } disp_{t,\tilde{H}_t} > \tilde{g}_t
\end{cases}
$$

where $p_t^{(k^*)}$ *is the price if types* $1, ..., k^*$ *are short-selling constrained,* p_t^* *is the corresponding price if short-selling constraints were absent (which satisfies* $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), *and*

$$
p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{25}
$$

Proof. See Section 6.4 of this appendix. ∎

Given Corollary 1, the computational algorithm needs to be amended as shown below.

3.3.1 Computational algorithm (Market maker and benchmark demands)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \frac{\mu\lambda}{a\sigma^2} \sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1}) - \frac{1}{1+r} f_{t,1}$. If $disp_{t,1} \leq \tilde{g}_t$, accept $p_t = p_t^*$ as the price and move to period $t + 1$. Otherwise, move to Step 3.

3. Compute $disp_{t,\tilde{H}_t} = -\frac{1}{1+r} f_{t,\tilde{H}_t}$. If $disp_{t,\tilde{H}_t} > \tilde{g}_t$, accept $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ as the price and move to period $t + 1$. Else, move to Step 4.

4. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k} + a\sigma^2 \overline{Z} - (1+r)p_t^{guess}}{a\sigma^2} < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq \tilde{g}_t < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a $k^* \in \{1, ..., \tilde{H}_t - 1\}$ is found such that $disp_{t,k^*+1} \leq \tilde{g}_t < disp_{t,k^*}$ and go to step 5.

5. Accept k^* as the number of short-selling constrained types, such that the price is
$$
p_t = p_t^{(k^*)} := \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2} \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1} \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}}, \text{ move to period } t + 1.
$$

3.4 Market maker and heterogeneous slope coefficients

Similar to Westerhoff (2004) we consider demands of the form $\tilde{a}_h(f_{t,h} - p_t)$, where $\tilde{a}_h > 0$ for all h. With a short-selling constraint $z_{t,h} \geq 0$, the demands are adjusted to

$$z_{t,h} = \begin{cases} \tilde{a}_h(\tilde{E}_{t,h}[p_{t+1}] - p_t) & \text{if } p_t \leq \tilde{E}_{t,h}[p_{t+1}] \\ 0 & \text{if } p_t > \tilde{E}_{t,h}[p_{t+1}]. \end{cases} \tag{26}$$

Given price beliefs in Assumption 1, we have $\tilde{E}_{t,h}[p_{t+1}] = \bar{c}p_t + \tilde{f}_{t,h}$, where we now assume $\bar{c} \in [0,1)$ (since the interest rate is zero). We define $f_{t,h} := \tilde{f}_{t,h}$ and write the demands as

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} - (1-\bar{c})p_t) & \text{if } p_t \leq \frac{f_{t,h}}{1-\bar{c}} \\ 0 & \text{if } p_t > \frac{f_{t,h}}{1-\bar{c}}. \end{cases} \tag{27}$$

Since $\bar{c} \in [0,1)$, demands are decreasing in the current price. By (27) and the price equation, $p_t = p_{t-1} + \mu[\lambda Z_t + (1-\lambda)Z_{t-1} - \overline{Z})]$ where $Z_t := \sum_{h \in \mathcal{H}} n_{t,h} z_{t,h}$ is aggregate demand.

Proposition 3 (Proposition 1 (main) adapted to new demand specification) *Let p_t be the price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Further, let $\tilde{n}_{t,h} := n_{t,h}\tilde{a}_h$ and $f_{t,h} := \tilde{f}_{t,h}$. Then the following holds:*

1. If $p_{t-1} - \frac{1}{1-\bar{c}} \min_{h \in \mathcal{H}}\{f_{t,h}\} + \mu\lambda \sum_{h \in \mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq \mu\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$, $z_{t,h} \geq 0\ \forall h$) and the price is

$$p_t = \frac{p_{t-1} + \mu\lambda \sum_{h \in \mathcal{H}} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h \in \mathcal{H}} \tilde{n}_{t,h}}.$$

2. If $p_{t-1} - \frac{1}{1-\bar{c}} \min_{h \in \mathcal{H}}\{f_{t,h}\} + \mu\lambda \sum_{h \in \mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > \mu\overline{Z}$, then one or more types $h \in \mathcal{H}$ are short-selling constrained with $z_{t,h} = 0$ and we have the following:

(i) If $\exists \mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\mu\lambda \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}} \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < \mu\lambda \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}} \max_{h \in \mathcal{S}_t^*}\{f_{t,h}\}$, price is given by

$$p_t = \frac{p_{t-1} + \mu\lambda \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}}$$

and demands are $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\bar{c})p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

(ii) Else, $\exists \mathcal{B}_t^* = \emptyset, \mathcal{S}_t^* = \mathcal{H}$ such that $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1-\bar{c}} \max_{h \in \mathcal{S}_t^*}\{f_{t,h}\} > \mu\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h$), and price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See Section 6.3 of this appendix. ∎

Corollary 2 (amended) *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set such that types are ordered as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. Let $disp_{t,k} := \mu\lambda \sum_{h=k+1}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - f_{t,k}) - \frac{f_{t,k}}{1-\bar{c}}$ and $g(p_{t-1}, Z_{t-1}) := \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, for $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The price solution is:*

$$p_t = \begin{cases} \dfrac{p_{t-1} + \mu\lambda \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq g(p_{t-1}, Z_{t-1}) \\[2ex] \dfrac{p_{t-1} + \mu\lambda \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,1} \\[2ex] \dfrac{p_{t-1} + \mu\lambda \sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,2} \\[1ex] \qquad\qquad\vdots & \qquad\vdots \\[1ex] \dfrac{p_{t-1} + \mu\lambda\tilde{n}_{t,\tilde{H}_t} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } disp_{t,\tilde{H}_t} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,\tilde{H}_t - 1} \\[2ex] p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}] := p_t^{(\tilde{H}_t)} & \text{if } disp_{t,\tilde{H}_t} > g(p_{t-1}, Z_{t-1}) \end{cases}$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{28}$$

Proof. See Section 6.2 of this appendix. ∎

Given Corollary 2, the computational algorithm needs to be amended as shown below.

3.4.1 Computational algorithm (Market maker and heterogeneous slopes)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \mu\lambda \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - f_{t,1}) - \frac{f_{t,1}}{1-\bar{c}}$. If $disp_{t,1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, accept $p_t = p_t^*$ as the price and move to period $t+1$. Otherwise, move to Step 3.

3. Compute $disp_{t,\tilde{H}_t} = -\frac{f_{t,\tilde{H}_t}}{1-\bar{c}}$. If $disp_{t,\tilde{H}_t} > \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, accept $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ as the price and move to period $t+1$. Else, move to Step 4.

4. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \tilde{a}_k(f_{t,k} - (1-\bar{c})p_t^{guess}) < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a $k^* \in \{1, ..., \tilde{H}_t - 1\}$ is found such that $disp_{t,k^*+1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < disp_{t,k^*}$ and go to step 5.

5. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{p_{t-1} + \mu\lambda \sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, and move to period $t+1$.

4 Additional heterogeneity

4.1 Heterogeneous forecast coefficients on p_t

In the case of heterogeneous forecast coefficients on p_t, price beliefs are amended to

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \bar{c}_h p_t + \tilde{f}_{t,h} \tag{29}$$

where $\bar{c}_h \in [0, 1 + \tilde{r})$ for all $h \in \mathcal{H}$.

As discussed below, this case requires an amendment to the computational algorithm because ranking types in terms of $f_{t,h}$ is no longer sufficient. We first provide an amended version of Proposition 1 before discussing the necessary changes to the algorithm.

Defining $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r_h := \tilde{r} - \bar{c}_h$, the demands are amended to:

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r_h)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}. \end{cases} \tag{30}$$

Using the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$, we have the following.

Proposition 4 (Heterogeneous p_t coefficients) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types). Let $f_{t,h}$, r_h be defined as above and $\tilde{n}_{t,h} := n_{t,h}(1+r_h)$. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} \leq 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}} \tag{31}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} > 0$, at least one type is short-selling constrained and $\exists$ unique $\mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \min_{h\in\mathcal{B}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \leq (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \max_{h\in\mathcal{S}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$, and the price is

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}} \tag{32}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. See Section 6.5 of this appendix. ∎

Note that types with the lowest values of $(f_{t,h} + a\sigma^2\overline{Z})/(1 + r_h)$ should be considered least optimistic, as they are more likely to be short-selling constrained at any given price. We thus define $\hat{f}_{t,h} := (f_{t,h} + a\sigma^2\overline{Z})/(1 + r_h)$, which allows us to state the following result.

Corollary 3 *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set of types such that $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} := \frac{f_{t,h} + a\sigma^2 \overline{Z}}{1 + r_h}$. Let $disp_{t,k} := \sum_{h=k}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left(\frac{f_{t,k} + a\sigma^2 \overline{Z}}{1 + r_k} \right) \sum_{h=k}^{\tilde{H}_t} \tilde{n}_{t,h}$ and $\tilde{disp}_{t,k} := disp_{t,k} + n_{t,k} a\sigma^2 \overline{Z}$ for $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := n_{t,h}(1 + r_h)$. The market-clearing price is:*

$$
p_t = \begin{cases}
\dfrac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \le (1 - \sum_{h=1}^{\tilde{H}_t} n_{t,h}) a\sigma^2 \overline{Z} \, (= 0) \\[2ex]
\dfrac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2 \overline{Z}}{\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \le (1 - \sum_{h=2}^{\tilde{H}_t} n_{t,h}) a\sigma^2 \overline{Z} < \tilde{disp}_{t,1} \\[2ex]
\dfrac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} - (n_{t,1} + n_{t,2}) a\sigma^2 \overline{Z}}{\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \le (1 - \sum_{h=3}^{\tilde{H}_t} n_{t,h}) a\sigma^2 \overline{Z} < \tilde{disp}_{t,2} \\[1ex]
\quad\quad\quad \vdots & \quad\quad\quad \vdots \\[1ex]
\dfrac{n_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t - 1} n_{t,h}) a\sigma^2 \overline{Z}}{\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } \tilde{disp}_{t,\tilde{H}_t - 1} > (1 - \sum_{h=\tilde{H}_t - 1}^{\tilde{H}_t} n_{t,h}) a\sigma^2 \overline{Z}
\end{cases}
$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \le k^*$), and

$$
p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{33}
$$

Proof. It follows from Proposition 4 and the proof of Corollary 1 (see main paper). ∎

In light of the changes in Corollary 3, our algorithm needs to be amended as shown below. We stick with the above notation for which $\tilde{n}_{t,h} = n_{t,h}(1 + r_h)$ for all $h \in \{1, ..., \tilde{H}_t\}$.

4.1.1 Computational algorithm (Heterogeneous coefficients on p_t)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} = \frac{f_{t,h} + a\sigma^2 \overline{Z}}{1 + r_h}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left(\frac{f_{t,1} + a\sigma^2 \overline{Z}}{1 + r_1} \right) \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}$. If $disp_{t,1} \le 0$, accept $p_t = p_t^*$ as the date t price solution and move to period $t + 1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k} + a\sigma^2 \overline{Z} - (1 + r_k) p_t^{guess}}{a\sigma^2} < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \le (1 - \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2 \overline{Z} < \tilde{disp}_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \le (1 - \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2 \overline{Z} < \tilde{disp}_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h} \right] a\sigma^2 \overline{Z}}{\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, and move to period $t + 1$.

4.1.2 Numerical example

We now consider a numerical example as in Section 3.4 of the main paper. We allow hetero-geneous coefficients on p_t in the beliefs of each type:

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \bar{c}_h p_t + \tilde{f}_{t,h}$$

where $\bar{c}_h \sim \mathcal{U}(\bar{c}_{min}, \bar{c}_{max})$ and $\mathcal{U}$ denotes the uniform distribution.

The $\bar{c}_h$ parameters are drawn at date 0. We consider 'high' and 'low' heterogeneity and the case of zero heterogeneity (i.e. $\bar{c}_{min} = \bar{c}_{max} = \bar{c}$). The idiosyncratic shocks to beliefs are set at zero in the first 10 periods and are normally distributed with standard deviation 0.04 thereafter.[5] In the high heterogeneity case we set $c_{min} = 0.95, c_{max} = 1.05$; in the low heterogeneity case $c_{min} = 0.995, c_{max} = 1.005$. For zero heterogeneity we set $\bar{c}_{min} = \bar{c}_{max} = 1$.
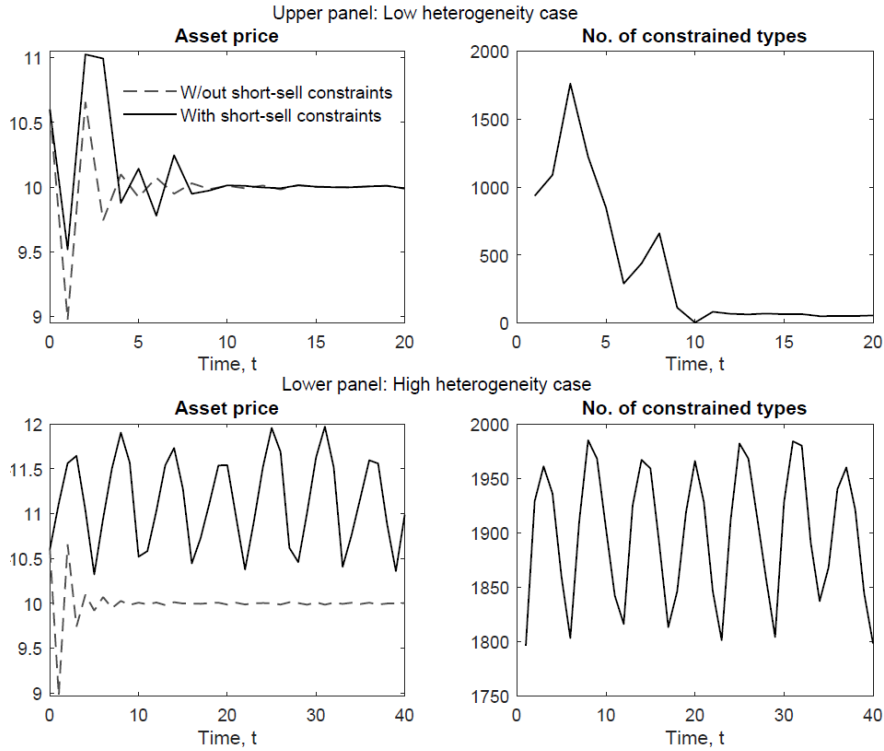

Figure 1: Simulation with low and high heterogeneity in $\bar{c}_h$ ($H = 3,000$ types)

Figure 1 shows the time series for the asset price and number of short-selling constrained types for the cases of high and low heterogeneity. The time series for zero heterogeneity (not shown) are very similar to those in the low heterogeneity case. We see that heterogeneity has a much bigger impact when short-selling constraints are present; intuitively, this is because heterogeneity largely 'washes out' in the no-constraints case where price depends on the

---

[5]To ensure the same sequence of shocks is used we set the random seed at 5 in all simulations.

average belief, whereas if short-selling is banned, additional heterogeneity means constraints bind more often. Table 1 of the main text shows the solutions with short-selling constraints are fast and accurate: the speed and accuracy are comparable to the known solution in the absence of short-selling constraints which does not require the use of any search procedure.

4.2   Heterogeneous subjective variances

In the case of heterogeneous subjective return variances, demands are given by

$$
z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\,[p_{t+1}] + \overline{d} - (1+\tilde{r})p_t}{a\sigma_h^2} & \text{if } p_t \le \dfrac{\tilde{E}_{t,h}\,[p_{t+1}] + \overline{d}}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\,[p_{t+1}] + \overline{d}}{1+\tilde{r}} \end{cases}
\tag{34}
$$

where $\sigma_h^2 > 0$ is the subjective return variance of type h and $\tilde{E}_{t,h}\,[p_{t+1}] = \overline{c}p_t + \tilde{f}_{t,h}$.

Defining $\tilde{a}_h := (a\sigma_h^2)^{-1}$, $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - \overline{Z}/\tilde{a}_h$ and $r := \tilde{r} - \overline{c}$, the demands in (34) are

$$
z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) & \text{if } p_t \le \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r}. \end{cases}
\tag{35}
$$

In (34) and (35), the subjective variances are heterogeneous, but it should be clear that heterogeneity in risk aversion is also nested by this approach. Time variation is also straightforward: add time subscripts. By (35) and market-clearing, we have the following.

Proposition 5 (Heterogeneous subjective variances) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \backslash \mathcal{B}_t$) be the set of unconstrained types (constrained types). Let $\tilde{a}_h = (a\sigma_h^2)^{-1}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$ and $f_{t,h}$ as above. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \le 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$
p_t = \frac{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h} f_{t,h}}{(1+r)\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}}
\tag{36}
$$

with demands $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \ge 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) > 0$, at least one type is short-selling constrained and $\exists$ unique $\mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \le (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z} < \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\})$, and the price is

$$
p_t = \frac{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}}
\tag{37}
$$

with demands $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \ge 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

13

Proof. See Section 6.6 of this appendix. ∎

Note that types with the lowest values of $f_{t,h} + \overline{Z}/\tilde{a}_h$ should be considered least optimistic, as they are more likely to be short-selling constrained at any given price. We thus define $\hat{f}_{t,h} := f_{t,h} + \overline{Z}/\tilde{a}_h$ to rank types by optimism, which allows us to state the following result.

Corollary 4 *Let* $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ *be the set such that types are ordered as* $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, *where* $\hat{f}_{t,h} := f_{t,h} + \overline{Z}/\tilde{a}_h$. *Let* $disp_{t,k} := \sum_{h=k}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - [f_{t,k} + \overline{Z}/\tilde{a}_k])$ *and* $\tilde{disp}_{t,k} := disp_{t,k} + n_{t,k}\overline{Z}$ *for* $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. *The market-clearing price is:*

$$
p_t = \begin{cases}
\dfrac{\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h}}{(1+r)\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq (1 - \sum_{h=1}^{\tilde{H}_t} n_{t,h})\overline{Z} \ (= 0) \\[2ex]
\dfrac{\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - n_{t,1}\overline{Z}}{(1+r)\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq (1 - \sum_{h=2}^{\tilde{H}_t} n_{t,h})\overline{Z} < \tilde{disp}_{t,1} \\[2ex]
\dfrac{\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - (n_{t,1} + n_{t,2})\overline{Z}}{(1+r)\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq (1 - \sum_{h=3}^{\tilde{H}_t} n_{t,h})\overline{Z} < \tilde{disp}_{t,2} \\[2ex]
\quad\quad\vdots & \quad\quad\vdots \\[1ex]
\dfrac{\tilde{n}_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t - 1} n_{t,h})\overline{Z}}{(1+r)\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } \tilde{disp}_{t,\tilde{H}_t - 1} > (1 - \sum_{h=\tilde{H}_t-1}^{\tilde{H}_t} n_{t,h})\overline{Z}
\end{cases}
$$

where $p_t^{(k^*)}$ *is the price if types* $1, ..., k^*$ *are short-selling constrained,* p_t^* *is the corresponding price if short-selling constraints were absent (which satisfies* $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), *and*

$$
p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{38}
$$

Proof. It follows from Proposition 5 and from the same steps used in the proof of Corollary 1 (see main paper) with the necessary alterations being made. ∎

4.2.1 Computational algorithm (Heterogeneous subjective variances)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} = f_{t,h} + \overline{Z}/\tilde{a}_h$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - [f_{t,1} + \overline{Z}/\tilde{a}_1])$. If $disp_{t,1} \leq 0$, accept $p_t = p_t^*$ as the date t price solution and move to period $t + 1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \tilde{a}_k(f_{t,k} + \overline{Z}/\tilde{a}_k - (1+r)p_t^{guess}) < 0$, say $\underline{k}$. Starting at $k = \underline{k}$, check if $disp_{t,k+1} \leq (1 - \sum_{h=k+1}^{\tilde{H}_t} n_{t,h})a\sigma^2 \overline{Z} < \tilde{disp}_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \leq (1 - \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h})a\sigma^2 \overline{Z} < \tilde{disp}_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is
$$
p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right]\overline{Z}}{(1+r)\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}, \text{ and move to period } t + 1.
$$

4.2.2 Numerical example

We again stick with Example 1 in the main paper except that we draw the subjective variances of different types from a uniform distribution, such that $\sigma_h^2 \sim \mathcal{U}(\sigma_{min}^2, \sigma_{max}^2)$. The results in Table 1 show that the solutions with short-selling constraints are fast and accurate, being comparable to those when short-selling constraints are absent. The corresponding time series for the simulations (in the first 20 periods) are plotted in Figure 2.
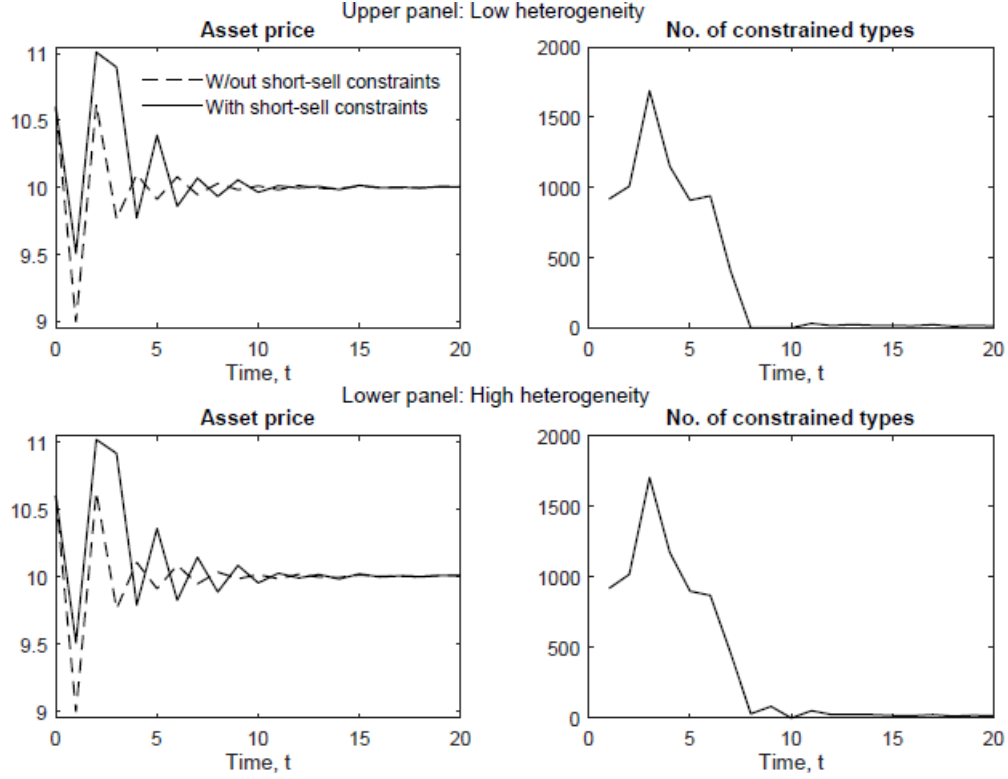

Figure 2: Simulation with low and high heterogeneity in σ_h^2 ($H = 3,000$ types)

Table 1: Computation times and accuracy: $T = 500$ periods, $H = 3,000$ types

Heterogeneity	Short-sale constraints	Time (s)	Bind freq.	max($Error_t$)
High case:	No	0.16	-	9.7e-16
$\sigma_h^2 \in (0.9, 1.1)$	Yes	0.23	499/500	3.4e-15
Low case:	No	0.14	-	8.1e-16
$\sigma_h^2 \in (0.99, 1.01)$	Yes	0.22	497/500	4.0e-15

Note: $\mathrm{max}(Error_t) = \mathrm{max}\{Error_1, ..., Error_T\}$, $Error_t := |\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$.

5   Generalization to population shares $n_{t,h} \in [0,1]$

In this section we relax the assumption $n_{t,h} \in (0,1)$ to allow $n_{t,h} \in [0,1]$ at dates $t \in \mathbb{N}_+$, subject to $\sum_{h \in \mathcal{H}} n_{t,h} = 1$. We show how our analytical results and algorithm can be adjusted for this weaker assumption. The cases $n_{t,h} = 1$ and $n_{t,h} = 0$ can be dealt with together because if $n_{t,h'} = 1$ for some h', then $n_{t,h} = 0$ for all $h \neq h'$;[6] and if $n_{t,h'} = 0$ for some types $h' \in \mathcal{H}$, then the sum of population shares across all other types is equal to 1. Let $\mathcal{H}_t^{sub} \subset \mathcal{H}$ be the subset containing all types with a zero population share at date t, such that $n_{t,h} = 0$ for all $h \in \mathcal{H}_t^{sub}$ and $n_{t,h} \in (0,1]$ for all $h \in \mathcal{H} \setminus \mathcal{H}_t^{sub}$, with $\sum_{\mathcal{H} \setminus \mathcal{H}_t^{sub}} n_{t,h} = 1$. Given that $n_{t,h} = 0$ for all $h \in \mathcal{H}_t^{sub}$, market-clearing at date t is given by

$$\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \sum_{h \in \mathcal{H}_t^*} n_{t,h} z_{t,h} = \overline{Z}, \quad \text{where } \mathcal{H}_t^* := \mathcal{H} \setminus \mathcal{H}_t^{sub} \neq \emptyset, \tag{39}$$

which is analogous to the problem we solve in the main text, except that the set of types under consideration $(\mathcal{H}_t^*)$ is not restricted to be fixed over time.

Since $\mathcal{H}_t^*$ can be found without knowledge of the *current* (or future) market-clearing price p_t, nothing changes except $\mathcal{H}$ is replaced by $\mathcal{H}_t^*$ in Proposition 1. Formally, we have:

Proposition 6 (Proposition 1 adapted for $0 \leq n_{t,h} \leq 1$) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, let $\mathcal{H}_t^*$ be the set of types (with non-zero pop. shares) defined above, let $\mathcal{B}_t \subseteq \mathcal{H}_t^*$ ($\mathcal{S}_t := \mathcal{H}_t^* \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types) at date t, . Then the following holds:*

(i) *If $\sum_{h \in \mathcal{H}_t^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}_t^*}\{f_{t,h}\}) \leq a\sigma^2 \overline{Z}$, then no type is short-selling constrained at date t ($\mathcal{B}_t^* = \mathcal{H}_t^*$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is*

$$p_t = \frac{\sum_{h \in \mathcal{H}_t^*} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{40}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \overline{Z} - (1+r)p_t) \geq 0 \; \forall h \in \mathcal{H}_t^*$.

(ii) *If $\sum_{h \in \mathcal{H}_t^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}_t^*}\{f_{t,h}\}) > a\sigma^2 \overline{Z}$, at least one type is short-selling constrained and $\exists$ unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}_t^*$, $\mathcal{S}_t^*$ such that $\sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2 \overline{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_t^*}\{f_{t,h}\})$, and the price and demands are given by*

$$p_t = \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})a\sigma^2 \overline{Z}}{(1+r)\sum_{h \in \mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{41}$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \overline{Z} - (1+r)p_t) \geq 0 \; \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \; \forall h \in \mathcal{S}_t^*$.

Proof. It follows from Proposition 1 in the main text when the set $\mathcal{H}$ is replaced by $\mathcal{H}_t^*$. ∎

Corollary 1 does not require any amendments since we may re-define the function $\tilde{h}_t$ in the main text as a function from the set $\mathcal{H}_t^*$ to the set $\tilde{\mathcal{H}}_t$, i.e. $\tilde{h}_t : \mathcal{H}_t^* \to \tilde{\mathcal{H}}_t$. Finally, since we keep the set $\tilde{\mathcal{H}}_t$, the computational algorithm also does not require any changes.

---

[6]Recall that $n_{t,h} \in [0,1]$ and $\sum_{h \in \mathcal{H}} n_{t,h} = 1$. Hence, if $n_{t,h'} = 1$ for a type $h' \in \mathcal{H}$, then this implies that $\sum_{h \neq h'} n_{t,h} = 0$. The latter is possible only if $n_{t,h} = 0$ for all $h \neq h'$, since negative shares are ruled out.

6 Proofs

6.1 Proof of Proposition 1

By Section 3.1 the indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K}) \leq 0\}}$ is equal to 1 if the short-selling constraint is in place at date t (i.e. if $g(p_{t-1}, ..., p_{t-K}) \leq 0$) and is 0 otherwise.

Case 1: $\mathbb{1}_t = 1$

If the short-selling constraint is in place at date t (i.e. if $\mathbb{1}_t = 1$), the cases for price and demands are equivalent to those for an unconditional short-selling constraint, as in Proposition 1 in the main paper and its proof. The short-selling constraint is slack for all types if $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2 \overline{Z}$ and binds for one or more types (but fewer than H) otherwise, i.e. if $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) > a\sigma^2 \overline{Z}$.

Case 2: $\mathbb{1}_t = 0$

If the short-selling constraint is not in place at date t (i.e. if $\mathbb{1}_t = 0$) then demands are given by $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \overline{Z} - (1+r)p_t) \in \mathbb{R}$ for all $h \in \mathcal{H}$, and thus the market-clearing condition $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ gives $p_t = p_t^* := \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{1+r}$ (which is the same expression as for $\mathbb{1}_t = 1$, $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2 \overline{Z}$). This conclusion holds regardless of whether $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2 \overline{Z}$ or $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) > a\sigma^2 \overline{Z}$. ∎

6.2 Proof of Corollary 1

The first (last) 'if' statement follows from Proposition 2 main paper as $disp_{t,1} \leq \tilde{g}_t$ ($disp_{t,\tilde{H}_t} > \tilde{g}_t$) is equivalent to $p_{t-1} - \frac{1}{1+r}\min_{h \in \mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq (\mu + (1+r)^{-1}a\sigma^2)\overline{Z}$ (resp. $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1+r}\max_{h \in \mathcal{S}_t^*}\{f_{t,h}\} > \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z}$). The other cases follow as there are $\tilde{H}_t - 1$ other candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, i.e. $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, ..., \tilde{H}_t - 1\}$; $\mathcal{S}_t = \{1, 2\}, \mathcal{B}_t = \{3, ..., \tilde{H}_t - 1\};...\mathcal{S}_t = \{1, ..., \tilde{H}_t\}, \mathcal{B}_t = \emptyset$. For arbitrary sets $\mathcal{S}_t = \{1, ..., k\}$, $B_t = \{k + 1, ..., \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$, we have by Proposition 2 (main paper) that $p_t = p_t^{(k)}$ and the guess is verified if and only if $disp_{t,k+1} \leq \tilde{g}_t < disp_{t,k}$.

It remains to show $p_t^* < p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}$ $\forall k < k^*$. Recall that the demands $z_{t,h}(p_t)$ are decreasing in the price for all h. Note that p_t^* satisfies $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^* - p_t^*$, where $\tilde{Z}_{t-1} := p_{t-1} + \mu(1-\lambda)Z_{t-1}$, $V_t^* := \mu\lambda\sum_{h=1}^{\tilde{H}_t} n_{t,h}z_{t,h}(p_t^*)$, and $z_{t,1} < 0$ since p_t^* is not consistent with short-selling constraints (or else $p_t = p_t^*$). Similarly, $p_t^{(1)}$ satisfies $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^{(1)} - p_t^{(1)}$, where $V_t^{(1)} := \mu\lambda\sum_{h=2}^{\tilde{H}_t} n_{t,h}z_{t,h}(p_t^{(1)})$. Suppose $p_t^{(1)} \leq p_t^*$, which implies that $V_t^{(1)} > V_t^*$. This leads to a contradiction since $V_t^{(1)} - p_t^{(1)} > \mu\overline{Z} - \tilde{Z}_{t-1}$; therefore $p_t^{(1)} > p_t^*$.

For arbitrary $k < k^*$ and $j = k - 1, k$, note that $p_t^{(j)}$ satisfies $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^{(j)} - p_t^{(j)}$, where $V_t^{(j)} := \mu\lambda \sum_{h>j}^{\tilde{H}_t} n_{t,h} z_{t,h}(p_t^{(j)})$. Suppose $p_t^{(k)} \leq p_t^{(k-1)}$. This leads to a contradiction since $\mu\overline{Z} - \tilde{Z}_{t-1} = V_t^{(k-1)} - p_t^{(k-1)} < V_t^{(k)} - p_t^{(k)}$; therefore $p_t^{(k)} > p_t^{(k-1)}$. Finally, $p_t^{(k^*)} > p_t^{(k)}$ $\forall k < k^*$ by applying the above argument for $j = k^* - 1, k^*$. ∎

6.3   Proof of Proposition 3

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} - (1 - \overline{c})p_t) \geq 0$ $\forall h \in \mathcal{H}$, which implies by the price equation that $p_t = \frac{p_{t-1} + \mu\lambda \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}}{1 + \mu\lambda(1-\overline{c}) \sum_{h\in\mathcal{H}} \tilde{n}_{t,h}} := p_t^*$, where $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The guess is verified if and only if $f_{t,h} \geq (1-\overline{c})p_t^*$ $\forall h \in \mathcal{H}$, which requires $(1-\overline{c})^{-1} \left(1 + \mu\lambda(1-\overline{c}) \sum_{h\in\mathcal{H}} \tilde{n}_{t,h}\right) f_{t,h} \geq p_{t-1} + \mu\lambda \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}$ $\forall h \in \mathcal{H}$. Note that this is equivalent to $(1-\overline{c})^{-1} \left(1 + \mu\lambda(1-\overline{c}) \sum_{h\in\mathcal{H}} \tilde{n}_{t,h}\right) \min_{h\in\mathcal{H}}\{f_{t,h}\} \geq p_{t-1} + \mu\lambda \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}$, which simplifies to the inequality in Proposition 3 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} - (1 - \overline{c})p_t) \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0$ $\forall h \in \mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$, such that $p_t = \frac{p_{t-1} + \mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}}{1 + \mu\lambda(1-\overline{c}) \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}}$ by the price equation, with $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The guess is verified iff $f_{t,h} \geq (1 - \overline{c})p_t$ $\forall h \in \mathcal{B}_t^*$ and $f_{t,h} < (1 - \overline{c})p_t$ $\forall h \in \mathcal{S}_t^*$, i.e. $(1-\overline{c})^{-1} \left(1 + \mu\lambda(1-\overline{c}) \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}\right) f_{t,h} - (p_{t-1} + \mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}) \geq 0$ (< 0) $\forall h \in \mathcal{B}_t^*$ $(\forall h \in \mathcal{S}_t^*)$, which are equivalent to $\mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1-\overline{c}} \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \mu[\overline{Z} - (1 - \lambda)Z_{t-1}] - p_{t-1} < \mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1-\overline{c}} \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, as stated in Proposition 3 Part 2(i).

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0$ $\forall h \in \mathcal{H}$, which implies $p_t = p_{t-1} + \mu[(1 - \lambda)Z_{t-1} - \overline{Z}]$ by the price equation. The guess is verified if and only if $f_{t,h} < (1 - \overline{c})p_t$ $\forall h \in \mathcal{H}$, i.e. if and only if $\frac{1}{1-\overline{c}} \max_{h\in\mathcal{H}}\{f_{t,h}\} < p_{t-1} + \mu[(1 - \lambda)Z_{t-1} - \overline{Z}]$, as stated in Proposition 3 Part 2(ii). ∎

6.4   Proof of Corollary 2

The first (last) 'if' statement follows from Proposition 2 main paper as $disp_{t,1} \leq g(p_{t-1}, Z_{t-1})$ $(disp_{t,\tilde{H}_t} > g(p_{t-1}, Z_{t-1}))$ is equivalent to $p_{t-1} + \mu(1-\lambda)Z_{t-1} + \mu\lambda \sum_{h\in\mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) - \frac{1}{1-\overline{c}} \min_{h\in\mathcal{H}}\{f_{t,h}\} \leq \mu\overline{Z}$ (resp. $p_{t-1} + \mu(1 - \lambda)Z_{t-1} - \frac{1}{1-\overline{c}} \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \mu\overline{Z}$). The other cases follow as there are $\tilde{H}_t - 1$ other candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, i.e. $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, ..., \tilde{H}_t - 1\}$;

18

$\mathcal{S}_t = \{1,2\}, \mathcal{B}_t = \{3, ..., \tilde{H}_t - 1\}; ... \mathcal{S}_t = \{1, ..., \tilde{H}_t\}, \mathcal{B}_t = \emptyset$. For arbitrary sets $\mathcal{S}_t = \{1, ..., k\}$, $B_t = \{k+1, ..., \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$ and by Proposition 2 (main paper) $p_t = p_t^{(k)}$ and the guess is verified iff $disp_{t,k+1} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,k}$. The proof that $p_t^* < p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)} \; \forall k < k^*$ follows the proof of Corollary 1 in Section 6.2. ∎

6.5   Proof of Proposition 4

Existence of a unique price follows from Anufriev and Tuinstra (2013, Proposition 2.1) when an appropriate relabelling of variables is used. We define $r_h := \tilde{r} - \bar{c}_h$ and $\tilde{n}_{t,h} := n_{t,h}(1 + r_h)$.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \; \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{\sum_{h \in \mathcal{H}} \tilde{n}_{t,h}}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^* \geq 0 \; \forall h \in \mathcal{H}$, i.e. $\left[\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\right] \sum_{h \in \mathcal{H}} \tilde{n}_{t,h} \geq \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$ $\forall h \in \mathcal{H}$, which simplifies to $\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} - \min_{h \in \mathcal{H}}\{\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\} \sum_{h \in \mathcal{H}} \tilde{n}_{t,h} \leq 0$ as stated in Proposition 4.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \; \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0$ $\forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h \in \mathcal{B}_t^*}\{f_{t,h} - (1+r_h)p_t\} > \max_{h \in \mathcal{S}_t^*}\{f_{t,h} - (1+r_h)p_t\}$. Under the above guess, $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h}$, so market-clearing is $\sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h} = \overline{Z}$, giving $p_t = \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{\sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^{\mathcal{B}_t^*} \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^{\mathcal{B}_t^*} < 0 \; \forall h \in \mathcal{S}_t^*$, i.e. $\left[\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\right] \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h} \geq (<)$ $\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} \; \forall h \in \mathcal{B}_t^* \; (\forall h \in \mathcal{S}_t^*)$, which simplify to $\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h} \min_{h \in \mathcal{B}_t^*}\{\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\} \leq (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h} \max_{h \in \mathcal{S}_t^*}\{\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\}$, which is the inequality given in Proposition 4. ∎

6.6   Proof of Proposition 5

Existence of a unique price follows from Anufriev and Tuinstra (2013, Proposition 2.1) when an appropriate relabelling of variables is used. We define $\tilde{a}_h = (a\sigma_h^2)^{-1}$ and $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0 \; \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h \in \mathcal{H}} \tilde{n}_{t,h} f_{t,h}}{(1+r)\sum_{h \in \mathcal{H}} \tilde{n}_{t,h}}$. The guess is verified if and only if $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^* \geq 0 \; \forall h \in \mathcal{H}$, i.e. $\left[f_{t,h} + \overline{Z}/\tilde{a}_h \right] \sum_{h \in \mathcal{H}} \tilde{n}_{t,h} \geq \sum_{h \in \mathcal{H}} \tilde{n}_{t,h} f_{t,h}$ $\forall h \in \mathcal{H}$. This inequality simplifies to $\sum_{h \in \mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq 0$, as stated in Proposition 5.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0 \; \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0 \; \forall h \in \mathcal{H} \backslash \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h \in \mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\} > \max_{h \in \mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}$. Under the above guess, $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h}$, so market-clearing is $\sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h} = \overline{Z}$, giving $p_t = \frac{\sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})\overline{Z}}{(1+r)\sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^{\mathcal{B}_t^*} \geq 0 \; \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^{\mathcal{B}_t^*} < 0$ $\forall h \in \mathcal{S}_t^*$, i.e. $\left[f_{t,h} + \overline{Z}/\tilde{a}_h \right] \sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h} \geq (<) \sum_{h \in \mathcal{H}} \tilde{n}_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})\overline{Z} \; \forall h \in \mathcal{B}_t^*$ $(\forall h \in \mathcal{S}_t^*)$, which simplify to $\sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h})\overline{Z} <$ $\sum_{h \in \mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\})$, which is the inequality given in Proposition 5. $\blacksquare$

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