Supplementary Appendix (For online publication only)

"Solving heterogeneous-belief asset pricing models with short selling constraints and many agents"

This appendix provides (i) derivations of optimal demands with short-selling constraints, supporting Sections 2.1 and 4.1 of the main text; and (ii) supporting material for the generalizations and extensions mentioned in Sections 3.2, 4.1, 4.2 and 4.3.2 of the main text.

1 Derivations

In this section we derive the optimal demand schedules for the cases of unconditional short-selling constraints (benchmark model) and conditional short-selling constraints (Section 4.1).

1.1 Derivation of demands in the benchmark model

The optimal portfolio choice of type $h \in \mathcal{H}$ solves the problem:[1]

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2} V\tilde{a}r_{t,h}[w_{t+1,h}] \qquad \text{s.t.} \qquad z_{t,h} \geq 0 \tag{1}$$

where $w_{t+1,h} = (p_{t+1} + d_{t+1})z_{t,h} + (1 + \tilde{r})(w_{t,h} - p_t z_{t,h})$ is the future wealth, $w_{t,h} - p_t z_{t,h}$ is holdings of the risk-free asset, and we assume $V\tilde{a}r_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$, with $\sigma^2 > 0$.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h}, \lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2} V\tilde{a}r_{t,h}\left[w_{t+1,h}\right] + \lambda_{t,h} z_{t,h} \tag{2}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint $z_{t,h} \geq 0$.

The first-order conditions are

$$z_{t,h}: \quad \tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1 + \tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h} = 0 \tag{3}$$

$$\lambda_{t,h}: \quad z_{t,h} \geq 0 \tag{4}$$

and the complementary slackness condition is:

$$\lambda_{t,h} z_{t,h} = 0. \tag{5}$$

By (3), $\lambda_{t,h} = -(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1 + \tilde{r})p_t - a\sigma^2 z_{t,h})$. Note that $\lambda_{t,h} = 0$ if and only if $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1 + \tilde{r})p_t)$. This $z_{t,h}$ satisfies (5) (given $\lambda_{t,h} = 0$) and therefore it is an optimal demand provided $\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] \geq (1 + \tilde{r})p_t$; see (4).

[1] We assume (as is standard) that these operators satisfy some basic properties of conditional expectation operators, namely, $\tilde{E}_{t,h}[y_t] = y_t$ and $V\tilde{a}r_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $V\tilde{a}r_{t,h}[x_t y_{t+1}] = x_t^2 V\tilde{a}r_{t,h}[y_{t+1}]$.

If $\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]<(1+\tilde{r})p_t$. we can reject $\lambda_{t,h}=0$, since the condition in (4) is not satisfied. It follows that $\lambda_{t,h}>0$ when $\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]<(1+\tilde{r})p_t$, and by the complementary slackness condition (5) it follows that $z_{t,h}=0$ (which satisfies (4)).

Therefore, the demand schedule of type $h\in\mathcal{H}$ is given by

$$z_{t,h}=\begin{cases}\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t\leq\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]}{1+\tilde{r}}\\ 0 & \text{if } p_t>\dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]}{1+\tilde{r}}.\end{cases}\tag{6}$$

as stated in Equation (2) of the main text.

1.2 Derivation of demands for conditional short-selling constraint

If $g(p_{t-1},...,p_{t-K})>0$, the short-selling constraint is absent at date t; if $g(p_{t-1},...,p_{t-K})\leq 0$ the short-selling constraint is present. We introduce an indicator variable $\mathbb{1}_t:=\mathbb{1}_{\{g(p_{t-1},...,p_{t-K})\leq 0\}}$ equal to 1 if the short-selling constraint is present at date t (i.e. if $g(p_{t-1},...,p_{t-K})\leq 0$), and equal to 0 otherwise. The problem of type $h\in\mathcal{H}$ at date t is thus amended from (1) to:

$$\max_{z_{t,h}}\ \tilde{E}_{t,h}[w_{t+1,h}]-\frac{a}{2}V\tilde{a}r_{t,h}[w_{t+1,h}]\qquad\text{s.t.}\qquad \mathbb{1}_t z_{t,h}\geq 0\tag{7}$$

where $w_{t+1,h}=(p_{t+1}+d_{t+1})z_{t,h}+(1+\tilde{r})(w_{t,h}-p_t z_{t,h})$ as above.

Note that if $\mathbb{1}_t=0$, the portfolio choice of all types $h\in\mathcal{H}$ is unconstrained at date t, since $\mathbb{1}_t z_{t,h}=0\geq 0$ is satisfied for any $z_{t,h}\in\mathbb{R}$. On the other hand, if $\mathbb{1}_t=1$ then all types face the same maximization problem as in (1), i.e. short-selling is ruled out at date t.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}}\mathcal{L}_{t,h}=\tilde{E}_{t,h}\left[w_{t+1,h}\right]-\frac{a}{2}V\tilde{a}r_{t,h}\left[w_{t+1,h}\right]+\lambda_{t,h}\mathbb{1}_t z_{t,h}\tag{8}$$

where $\lambda_{t,h}\geq 0$ is the Lagrange multiplier on the constraint $\mathbb{1}_t z_{t,h}\geq 0$.[2]

The first-order conditions are

$$z_{t,h}:\ \ \tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t-a\sigma^2 z_{t,h}+\lambda_{t,h}\mathbb{1}_t=0\tag{9}$$

$$\lambda_{t,h}:\ \ \mathbb{1}_t z_{t,h}\geq 0\tag{10}$$

and the complementary slackness condition is:

$$\lambda_{t,h}\mathbb{1}_t z_{t,h}=0.\tag{11}$$

If $\mathbb{1}_t=0$, then (10)–(11) are satisfied and $z_{t,h}=(a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t)\in\mathbb{R}$ by (9). Hence, demands are unconstrained if $\mathbb{1}_t=0$. If $\mathbb{1}_t=1$, the first-order conditions (9)–(11) are identical to (3)–(5), so the cases are the same as discussed below (5).

Therefore, demands are $z_{t,h}=(a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right]+\tilde{E}_{t,h}\left[d_{t+1}\right]-(1+\tilde{r})p_t)$ if $\mathbb{1}_t=0$ or $p_t\leq\frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}$; and $z_{t,h}=0$ otherwise (i.e. if $\mathbb{1}_t=1$ and $p_t>\frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+\tilde{r}}$).

[2] Note $\lambda_{t,h}$ is irrelevant if $\mathbb{1}_t=0$ (see (8)), but not if $\mathbb{1}_t=1$, in which case it has the usual interpretation.

2 Nested cases

In this section we show some cases (housing as the risky asset; constraints $z_{t,h} \geq L$) which are essentially nested by the benchmark model as discussed in Section 3.2 of the paper.

2.1 Housing as the risky asset

In this section we demonstrate how our results can be applied when the risky asset is housing as in Bolt et al. (2019) and Hatcher (2021). In these models, housing is an investment asset that differs from shares due to the interpretation of 'dividends'. In Bolt et al. (2019) dividends are replaced by imputed rent based on an arbitrage condition between the user and rental costs, whereas Hatcher (2021) assumes linear housing utility scaled by a housing preference variable.[3] In both models, these additional variables are exogenous processes whose properties are known to the investors. We assume a fixed supply of housing $\overline{Z} > 0$.

With linear excess returns and an unconditional short-selling constraint, demands are:

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1+\hat{R}) - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1+\hat{R})}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + Q_t(1+\hat{R})}{1+\tilde{r}} \end{cases} \tag{BDDHL1}$$

where Q_t is the exogenous rental price and $\hat{R}$ is the fixed risk-free mortgage rate; and

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \Theta_t\overline{U}_z}{1+\tilde{r}} \end{cases} \tag{Hatcher1}$$

where $\Theta_t > 0$ is an exogenous preference for housing utility versus financial wealth, and $\overline{U}_z > 0$ is a fixed marginal utility of housing (which does not depend on $z_{t,h}$).

Defining $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2\overline{Z}$ and $r := \tilde{r} - \overline{c}$, with $\tilde{E}_{t,h}\left[d_{t+1}\right] = Q_t(1+\hat{R})$ (or $\tilde{E}_{t,h}\left[d_{t+1}\right] = \Theta_t\overline{U}_z$), the above demands can be written as:

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \end{cases} \tag{12}$$

as in Equation (4) of the main text.

Hence, our approach can easily accommodate these cases. Note that the above specifications will imply a change in the fundamental price even if shocks are IID, since the *current* 'dividend', rather than the expected future value, matters for demand. Nevertheless, it remains straightforward to write demands in deviations from the fundamental price.

[3] Using quadratic utility from housing in the framework of Hatcher (2021) is also possible as this mirrors the mean-variance assumption in the benchmark model.

2.2 Short-selling constraints of the form: $z_{t,h} \geq L$

Suppose that negative positions beyond some limit $L \leq 0$ are not permitted in any period. We can formulate this assumption as an unconditional short-selling limit $z_{t,h} \geq L$, $\forall t, h$.[4]

Formulating the maximization problem of type h as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2} V\tilde{a}r_{t,h}\left[w_{t+1,h}\right] + \lambda_{t,h}(z_{t,h} - L) \tag{13}$$

where $\lambda_{t,h} \geq 0$ is the Lagrange multiplier on the short-selling constraint $z_{t,h} \geq L$.

The first-order conditions are

$$z_{t,h} : \tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t - a\sigma^2 z_{t,h} + \lambda_{t,h} = 0, \qquad \lambda_{t,h} : z_{t,h} \geq L \tag{14}$$

and the complementary slackness condition is: $\lambda_{t,h}(z_{t,h} - L) = 0$.

Analogous to the discussion after (3)–(5), $\lambda_{t,h} = 0$ and $z_{t,h} = (a\sigma^2)^{-1}(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t)$ is an optimal demand provided $\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t \geq a\sigma^2 L$, while if $\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t < a\sigma^2 L$, then $\lambda_{t,h} > 0$ and $z_{t,h} = L$.

Hence, the demand schedule of type $h \in \mathcal{H}$ is given by

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+\tilde{r})p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2 L}{1+\tilde{r}} \\ L & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2 L}{1+\tilde{r}}. \end{cases} \tag{15}$$

Defining $f_{t,h} := \tilde{f}_{t,h} + \tilde{E}_{t,h}\left[d_{t+1}\right] - a\sigma^2\overline{Z}$, $r := \tilde{r} - \overline{c}$ and $\tilde{z}_{t,h} := z_{t,h} - L$, the demands in (15) can be written in the form:

$$\tilde{z}_{t,h} = \begin{cases} \dfrac{f_{t,h} + a\sigma^2\tilde{Z} - (1+r)p_t}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\tilde{Z}}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\tilde{Z}}{1+r} \end{cases} \tag{16}$$

where $\tilde{Z} := \overline{Z} - L$.

Note that the demands in (29) have the same form as in Eq. (4) in the main paper, except that $z_{t,h}$ is replaced by $\tilde{z}_{t,h}$ and $\overline{Z}$ is replaced by $\tilde{Z}$. Similarly, market-clearing is:

$$\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z} \quad \Longrightarrow \quad \sum_{h\in\mathcal{H}} n_{t,h}\tilde{z}_{t,h} = \tilde{Z}. \tag{17}$$

where $\sum_{h\in\mathcal{H}} n_{t,h} = 1$ is used.[5]

We can therefore state the following equivalence result.

Remark 1 *In the above model, demands and market-clearing are as in the benchmark model, except that $z_{t,h}$ is replaced by $\tilde{z}_{t,h}$ and $\overline{Z}$ is replaced by $\tilde{Z}$. Therefore, the market-clearing price and demands follow Proposition 1 with $\tilde{z}_{t,h}$ replacing $z_{t,h}$ and $\tilde{Z}$ replacing $\overline{Z}$.*

[4] The case of a conditional short-selling limit is also straightforward; see Section 1.2.

[5] The 'deviations' representation of market-clearing in (17) holds regardless of whether some types (those in the set $\mathcal{S}_t^* = \mathcal{H}\backslash\mathcal{B}_t^*$) are short-selling constrained with $z_{t,h} = L$ while the others (in $\mathcal{B}_t^*$) are unconstrained.

3 Supporting results for Remarks in the main paper

This section provides propositions supporting the Remarks given in the 'Extensions' section of the main paper. Any non-trivial proofs appear in Section 5.

3.1 Conditional short-selling constraints

Recall from Section 1.2 that we introduced an indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},\ldots,p_{t-K})\leq 0\}}$ that is equal to 1 if the short-selling constraint is in place at date t (i.e. if $g(p_{t-1},...,p_{t-K}) \leq 0$) and is equal to 0 otherwise (i.e. if $g(p_{t-1},...,p_{t-K}) > 0$).

The demands for types $h \in \mathcal{H}$ are given by:

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ or } \mathbb{1}_t = 0 \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r} \text{ and } \mathbb{1}_t = 1. \end{cases} \tag{18}$$

where $r := \tilde{r} - \overline{c}$ and $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$.

Proposition 1 (Proposition 1 (main) adapted to conditional constraint) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$ and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date t. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ or $\mathbb{1}_t = 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{1+r} := p_t^* \tag{19}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t)$ $\forall h \in \mathcal{H}$ with $z_{t,h} \in \mathbb{R}$ if $\mathbb{1}_t = 0$, and $z_{t,h} \geq 0$ otherwise (i.e. if $\mathbb{1}_t = 1$ and $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$).

(ii) If $\mathbb{1}_t = 1$ and $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$, then at least one type is short-selling constrained and there exist unique non-empty sets $\mathcal{B}_t^*$ and $\mathcal{S}_t^*$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\})$, and the market-clearing price and demands are

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}} > p_t^* \tag{20}$$

and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \geq 0$ $\forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0$ $\forall h \in \mathcal{S}_t^*$.

Proof. See Section 5.1 of this appendix. ■

3.2 Multiple markets and endogenous participation

In this section we adapt Proposition 1 for the case of multiple risky assets $m \in \{1, ..., M\}$ when participation shares $w_t^m \in (0,1)$ in each market are determined by attractiveness relative to other markets; see Westerhoff (2004) and Section 4.2 of the paper.

We show in the main text (Section 4.2) that demand of type h in market m is given by

$$z_{t,h}^m = \begin{cases} w_t^m \left(\frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1+r^m) p_t^m}{a\sigma_m^2} \right) & \text{if } p_t^m \leq \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m} \\ 0 & \text{if } p_t^m > \frac{f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m}{1+r^m}. \end{cases} \tag{21}$$

where $f_{t,h}^m := \tilde{f}_{t,h}^m + \overline{d}^m - a\sigma_m^2 \overline{Z}_m / w_t^m$ and $r^m := \tilde{r} - \overline{c}^m$.

Market-clearing in market m is given by

$$\sum_{h \in \mathcal{H}} n_{t,h}^m \tilde{z}_{t,h}^m = \overline{Z}_m / w_t^m, \qquad \text{where } \tilde{z}_{t,h}^m := z_{t,h}^m / w_t^m. \tag{22}$$

With this change in variables, the market-clearing condition has the same form as in the benchmark model (aside from a scaling of supply by $1/w_t^m$). We therefore have the following.

Proposition 2 (Proposition 1 adapted to multiple markets) *Let p_t^m be the market-clearing price at date t and let $\mathcal{B}_t^m \subseteq \mathcal{H}$ ($\mathcal{S}_t^m := \mathcal{H} \setminus \mathcal{B}_t^m$) be the set of unconstrained types (short-selling constrained types) in market m at date $t \in \mathbb{N}_+$. Then the following holds:*

(i) If $\sum_{h \in \mathcal{H}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{H}} \{f_{t,h}^m\}) \leq a\sigma_m^2 \overline{Z}_m / w_t^m$, then no type is short-selling constrained ($\mathcal{B}_t^{m*} = \mathcal{H}$, $S_t^{m*} = \emptyset$) and the market-clearing price is

$$p_t^m = \frac{\sum_{h \in \mathcal{H}} n_{t,h}^m f_{t,h}^m}{1 + r^m} := p_t^{m*} \tag{23}$$

with demands $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1} (f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1 + r^m) p_t^m) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h \in \mathcal{H}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{H}} \{f_{t,h}^m\}) > a\sigma_m^2 \overline{Z}_m / w_t^m$, then at least one type is short-selling constrained at date t and there exist unique non-empty sets $\mathcal{B}_t^{m*} \subset \mathcal{H}$ and $\mathcal{S}_t^{m*}$ such that $\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (f_{t,h}^m - \min_{h \in \mathcal{B}_t^{m*}} \{f_{t,h}^m\}) \leq a\sigma_m^2 \overline{Z}_m / w_t^m < \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m (f_{t,h}^m - \max_{h \in \mathcal{S}_t^{m*}} \{f_{t,h}^m\})$, and the market-clearing price and demands are

$$p_t^m = \frac{\sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m f_{t,h}^m - (1 - \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m) a\sigma_m^2 \overline{Z}_m / w_t^m}{(1 + r^m) \sum_{h \in \mathcal{B}_t^{m*}} n_{t,h}^m} > p_t^{m*} \tag{24}$$

and $z_{t,h}^m = w_t^m (a\sigma_m^2)^{-1} (f_{t,h}^m + a\sigma_m^2 \overline{Z}_m / w_t^m - (1 + r^m) p_t^m) \geq 0 \ \forall h \in \mathcal{B}_t^{m*}$, $z_{t,h}^m = 0 \ \forall h \in \mathcal{S}_t^{m*}$.

Proof. It follows from the Proposition 1 Proof (main paper) when p_t, $f_{t,h}$, r and $\overline{Z}$ are replaced by p_t^m, $f_{t,h}^m$, r^m and $\overline{Z}_m / w_t^m$, and the demands $z_{t,h}$ are replaced by $z_{t,h}^m$ in (21). ■

3.3 Market maker and heterogeneous slope coefficients

Similar to Westerhoff (2004) we consider demands of the form $\tilde{a}_h(f_{t,h} - p_t)$, where $\tilde{a}_h > 0$. With a short-selling constraint $z_{t,h} \geq 0$, the demands are adjusted to:

$$z_{t,h} = \begin{cases} \tilde{a}_h(\tilde{E}_{t,h}[p_{t+1}] - p_t) & \text{if } p_t \leq \tilde{E}_{t,h}[p_{t+1}] \\ 0 & \text{if } p_t > \tilde{E}_{t,h}[p_{t+1}]. \end{cases} \tag{25}$$

Given price beliefs in Assumption 1, we have $\tilde{E}_{t,h}[p_{t+1}] = \bar{c}p_t + \tilde{f}_{t,h}$, where we assume $\bar{c} \in [0,1)$ since the effective interest rate is zero. We define $f_{t,h} := \tilde{f}_{t,h}$ and write the demands as

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} - (1-\bar{c})p_t) & \text{if } p_t \leq \frac{f_{t,h}}{1-\bar{c}} \\ 0 & \text{if } p_t > \frac{f_{t,h}}{1-\bar{c}}. \end{cases} \tag{26}$$

Since $\bar{c} \in [0,1)$, demands are decreasing in the current price. By (26) and the price equation $p_t = p_{t-1} + \mu[\lambda Z_t + (1-\lambda)Z_{t-1} - \overline{Z})]$, where $Z_t - \overline{Z} := \sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}$ is excess demand.

Proposition 3 (Proposition 1 (main) adapted to new demand specification) *Let p_t be the price determined by $p_t = p_{t-1} + \mu[\lambda Z_t + (1-\lambda)Z_{t-1} - \overline{Z})]$ and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H}\setminus\mathcal{B}_t$) be the set of types that are unconstrained (short-selling constrained) at date $t \in \mathbb{N}_+$. Further, let $\tilde{n}_{t,h} := n_{t,h}\tilde{a}_h$ and $f_{t,h} := \tilde{f}_{t,h}$. Then the following holds:*

1. If $p_{t-1} - \frac{1}{1-\bar{c}}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} \leq \mu\overline{Z}$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$, $z_{t,h} \geq 0\ \forall h$) and the price is

$$p_t = \frac{p_{t-1} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}}.$$

2. If $p_{t-1} - \frac{1}{1-\bar{c}}\min_{h\in\mathcal{H}}\{f_{t,h}\} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) + \mu(1-\lambda)Z_{t-1} > \mu\overline{Z}$, then one or more types $h \in \mathcal{H}$ are short-selling constrained with $z_{t,h} = 0$ and we have the following:

(i) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}}\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < \mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, price is given by

$$p_t = \frac{p_{t-1} + \mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c})\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}}.$$

and demands are $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\bar{c})p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

(ii) Else, $\exists \mathcal{B}_t^* = \emptyset, \mathcal{S}_t^* = \mathcal{H}$ such that $p_{t-1} + \mu(1-\lambda)Z_{t-1} - \frac{1}{1-\bar{c}}\max_{h\in\mathcal{S}_t^*}\{f_{t,h}\} > \mu\overline{Z}$, all types are constrained ($z_{t,h} = 0\ \forall h$), and price is $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$.

Proof. See Section 5.2 of this appendix. ■

Corollary 1 (amended) *Let* $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ *be the set such that types are ordered as* $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$. *Let* $disp_{t,k} := \mu\lambda \sum_{h=k+1}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - f_{t,k}) - \frac{f_{t,k}}{1-\bar{c}}$ *and* $g(p_{t-1}, Z_{t-1}) := \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, *for* $k \in \{1, ..., \tilde{H}_t - 1\}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. *The price solution is:*

$$p_t = \begin{cases} \dfrac{p_{t-1} + \mu\lambda \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq g(p_{t-1}, Z_{t-1}) \\ \dfrac{p_{t-1} + \mu\lambda \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,1} \\ \dfrac{p_{t-1} + \mu\lambda \sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,2} \\ \vdots & \vdots \\ \dfrac{p_{t-1} + \mu\lambda \tilde{n}_{t,\tilde{H}_t} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t)} & \text{if } disp_{t,\tilde{H}_t} \leq g(p_{t-1}, Z_{t-1}) < disp_{t,\tilde{H}_t - 1} \\ p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}] & \text{if } disp_{t,\tilde{H}_t} > g(p_{t-1}, Z_{t-1}) \end{cases}$$

where $p_t^{(k^*)}$ *is the price if types* $1, ..., k^*$ *are short-selling constrained,* p_t^* *is the corresponding price if short-selling constraints were absent (which satisfies* $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$*), and*

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{27}$$

Proof. See Section 5.2 of this appendix. ■

Given Corollary 1, the computational algorithm needs to be amended as shown below.

3.3.1 Computational algorithm (Market maker and heterogeneous slopes)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $f_{t,1} < f_{t,2} < ... < f_{t,\tilde{H}_t}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \mu\lambda \sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}(f_{t,h} - f_{t,1}) - \frac{f_{t,1}}{1-\bar{c}}$. If $disp_{t,1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, accept $p_t = p_t^*$ as the price and move to period $t+1$. Otherwise, move to Step 3.

3. Compute $disp_{t,\tilde{H}_t} = -\frac{f_{t,\tilde{H}_t}}{1-\bar{c}}$. If $disp_{t,\tilde{H}_t} > \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1}$, accept $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ as the price and move to period $t+1$. Else, move to Step 4.

4. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \tilde{a}_k(f_{t,k} - (1-\bar{c})p_t^{guess}) < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a $k^* \in \{1, ..., \tilde{H}_t - 1\}$ is found such that $disp_{t,k^*+1} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < disp_{t,k^*}$ and go to step 5.

5. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{p_{t-1} + \mu\lambda \sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]}{1 + \mu\lambda(1-\bar{c}) \sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, and move to period $t+1$.

4 Additional heterogeneity

4.1 Heterogeneous forecast coefficients on p_t

In the case of heterogeneous forecast coefficients on p_t, price beliefs are amended to

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{c}_h p_t + \tilde{f}_{t,h} \tag{28}$$

where $\overline{c}_h \in [0, 1+\tilde{r})$ for all $h \in \mathcal{H}$.

As discussed below, this case requires an amendment to the computational algorithm because ranking types in terms of $f_{t,h}$ is no longer sufficient. We first provide an amended version of Proposition 1 before discussing the necessary changes to the algorithm.

Defining $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - a\sigma^2\overline{Z}$ and $r_h := \tilde{r} - \overline{c}_h$, the demands are amended to:

$$z_{t,h} = \begin{cases} \dfrac{f_{t,h} - (1+r_h)p_t + a\sigma^2\overline{Z}}{a\sigma^2} & \text{if } p_t \leq \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}. \end{cases} \tag{29}$$

Using the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$, we have the following.

Proposition 4 (Heterogeneous p_t coefficients) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$ and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types). Let $f_{t,h}$, r_h be defined as above and $\tilde{n}_{t,h} := n_{t,h}(1+r_h)$. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} \leq 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}} \tag{30}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0\ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} > 0$, at least one type is short-selling constrained and $\exists$ unique $\mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \min_{h\in\mathcal{B}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \leq (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \max_{h\in\mathcal{S}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$, and the price is

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}} \tag{31}$$

with demands $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^*$.

Proof. See Section 5.3 of this appendix. ■

Note that types with the lowest values of $(f_{t,h} + a\sigma^2\overline{Z})/(1+r_h)$ should be considered least optimistic, as they are more likely to be short-selling constrained at any given price. We thus define $\hat{f}_{t,h} := (f_{t,h} + a\sigma^2\overline{Z})/(1+r_h)$, which allows us to state the following result.

Corollary 2 *Let $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ be the set such that types are ordered as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} := \frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$. Let $disp_{t,k} := \sum_{h=k}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left(\frac{f_{t,k}+a\sigma^2\overline{Z}}{1+r_{t,k}}\right) \sum_{h=k}^{\tilde{H}_t} \tilde{n}_{t,h}$, where $k \in \{1, ..., \tilde{H}_t - 1\}$, and $\tilde{n}_{t,h} := n_{t,h}(1 + r_h)$. The market-clearing price is:*

$$p_t = \begin{cases} \frac{\sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h}}{\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq (1 - \sum_{h=1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} \ (= 0) \\ \frac{\sum_{h=2}^{\tilde{H}_t} n_{t,h} f_{t,h} - n_{t,1} a\sigma^2\overline{Z}}{\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq (1 - \sum_{h=2}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < disp_{t,1} \\ \frac{\sum_{h=3}^{\tilde{H}_t} n_{t,h} f_{t,h} - (n_{t,1} + n_{t,2}) a\sigma^2\overline{Z}}{\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq (1 - \sum_{h=3}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < disp_{t,2} \\ \vdots & \vdots \\ \frac{n_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t-1} n_{t,h}) a\sigma^2\overline{Z}}{\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t-1)} & \text{if } disp_{t,\tilde{H}_t-1} > (1 - \sum_{h=\tilde{H}_t-1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} \end{cases}$$

where $p_t^{(k^*)}$ is the price if types $1, ..., k^*$ are short-selling constrained, p_t^* is the corresponding price if short-selling constraints were absent (which satisfies $p_t^* < p_t^{(k)}$, $\forall k \leq k^*$), and

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{32}$$

Proof. It follows from Proposition 4 and the proof of Corollary 1 (see main paper). ■

In light of the changes in Corollary 2, the computational algorithm needs to be amended as shown below.

4.1.1 Computational algorithm (Heterogeneous coefficients on p_t)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} = \frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left(\frac{f_{t,1}+a\sigma^2\overline{Z}}{1+r_{t,1}}\right) \sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}$. If $disp_{t,1} \leq 0$, accept $p_t = p_t^*$ as the date t price solution and move to period $t + 1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = \frac{f_{t,k}+a\sigma^2\overline{Z}-(1+r_k)p_t^{guess}}{a\sigma^2} < 0$, say $\underline{k}$. Starting from $k = \underline{k}$, check if $disp_{t,k+1} \leq (1 - \sum_{h=k+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < disp_{t,k}$; if not, try $k = k_{prev}+1$ until a k^* is found such that $disp_{t,k^*+1} \leq (1-\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h}) a\sigma^2\overline{Z} < disp_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right] a\sigma^2\overline{Z}}{\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, and move to period $t + 1$.

4.1.2 Numerical example

We now consider a numerical example. The exercise is the same as for Example 1 (main paper), except that we allow heterogeneous coefficients on p_t in the beliefs of each type:

$$\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{c}_h p_t + \tilde{f}_{t,h}$$

where $\overline{c}_h \sim \mathcal{U}(\overline{c}_{min}, \overline{c}_{max})$ and $\mathcal{U}$ denotes the uniform distribution.

The $\overline{c}_h$ parameters are drawn at date 0. We consider both 'high' and 'low' heterogeneity and the same sequence of shocks as in Example 1 (main paper) – i.e. idiosyncratic shocks to beliefs are set at zero in the first 10 periods and are normally distributed with standard deviation 0.04 thereafter.[6] In the high heterogeneity case we set $c_{min} = 0.95, c_{max} = 1.05$; in the low heterogeneity case $c_{min} = 0.995, c_{max} = 1.005$.

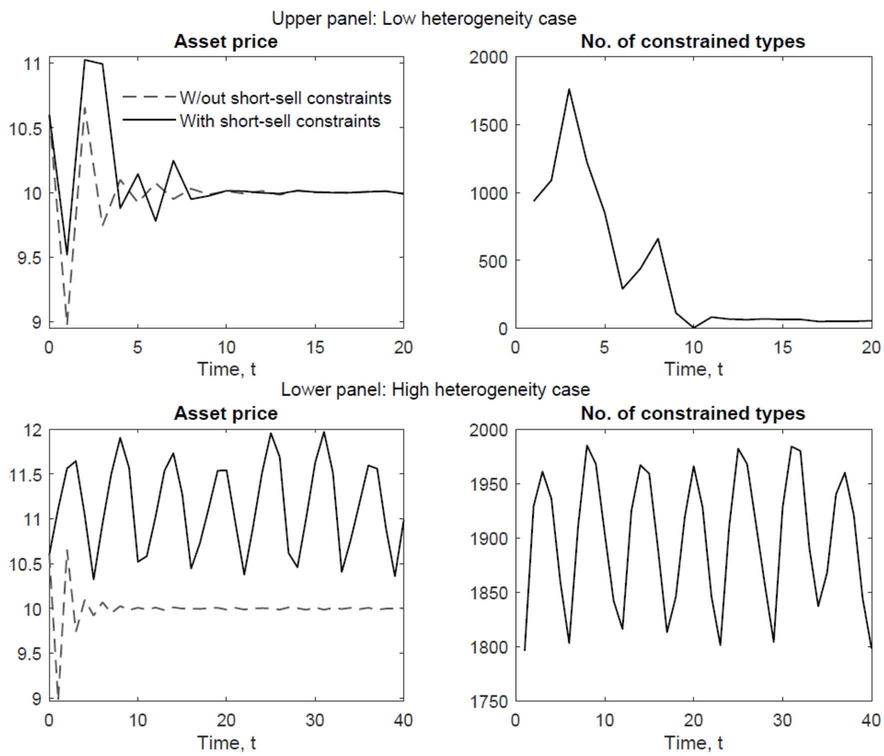

Figure 1: Simulation with low and high heterogeneity in $\overline{c}_h$ ($H = 3,000$ types)

Figure 1 shows the results from two simulations. We see that heterogeneity has a much bigger impact when short-selling constraints are present; intuitively, this is because heterogeneity largely 'washes out' in the no-constraints case where the average belief matters, whereas if short-selling is prevented the heterogeneity means constraints bind more often. The results in Table 1 show that the solution of the model with short-selling constraints is fast and accurate. In terms of speed and accuracy, it is comparable to the known solution in the absence of short-selling constraints which does not require any search procedure.

[6]To ensure the same sequence of shocks is used with set the random seed at 5 in all simulations.

Table 1: Computation times and accuracy: $T = 500$ periods, $H = 3,000$ types

Heterogeneity	Case	Time (s)	Bind freq.	$\max(Error_t)$
High case:	W/out short-sell constraints	0.40	-	2.0e-15
$\overline{c}_h \sim \mathcal{U}(0.95, 11.05)$	With short-sell constraints	0.66	500/500	8.9e-16
Low case:	W/out short-sell constraints	0.13	-	5.3e-16
$\overline{c}_h \sim \mathcal{U}(0.995, 1.005)$	With short-sell constraints	0.22	500/500	4.1e-15

Note: $\max(Error_t) = \max\{Error_1, ..., Error_T\}$, $Error_t := |\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} - \overline{Z}|$.

4.2 Heterogeneous subjective variances

In the case of heterogeneous subjective return variances, demands are given by

$$z_{t,h} = \begin{cases} \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d} - (1+\tilde{r})p_t}{a\sigma_h^2} & \text{if } p_t \leq \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d}}{1+\tilde{r}} \\ 0 & \text{if } p_t > \dfrac{\tilde{E}_{t,h}\left[p_{t+1}\right] + \overline{d}}{1+\tilde{r}}. \end{cases} \tag{33}$$

where $\sigma_h^2 > 0$ is the subjective return variance of type h and $\tilde{E}_{t,h}\left[p_{t+1}\right] = \overline{c}p_t + \tilde{f}_{t,h}$.

Defining $\tilde{a}_h := (a\sigma_h^2)^{-1}$, $f_{t,h} := \tilde{f}_{t,h} + \overline{d} - \overline{Z}/\tilde{a}_h$ and $r := \tilde{r} - \overline{c}$, the demands in (33) are

$$z_{t,h} = \begin{cases} \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) & \text{if } p_t \leq \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r} \\ 0 & \text{if } p_t > \dfrac{f_{t,h} + \overline{Z}/\tilde{a}_h}{1+r}. \end{cases} \tag{34}$$

In (33) and (34), subjective variances are heterogeneous but fixed over time and exogenous; however, allowing them to be endogenous and time-varying is straightforward as it simply requires the addition of time subscripts. By (34) and market-clearing, we have the following.

Proposition 5 (Heterogeneous subjective variances) *Let p_t be the market-clearing price at date $t \in \mathbb{N}_+$ and let $\mathcal{B}_t \subseteq \mathcal{H}$ ($\mathcal{S}_t := \mathcal{H} \setminus \mathcal{B}_t$) be the set of unconstrained types (constrained types). Let $\tilde{a}_h = (a\sigma_h^2)^{-1}$, $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$ and $f_{t,h}$ as above. Then the following holds:*

(i) If $\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq 0$, then no type is short-selling constrained ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_t^* = \emptyset$) and the market-clearing price is

$$p_t = \frac{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h} f_{t,h}}{(1+r)\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}} \tag{35}$$

with demands $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{H}$.

(ii) If $\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) > 0$, at least one type is short-selling constrained and $\exists$ unique $\mathcal{B}_t^*, \mathcal{S}_t^* \subset \mathcal{H}$ such that $\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z} < \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\})$, and the price is

$$p_t = \frac{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}} \tag{36}$$

with demands $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*, \ z_{t,h} = 0 \ \forall h \in \mathcal{S}_t^*$.

Proof. See Section 5.4 of this appendix. ■

Note that types with the lowest values of $f_{t,h} + \overline{Z}/\tilde{a}_h$ should be considered least optimistic, as they are more likely to be short-selling constrained at any given price. We thus define $\hat{f}_{t,h} := f_{t,h} + \overline{Z}/\tilde{a}_h$ to rank types by optimism, which allows us to state the following result.

Corollary 3 *Let* $\tilde{\mathcal{H}}_t = \{1, ..., \tilde{H}_t\}$ *be the set such that types are ordered as* $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$*, where* $\hat{f}_{t,h} := f_{t,h} + \overline{Z}/\tilde{a}_h$*. Let* $disp_{t,k} := \sum_{h=k}^{\tilde{H}_t} n_{t,h}(f_{t,h} - [f_{t,k} + \overline{Z}/\tilde{a}_k])$*, where* $k \in \{1, ..., \tilde{H}_t - 1\}$*, and* $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$*. The market-clearing price is:*

$$p_t = \begin{cases} \dfrac{\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h}}{(1+r)\sum_{h=1}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^* & \text{if } disp_{t,1} \leq (1 - \sum_{h=1}^{\tilde{H}_t} n_{t,h})\overline{Z} \ (= 0) \\ \dfrac{\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - n_{t,1}\overline{Z}}{(1+r)\sum_{h=2}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(1)} & \text{if } disp_{t,2} \leq (1 - \sum_{h=2}^{\tilde{H}_t} n_{t,h})\overline{Z} < disp_{t,1} \\ \dfrac{\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - (n_{t,1} + n_{t,2})\overline{Z}}{(1+r)\sum_{h=3}^{\tilde{H}_t} \tilde{n}_{t,h}} := p_t^{(2)} & \text{if } disp_{t,3} \leq (1 - \sum_{h=3}^{\tilde{H}_t} n_{t,h})\overline{Z} < disp_{t,2} \\ \vdots & \vdots \\ \dfrac{\tilde{n}_{t,\tilde{H}_t} f_{t,\tilde{H}_t} - (\sum_{h=1}^{\tilde{H}_t - 1} n_{t,h})\overline{Z}}{(1+r)\tilde{n}_{t,\tilde{H}_t}} := p_t^{(\tilde{H}_t - 1)} & \text{if } disp_{t,\tilde{H}_t - 1} > (1 - \sum_{h=\tilde{H}_t - 1}^{\tilde{H}_t} n_{t,h})\overline{Z} \end{cases}$$

where $p_t^{(k^*)}$ *is the price if types* $1, ..., k^*$ *are short-selling constrained,* p_t^* *is the corresponding price if short-selling constraints were absent (which satisfies* $p_t^* < p_t^{(k)}$*,* $\forall k \leq k^*$*), and*

$$p_t^{(k-1)} < p_t^{(k)} < p_t^{(k^*)}, \qquad \text{for all } k < k^*. \tag{37}$$

Proof. It follows from Proposition 5 and from the same steps as used in the proof of Corollary 1 (see main paper) with necessary alterations being made. ■

4.2.1 Computational algorithm (Heterogeneous subjective variances)

1. Construct the set $\tilde{\mathcal{H}}_t$ by ordering types by optimism as $\hat{f}_{t,1} < \hat{f}_{t,2} < ... < \hat{f}_{t,\tilde{H}_t}$, where $\hat{f}_{t,h} = f_{t,h} + \overline{Z}/\tilde{a}_h$, and find the associated population shares $n_{t,h}$ of types $h = 1, ..., \tilde{H}_t$.

2. Compute $disp_{t,1} = \sum_{h=1}^{\tilde{H}_t} n_{t,h}(f_{t,h} - [f_{t,1} + \overline{Z}/a_1])$. If $disp_{t,1} \leq 0$, accept $p_t = p_t^*$ as the date t price solution and move to period $t+1$. Otherwise, move to Step 3.

3. Set $p_t^{guess} = p_t^*$. Find the largest k such that $z_{t,k}^{guess} = a_k(f_{t,k} + \overline{Z}/a_k - (1+r)p_t^{guess}) < 0$, say $\underline{k}$. Starting at $k = \underline{k}$, check if $disp_{t,k+1} \leq (1 - \sum_{h=k+1}^{\tilde{H}_t} n_{t,h})a\sigma^2\overline{Z} < disp_{t,k}$; if not, try $k = k_{prev} + 1$ until a k^* is found such that $disp_{t,k^*+1} \leq (1 - \sum_{h=k^*+1}^{\tilde{H}_t} n_{t,h})a\sigma^2\overline{Z} < disp_{t,k^*}$.

4. Accept k^* as the number of short-selling constrained types, such that the price is $p_t = p_t^{(k^*)} := \frac{\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h} f_{t,h} - \left[\sum_{h=1}^{k^*} n_{t,h}\right]\overline{Z}}{(1+r)\sum_{h=k^*+1}^{\tilde{H}_t} \tilde{n}_{t,h}}$, and move to period $t+1$.

4.2.2 Numerical example

We again stick with Example 1 in the main paper except that we draw the subjective variances of different types from a uniform distribution, such that $\sigma_h^2 \sim \mathcal{U}(\sigma_{min}^2, \sigma_{max}^2)$. The results in Table 2 show that the solutions with short-selling constraints are fast and accurate, being comparable to those when short-selling constraints are absent. The corresponding time series for the simulations (in the first 20 periods) are plotted in Figure 2.

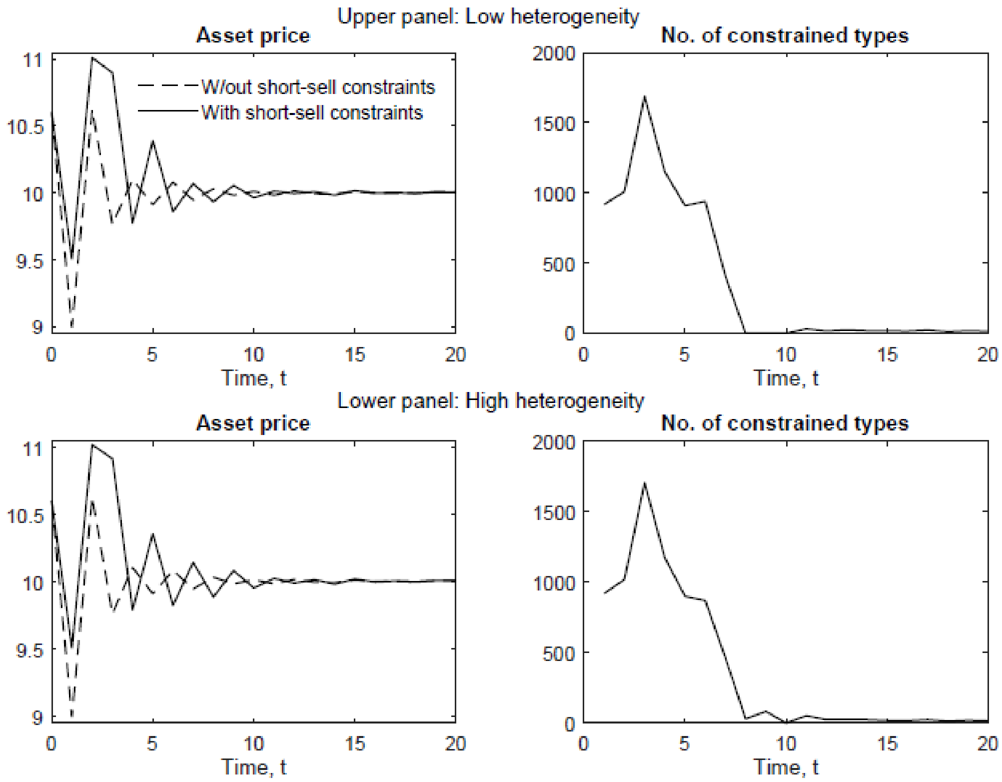

Figure 2: Simulation with low and high heterogeneity in σ_h^2 ($H = 3,000$ types)

Table 2: Computation times and accuracy: $T = 500$ periods, $H = 3,000$ types

Heterogeneity	Case	Time (s)	Bind freq.	$\max(Error_t)$
High case:	W/out short-sell constraints	0.16	-	9.7e-16
$\sigma_h^2 \sim \mathcal{U}(0.9, 1.1)$	With short-sell constraints	0.23	499/500	3.4e-15
Low case:	W/out short-sell constraints	0.14	-	8.1e-16
$\sigma_h^2 \sim \mathcal{U}(0.99, 1.01)$	With short-sell constraints	0.22	497/500	4.0e-15

Note: $\max(Error_t) = \max\{Error_1, ..., Error_T\}$, $Error_t := |\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$.

5 Proofs

5.1 Proof of Proposition 1

By Section 3.1 the indicator variable $\mathbb{1}_t := \mathbb{1}_{\{g(p_{t-1},...,p_{t-K})\leq 0\}}$ is equal to 1 if the short-selling constraint is in place at date t (i.e. if $g(p_{t-1},...,p_{t-K}) \leq 0$) and is 0 otherwise.

Case 1: $\mathbb{1}_t = 1$

If the short-selling constraint is in place at date t (i.e. if $\mathbb{1}_t = 1$), the cases for price and demands are equivalent to those for an unconditional short-selling constraint, as in Proposition 1 in the main paper and its proof. Hence, the short-selling constraint is slack for all types if $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ and binds for one or more types (but fewer than H) otherwise, i.e. if $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$.

Case 2: $\mathbb{1}_t = 0$

If the short-selling constraint is not in place at date t (i.e. if $\mathbb{1}_t = 0$) then demands are given by $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)p_t) \in \mathbb{R}$ for all $h \in \mathcal{H}$, and thus the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ gives $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}}{1+r}$ (which is the same expression as for $\mathbb{1}_t = 1$, $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$). This conclusion holds regardless of whether $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$ or $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h}\}) > a\sigma^2\overline{Z}$. ■

5.2 Proof of Proposition 3

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\overline{c})p_t) \geq 0\ \forall h \in \mathcal{H}$, which implies by the price equation that $p_t = \frac{p_{t-1}+\mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h}+\mu(1-\lambda)Z_{t-1}-\mu\overline{Z}}{1+\mu\lambda(1-\overline{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}} := p_t^*$, where $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The guess is verified if and only if $f_{t,h} \geq (1-\overline{c})p_t^*\ \forall h \in \mathcal{H}$, which requires $(1-\overline{c})^{-1}\left(1+\mu\lambda(1-\overline{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}\right)f_{t,h} \geq p_{t-1} + \mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}\ \forall h \in \mathcal{H}$. Note that this is equivalent to $(1-\overline{c})^{-1}\left(1+\mu\lambda(1-\overline{c})\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}\right)\min_{h\in\mathcal{H}}\{f_{t,h}\} \geq p_{t-1}+\mu\lambda\sum_{h\in\mathcal{H}}\tilde{n}_{t,h}f_{t,h}+\mu(1-\lambda)Z_{t-1}-\mu\overline{Z}$, which simplifies to the inequality in Proposition 3 Part 1.

Case 2(i): Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ **and binds for all** $h \in \mathcal{H}\setminus\mathcal{B}_t^*$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} - (1-\overline{c})p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{S}_t^* = \mathcal{H}\setminus\mathcal{B}_t^*$, such that $p_t = \frac{p_{t-1}+\mu\lambda\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}f_{t,h}+\mu(1-\lambda)Z_{t-1}-\mu\overline{Z}}{1+\mu\lambda(1-\overline{c})\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}}$ by the price equation, with $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$. The guess is verified iff $f_{t,h} \geq (1-\overline{c})p_t\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} < (1-\overline{c})p_t\ \forall h \in \mathcal{S}_t^*$, i.e. $(1-\overline{c})^{-1}\left(1+\mu\lambda(1-\overline{c})\sum_{h\in\mathcal{B}_t^*}\tilde{n}_{t,h}\right)f_{t,h}-$

$(p_{t-1} + \mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} f_{t,h} + \mu(1-\lambda)Z_{t-1} - \mu\overline{Z}) \geq 0 \ (< 0) \ \forall h \in \mathcal{B}_t^* \ (\forall h \in \mathcal{S}_t^*)$, which are equivalent to $\mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}} \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} \leq \mu[\overline{Z} - (1-\lambda)Z_{t-1}] - p_{t-1} < \mu\lambda \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}) - \frac{1}{1-\bar{c}} \max_{h\in\mathcal{S}_t^*}\{f_{t,h}\}$, as stated in Proposition 3 Part 2(i).

Case 2(ii): Short-selling constraint binds for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = 0 \ \forall h \in \mathcal{H}$, which implies $p_t = p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$ by the price equation. The guess is verified if and only if $f_{t,h} < (1-\bar{c})p_t \ \forall h \in \mathcal{H}$, i.e. if and only if $\frac{1}{1-\bar{c}} \max_{h\in\mathcal{H}}\{f_{t,h}\} < p_{t-1} + \mu[(1-\lambda)Z_{t-1} - \overline{Z}]$, as stated in Proposition 3 Part 2(ii). ■

Proof of Corollary 1

The proof is similar to Corollary 1 (main paper). The first (final) line of Corollary 1 is equivalent to Part 1 (Part 2(ii)) in Proposition 3. The remaining lines follow from Proposition 3 since the result is shown for arbitrary sets $\mathcal{B}_t^*, \mathcal{S}_t^*$. Finally, the market-maker sets the price $p_t = p_{t-1} + \mu[\lambda(Z_t - \overline{Z}) + (1-\lambda)(Z_{t-1} - \overline{Z})]$, where Z_{t-1}, p_{t-1} predetermined and Z_t is smaller the larger the number of types who short-sell. Short-selling occurs at price p_t^* for any $k^* \geq 1$ and at prices $p_t^{(k)}$ for $1 \leq k < k^*$, with $k^* - k$ short-sellers (since the prices were not verified). Hence, $p_t^* < p_t^{(1)} < ... < p_t^{(k^*)}$. ■

5.3 Proof of Proposition 4

Existence of a unique price follows from Anufriev and Tuinstra (2013, Proposition 2.1) when an appropriate relabelling of variables is used. We define $r_{t,h} := \tilde{r} - \bar{c}_{t,h}$ and $\tilde{n}_{t,h} := n_{t,h}(1 + r_h)$.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \ \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h}}{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^* \geq 0 \ \forall h \in \mathcal{H}$, i.e. $\left[\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\right] \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} \geq \sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} \ \forall h \in \mathcal{H}$, which simplifies to $\sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - \min_{h\in\mathcal{H}}\{\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\} \sum_{h\in\mathcal{H}} \tilde{n}_{t,h} \leq 0$ as stated in Proposition 4.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0 \ \forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h} - (1 + r_h)p_t\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h} - (1+r_h)p_t\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h} z_{t,h}$, so market-clearing is $\sum_{h\in\mathcal{B}_t^*} n_{t,h} z_{t,h} = \overline{Z}$, giving $p_t = \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^{\mathcal{B}_t^*} \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r_h)p_t^{\mathcal{B}_t^*} < 0 \ \forall h \in \mathcal{S}_t^*$, i.e. $\left[\frac{f_{t,h} + a\sigma^2\overline{Z}}{1+r_h}\right] \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \geq (<) \sum_{h\in\mathcal{H}} n_{t,h} f_{t,h} - (1 - \sum_{h\in\mathcal{B}_t^*} n_{t,h}) a\sigma^2\overline{Z} \ \forall h \in \mathcal{B}_t^* \ (\forall h \in$

$\mathcal{S}_t^*$), which simplify to $\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \min_{h\in\mathcal{B}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\} \leq (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \max_{h\in\mathcal{S}_t^*}\{\frac{f_{t,h}+a\sigma^2\overline{Z}}{1+r_h}\}$, which is the inequality given in Proposition 4. ■

5.4 Proof of Proposition 5

Existence of a unique price follows from Anufriev and Tuinstra (2013, Proposition 2.1) when an appropriate relabelling of variables is used. We define $\tilde{a}_h = (a\sigma_h^2)^{-1}$ and $\tilde{n}_{t,h} := \tilde{a}_h n_{t,h}$.

Case 1: Short-selling constraint is slack for all $h \in \mathcal{H}$

Let us guess $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0\ \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ that $p_t = p_t^* := \frac{\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}f_{t,h}}{(1+r)\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}}$. The guess is verified if and only if $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^* \geq 0\ \forall h \in \mathcal{H}$, i.e. $\left[f_{t,h} + \overline{Z}/\tilde{a}_h\right]\sum_{h\in\mathcal{H}} \tilde{n}_{t,h} \geq \sum_{h\in\mathcal{H}} \tilde{n}_{t,h}f_{t,h}\ \forall h \in \mathcal{H}$. This inequality simplifies to $\sum_{h\in\mathcal{H}} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{H}}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq 0$, as stated in Proposition 5.

Case 2: Short-selling constraint slack for all $h \in \mathcal{B}_t^*$ and binds for all $h \in \mathcal{H} \setminus \mathcal{B}_t^*$

Let us guess that $z_{t,h} = \tilde{a}_h(f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t) \geq 0\ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0\ \forall h \in \mathcal{H} \setminus \mathcal{B}_t^* := \mathcal{S}_t^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ is the set of investor types for which the short-selling constraint is slack, and $\mathcal{S}_t^*$ is the set of all other types. Clearly, the above conditions imply that $\min_{h\in\mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\} > \max_{h\in\mathcal{S}_t^*}\{f_{t,h}+\overline{Z}/\tilde{a}_h\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$, so market-clearing is $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$, giving $p_t = \frac{\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}}{(1+r)\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}} := p_t^{\mathcal{B}_t^*}$. The guess is verified iff $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^{\mathcal{B}_t^*} \geq 0\ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + \overline{Z}/\tilde{a}_h - (1+r)p_t^{\mathcal{B}_t^*} < 0\ \forall h \in \mathcal{S}_t^*$, i.e. $\left[f_{t,h} + \overline{Z}/\tilde{a}_h\right]\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h} \geq (<) \sum_{h\in\mathcal{H}} \tilde{n}_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z}\ \forall h \in \mathcal{B}_t^*\ (\forall h \in \mathcal{S}_t^*)$, which simplify to $\sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\}) \leq (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})\overline{Z} < \sum_{h\in\mathcal{B}_t^*} \tilde{n}_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_t^*}\{f_{t,h} + \overline{Z}/\tilde{a}_h\})$, which is the inequality given in Proposition 5. ■

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