Solving heterogeneous-belief asset pricing models with short-selling constraints and many agents

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Abstract

Short-selling constraints are common in financial markets, while physical assets such as housing often lack markets for short-selling altogether. As a result, investment decisions are often restricted by such constraints. This paper studies asset prices in behavioural heterogeneous-belief models with short-selling constraints. We provide analytical expressions for equilibrium price and demands in a market with an arbitrarily large number of belief types, along with computationally-efficient solution algorithms. Several extensions of the benchmark model are considered, including multiple assets, trading costs for selling short, and the market-maker approach. An application studies an alternative uptick rule, as in the United States, in a market with many belief types.

Keywords: Asset pricing, heterogeneous beliefs, short-selling constraints, updating.

JEL-Classification: C63, D84, G12, G18, G40.

1 Introduction

The practice of short selling is common in financial markets but also widely regulated. When investors go short, they borrow and immediately sell a financial asset before repurchasing and returning the asset to the lender, closing their position. Whereas a long position can be thought of as a bet that asset prices will increase, short-selling allows investors to bet on a fall in asset prices. It has been argued that such betting may increase volatility in financial markets. A common policy response among regulators has been to restrict short-selling; for example, during the 2008-9 financial crisis many countries introduced short-selling bans

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following sharp declines in asset prices. Similar short-selling bans were reinstated in some European economies during the 2011-12 sovereign debt crisis and the Covid-19 outbreak (see Siciliano and Ventoruzzo, 2020). It is therefore important that researchers be able to solve asset pricing models with short-selling constraints in an efficient manner.

In this paper, we show how to solve behavioural asset pricing models with short-selling constraints and arbitrarily many heterogeneous beliefs. For the case of such dynamic models (see Brock and Hommes, 1998; Hommes, 2021), we derive expressions for the equilibrium price and demands and show how these results can be used to speed up computational algorithms. Our results allow researchers to incorporate short-selling constraints in models with many agents or belief types – as in real-world asset markets – thereby facilitating work on the effectiveness of such regulations. Hence, the paper is essentially a methodological contribution to the literature on heterogeneous-belief asset pricing models (HAMs).

We provide results for a benchmark asset pricing model, as well as several generalizations studied in the literature: the case of multiple risky assets; the market-maker approach to price determination; and costly rather than prohibited short-selling. We also show that, with minor amendments, the benchmark results extend to many individual investors who update using social networks (or other idiosyncratic factors) and the case of certain physical investment assets like housing which are also subject to short-selling constraints.

Our main analysis is built around the Brock and Hommes (1998) asset pricing model. Differently from the bulk of literature, we restrict short-selling of the risky asset, such that the equilibrium price function is piecewise-linear, with the solution depending on how many types are short-selling constrained. For a market with a large number of investor types it is computationally intensive to solve for an equilibrium price and demands; however, our analytical results enable computation to be streamlined by using the price under unconstrained short-selling in conjunction with an adjusted set of types in which types are indexed by their level of optimism. As a result, it becomes computationally feasible to simulate models with large numbers of heterogeneous beliefs while retaining solution accuracy.

We provide several analytical examples to help build intuition and present an application that studies an alternative uptick rule, as currently in place in the United States. The alternative uptick rule is a 'circuit breaker' that bans short-selling if prices fall 10% or more in the previous trading period; surprisingly, there do not appear to be any assessments of this rule in the HAMs literature, and we therefore study such a rule in a market with a large number of heterogeneous belief types. The results indicate that such regulations can attenuate falls in prices and may reduce price volatility; however, we also find that such constraints can hinder price discovery and cause price fluctuations that would otherwise be absent, including explosive price paths. In addition, we find that an alternative uptick rule can have substantive distributional (wealth) implications.

The closest papers in the literature are Anufriev and Tuinstra (2013) and Dercole and Radi (2020). Anufriev and Tuinstra (2013) add trading costs for short-selling into an asset pricing model with two types and find that this affects the global dynamics and leads to additional (non-fundamental) steady states as beliefs are updated more aggressively; in a similar vein, but with the addition of a leverage constraint, see in't Veld (2016).[1] By comparison, Dercole and Radi (2020) study the original 'uptick rule' in the United States from 1938 until 2007, which banned short-selling of shares at lower prices, and find that there is no clear-cut impact on price volatility. There is also a wider literature on non-smooth financial-market models (see e.g. Tramontana et al., 2010); short-selling constraints can be thought of as a specific application that gives rise to such models.

All the above papers consider a small number of investor types and solve for prices and demands in specific cases. The present paper contributes to the literature by solving for asset prices and demands when there are arbitrarily many belief types with general price predictors, and by constructing computationally efficient algorithms for numerical analysis. We make our results accessible by considering several generalizations and extensions of the benchmark model, such as costly rather than prohibited short-selling, multiple risky assets, the market-maker approach, and individual investors with different beliefs.

Our paper is part of a growing literature studying heterogeneous beliefs, asset prices and the effectiveness of regulatory policies in financial markets (Westerhoff, 2016). In financial market models it is known that differences in beliefs combined with short-selling constraints can lead to price bubbles (see e.g. Scheinkman and Xiong, 2003), but such regulations could also aid market stability as noted above. There has also been interest in the impact of short-selling restrictions in markets for *physical* investment assets like housing which are subject to boom and bust (see Shiller, 2015; Fabozzi et al., 2020). The results in this paper are thus of potential relevance for physical as well as financial assets, and we explain how our findings may be applied in this direction using housing as an example.

The paper proceeds as follows. Section 2 presents a baseline model for which analytical results are presented in Section 3. Section 4 presents three extensions of the baseline model, and Section 5 presents our numerical application. Finally, Section 6 concludes.

2 Model

Consider a finite set of myopic, risk-averse investor types $\mathcal{H} = \{h_1, ..., h_H\}$. At each date $t \in \mathbb{N}$, each type $h \in \mathcal{H}$ chooses a portfolio of a risky asset $z_{t,h}$ and a riskless bond paying $r > 0$ to maximize a mean-variance utility function over future wealth with risk-aversion parameter $a > 0$. The risky asset has current price p_t, future price p_{t+1}, and pays stochastic dividends d_{t+1}; hence future wealth is $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1 + r)(w_{t,h} - p_t z_{t,h})$. We denote the subjective expectation of type h at date t by $\tilde{E}_{t,h}[.]$. Investors form subjective expectations of the future price and future dividends of the risky asset as described below. The basic model follows Brock and Hommes (1998), except that the risky asset is in positive net supply $\overline{Z} > 0$ and short-selling is ruled out at all dates, such that $z_{t,h} \geq 0$ for all t and h.

[1] Anufriev and Tuinstra (2013) do present an algorithm for finding the equilibrium price and demands in an Appendix, but they do not present any analytical results for an arbitrary number of types. By contrast, we use analytical results to speed up our algorithm.

2.1 Asset demand

Given short-selling constraints, the date t demand of each investor type $h \in \mathcal{H}$ is:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]-(1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+r} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+r}. \end{cases} \quad (1)$$

If the price p_t is small enough, then type h's short-selling constraint is slack and their demand for the risky asset decreases with the price and the conditional variance of the return σ^2 (assumed fixed and given) times the risk-aversion parameter a; this is the standard demand function in Brock and Hommes (1998) where short-selling constraints are absent. However, if price is high enough to make the expected excess return of type h negative, then the short-selling constraint will bind on type h, and their position in risky asset is zero.

Dividends follow an IID process $d_t = \overline{d} + \epsilon_t$, where $\overline{d} \geq 0$ and ϵ_t is a zero-mean shock with constant variance. We assume that all investor types know the dividend process, such that $\tilde{E}_{t,h}\left[d_{t+1}\right] = \overline{d}$ for all t and h, leaving us with heterogeneity in price beliefs $\tilde{E}_{t,h}\left[p_{t+1}\right]$ only. Note from Equation (1) that, other things being equal, the short-selling constraint is more likely to bind on type h the more pessimistic their price expectation at date t.

2.2 Price beliefs

To keep our results as general as possible, we do not specify particular price beliefs in the analysis that follows. We assume only that price beliefs are boundedly-rational and have the same form as in Brock and Hommes (1998), as summarized in Assumption 1.

Assumption 1 *All price beliefs are of the form:*

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = E_t[p^*_{t+1}] + f_h(x_{t-1}, ..., x_{t-L})$$

where p^*_t is the fundamental price, E_t is the conditional expectations operator, $x_t := p_t - p^*_t$ is the deviation of price from the fundamental price, and $f_h : \mathbb{R}^L \to \mathbb{R}$ is a deterministic function that can differ across investor types h.

Note that the above specification restricts beliefs to deterministic functions of past deviations from the fundamental price. Nevertheless, this specification is quite general as it includes general trend-chasing rules, price beliefs that may be 'biased' upwards or downwards relative to the fundamental price, and contrarian beliefs that go 'against the grain' of other belief types. Given our assumption that dividends are IID, the fundamental price is constant at $p^*_t = \overline{p} := \frac{\overline{d}-a\sigma^2\overline{Z}}{r}$, and hence $E_t[p^*_{t+1}] = \overline{p}$. Notice that the fundamental price $\overline{p}$ is lower than in Brock and Hommes (1998) since the risky asset is in positive net supply, $\overline{Z} > 0$.[2]

[2]The fundamental price can be derived as in Brock and Hommes (1998) – i.e. by solving for the equilibrium price when all in investors are fundamentalists with rational expectations and speculative bubbles are absent.

By Assumption 1, beliefs in terms of deviations from the fundamental price are

$$\tilde{E}_{t,h}\left[x_{t+1}\right] = f_h(x_{t-1}, ..., x_{t-L}) \tag{2}$$

where $\tilde{E}_{t,h}\left[x_{t+1}\right] := \tilde{E}_{t,h}\left[p_{t+1}\right] - E_t[p^*_{t+1}]$.

Thus, the demands in (1) can be written in terms of price deviations as

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}]-(1+r)x_t+a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}]+a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}]+a\sigma^2\overline{Z}}{1+r}. \end{cases} \tag{3}$$

2.3 Population shares

The aggregate demand for the risky asset is $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h}$, where $n_{t,h} \in (0,1)$ is the population share of type h at date t and $\sum_{h\in\mathcal{H}} n_{t,h} = 1$. Therefore, before discussing asset market equilibrium, we need to explain how the population shares are determined. We allow the population shares $n_{t,h}$ to be potentially time-varying, but we rule out dependence on the contemporaneous price x_t (or p_t). Following Brock and Hommes (1997), we assume the population shares are given by a discrete choice model, as summarized in Assumption 2.

Assumption 2 *For all $t \geq 0$, the population shares of each type are updated using a discrete choice logistic model as follows:*

$$n_{t+1,h} = \frac{exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}} exp(\beta U_{t,h})} \tag{4}$$

where $\beta \geq 0$ is the intensity of choice and $U_{t,h} \in \mathbb{R}$ is the fitness of predictor h at date t.

Assumption 2 states that the fraction of the population using predictor h at date $t+1$ depends on the relative performance of predictor h against all other predictors, as judged by past observed levels of fitness $U_{t,h}$. The intensity of choice parameter β determines how fast agents switch to better-performing predictors. In the special case $\beta = 0$, the population shares of each type are fixed and equal to $1/H$, whereas for $\beta \to \infty$ all investors will adopt in period $t+1$ the best-performing predictor(s) in period t. In order to preserve generality, we do not specify a particular functional form for the fitness $U_{t,h}$.

A popular choice in the literature is to make fitness a linear function of realized profits (i.e. realized excess returns, scaled by asset holdings) net of some predictor cost, with past fitness potentially entering via a memory parameter (see Brock and Hommes, 1998). Alternative specifications allow fitness to depend partly (or entirely) on forecast accuracy (Ap Gwilym, 2010) or take into account social factors influencing belief adoption, as in Chang (2007) where discrete-choice social interactions (see Brock and Durlauf, 2001) are added in the Brock and Hommes (1998) asset pricing model. We choose to keep the fitness function general as in (4) to make clear that our results also apply to these cases.

3 Benchmark results

The asset market is in equilibrium when the aggregate demand per investor equals the fixed outside supply of the risky asset, i.e. when $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ subject to (3),(4).

Given positive outside supply of the risky asset $\overline{Z} > 0$, there exists a unique price x_t that satisfies this equation (see Anufriev and Tuinstra, 2013, Proposition 2.1). We now characterize the market-clearing price and demands in this setting (see Proposition 1).

Proposition 1 *Let $x_t = p_t - \overline{p}$ be the equilibrium price at date t and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H}\backslash\mathcal{B}_t$) be the set of unconstrained types (constrained types) at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\}) \leq a\sigma^2\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $S_t^* = \emptyset$) and the equilibrium price is

$$x_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_h(x_{t-1}, ..., x_{t-L})}{1+r} := x_t^* \tag{5}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\}) > a\sigma^2\overline{Z}$, then at least one type is short-selling constrained and there exist unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}$, $\mathcal{S}_t^*$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{B}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\})$, and the associated equilibrium price and demands are

$$x_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > x_t^* \tag{6}$$

and $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*,\ z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. See the Appendix. ■

Proposition 1 gives the equilibrium price and demands in a market with an arbitrarily large (but finite) set of investor types subject to short-selling constraints. Part (i) gives a simple condition that can be used to check, in a single computation, whether short-selling constraints are slack for all types. If so, the price is given by x_t^* in (5), which is the usual expression in the Brock and Hommes (1998) model in the absence of short-selling constraints.

Whether short-selling constraints bind depends on average belief dispersion relative to the most pessimistic type, i.e. $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\})$. If belief dispersion is small enough relative to the (risk-adjusted) outside supply of the asset $\overline{Z}$, then no types are short-selling constrained at date t. Otherwise, we are in Part (ii) of Proposition 1, such that at least one type (and at most $H-1$ types) are short-selling constrained. In this case, the sets of unconstrained and short-selling constrained types ($\mathcal{B}_t^*$, $\mathcal{S}_t^*$) are determined by 'cut-off' conditions which require that for unconstrained types $h \in \mathcal{B}_t^*$, the average belief dispersion *within the group* is sufficiently small relative to outside supply, whereas for the short-selling constrained types $h \in \mathcal{S}_t^*$ this condition of sufficiently small belief dispersion is not met for any type in the set. We now illustrate these results using a simple two-type example.

Example 1 *Consider two types h_1 and h_2 with population shares n_{t,h_1}, $n_{t,h_2} = 1 - n_{t,h_1}$. By Proposition 1, if $\sum_{h\in\{h_1,h_2\}} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] - \min\{\tilde{E}_{t,h_1}[x_{t+1}], \tilde{E}_{t,h_2}[x_{t+1}]\} \leq a\sigma^2\overline{Z}$ neither type is short-selling constrained, and $x_t = (1+r)^{-1}\sum_{h\in\{h_1,h_2\}} n_{t,h}\tilde{E}_{t,h}[x_{t+1}]$ by (5). If the above condition is not met, then either $\tilde{E}_{t,h_1}[x_{t+1}] - \tilde{E}_{t,h_2}[x_{t+1}] > a\sigma^2\overline{Z}/n_{t,h_1}$ (if h_1 is more optimistic) or $\tilde{E}_{t,h_2}[x_{t+1}] - \tilde{E}_{t,h_1}[x_{t+1}] > a\sigma^2\overline{Z}/n_{t,h_2}$ (if h_2 is more optimistic). In the former case, $\mathcal{B}_t^* = \{h_1\}$, $\mathcal{S}_t^* = \{h_2\}$, and by (6) the equilibrium price is $x_t = [(1+r)n_{t,h_1}]^{-1}(n_{t,h_1}\tilde{E}_{t,h_1}[x_{t+1}] - (1-n_{t,h_1})a\sigma^2\overline{Z})$, with demands $z_{t,h_1} = \overline{Z}/n_{t,h_1}$, $z_{t,h_2} = 0$. In the latter case, $\mathcal{B}_t^* = \{h_2\}$, $\mathcal{S}_t^* = \{h_1\}$, so $x_t = [(1+r)n_{t,h_2}]^{-1}(n_{t,h_2}\tilde{E}_{t,h_2}[x_{t+1}] - (1-n_{t,h_2})a\sigma^2\overline{Z})$, along with $z_{t,h_1} = 0$ and $z_{t,h_2} = \overline{Z}/n_{t,h_2}$. Therefore, if type h_1 is a fundamentalist with $\tilde{E}_{t,h_1}[x_{t+1}] = 0$ and h_2 is a chartist with $\tilde{E}_{t,h_2}[x_{t+1}] = \overline{g}x_{t-1}$, and $x_{t-1}, \overline{g} > 0$, we have*

$$x_t = \begin{cases} (1+r)^{-1} n_{t,h_2}\overline{g}x_{t-1} & \text{if } \overline{g}x_{t-1} \leq a\sigma^2\overline{Z}/n_{t,h_2} \\ [n_{t,h_2}(1+r)]^{-1}(n_{t,h_2}\overline{g}x_{t-1} - (1-n_{t,h_2})a\sigma^2\overline{Z}) & \text{if } \overline{g}x_{t-1} > a\sigma^2\overline{Z}/n_{t,h_2}. \end{cases}$$

The above two-type example is especially simple: if belief dispersion is large enough that some type is constrained, then ranking types by optimism immediately determines the sets of unconstrained types ($\mathcal{B}_t^*$) and short-selling constrained types ($\mathcal{S}_t^*$). In a general setting with many types, however, there are many candidates for the sets $\mathcal{B}_t^*$, $\mathcal{S}_t^*$, and this number increases exponentially as the number of types H is increased. In fact, including the case where short-selling constraints are slack for all types, there are $2^H - 1$ different candidates for $\mathcal{B}_t^*$, $\mathcal{S}_t^*$.[3] As a result, the task of finding an equilibrium is computationally intensive when there are a large number of types H, as seems plausible in many real-world asset markets.

To overcome this problem, we now set out a version of the solution in Proposition 1 that reduces the number of candidates that need to be checked and hence is useful for computational purposes. To do so, we use the fact that types who are short-selling constrained in a given period t must have more pessimistic expectations than those agents who were unconstrained (see (3)), such that ranking types in terms of optimism is useful. In fact, we have already seen the usefulness of ranking type by optimism in Example 1, where the information that the chartist type was more optimistic allowed us to narrow down to 2 cases for the equilibrium price rather than 3 ($= 2^2 - 1$) if beliefs were left unordered. We now show how this principle can be applied in a general setting with many types.

To order types in terms of optimism, we consider the function $\tilde{h}_t : \mathcal{H} \to \tilde{\mathcal{H}}_t$, where $\tilde{\mathcal{H}}_t := \{1, ..., \tilde{H}_t\}$ is an adjusted set of types with the property that the most optimistic type(s) in $\mathcal{H}$ get label $\tilde{H}_t$, the next most optimistic type(s) gets label $\tilde{H}_t - 1$, and so on,

[3] The number of candidate sets corresponds to the number of members of the power set of $\mathcal{H}$ minus 1. Intuitively, the power set of $\mathcal{H}$ is the set of all subsets of $\mathcal{H}$, including the empty set. The 'minus 1' correction arises because the asset market cannot clear if $\mathcal{B}_t^*$ were an empty set (i.e. if no agent held the asset).

down to the least optimistic type(s) in $\mathcal{H}$ with label 1. Note that types with equal optimism get the *same* label, so $\tilde{H}_t \leq H$, which implies that $|\tilde{\mathcal{H}}_t| \leq |\mathcal{H}|$. In the case of ties, the period t population share of the 'group' is the sum of the population shares of the individual types.

We first present a corollary based on the ordered types $h \in \tilde{\mathcal{H}}_t$ and their associated population shares $n_{t,h}$, before presenting a computationally-efficient algorithm for finding an equilibrium price and demands in a market with many belief types.

Corollary 1 *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set defined above, such that beliefs are ordered as $\tilde{E}_{t,1}[x_{t+1}] < \tilde{E}_{t,2}[x_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[x_{t+1}]$. Let $disp_{t,k} := \sum_{h>k}^{\tilde{H}_t} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \tilde{E}_{t,k}[x_{t+1}])$ be a measure of belief dispersion, where $k \in \{1, ..., \tilde{H}_t - 1\}$. Then the equilibrium price is:*

$$x_t = \begin{cases} \frac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_h(x_{t-1},...,x_{t-L})}{1+r} := x_t^* & \text{if } disp_{t,1} \leq a\sigma^2\overline{Z} \\ \frac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_h(x_{t-1},...,x_{t-L}) - n_{t,1}a\sigma^2\overline{Z}}{(1-n_{t,1})(1+r)} := x_t^{(1)} & \text{if } disp_{t,2} \leq a\sigma^2\overline{Z} < disp_{t,1} \\ \frac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_h(x_{t-1},...,x_{t-L}) - (n_{t,1}+n_{t,2})a\sigma^2\overline{Z}}{(1-n_{t,1}-n_{t,2})(1+r)} := x_t^{(2)} & \text{if } disp_{t,3} \leq a\sigma^2\overline{Z} < disp_{t,2} \\ \vdots & \vdots \\ \frac{n_{t,\tilde{H}_t} f_{\tilde{H}_t}(x_{t-1},...,x_{t-L}) - (\sum_{h=1}^{\tilde{H}_t-1} n_{t,h})a\sigma^2\overline{Z}}{(1-\sum_{h=1}^{\tilde{H}_t-1} n_{t,h})(1+r)} := x_t^{(\tilde{H}_t-1)} & \text{if } disp_{t,\tilde{H}_t-1} > a\sigma^2\overline{Z} \end{cases} \tag{7}$$

where $x_t^{(k)} > x_t^*$ is the price when types $1, ..., k$ are short-selling constrained.

Proof. See the Appendix. ■

Corollary 1 streamlines the task of finding the equilibrium price considerably. In Proposition 1, where beliefs are unordered, there are $2^H - 1$ cases (regions) to check, as compared to only $\tilde{H}_t \leq H$ when belief types are ordered as in Corollary 1. Clearly, this amounts to a substantial reduction in computational burden in models with a large number of types H.

Example 2 *Suppose there are 10 types: a biased fundamentalist (h_1), a pure fundamentalist (h_2), and 8 chartists (h_i, $i = 3, ..., 10$). The beliefs of the types are: $\tilde{E}_{t,h_1}[x_{t+1}] = \overline{b} < 0$, $\tilde{E}_{t,h_2}[x_{t+1}] = 0$, and $\tilde{E}_{t,h_i}[x_{t+1}] = g_i x_{t-1}$, where $g_i > g_j$ for all $i > j$ and $g_i > 0$ for all $i \in \{3, ..., 10\}$. We also assume that $x_{t-1} > 0$. If we do not rank types according to optimism, then by Proposition 1 there are $2^{10} - 1 = 1023$ different candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, each implying a different expression for the price at date t. However, if we rank types from least optimistic to most optimistic as $\tilde{E}_{t,h_1}[x_{t+1}] < \tilde{E}_{t,h_2}[x_{t+1}] < \tilde{E}_{t,h_3}[x_{t+1}] ... < \tilde{E}_{t,h_{10}}[x_{t+1}]$ and construct the set $\tilde{\mathcal{H}}_t = \{1, 2, ..., 10\}$, where 1 is type h_1, 2 is type h_2, and 3,...,10 are types $h_3, ..., h_{10}$, then there are only 10 candidates for the sets of constrained and unconstrained types and 10 candidate expressions for the equilibrium price, corresponding to Corollary 1 when $\tilde{H}_t = 10$.*

We now present a computational algorithm that can be used to efficiently compute the equilibrium asset price and demands using the approach in Corollary 1.

3.1 Computational algorithm

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering beliefs as $\tilde{E}_{t,1}[x_{t+1}] < \tilde{E}_{t,2}[x_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[x_{t+1}]$ and find the associated population shares $n_{t,h}$ of types $h = 1, 2, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=2}^{\tilde{H}_t} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \tilde{E}_{t,1}[x_{t+1}])$. If $disp_{t,1} \leq a\sigma^2\overline{Z}$, accept $x_t = x_t^*$ as the date t solution and move to period $t+1$. Otherwise, move to Step 3.

3. Set $x_t^{guess} = x_t^*$ and find the largest k such that $z_{t,k}^{guess} = \frac{\tilde{E}_{t,k}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t^{guess}}{a\sigma^2} < 0$, and denote this value $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \leq a\sigma^2\overline{Z} < disp_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the equilibrium price is $x_t = x_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) - \left[\sum_{h=1}^{k^*} n_{t,h}\right] a\sigma^2\overline{Z}}{(1 - \sum_{h=1}^{k^*} n_{t,h})(1+r)}$, and move to period $t+1$.

The above algorithm is efficient for two reasons. First, if the condition in Step 2 is met, no computation time is wasted checking cases where the short-selling constraint is binding. Second, if that condition is not met, then by using the unconstrained solution x_t^* as a guess we get a lower bound $\underline{k}$ on the number of types k^* who will be short-selling constrained in equilibrium, and hence all cases $k < \underline{k}$ need not be checked. Note that $\underline{k}$ is a lower bound for k^* since $x_t^{(k)} > x_t^*$ for all k (see Corollary 1); that is, if types $1, ..., k$ are short-selling constrained at price x_t^* then they must be short-selling constrained at price $x_t^{(k)}$ also. Because some types who are not short-selling constrained at price x_t^* may be short-selling constrained at price $x_t^{(k)}$, we have that $k^* \geq \underline{k}$. Note that this approach of using the unconstrained solution to improve computational efficiency is used, for example, in the OccBin Toolkit of Guerrieri and Iacoviello (2015) that solves log-linearized rational expectations models.

In practice, we have found that the speed of the computational algorithm may be improved in various ways. For example, rather than increasing k in steps of 1 from the initial value $\underline{k}$ (when $\underline{k}$ is not a solution), the algorithm can typically 'jump' closer to the final k^* by repeatedly replacing x_t^{guess} with $x_t^{(k)}$ (i.e. price based on the current guess) in Step 3 and recomputing an updated value of k, say $k = k'$, such that $z_{t,k}^{guess}$ is the largest value of k for which $z_{t,k}^{guess} < 0$. Our simulations suggest that with a large number of types such as several thousand or more, there is a considerable speed-up with 5-10 iterations of this procedure.[4] Moreover, our algorithm also has an option to work backwards from the maximum possible k (i.e. $\tilde{H}_t - 1$) in Step 3. Typically, this option is not needed, but it can be useful in models with large numbers of types and a small number of types with extreme beliefs, since in periods where the extreme belief types are the most optimistic, they may be the only ones in the market who are not constrained short-sellers with positions of zero.

[4]Note that this procedure will not overshoot k^* because the guessed price will remain below the equilibrium price. If $k' = k_{prev}$ or if k^* is reached, then the iterations are terminated early using a 'break' command.

3.2 Generalizations and nested cases

In this section we note some cases which are nested by the above results or require only minor amendments to the model. Any formal results appear in the *Supplementary Appendix*.[5]

First, the results extend straightforwardly to short-selling constraints which are 'circuit breakers' that are triggered only if price falls meet some condition. For example, for an uptick rule (as in the U.S. from 1938-2007), one or more types will be short-selling constrained only if belief dispersion is large enough (see Proposition 1) *and* price fell last period ($x_{t-1} < x_{t-2}$); see the *Supplementary Appendix* for a formal statement. It is thus straightforward to amend the Computational Algorithm for rules with price conditions. Using this approach, we study an alternative uptick rule (as currently in place in the U.S.) in Section 5.

Second, note that the types $h_1, h_2, ..., h_H$ may be interpreted as individual investors with population shares $n_{t,h} = 1/H$. Clearly, the analytical expressions in Proposition 1 and Corollary 1 apply here since population shares enter in the general form $n_{t,h}$. An application is *individual investors* who update from *social networks* as in Hatcher and Hellmann (2022). In this case, the beliefs of *investor* h, $\tilde{E}_{t,h}[x_{t+1}]$, depend in part on the past performance of other investors observed via an exogenously given social network. An alternative approach to social networks is taken in Panchenko et al. (2013), where updating follows the Brock and Hommes (1998) model except that only the types (and performance) of investors in an agent's local network can be observed and adopted in type updating. This case is also nested by the benchmark results, though the determination of population shares changes slightly relative to (4) (see Panchenko et al., 2013, Eq. 10). In both cases, our results are applicable because the demand schedules in the above papers have the same functional form as in (3); only determination of the beliefs $\tilde{E}_{t,h}[x_{t+1}]$ changes.

Third, there has been some interest in the inability to short housing as a possible explanation for rising house prices and market volatility (see Shiller, 2015; Fabozzi et al., 2020). One simple approach to modelling housing as an investment asset is to replace dividends with either exogenous imputed rents (see Bolt et al., 2019) or marginal utility from housing (see Hatcher, 2021). In these cases, the analytics are essentially unchanged since we just need a re-labelling of variables. For interested readers, we provide some technical details and re-worked versions of Proposition 1 in the *Supplementary Appendix*.

Finally, note that price beliefs different to Assumption 1 can be handled provided they do not depend on the current price, i.e. p_t or x_t. For example, if a type has beliefs that do not depend on the (expected) fundamental price, such as $\tilde{E}_{t,h}[p_{t+1}] = p_{t-1} + g(p_{t-1} - p_{t-2})$ (Gaunersdorfer and Hommes, 2007), then we may write this in terms of price deviations as $\tilde{E}_{t,h}[x_{t+1}] = x_{t-1} + g(x_{t-1} - x_{t-2})$ since $x_t = p_t - \bar{p}$. Note that it makes little difference whether the model is written in terms of the raw price p_t or the price deviation x_t, since we can switch between these cases with an appropriate re-labelling of variables. This observation may be useful, for example, in the context of agent-based models in which agents have price expectations that do not depend on a fundamental benchmark.

[5] The Supplementary Appendix is available at: https://github.com/MCHatcher.

4 Extensions

We now present results for several extensions of the baseline model above, including the case of multiple risky assets; trading costs for selling short; and the case where the price is determined by a market-maker who adjusts price in response to excess demand.

4.1 Multiple asset markets

Suppose there are $M \geq 2$ risky assets in fixed supply. Let $z^m_{t,h}$ be the date t demand of type h for asset $m \in \{1, ..., M\}$. We consider two specifications. In the first, demand for each asset depends on the extent of *endogenous participation* in each market as determined by its attractiveness compared to all other markets. In the second specification, demands are determined by a mean-variance problem with multiple risky assets, such that the demand of type h for a given asset may depend on their demands for all other risky assets.

4.1.1 Endogenous market participation

Following Westerhoff (2004), suppose that type h's demand for asset m depends not only on the (risk-adjusted) expected excess return on the asset, but also on its attractiveness. In particular, suppose a fraction w^m_t of each investor type participates in a given market m, with this fraction determined by the relative attractiveness of the market as compared to all other markets (see below). We also assume, differently from Westerhoff, that all M asset markets have short-selling constraints. Each asset market is assumed to have IID dividends $d^m_t = \overline{d}^m + \epsilon^m_t$, such that the expected dividend in a given market m is $\tilde{E}_{t,h}[d^m_{t+1}] = \overline{d}^m \geq 0$.

Then, analogous to (1), the demand of type $h \in \mathcal{H}$ in market $m \in \{1, ..., M\}$ is

$$z^m_{t,h} = \begin{cases} w^m_t \left(\frac{\tilde{E}_{t,h}\left[p^m_{t+1}\right] + \overline{d}^m - (1+r)p^m_t}{a\sigma^2_m} \right) & \text{if } p^m_t \leq \frac{\tilde{E}_{t,h}\left[p^m_{t+1}\right] + \overline{d}^m}{1+r} \\ 0 & \text{if } p^m_t > \frac{\tilde{E}_{t,h}\left[p^m_{t+1}\right] + \overline{d}^m}{1+r} \end{cases} \tag{8}$$

where p^m_t is the price and σ^2_m is the conditional return variance (assumed constant).

Note the demand function (8) has the same form as in the benchmark case (see (1)), except for the scaling by the share w^m_t that participates in the market. As in Westerhoff (2004), we assume that the attractiveness of a market A^m_t depends on the deviation of the market price from the fundamental price and that the shares w^m_t depend on attractiveness, analogous to the determination of population shares based on fitness (see (4)):

$$A^m_t = f([p^m_t - p^{m,*}_t]), \qquad w^m_{t+1} = \frac{\exp(\beta A^m_t)}{\sum_{m=1}^{M} \exp(\beta A^m_t)} \tag{9}$$

where $p^{m,*}_t$ is the fundamental price in market m, and $f : \mathbb{R} \to \mathbb{R}$ is a function with $f(0) = 0$.[6]

[6] Westerhoff (2004) sets $f([p^m_t - p^{m,*}_t]) = \ln\left[(1 + c[p^m_t - p^{m,*}_t]^2)^{-1}\right]$, where $c > 0$, such that attractiveness

The fundamental price in market m is the price that would clear the market if all types $h \in \mathcal{H}$ were fundamental investors with common expectations $E_t[.]$. Given equations (8)–(9), the fundamental price in market $m \in \{1, ..., M\}$ is:[7]

$$p_t^{m,*} = \overline{p}^m := \frac{\overline{d}^m - a\sigma_m^2 M\overline{Z}_m}{r} \tag{10}$$

where $\overline{Z}_m > 0$ is the fixed supply of asset m per investor.

Therefore, the demand of type h in market m can be written in terms of deviations from the fundamental price, $x_t^m := p_t^m - \overline{p}^m$, as

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m - (1+r)x_t^m}{a\sigma_m^2} \right) & \text{if } x_t^m \leq \frac{\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m}{1+r} \\ 0 & \text{if } x_t^m > \frac{\tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 M\overline{Z}_m}{1+r}. \end{cases} \tag{11}$$

We assume that the population shares in market m, denoted by $n_{t,h}^m$, are determined by a market-specific version of (4). Hence, market-clearing in market m is given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m / w_t^m \tag{12}$$

where $\tilde{z}_{t,h}^m := (a\sigma_m^2)^{-1}(\hat{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 \overline{Z}_m / w_t^m - (1+r)x_t^m)$ and $\hat{E}_{t,h}\left[x_{t+1}^m\right] := \tilde{E}_{t,h}\left[x_{t+1}^m\right] + a\sigma_m^2 \overline{Z}_m (M - 1/w_t^m)$. Note that with this change in variables, the market-clearing condition is in the same form as in the benchmark model (aside from a scaling of supply by $1/w_t^m$). We therefore have the following simple result.

Remark 1 *In the model above with M risky assets subject to short-selling constraints, the expressions for the equilibrium prices x_t^m and demands $z_{t,h}^m$ $\forall h \in \mathcal{H}$ are given by Proposition 1, except that x_t and $\tilde{E}_{t,h}[x_{t+1}]$ must be replaced by x_t^m and $\hat{E}_{t,h}\left[x_{t+1}^m\right]$ in (12). A re-worked version of Proposition 1 following this approach is provided in the Supplementary Appendix.*

4.1.2 Hedging demands

We now allow demands to depend directly on the demands for all other risky assets due to the 'hedging demands' of risk-averse investor types. In particular, note that with $M \geq 2$ risky assets, the demands for each asset $m \in \{1, ..., M\}$ will generally depend on the demands for all other assets $j \in \{1, ..., M\} \setminus \{m\}$, since positive (negative) return correlations imply

declines with distance from the fundamental price due to the risk of being caught in a bubble that collapses.

[7]By definition, the market price equals the fundamental price if all investors are fundamentalists, and hence $A_t^m = f(0) = 0$ $\forall m$, such that $w_t^m = 1/M$ for all m and all $t \geq 0$ (given an initial condition $A_{-1}^m = 0$). Using this result in conjunction with the demands (8), common expectations $E_t[p_{t+1}^m]$ and market-clearing leads to the equation $p_t^m = (1+r)^{-1}[E_t[p_{t+1}^m] + \overline{d}^m - a\sigma_m^2 M\overline{Z}_m]$, which can be solved forwards to give (10).

covariance risk (risk-hedging) for the asset portfolio. We assume the covariances are constant and denote the covariance in returns of assets m and j by $\sigma_{mj} \in [-\sigma_m\sigma_j, \sigma_m\sigma_j]$ (given).[8]

To obtain tractable analytical results, we make two additional assumptions. First, we assume there is one short-selling constrained asset, which we take to be Asset 1. We may interpret such a restriction as a particular asset market imposing short-selling regulations due to asset- or country-specific shocks. Second, we assume that each type $h \in \mathcal{H}$ has a common population share $n_{t,h}$ in *every* asset market. Note that this assumption may be justified as a rule-of-thumb investment strategy for which investors look at *joint fitness* of predictors across all M asset markets, rather than choosing different predictors in each market.

Given our assumption of a short-selling constraint in market 1, the first-order conditions of investor $h \in \mathcal{H}$ with respect to assets 1 to M yield:

$$\begin{cases} \tilde{E}_{t,h}\left[p^1_{t+1}\right] + \overline{d}^1 - (1+r)p^1_t - a\sigma_1^2 z^1_{t,h} - a\sum\limits_{j\neq 1}\sigma_{1j}z^j_{t,h} + \lambda_{t,h} = 0 \\ z^m_{t,h} = \frac{\tilde{E}_{t,h}\left[p^m_{t+1}\right] + \overline{d}^m - (1+r)p^m_t - a\sum\limits_{j\neq m}\sigma_{mj}z^j_{t,h}}{a\sigma_m^2} \qquad \text{for } m = 2, ..., M \end{cases} \tag{13}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on $z^1_{t,h} \geq 0$, with slackness condition $\lambda_{t,h}z^1_{t,h} = 0$.

The demand schedules in markets 2 to M are standard, except for the dependence of demand on demands in other markets via the covariances σ_{mj}. In market 1, the first-order condition is non-linear due to the Lagrange multiplier on the short-selling constraint.

Given common fractions in each market, the fundamental prices are as follows:[9]

$$\overline{p}^m = \frac{\overline{d}^m - a\sum\limits_{j\neq m}\sigma_{mj}\overline{Z}_j - a\sigma_m^2\overline{Z}_m}{r} \qquad \text{for } m = 1, ..., M. \tag{14}$$

Using (14) in (13), the first-order conditions in price deviations $x^m_t := p^m_t - \overline{p}^m$ are:

$$\begin{cases} \tilde{E}_{t,h}\left[x^1_{t+1}\right] - a\sigma_1^2(z^1_{t,h} - \overline{Z}_1) - (1+r)x^1_t - a\sum\limits_{j\neq 1}\sigma_{1j}(z^j_{t,h} - \overline{Z}_j) + \lambda_{t,h} = 0 \\ z^m_{t,h} = \frac{\tilde{E}_{t,h}\left[x^m_{t+1}\right] + a\sigma_m^2\overline{Z}_m - (1+r)x^m_t - a\sum\limits_{j\neq m}\sigma_{mj}(z^j_{t,h} - \overline{Z}_j)}{a\sigma_m^2} \qquad \text{for } m = 2, ..., M. \end{cases} \tag{15}$$

Market-clearing determines the prices of assets 2 to M as $x^m_t = (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}\tilde{E}_{t,h}\left[x^m_{t+1}\right]$. Therefore, by (15), the equilibrium demands of assets 2 to M by type $h \in \mathcal{H}$ satisfy

$$z^m_{t,h} - \overline{Z}_m = E^m_{t,h} - \sum_{j\neq m}\frac{\sigma_{mj}}{\sigma_m^2}(z^j_{t,h} - \overline{Z}_j) \qquad \text{for } m = 2, ..., M \tag{16}$$

where $E^m_{t,h} := \frac{1}{\sigma_m^2}(\tilde{E}_{t,h}\left[x^m_{t+1}\right] - \sum_{h\in\mathcal{H}} n_{t,h}\tilde{E}_{t,h}\left[x^m_{t+1}\right])$.

[8]Note that the interval endpoints correspond to return correlations of -1 and $+1$ respectively.

[9]Terms with covariances simplify to $a\sigma_{mj}\sum_{h\in\mathcal{H}} n_{t,h}z^j_{t,h} = a\sigma_{mj}\overline{Z}_j$ by market-clearing in market j.

We can write the above expressions in matrix form as

$$\vec{z}_{t,h} = \vec{E}_{t,h} + \vec{\gamma}(z^1_{t,h} - \overline{Z}_1) - A\vec{z}_{t,h} \tag{17}$$

where $\vec{z}_{t,h} := \left[(z^2_{t,h} - \overline{Z}^2) \dots (z^M_{t,h} - \overline{Z}_m)\right]^\intercal$, $\vec{E}_{t,h} := \left[E^2_{t,h} \dots E^M_{t,h}\right]^\intercal$, $\vec{\gamma} := -\left[\frac{\sigma_{12}}{\sigma_2^2} \dots \frac{\sigma_{1M}}{\sigma_M^2}\right]^\intercal$ are $M-1$ vectors, and $A = [a_{ij}]$ is a square matrix with $a_{ii} = 0$ and $a_{ij} = \frac{\sigma_{ij}}{\sigma_i^2}$, $i \neq j$.

Provided $\det[I_{M-1} + A] \neq 0$, we have:

$$\vec{z}_{t,h} = (I_{M-1} + A)^{-1}[\vec{E}_{t,h} + \vec{\gamma}(z^1_{t,h} - \overline{Z}_1)] \tag{18}$$

where I_{M-1} is the identity matrix of size $M-1$.

Substituting (18) into the first-order condition (15) and defining $\sigma_{vec} := [\sigma_{12} \dots \sigma_{1M}]$:

$$\tilde{E}_{t,h}\left[x^1_{t+1}\right] - a\sigma_{vec}(I_{M-1}+A)^{-1}\vec{E}_{t,h} - a[\sigma_1^2 + \sigma_{vec}(I_{M-1}+A)^{-1}\vec{\gamma}](z^1_{t,h} - \overline{Z}_1) - (1+r)x^1_t + \lambda_{t,h} = 0$$

such that the demand for asset 1 by type $h \in \mathcal{H}$ is given by

$$z^1_{t,h} = \begin{cases} \frac{\hat{E}_{t,h}\left[x^1_{t+1}\right] + a\Sigma^1\overline{Z}_1 - (1+r)x^1_t}{a\Sigma^1} & \text{if } x^1_t \leq \frac{\hat{E}_{t,h}\left[x^1_{t+1}\right] + a\Sigma^1\overline{Z}_1}{1+r} \\ 0 & \text{if } x^1_t > \frac{\hat{E}_{t,h}\left[x^1_{t+1}\right] + a\Sigma^1\overline{Z}_1}{1+r} \end{cases} \tag{19}$$

where $\hat{E}_{t,h}\left[x_{t+1}\right] := \tilde{E}_{t,h}\left[x_{t+1}\right] - a\sigma_{vec}(I_{M-1} + A)^{-1}\vec{E}_{t,h}$ and $\Sigma^1 := \sigma_1^2 + \sigma_{vec}(I_{M-1} + A)^{-1}\vec{\gamma}$.

Note that (19) is of the same form as the demand schedule (3) in the benchmark model, and is well-defined provided $\det[I_{M-1} + A] \neq 0$, $\Sigma^1 \neq 0$. In this case, the equilibrium price and demands for asset 1 are nested by Proposition 1, as clarified in the following remark.

Remark 2 *In the model above with M risky assets and asset 1 subject to a short-selling constraint, the expressions for the equilibrium price x^1_t and demands $z^1_{t,h}$ $\forall h \in \mathcal{H}$ are given by Proposition 1, except that x_t must be replaced by x^1_t and $\tilde{E}_{t,h}\left[x_{t+1}\right]$, σ^2 must be replaced by $\hat{E}_{t,h}\left[x^1_{t+1}\right]$, Σ^1 defined in (19). Provided $\det[I_{M-1} + A] \neq 0$, $\Sigma^1 \neq 0$, demand for asset 1 is well-defined and equilibrium prices and demands are uniquely determined in all M markets.*

4.2 Costly short selling

We now return to a one-asset model but allow short-selling to be costly rather than banned. Following Anufriev and Tuinstra (2013), if investors short the risky asset they face a cost (or tax) per share $T > 0$, such that total transaction costs are $T|z_{t,h}|$. If investors do not sell short, no cost is paid. The demand of type $h \in \mathcal{H}$ is thus amended from (3) to

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \leq x^h_t \\ 0 & \text{if } x_t \in (x^h_t, x^h_t + T]. \\ \frac{\tilde{E}_{t,h}[x_{t+1}] - (1+r)(x_t - T) + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t > x^h_t + T \end{cases} \tag{20}$$

where $x_t^h := \frac{\tilde{E}_{t,h}[x_{t+1}]+a\sigma^2\overline{Z}}{1+r}$ and x_t is the deviation from the fundamental price (see Sec.2.2).

The intuition for (20) is quite simple. If $T \to \infty$, any type whose participation price x_t^h is exceeded by the market price x_t would be short-selling constrained with demand of zero, as in the benchmark model. Given a finite cost $T \in (0, \infty)$ for selling short, those with non-positive demand may have either zero demand (if their after-cost demand is non-negative) or short sell (if they prefer a short position despite facing the transaction cost).

The asset market clears when $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$. By Proposition 2.1 in Anufriev and Tuinstra (2013), there is a unique equilibrium price. The solution is stated in Proposition 2.

Proposition 2 *Let $x_t = p_t - \overline{p}$ be the equilibrium price in period t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ be the non-empty set of types that are unconstrained and do not short the risky asset. Further, denote the date t belief of type h by $f_{t,h} := \tilde{E}_{t,h}[x_{t+1}]$, and let $\mathcal{S}_{1,t} \subset \mathcal{H} \setminus \mathcal{B}_t$ ($\mathcal{S}_{2,t} = \mathcal{H} \setminus (\mathcal{B}_t \cup \mathcal{S}_{1,t})$) be the sets of constrained types (short-selling types) at date t. Then the following holds:*

1. If $\sum_{h\in\mathcal{H}} n_{t,h}\left(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}\right) \le a\sigma^2\overline{Z}$, then no type $h \in \mathcal{H}$ is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_{1,t}^* = \mathcal{S}_{2,t}^* = \emptyset$, $z_{t,h} \ge 0\ \forall h \in \mathcal{H}$) and the price is

$$x_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_h(x_{t-1}, ..., x_{t-L})}{1+r} := x_t^*. \tag{21}$$

2. If $\sum_{h\in\mathcal{H}} n_{t,h}\left(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}\right) > a\sigma^2\overline{Z}$, then one or more types are short-selling constrained, or short-sell the asset, at date t (i.e. $\mathcal{B}_t^* \subset \mathcal{H}$) and we have the following:

(i) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_{1,t}^* = \mathcal{H}\setminus\mathcal{B}_t^*$ such that $\max\{d_{1,t}, d_{2,t}\} \le a\sigma^2\overline{Z} < \sum_{h\in B_t^*} n_{t,h}(f_{t,h} - \max_{h\in S_t^*}\{f_{t,h}\})$, then $z_{t,h} \ge 0\ \forall h \in \mathcal{B}_t^* \ne \emptyset$, $z_{t,h} = 0\ \forall h \in \mathcal{S}_{1,t}^* \ne \emptyset$ and the price is

$$x_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > x_t^* \tag{22}$$

where $d_{1,t} = \sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\})$, $d_{2,t} = -\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}(1+r)T + \sum\limits_{h\in B_t^*} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\})$.

(ii) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_{2,t}^* = \mathcal{H}\setminus\mathcal{B}_t^*$ such that $d_{1,t} \le a\sigma^2\overline{Z} - \sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}(1+r)T < d_{2,t}$, then $z_{t,h} \ge 0$ $\forall h \in \mathcal{B}_t^* \ne \emptyset$, $z_{t,h} < 0\ \forall h \in \mathcal{S}_{2,t}^* \ne \emptyset$ and the price is

$$x_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) + \sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}(1+r)T}{1+r} > x_t^* \tag{23}$$

where $d_{1,t} = \sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\})$ and $d_{2,t} = \sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \max\limits_{h\in\mathcal{S}_{2,t}^*}\{f_{t,h}\}) - (1+r)T$.

(iii) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^*$ such that $\max\{d_{1,t}, d_{2,t}\} \le a\sigma^2\overline{Z} - \sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}(1+r)T < \min\{d_{3,t}, d_{4,t}\}$,

then $z_{t,h} \geq 0 \; \forall h \in \mathcal{B}_t^* \neq \emptyset$, $z_{t,h} = 0 \; \forall h \in \mathcal{S}_{1,t}^* \neq \emptyset$, $z_{t,h} < 0 \; \forall h \in \mathcal{S}_{2,t}^* \neq \emptyset$ *and price is*

$$x_t = \frac{\sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}[f_h(x_{t-1}, ..., x_{t-L}) + a\sigma^2 \overline{Z}] + (1+r)T \sum\limits_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2 \overline{Z}}{(1+r)\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} > x_t^* \quad (24)$$

where $d_{1,t} = \sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \min\limits_{h \in \mathcal{B}_t^*}\{f_{t,h}\})$, $d_{2,t} = \sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \min\limits_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\} - (1+r)T)$, $d_{3,t} = \sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \max\limits_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\})$, $d_{4,t} = \sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} (f_{t,h} - \max\limits_{h \in \mathcal{S}_{2,t}^*}\{f_{t,h}\} - (1+r)T)$.

Proof. See the Appendix. ■

Proposition 2 shows that, under costly short-selling, there are additional regions for which the price has a different expression to Proposition 1. Intuitively, this is because demands are not restricted to be non-negative, such that types who find it optimal to short-sell are permitted to do so but face the transaction cost T. As a result, we may have some types short-selling and paying the transaction cost but none constrained (Prop. 2, part (ii)); some types constrained but none short-selling (Prop. 2, part (i)); or some types constrained and other types (who are more pessimistic) short-selling (Prop. 2, part (iii)). As noted by Anufriev and Tuinstra (2013), there are $3^H - 2^H$ different cases. Finally, note that the prices in (22)–(24) are strictly larger than x_t^*, i.e. if one or more agents short-sell or are short-selling constrained in period t, then the date-t price is higher than in absence of short-selling costs.

Example 3 *Suppose that as in Section 2.2.2 of Anufriev and Tuinstra (2013) there are* $H = 2$ *types, which we denote* h_1 *and* h_2*. Then, using the result in Proposition 2, there are* $3^2 - 2^2 = 5$ *different regions and the equilibrium price is given by*

$$x_t = \begin{cases} \frac{\sum_{h \in \{h_1,h_2\}} n_{t,h} f_h(x_{t-1},...,x_{t-L})}{1+r} & \text{if } -\frac{a\sigma^2\overline{Z}}{n_{h_2,t}} \leq \Delta\tilde{E}_t \leq \frac{a\sigma^2\overline{Z}}{n_{h_1,t}} \\ \frac{n_{t,h_1} f_{h_1}(x_{t-1},...,x_{t-L}) - (1-n_{t,h_1})a\sigma^2\overline{Z}}{(1+r)n_{t,h_1}} & \text{if } \frac{a\sigma^2\overline{Z}}{n_{t,h_1}} < \Delta\tilde{E}_t \leq \frac{a\sigma^2\overline{Z}}{n_{t,h_1}} + (1+r)T \\ \frac{n_{t,h_2} f_{h_2}(x_{t-1},...,x_{t-L}) - (1-n_{t,h_2})a\sigma^2\overline{Z}}{(1+r)n_{t,h_2}} & \text{if } -\frac{a\sigma^2\overline{Z}}{n_{t,h_2}} - (1+r)T \leq \Delta\tilde{E}_t < -\frac{a\sigma^2\overline{Z}}{n_{t,h_2}} \\ \frac{\sum_{h \in \{h_1,h_2\}} n_{t,h} f_h(x_{t-1},...,x_{t-L}) + n_{t,h_1}(1+r)T}{1+r} & \text{if } \Delta\tilde{E}_t > \frac{a\sigma^2\overline{Z}}{n_{t,h_1}} + (1+r)T \\ \frac{\sum_{h \in \{h_1,h_2\}} n_{t,h} f_h(x_{t-1},...,x_{t-L}) + n_{t,h_2}(1+r)T}{1+r} & \text{otherwise} \end{cases} \quad (25)$$

where $\Delta\tilde{E}_t := \tilde{E}_{t,h_1}[x_{t+1}] - \tilde{E}_{t,h_2}[x_{t+1}]$.

The solution in (25) is consistent with Proposition 2.2 in Anufriev and Tuinstra (2013). Note that the case listed in Part (iii) of Proposition 2 does not arise in this example, since with two types there can be at most one non-buyer. The different regions correspond to the five possible permutations of 'buyers' (with unrestricted demands $z_{t,h} \geq 0$), constrained types (with $z_{t,h} = 0$), and short-selling types (with $z_{t,h} < 0$) when there are two types h_1, h_2.

Since there are $3^H - 2^H$ cases, finding a solution using Proposition 2 is very computationally intensive even for small H. We therefore seek, as in Corollary 1, a solution with types ordered in terms of optimism. Sticking with $\tilde{\mathcal{H}}_t = \{1, 2, ..., \tilde{H}_t\}$ as the set of ordered types such that $\tilde{E}_{t,1}[x_{t+1}] < \tilde{E}_{t,2}[x_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[x_{t+1}]$, we have the following result.

Corollary 2 *Let $\tilde{\mathcal{H}}_t = \{1, 2..., \tilde{H}_t\}$ be the set defined above, such that beliefs are ordered as $\tilde{E}_{t,1}[x_{t+1}] < \tilde{E}_{t,2}[x_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[x_{t+1}]$. Let $disp_{t,k} := \sum_{h>k}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,k})$ be a measure of belief dispersion, where $k \in \{1, ..., \tilde{H}_t - 1\}$. Further, let $\tilde{disp}_{t,k} := \sum_{h \neq k} n_{t,h}(f_{t,h} - f_{t,k})$ and $\hat{disp}_{t,k}^{(\mathcal{S}_{1,t})} := \sum_{h \notin \mathcal{S}_{1,t}} n_{t,h}(f_{t,h} - f_{t,k})$, where $\mathcal{S}_{1,t}$ is defined below. Then we have the following:*

1. *If $disp_{t,1} \leq a\sigma^2\overline{Z}$, then $x_t = x_t^* := (1+r)^{-1} \sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}$.*

2. *If $disp_{t,2} \leq a\sigma^2\overline{Z} < disp_{t,1}$, then*

$$x_t = \begin{cases} \hat{x}_t^{(\emptyset,\{1\})} & \text{if } disp_{t,1} \leq a\sigma^2\overline{Z} + (1 - n_{t,1})(1+r)T \\ \overline{x}_t^{(\{1\},\emptyset)} & \text{otherwise} \end{cases}$$

where $\hat{x}_t^{(\emptyset,\{1\})} := \frac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2\overline{Z}}{(1-n_{t,1})(1+r)}$ and $\overline{x}_t^{(\{1\},\emptyset)} := \frac{\sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h} f_{t,h} + (1-n_{t,1})(1+r)T}{1+r}$.

3. *If $disp_{t,\overline{k}+1} \leq a\sigma^2\overline{Z} < disp_{t,\overline{k}}$ for some $\overline{k} \in \{2, ..., \tilde{H}_t - 1\}$, then*

$$x_t = \begin{cases} \hat{x}_t^{(\emptyset,\{1,..,\overline{k}\})} & \text{if } \tilde{disp}_{t,1} \leq a\sigma^2\overline{Z} + (1 - \sum_{h=1}^{\overline{k}} n_{t,h})(1+r)T \\ \tilde{x}_t^{(\mathcal{S}_{2,t},\mathcal{S}_{1,t})} & \begin{array}{l} \text{if } \exists\ \mathcal{S}_{2,t} = \{1, .., \underline{k}\},\ \mathcal{S}_{1,t} = \{\underline{k}+1, ..., \overline{k}\}, \text{ where } 1 \leq \underline{k} < \overline{k}, \\ \text{such that } \hat{disp}_{t,\overline{k}+1}^{(\mathcal{S}_{1,t})} \leq a\sigma^2\overline{Z} - \sum_{h \in \mathcal{S}_{2,t}} n_{t,h}(1+r)T < \hat{disp}_{t,\overline{k}}^{(\mathcal{S}_{1,t})} \\ \text{and } \hat{disp}_{t,\overline{k}}^{(\mathcal{S}_{1,t})} \leq a\sigma^2\overline{Z} + (1 - \sum_{h \in \mathcal{S}_{2,t}} n_{t,h})(1+r)T < \hat{disp}_{t,\overline{k}-1}^{(\mathcal{S}_{1,t})} \end{array} \\ \overline{x}_t^{(\{1,..,\overline{k}\},\emptyset)} & \text{otherwise: with } \overline{x}_t^{(\{1,...,\overline{k}\},\emptyset)} := \frac{\sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h} f_{t,h} + \sum_{h>\overline{k}} n_{t,h}(1+r)T}{1+r} \end{cases}$$

where $\hat{x}_t^{(\emptyset,\{1,...,\overline{k}\})} := \frac{\sum_{h>\overline{k}} n_{t,h} f_{t,h} - (1 - \sum_{h>\overline{k}} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h>\overline{k}}^{\tilde{H}_t} n_{t,h}}$, $\tilde{x}_t^{(\mathcal{S}_{2,t},\mathcal{S}_{1,t})} := \frac{\sum_{h \notin [\underline{k}]} n_{t,h}[f_{t,h} + a\sigma^2\overline{Z}] + \sum_{h=1}^{\underline{k}} (1+r)T - a\sigma^2\overline{Z}}{(1+r)\sum_{h \notin [\underline{k}]} n_{t,h}}$.

Proof. See the Appendix. ■

Corollary 2 provides a streamlined version of Proposition 2. In particular, by ordering types by optimism, the number of cases for the price is reduced from $3^H - 2^H$ to $\tilde{H}_t(\tilde{H}_t+1)/2$, where $\tilde{H}_t \leq H$. The intuition for this result is quite simple. One case corresponds to the price when no types are constrained or short-sell (Corollary 2 Part 1), whereas the other cases correspond to type 1 short-selling or being constrained, types 1 and 2 short-selling or being constrained,..., types $1, ..., \tilde{H}_t - 1$ short-selling or being constrained.

In general, for each 'branch' with $\overline{k} \leq \tilde{H}_t - 1$ types short-selling or constrained, there are $\overline{k}+1$ expressions for price that correspond to: types $1, ..., \overline{k}$ short-selling; types $1, ..., \overline{k}$ short-selling constrained; and types $1, ..., \underline{k}$ short-selling, types $\underline{k}+1, ..., \overline{k}$ short-selling constrained, for some $1 \leq \underline{k} < \overline{k}$. Note that these permutations arise because sufficiently pessimistic types will short sell and pay the transaction cost T. Summing over the different cases we have: 1 (no types short or constrained) + 2 (type 1 short or constrained) + 3 (types 1,2 short or constrained) +... + $\tilde{H}_t$ (types $1, ..., \tilde{H}_t - 1$ short or constrained) = $\tilde{H}_t(\tilde{H}_t + 1)/2$.

Given these additional cases (i.e. regions to consider), the benchmark computational algorithm for finding the price needs to be extended as shown below.

Extended computational algorithm

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering beliefs as $\tilde{E}_{t,1}[x_{t+1}] < \tilde{E}_{t,2}[x_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[x_{t+1}]$ and find the associated population shares $n_{t,h}$ of types $h = 1, 2, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=2}^{\tilde{H}_t} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \tilde{E}_{t,1}[x_{t+1}])$. If $disp_{t,1} \leq a\sigma^2\overline{Z}$, accept $x_t = x_t^*$ as the date t solution and move to period $t+1$. Otherwise, move to Step 3.

3. Set $x_t^{guess} = x_t^*$ and find the largest k such that $z_{t,k}^{guess} = \frac{\tilde{E}_{t,k}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t^{guess}}{a\sigma^2} < 0$, and denote this value k_{min}. Starting from $k = k_{min}$, check if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a $\overline{k}$ is found such that $disp_{t,\overline{k}+1} \leq a\sigma^2\overline{Z} < disp_{t,\overline{k}}$.

4. If $\overline{k} = 2$ (Cor. 2, Part 2), then $x_t = \hat{x}_t^{(\emptyset,\{1\})}$ if $disp_{t,1} \leq a\sigma^2\overline{Z} + (1 - n_{t,1})(1+r)T$ (Type 1 is short-selling constrained), and $x_t = \overline{x}_t^{(\{1\},\emptyset)}$ otherwise (Type 1 is a short-seller).

5. If $\overline{k} \geq 3$ (Cor. 2, Part 3), then $x_t = \hat{x}_t^{(\emptyset,\{1,..,\overline{k}\})}$ if $\tilde{disp}_{t,1} \leq a\sigma^2\overline{Z} + (1 - \sum_{h=1}^{\overline{k}} n_{t,h})(1+r)T$ (i.e. $\overline{k}$ types short-selling constrained), and $x_t = \tilde{x}_t^{(\mathcal{S}_{2,t},\mathcal{S}_{1,t})}$ if $\exists\ \mathcal{S}_{2,t} = \{1, .., \underline{k}\}$, $\mathcal{S}_{1,t} = \{\underline{k}+1, ..., \overline{k}\}$ such that $\hat{disp}_{t,\overline{k}+1}^{(\mathcal{S}_{1,t})} \leq a\sigma^2\overline{Z} - \sum_{h\in\mathcal{S}_{2,t}} n_{t,h}(1+r)T < \hat{disp}_{t,\overline{k}}^{(\mathcal{S}_{1,t})}$. Check the latter cases by iterating on $\underline{k}$, starting from $\underline{k} = 1$. If above conditions are not met for some $\underline{k} \in \{1, ..., \overline{k} - 1\}$, then $x_t = \overline{x}_t^{(\{1,..,\overline{k}\},\emptyset)}$ (i.e. $\overline{k}$ types short-sell and pay cost T).

4.3 Market-maker approach

As a final extension we consider asset prices determined by a market-maker rather than a market-clearing mechanism (see Beja and Goldman, 1980; Chiarella, 1992; Farmer and Joshi, 2002; Chiarella et al., 2009). As is standard in the literature, we consider price impact functions which are linear in excess demand. We allow the asset price to potentially depend on both current and past excess demand as follows:

$$p_t = p_{t-1} + \mu[\lambda(Z_t - \overline{Z}) + (1-\lambda)(Z_{t-1} - \overline{Z})] \tag{26}$$

where $\mu > 0$, $\lambda \in [0,1]$ and $Z_t := \sum_{h\in\mathcal{H}} n_{t,h} z_{t,h}$ is aggregate demand per investor at date t.

We assume that demands are still given by (1), such that the fundamental price remains at $\overline{p} = \frac{\overline{d} - a\sigma^2\overline{Z}}{r}$. Hence, as in the benchmark model of Section 2, the demand of investor type $h \in \mathcal{H}$ can be written in terms of deviations from the fundamental price, $x_t := p_t - \overline{p}$, as

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r}. \end{cases} \tag{27}$$

Similarly, the price impact function (26) can be written in terms of price deviations as

$$x_t = x_{t-1} + \mu[\lambda(Z_t - \overline{Z}) + (1-\lambda)(Z_{t-1} - \overline{Z})] \tag{28}$$

where Z_t, Z_{t-1} depend on deviations from the fundamental price via (27). We can easily solve for the price and demands in this setting, as summarized in Proposition 3.

Proposition 3 *Let $x_t = p_t - \overline{p}$ be the price determined by (28) at date t and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Further, let $Z_{t-1} - \overline{Z} = \sum_{h\in\mathcal{B}_{t-1}} n_{t-1,h} z_{t-1,h} - \overline{Z}$ be excess demand at date $t-1$, and denote the date t belief of type h by $f_{t,h} := \tilde{E}_{t,h}[x_{t+1}]$. Then the following holds:*

1. *If $x_{t-1} - \frac{1}{1+r}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $S_t^* = \emptyset$, $z_{t,h} \geq 0\ \forall h$) and the price is*

$$x_t = \frac{x_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}}.$$

2. *If $x_{t-1} - \frac{1}{1+r}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then one or more types is short-selling constrained with $z_{t,h} = 0$ and we have the following:*

   (i) If there exist non-empty $\mathcal{B}_t^*, \mathcal{S}_t^*$ such that $\frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z} - x_{t-1} - \mu(1-\lambda)Z_{t-1} < \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{S}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, price is

$$x_t = \frac{x_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_h(x_{t-1}, ..., x_{t-L}) + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}}.$$

   (ii) Otherwise, $\exists \mathcal{S}_t^* = \mathcal{H}$ such that $x_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h$), and the price is $x_t = x_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See the Appendix. ■

Note that there are *three* distinct cases in Proposition 3, since it is possible that *all types* will be short-selling constrained at the price set by the market maker. By contrast, under equilibrium asset pricing (see Proposition 1) at least one type must purchase the risky asset. Further, Proposition 3 nests some special cases. When $\lambda = 0$, only past excess demand matters for the current price, and hence the price solution is $x_t = x_{t-1} + \mu(Z_{t-1} - \overline{Z})$ irrespective of current demands (and hence current beliefs). In this case, the short-selling constraint influences price only with a lag through its past impact on the aggregate demand, Z_{t-1} or the past price x_{t-1}. On the other hand, when $\lambda = 1$ past excess demand does not influence price, such that only current aggregate demand $Z_t = \sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h}$ matters. In this case, the price depends on how many types are currently short-selling constrained.

Analogous to the results in Proposition 1, it can be shown that cases 2(i) and 2(ii) in Proposition 3 (for which the short-selling constraint is binding) imply a higher price than when short-selling constraints are absent (in which case price is given by the expression in Part 1 of Proposition 3). The intuition for this result is quite simple. Looking at Eq. (28) we see that, given predetermined past excess demand $Z_{t-1} - \overline{Z}$ and the past price x_{t-1}, the current price is higher the larger the current excess demand, $Z_t - \overline{Z}$. Since excess demand is unambiguously smaller if short-selling is permitted, the current price is also smaller.

The benchmark computational algorithm can easily be amended to fit the market-maker approach. In particular, to check for short-sellers in period t, we obtain the set $\mathcal{H}_t$ and then Step 2 of the algorithm is amended to $disp_{t,1} \leq (\mu + (1+r)^{-1} a\sigma^2)\,\overline{Z} - x_{t-1} - \mu(1-\lambda)Z_{t-1}$, where $disp_{t,1} := \frac{\mu\lambda}{a\sigma^2} \sum_{h \in \tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,1}) - \frac{1}{1+r} f_{t,1}$ (see Proposition 3, Part 1). If this condition is satisfied, then the price follows Proposition 3 Part 1.

If the above condition is not satisfied, further steps are needed. Following Proposition 3 Part 2(i), we first search for a $k^* \in \{1, ..., \tilde{\mathcal{H}}_t - 1\}$ such that $disp_{t,k^*+1} \leq (\mu + (1+r)^{-1} a\sigma^2)\,\overline{Z} - x_{t-1} - \mu(1-\lambda)Z_{t-1} < disp_{t,k^*}$, where $disp_{t,k} := \frac{\mu\lambda}{a\sigma^2} \sum_{h>k} n_{t,h}(f_{t,h} - f_{t,k}) - \frac{1}{1+r} f_{t,k}$. If such a k^* exists, the price x_t is given by the expression in Proposition 3 Part 2(i) .

Finally, if there is no k^* that satisfies the above condition, then *all types* are short-selling constrained in period t. By Proposition 3, Part 2(ii), this is the case if $disp_{t,\tilde{\mathcal{H}}_t} > (\mu + (1+r)^{-1} a\sigma^2)\,\overline{Z} - x_{t-1} - \mu(1-\lambda)Z_{t-1}$ and the price is $x_t = x_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

5 Application: Alternative uptick rule

We now consider a application based on an alternative uptick rule, as currently in place in the United States. Under the rule, short-selling is banned following price falls of 10% or more. This contrasts with the original uptick rule in place from 1938 to 2007, which banned short-selling of shares after a fall in prices (regardless of the magnitude). While the original uptick rule has been studied in previous works (see Dercole and Radi, 2020), the alternative uptick rule does not seem to have been studied in the heterogeneous-beliefs literature.

Since the alternative uptick rule bans short-selling following price falls of 10% or more, equilibrium prices and demands are covered by Proposition 1 and Corollary 1, except that the short-selling constraint is present in period t if and only if $p_{t-1} \leq (1-\kappa)p_{t-2}$ where $\kappa = 0.1$; note that this amounts to $x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p}$ in terms of price deviations. It is thus straightforward to amend Proposition 1, Corollary 1 and the Computational Algorithm to deal with this case.[10]

The demand of type $h \in \mathcal{H}$ in period t is given by

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[x_{t+1}]-(1+r)x_t+a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \leq \frac{\tilde{E}_{t,h}[x_{t+1}]+a\sigma^2\overline{Z}}{1+r} \text{ or } x_{t-1} > (1-\kappa)x_{t-2} - \kappa\overline{p} \\ 0 & \text{if } x_t > \frac{\tilde{E}_{t,h}[x_{t+1}]+a\sigma^2\overline{Z}}{1+r} \text{ and } x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p}. \end{cases} \quad (29)$$

Equation (29) shows that short-selling is banned only if we enter period t with a past price $x_{t-1} \leq (1-\kappa)x_{t-2} - \kappa\overline{p}$, such that Proposition 1 is modified (see *Supplementary Appendix*).

We consider linear predictors of the form:

$$\tilde{E}_{t,h}\left[x_{t+1}\right] = b_h + g_h x_{t-1} \quad (30)$$

where $b_h \in \mathbb{R}$ and $g_h \geq 0$.

Equation (30) is a standard specification in the literature. The intercept terms b_h represent 'bias' in the price forecast of type h (relative to a fundamental benchmark), whereas the g_h parameter represents the degree of trend-following of type h. Note that type h is a pure fundamentalist investor if $b_h = g_h = 0$, while larger values of g_h or $|b_h|$ imply, respectively, stronger trend-following and forecast bias.

We consider a standard specification for fitness $U_{t,h}$ whereby performance is a linear function of past profits net of predictor costs $C_h \geq 0$. Profits are given by scaling the realized excess return $R_t := p_t + d_t - (1+r)p_{t-1} = x_t + a\sigma^2\overline{Z} - (1+r)x_{t-1} + \epsilon_t$ by demand $z_{t-1,h}$, where ϵ_t is the IID dividend shock. For simplicity, we abstract from memory of past performance, such that for all $t \geq 1$ fitness and population shares are given by

$$U_{t,h} = R_t z_{t-1,h} - C_h, \qquad n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}} \exp(\beta U_{t,h})}. \quad (31)$$

Note that the fitness levels $U_{t,h}$ determine the population shares $n_{t+1,h}$ of each type according to the logistic model (see (4)) with intensity of choice parameter $\beta \geq 0$. Recall that the intensity of choice determines how fast agents switch toward the better-performing predictors, i.e. those with higher past profit net of predictor costs. In the special case $\beta = 0$ no such switching occurs; increasing β implies more switching to profitable predictors.

The above specifications of beliefs and fitness allow us to keep contact with the literature. We give the model the same parameters as in Section 3.1 of Anufriev and Tuinstra (2013):

[10] For computation purposes the rule is specified as $p_{t-1} - p_{t-2} \leq -\kappa|p_{t-2}|$ to ensure that if price turns negative (i.e. if x_t falls below $-\overline{p}$) then short-selling is banned only if the price falls more than 10%.

$\overline{Z} = 0.1$, $a\sigma^2 = 1$, $r = 0.1$, and we set $\overline{d} = 0.6$, giving a fundamental price $\overline{p} = \frac{\overline{d} - a\sigma^2\overline{Z}}{r} = 5$. In their model there are two types: a fundamentalist type with $\tilde{E}_{t,f}[x_{t+1}] = 0$ and cost parameter $C = 1$, and a chartist type with $\tilde{E}_{t,c}[x_{t+1}] = \overline{g}x_{t-1}$, where $\overline{g} = 1.2$, and cost 0. We consider a large population of investor types $H = 1,000$, but we initially give 500 types the fundamental predictor (at cost $C = 1$) and the remaining 500 types the chartist predictor $\overline{g} = 1.2$ at no cost, such that there are two homogeneous groups in the population.

5.1 Long run price dynamics

We first consider a bifurcation diagram for the price deviation as the intensity of choice parameter β is increased. As is standard in the literature, we focus here on the deterministic skeleton with $d_t = \overline{d}$, such that $\epsilon_t = 0$ for all t. Following Anufriev and Tuinstra (2013), we study no short-selling constraint and negative initial values, such that there is a 'lower attractor' for the price (see Figure 1). We do not present a separate diagram for an alternative uptick rule because, with two groups, there is no substantive difference in the attractors.

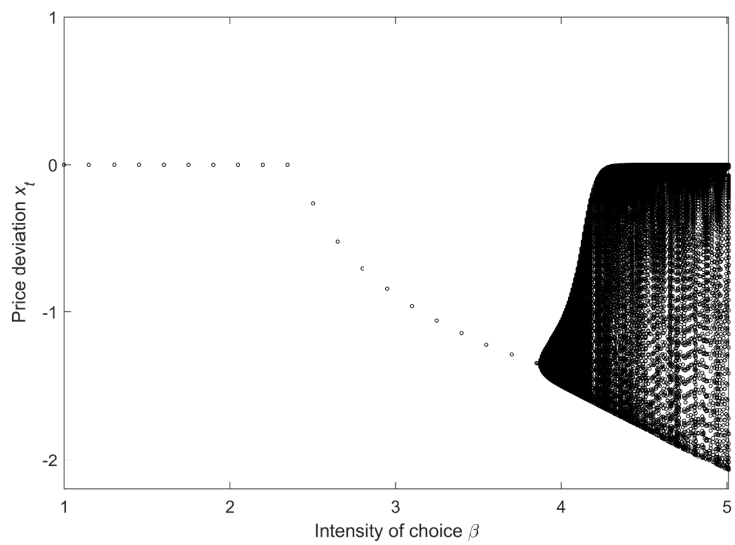

Figure 1: Bifurcation diagram in the absence of short-selling constraints. For each β, we plot 300 points following a transitory of 3,000 periods from given initial values $x_{-1} \in (-4, 0)$.

Figure 1 shows that for sufficiently low values of the intensity of choice, the fundamental steady state $x = 0$ is the unique price attractor. Intuitively, this is because we are in the case of $(1+r) < \overline{g} < 2(1+r)$ and positive outside supply, for which Anufriev and Tuinstra (2013) (Proposition 3.1) show that the fundamental steady state is globally stable for sufficiently small values of the intensity of choice, β. Once a critical value of β is exceeded, there initially exist two non-fundamental steady states in addition to the fundamental steady state, which is locally stable. As β is increased further, however, the fundamental steady state becomes unstable, while the non-fundamental steady states are locally stable if β is not too large.

Given negative initial values, only the non-fundamental steady state with $x < 0$ is an attractor for the price dynamics at intermediate values of β; this amounts to the lower 'fork' seen for β between (approx.) 2.4 and 3.8 in the left panel of Figure 1. Increasing β further causes the non-fundamental steady states to lose their stability through a Neimark-Sacker bifurcation, leading to an invariant closed curve and (quasi-)periodic dynamics. The results in Figure 1 (top panel) are consistent with those in Anufriev and Tuinstra (2013) for the same parameter values. Note that while we obtained the above diagram using $H = 1,000$ types rather than two, we effectively have a two-type model as groups are homogeneous.

We now introduce heterogeneity proper by having many different types. We first consider a small deviation from the two homogeneous groups of investors considered above by allowing the 500 'fundamentalists' with $g_h = 0$ to differ in terms of their bias parameters b_h, which we assume are linearly spaced on interval $[-0.2, 0.2]$. Further, we assume that fundamental types with more bias pay a lower cost, which we model as $C_h = 1 - |b_h|$; the idea is that financial advisers with better information on fundamentals will face higher costs (e.g. search) and can charge more for their services. The initial values are kept fixed relative to Figure 1. The bifurcation diagrams are plotted in Figure 2 for the case of unrestricted short-selling (top left) and an alternative uptick rule (top right), along with the relative volatility of price under an alternative uptick rule, which is defined as the ratio of the standard deviation to the no-constraints case (lower panel).

We see from Figure 2 that the addition of heterogeneity between the fundamentalists leads to both upper and lower price attractors, even though the initial price x_{-1} is negative; intuitively, this is because fundamentalists with positive bias (i.e. $b_h > 0$) may perform well and hence their optimism (expecting an asset overvaluation) can spread through the population and raise the price to positive values. As the intensity of choice β is increased the original steady-state loses its stability via a bifurcation to two stable non-fundamental steady states $x < 0$ and $x > 0$.[11] For sufficiently large β there is a further bifurcation in which the stable non-fundamental steady $x < 0$ gives way to quasi-periodic dynamics; this is similar to Figure 1, except that price volatility is higher in the sense that price fluctuates in a wide range that includes a large region of positive price deviations.

The attractors under unconstrained short-selling (left panel) and the alternative uptick rule (right panel) are very similar, though not identical. From the lower panels, we see that the impact of an alternative uptick rule on long run price volatility is somewhat mixed: in some simulations price volatility is much higher (bottom left), but in others it is much lower (right panel). An alternative uptick rule performs quite poorly for intermediate values of the intensity of choice β but has much improved performance in terms of price volatility when type updating responds more aggressively to past performance (see bottom right).

Finally, we also add heterogeneity within the group of 500 chartists (i.e. types with

[11] Note that the steady state price when the short-selling constraint is not binding satisfies $x\left(1 - \frac{\sum_{h\in\mathcal{H}} n_h g_h}{1+r}\right) = \frac{\sum_{h\in\mathcal{H}} n_h b_h}{1+r}$, where the population shares n_h are determined by (31).

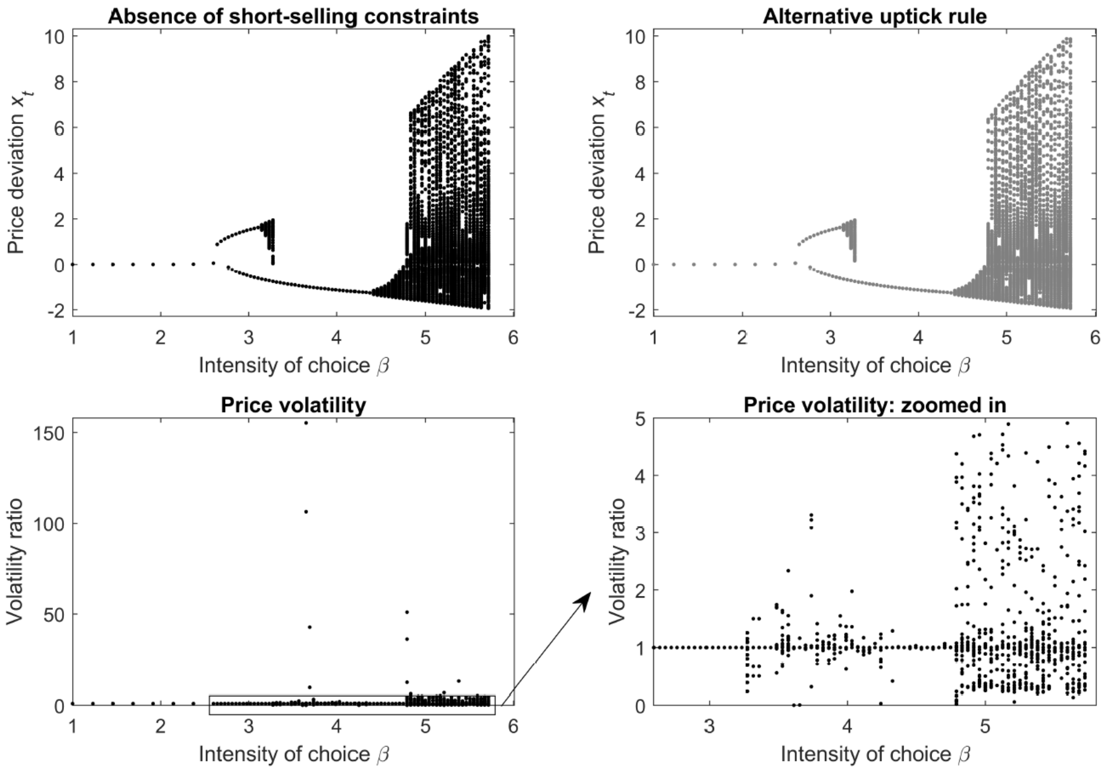

Figure 2: Bifurcation diagrams: heterogeneous fundamentalists. For each β, we plot 300 points following a transitory of 3,000 periods from given initial values $x_{-1} \in (-4, 0)$ under unrestricted short-selling (black dots) and an alternative uptick rule (grey circles). The lower panel plots the ratio of the standard deviation under an alternative uptick rule to that under no constraints (based on final 50 periods) in all simulations where this ratio is well-defined.

$g_h > 0$ and $C_h = 0$). We assume these types have trend-following parameters g_h drawn from a uniform distribution on $(1.1, 1.4)$. Both initial values and the parameters of fundamental types are kept unchanged relative to Figure 2. We study the bifurcation diagram under the deterministic skeleton $d_t = \overline{d}$ for all t as in the previous examples; however, we now allow positive and negative values for the initial price x_{-1} in contrast to the previous diagrams.

Figure 3 shows the attractors under an alternative uptick rule and unrestricted short-selling. In both cases there is an initially stable steady state on which price trajectories converge if the intensity of choice, β, is small enough. As β is increased, however, there is a bifurcation at which (initially-)stable non-fundamental steady states emerge before quasi-periodic dynamics as β is increased further. For large enough β, there is no attractor as the price dynamics are explosive or simulations do not reach completion.

The upper and lower attractors under an alternative uptick rule (grey dots, right panel) are quite similar to those under unrestricted short-selling (black dots, left panel). However, for an upper region of β values, an alternative uptick rule leads to explosive price dynamics

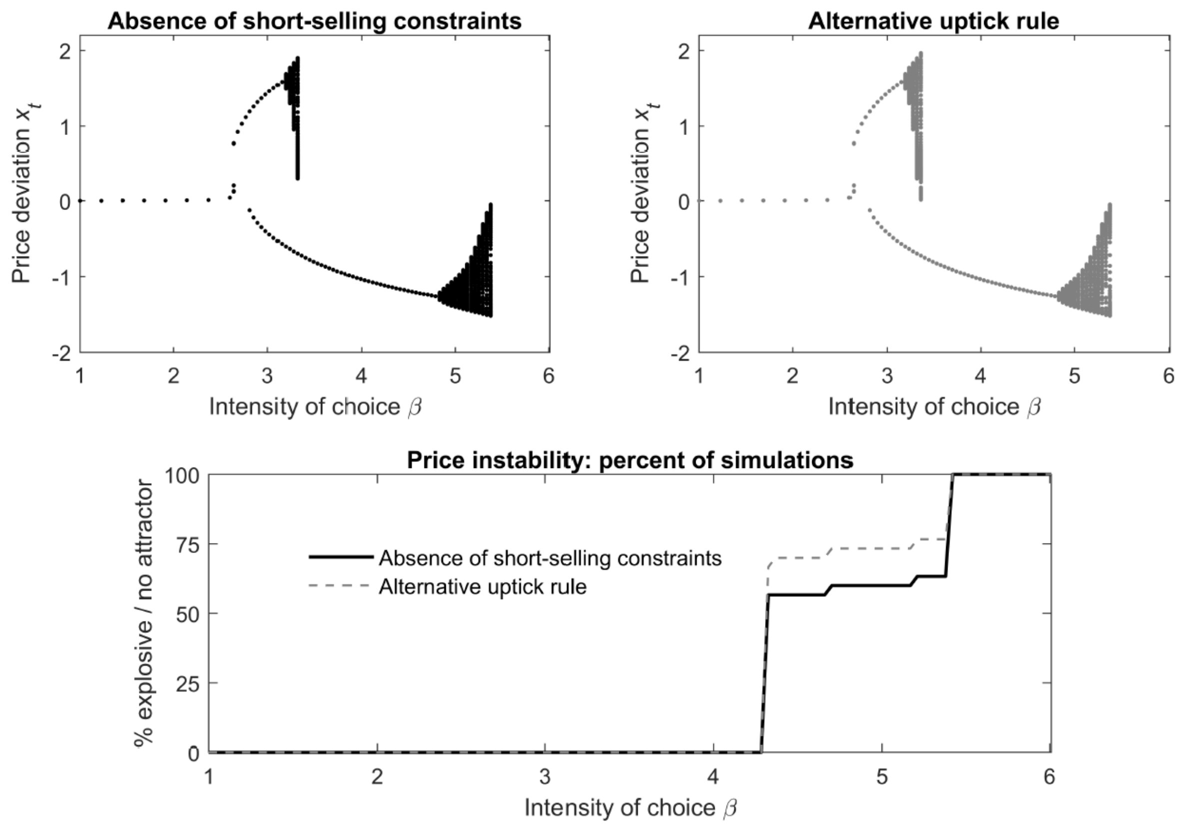

Figure 3: Bifurcation diagrams: heterogeneous fundamentalists and chartists. For each β, we plot 450 points following a transitory of 3,000 periods from given initial values $x_{-1} \in (-4, 4)$ under unrestricted short-selling (black dots) and an alternative uptick rule (grey circles). The lower panel records the % of simulations with no attractor at each β.

in simulations where it would otherwise be absent (lower panel, Figure 3). The reason for the poor performance of the alternative uptick rule seems to be that strong trend-following rules are more likely to gain a foothold; we provide an example in the next section.

5.2 Simulated time series: four scenarios

In this section we present some simulated time series generated by the model. We consider four different scenarios, or cases, which are instances of the within-group heterogeneities considered above. In particular, we present simulations for some specific cases where the initial price x_{-1} is held fixed and only the intensity of choice β or the degree of heterogeneity (in g_h, b_h and C_h) are changed.[12] We first present some simulated price series in the four scenarios, and we then report some results on computation speed and accuracy when stochastic dividend shocks are present. Finally, we consider some distributional implications

[12]All other parameters are the same as in the previous section, so $a\sigma^2 = 1$, $\overline{d} = 0.6$, $r = 0.1$ and $\overline{Z} = 0.1$.

of an alternative uptick rule by simulating the Gini coefficient for wealth across types.

5.2.1 Four price simulations

The simulated price series in the four scenarios (S1–S4) are presented in Figure 4. All four time series are started from the same initial price $x_0 = 3$ and assume deterministic dividends $d_t = \overline{d} = 0.6$ for all t in order to focus on the underlying dynamics. The four scenarios correspond to: heterogeneity among the 500 fundamental types with $g_h = 0$ due to bias b_h which is linearly-spaced on the interval $[-0.2, 0.2]$ and predictor costs $C_h = 1 - |b_h|$ for such types (S1); the same setting as S1 except that heterogeneity is increased such that $b_h \in [-0.4, 0.4]$ (S2); the same setting as S1 except that the intensity of choice is increased from $\beta = 3$ to $\beta = 4.5$ (S3); and the same setting as S3 except that chartists are also heterogeneous with g_h drawn from a uniform distribution on the interval $(1, 1.4)$.

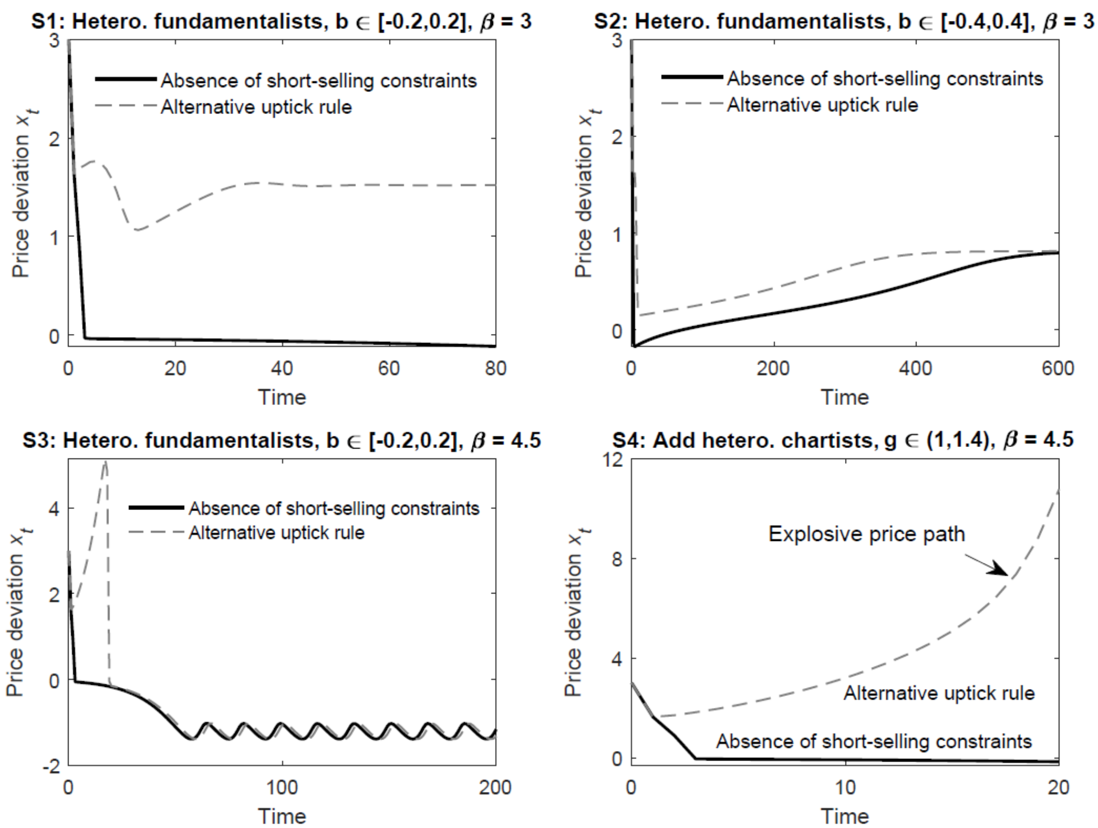

Figure 4: Simulated price series in four scenarios from an initial value $x_0 = 3$.

We see that the price paths in these cases are quite different even though the additional heterogeneities are fairly small. In Scenario 1 (Figure 4, top left), we see that if short-selling constraints are absent, then the price quickly falls towards its fundamental value and then slowly converges on a non-fundamental steady state $x < 0$ (black line). Under an alternative uptick rule, by comparison, the initial drop in price is halted because the short-selling constraint binds; the price then oscillates around this higher value before converging on a non-fundamental steady state with $x > 0$. Thus, the alternative uptick rule leads to a different long run outcome and convergence to quite a different steady state – in this case one where the asset is somewhat overvalued. In Scenario 2, only the degree of bias among fundamentalists is increased, but this is enough to ensure that price converges on the same non-fundamental steady state in both cases (top right). Thus, long run price implications of the alternative uptick rule are absent in this case, though in the short run the drop in price is less severe and price stays higher under the alternative uptick rule (dashed line).

In Scenario 3 (bottom left), the intensity of choice β is set at 4.5 rather than at 3, and this is the only difference relative to Scenario 1. In this case the dynamics settle on permanent price oscillations as in indicated in Figure 2 (top panel). However, the short run price dynamics under an alternative uptick rule are quite different, with an initial price spike after the short-selling constraint first binds, which arises because the short-selling constraint binds on many types simultaneously. Lastly, in Scenario 4, heterogeneity in chartists is added on top of Scenario 3. In this case the reversal in price under an alternative uptick rule is reinforced by trend-following into a permanent price 'bubble' where the asset price diverges to $+\infty$. By comparison, price converges on a non-fundamental steady state $x < 0$ when short-selling constraints are absent, and hence explosive price dynamics are avoided. Thus, the alternative uptick rule causes instability, consistent with Figure 3 (lower panel).

Computation speed and accuracy. To give an idea of computation speed and accuracy, Table 1 reports simulation times for Scenario 3 when the simulation length is 500 periods and the number of types increased from $H = 1,000$ to $H = 10,000$ and then to $H = 50,000$. We simulate with stochastic dividend shocks $d_t = \overline{d} + \epsilon_t$ and allow the coefficient κ to also take on the value of $\kappa = 0$ (original uptick rule), so that short-selling constraints bind more frequently.[13] We also include a measure of accuracy based on the absolute difference between demand and supply at the computed equilibrium price, i.e. $Error_t := |\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$.

The results show that the solution algorithm is fast and accurate. The final column confirms that excess demand is essentially zero in all simulations, and the accuracy here is similar to when short-selling constraints are absent (top rows), in which case the standard analytical solution $x_t = x_t^* := (1+r)^{-1} \sum_{h\in\mathcal{H}} n_{t,h} \tilde{E}_{t,h}[x_{t+1}]$ is used to compute the price and the simulation error. Simulation times are below one second in all cases, increase with the number of types H, and are higher under the original uptick rule (where $\kappa = 0$), since this causes the short-selling constraint to bind in a much larger number of periods, as shown in the fourth column.[14] Even when the short-selling constraint binds much more frequently,

[13] Dividend shocks ϵ_t are drawn from a truncated-normal distribution with mean zero, standard deviation $\sigma_d = 0.01$ and support $[-\overline{d}, \overline{d}]$, such that dividends are guaranteed to be non-negative. We use the same draws of shocks in each simulation in Table 1.

[14] Recall that $\kappa = 0.1$ implies that the short-selling constraint is not present in period t unless the price fell by 10% or more in the previous period. For $\kappa = 0$, the short-selling constraint will be present following

Table 1: Computation times and accuracy in Scenario 3

No. of types	Regime	Time (s)	Bind freq.	$\max(Error_t)$
$H = 1,000$	No short-sell constraints	0.02	-	2.4e-14
	Alt. uptick rule: $\kappa = 0.1$	0.03	1/500	3.2e-14
	Orig. uptick rule: $\kappa = 0$	0.05	34/500	4.8e-14
$H = 10,000$	No short-sell constraints	0.17	-	2.3e-13
	Alt. uptick: $\kappa = 0.1$	0.18	1/500	6.3e-13
	Orig. uptick: $\kappa = 0$	0.25	43/500	3.0e-13
$H = 50,000$	No short-sell constraints	0.82	-	1.2e-12
	Alt. uptick: $\kappa = 0.1$	0.84	1/500	2.3e-12
	Orig. uptick: $\kappa = 0$	0.94	36/500	1.8e-12

Notes: Simulation length $T = 500$ periods. $\max(Error_t) = \max\{Error_0, Error_1, ..., Error_T\}$, where we define the date t simulation error as $Error_t = |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$.

computation times do not increase much and accuracy of the solution is preserved.

5.2.2 Distributional implications

We now consider some distributional effects of an alternative uptick rule. Recall that the evolution of wealth of type h is $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1 + r)(w_{t,h} - p_t z_{t,h})$, such that an alternative uptick rule will affect wealth distribution though its impact on price and equilibrium demands $z_{t,h}$. Further, note that if the short-selling constraint binds on type h at date t, then $z_{t,h} = 0$ and hence their wealth evolves as $w_{t+1,h} = (1 + r)w_{t,h}$. By being out of the market in period t, type h foregoes potential returns but also avoids the possibility of losses; hence the overall implications for their wealth will thus depend on whether they would have on average made returns or losses in the absence of an alternative uptick rule.

We stick with the same four scenarios as in Figure 4 but we focus on a measure of wealth inequality across different types.[15] In particular, we plot the Gini coefficient of the wealth distribution across investor types at each date t. We assume all investor types have equal initial wealth, which we set at $W_{0,h} = 50$ for all h. The results are shown in Figure 5.

An alternative uptick rule has mixed effects on wealth inequality between types. In Scenario 1 (Figure 5, top left), the Gini coefficient initially increases and then settles, but there is a smaller increase in inequality if the alternative uptick rule is present because price does not fall sharply for several periods (see Figure 4, top left), which benefits fundamental types at the expense of chartist types. Such redistribution is smaller and more gradual under the alternative uptick rule because the fall in asset prices is smaller and, since price stabilizes, inequality is lower in the long run. The dynamics are similar in Scenario 2 (top

any previous fall in price (regardless of the magnitude), and short-selling is permitted only on an uptick.

[15] That is, we do not describe the wealth distribution of the population, but rather differentials in wealth due to differences in performance of different forecasting strategies.

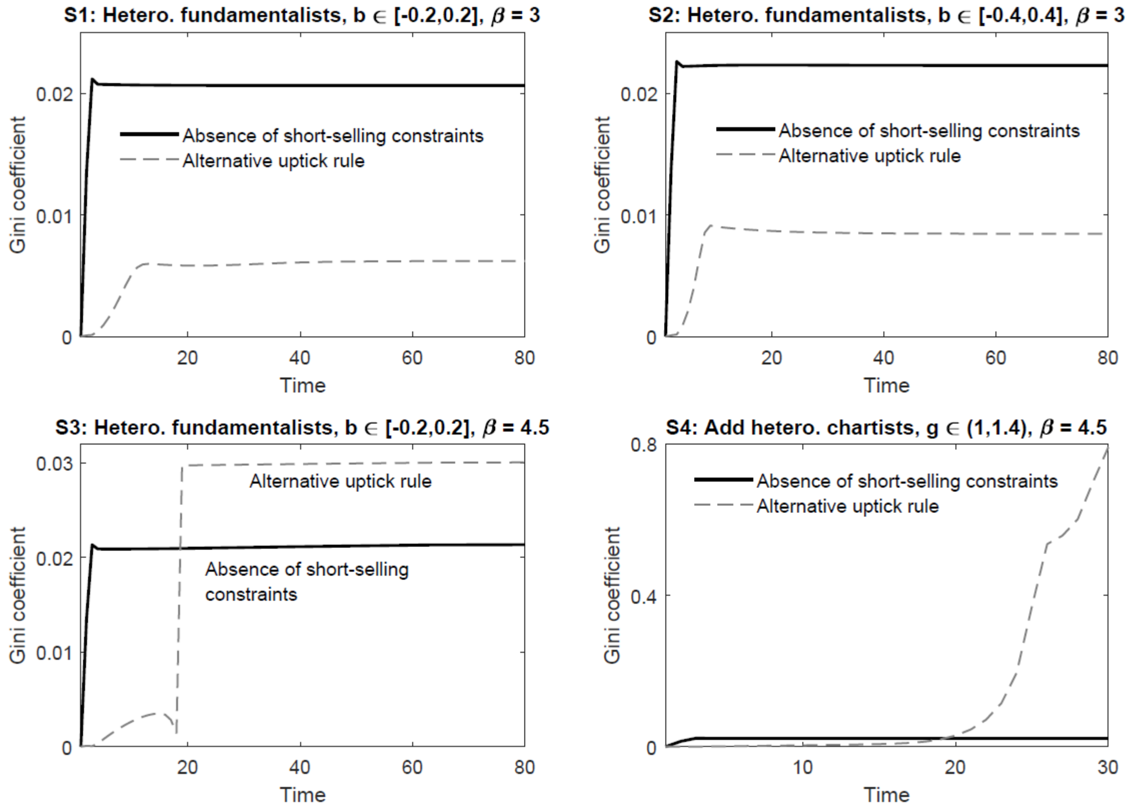

Figure 5: Simulated Gini coefficient of wealth in Scenarios 1 to 4.

right) because the price paths are quite similar to those in Scenario 1.

In Scenario 3, wealth inequality is initially muted under an alternative uptick rule because price rises rather than falls (Figure 4, bottom left). However, this initial period is followed by a severe drop in price, such that more fundamental types outperform more chartist types, and wealth inequality increases before stabilizing (see Figure 5, bottom left). As a result, wealth inequality across types is initially lower under an alternative uptick rule but ends up higher in the long run. Finally, in Scenario 4 (Figure 5, bottom right), wealth inequality across types is initially lower under an alternative uptick rule since the initial period of falling prices is ended as in Scenario 1 (Figure 4, bottom right). However, because price then explodes, chartist types earn profits and fundamental types make losses, such that wealth inequality across types increases and the Gini is around 0.8 by period 30.[16]

To better understand the wealth dynamics in Scenario 4, in Figure 6 we plot the wealth distribution across types in periods $t = 3$, $t = 6$ and $t = 24$ under both unrestricted short-selling (top panel) and an alternative uptick rule (bottom panel). We see that wealth inequalities across types appear rather quickly under unrestricted short-selling, but not under

[16]The 'kink' in period 25 arises because we assume that types that hit negative wealth (in this case more fundamental types) have it reset to zero, and period 25 is the first period in which this rule is triggered.

an alternative uptick rule where the initial fall in price is halted. However, as time increases, the price bubble under the alternative uptick rule soon leads to much greater inequality across types than if short-selling constraints are absent, and by period 24 an extremely large number of types have wealth levels that are a small fraction of the highest wealth type. These results are consistent with the rapid and sustained increase in the Gini coefficient that is observed in Figure 5.

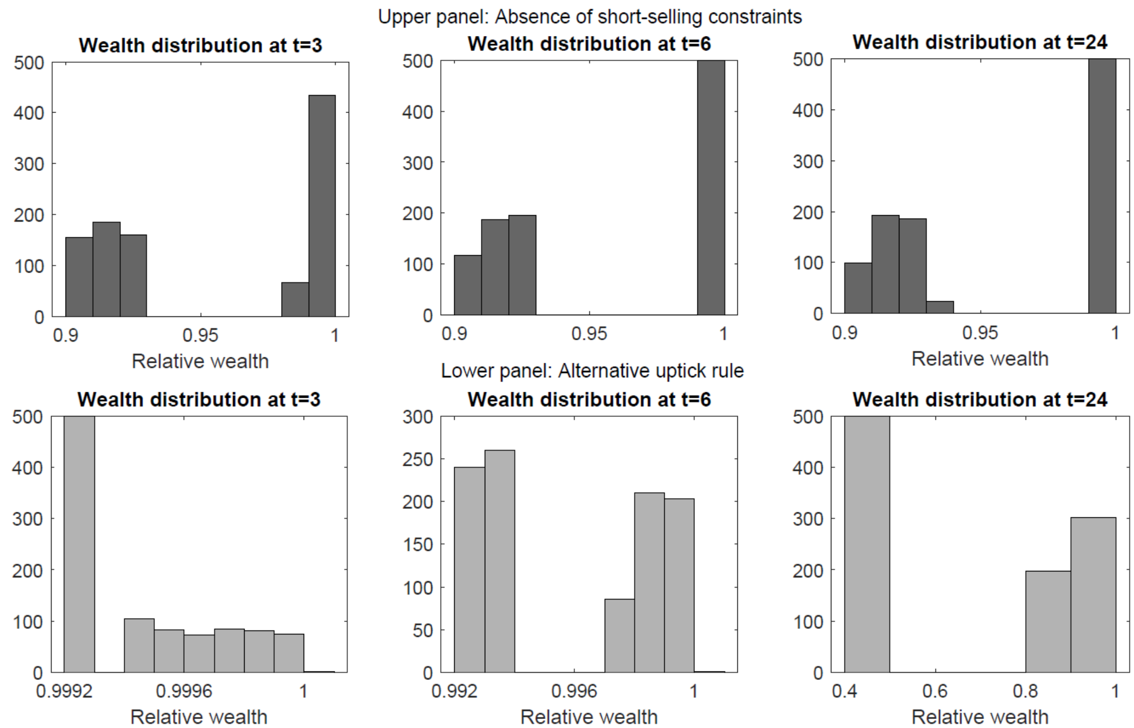

Figure 6: Simulated wealth distribution across types in Scenario 4

6 Conclusion

This paper has studied asset pricing in behavioural heterogeneous-belief models with short-selling constraints and many belief types. Our results provide analytical expressions for asset prices along with conditions on beliefs such that short-selling constraints bind on different types, allowing us to construct computationally-efficient solution algorithms. The analysis is built around a version of the Brock and Hommes (1998) model with short-selling constraints, but we also presented extensions for the cases of multiple risky assets; costly short-selling as in Anufriev and Tuinstra (2013); and pricing by a market-maker.

The utility of these results was illustrated using analytical examples and a numerical application that studied an alternative uptick rule, as currently in place in the United States, in a market with a large number of belief types. The results highlight the complex relationship between the design of short-selling regulations and their implications for asset price stabilization, as judged by mispricing relative to fundamentals and price volatility. We also saw that belief heterogeneity matters for the impact of short-selling constraints on price stability, and that such restrictions can have non-trivial distributional consequences.

There are several promising avenues for future research. First, it would be of interest to investigate whether adding short-selling restrictions in asset pricing models with many agents improves the ability of models to reproduce empirical stylized facts, especially during times of market turmoil, when such constraints are more likely to be active. In a similar vein, it may be feasible to estimate such models in order to evaluate the relative empirical contribution of adding short-selling restrictions. Second, from a policy perspective, there has been interest in whether short-selling restrictions increase price volatility and might cause or exacerbate price bubbles, both in the context of financial markets and other important asset classes such as housing (Scheinkman and Xiong, 2003; Fabozzi et al., 2020). While the main focus in such work has been on implications for price volatility, it would also be of interest to investigate in detail the distributional implications of short-selling restrictions for income and wealth inequality in models with many agents.

Finally, from a technical perspective, there are some modelling specifications of interest which are not covered by the results presented in this paper. For instance, one could confront a large number of investor types with additional restrictions such as a leverage constraint (see in't Veld, 2016), the elimination of investors who hit low or negative wealth, or margin calls that prevent a short position being maintained in future periods. These approaches might have important implications not just for the price effects of short-selling restrictions, but also their distributional implications that have received little attention so far.

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Appendix

Proof of Proposition 1

Existence of a unique equilibrium price follows from Proposition 2.1 of Anufriev and Tuinstra (2013) when $T \to \infty$, such that short-selling becomes prohibitively costly.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \; \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ that $x_t = x_t^* := \frac{\sum_{h\in\mathcal{H}} n_{t,h}\tilde{E}_{t,h}[x_{t+1}]}{1+r}$. The guess is verified if and only if $\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t^* \geq 0 \; \forall h \in \mathcal{H}$, which amounts to $\sum_{h\in\mathcal{H}} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] \leq \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\} + a\sigma^2\overline{Z}$. Given $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, the above inequality simplifies to $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\}) \leq a\sigma^2\overline{Z}$, as stated in Proposition 1.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \; \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0$ $\forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\} > \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$ and hence the market-clearing condition is $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$, which gives $x_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} := x_t^{\mathcal{B}_t^*}$. The guess is verified if and only if $\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t^{\mathcal{B}_t^*} \geq 0 \; \forall h \in \mathcal{B}_t^*$ and $\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z} - (1+r)x_t^{\mathcal{B}_t^*} < 0 \; \forall h \in \mathcal{S}_t^*$, i.e. iff $(\tilde{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z})\sum_{h\in\mathcal{B}_t^*} n_{t,h} \geq (<) \sum_{h\in\mathcal{B}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}$ $\forall h \in \mathcal{B}_t^*$ $(\forall h \in \mathcal{S}_t^*)$, which simplify to $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{B}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\})$, as stated in Proposition 1.

Note that $(1+r)(x_t^{\mathcal{B}_t^*} - x_t^*) = (1 - \frac{1}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}})a\sigma^2\overline{Z} + \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}]}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \sum_{h\in\mathcal{H}} n_{t,h}\tilde{E}_{t,h}[x_{t+1}]$. Since $\sum_{h\in\mathcal{H}} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] = \sum_{h\in\mathcal{B}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] + \sum_{h\in\mathcal{S}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}]$, we have:

$$(1+r)(x_t^{\mathcal{B}_t^*} - x_t^*) = (\sum_{h\in\mathcal{S}_t^*} n_{t,h})\left[\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h\in\mathcal{S}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}]}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}}\right] > 0$$

where $\sum_{h\in\mathcal{S}_t^*} \frac{n_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}}\tilde{E}_{t,h}[x_{t+1}] \leq \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}$ and $\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}\tilde{E}_{t,h}[x_{t+1}] - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}$ is implied by the condition $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \max_{h\in\mathcal{S}_t^*}\{\tilde{E}_{t,h}[x_{t+1}]\}) > a\sigma^2\overline{Z}$ above. ■

Proof of Corollary 1

The first 'if' statement (5) follows from Proposition 1 as $\sum_{h=2}^{\tilde{H}_t} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \tilde{E}_{t,1}[x_{t+1}]) \leq a\sigma^2\overline{Z}$ is equivalent to $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\} \leq a\sigma^2\overline{Z}$ when beliefs are ordered and types with equal optimism receive a population share equal to the sum of the individual population shares. The remaining cases follow as there are $\tilde{H}_t - 1$ candidates for the sets B_t^*, S_t^*, i.e. $S_t = \{1\}, B_t = \{2,..,\tilde{H}_t - 1\}$; $S_t = \{1,2\}, B_t = \{3,..,\tilde{H}_t - 1\}$;...$S_t = \{\tilde{H}_t - 1\}, B_t = \{\tilde{H}_t\}$. For arbitrary $S_t = \{1,..,k\}$, $B_t = \{k+1,...,\tilde{H}_t\}$, where $k \in \{1,..,\tilde{H}_t - 1\}$, we have $x_t = \frac{\sum_{h=k+1}^{\tilde{H}_t} n_{t,h} f_h(x_{t-1},...,x_{t-L}) - \left[\sum_{h=1}^{k} n_{t,h}\right] a\sigma^2\overline{Z}}{(1-\sum_{h=1}^{k} n_{t,h})(1+r)} := x_t^{(k)}$ by the market-clearing condition, and by the expressions in Proposition 1 this guess is verified if and only if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$. ■

Proof of Proposition 2

Existence of a unique equilibrium price is shown in Proposition 2.1 of Anufriev and Tuinstra (2013). The first part of Proposition 2 follows from Case 1 in the Proposition 1 proof. For the remaining cases 2(i)–2(iii) in Proposition 2, we have the following, where we define $f_{t,h} := \tilde{E}_{t,h}[x_{t+1}]$.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

The proof follows Case 2 Proposition 1, except that for the guess to be verified we also require that $\forall h \in \mathcal{S}_{1,t}^*$, $f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T) \geq 0$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t < 0$, where $x_t = ((1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h})^{-1}\left[\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}\right]$. Thus, the guess is verified iff $\sum_{h\in B_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}) - \sum_{h\in\mathcal{B}_t^*} n_{t,h}(1+r)T \leq a\sigma^2\overline{Z} < \sum_{h\in B_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\})$ and (by Prop. 1) $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\})$, as stated in Proposition 2 Part 2(i). Note that $x_t > x_t^*$ since the expression for the price differential $(1+r)(x_t - x_t^*)$ matches that in the Proposition 1 Case 2 proof, and the stated inequalities hold.

Case 2(ii): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and short-sellers $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T)) < 0\ \forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_{2,t}^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of buyers and $\mathcal{S}_{2,t}^*$ is the set of short-sellers. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} > \max_{h\in\mathcal{S}_{2,t^*}}\{f_{t,h}\}$. Under the above guess, the market-clearing condition is $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$, which implies that $x_t = (1+r)^{-1}\left[\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}\right] + \sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}T := \hat{x}_t$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)\hat{x}_t \geq 0\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)(\hat{x}_t - T) < 0\ \forall h \in \mathcal{S}_{2,t}^*$, i.e. if and only if $f_{t,h} + a\sigma^2\overline{Z} \geq \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} + (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} + (1+r)T\sum_{h\in\mathcal{B}_t^*} n_{t,h} < \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}\ \forall h \in \mathcal{S}_{2,t}^*$, which simplify to $\sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}$ and $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - (1+r)T > a\sigma^2\overline{Z}$, as stated in Proposition 2 Part 2(ii). Note that $\hat{x}_t > x_t^* := (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$ since $(1+r)(\hat{x}_t - x_t^*) = \sum_{h\in S_{2,t}^*} n_{t,h}(1+r)T > 0$.

Case 2(iii): Buyers $h \in \mathcal{B}_t^*$, constrained types $h \in \mathcal{S}_{1,t}^*$, and short-sellers $h \in \mathcal{S}_{2,t}^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_{1,t}^*$, and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T)) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of buyers, $\mathcal{S}_{1,t}^*$ is the set of short-selling constrained types, and $\mathcal{S}_{2,t}^*$ is the set of short-sellers. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} > \max_{h\in\mathcal{S}_{1,t^*}}\{f_{t,h}\} > \max_{h\in\mathcal{S}_{2,t^*}}\{f_{t,h}\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}} n_{t,h}z_{t,h}$ and hence market-clearing condition requires $\sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}} n_{t,h}z_{t,h} = \overline{Z}$, giving $x_t = \frac{\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}[f_{t,h}+a\sigma^2\overline{Z}]+(1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}} := \tilde{x}_t$ where $\mathcal{B}_t^* \cup \mathcal{S}_{2,t} = \mathcal{H} \setminus \mathcal{S}_{1,t}^*$ is used. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)\tilde{x}_t \geq 0 \ \forall h \in \mathcal{B}_t^*$, $f_{t,h} + a\sigma^2\overline{Z} - (1+r)(\tilde{x}_t - T) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$, and $0 \leq f_{t,h} + a\sigma^2\overline{Z} - (1+r)(\tilde{x}_t - T) < (1+r)T \ \forall h \in \mathcal{S}_{1,t}^*$, i.e. iff $\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}$; $\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_{2,t}^*}\{f_{t,h}\} - (1+r)T) > a\sigma^2\overline{Z} - (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}$; $\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{S}_{2,t}^*}\{f_{t,h}\} - (1+r)T) \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}$ and $\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}) > a\sigma^2\overline{Z} - (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}$, which can be simplified to the inequality stated in Proposition 2 Part 2(iii).

Note that $(1+r)(\tilde{x}_t - x_t^*) = \frac{\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}[f_{t,h}+a\sigma^2\overline{Z}]+(1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}} - \sum\limits_{h\in\mathcal{H}} n_{t,h}f_{t,h}$. Using $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} = \sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h} + \sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h}$ and simplifying, we have:

$$(1+r)(\tilde{x}_t - x_t^*) = (\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}) \left[\frac{\sum\limits_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h} + \frac{(1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}}{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}} \right] > 0$$

where $\sum\limits_{h\in\mathcal{S}_{1,t}^*} \frac{n_{t,h}}{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}} f_{t,h} \leq \max\limits_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}$ and $\frac{\sum\limits_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h} + \frac{(1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}}{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}} > \max\limits_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}$ is implied by condition $\sum_{h\in\mathcal{H}\setminus\mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \max\limits_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}) > a\sigma^2\overline{Z} - (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}$ above. ■

Proof of Corollary 2

Part 1 follows immediately from Proposition 2 as $\sum_{h=2}^{\tilde{H}_t} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \tilde{E}_{t,1}[x_{t+1}]) \leq a\sigma^2\overline{Z}$ is equivalent to $\sum_{h\in\mathcal{H}} n_{t,h}(\tilde{E}_{t,h}[x_{t+1}] - \min\limits_{h\in\mathcal{H}}\{\tilde{E}_{t,h}[x_{t+1}]\} \leq a\sigma^2\overline{Z}$ when beliefs are ordered and types with equal optimism receive a population share equal to the sum of the individual population shares. Part 2 of Corollary 2 applies when *one type* does not buy ($\overline{k} = 1$), so we are in either Case 2(i) or Case 2(ii) of Proposition 2. By Proposition 2, we are in Case 2(i) if $\exists \ \mathcal{B}_t^* = \{2, ..., \tilde{\mathcal{H}}_t\}, \mathcal{S}_{1,t}^* = \{1\}$ such that $\max\{d_{1,t}, d_{2,t}\} \leq a\sigma^2\overline{Z} < \sum_{h\in B_t^*} n_{t,h}(f_{t,h} - f_{t,1})$, where $d_{1,t} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - f_{t,2})$, $d_{2,t} = -\sum_{h\in\mathcal{B}_t^*} n_{t,h}(1+r)T + \sum_{h\in B_t^*} n_{t,h}(f_{t,h} - f_{t,1})$, in which case $x_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$. By Proposition 2, we are in Case 2(ii) if $\exists$

$\mathcal{B}_t^* = \{2, ..., \tilde{\mathcal{H}}_t\}, \mathcal{S}_{2,t}^* = \{1\}$ such that $d_{1,t} \leq a\sigma^2\overline{Z} - \sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}(1+r)T < d_{2,t}$, where $d_{1,t} = \sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,2})$, $d_{2,t} = \sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,1}) - (1+r)T$, and $x_t = (1+r)^{-1}\sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h}f_{t,h} + n_{t,1}T$.

Part 3 of Corollary 2 applies when two or more types do not buy, so we are either in Case 2(i), Case 2(ii) or Case 2(iii) of Proposition 2. For Case 2(i), we have the same as above, except that $\mathcal{B}_t^* = \{\overline{k}+1, .., \tilde{\mathcal{H}}_t\}, \mathcal{S}_{1,t}^* = \{1, ..., \overline{k}\}$ (for some $\overline{k} \in \{2, .., \tilde{\mathcal{H}}_t - 1\}$) and $d_{1,t} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - f_{t,\overline{k}+1})$. For Case 2(ii), we have the same as above, except that $\mathcal{B}_t^* = \{\overline{k}+1, .., \tilde{\mathcal{H}}_t\}, \mathcal{S}_{2,t}^* = \{1, ..., \overline{k}\}$ (for some $\overline{k} \in \{2, .., \tilde{\mathcal{H}}_t - 1\}$), $d_{1,t} = \sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,\overline{k}+1})$, and $d_{2,t} = \sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,\overline{k}}) - (1+r)T$, and $x_t = \frac{\sum_{h=1}^{\tilde{\mathcal{H}}_t} n_{t,h}f_{t,h} + \sum_{h=1}^{\overline{k}} n_{t,h}(1+r)T}{1+r}$. We are in Case 2(iii) if $\exists\, \mathcal{B}_t^* = \{\overline{k}+1, .., \tilde{\mathcal{H}}_t\}$, $\mathcal{S}_{1,t}^* = \{\underline{k}+1, ..., \overline{k}\}$, $\mathcal{S}_{2,t}^* = \{1, ..., \underline{k}\}$ (for some $\underline{k} \in \{1, ..., \overline{k}-1\}$, $\overline{k} \in \{2, .., \tilde{\mathcal{H}}_t - 1\}$) such that $\max\{d_{1,t}, d_{2,t}\} \leq a\sigma^2\overline{Z} - \sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h}(1+r)T < \min\{d_{3,t}, d_{4,t}\}$, where $d_{1,t} = \sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}^*} n_{t,h}(f_{t,h} - f_{t,\overline{k}+1})$, $d_{2,t} = \sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}^*} n_{t,h}(f_{t,h} - f_{t,\underline{k}} - (1+r)T)$, $d_{3,t} = \sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}^*} n_{t,h}(f_{t,h} - f_{t,\overline{k}})$, $d_{4,t} = \sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}^*} n_{t,h}(f_{t,h} - f_{t,\underline{k}} - (1+r)T)$, in which case $x_t = \frac{\sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}^*} n_{t,h}[f_{t,h} + a\sigma^2\overline{Z}] + (1+r)T\sum_{h\in\mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*\cup\mathcal{S}_{2,t}^*} n_{t,h}}$ by Prop. 2. ■

Proof of Proposition 3

Uniqueness of price follows from price impact equation. We let $f_{t,h} := \tilde{E}_{t,h}[x_{t+1}]$ to ease notation.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t)\ \forall h \in \mathcal{H}$, which implies by the price equation $x_t = \frac{x_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}} := x_t^*$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t^* \geq 0\ \forall h \in \mathcal{H}$, i.e. iff $\left(\frac{1}{1+r} + \frac{\mu\lambda}{a\sigma^2}\right)(a\sigma^2\overline{Z} + \min_{h\in\mathcal{H}}\{f_{t,h}\}) \geq x_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})$, which simplifies to the inequality in Proposition 3 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t)\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$, so $x_t = \frac{x_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}}$ by the price equation. Guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t \geq 0 (< 0)\ \forall h \in \mathcal{B}_t^*\ (\forall h \in \mathcal{S}_t^*)$, i.e. iff $\frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z} - x_{t-1} - \mu(1-\lambda)Z_{t-1} < \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{S}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$.

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0\ \forall h \in \mathcal{H}$, which implies by price equation that $x_t = x_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t < 0\ \forall h \in \mathcal{H}$, i.e. iff $\max_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z} < (1+r)(x_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}])$, which simplifies to the inequality in Proposition 3 Part 2(ii). ■

***Supplementary Material can be found at:** https://github.com/MCHatcher.