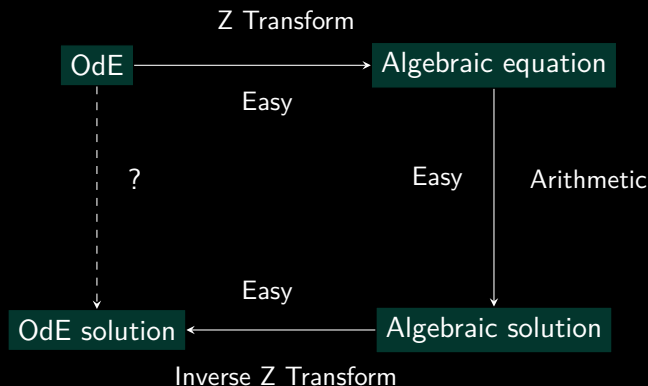


# Introduction to Modern Controls

## Z transform



# The Z transform approach to Ordinary difference Equations (OdEs)



- analogous to Laplace transform for continuous-time signals

# Definition

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- the series  $1 + \gamma + \gamma^2 + \dots$  converges to  $\frac{1}{1-\gamma}$  for  $|\gamma| < 1$  [region of convergence (ROC)]
- (also, recall that  $\sum_{k=0}^N \gamma^k = \frac{1-\gamma^{N+1}}{1-\gamma}$  if  $\gamma \neq 1$ )

Example: geometric sequence  $\{a^k\}_{k=0}^{\infty}$

$$\sum_{k=0}^{\infty} \gamma^k = \frac{1}{1 - \gamma}$$

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- $x(k) = a^k$

- $\mathcal{Z}\{a^k\} = \sum_{k=0}^{\infty} a^k z^{-k} = \boxed{\frac{1}{1 - az^{-1}}} = \frac{z}{z-a}$

## Example: step sequence (discrete-time unit step function)

$$\mathcal{Z}\{a^k\} = \frac{1}{1 - az^{-1}}$$

- $1(k) = \begin{cases} 1, & \forall k = 1, 2, \dots \\ 0, & \forall k = \dots, -1, 0 \end{cases}$

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- $\mathcal{Z}\{1(k)\} = \mathcal{Z}\{a^k\}|_{a=1} = \boxed{\frac{1}{1 - z^{-1}}} = \frac{z}{z-1}$

## Example: discrete-time impulse

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- $\mathcal{Z}\{\delta(k)\} = 1$

## Exercise: $\cos(\omega_0 k)$

$f(k)$	$F(z)$	ROC
$\delta(k)$	1	All $z$
$a^k 1(k)$	$\frac{1}{1 - az^{-1}}$	$ z  >  a $
$-a^k 1(-k - 1)$	$\frac{1}{1 - az^{-1}}$	$ z  <  a $
$ka^k 1(k)$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  >  a $
$-ka^k 1(-k - 1)$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$ z  <  a $
$\cos(\omega_0 k)$	$\frac{1 - z^{-1} \cos(\omega_0)}{1 - 2z^{-1} \cos(\omega_0) + z^{-2}}$	$ z  > 1$
$\sin(\omega_0 k)$	$\frac{z^{-1} \sin(\omega_0)}{1 - 2z^{-1} \cos(\omega_0) + z^{-2}}$	$ z  > 1$
$a^k \cos(\omega_0 k)$	$\frac{1 - az^{-1} \cos(\omega_0)}{1 - 2az^{-1} \cos(\omega_0) + a^2 z^{-2}}$	$ z  >  a $
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$$\begin{aligned}\mathcal{Z}\{x(k-1)\} &= \sum_{k=0}^{\infty} x(k-1)z^{-k} = \sum_{k=1}^{\infty} x(k-1)z^{-k} + x(-1) \\ &= \sum_{k=1}^{\infty} x(k-1)z^{-(k-1)}z^{-1} + x(-1) \\ &= z^{-1}X(z) + \cancel{x(-1)} = \boxed{z^{-1}X(z)}\end{aligned}$$

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- analogously,  $\underline{\mathcal{Z}\{x(k+1)\}} = \sum_{k=0}^{\infty} x(k+1)z^{-k} = \boxed{zX(z) - zx(0)}$

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- analogously,  $\mathcal{Z}\{x(k+1)\} = \sum_{k=0}^{\infty} x(k+1)z^{-k} = \boxed{zX(z) - zx(0)}$
- thus, if  $x(k+1) = Ax(k) + Bu(k)$  and  $x(0) = 0$ ,

$$zX(z) = AX(z) + BU(z) \Rightarrow X(z) = (zI - A)^{-1}BU(z)$$

provided that  $(zI - A)$  is invertible

# Solving difference equations

Solve the difference equation

$$y(k) + 3y(k-1) + 2y(k-2) = u(k-2)$$

where  $y(-2) = y(-1) = 0$  and  $u(k) = 1(k)$ .

- $\mathcal{Z}\{y(k-1)\} = z^{-1}\mathcal{Z}\{y(k)\} = z^{-1}Y(z)$

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- $\Rightarrow \boxed{Y(z) = \frac{1}{z^2 + 3z + 2}U(z)}$



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- inverse Z transform then gives  
 $y(k) = \frac{1}{6}1(k) + \frac{1}{3}(-2)^k - \frac{1}{2}(-1)^k, \quad k \geq 0$

# From difference equation to transfer functions

- general discrete-time OdE:

$$y(k) + a_{n-1}y(k-1) + \cdots + a_0y(k-n) = b_mu(k+m-n) + \cdots + b_0u(k-n)$$

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- applying Z transform to the OdE yields

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$$Y(z) = \underbrace{\frac{b_mz^m + b_{m-1}z^{m-1} \dots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \dots + a_1z + a_0}}_{G_{yu}(z): \text{ discrete-time transfer function}} U(z)$$

# DC gain of discrete-time transfer functions

- general discrete-time OdE and transfer function:

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- assuming constant input and convergent output, then at steady state,
  - ▶  $y(k) = y(k-1) = \cdots = y(k-n) \triangleq y_{ss}$  and  
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  - ▶  $y_{ss} + a_{n-1}y_{ss} + \cdots + a_0y_{ss} = b_mu_{ss} + \cdots + b_0u_{ss}$
- thus,

$$\underline{\text{DC gain of } G_{yu}(z)} = \frac{b_m + b_{m-1} + \cdots + b_0}{1 + a_{n-1} + \cdots + a_0} = \underline{G_{yu}(z)|_{z=1}}$$

# Transfer functions in two domains

$$y(k) + a_{n-1}y(k-1) + \cdots + a_0y(k-n) = b_mu(k+m-n) + \cdots + b_0u(k-n)$$
$$\iff G_{yu}(z) = \frac{B(z)}{A(z)} = \frac{b_mz^m + b_{m-1}z^{m-1} \cdots + b_1z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0}$$

v.s.

$$\frac{d^ny(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_0y(t) = b_m\frac{d^mu(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_0u(t)$$
$$\iff G_{yu}(s) = \frac{B(s)}{A(s)} = \frac{b_ms^m + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}$$

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Properties	$G_{yu}(s)$	$G_{yu}(z)$
poles and zeros	roots of $A(s)$ and $B(s)$	roots of $A(z)$ and $B(z)$
causality condition	$n \geq m$	$n \geq m$
DC gain / steady-state response to unit step	$G_{yu}(0)$	$G_{yu}(1)$

# Coding a discrete-time transfer function

```
num = [0.09952, -0.08144];
den = [1, -1.792, 0.8187];
Ts = 0.1;
sys_tf = tf(num,den,Ts)
poles = pole(sys_tf);
zeros = zero(sys_tf);
disp(['System Poles = ',num2str(poles)])
disp(['System Zeros = ',num2str(zeros)])

[yout, T] = step(sys_tf);
figure, stairs(T, yout)
figure, impulse(sys_tf)

u1 = 2*ones(length(T),1);
u2 = sin(T);
figure, lsim(sys_tf,u1,T)
figure, lsim(sys_tf,u2,T)
```

```

import control as co
import matplotlib.pyplot as plt
import numpy as np
Ts = 0.1 # sampling time
num = [0.09952, -0.08144] # Numerator co-efficients
den = [1, -1.792, 0.8187] # Denominator co-efficients
sys_tf = co.tf(num,den, Ts)
print(sys_tf)

poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout = co.step_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.step(T,np.append(0,yout[0:-1]))
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()

```



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print(sys_tf)

poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout_i = co.impulse_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.step(T,np.append(0,yout_i[0:-1]))
plt.grid(True)
plt.ylabel("y")
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plt.show()

```

## Additional useful properties of Z transform

- time shifting (assuming  $x(k) = 0$  if  $k < 0$ ):

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- differentiation:  $\mathcal{Z} \{kx(k)\} = -z \frac{dX(z)}{dz}$

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- $\Rightarrow b = \frac{a^N y(0)(a-1)}{a^N - 1} = \$477.42$