Solving heterogeneous-belief asset pricing models with short selling constraints and many agents

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Abstract

Short-selling constraints are common in financial markets, while physical assets such as housing often lack markets for short-selling altogether. As a result, investment decisions are often restricted by such constraints. This paper studies asset prices in behavioural heterogeneous-belief models with short-selling constraints. We provide conditions on beliefs such that short-selling constraints are binding with arbitrarily many belief types, along with analytical expressions for price and demands that allow us to construct fast solution algorithms relevant for a wide range of models. Extensions include *conditional* short-selling constraints, multiple asset markets, and a market maker. An application studies how an alternative uptick rule, as in the United States, affects price dynamics and wealth distribution in a market with many belief types in *evolutionary competition*. In a numerical example we show a scenario in which a modified version of the alternative uptick rule, triggered by smaller falls in price, reduces both asset mispricing and wealth inequality relative to the current regulation.

Keywords: Asset pricing, heterogeneous beliefs, short-selling constraints, updating.

1 Introduction

The practice of short-selling is common in financial markets but also widely regulated. When investors go short, they borrow and immediately sell a financial asset before repurchasing and returning the asset to the lender, closing their position. Whereas a long position can be thought of as a bet that asset prices will increase, short-selling allows investors to bet on a fall in asset prices. It has been argued that such betting may increase volatility in financial markets. A common policy response among regulators has been to restrict short-selling; for example, during the 2008-9 financial crisis many countries introduced short-selling bans following sharp declines in asset prices. Similar short-selling bans were reinstated in some

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European economies during the 2011-12 sovereign debt crisis and the Covid-19 outbreak (see Siciliano and Ventoruzzo, 2020). It is therefore important that researchers be able to solve asset pricing models with short-selling constraints in an efficient manner.

In this paper, we show how to efficiently solve dynamic, behavioural asset pricing models with short-selling constraints and arbitrarily many heterogeneous beliefs.[1] We are thinking here of discrete time heterogeneous-belief asset pricing models such as Brock and Hommes (1998), LeBaron et al. (1999), and Westerhoff (2004); for instance, the population shares of different types may be endogenously determined by *evolutionary competition*. We also allow beliefs to potentially depend on the current asset price (in a linear fashion), which is a generalization relative to some models in the literature.[2] For the case of such dynamic models, we derive expressions for the market-clearing price and demands and show how these results can be used to construct fast solution algorithms, such that researchers can add short-selling constraints in models with thousands of agents or belief types. Our algorithm is supported by analytical results that do not seem to have been documented previously.

We provide results for a benchmark asset pricing model, as well as several other cases studied in the literature: *conditional* short-selling constraints, the case of multiple asset markets with short-selling constraints; and the market-maker approach to price determination. We also show how our benchmark results can be related to certain physical investment assets (housing) or to models in which beliefs are determined by an individual's social network.

Our analysis is built around the behavioural heterogeneous-beliefs asset pricing model (Brock and Hommes, 1998). We allow arbitrarily many belief types whose population shares may be exogenous or determined by *evolutionary competition*. The model with evolutionary competition has already been studied in the many-types case by Brock et al. (2005), who allow short-selling by investors. We show that when investors face short-selling constraints, the market-clearing price and demands depend on *belief dispersion* across types. We also provide a fast solution algorithm that exploits these analytical results.

The difficulty in the many-types case stems from the demand functions being *piecewise-linear*, such that the market-clearing price depends on how many types are short-selling constrained. For a market with a large number of investor types, it is computationally intensive to solve for a price and demands. To overcome this problem, we exploit the fact that types who are short-selling constrained in a given period must be *more pessimistic* than those agents who were unconstrained, such that ordering types in terms of optimism reduces computational burden and solution time. As a result, it becomes feasible to simulate models with many heterogeneous belief types over many periods while retaining solution accuracy.

We provide both analytical and numerical examples; hence we keep contact with known models in the literature while demonstrating the speed and accuracy of our algorithm. In Section 5 we provide a *policy application* which studies an *alternative uptick rule*, as currently in place in the United States, in a model with a large number of belief types whose

[1] Short-selling constraints appear to have first been studied, in a static model, by Miller (1977).

[2] For example, in Brock and Hommes (1998) beliefs do not depend on the current asset price.

population shares are determined by *evolutionary competition*. The alternative uptick rule is a 'circuit breaker' that bans short-selling if prices fall 10% or more in the previous trading period; surprisingly, there do not appear to be any previous assessments of this rule in the heterogeneous-beliefs literature. Our results indicate that an alternative uptick rule may attenuate (or prevent) falls in price, but we also find that such rules can hinder price discovery, increase price volatility and lead to explosive price paths. An alternative uptick rule can also have substantive distributional implications in terms of wealth inequality. In a numerical example we find that a modified rule, triggered by price falls smaller than 10%, may reduce both mispricing and wealth inequality relative to the current regulation.

The closest papers in the literature are Anufriev and Tuinstra (2013) and Dercole and Radi (2020). Anufriev and Tuinstra (2013) add trading costs for short-selling into a two-type asset pricing model and find that this leads to additional (non-fundamental) steady states as beliefs are updated more aggressively; in a similar vein, but with the addition of a leverage constraint, see in't Veld (2016). By comparison, Dercole and Radi (2020) study the original 'uptick rule' in the United States from 1938–2007, which banned short-selling at lower prices, and find that there is no clear-cut impact on price volatility. There is also a wider literature on non-smooth asset pricing models (see e.g. Tramontana et al., 2010); short-selling constraints are a specific application that gives rise to such models.

The above papers all consider a small number of investor types and solve for prices and demands in specific cases. The present paper contributes to the literature by (i) solving for price and demands when there are arbitrarily many belief types with general price predictors who can be in evolutionary competition, and (ii) providing efficient solution algorithms. We make our results accessible by providing examples and highlighting several extensions, such as *conditional* short-selling constraints, multiple risky assets, and the market-maker approach.

Our paper is part of a growing literature studying heterogeneous beliefs, asset prices and the effectiveness of regulatory policies in financial markets (Westerhoff, 2016). In financial market models it is known that differences in beliefs combined with short-selling constraints can lead to price bubbles (see e.g. Scheinkman and Xiong, 2003), but such regulations could also aid market stability as noted above. There has also been interest in the impact of short-selling restrictions in markets for *physical* investment assets like housing which are subject to boom and bust (see Shiller, 2015; Fabozzi et al., 2020). Our results are thus of potential relevance for physical as well as financial assets, as we explain using housing as an example.

Section 2 presents a baseline model for which analytical results are presented in Section 3. along with some analytical and numerical examples. Section 4 presents three extensions of the baseline model, and Section 5 presents our policy application. Section 6 concludes.

2 Model

Consider a finite set of myopic, risk-averse investor types $\mathcal{H} = \{h_1, ..., h_H\}$. At each date $t \in \mathbb{N}_+$, each type $h \in \mathcal{H}$ chooses a portfolio of a risky asset $z_{t,h}$ and a riskless bond paying

$\tilde{r} > 0$ to maximize a mean-variance utility function over future wealth with risk-aversion parameter $a > 0$. The risky asset has current price p_t, future price p_{t+1}, and pays stochastic dividends d_{t+1}, which are exogenous. Investors form subjective expectations of the future price and future dividends of the risky asset as described below. The underlying model follows Brock and Hommes (1998), except the risky asset is in positive net supply $\overline{Z} > 0$ and short-selling is ruled out by constraints of the form $z_{t,h} \geq 0$ for all t and h.

2.1 Asset demand

We denote the subjective expectation of type h at date t by $\tilde{E}_{t,h}[.]$, and the subjective variance by $\tilde{V}_{t,h}[.]$. The optimal portfolio choice of type h solves the problem:[3]

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad z_{t,h} \geq 0 \tag{1}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+\tilde{r})(w_{t,h} - p_t z_{t,h})$ is future wealth, $w_{t,h} - p_t z_{t,h}$ is holdings of the risk-free asset, and $\tilde{V}_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$, with $\sigma^2 > 0$ and weight $a/2 > 0$.

Given short-selling constraints, the demand of each investor type $h \in \mathcal{H}$ is:

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \frac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right]}{1+\tilde{r}} \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right]}{1+\tilde{r}}. \end{cases} \tag{2}$$

If the price p_t is small enough, then type h's short-selling constraint is slack and their demand for the risky asset decreases with the price; this is the standard demand function that arises in Brock and Hommes (1998), where short-selling constraints are absent. However, if the current price is high enough to make the expected excess return of type h *negative*, then the short-selling constraint will bind on type h and their position in the risky asset is zero.

Dividends follow an IID process $d_t = \overline{d} + \epsilon_t$, where $\overline{d} > 0$ and ϵ_t is a zero-mean shock with constant variance. All investor types know the dividend process, such that $\tilde{E}_{t,h}\left[d_{t+1}\right] = \overline{d}$ for all t and h; note that there is no loss of generality as our solution nests a generic specification of $\tilde{E}_{t,h}\left[d_{t+1}\right]$ at no extra cost.[4] From Equation (2) we see that the short-selling constraint is more likely to bind on type h the more *pessimistic* their price expectation $\tilde{E}_{t,h}\left[p_{t+1}\right]$.

2.2 Price beliefs

We consider generic price beliefs which are *boundedly-rational* and can depend linearly on the current price p_t via a common coefficient (we relax the latter assumption later on).

[3]We assume (as is standard) that these operators satisfy some of the basic properties of conditional expectation operators, namely, $\tilde{E}_{t,h}[y_t] = y_t$ and $\tilde{V}_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $\tilde{V}_{t,h}[x_t y_{t+1}] = x_t^2 \tilde{V}_{t,h}[y_{t+1}]$.

[4]See the definition of $f_{t,h}$ in (4), which potentially allows $\tilde{E}_{t,h}\left[d_{t+1}\right]$ to vary over time and across types.

Assumption 1 *All price beliefs are of the form:*

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{c}p_t + \tilde{f}_{t,h} \tag{3}$$

where $\overline{c} \in [0, 1+\tilde{r})$ *and* $\tilde{f}_{t,h} \in \mathbb{R}$ *is a generic forecast that cannot depend on current price* p_t.

Assumption 1 allows a wide range of boundedly-rational beliefs. The coefficient $\overline{c}$ allows linear dependence of price expectations on the current price; for example, investors may extrapolate one-for-one on top of the current price like the extrapolators in Barberis et al. (2018) or they may use the information of the current price with some weight (see, for instance, LeBaron et al., 1999; Westerhoff, 2004). We assume $\overline{c} \geq 0$ to allow the case of no dependence on the current price (i.e. $\overline{c} = 0$) and we assume $\overline{c} < 1+\tilde{r}$ to ensure that individual demands are *decreasing* in the current price p_t; for reference, see (4) below. *Time-varying or heterogeneous values of* $\overline{c}$ are discussed as an extension in Section 3.3.

The generic forecast $\tilde{f}_{t,h}$ (which can differ across types and over time) permits a potentially non-linear response to past prices, such as non-linear trend-following rules. In addition, $\tilde{f}_{t,h}$ may contain type-specific 'fixed effects', be subject to random disturbances, or be influenced by social networks as in Yang (2009) or Panchenko et al. (2013). Assumption 1 in Brock and Hommes (1998) is nested by (3) when $\overline{c} = 0$ and $\tilde{f}_{t,h} = E_t[p^*_{t+1}] + g_h(x_{t-1}, ..., x_{t-L_h})$, where $g_h : \mathbb{R}^{L_h} \to \mathbb{R}$ is a function that can differ across types, L_h is the lag of type h, and $x_t := p_t - p^*_t$ is the price deviation from the fundamental price p^*_t.

For convenience, let $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2\overline{Z}$ and $r := \tilde{r} - \overline{c}$. Given IID dividends, $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and the demands in (2) can be written as

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r}. \end{cases} \tag{4}$$

Writing demands in terms of $f_{t,h}$ is convenient because the latter does *not* depend on the current price p_t and allows us to add the term $a\sigma^2\overline{Z}$ in the numerator of the demand function (see top line of (4)), which simplifies the algebra of the price solution; see Proposition 1. Writing demands this way is also consistent with a 'deviation from fundamentals' representation; see Brock and Hommes (1998) and several other papers in the related literature.[5]

2.3 Population shares

We allow the population shares $n_{t,h}$ of investor types to be endogenous and time-varying, but we rule out any dependence on the current price p_t (see Assumption 2).

[5]Given our assumptions, the fundamental price is $p^*_t = \overline{p} := (\overline{d} - a\sigma^2\overline{Z})/\tilde{r}$, so $f_{t,h} = \tilde{f}_{t,h} + \tilde{r}\overline{p}$ (see (4)). Thus, $f_{t,h} - (1+r)p_t = \hat{E}_{t,h}\left[x_{t+1}\right] - (1+r)x_t$, where $r = \tilde{r} - \overline{c}$, $x_t := p_t - \overline{p}$, and $\hat{E}_{t,h}\left[x_{t+1}\right] := \tilde{f}_{t,h} - (1-\overline{c})\overline{p}$. Demands follow (4), with x_t replacing p_t and $\hat{E}_{t,h}\left[x_{t+1}\right]$ replacing $f_{t,h}$. For an applied example, see Sec. 5.

Assumption 2 *We consider population shares of the form $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ where $\hat{n}_h$ is a real function such that $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, $n_{t,h} \in (0,1)$ $\forall t, h$, and $\mathbf{n}_{t-1}$ ($\mathbf{u}_{t-1}$) is the vector of past population shares (resp. past fitness levels). In particular, we rule out dependence of $n_{t,h}$ on the current asset price p_t (though not dependence on lagged prices p_{t-1}, p_{t-2} etc.).*

Assumption 2 is quite general. For instance, population shares may be *endogenously* determined by evolutionary competition as in Brock and Hommes (1998). Following Brock and Hommes (1997), a popular approach is a discrete choice logistic model $n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}} \exp(\beta U_{t,h})}$, where the intensity of choice $\beta \in [0,\infty)$ determines how fast agents switch to better-performing predictors. Various fitness measures $U_{t,h}$ are used in the literature, including realized profits net of a predictor cost (Brock and Hommes, 1998) and forecast accuracy (e.g. Ap Gwilym, 2010), potentially in a risk-adjusted measure of profits (De Grauwe and Grimaldi, 2006). While Assumption 2 rules out the 'extreme' population shares of 0 or 1, it is straightforward (though analytically cumbersome) to relax this assumption, and we show how this can be done in Section 5 of the *Supplementary Appendix*.[6]

The presence of past fitness levels $\mathbf{u}_{t-1}$ in the function $\hat{n}_h$ allows evolutionary competition mechanisms such as the logit specification above, while the inclusion of past population shares $\mathbf{n}_{t-1}$ allows asynchronous updating (see Hommes, 2013, Ch. 5). It would not impose any extra burden to allow the function $\hat{n}_h$ to be time-varying or to include additional endogenous variables in the vector $\mathbf{u}_{t-1}$; however, as emphasized in Assumption 2, we rule out dependence of population shares on the *current* price p_t (or any future values).

The special case of fixed population shares $n_{t,h} = 1/H$ is relevant for agent-based or social network models where types are *individuals*, while exogenous time-varying population shares may be used to study herding in beliefs, as in the models of Kirman (1991, 1993).

3 Solving the model

The asset market clears when $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ subject to (4) and Assumptions 1 and 2. Given positive outside supply $\overline{Z} > 0$, there exists a unique market-clearing price p_t (see Anufriev and Tuinstra, 2013, Proposition 2.1). We now characterize the price and demands.

3.1 Benchmark results

Proposition 1 *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types) at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$, then no type is short-selling constrained at

[6] Allowing $n_{t,h} \in [0,1]$ means the set of types relevant for price determination (i.e. market-clearing) can be time-varying, which reduces analytical tractability of the results relative to Proposition 1.

date t ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{5}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$, at least one type is short-selling constrained and $\exists$ unique non-empty sets $\mathcal{B}_t^* \subset \mathcal{H}$, $\mathcal{S}_t^*$ such that $\sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_t^*}\{f_{t,h}\})$, and the price and demands are given by

$$p_t = \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h}) a\sigma^2 \overline{Z}}{(1+r)\sum_{h \in \mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{6}$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. See the Appendix. ■

Proposition 1 gives the market-clearing price and demands for an arbitrarily large set of belief types whose population shares may be endogenously determined. Since the results apply at any date $t \in \mathbb{N}_+$, we can find a solution for $t = 1, 2, ...$, starting from period 1. We see that if *no* types are short-selling constrained, the asset price depends on *all beliefs* as in (5); however, when short-selling constraints *bind*, only the beliefs of the unconstrained types (i.e. 'buyers') matter for price determination; see (6). Part (i) of Proposition 1 gives a simple condition on beliefs that can be used to check, in one computation, whether short-selling constraints are slack for all types. If so, the price is given by p_t^* in (5), which is the standard solution in the absence of short-selling constraints; see e.g. Brock and Hommes (1998).

Whether short-selling constraints bind depends on *belief dispersion* relative to the most pessimistic type, $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\})$; see Part (i). If belief dispersion is small enough relative to the (risk-adjusted) outside supply $a\sigma^2\overline{Z}$, then no types are short-selling constrained at date t. Otherwise, we are in Part (ii) of Proposition 1, such that at least one type (and at most $H-1$ types) are short-selling constrained. In this case, the sets of unconstrained and short-selling constrained types $\mathcal{B}_t^*$, $\mathcal{S}_t^*$ are determined by 'cut-off' conditions which require that for unconstrained types $h \in \mathcal{B}_t^*$, the average belief dispersion *within the group* is sufficiently small, whereas for the short-selling constrained types $h \in \mathcal{S}_t^*$ this condition of sufficiently small belief dispersion is not met for any type in the set.

Finally, notice that when one or more types are short-selling constrained, the market-clearing price p_t is higher than the (hypothetical) price p_t^* if short-selling constraints were absent – i.e. short-selling constraints raise the asset price, as argued by Miller (1977).

We now give an application of these results using a simple two-type example.

Example 1 *Consider two types h_1, h_2 with generic beliefs f_{t,h_1}, f_{t,h_2} that satisfy Assumption 1 and have the form $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$, as in (4), for $h = h_1, h_2$. The population shares* *are given by a discrete-choice logistic model:* $n_{t,h} = \frac{exp(\beta U_{t-1,h})}{\sum_{h \in \{h_1,h_2\}} exp(\beta U_{t-1,h})}$, *where* $\beta \in [0, \infty)$ *is the intensity of choice,* $U_{t,h} = R_t z_{t-1,h}$ *is the profit of type* h *at date* t, $R_t = p_t + d_t - (1+\tilde{r})p_{t-1}$ *is the realized return, and* $z_{t-1,h}$ *is the demand of type* h *at date* $t-1$*; see* (4)*.*

By Proposition 1, if $\sum_{h \in \{h_1,h_2\}} n_{t,h}(f_{t,h} - \min\{f_{t,h_1}, f_{t,h_2}\}) \leq a\sigma^2 \overline{Z}$ *neither type is short-selling constrained, and* $p_t = \sum_{h \in \{h_1,h_2\}} n_{t,h} f_{t,h}/(1+r)$ *by* (5)*, where* $r = \tilde{r} - \bar{c}$*. If the above condition is not met, then either* $f_{t,h_1} - f_{t,h_2} > a\sigma^2 \overline{Z}/n_{t,h_1}$ *(if* h_1 *is more optimistic) or* $f_{t,h_2} - f_{t,h_1} > a\sigma^2 \overline{Z}/n_{t,h_2}$ *(if* h_2 *is more optimistic). In the former case,* $\mathcal{B}_t^* = \{h_1\}$, $\mathcal{S}_t^* = \{h_2\}$*, and by* (6) *the market-clearing price is* $p_t = [(1+r)n_{t,h_1}]^{-1}(n_{t,h_1} f_{t,h_1} - (1-n_{t,h_1})a\sigma^2 \overline{Z})$*, with demands* $z_{t,h_1} = \overline{Z}/n_{t,h_1}$, $z_{t,h_2} = 0$*. In the latter case,* $\mathcal{B}_t^* = \{h_2\}$, $\mathcal{S}_t^* = \{h_1\}$*, so* $p_t = [(1+r)n_{t,h_2}]^{-1}(n_{t,h_2} f_{t,h_2} - (1-n_{t,h_2})a\sigma^2 \overline{Z})$ *and* $z_{t,h_1} = 0$, $z_{t,h_2} = \overline{Z}/n_{t,h_2}$*.*

Suppose that beliefs follow the two-type Brock and Hommes (1998) model, where $\bar{c} = 0$ *such that* $r = \tilde{r}$*. Type* h_1 *is a fundamentalist with* $\tilde{E}_{t,h_1}[p_{t+1}] = \overline{p}$*, where* $\overline{p} = (\overline{d} - a\sigma^2 \overline{Z})/r$ *is the fundamental price, and* h_2 *is a 1-lag chartist:* $\tilde{E}_{t,h_2}[p_{t+1}] = \overline{p} + \overline{g}(p_{t-1} - \overline{p})$*, where* $\overline{g} > 0$*. Note that these beliefs imply that* $f_{t,h_1} = (1+r)\overline{p}$ *and* $f_{t,h_2} = (1+r)\overline{p} + \overline{g}(p_{t-1} - \overline{p})$*; see* (4)*. Assuming* $p_{t-1} > \overline{p}$*, the chartist is more optimistic at date* t*, and hence by Proposition 1:*

$$p_t = \begin{cases} \overline{p} + \dfrac{n_{t,h_2}\overline{g}(p_{t-1} - \overline{p})}{1+r} & \text{if } \overline{g}(p_{t-1} - \overline{p}) \leq a\sigma^2 \overline{Z}/n_{t,h_2} \\ \overline{p} + \dfrac{n_{t,h_2}\overline{g}(p_{t-1} - \overline{p}) - (1-n_{t,h_2})a\sigma^2 \overline{Z}}{n_{t,h_2}(1+r)} & \text{if } \overline{g}(p_{t-1} - \overline{p}) > a\sigma^2 \overline{Z}/n_{t,h_2} \end{cases} \tag{7}$$

which replicates a known result in the literature.[7]

The above example is particularly simple: if belief dispersion is large enough that some type is constrained, then ranking the two types by optimism immediately determines the set of short-selling constrained types ($\mathcal{S}_t^*$) and the set of unconstrained types ($\mathcal{B}_t^*$). In a general setting with many types, however, there are many candidates for the sets $\mathcal{B}_t^*$, $\mathcal{S}_t^*$, and this number increases *exponentially* as the number of types H is increased. In fact, including the case where short-selling constraints are slack for all types, there are $2^H - 1$ candidates for $\mathcal{B}_t^*$, $\mathcal{S}_t^*$.[8] As a result, the task of finding the price is *computationally intensive* when there are a large number of types H, as seems plausible in many real-world asset markets.

To overcome this problem, we now present a version of Proposition 1 that reduces the number of candidates that need to be checked and is useful for computational purposes. Following Anufriev and Tuinstra (2013), we use the fact that types who are short-selling constrained in a given period t must be *more pessimistic* than those who were unconstrained (see (4)), such that ranking types in terms of optimism can speed up discovery of the set of short-selling constrained types $\mathcal{S}_t^*$. We have already seen a relevant case in Example 1:

[7] In particular, given our assumption that $p_{t-1} > \overline{p}$, Equation (7) is consistent with Proposition 2.2 in Anufriev and Tuinstra (2013) when short-selling is prohibitively costly.

[8] The number of candidate sets equals the cardinality of the power set of $\mathcal{H}$ minus 1. The 'minus 1' correction arises because the asset market cannot clear if $\mathcal{B}_t^*$ is an empty set (i.e. if no type holds the asset).

knowing that the chartist type was more optimistic than the fundamental type allowed us to narrow down to 2 cases for the price rather than 3 ($= 2^2 - 1$) if beliefs were left unordered. We now show how this principle can be applied in a general setting with many belief types.

Consider the function $\tilde{h}_t : \mathcal{H} \to \tilde{\mathcal{H}}_t$, where $\tilde{\mathcal{H}}_t := \{1, ..., \tilde{H}_t\}$ is an adjusted set of types with the property that the most optimistic type(s) in $\mathcal{H}$ get label $\tilde{H}_t$, the next most optimistic type(s) gets label $\tilde{H}_t - 1$, and so on, down to the least optimistic type(s) in $\mathcal{H}$ with label 1. Types with equal optimism get the *same* label, so $\tilde{H}_t \leq H$, which implies that $|\tilde{\mathcal{H}}_t| \leq |\mathcal{H}|$. In the case of ties in terms of optimism, the period t population share of the 'group' is the sum of the population shares of the individual types. We first present a corollary based on the adjusted set of types $\tilde{\mathcal{H}}_t$, before presenting a computationally-efficient algorithm.

Corollary 1 *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set defined above, such that beliefs are ordered as $\tilde{E}_{t,1}[p_{t+1}] < \tilde{E}_{t,2}[p_{t+1}] < ... < \tilde{E}_{t,\tilde{H}_t}[p_{t+1}]$, or equivalently $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. Let $disp_{t,k} := \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,k})$, where $k \in \{1, ..., \tilde{H}_t - 1\}$. Then we have the following:*

$$p_t = \begin{cases} \dfrac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{1+r} := p_t^* & \text{if } disp_{t,1} \leq a\sigma^2\overline{Z} \\ \dfrac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2\overline{Z}}{(1-n_{t,1})(1+r)} := p_t^{(1)} & \text{if } disp_{t,2} \leq a\sigma^2\overline{Z} < disp_{t,1} \\ \dfrac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} - (n_{t,1}+n_{t,2}) a\sigma^2\overline{Z}}{(1-n_{t,1}-n_{t,2})(1+r)} := p_t^{(2)} & \text{if } disp_{t,3} \leq a\sigma^2\overline{Z} < disp_{t,2} \\ \qquad\vdots & \qquad\vdots \\ \dfrac{n_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t-1} n_{t,h}) a\sigma^2\overline{Z}}{(1-\sum_{h=1}^{\tilde{H}_t-1} n_{t,h})(1+r)} := p_t^{(\tilde{H}_t-1)} & \text{if } disp_{t,\tilde{H}_t-1} > a\sigma^2\overline{Z} \end{cases} \tag{8}$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } 2 \leq k < k^*. \tag{9}$$

Proof. It follows from Proposition 1. See the Appendix. ■

Corollary 1 streamlines the task of finding the market-clearing price. In Proposition 1, where beliefs are unordered, there are $2^H - 1$ cases to check, as compared to $\tilde{H}_t \leq H$ when belief types are ordered as in Corollary 1. Clearly, this amounts to a substantial reduction in computational burden in models with a large number of types H. For example, with only 15 distinct beliefs (types) at date t, there are $2^{15} - 1 = 32,767$ candidates for the sets $\mathcal{B}_t^*, \mathcal{S}_t^*$ when types are not ordered by optimism. However, if we order types from least to most optimistic and construct the set $\tilde{\mathcal{H}}_t = \{1, , ..., 15\}$, then there are only 15 candidates for the sets and the market-clearing price, corresponding to Corollary 1 when $\tilde{H}_t = 15$. Ranking types in terms of optimism has previously been suggested by Anufriev and Tuinstra (2013); the main difference with our algorithm is that it draws on analytical results that facilitate fast computation of solutions with many types; see Tables 1 and 2, Sections 3.4 and 5.2.1.

The final part of Corollary 1 is important. It tells us that the market price when one or more short-selling constraints are binding is higher than the (hypothetical) price p_t^* if short-selling constraint are absent, and that $p_t^* < p_t^{(1)}$ and $p_t^{(1)} < p_t^{(2)} < ... < p_t^{(k^*-1)} < p_t^{(k^*)}$, i.e. the price is smaller the fewer short-selling constraints are assumed to be binding. These properties are useful because we can use the unconstrained solution p_t^* to obtain a *lower bound* $\underline{k}$ for the actual number of types k^* who are short-selling constrained, by counting the number of negative (unconstrained) demands at price p_t^*. This gives our algorithm a good start. In a similar way, counting the number of negative (unconstrained) demands at prices $p_t^{(k)}$, for $k < k^*$, will give an improved estimate of k^* when it lies above the lower bound $\underline{k}$.

We now present a computational algorithm which efficiently finds the number of short-selling constrained types k^* and hence the market-clearing price and demands.

3.2 Computational algorithm

Our computational algorithm is easy to use and exploits analytical results in Corollary 1. The main steps in the algorithm are as follows:

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering beliefs as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$ and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1})$. If $disp_{t,1} \leq a\sigma^2\overline{Z}$, accept $p_t = p_t^*$ as the date t price, compute demands and move to period $t + 1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$ and find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k}+a\sigma^2\overline{Z}-(1+r)p_t^{guess}}{a\sigma^2} < 0$, and denote this value $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \leq a\sigma^2\overline{Z} < disp_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right] a\sigma^2\overline{Z}}{(1+r)\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}}$, compute demands and move to period $t+1$.

The above algorithm is efficient for two reasons. First, if the condition in Step 2 is met, no computation time is wasted checking cases where one of more types have binding short-selling constraints. Second, if the condition in Step 2 is not met, using the unconstrained solution p_t^* as a guess immediately gives a lower bound $\underline{k}$ on the number of short-selling constrained types k^*. Note that $\underline{k}$ is a lower bound for k^* since $p_t^{(k)} > p_t^*$ for all $k \leq k^*$ (see Corollary 1); that is, binding short-selling constraints *raise* price relative to the counterfactual scenario of no constraints. Therefore, if types $1, ..., k$ would like to short-sell at price p_t^*, they *must* also be short-selling constrained at price $p_t^{(k^*)} > p_t^*$ (see (4)), implying that $k^* \geq \underline{k}$.

In practice, we have found that the speed of the computational algorithm can be improved using an iterative procedure. Specifically, rather than increasing k in steps of 1 from the initial value $\underline{k}$ as in Step 3 (whenever $\underline{k}$ is not a solution), the algorithm will 'jump' closer to the true number of constrained types k^* by repeatedly replacing p_t^{guess} with the price $p_t^{(k)}$ based on the current guess of k in Step 3 and then generating an updated value of k, say $k = k'$, that equals the number of negative (unconstrained) demands at this price. In other words, we exploit the property that $p_t^{(1)} < ... < p_t^{(k^*-1)} < p_t^{(k^*)}$ (Corollary 1) to find k^* faster. Our simulations suggest that with a large number of types such as several thousand or more, there is a considerable speed-up with 5-10 iterations of this procedure.

Our solution algorithm is fast even with very large numbers of belief types. Computation speed is thus an important advantage of our approach relative to existing algorithms, such as Anufriev and Tuinstra (2013), that do not utilize our analytical results.[9]

3.3 Generalizations and nested cases

Before turning to a numerical example, we briefly discuss some cases which are either nested or require only minor extensions. Any formal results appear in the *Supplementary Appendix*.

3.3.1 Nested cases

First, note that if the types $h_1, h_2, ..., h_H$ are *individual investors* we may fix the population shares at $n_{t,h} = 1/H$. This interpretation is relevant for asset pricing models with many agents that differ in beliefs; for example, agent-based models as in LeBaron et al. (1999) or the *social network* model with type updating in Hatcher and Hellmann (2022). In the social network model of Panchenko et al. (2013), type updating follows the Brock and Hommes (1998) model, except that only the types (and performance) of investors in an agent's *social network* can be observed and adopted; see Panchenko et al. (2013, Eq. 10). These cases are nested by the benchmark results because the demand schedules in these papers have the same functional form as in (2) and (4), and the beliefs are nested by Assumption 1.

Second, there has been some interest in the inability to short housing as a possible explanation for rising house prices and price volatility (Shiller, 2015; Fabozzi et al., 2020). Our model may be reinterpreted as a housing investment model if we replace the exogenous expected dividends with exogenous imputed rents from housing, as in Bolt et al. (2019). In this case, the analytical results are essentially unchanged and we just need a re-labelling of variables, as shown in Section 2.1 of the *Supplementary Appendix*.

Third, several other cases of interest are essentially nested by the benchmark model, including (i) heterogeneity in expected dividends $\tilde{E}_{t,h}[d_{t+1}]$, which is nested by defining the forecast as $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}[d_{t+1}] - a\sigma^2\overline{Z}$ (i.e. without assuming $\tilde{E}_{t,h}[d_{t+1}] = \overline{d}$); and (ii) a time-varying response of beliefs to the current price, such that $\overline{c}$ is replaced with some time-varying scalar $\overline{c}_t \in [0, 1 + \tilde{r})$ (see (3)) which may be exogenous or endogenous but

[9] The existing papers in the literature all consider small numbers of types in numerical simulations (see Anufriev and Tuinstra, 2013; in't Veld, 2016; Dercole and Radi, 2020). Anufriev and Tuinstra (2013, Appendix A) do present an algorithm for an arbitrary number of belief types, but their approach requires evaluating the value of aggregate demand at each guess due to the absence of analytical results. Our algorithm does not require such a check and can quickly find the solution as described above.

cannot depend on the current price p_t. Lastly, consider short-selling constraints of the form $z_{t,h} \geq L$, where $L < 0$, such that negative positions are permitted up to some limit. We show formally how the case $z_{t,h} \geq L$ is nested in Section 2.2 of the *Supplementary Appendix*.

3.3.2 Additional heterogeneity

We now consider additional sources of heterogeneity. Assumption 1 allows a common response of beliefs to price via the term $\overline{c}p_t$ in (3). Allowing a time-varying $\overline{c}$ is straightforward (see above), and we now allow *heterogeneity across types*. In this case, price beliefs are

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{c}_h p_t + \tilde{f}_{t,h} \tag{10}$$

where $\overline{c}_h \in [0, 1+\tilde{r})$. Defining $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r_h := \tilde{r} - \overline{c}_h$, demands are now

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \end{cases} \tag{11}$$

where the only difference relative to (4) is that r is now *type-specific*.

Equation (11) shows that optimism is no longer determined solely by $f_{t,h}$; however, we can distinguish least and most optimistic types by looking at the term $\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$, since a given type h will short-sell only if this term is sufficiently small. As a result, we can give an amended version of Proposition 1 in which the sets of unconstrained and short-selling constrained types $\mathcal{B}_t^*, \mathcal{S}_t^*$ depend on $\min_{h\in\mathcal{B}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$ and $\max_{h\in\mathcal{S}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$, rather than $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}$ and $\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ as in the benchmark case (see Proposition 1, part (ii)). Likewise we can amend Corollary 1 and the computational algorithm of Section 3.2 for heterogeneous $\overline{c}_h$. We provide the analytical results and an updated algorithm in Section 4.1 of the *Supplementary Appendix*. We also give a numerical example in Section 3.4 below.

A similar approach can be used when there is heterogeneity in *subjective return variances* (or risk aversion); see (1). In this case, the terms $a\sigma^2$ in the denominator of the demand function (2) become type-specific, i.e. $a\sigma_h^2$, and it is convenient to define $\tilde{a}_h := (a\sigma_h^2)^{-1}$ and $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - \overline{Z}/\tilde{a}_h$. The demands of types $h \in \mathcal{H}$ can then be written as

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) & \text{if } p_t \leq \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r} \end{cases} \tag{12}$$

where $r = \tilde{r} - \overline{c}$ as before.

In this case, optimism depends on $f_{t,h} + \overline{Z}/\tilde{a}_h$ – i.e. types who short-sell must have lower values of $f_{t,h}+\overline{Z}/\tilde{a}_h$ than those who do not. As a result, it is easy to provide amended versions of Proposition 1 and Corollary 1 and to adjust the computational algorithm. We include these results, and a numerical example, in Section 4.2 of the *Supplementary Appendix*.

3.4 Numerical example

We now turn to a numerical example. Consider $H = 3{,}000$ belief types with heterogeneity in the weights $\bar{c}_h$ on the current price (as in Section 3.3.2 above) and fixed population shares $n_{t,h} = 1/H$ for all t, h (we study an evolutionary competition example in Sec. 5). There are three groups of investors consisting of 1,000 types each; within each group individuals use the same forecasting method, but their individual forecasts (i.e. beliefs) differ. Trend-followers expect the future change in price to be linked to past changes in price; contrarians believe the recent trend in prices will be reversed; and arbitrageurs base their expectations on the deviation of the current price from a fundamental price $\bar{p}$. The first 1,000 types are trend-followers, types 1,001–2,000 are contrarians, and types 2,001–3,000 are arbitrageurs.

All investors use the *current* price as a reference point, but we allow heterogeneity in the weights $\bar{c}_h$, as in Section 3.3.2. In addition, each type has an idiosyncratic random component to beliefs $u_{t,h}$. Beliefs of trend-followers have the form $\tilde{E}_{t,h}[p_{t+1}] = \bar{c}_h p_t + g_h^1 \Delta p_{t-1} + g_h^2 \Delta p_{t-2} + u_{t,h}$, where $g_h^1, g_h^2 > 0$ and $\Delta p_t = p_t - p_{t-1}$. Contrarians have beliefs $\tilde{E}_{t,h}[p_{t+1}] = \bar{c}_h p_t + g_h^3 \Delta p_{t-1} + g_h^4 \Delta p_{t-2} + u_{t,h}$, where $g_h^3, g_h^4 < 0$. For arbitrageurs, $\tilde{E}_{t,h}[p_{t+1}] = \bar{c}_h p_t - g_h^5 (p_{t-1} - \bar{p}) + u_{t,h}$, where $g_h^5 > 0$. We set $d_t = \bar{d} = 1.1$, $\tilde{E}_{t,h}[d_{t+1}] = \bar{d}$ for all h, $\tilde{r} = 0.1$, $a = \sigma^2 = 1$ and $\bar{Z} = 0.1$. The fundamental price is therefore $\bar{p} = \frac{\bar{d} - a\sigma^2 \bar{Z}}{\tilde{r}} = 10$.

Prior to period 1, the parameters g_h^1 and g_h^2 are drawn from uniform distributions on (0,0.5) and (0,0.2), g_h^3, g_h^4 are drawn from a uniform distribution on (-0.1,0), and g_h^5 is drawn from a uniform distribution on (0.2,0.8). The $\bar{c}_h$ parameters are drawn from a uniform distribution on $(0.95, 1.05)$. The idiosyncratic shocks $u_{t,h}$ are set at zero in periods 1–10 and are drawn from a normal distribution $\mathcal{N}(0, 0.04^2)$ in all later periods. Initial prices are set at $\bar{p} + 0.6 = 10.6$. Figure 1 shows time series of the price and number of short-selling constrained types in a simulation of $T = 500$ periods, of which the first 40 periods are shown.

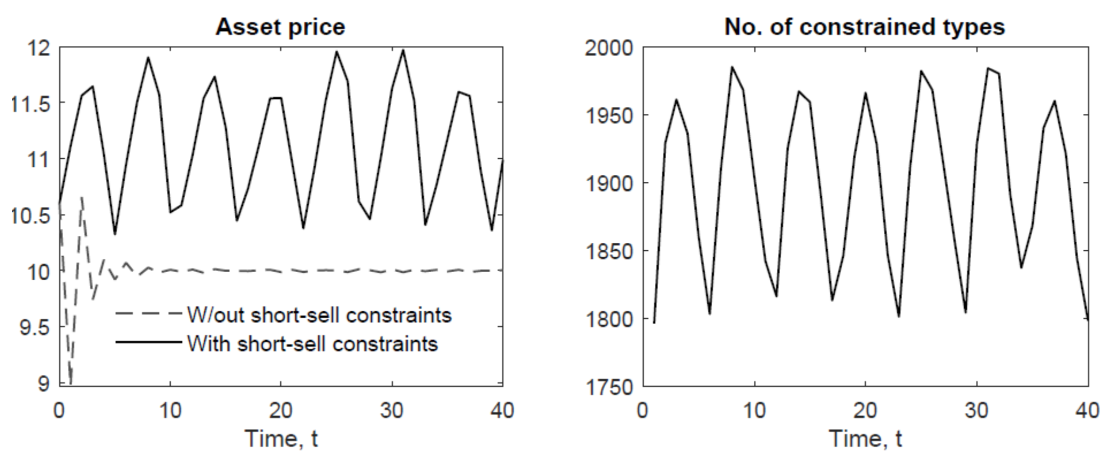

Figure 1: Simulation paths plotted over the first 40 periods ($H = 3{,}000$ types). The left panel shows the dynamics of the asset price p_t from periods $t = 1, .., 40$ and the right panel shows the number of short-selling constrained types, $|\mathcal{S}_t^*|$, in each period.

Figure 1 (left panel) shows that short-selling constraints have a substantial impact on the asset price dynamics. With short-selling constraints there are persistent price cycles (solid line); by contrast, when short-selling constraints are absent there are dampening price oscillations that rapidly converge toward the fundamental price (dashed line, left panel). The price is higher with short-selling constraints, and it oscillates depending the number of types which are short-selling constrained in a given period (see right panel).

Table 1: Computation times and accuracy: $H = 3,000$ types and $T = 500$ periods

Case	Short-sale constraints	Time (s)	Bind freq.	$\max(Error_t)$
No heterogeneity:	No	0.09	-	3.8e-16
$\overline{c}_h = 1$ for all h	Yes	0.16	497/500	4.3e-14
Heterogeneity 1:	**No**	**0.40**	**-**	**2.0e-15**
$\boldsymbol{\overline{c}_h \in (0.95, 1.05)}$	**Yes**	**0.66**	**500/500**	**8.9e-16**
Heterogeneity 2:	No	0.13	-	5.3e-16
$\overline{c}_h \in (0.995, 1.005)$	Yes	0.22	500/500	4.1e-15

Note: $\max(Error_t) = \max\{Error_1, ..., Error_T\}$, $Error_t := |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$. The middle row of the table (bold font) shows the results for the case in Figure 1 and described above. The other cases change the amount of heterogeneity in $\overline{c}_h$ with all other parameters and shocks fixed.

In Table 1 we report computation times and a measure of accuracy for the example in Figure 1, as well as two variations on this example. The first variant (bottom row) reduces the heterogeneity in $\overline{c}_h$, while the second case (top row) eliminates heterogeneity in $\overline{c}_h$ altogether. Our measure of accuracy (final column) is based on the maximum deviation from market clearing across all periods, i.e. $\max(Error_t) := \max_{1\le t\le T} |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$. The results in Table 1 show that the solutions with short-selling constraints are computed quickly using our algorithm: computation times for a 500 period simulation with 3,000 types are below 1 second in all three cases. Further, both computation time and accuracy are comparable to the case where short-selling constraints are absent (see top rows in Table 1), which is based on the standard analytical solution p_t^* and does not require any search procedure.[10]

4 Extensions

We now present several extensions of the baseline model set out above, including *conditional* short-selling constraints; the case of multiple risky assets; and the case where the price is determined by a market-maker who adjusts price in response to excess demand.

[10]The simulations were run in Matlab 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

4.1 Conditional short-selling constraints

Thus far we studied *unconditional* short-selling constraints: $z_{t,h} \geq 0$ for all t and h. Such constraints apply in *all periods* regardless of the evolution of the price. However, in practice many short-selling restrictions are *conditional*: short-selling is banned at date t only if a certain price condition is met. For instance, the 'uptick rule' in the United States from 1938-2007 banned short-selling only if price fell in the last trading interval; in 2010 this was replaced by an alternative uptick rule which bans short-selling when price falls 10% or more.

To model *conditional* short-selling constraints, let $g(p_{t-1}, ..., p_{t-K})$ be the 'trigger' for the short-selling constraint, with K being the longest lag in the price that is considered. If $g(p_{t-1}, ..., p_{t-K}) \leq 0$ the short-selling constraint is present at date t; if $g(p_{t-1}, ..., p_{t-K}) > 0$, the short-selling constraint is lifted. Sticking with our two examples, the original uptick rule has a trigger of the form $g^{UR}(p_{t-1}, p_{t-2}) = p_{t-1} - p_{t-2}$, whereas the alternative uptick rule has $g^{AUR}(p_{t-1}, p_{t-2}) = p_{t-1} - (1-\kappa)p_{t-2}$, where $\kappa = 0.1$ (i.e. 10%).

To nest generic rules, we introduce an indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K}) \leq 0\}}$ which is equal to 1 if the short-selling constraint is present at date t (i.e. if $g(p_{t-1}, ..., p_{t-K}) \leq 0$), and equal to 0 otherwise. With this formulation, investors may take negative positions in periods where the indicator variable is zero (short-selling constraints absent) but are restricted to non-negative positions in periods where the indicator variable is 1 (short-selling ban); for further details, see Section 1.2 of the *Supplementary Appendix*.

The demand of type $h \in \mathcal{H}$ is thus given by

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d} - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d}}{1+\tilde{r}} \text{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d}}{1+\tilde{r}} \text{ and } \mathbb{1}_t = 1. \end{cases} \tag{13}$$

Note that the only difference in the demand schedules relative to (2) is compound 'if-or' and 'if-and' statements, whose second part depends on the value of the indicator variable. As a result, it is straightforward to use the same approach as in Proposition 1 and Corollary 1 to find the market-clearing price and demands, as explained in the following remark.

Remark 1 *In the above model with a conditional short-selling constraint, the market-clearing price and demands follow Proposition 1, except that in part (i) the 'if...' statement is replaced by 'if...or $\mathbb{1}_t = 0$', and in part (ii) the 'if...' statement is replaced by 'if...and $\mathbb{1}_t = 1$'. A proposition and proof are provided in Section 3.1 of the Supplementary Appendix.*

4.2 Multiple asset markets

We now consider multiple risky assets $M \geq 2$ in positive net supply. Let $z_{t,h}^m$ be the date t demand of type h for asset $m \in \{1, ..., M\}$. Following Westerhoff (2004), we assume type h's demand for asset m depends not only on the expected excess return on asset m, but also on the relative attractiveness of that asset. In particular, suppose a fraction $w_t^m \in (0, 1)$

of each investor type participates in a given market m, with this fraction determined by comparison with all other markets (see below). Differently from Westerhoff, we allow all M asset markets to have unconditional short-selling constraints, such that $z_{t,h}^m \geq 0$ for all m. Each asset market m has IID dividends $d_t^m = \overline{d}^m + \epsilon_t^m$, and we assume $\tilde{E}_{t,h}[d_{t+1}^m] = \overline{d}^m > 0$.

Analogous to (2), the demand of type $h \in \mathcal{H}$ in market $m \in \{1, ..., M\}$ is

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{\tilde{E}_{t,h}\left[p_{t+1}^m\right] + \overline{d}^m - (1+\tilde{r}) p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{\tilde{E}_{t,h}\left[p_{t+1}^m\right] + \overline{d}^m}{1+\tilde{r}} \\ 0 & \text{if } p_t^m > \frac{\tilde{E}_{t,h}\left[p_{t+1}^m\right] + \overline{d}^m}{1+\tilde{r}} \end{cases} \tag{14}$$

where p_t^m is price in market m and σ_m^2 is the subjective return variance (assumed constant).

The demand function (14) has the same form as in the benchmark case (see (2)), except for the scaling by the share w_t^m that participates in the market. As in Westerhoff (2004), we assume the participation shares w_t^m depend on relative attractiveness of each market A_t^m:

$$w_{t+1}^m = \frac{\exp(\beta A_t^m)}{\sum_{m=1}^M \exp(\beta A_t^m)}, \qquad A_t^m = f([p_t^m - \overline{p}^m]) \tag{15}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function with $f(0) = 0$ and $\beta \in [0, \infty)$ is the intensity of choice.

Equation (15) states that the participation shares w_t^m are determined by *evolutionary competition*, with the fitness of market m, A_t^m, depending on the deviation of the market price from the fundamental price $\overline{p}^m$.[11] The fundamental price in market m is the price that would clear the market if all types $h \in \mathcal{H}$ had common rational expectations $E_t[.]$. Given equations (14)–(15), the fundamental price in market $m \in \{1, ..., M\}$ is:[12]

$$\overline{p}^m = \frac{\overline{d}^m - a\sigma_m^2 M \overline{Z}_m}{\tilde{r}} \tag{16}$$

where $\overline{Z}_m > 0$ is the fixed supply of asset m per investor.

Belief types in each market m follow a market-specific version of Assumption 1 (see (3)):

$$\tilde{E}_{t,h}\left[p_{t+1}^m\right] = \overline{c}^m p_t^m + \tilde{f}_{t,h}^m, \qquad \overline{c}^m \in [0, 1+\tilde{r}) \tag{17}$$

where $\tilde{f}_{t,h}^m$ is a generic price forecast of type h in market m that does not depend on p_t^m.

Let $f_{t,h}^m := \tilde{f}_{t,h}^m + \overline{d}^m - a\sigma_m^2 \overline{Z}_m / w_t^m$ and $r^m := \tilde{r} - \overline{c}^m$ (see (3)–(4)). Then, by (14), the

[11] Westerhoff (2004) sets $f([p_t^m - \overline{p}^m]) = \ln\left[(1 + c[p_t^m - \overline{p}^m]^2)^{-1}\right]$, where $c > 0$, such that attractiveness declines with distance from the fundamental price due to the risk of being caught in a bubble that collapses.

[12] If all investors are fundamentalists, then $A_t^m = f(0) = 0$ for all m, such that $w_t^m = 1/M$ for all m. Using this result in conjunction with the demands (14), common expectations $E_t[p_{t+1}^m]$ and market-clearing leads to the equation $p_t^m = (1+\tilde{r})^{-1}[E_t[p_{t+1}^m] + \overline{d}^m - a\sigma_m^2 M \overline{Z}_m]$, which can be solved forwards to give (16).

demand of type h in market m can be written as

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1+r^m)p_t^m}{a\sigma_m^2}\right) & \text{if } p_t^m \leq \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m} \\ 0 & \text{if } p_t^m > \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m}. \end{cases} \tag{18}$$

We assume the population shares in each market m are determined by Assumption 2. Market-clearing in each market is thus given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m / w_t^m, \qquad \text{where } \tilde{z}_{t,h}^m := z_{t,h}^m / w_t^m. \tag{19}$$

With the change in variables in (19), the market-clearing condition has the same form as in the benchmark model, except for a scaling of supply by $1/w_t^m$. Hence, we have the following.

Remark 2 *In the above model with M risky assets subject to short-selling constraints, the expressions for the market-clearing prices p_t^m and demands $z_{t,h}^m$ in each market $m \in \{1, ..., M\}$ are given by Proposition 1, except that p_t, $f_{t,h}$, r, $\overline{Z}$ must be replaced by p_t^m, $f_{t,h}^m$, r^m, $\overline{Z}_m / w_t^m$, and the demands $z_{t,h}$ are replaced by market-specific demands $z_{t,h}^m$ in* (18)*. A re-worked version of Proposition 1 for this case is provided in Section 3.2 of the Supplementary Appendix.*

4.3 Market-maker approach

We now return to one risky asset and let price be determined by a market-maker rather than market-clearing; see, for example, Beja and Goldman (1980), Chiarella (1992), Farmer and Joshi (2002) and Westerhoff (2003). As is standard in the literature, we consider price impact functions which are linear in excess demand. We allow the price set by the market maker to potentially depend on both current and past excess demand as follows:[13]

$$p_t = p_{t-1} + \mu[\lambda(Z_t - \overline{Z}) + (1-\lambda)(Z_{t-1} - \overline{Z})] \tag{20}$$

where $\mu > 0$, $\lambda \in (0, 1]$ and $Z_t := \sum_{h \in \mathcal{H}} n_{t,h} z_{t,h}$ is aggregate demand per investor at date t, such that $Z_t - \overline{Z}$ can be interpreted as (average) excess demand per investor.

When $\lambda \in (0, 1)$, past demand matters for the current price, whereas if $\lambda = 1$ only current demand Z_t matters. We stick with the beliefs $\tilde{E}_{t,h}[p_{t+1}] = \bar{c}p_t + \tilde{f}_{t,h}$ in Assumption 1 and consider two specifications of asset demands. In the first case we work with the demand functions considered thus far; see (2) and (4). In the second case we allow a different demand specification as in some models that use the market-maker approach.

[13] Allowing price to be a nonlinear function of *past* excess demand $Z_{t-1} - \overline{Z}$ does not pose any difficulty as this variable is predetermined at date t; however, we use linearity in *current* excess demand to solve for p_t.

4.3.1 Benchmark demand specification

We first consider demands as in (4), with $f_{t,h} = \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r = \tilde{r} - \overline{c}$. We can easily solve for the price and demands in this case as summarized in Proposition 2.

Proposition 2 *Let p_t be the price given by (20) at date $t \in \mathbb{N}_+$, let $n_{t,h} = \hat{n}_h(\mathbf{n}_{t-1}, \mathbf{u}_{t-1})$ be the population share of type h at date t, and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained (short-selling constrained) types at date t. Then the following holds:*

1. *If $p_{t-1} - \frac{1}{1+r}\min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$, $z_{t,h} \geq 0\ \forall h$) and price is given by*

$$p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}}.$$

2. *If $p_{t-1} - \frac{1}{1+r}\min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + \frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z}$, then one or more types are short-selling constrained with $z_{t,h} = 0$ and we have the following:*

   (i) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\min\limits_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1} < \frac{\mu\lambda}{a\sigma^2}\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1+r}\max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, price is

$$p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1 - \lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$$

   with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

   (ii) Else, $\exists \mathcal{B}_t^* = \emptyset, \mathcal{S}_t^* = \mathcal{H}$ such that $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1+r}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \left(\mu + \frac{a\sigma^2}{1+r}\right)\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h$), and price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See the Appendix. ■

There are *three* distinct cases in Proposition 2, in contrast to Proposition 1. The reason is that *all types* may be short-selling constrained at the price set by the market maker. By contrast, under market-clearing as in Proposition 1, at least one type must buy the risky asset in each period. Analogous to the results in Proposition 1, we have that cases 2(i) and 2(ii) in Proposition 2 imply a higher price than when short-selling constraints are absent.

Corollary 1 and the Computational Algorithm can easily be amended for this case.[14] To check for short-sellers in period t, we find the set $\tilde{\mathcal{H}}_t$ as before and Step 2 of the algorithm is amended to $disp_{t,1} \leq (\mu + (1+r)^{-1}a\sigma^2)\,\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$, where $disp_{t,1} := \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\tilde{\mathcal{H}}_t} n_{t,h}(f_{t,h} - f_{t,1}) - \frac{1}{1+r}f_{t,1}$ (Proposition 2, Part 1). If this condition is satisfied, price follows Proposition 2 Part 1; if not, then following Proposition 2 Part 2(i), we search for a

[14] We provide the full details in Section 3.3 of the *Supplementary Appendix*.

$k^* \in \{1, ..., \tilde{H}_t - 1\}$ such that $disp_{t,k^*+1} \le (\mu + (1+r)^{-1} a\sigma^2)\,\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1} < disp_{t,k^*}$, where $disp_{t,k} := \frac{\mu\lambda}{a\sigma^2} \sum_{h>k} n_{t,h}(f_{t,h} - f_{t,k}) - \frac{1}{1+r} f_{t,k}$. If such a k^* exists, p_t is given by Proposition 2 Part 2(i) . Finally, if there is no k^* that satisfies the above condition, then *all types* are short-selling constrained in period t. By Proposition 2, Part 2(ii), this is the case if $disp_{t,\tilde{H}_t} > (\mu + (1+r)^{-1} a\sigma^2)\,\overline{Z} - p_{t-1} - \mu(1-\lambda)Z_{t-1}$ and the price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

4.3.2 Alternative demand specification

We now consider an alternative demand specification, as used in several papers in the market-maker literature; this section describes the solution for the price and demands in this case.

A common specification for demand is $\tilde{a}_h(\tilde{E}_{t,h}[p_{t+1}] - p_t)$, where $\tilde{a}_h > 0$; see, for example, Westerhoff (2004). With a short-selling constraint $z_{t,h} \ge 0$, the demands are adjusted to

$$z_{t,h} = \begin{cases} \tilde{a}_h(\tilde{E}_{t,h}[p_{t+1}] - p_t) & \text{if } p_t \le \tilde{E}_{t,h}[p_{t+1}] \\ 0 & \text{if } p_t > \tilde{E}_{t,h}[p_{t+1}]. \end{cases} \tag{21}$$

The key difference relative to (2) is that demand is scaled by the *type-specific* coefficient $\tilde{a}_h$. Note that more pessimistic types – i.e. those with lower expectations $\tilde{E}_{t,h}[p_{t+1}]$ – are more likely to be short-selling constrained at a given price p_t set by the market-maker.

We let $f_{t,h} := \tilde{f}_{t,h} = \tilde{E}_{t,h}[p_{t+1}] - \overline{c}p_t$ (see (3)) and write the demands in (21) as:

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} - (1+r)p_t) & \text{if } p_t \le \frac{f_{t,h}}{1+r} \\ 0 & \text{if } p_t > \frac{f_{t,h}}{1+r} \end{cases} \tag{22}$$

where $r := -\overline{c}$ and we assume $\overline{c} \in [0, 1)$ to ensure demands are decreasing in the price.

The demands in (22) match (4), except the 'intercept' $a\sigma^2\overline{Z}$ is absent, the expected dividend $\overline{d}$ is absent, and the scaling $\tilde{a}_h$ is type-specific. As a result, the price solution and demands are similar to Proposition 2, as summarized in the following remark.

Remark 3 *When demands are given by* (22) *and a market-maker sets price following* (20)*, the price solution and demands follow Proposition 2, except that $n_{t,h}$ is replaced by $\tilde{n}_{t,h} := n_{t,h}\tilde{a}_h$ (for all h), all terms $a\sigma^2$ are set at 1, and whenever $\overline{Z}$ appears it is multiplied only by the coefficient μ. A proposition and proof are given in Section 3.4 of the Supplementary Appendix, along with an amended version of Corollary 1 and the Computational Algorithm.*

We now turn to our policy application in which heterogeneous belief types are subject to conditional short-selling constraints and engage in *evolutionary competition*; we draw here on the results in Sections 3.1, 3.2 and 4.1, including Proposition 1, Corollary 1 and the Computational Algorithm. Codes are available on the author's Github page (https://github.com/MCHatcher), along with the *Supplementary Appendix.*

5 Application: Alternative uptick rule

We consider an application with an alternative uptick rule, as currently in place in the United States. Under the rule, short-selling is banned next period following price falls of 10% or more. This contrasts with the original uptick rule in place from 1938 to 2007, which banned short-selling of shares following any fall in price (regardless of the magnitude). We work with a version of the Brock and Hommes (1998) model with a large number of types and an alternative uptick rule; this case does not seem to have been studied previously.

Since the alternative uptick rule bans short-selling following price falls of 10% or more (but *not* otherwise) it is a *conditional* short-selling constraint; see Section 4.1. Accordingly, the indicator variable has the form $\mathbb{1}_t := \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\le 0\}}$ for $\kappa = 0.1$, and the solution is described by Remark 1. Demands of types $h \in \mathcal{H}$ are given by (13):

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d} - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \le \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d}}{1+\tilde{r}} \text{ or } \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\le 0\}} = 0 \\ 0 & \text{if } p_t > \frac{\tilde{E}_{t,h}[p_{t+1}] + \overline{d}}{1+\tilde{r}} \text{ and } \mathbb{1}_{\{p_{t-1}-(1-\kappa)p_{t-2}\le 0\}} = 1 \end{cases} \tag{23}$$

where we have assumed IID dividends $d_t = \overline{d} + \epsilon_t$ with $\tilde{E}_{t,h}[d_{t+1}] = \overline{d}\ \forall t, h$.

Equation (23) shows that short-selling is banned in period t only if $p_{t-1} \le (1-\kappa)p_{t-2}$. Following Brock and Hommes (1998), we consider linear predictors of the form:

$$\tilde{E}_{t,h}[p_{t+1}] = \overline{p} + b_h + g_h(p_{t-1} - \overline{p}), \qquad b_h \in \mathbb{R},\ g_h \ge 0. \tag{24}$$

Equation (25) is a standard specification in the literature. The intercept term consists of the fundamental price $\overline{p}$ plus 'bias' b_h in the price forecast of type h, whereas g_h is trend-following parameter of type h. Type h is a pure fundamentalist investor if $b_h = g_h = 0$, while larger values of g_h or $|b_h|$ imply, respectively, stronger trend-following and stronger forecast bias.

The fundamental price $\overline{p}$ is the unique fundamental solution under common rational expectations; see Brock and Hommes (1998). Given that the risky asset is in positive net supply $\overline{Z} > 0$, the fundamental price is $\overline{p} = (\overline{d} - a\sigma^2\overline{Z})/r$, where $r := \tilde{r}$ is the interest rate on the riskless asset. Writing the predictor in (24) in price deviations $x_t := p_t - \overline{p}$ gives:

$$\hat{E}_{t,h}[x_{t+1}] = b_h + g_h x_{t-1}, \qquad \text{where } \hat{E}_{t,h}[x_{t+1}] := \tilde{E}_{t,h}[p_{t+1}] - \overline{p}. \tag{25}$$

Given the indicator $\mathbb{1}_t = \mathbb{1}_{\{x_{t-1}+\kappa\overline{p}\le(1-\kappa)x_{t-2}\}}$, the demands in (23) can be written as:

$$z_{t,h} = \begin{cases} \frac{\hat{E}_{t,h}[x_{t+1}] - (1+r)x_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } x_t \le \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \vee \mathbb{1}_{\{x_{t-1}+\kappa\overline{p}\le(1-\kappa)x_{t-2}\}} = 0 \\ 0 & \text{if } x_t > \frac{\hat{E}_{t,h}[x_{t+1}] + a\sigma^2\overline{Z}}{1+r} \wedge \mathbb{1}_{\{x_{t-1}+\kappa\overline{p}\le(1-\kappa)x_{t-2}\}} = 1. \end{cases} \tag{26}$$

Fitness $U_{t,h}$ is a linear function of past profits net of predictor costs $C_h \ge 0$. Profits at date t are given by scaling demand $z_{t-1,h}$ by the realized excess return $R_t := p_t + d_t - (1+r)p_{t-1} =$

$x_t + a\sigma^2\overline{Z} - (1+r)x_{t-1} + \epsilon_t$, where ϵ_t is the IID dividend shock, and we abstract from memory of past performance. For all $t \geq 1$ fitness and population shares are given by

$$U_{t,h} = R_t z_{t-1,h} - C_h, \qquad n_{t+1,h} = \frac{\exp(\beta U_{t,h})}{\sum_{h\in\mathcal{H}} \exp(\beta U_{t,h})}, \qquad \text{where } \beta \in [0, \infty). \tag{27}$$

The fitness levels $U_{t,h}$ determine the population shares $n_{t+1,h}$ of each type via a discrete-choice logistic model with intensity of choice β. The intensity of choice determines how fast agents switch to better-performing predictors. In the special case $\beta = 0$ no switching occurs; increasing β implies more switching to relatively profitable predictors. Following Brock and Hommes (1997, 1998), this *evolutionary competition* mechanism has been widely studied.

We use the same parameters as in Section 3.1 of Anufriev and Tuinstra (2013): $\overline{Z} = 0.1$, $a\sigma^2 = 1$, $r = 0.1$, and we set $\overline{d} = 0.6$, giving a fundamental price $\overline{p} = \frac{\overline{d} - a\sigma^2\overline{Z}}{r} = 5$. In their model there are two types: a fundamentalist type with $\hat{E}_{t,f}[x_{t+1}] = 0$ and cost $C = 1$, and a chartist type with $\hat{E}_{t,c}[x_{t+1}] = \overline{g}x_{t-1}$, where $\overline{g} = 1.2$, and cost 0. We consider a large number of types $H = 1,000$, with predictors described by (25), population shares $n_{t,h}$ given by (27), and predictor costs C_h depending on the 'closeness' of beliefs to a pure fundamentalist.

5.1 Benchmark exercise

We first perform a sanity check by giving 500 types the same (pure) fundamental predictor (at cost $C = 1$) and the remaining 500 types the same chartist predictor $\overline{g} = 1.2$ at no cost. In this case there are two 'groups' in the population whose (aggregate) population shares are endogenously determined based on fitness. As a result, we should replicate the numerical bifurcation results in Anufriev and Tuinstra (2013) for the case of a *two-type* model in which the price deviation x_t is studied as intensity of choice parameter β is increased. Following Anufriev and Tuinstra (2013, Fig. 5), we study the deterministic skeleton with $d_t = \overline{d}$ and the case of no short-selling constraint; see Figure 2.

For sufficiently low values of the intensity of choice, the fundamental steady state $x = 0$ is the unique price attractor. Intuitively, we are in the case $(1 + r) < \overline{g} < 2(1 + r)$ and positive outside supply in Figure 2, for which Anufriev and Tuinstra (2013, Proposition 3.1) show that the fundamental steady state is globally stable for sufficiently small values of the intensity of choice β. Once a critical value of β is exceeded, there exist two non-fundamental steady states in addition to the fundamental steady state, which is locally stable. As β is increased further, however, the fundamental steady state becomes unstable, while the non-fundamental steady states are locally stable if β is not too large.

Given negative initial price, only the non-fundamental steady state with $x < 0$ is an attractor for the price dynamics at intermediate values of β; this amounts to the lower 'fork' seen for β between (approx.) 2.4 and 3.8 in Figure 2. Increasing β further causes the non-fundamental steady states to lose their stability through a Neimark-Sacker bifurcation, leading to an invariant closed curve and (quasi-)periodic dynamics. The results in Figure 2

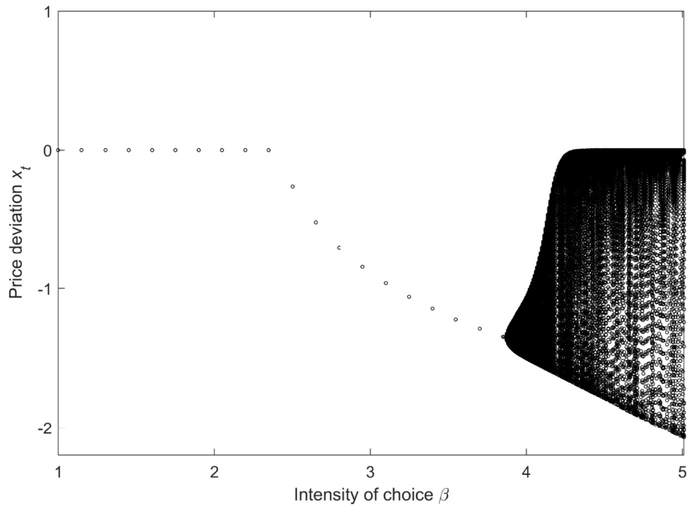

Figure 2: Bifurcation diagram in the absence of short-selling constraints. For each β, we plot 300 points following a transitory of 3,000 periods from given initial values $x_0 \in (-4, 0)$.

are consistent with those in Anufriev and Tuinstra (2013) for the same parameter values. Note that while we obtained the above diagram using $H = 1,000$ types rather than two, we effectively have a two-type model as the groups consist of homogeneous investor types.

5.2 Simulated time series: four scenarios

We now introduce heterogeneity proper by having many different types. We consider four different scenarios where the initial price x_0 is held fixed and only the intensity of choice β or the degree of heterogeneity (in g_h, b_h and C_h) are changed.[15] We first present simulated price series in four scenarios (without any noise) and then provide some results on computation speed and accuracy with stochastic dividends. We then consider some distributional implications of an alternative uptick rule by simulating the wealth distribution.

5.2.1 Four price simulations

The simulated price series in the four scenarios (S1–S4) are presented in Figure 3. All four time series are started from the same initial price $x_0 = 3$ and we assume deterministic dividends $d_t = \overline{d} = 0.6$ for all t in order to focus on the underlying dynamics. The four scenarios correspond to: heterogeneity among the 500 fundamental types (with $g_h = 0$) due to bias b_h which is linearly-spaced on the interval $[-0.2, 0.2]$ and predictor costs $C_h = 1 - |b_h|$ for such types (S1); the same setting as S1 except that heterogeneity is increased such that $b_h \in [-0.4, 0.4]$ (S2); the same setting as S1 except that the intensity of choice is increased

[15] All other parameters are the same as in the previous section, so $a\sigma^2 = 1$, $\overline{d} = 0.6$, $r = 0.1$ and $\overline{Z} = 0.1$.

from $\beta = 3$ to $\beta = 4.5$ (S3); and the same setting as S3 except that chartists are also heterogeneous with g_h drawn from a uniform distribution on the interval $(1, 1.4)$.

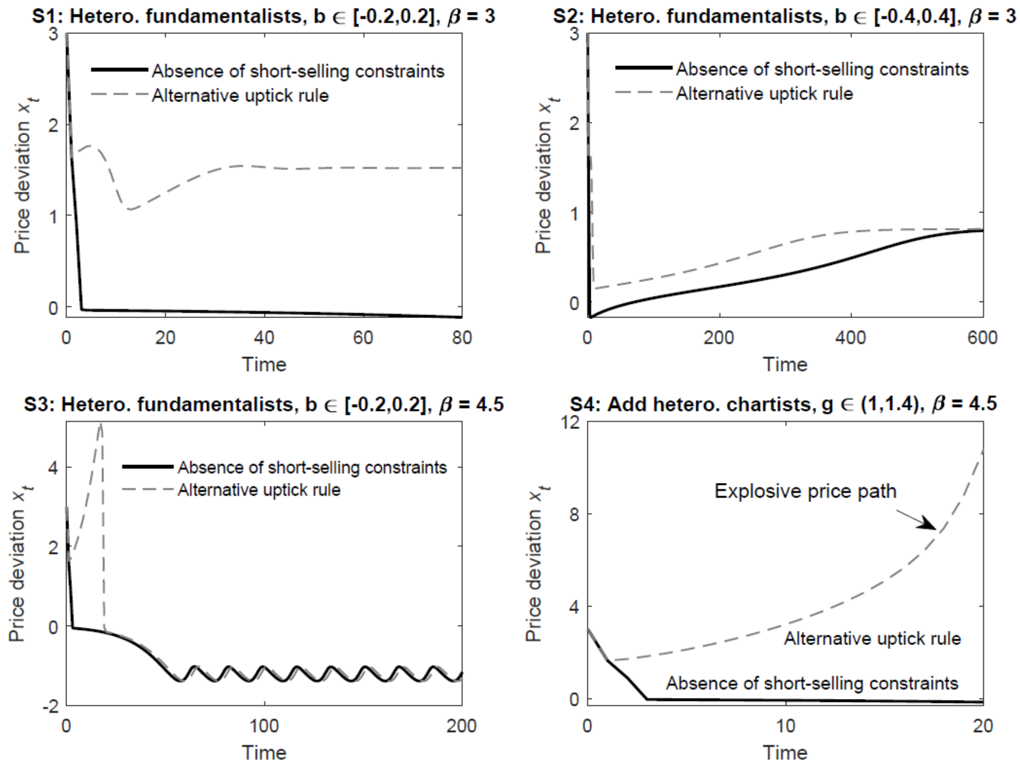

Figure 3: Simulated price series in four scenarios from an initial value $x_0 = 3$.

The price paths in these scenarios are quite different, even though the additional heterogeneities are small (see Figure 3). In Scenario 1 (top left), we see that if short-selling constraints are absent, the price quickly falls and then slowly converges on a non-fundamental steady state $x < 0$ (black line). Under an alternative uptick rule, by comparison, the initial drop in price is halted because the short-selling constraint binds; the price then oscillates around this higher value before converging on a non-fundamental steady state with $x > 0$. Thus, the alternative uptick rule leads to a quite different long run outcome, with convergence to a steady state in which the asset is somewhat overvalued. In Scenario 2, only the degree of bias among fundamentalists is increased, but this is enough to ensure that price converges on the same non-fundamental steady state in both cases (Figure 3, top right). Thus, in contrast to Scenario 1, long run price implications of the alternative uptick rule are absent in this case. This difference in results seems to be related to differences in performance across types when price is initially falling in the two scenarios; intuitively, greater heterogeneity implies more belief dispersion, but also larger differences in performance.

In Scenario 3, the intensity of choice β is set at 4.5 rather than at 3, and this is the only difference relative to Scenario 1. In this case there are permanent price oscillations in both cases (bottom left); however, the short run price dynamics under an alternative uptick rule are quite different, with an initial price spike after the short-selling constraint first binds, since the short-selling constraint binds on many types simultaneously. Lastly, in Scenario 4, (bottom right) heterogeneity in chartists is added on top of Scenario 3. In this case the reversal in price under an alternative uptick rule is reinforced by trend-following into a permanent price 'bubble' where the asset price diverges to $+\infty$. By comparison, when short-selling constraints are absent the price converges on a non-fundamental steady state $x < 0$, and hence the explosive price dynamics can be attributed to the short-selling regulation.

Computation speed and accuracy. Table 2 reports simulation times for Scenario 3 when the simulation length is 500 periods and the number of types increased from $H = 1,000$ to $H = 10,000$ and $H = 50,000$. Dividend shocks $d_t = \overline{d} + \epsilon_t$ are stochastic and we consider the cases $\kappa = 0.1$ and $\kappa = 0$ (original uptick rule) because short-selling constraints bind more frequently in the latter case. We also include a measure of accuracy based on the distance between demand and supply at the computed price, i.e. $Error_t := |\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$; in particular, we report the largest absolute deviation recorded across all 500 periods.[16]

Table 2: Computation times and accuracy in Scenario 3: $T = 500$ periods

No. of types	Regime	Time (s)	Bind freq.	$\max(Error_t)$
$H = 1,000$	No short-sell constraints	0.02	-	2.4e-14
	Alt. uptick rule: $\kappa = 0.1$	0.03	1/500	3.2e-14
	Orig. uptick rule: $\kappa = 0$	0.05	34/500	4.8e-14
$H = 10,000$	No short-sell constraints	0.17	-	2.3e-13
	Alt. uptick: $\kappa = 0.1$	0.18	1/500	6.3e-13
	Orig. uptick: $\kappa = 0$	0.25	43/500	3.0e-13
$H = 50,000$	No short-sell constraints	0.82	-	1.2e-12
	Alt. uptick: $\kappa = 0.1$	0.84	1/500	2.3e-12
	Orig. uptick: $\kappa = 0$	0.94	36/500	1.8e-12

Notes: $\max(Error_t) := \max\{Error_1, ..., Error_T\}$, where we define the date t simulation error as $Error_t = |\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$. Demands $z_{t,h}$ depend on the computed market-clearing price.

The results in Table 2 show that the solution algorithm is fast and accurate. The final column confirms that excess demand is essentially zero in all simulations, and the accuracy here is similar to when short-selling constraints are absent (top rows), in which case the standard analytical solution $x_t = (1 + r)^{-1} \sum_{h\in\mathcal{H}} n_{t,h} \hat{E}_{t,h}[x_{t+1}]$ is used to compute the price and the simulation error. Simulation times are below one second in all cases, increase with the number of types H, and are higher under the original uptick rule (where $\kappa = 0$), since this

[16]Dividend shocks ϵ_t were drawn at date 0 from a truncated-normal distribution with mean zero, standard deviation $\sigma_d = 0.01$ and support $[-\overline{d}, \overline{d}]$. Simulations were run in Matlab 2020a (Windows version) on a Viglen Genie desktop PC with Intel(R) Core(TM) i5-4570 CPU 3.20GHz processor and 8GB of RAM.

causes the short-selling constraint to bind in a much larger number of periods, as shown in the fourth column.[17] Even when short-selling constraints bind more frequently, computation times do not increase much and accuracy of the solution is preserved.

5.2.2 Distributional implications

We now consider some distributional effects of an alternative uptick rule. Recall that wealth of type h evolves as $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1+r)(w_{t,h} - p_t z_{t,h})$, such that an alternative uptick rule will affect wealth distribution through its impact on price and demands $z_{t,h}$. Note that if the short-selling constraint binds on type h at date t, then $z_{t,h} = 0$ and hence their wealth evolves as $w_{t+1,h} = (1+r)w_{t,h}$. By being out of the market in period t, type h foregoes potential returns but also avoids potential losses. Thus, the implications of an alternative uptick rule for wealth distribution will depend on the distribution of returns and losses.

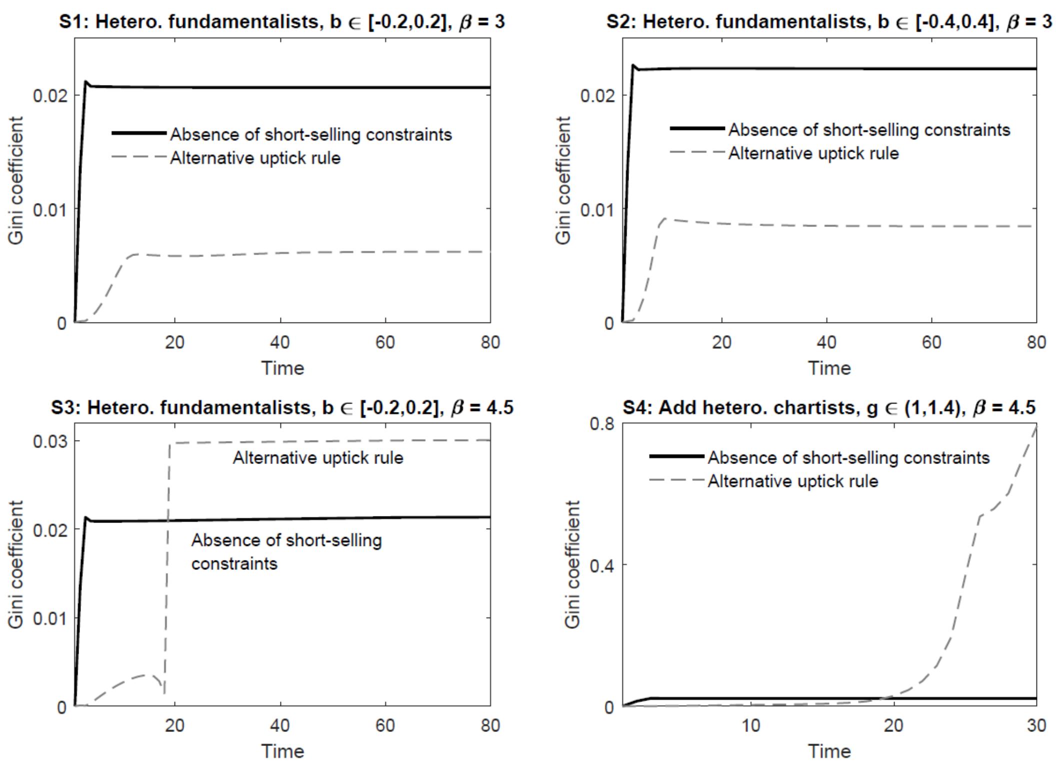

Figure 4: Simulated Gini coefficient of wealth in Scenarios 1 to 4.

We stick with the same four scenarios as in Figure 3 but we now focus on a measure of wealth inequality across investor types. In particular, we plot the Gini coefficient of the wealth distribution across types at each date t of our simulations. We assume all investor types have equal initial wealth, which we set at 50. The results are shown in Figure 4.

[17] Recall that $\kappa = 0.1$ means short-selling constraint is banned in period t only if price fell by 10% or more in the previous period. For $\kappa = 0$, short-selling constraint is banned following any previous fall in price.

An alternative uptick rule has mixed effects on wealth inequality. In Scenario 1 (Figure 4, top left), the Gini coefficient initially increases and then settles, but there is a smaller increase in inequality with an alternative uptick rule since price does not fall sharply for several periods (Figure 3, top left), which benefits more fundamental types. Such redistribution is smaller and more gradual under the alternative uptick rule because the fall in asset prices is smaller and, since price stabilizes, inequality remains lower in the long run. We see similar results in Scenario 2 (top right) where price also falls less under the alternative uptick rule.

In Scenario 3, wealth inequality is initially muted under an alternative uptick rule because price rises rather than falls (Figure 3, bottom left). However, this initial period is followed by a severe drop in price, such that more fundamental types outperform more chartist types, and wealth inequality increases before stabilizing (see Figure 4, bottom left). As a result, wealth inequality across types is initially lower under an alternative uptick rule but ends up higher in the long run. Finally, in Scenario 4 (Figure 4, bottom right), wealth inequality across types is *initially* lower under an alternative uptick rule, as the initial period of falling prices is ended as in Scenario 1 (Figure 3, bottom right). However, since price then explodes, strong chartist types make large profits and fundamental types losses, such that wealth inequality across types increases dramatically, with a Gini coefficient of around 0.8 by period 30.[18]

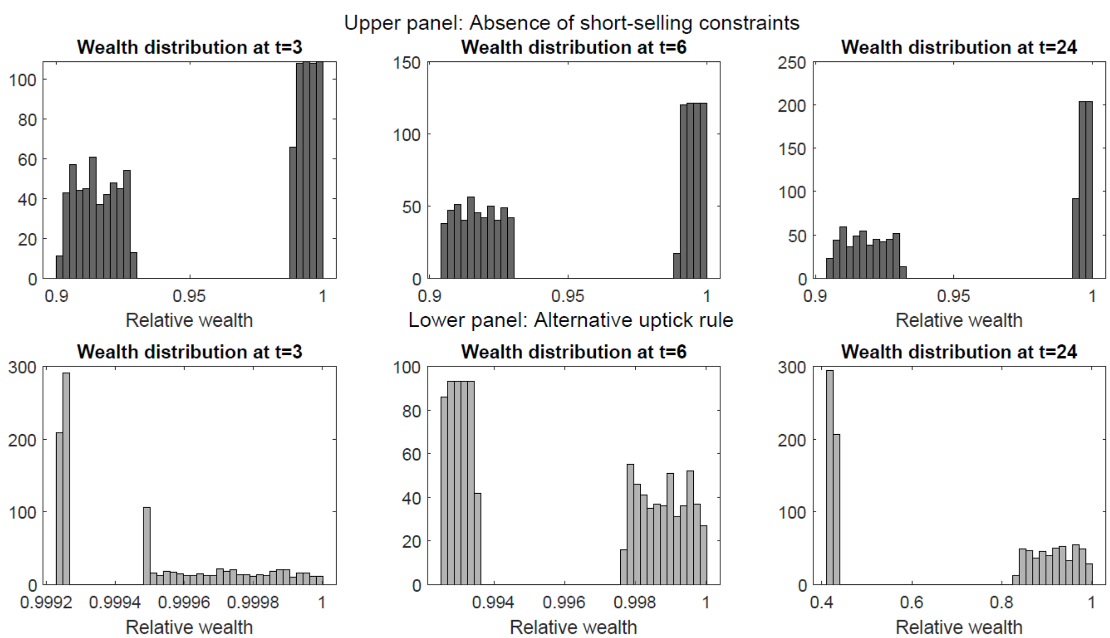

Figure 5: Simulated wealth distribution across types in Scenario 4

To better understand the wealth dynamics in Scenario 4, Figure 5 plots the wealth distribution across types in periods $t = 3$, $t = 6$ and $t = 24$ under both unrestricted short-selling

[18]The 'kink' in period 26 arises because we assume that types that hit negative wealth (in this case more fundamental types) have it reset to zero, and period 26 is the first period in which this rule is triggered.

(top panel) and an alternative uptick rule (bottom panel). We see that wealth inequalities appear rather quickly under unrestricted short-selling, but not under an alternative uptick rule, where the initial fall in price is halted. However, as time increases, the price bubble under the alternative uptick rule soon leads to much greater inequality than if short-selling constraints are absent, and by period 24 an extremely large number of types have wealth levels that are a small fraction of the highest wealth type. These results are consistent with the rapid and sustained increase in the Gini coefficient observed in Figure 4.

5.3 Policy exercise: varying κ

Having studied price dynamics and wealth distribution, we now ask whether 'fine tuning' the policy parameter κ in the alternative uptick rule could lead to a better mix between wealth inequality and mispricing from a policy perspective. Recall that $\kappa \geq 0$ represents the minimum percentage fall in price in period $t-1$ that will trigger a ban on short-selling in period t; and under the current U.S. short-selling regulation, $\kappa = 0.1$.

We consider an *ad hoc* loss function which says policymakers dislike mispricing (deviations from the fundamental price) and wealth inequality measured by the Gini coefficient:

$$L_\kappa = \sum_{t=1}^{T} |x_t| + \lambda \sum_{t=1}^{T} Gini_t \tag{28}$$

where $\lambda > 0$ is the relative weight on wealth inequality.

The loss function (28) treats positive and negative price deviations equally and we assume policymakers care about the sum of mispricings and the sum of Gini coefficients over a horizon of T periods. The loss is denoted L_κ because it depends on the parameter κ in the alternative uptick rule. Recall that $\kappa = 0.1$ corresponds to the current alternative uptick rule in the U.S., while $\kappa = 0$ corresponds to the original uptick rule that was followed until 2007. We restrict our analysis to values that satisfy $\kappa \in [0, 0.1]$, i.e. the original uptick rule, the alternative uptick rule, and intermediate options in between these two cases.

In Figure 6 we plot mispricing, wealth inequality and the loss function L_κ for 50 different values of κ equally spaced in the interval [0,0.1]. We focus on Scenario 4, in which both heterogeneous fundamentalists and heterogeneous chartists are present, and we set $\beta = 3.5$. Mispricing and inequality (as in (28)) are normalized by dividing by their values when short-selling constraints are absent in all periods; hence the y-axis values tell us whether *for a given κ value* the rule lowers mispricing and inequality. Note that if mispricing (inequality) is smaller than 1, then for λ is sufficiently small (large) the loss in (28) will be lower with an uptick rule than under unfettered short-selling. Glancing at Figure 6, we see that the short-selling rule reduces inequality but raises mispricing relative to unrestricted short-selling.

As κ is increased (so that larger drops in price are needed to trigger a short-selling ban), the impact on mispricing and inequality is quite different (left and middle panel). Increasing κ from 0 (original uptick rule) to a small positive value lowers both mispricing and wealth

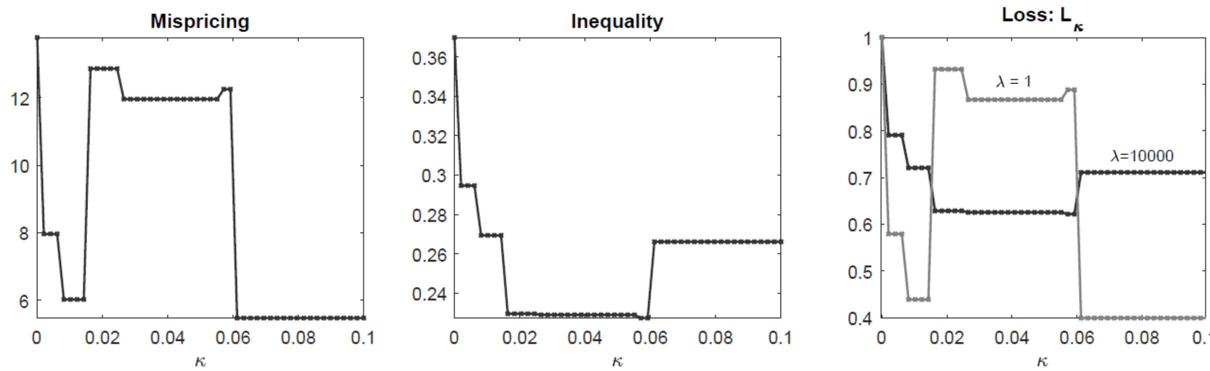

Figure 6: Mispricing, wealth inequality and the loss L_κ in Scenario 4 when $\beta = 3.5$. Mispricing and inequality are ratios to the values when short-selling is unrestricted; we normalize $\max(Loss)$ to 1. The parameter κ takes on 50 values linearly spaced on $[0, 0.1]$. The plot of the loss is shown in the final panel for two different λ values, $\lambda = 1$ and $\lambda = 10,0000$.

inequality; however, increasing κ further leads to a *trade-off*: mispricing increases while inequality goes on falling; then, at the highest values of κ, inequality is increased whereas mispricing is minimized. Because inequality is minimized at intermediate values of κ (close to 0.06) while mispricing is minimized at the highest values of κ (including $\kappa = 0.1$), the loss is sensitive to the value of λ, the relative weight on wealth inequality.[19] In particular, for relatively low values such as $\lambda = 1$ the loss is minimized by the alternative uptick rule ($\kappa = 0.1$), whereas for sufficiently high weight λ on inequality the loss is minimized at intermediate κ (≈ 0.06) because inequality is also (Figure 6, right panel). As a first pass, the results in Figure 6 suggest that policymakers might improve on the current alternative uptick rule by choosing a value of κ which is positive but smaller than 10%.

We now consider robustness. We start by computing the optimal (i.e. loss-minimizing) value of κ, among our 50 values linearly spaced on the interval $[0, 0.1]$, for two different values of λ when the intensity of choice β is increased; see Figure 7. Here we follow the rule that if the loss-minimizing value of κ is unique, we report that value; otherwise, we report as optimal the highest (lowest) value of κ in the set of loss-minimizing values when $\kappa = 0.1$ is (is not) included in the set. This rule means that $\kappa = 0.1$ is reported as optimal whenever we find the current alternative uptick rule cannot be improved upon. From Figure 7 we see that the optimal value of κ is sensitive even to small changes in β once the intensity of choice reaches a sufficiently large value. The conclusion that intermediate values of κ can minimize the loss (28) seems robust: $\kappa \in (0, 0.1)$ is optimal in around one-half of the cases in each panel. For a relatively low weight on inequality (left panel), the optimal κ is either 0 (original uptick rule), 0.1 (alternative uptick rule), or slightly above 0. By comparison, with a large

[19]Note that mispricing, inequality and the loss can remain unchanged as κ is increased since falls in price may exceed the threshold set by κ and this can remain true even when κ increases.

relative weight on inequality an original uptick rule is never optimal and intermediate κ close to 2% and 6% are optimal in some cases (right panel). In short, a high weight on inequality seems to justify moving away from the polar cases of the original and the alternative uptick rules, such that a short-selling ban is triggered by price falls much smaller than 10%.

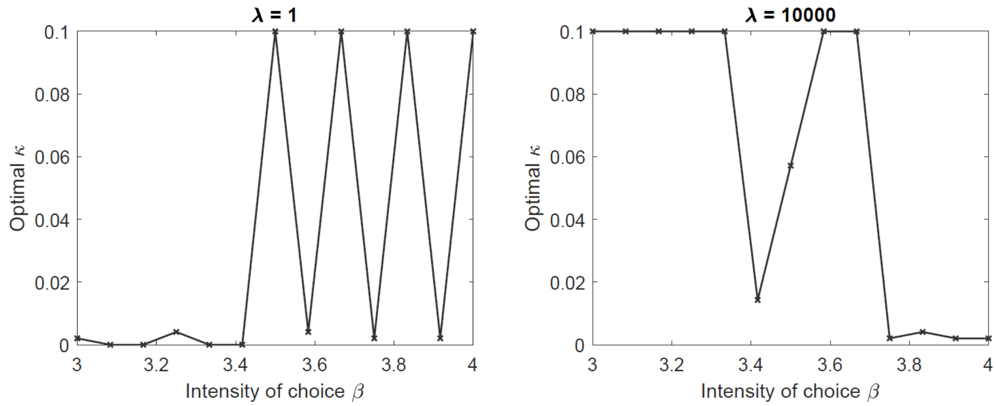

Figure 7: Optimal κ in Scenario 4 for various β and λ. The relative weight on inequality is set at either $\lambda = 1$ (left panel) and $\lambda = 10,000$ (right panel). Results are based on 50 values of κ linearly spaced on the interval $[0, 0.1]$ and 13 values of β linearly spaced on $[3, 4]$.

To illustrate how κ influences the mix between mispricing and wealth inequality, we now present plots of this relationship as κ is varied. We do this for three different values of the intensity of choice β to highlight some different cases; see Figure 8. In the first case (left panel) where $\beta = 1.55$ the original uptick rule is clearly dominated since setting $\kappa > 0$ lowers both mispricing and inequality. Once $\kappa > 0$ we see a clear *trade-off*: increasing κ lowers mispricing but raises inequality, such that the optimal value of κ will get smaller as λ is increased. In the second case ($\beta = 2.90$, middle panel), there is no clear relationship between mispricing and inequality. The original uptick rule ($\kappa = 0$) minimizes mispricing in this case, but inequality is reduced by setting $\kappa > 0$. Setting κ close to zero (0.006) raises inequality slightly but reduces mispricing by around one-fifth compared to higher values of κ (0.008–0.1), so will be optimal for policymakers who prefer to balance mispricing and inequality. Finally, in the right panel of Figure 8 we see a case where intermediate values of κ (0.002–0.0265) are optimal *irrespective* of the value of λ because such a policy *minimizes mispricing and inequality*, such that both the original uptick rule and the alternative uptick rule are dominated. While this is just a single numerical example, it shows that policies which lie *between* the original uptick rule ($\kappa = 0$) and the current alternative uptick rule ($\kappa = 0.1$) may improve price discovery *and* reduce wealth inequality.

In summary, our results suggest it might be possible to improve on the current alternative uptick rule by having short-selling bans triggered by smaller falls in price. While we should be cautious about drawing general conclusions from a small number of policy experiments, these results clearly raise question marks about whether current regulation could be improved.

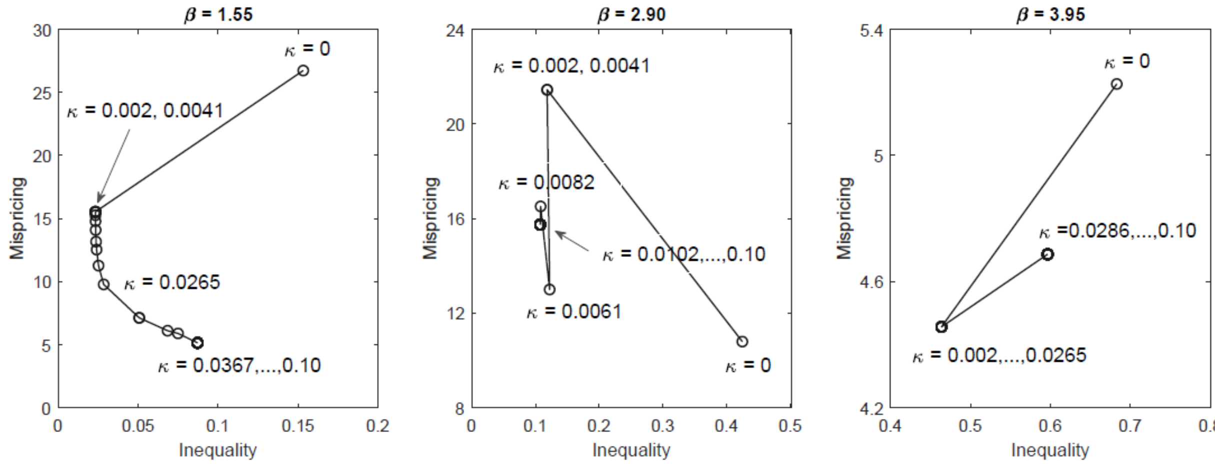

Figure 8: Trade-offs between mispricing and inequality: various β. Each panel plots the relationship between mispricing and wealth inequality for a given β as the policy parameter κ is varied. Mispricing and inequality are ratios to the values when short-selling is unrestricted. The parameter κ takes on 50 values linearly spaced on the interval $[0, 0.1]$.

6 Conclusion

This paper has studied dynamic behavioural asset pricing models with short-selling constraints and many investor types with heterogeneous beliefs. Our results provide analytical expressions for asset prices along with conditions on beliefs such that short-selling constraints bind on different types, allowing us to construct computationally-efficient solution algorithms. The analysis is built around a Brock and Hommes (1998) model with short-selling constraints and generic price predictors; we also presented extensions for conditional short-selling constraints, multiple risky assets, and pricing by a market-maker.

The utility of these results was shown via examples and a numerical application that studied an alternative uptick rule, as currently in place in the United States, in a market with a large number of belief types in *evolutionary competition*. The results highlight the complicated relationship between the design of short-selling regulations and their implications for asset mispricing and wealth distribution. In particular, an alternative uptick rule may attenuate (or prevent) falls in price, but we also found that such rules can hinder price discovery, increase price volatility and lead to explosive price paths. An alternative uptick rule can also have substantive distributional (wealth) implications, and we showed a scenario in which a modified alternative uptick rule, which bans short-selling following smaller falls in price, reduces both mispricing and wealth inequality relative to the current regulation.

There are several promising avenues for future research. First, it would be of interest to investigate whether adding short-selling constraints in models with many belief types improves the ability of models to reproduce empirical stylized facts, especially during times of market turmoil, when such constraints are more likely to be present. In a similar vein, it may be feasible to estimate such models in order to evaluate the relative empirical contribution of adding short-selling constraints. Second, from a policy perspective, there has been interest in whether short-selling restrictions lead to mispricing and might cause or exacerbate price bubbles, both in financial markets and housing markets (Shiller, 2015; Fabozzi et al., 2020). The main focus in the literature has been on mispricing and price volatility, but it would be of interest to investigate further the distributional implications of short-selling restrictions for traders' inequality in models with many agents, along the lines of our application.

Finally, from a technical perspective, there are some modelling specifications of interest which are not covered by the results presented in this paper. For instance, one could confront a large number of investor types with additional restrictions such as leverage constraints (see in't Veld, 2016), the elimination of investors who hit low or negative wealth, or margin calls that prevent a short position being maintained in future periods. These approaches might have important implications not just for the price effects of short-selling restrictions, but also their distributional implications that have received little attention so far.

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Appendix

Proof of Proposition 1

A unique market-clearing price exists by Proposition 2.1 in Anufriev and Tuinstra (2013).

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ that $p_t = p_t^* := (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^* \geq 0 \ \forall h \in \mathcal{H}$, which amounts to $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} \leq \min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z}$. Given $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, the above inequality simplifies to $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$, as stated in Proposition 1.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0$ $\forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$ and hence the market-clearing condition is $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$, which gives $p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^{\mathcal{B}_t^*} \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^{\mathcal{B}_t^*} < 0 \ \forall h \in \mathcal{S}_t^*$, i.e. iff $(f_{t,h} + a\sigma^2\overline{Z})\sum_{h\in\mathcal{B}_t^*} n_{t,h} \geq (<) \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} \ \forall h \in \mathcal{B}_t^* \ (\forall h \in \mathcal{S}_t^*)$, which simplifies to $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\})$, as stated in Proposition 1.

It remains to show $p_t^{\mathcal{B}_t^*} > p_t^* = \frac{\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}}{1+r}$, where p_t^* is the price if short-selling constraints are absent. Note $(1+r)(p_t^{\mathcal{B}_t^*} - p_t^*) = (1 - \frac{1}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}})a\sigma^2\overline{Z} + \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$ and $\sum_{h\in\mathcal{B}_t^*} n_{t,h} = 1 - \sum_{h\in\mathcal{S}_t^*} n_{t,h}$. Since $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} + \sum_{h\in\mathcal{S}_t^*} n_{t,h}f_{t,h}$, we obtain:

$$(1+r)(p_t^{\mathcal{B}_t^*} - p_t^*) = (\sum_{h\in\mathcal{S}_t^*} n_{t,h})\left[\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h\in\mathcal{S}_t^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}}\right] > 0$$

where $\sum\limits_{h\in\mathcal{S}_t^*} \frac{n_{t,h}}{\sum_{h\in\mathcal{S}_t^*} n_{t,h}} f_{t,h} \leq \max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ and $\frac{\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - a\sigma^2\overline{Z}}{\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}} > \max\limits_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$ is implied by the condition $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) > a\sigma^2\overline{Z}$ above. ■

Proof of Corollary 1

The first 'if' statement follows from Proposition 1 as $\sum_{h=2}^{\tilde{H}_t} n_{t,h}(f_{t,h} - f_{t,1}) \leq a\sigma^2\overline{Z}$ is equivalent to $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$. The other cases follow as there are $\tilde{H}_t - 1$ other candidates for $\mathcal{B}_t^*, \mathcal{S}_t^*$, i.e. $\mathcal{S}_t = \{1\}, \mathcal{B}_t = \{2, ..., \tilde{H}_t - 1\}$; $\mathcal{S}_t = \{1, 2\}, \mathcal{B}_t = \{3, ..., \tilde{H}_t - 1\}$;...$\mathcal{S}_t = \{1, ..., \tilde{H}_t - 1\}, \mathcal{B}_t = \{\tilde{H}_t\}$. For arbitrary sets $\mathcal{S}_t = \{1, ..., k\}$, $B_t = \{k+1, ..., \tilde{H}_t\}$, where $k \in \{1, .., \tilde{H}_t - 1\}$, by market-clearing $p_t = \frac{\sum_{h>k} n_{t,h} f_{t,h} - \left[\sum_{h=1}^{k} n_{t,h}\right] a\sigma^2\overline{Z}}{(1-\sum_{h=1}^{k} n_{t,h})(1+r)} := p_t^{(k)}$ and by Proposition 1 the guess is verified if and only if $disp_{t,k+1} \leq a\sigma^2\overline{Z} < disp_{t,k}$. Note that $p_t^{(k^*)} > p_t^* = \frac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{1+r}$ for any $k^* \in \{1, ..., \tilde{H}_t - 1\}$ follows from the proof of Proposition 1.

It remains to show $p_t^* < p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}$ $\forall k \in [2, k^*)$. Note that $p_t^{(1)}$ solves $\sum_{h>1} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(1)}) = a\sigma^2\overline{Z}$ and p_t^* solves $a\sigma^2\overline{Z} = \sum_{h>1} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^*) + n_{t,1}(f_{t,1} + a\sigma^2\overline{z} - (1+r)p_t^*)$, where the last term is < 0 since p_t^* is not verified. So $p_t^{(1)} > p_t^*$. For arbitrary k, $p_t^{(k)}$ solves $\sum_{h>k} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(k)}) = a\sigma^2\overline{Z}$ and $p_t^{(k-1)}$ solves $a\sigma^2\overline{Z} = \sum_{h>k} n_{t,h}(f_{t,h} + a\sigma^2\overline{z} - (1+r)p_t^{(k-1)}) + n_{t,k}(f_{t,k} + a\sigma^2\overline{z} - (1+r)p_t^{(k-1)})$, where the last term is < 0 since $p_t^{(k-1)}$ is not verified. So $p_t^{(k)} > p_t^{(k-1)}$ $\forall k \in [2, k^*)$. Finally, $p_t^{(k^*)} > p_t^{(k)}$ $\forall k < k^*$ follows from the above argument with $k = k^*$ and $k - 1 = k^* - 1$. ■

Proof of Proposition 2

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0$ $\forall h \in \mathcal{H}$, which implies by the price equation that $p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}} := p_t^*$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t^* \geq 0$ $\forall h \in \mathcal{H}$, which requires $\left(\frac{1}{1+r} + \frac{\mu\lambda}{a\sigma^2}\right)(a\sigma^2\overline{Z} + \min_{h\in\mathcal{H}}\{f_{t,h}\}) \geq p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} + \mu(1-\lambda)(Z_{t-1} - \overline{Z})$, giving the inequality in Proposition 2 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0$ $\forall h \in \mathcal{S}_t^* = \mathcal{H} \setminus \mathcal{B}_t^*$, so $p_t = \frac{p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]}{1 + \mu\lambda(1+r)(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h}}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t \geq 0$ $\forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t < 0$ $\forall h \in \mathcal{S}_t^*$, which requires $(\frac{1}{1+r} + \mu\lambda(a\sigma^2)^{-1}\sum_{h\in\mathcal{B}_t^*} n_{t,h})(a\sigma^2\overline{Z} + f_{t,h}) \geq (<)\; p_{t-1} + \frac{\mu\lambda}{a\sigma^2}\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - (1-\lambda\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}]$ $\forall h \in \mathcal{B}_t^*$ ($\forall h \in \mathcal{S}_t^*$), giving the inequality in Proposition Part 2(i).

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0$ $\forall h \in \mathcal{H}$, which implies that $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t < 0$ $\forall h \in \mathcal{H}$, i.e. iff $\max_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z} < (1+r)(p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}])$, which is the inequality in Proposition 2 Part 2(ii). ■