

# Lyapunov Stability



## 1. Definitions in Lyapunov stability analysis

## 2. Lyapunov's approach to stability

- Relevant tools

- Lyapunov stability theorems

- Instability theorem

- Discrete-time case

## 3. Recap

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default in this set of notes:  $\|\cdot\| = \|\cdot\|_2$

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- ▶ the condition must be satisfied by all  $t \geq 0$
- ▶ if a system starts at equilibrium state, it stays there

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- ▶ when  $A(t)$  is singular, multiple equilibrium states exist

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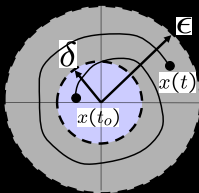


Figure: Stable s.i.L:  $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \forall t \geq t_0$ .

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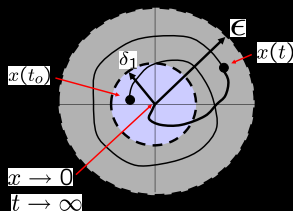


Figure: Asymptotically stable i.s.L:  $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \rightarrow 0$ .

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- ▶  $e^{\sigma t} \rightarrow 0$  if  $\sigma < 0$ ;  $e^{\lambda t} \rightarrow 0$  if  $\lambda < 0$
- ▶  $\lambda^k \rightarrow 0$  if  $|\lambda| < 1$ ;  $r^k \rightarrow 0$  if  $|r| = |\sqrt{\sigma^2 + \omega^2}| = |\lambda| < 1$

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- ▶ no need for explicit solutions to system responses
- ▶ an “energy” perspective
- ▶ fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

# Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \quad (x_1: \text{position}; x_2 : \text{velocity})$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2, \quad b > 0 \quad (\text{Newton's law})$$

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# Stability from an energy viewpoint: Generalization

Consider unforced, time-varying, nonlinear systems

$$\begin{aligned}\dot{x}(t) &= f(x(t), t), \quad x(t_0) = x_0 \\ x(k+1) &= f(x(k), k), \quad x(k_0) = x_0\end{aligned}$$

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- ▶ goal is to relate properties of the state through the Lyapunov function
- ▶ main tool: matrix formulation, linear algebra, positive definite functions

# Relevant tools

## Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- general quadratic functions in matrix form

$$Q(x) = x^T P x, \quad P^T = P$$



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$$\text{general case: } P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

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- ▶ a real square matrix  $A \in \mathbb{R}^{n \times n}$  is *orthogonal* if  $A^T A = A A^T = I$
- ▶ meaning that the columns of  $A$  form a orthonormal basis of  $\mathbb{R}^n$

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | & | \end{bmatrix}$$

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

namely,  $a_j^T a_j = 1$  and  $a_j^T a_m = 0 \ \forall j \neq m$ .

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*The eigenvalues of symmetric matrices are all real.*

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Also,  $\bar{u}^T u \in \mathbb{R}$ .

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Also,  $\bar{u}^T u \in \mathbb{R}$ . Thus  $\lambda = \frac{\bar{u}^T Au}{\bar{u}^T u}$  must also be a real number. □

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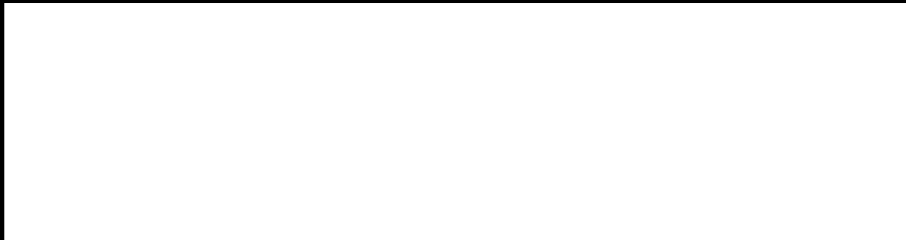
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matrix structure	analogy in complex plane
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skew-symmetric	imaginary line
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- ▶  $\Lambda = \text{diagonal}(\lambda_1, \lambda_2, \dots, \lambda_n)$

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- ▶ If  $A$  has  $r(< n)$  distinct eigenvalues, we can *choose* multiple orthogonal eigenvectors for the eigenvalues with none-unity multiplicities.

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Example:  $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5 - \lambda & \sqrt{3} \\ \sqrt{3} & 7 - \lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$
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choose  $t_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$ . No need to solve  $(A - \lambda_2 I) t_2 = 0$ !

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$$\lambda_{\max} = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T A x}{\|x\|_2^2} \quad (2)$$

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# Negative definite matrices

## Definition

A symmetric matrix  $Q \in \mathbb{R}^{n \times n}$  is called **negative-definite**, written  $Q \prec 0$ , if  $-Q \succ 0$ , i.e.,  $x^T Q x < 0$  for all  $x (\neq 0) \in \mathbb{R}^n$ .

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# Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	$\mathbb{R}_+$ axis
negative definite	negative	$\mathbb{R}_-$ axis

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## Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$  is positive-definite, as  $P = P^T$  and take any  $v = [x, y]^T$ , we have

$$\begin{aligned} v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\ &= x^2 + y^2 + (x - y)^2 \geq 0 \end{aligned}$$



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and the equality sign holds only when  $x = y = 0$ .

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$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 < 0$$

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## Theorem

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Since  $P$  is symmetric, we have

$$\lambda_{\max}(P) = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T A x}{\|x\|_2^2} \quad (4)$$

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which gives  $x^T A x \in [\lambda_{\min} \|x\|_2^2, \lambda_{\max} \|x\|_2^2]$ . Thus  
 $x^T A x > 0, x \neq 0 \Leftrightarrow \lambda_{\min} > 0$ . □

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## Definition

The leading principle minors of  $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$  are defined as

$$p_{11}, \det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \det P.$$

# Relevant tools

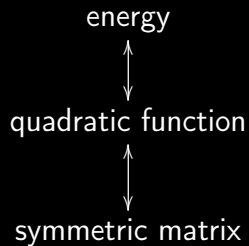
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## Example

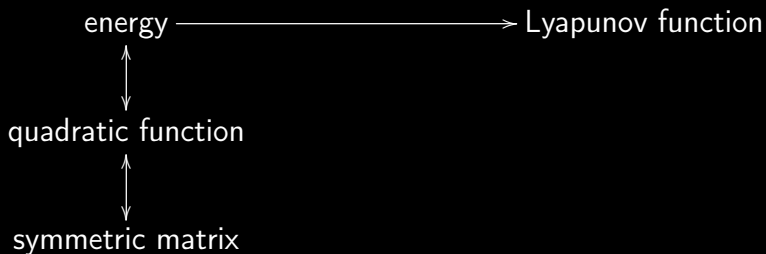
None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

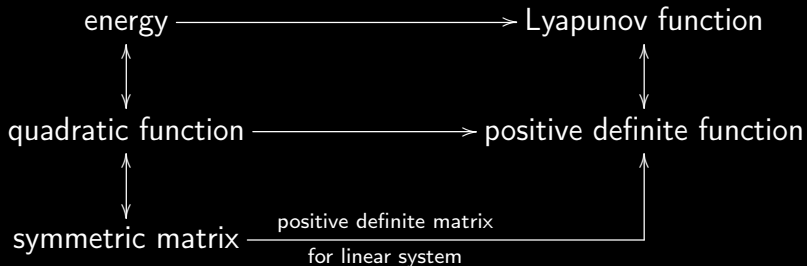
# Recap



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# Relevant tools

## Definition (Positive Definite Functions)

A continuous time function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , called to be PD, satisfying

- ▶  $W(x) > 0$  for all  $x \neq 0$
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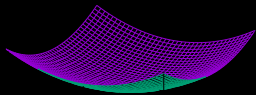
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In the 3D space, positive definite functions are “bowl-shaped”, e.g.,  $W(x_1, x_2) = x_1^2 + x_2^2$ .





# Relevant tools

## Definition (Locally Positive Definite Functions)

A continuous time function  $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$ , called to be LPD, satisfying

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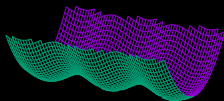
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In the 3D space, locally positive definite functions are “bowl-shaped” locally, e.g.,  $W(x_1, x_2) = x_1^2 + \sin^2 x_2$  for  $x_1 \in \mathbb{R}$  and  $|x_2| < \pi$



# Relevant tools

## Exercise

Let  $x = [x_1, x_2, x_3]^T$ . Check the positive definiteness of the following functions

1.  $V(x) = x_1^4 + x_2^2 + x_3^4$

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1.  $V(x) = x_1^4 + x_2^2 + x_3^4$  (PD)
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1. Definitions in Lyapunov stability analysis

2. Lyapunov's approach to stability

- Relevant tools

- Lyapunov stability theorems

- Instability theorem

- Discrete-time case

3. Recap

# Lyapunov stability theorems

- recall the spring mass damper example in matrix form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



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- energy function is PD:

$\mathcal{E}(t)$  = potential energy + kinetic energy =  $\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$   
and its derivative is NSD:

$$\dot{\mathcal{E}}(t) = \left[ \frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2} \right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 \quad (6)$$

$$= k_1 x_1 x_2 + m x_2 \left( -\frac{k}{m} x_1 - \frac{b}{m} x_2 \right) = \left[ \frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2} \right] A x \quad (7)$$

$$= -b x_2^2$$

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# Lyapunov stability concept for linear systems

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- ▶ and the origin is stable in the sense of Lyapunov



## Theorem (Lyapunov stability theorem for linear systems)

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► can stack the columns of  $A^T P + PA$  and  $Q$  to yield

$$\begin{aligned} \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11}/ & a_{21}/ \\ a_{12}/ & a_{22}/ \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ \underbrace{\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}/ & a_{21}/ \\ a_{12}/ & a_{22}/ \end{bmatrix} \right\}}_{L_A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \end{aligned}$$

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# The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- can simply write  $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$  using the Kronecker

product notation  $B \otimes C =$

$$\begin{bmatrix} b_{11}C & b_{12}C & \dots & b_{1n}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$$

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► e.g.,  $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{aligned} L_A &= I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix} \\ &= \left[ \begin{array}{cc|cc} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] = \left[ \begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] \end{aligned}$$



Example:  $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ ,  $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

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The eigenvalues of  $L_A$  are  $-1, -1, -1 - \sqrt{3}, -1 + \sqrt{3}$ , which are precisely  $\lambda_1 + \lambda_1, \lambda_1 + \lambda_2, \lambda_2 + \lambda_1, \lambda_2 + \lambda_2$ .

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Therefore,  $x \rightarrow 0$  as  $t \rightarrow \infty$ , regardless of the initial condition.  $\square$

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$$\begin{aligned} \cancel{x^T(\infty)Px(\infty)} - x^T(0)Px(0) &= \int_0^\infty \frac{d}{dt} x^T(t)Px(t) dt = \int_0^\infty x^T(t) (A^T P + PA) x(t) dt \\ &\Rightarrow x^T(0)Px(0) = \int_0^\infty x^T(t) Q x(t) dt = \int_0^\infty x^T(0) e^{A^T t} Q e^{At} x(0) dt \end{aligned}$$

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# Procedures of Lyapunov's direct method

1. Given  $A$ , select an arbitrary positive-definite symmetric matrix  $Q$  (e.g.,  $I$ ).
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# Lyapunov stability theorems

## Example

$\dot{x} = Ax$ ,  $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$ . The Lyapunov equation is

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Leading principle minors:  $p_{11} > 0$ ,  $p_{11}p_{22} - p_{12}^2 > 0$   
 $\Rightarrow P \succ 0 \Rightarrow$  asymptotically stable

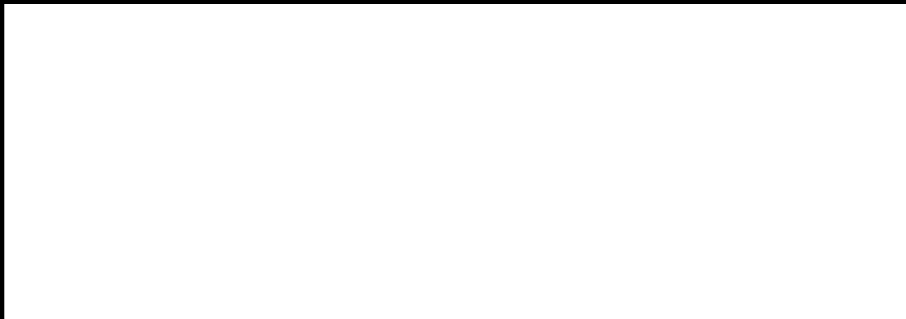
# Lyapunov analysis with Matlab

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# Lyapunov analysis with Python

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It suffices to select  $Q = I$

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- $\tilde{A} = N^{-1} A N$  and  $A$  are similar matrices and have the same eigenvalues.
- $\tilde{P} = N^T P N$  and  $P$  have the same definiteness. If we can find a positive definite solution  $P$  then the  $\tilde{P}$  will also be positive definite. Vice versa.

# Instability theorem

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## Theorem

*The equilibrium state  $0$  of  $\dot{x} = f(x)$  is unstable if there exists a function  $W(x)$  such that*

- ▶  *$\dot{W}(x)$  is PD locally:  $\dot{W}(x) > 0 \ \forall |x| < r$  for some  $r$  and  $\dot{W}(0) = 0$*
- ▶  *$W(0) = 0$*
- ▶ *there exist states  $x$  arbitrarily close to the origin such that  $W(x) > 0$*

# Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x, \quad P = P^T \succ 0$$

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Asymptotic stability desires  $\Delta V(x)$  to be negative.

# DT Lyapunov stability theorem for linear systems

## Theorem

*For system  $x(k+1) = Ax(k)$  with  $A \in \mathbb{R}^{n \times n}$ , the origin is asymptotically stable if and only if  $\exists Q \succ 0$ , such that the discrete-time Lyapunov equation*

$$A^T P A - P = -Q$$

*has a unique positive definite solution  $P \succ 0$ ,  $P^T = P$ .*



# The DT Lyapunov Eq.

$$\boxed{A^T P A - P = -Q}$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds ( $A$  is Schur stable), comes from:

$$\begin{aligned} \cancel{V(x(\infty))} - V(x(0)) &= \sum_{k=0}^{\infty} x^T(k) [A^T P A - P] x(k) \\ &= - \sum_{k=0}^{\infty} x^T(0) (A^T)^k Q A^k x(0) \\ \Rightarrow P &= \sum_{k=0}^{\infty} (A^T)^k Q A^k \end{aligned}$$

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- can show that the DT Lyapunov operator  $L_A = A^T P A - P$  is invertible if and only if  $\forall i, j \ (\lambda_A)_i (\lambda_A)_j \neq 1$

# DT Lyapunov analysis with MATLAB

## Example

$$x(k+1) = Ax(k), \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

# DT Lyapunov analysis with Python

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# Recap

- ▶ Internal stability
  - ▶ Stability in the sense of Lyapunov:  $\epsilon, \delta$  conditions
  - ▶ Asymptotic stability
- ▶ Stability analysis of linear time invariant systems ( $\dot{x} = Ax$  or  $x(k+1) = Ax(k)$ )
  - ▶ Based on the eigenvalues of  $A$ 
    - ▶ Time response modes
    - ▶ Repeated eigenvalues on the imaginary axis
  - ▶ Routh's criterion
    - ▶ No need to solve the characteristic equation
    - ▶ Discrete time case: bilinear transform ( $z = \frac{1+s}{1-s}$ )

# Recap

## ► Lyapunov equations

**Theorem:** All eigenvalues of  $A$  have negative real parts iff for any given  $Q \succ 0$ , the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique solution  $P$  and  $P \succ 0$ .

Given  $Q$ , the Lyapunov equation  $A^T P + PA = -Q$  has a unique solution when  $\lambda_{A,i} + \lambda_{A,j} \neq 0$  for all  $i$  and  $j$ .

**Theorem:** All eigenvalues of  $A$  are inside the unit circle iff for any given  $Q \succ 0$ , the Lyapunov equation

$$A^T P A - P = -Q$$

has a unique solution  $P$  and  $P \succ 0$ .

Given  $Q$ , the Lyapunov equation  $A^T P A - P = -Q$  has a unique solution when  $\lambda_{A,i} \lambda_{A,j} \neq 1$  for all  $i$  and  $j$ .

# Recap

- ▶  $P$  is positive definite if and only if any one of the following conditions holds:
  1. All the eigenvalues of  $P$  are positive.
  2. All the leading principle minors of  $P$  are positive.
  3. There exists a nonsingular matrix  $N$  such that  $P = N^T N$ .