Linear Systems

Controllable and Observable Subspaces
Kalman Canonical Decomposition

1. Controllable subspace
2. Observable subspace
3. Separating the uncontrollable subspace
    - Discrete-time version
    - Continuous-time version
    - Stabilizability
4. Separating the unobservable subspace
    - Discrete-time version
    - Detectability
    - Continuous-time version
5. Transfer-function perspective
6. Kalman decomposition

Controllable subspace: Introduction

Example

$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1(k+1) & = x_1(k) + u(k) \\ x_2(k+1) & = 0 \end{cases}$$

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1(k+1) & = x_1(k) + x_2(k) + u(k) \\ x_2(k+1) & = x_2(k) \end{cases}$$

- there exists controllable and uncontrollable states: x_1 controllable and x_2 uncontrollable
- how to compute the dimensions of the two for general systems?
- how to separate them?

Controllable subspace: Assumptions

Consider an uncontrollable LTI system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \;\; A \in \mathbb{R}^{n\times n} \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Let the controllability matrix

$$P = \left[B, AB, A^2B, \ldots, A^{n-1}B\right]$$

have rank $n_1 < n$.

Controllable subspace

- The controllable subspace χ_C is the set of all vectors $x \in \mathbb{R}^n$ that can be reached from the origin.
- From

$$x(n) - A^n x(0) = \underbrace{\left[B, AB, A^2B, \ldots, A^{n-1}B\right]}_{P} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix}$$

χ_C is the range space of P: $\chi_C = \mathcal{R}(P)$

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Observable subspace: Introduction

Example

$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \Leftrightarrow \begin{cases} x_1(k+1) & = x_1(k) + u(k) \\ x_2(k+1) & = x_1(k) + x_2(k) \\ y(k) & = x_1(k) \end{cases}$$

$$\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- exists observable and unobservable states: x_1 observable and x_2 unobservable
- how to separate the two?
- how to separate controllable but observable states, controllable but unobservable states, etc?

Observable subspace: Assumptions

Consider an unobservable LTI system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \;\; A \in \mathbb{R}^{n\times n} \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

Let the observability matrix

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

have rank $n_2 < n$.

Unobservable subspace

- The unobservable subspace χ_{uo} is the set of all nonzero initial conditions $x(0) \in \mathbb{R}^n$ that produce a zero free response.
- From

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(n-1) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q} x(0)$$

χ_{uo} is the null space of Q: $\chi_{uo} = \mathcal{N}(Q)$

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Separating the uncontrollable subspace

- recall 1: similarity transform $x = Mx^*$ preserves controllability

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \Rightarrow \begin{cases} x^*(k+1) = M^{-1}AMx^*(k) + M^{-1}Bu(k) \\ y(k) = CMx^*(k) + Du(k) \end{cases}$$

- recall 2: the uncontrollable system structure at introduction

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} x_1(k+1) & = x_1(k) + x_2(k) + u(k) \\ x_2(k+1) & = x_2(k) \end{cases}$$

- decoupled structure for generalized systems

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

$\bar{x}_{uc}$ impacted by neither u nor $\bar{x}_c$.

Theorem (Kalman canonical form (controllability))

Let $x \in \mathbb{R}^n$, $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$ be uncontrollable with rank of the controllability matrix, $\mathrm{rank}(P) = n_1 < n$. Let $M = \begin{bmatrix} M_c & M_{uc} \end{bmatrix}$, where $M_c = [m_1, \ldots, m_{n_1}]$ consists of n_1 linearly independent columns of P, and $M_{uc} = [m_{n_1+1}, \ldots, m_n]$ are added columns to complete the basis and yield a nonsingular M. Then $x = M\bar{x}$ transforms the system equation to

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

Furthermore, $(\bar{A}_c, \bar{B}_c)$ is controllable, and

$$C(zI - A)^{-1}B + D = \bar{C}_c(zI - \bar{A}_c)^{-1}\bar{B}_c + D$$

Theorem (Kalman canonical form (controllability))

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}^{M^{-1}B} u(k)$$

intuition: the "B" matrix after transformation

- columns of $B \in$ column space of P, which is equivalent to $\mathcal{R}(M_c)$
- columns of M_{uc} and M_c are linearly independent $\Rightarrow$ columns of $B \notin \mathcal{R}(M_{uc})$
- thus

$$B = \begin{bmatrix} M_c & M_{uc} \end{bmatrix} \begin{bmatrix} \overbrace{*}^{\text{denote as } \bar{B}_c} \\ 0 \end{bmatrix} \Rightarrow M^{-1}B = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

Theorem (Kalman canonical form (controllability))

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}^{M^{-1}AM} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

intuition: the "A" matrix after transformation

- the range space of M_c is "A-invariant":

$$\text{columns of } AM_c \in \left\{AB, A^2B, \ldots, A^nB\right\} \in \mathcal{R}(M_c)$$

  where columns of $A^nB \in \mathcal{R}(P) = \mathcal{R}(M_c)$ ($\because$ Cayley Halmilton Thm)

- i.e., $AM_c = M_c\bar{A}_c$ for some $\bar{A}_c \Rightarrow$

$$A\left[M_c, M_{uc}\right] = \left[M_c, M_{uc}\right] \underbrace{\begin{bmatrix} \bar{A}_c & \overbrace{*}^{\triangleq \bar{A}_{12}} \\ 0 & \overbrace{*}^{\triangleq \bar{A}_{uc}} \end{bmatrix}}_{\bar{A}} \Rightarrow M^{-1}AM = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}$$

Theorem (Kalman canonical form (controllability))

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}^{M^{-1}AM} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}^{M^{-1}B} u(k)$$

<u>$(\bar{A}_c, \bar{B}_c)$ *is controllable*</u>

- controllability matrix after similarity transform

$$\begin{aligned} \bar{P} &= \left[\begin{array}{cccc|cc} \bar{B}_c & \bar{A}_c\bar{B}_c & \dots & \bar{A}_c^{n_1-1}\bar{B}_c & \dots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \dots & 0 & \dots & 0 \end{array}\right] \\ &= \left[\begin{array}{c|ccc} \bar{P}_c & \bar{A}_c^{n_1}\bar{B}_c & \dots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \dots & 0 \end{array}\right] \end{aligned}$$

- similarity transform does not change controllability $\Rightarrow$ $\text{rank}(\bar{P}) = \text{rank}(P) = n_1$
- thus $\text{rank}(\bar{P}_c) = n_1 \Rightarrow (\bar{A}_c, \bar{B}_c)$ is controllable

Theorem (Kalman canonical form (controllability))

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

$$\underline{C(zI - A)^{-1}B + D = \bar{C}_c(zI - \bar{A}_c)^{-1}\bar{B}_c + D}$$

we can check that

$$\begin{aligned} & \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} zI - \bar{A}_c & -\bar{A}_{12} \\ 0 & zI - \bar{A}_{uc} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D \\ = & \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \left(zI - \bar{A}_c\right)^{-1} & * \\ 0 & \left(zI - \bar{A}_{uc}\right)^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + D \\ = & \bar{C}_c \left(zI - \bar{A}_c\right)^{-1} \bar{B}_c + D \end{aligned}$$

Matlab commands

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}}^{M^{-1}AM} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \overbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}^{M^{-1}B} u(k)$$

$x = M\bar{x}$ where $M = \begin{bmatrix} M_c & M_{uc} \end{bmatrix}$

- $M_c = [m_1, \ldots, m_{n_1}]$ consists of all the linearly independent columns of P: **Mc = orth(P)**
- $M_{uc} = [m_{n_1+1}, \ldots, m_n]$ are added columns to complete the basis and yield a nonsingular M
  - from linear algebra: the orthogonal complement of the range space of P is the null space of P^T:

$$\mathbb{R}^n = \mathcal{R}(P) \oplus \mathcal{N}\left(P^T\right)$$

  - hence **Muc = null(P')** (the transpose is important here)

The techniques apply to CT systems

Theorem (Kalman canonical form (controllability))

Let a n-dimensional state-space system $\dot{x} = Ax + Bu$, $y = Cx + Du$ *be uncontrollable with the rank of the controllability matrix* $rank(P) = n_1 < n$. *Let* $M = \begin{bmatrix} M_c & M_{uc} \end{bmatrix}$ *where* $M_c = [m_1, \ldots, m_{n_1}]$ *consists of* n_1 *linearly independent columns of* P, $M_{uc} = [m_{n_1+1}, \ldots, m_n]$ *are added columns to complete the basis for* $\mathbb{R}^n$ *and yield a nonsingular* M. *Then the similarity transformation* $x = M\bar{x}$ *transforms the system equation to*

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix}\begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix}\begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix} + Du$$

Example

$$\frac{d}{dt}\begin{bmatrix} v_m \\ F_{k_1} \\ F_{k_2} \end{bmatrix} = \begin{bmatrix} -b/m & -1/m & -1/m \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_m \\ F_{k_1} \\ F_{k_2} \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \\ 0 \end{bmatrix} F$$

Let $m = 1, b = 1$

$$P = \begin{bmatrix} 1 & -1 & 1-k_1-k_2 \\ 0 & k_1 & -k_1 \\ 0 & k_2 & -k_2 \end{bmatrix}, \; M = \begin{bmatrix} 1 & -1 & 0 \\ 0 & k_1 & 0 \\ 0 & k_2 & 1 \end{bmatrix}, \; M^{-1} = \begin{bmatrix} 1 & 1/k_1 & 0 \\ 0 & 1/k_1 & 0 \\ 0 & -k_2/k_1 & 1 \end{bmatrix}$$

$$\bar{A} = M^{-1}AM = \left[\begin{array}{cc|c} 0 & -(k_1+k_2) & 1 \\ 1 & -1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \; \bar{B} = M^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right]$$

Stabilizability

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

The system is *stabilizable* if

- all its unstable modes, if any, are controllable
- i.e., the uncontrollable modes are stable ($\bar{A}_{uc}$ is Schur, namely, all eigenvalues are in the unit circle)

Separating the unobservable subspace

- recall 1: similarity transform $x = O^{-1}x^*$ preserves observability

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \Rightarrow \begin{cases} x^*(k+1) = OAO^{-1}x^*(k) + OBu(k) \\ y(k) = CO^{-1}x^*(k) + Du(k) \end{cases}$$

- an unobservable system structure

$$\bar{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Leftrightarrow \begin{cases} x_1(k+1) & = x_1(k) + u(k) \\ x_2(k+1) & = x_1(k) + x_2(k) \\ y(k) & = x_1(k) \end{cases}$$

$$\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- decoupled structure for generalized systems

$$\begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k)$$

the "observed" $\bar{x}_o$ doesn't reflect $\bar{x}_{uc}$ $\left(\bar{x}_o(k+1) = \bar{A}_o\bar{x}_o(k) + \bar{B}_o u(k)\right)$

Theorem (Kalman canonical form (observability))

Let $x \in \mathbb{R}^n$, $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$ be unobservable with rank of the observability matrix, $rank(Q) = n_2 < n$. Let $O = \begin{bmatrix} O_o \\ O_{uo} \end{bmatrix}$ where O_o consists of n_2 linearly independent rows of Q, and $O_{uo} = \left[o_{n_1+1}^T, \dots, o_n^T\right]^T$ are added rows to complete the basis and yield a nonsingular O. Then $\bar{x} = Ox$ transforms the system equation to

$$\begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k)$$

Furthermore, $(\bar{A}_o, \bar{O}_o)$ is observable, and

$$C(zI - A)^{-1}B + D = \bar{C}_o(zI - \bar{A}_o)^{-1}\bar{B}_o + D$$

Theorem (Kalman canonical form)

Case for observability

$$\begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k)$$

v.s. case for controllability

$$\begin{bmatrix} \bar{x}_c(k+1) \\ \bar{x}_{uc}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_c & \bar{C}_{uc} \end{bmatrix} \begin{bmatrix} \bar{x}_c(k) \\ \bar{x}_{uc}(k) \end{bmatrix} + Du(k)$$

Intuition: duality between controllability and observability

$$(A, B) \text{ unconrollable} \Leftrightarrow \left(A^T, B^T\right) \text{ unobservable}$$

Detectability

$$\begin{bmatrix} \bar{x}_o(k+1) \\ \bar{x}_{uo}(k+1) \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o(k) \\ \bar{x}_{uo}(k) \end{bmatrix} + Du(k)$$

The system is *detectable* if

- all its unstable modes, if any, are observable
- i.e., the unobservable modes are stable ($\bar{A}_{uo}$ is Schur)

Continuout-time version

Theorem (Kalman canonical form (observability))

Let a n-dimensional state-space system $\dot{x} = Ax + Bu$, $y = Cx + Du$ be unobservable with the rank of the observability matrix $rank(Q) = n_2 < n$. Then there exists similarity transform $\bar{x} = Ox$ that transforms the system equation to

$$\frac{d}{dt}\begin{bmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{uo} \end{bmatrix}\begin{bmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{uo} \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_o \\ \bar{x}_{uo} \end{bmatrix} + Du$$

Furthermore, $(\bar{A}_o, \bar{C}_o)$ is observable, and $C(sI - A)^{-1}B + D = \bar{C}_o(sI - \bar{A}_o)^{-1}\bar{B}_o + D$.

Transfer-function perspective

uncontrollable system: $C(zI - A)^{-1}B + D = \bar{C}_c(zI - \bar{A}_c)^{-1}\bar{B}_c + D$

unobservable system: $C(zI - A)^{-1}B + D = \bar{C}_o(zI - \bar{A}_o)^{-1}\bar{B}_o + D$

where $A \in \mathbb{R}^{n\times n}$, $\bar{A}_c \in \mathbb{R}^{n_1\times n_1}$, $\bar{A}_o \in \mathbb{R}^{n_2\times n_2}$

- Order reduction exists

$$G(z) = C(zI - A)^{-1}B + D = \frac{B(z)}{A(z)},\ A(z) = \det(zI - A) \text{ order} : n$$

$$G(z) = \bar{C}_c(zI - \bar{A}_c)^{-1}\bar{B}_c + D = \frac{\bar{B}_c(z)}{\bar{A}_c(z)},\ \bar{A}_c(z) = \det\left(zI - \bar{A}_c\right) \text{ order} : n_1$$

- $\Rightarrow A(z)$ **and** $B(z)$ **are not co-prime | pole-zero cancellation exists**
- same applies to unobservable systems

Example

Consider

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The transfer function is

$$G(s) = \frac{s + c_1}{s^2 + 3s + 2} = \frac{s + c_1}{(s+1)(s+2)}$$

- System is in controllable canonical form and is controllable.
- observability matrix

$$Q = \begin{bmatrix} c_1 & 1 \\ -2 & c_1 - 3 \end{bmatrix}, \ \det Q = (c_1 - 1)(c_1 - 2)$$

$\Rightarrow$unobservable if $c_1 = 1$ or 2

Kalman decomposition

an extended example:

$$A = \left[\begin{array}{c|c|c|c} A_{11} & 0 & A_{13} & 0 \\ \hline A_{21} & A_{22} & A_{23} & A_{24} \\ \hline 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{array}\right], \quad B = \left[\begin{array}{c} B_1 \\ B_2 \\ \hline 0 \\ 0 \end{array}\right]$$

$$C = \left[\begin{array}{cccc} C_1 & 0 & C_3 & 0 \end{array}\right]$$

- A_{ij}, C_i and B_i are nonzero
- The A_{11} mode is controllable and observable. The A_{22} mode is controllable but not observable. The A_{33} mode is not controllable but observable. The A_{44} mode is not controllable and not observable.