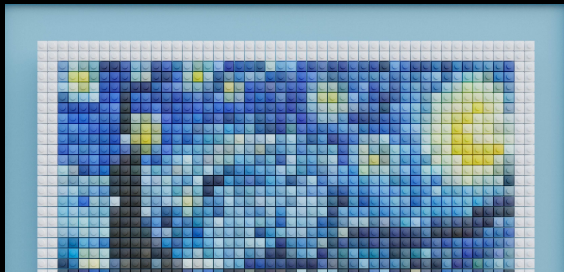


Introduction to Modern Controls

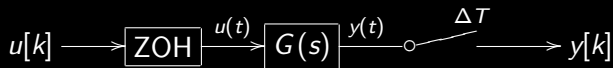
Discretization of Continuous-time Transfer-function Models



1 Discretization of transfer-function models

Overview

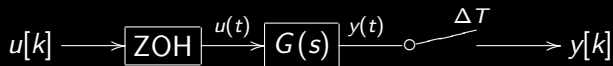
- consider the discrete-time controller implementation scheme



where $u[k]$ and $y[k]$ have the same sampling time

- for this note, we use $[k]$ to distinguish DT signals from their CT counter parts
- goal: to derive the transfer function from $u[k]$ to $y[k]$
- solution concept: let $u[k]$ be a discrete-time unit impulse (whose Z transform is 1) and obtain the Z transform of $y[k]$

Solution



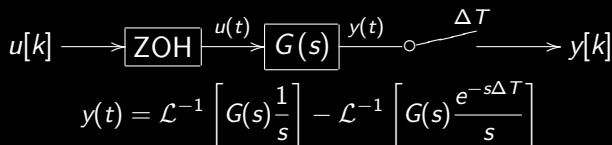
- $u[k]$ is a DT impulse \Rightarrow after ZOH

$$u(t) = \begin{cases} 1, & 0 \leq t < \Delta T \\ 0, & \text{otherwise} \end{cases} = 1(t) - 1(t - \Delta T) \Rightarrow U(s) = \frac{1 - e^{-s\Delta T}}{s}$$

- hence

$$y(t) = \mathcal{L}^{-1} \left[G(s) \frac{1 - e^{-s\Delta T}}{s} \right] = \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] - \mathcal{L}^{-1} \left[G(s) \frac{e^{-s\Delta T}}{s} \right]$$

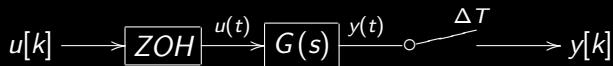
Solution



- sampling $y(t)$ at ΔT and performing Z transform give:

$$\begin{aligned} \underbrace{Y(z)}_{=G(z) \times 1} &= \mathcal{Z} \left\{ \underbrace{\underbrace{\left. \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] \right|_{t=k\Delta T}}_{\triangleq \tilde{y}[k]} - \underbrace{\left. \mathcal{L}^{-1} \left[G(s) \frac{e^{-s\Delta T}}{s} \right] \right|_{t=k\Delta T}}_{=\tilde{y}[k-1]!!!}}_{\tilde{y}(t) - \tilde{y}(t-\Delta T)} \right\} \\ &= \mathcal{Z} \left\{ \left. \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] \right|_{t=k\Delta T} \right\} - z^{-1} \mathcal{Z} \left\{ \left. \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] \right|_{t=k\Delta T} \right\} \end{aligned}$$

Solution



- $u[k] = \delta[k]$ ($U(z) = 1$)
- $Y(z) = \mathcal{Z} \left\{ \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] \Big|_{t=k\Delta T} \right\} - z^{-1} \mathcal{Z} \left\{ \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] \Big|_{t=k\Delta T} \right\}$
- \Rightarrow the zero order hold equivalent of $G(s)$ is

$$G(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \mathcal{L}^{-1} \left[G(s) \frac{1}{s} \right] \Big|_{t=k\Delta T} \right\}$$

where ΔT is the sampling time

Example: obtain the ZOH equivalent of

$$G(s) = \frac{a}{s + a}$$

- $\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$
- hence

$$\mathcal{L}^{-1} \left\{ \frac{G(s)}{s} \right\} = 1(t) - e^{-at}1(t)$$

- sampling at ΔT gives $1[k] - e^{-ak\Delta T}1[k]$, whose Z transform is

$$\frac{z}{z-1} - \frac{z}{z-e^{-a\Delta T}} = \frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})}$$

- hence the ZOH equivalent is

$$(1-z^{-1}) \frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})} = \frac{1-e^{-a\Delta T}}{z-e^{-a\Delta T}}$$

Matlab and Python commands

- $c2d(G, dt, 'zoh')$
- e.g., $G(s) = \frac{1}{s^2}$ and $\Delta T = 1$

```
% matlab
dt = 1;
num = 1;
den = [1,0,0];
G = tf(num,den);
Gd = c2d(G,dt,'zoh');
```

```
#Python
import control as ct
dt = 1
num = [1]
den = [1,0,0]
G = ct.tf(num,den)
Gd = ct.c2d(G,dt,'zoh')
print(Gd)
```



```

% MATLAB code to generate a single-stage HDD model
num_sector=420;           % Number of sector
num_rpm=7200;             % Number of RPM
Ts = 1/(num_rpm/60*num_sector); % Sampling time

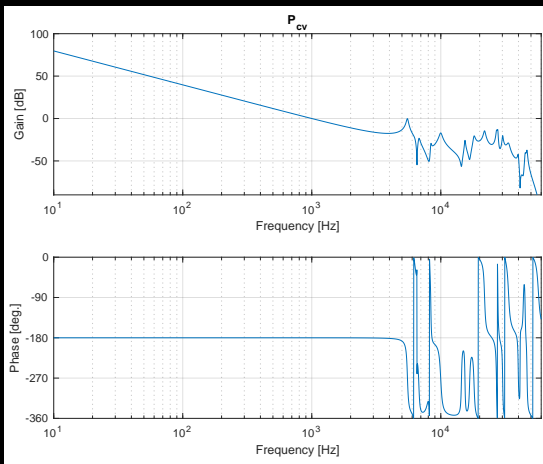
Kp_vcm=3.7976e+07;        % VCM gain
omega_vcm=[0, 5300 ,6100 ,6500 ,8050 ,9600 ,14800 ,17400 ,21000 ,26000
↪ ,26600 ,29000 ,32200 ,38300 ,43300 ,44800]*2*pi;
kappa_vcm=[1, -1.0 ,+0.1 , -0.1 ,0.04 , -0.7 , -0.2 , -1.0 ,+3.0 , -3.2
↪ ,2.1 , -1.5 ,+2.0 , -0.2 ,+0.3 , -0.5 ];
zeta_vcm =[0, 0.02 ,0.04 ,0.02 ,0.01 ,0.03 ,0.01 , 0.02 ,0.02 ,0.012
↪ ,0.007 ,0.01 ,0.03 ,0.01 ,0.01 ,0.01 ];

Sys_Pc_vcm_c1=0;
for i=1:length(omega_vcm)
    Sys_Pc_vcm_c1=Sys_Pc_vcm_c1+tf([0,0,kappa_vcm(i)]*Kp_vcm,[1,
↪ 2*zeta_vcm(i)*omega_vcm(i), (omega_vcm(i))^2]);
end

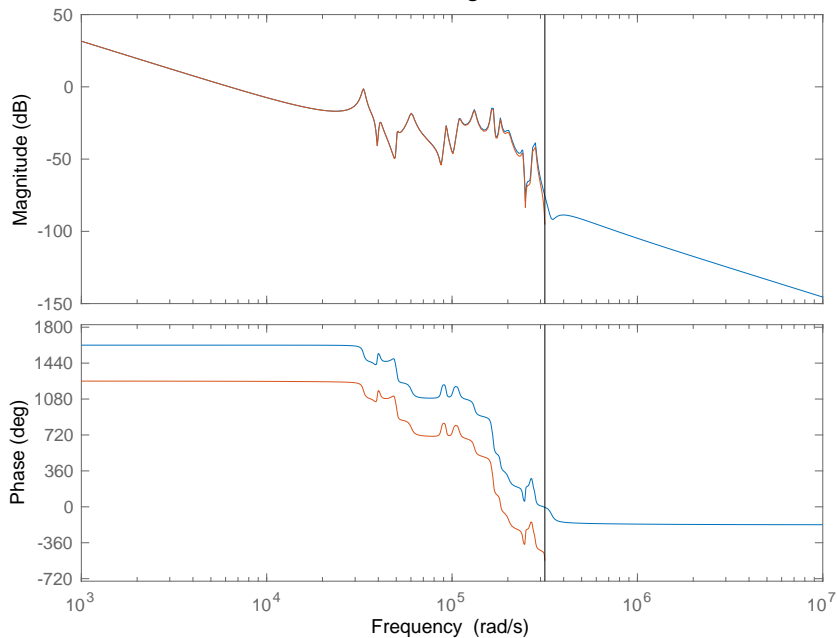
Sys_Pd_vcm_c1 = c2d(Sys_Pc_vcm_c1,Ts,'zoh')
Sys_Pd1_vcm_c1 = c2d(Sys_Pc_vcm_c1,Ts*2,'zoh')
Sys_Pd2_vcm_c1 = c2d(Sys_Pc_vcm_c1,Ts/2,'zoh')

```

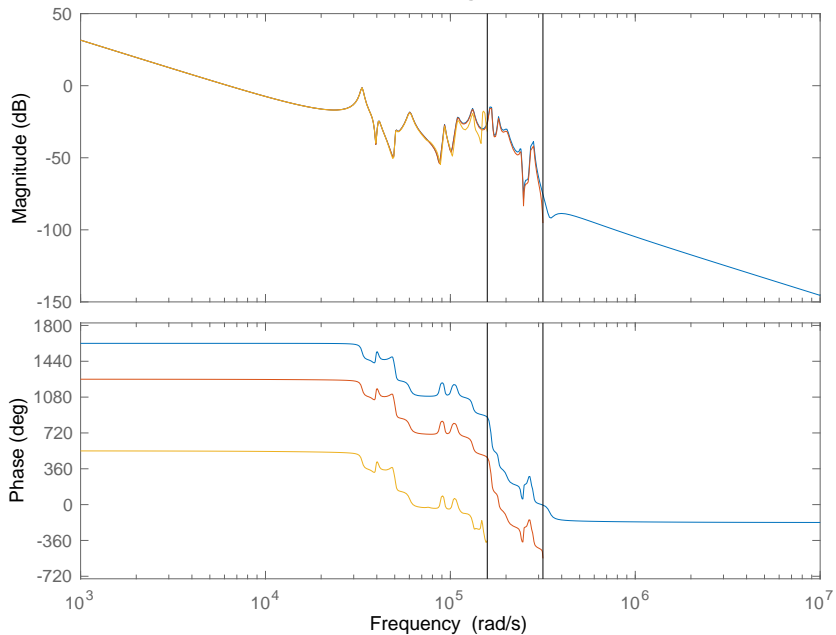
```
figure, bodeplot(Sys_Pc_vcm_c1)
figure, bodeplot(Sys_Pc_vcm_c1, Sys_Pd2_vcm_c1)
figure, bodeplot(Sys_Pc_vcm_c1, Sys_Pd2_vcm_c1, Sys_Pd_vcm_c1)
figure,
↳ bodeplot(Sys_Pc_vcm_c1, Sys_Pd2_vcm_c1, Sys_Pd_vcm_c1, Sys_Pd1_vcm_c1)
```



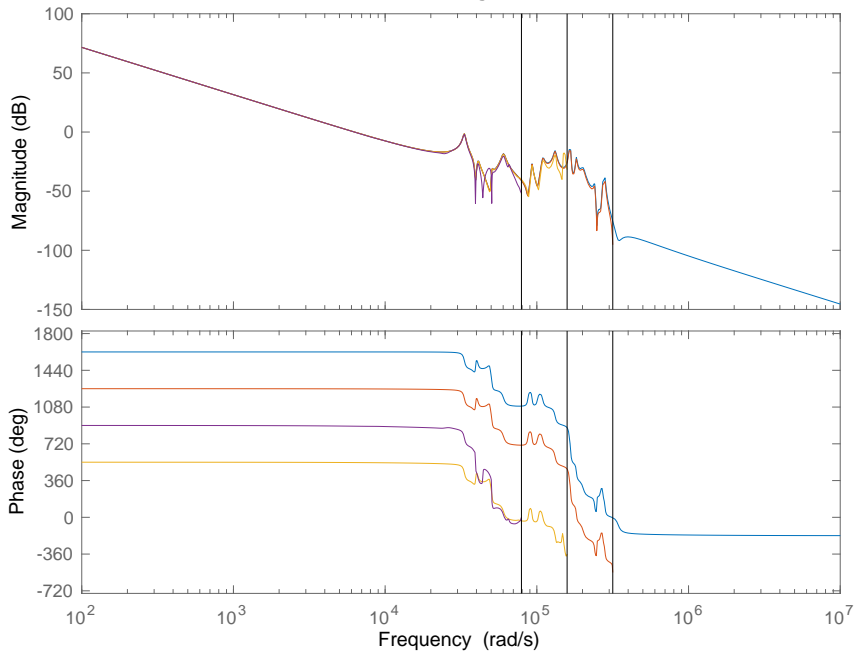
Bode Diagram



Bode Diagram



Bode Diagram



Exercise

Find the zero order hold equivalent of $G(s) = e^{-Ls}$, $2\Delta T < L < 3\Delta T$, where ΔT is the sampling time.