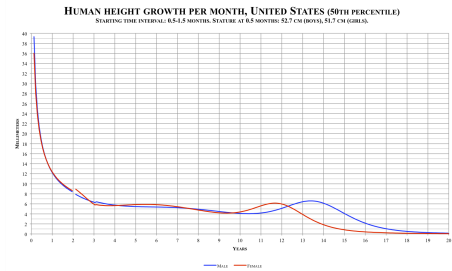


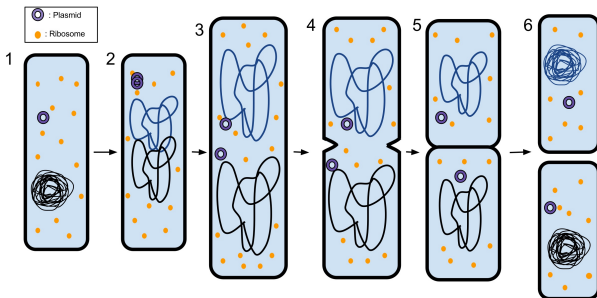
# Introduction to Modern Controls

## Solution of LTI State-Space Equations

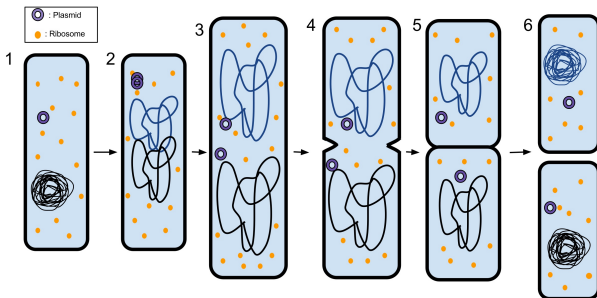


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# Population dynamics

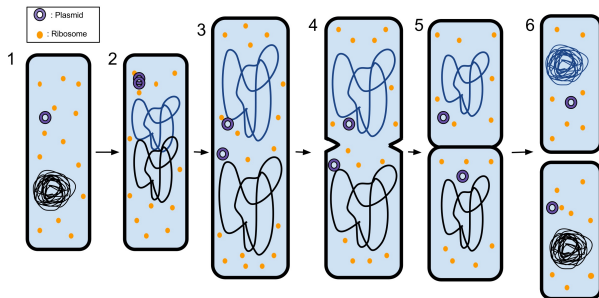


# Population dynamics



prokaryotic fission

# Population dynamics

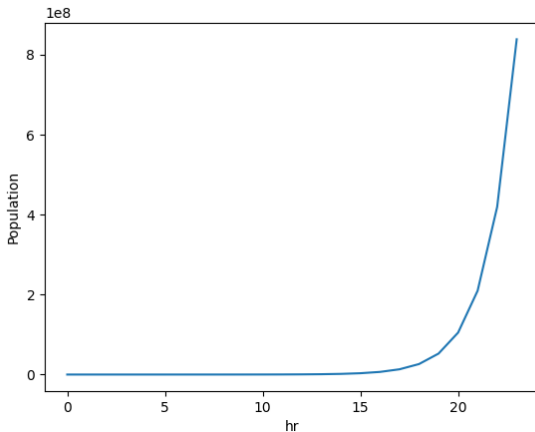
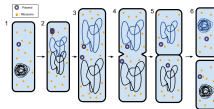


prokaryotic fission

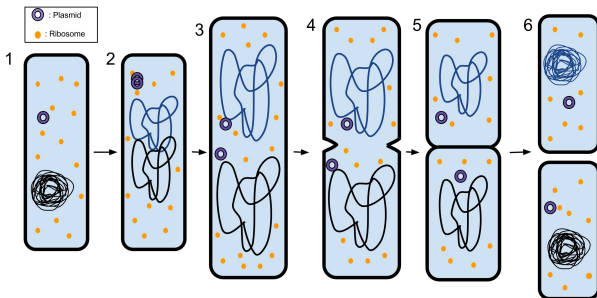
- ~1 hour / division with infinite resource

$$100 \xrightarrow{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} 800 \xrightarrow{1\text{hr}} \dots$$

# Population dynamics



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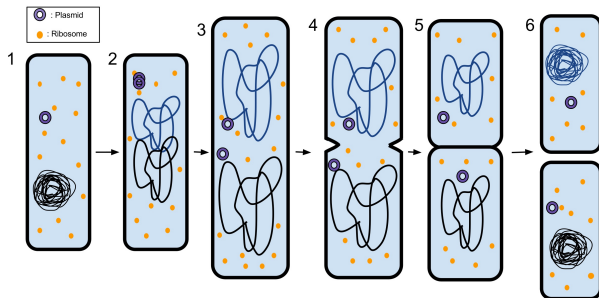


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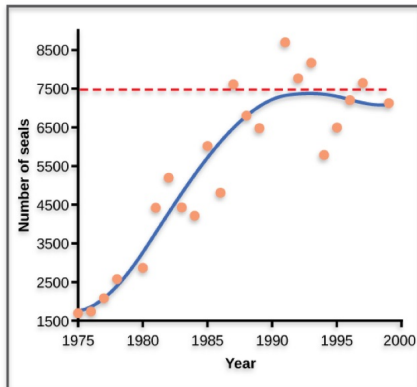
$$100 \xrightarrow{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} 800 \xrightarrow{1\text{hr}} \dots$$

- after 1 day:

$$100 \xrightarrow[\frac{\Delta N}{N}=1]{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} \dots \longrightarrow 100 \times 2^{24} = 1.7\text{B!}$$

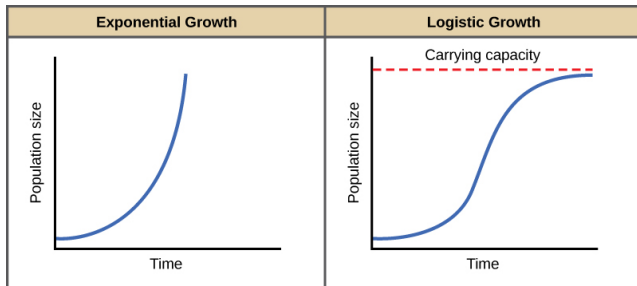


# Population dynamics



Environmental limits to population growth: Figure 1, by OpenStax College, Biology, CC BY 4.0.

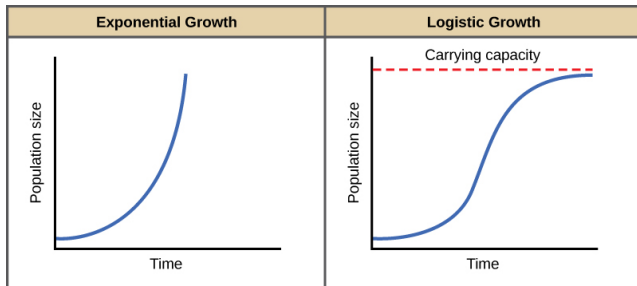
# The exponential function and population dynamics



- more general population dynamics (w/ infinite resources)

$$\frac{dN}{dt} = \overbrace{(\text{birth rate} - \text{death rate})}^r N \Rightarrow N(t) = e^{rt} N(0)$$

# The exponential function and population dynamics



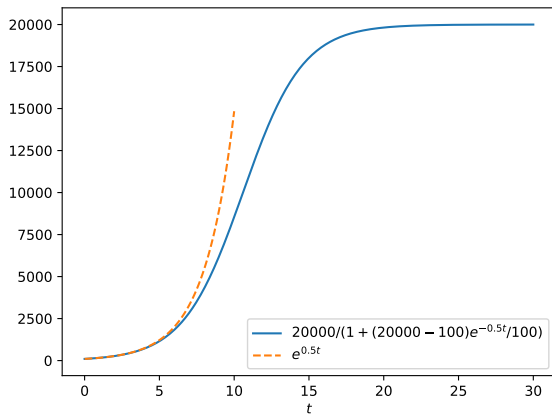
- more general population dynamics (w/ infinite resources)

$$\frac{dN}{dt} = \overbrace{(\text{birth rate} - \text{death rate})}^r N \Rightarrow N(t) = e^{rt} N(0)$$

- logistic growth (w/ limited resources in reality)

$$\frac{dN}{dt} = r \frac{K - N}{K} N \Rightarrow N(t) = \frac{KN_0 e^{rt}}{(K - N_0) + N_0 e^{rt}} = \frac{K}{1 + \frac{K - N_0}{N_0} e^{-rt}}$$

# The exponential function and the logistic S curve: example



# The logistic S curve

$$\frac{K}{1 + \frac{K - N_0}{N_0} e^{-rt}}$$

can also be written as

$$\frac{K}{1 + e^{-r(t-t_o)}}$$

- $K$ : final value
- $r$ : logistic growth rate
- $t_o$ : midpoint

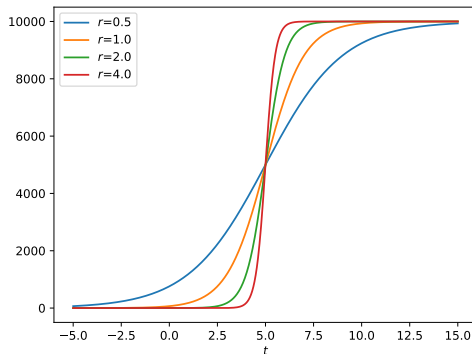
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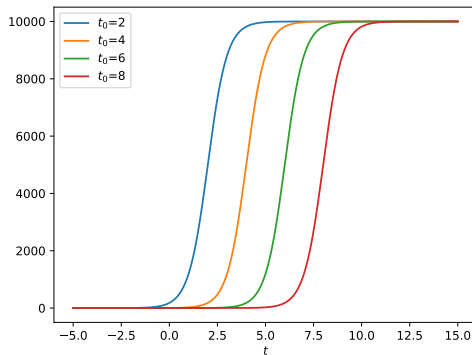
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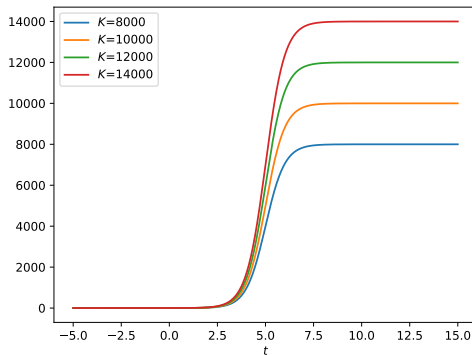
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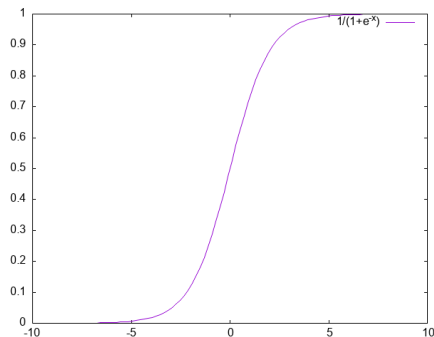
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# The logistic function in deep learning

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



- transforms the input variables into a probability value between 0 and 1
- represents the likelihood of the dependent variable being 1 or 0

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# General LTI continuous-time state equation

$$\frac{dx}{dt} = Ax + Bu$$
$$\Sigma = \left[ \begin{array}{c|c} A_{n \times n} & B_{n \times m} \\ \hline C_{n_y \times n} & D_{n_y \times m} \end{array} \right]$$

- to solve the vector equation  $\dot{x} = Ax + Bu$ , we start with the scalar case when  $x, a, b, u \in \mathbb{R}$ .

# The solution to $\dot{x} = ax + bu$

- fundamental property of exponential functions

$$\frac{d}{dt}e^{at} = ae^{at}, \quad \frac{d}{dt}e^{-at} = -ae^{-at}$$

- $\dot{x}(t) = ax(t) + bu(t)$ ,  $a \neq 0 \stackrel{\because e^{-at} \neq 0}{\implies} e^{-at}\dot{x}(t) - e^{-at}ax(t) = e^{-at}bu(t)$
- namely,

$$\frac{d}{dt} \{e^{-at}x(t)\} = e^{-at}bu(t) \Leftrightarrow d \{e^{-at}x(t)\} = e^{-at}bu(t) dt$$

$$\implies \boxed{e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^t e^{-a\tau}bu(\tau) d\tau}$$

The solution to  $\dot{x} = ax + bu$

$$e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^t e^{-a\tau}bu(\tau)d\tau$$

when  $t_0 = 0$ , we have

$$x(t) = \underbrace{e^{at}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}_{\text{forced response}}$$

# About $e$

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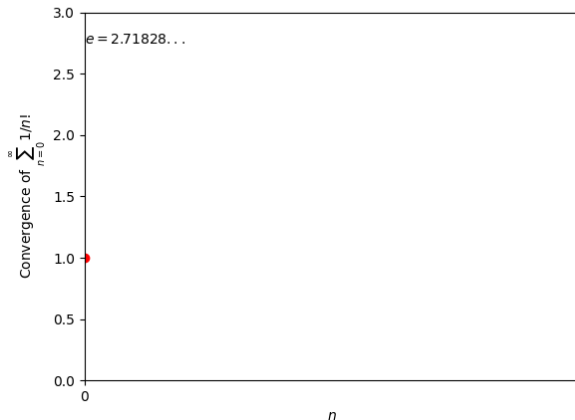
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- ▶ letting  $x = 1$  gives  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$
- Python demonstration:

```
import math
math.e
for ii in range(10):
    print(sum(1/math.factorial(k) for k in range(ii)))
```

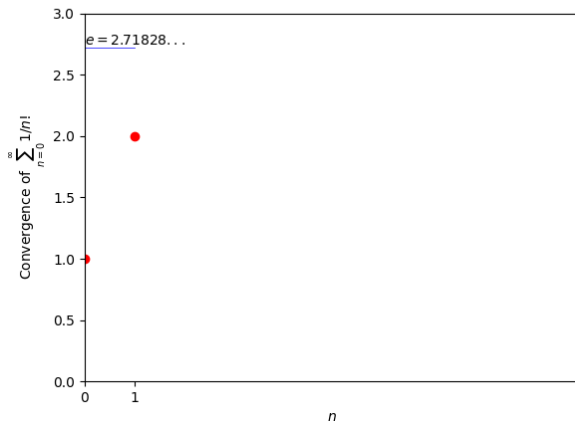
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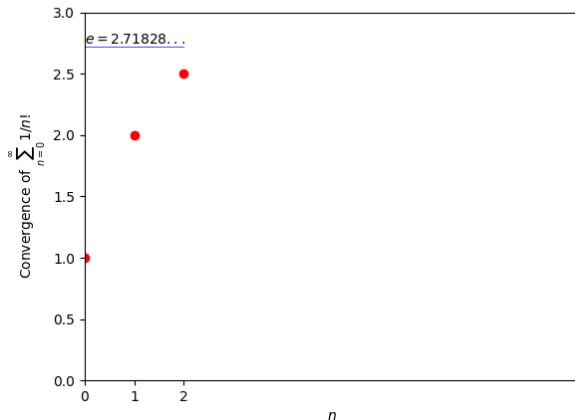
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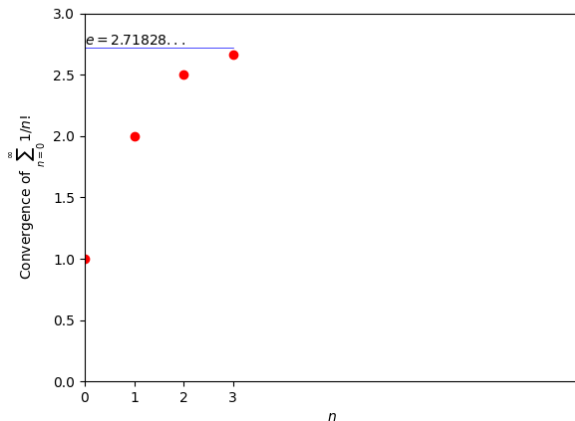
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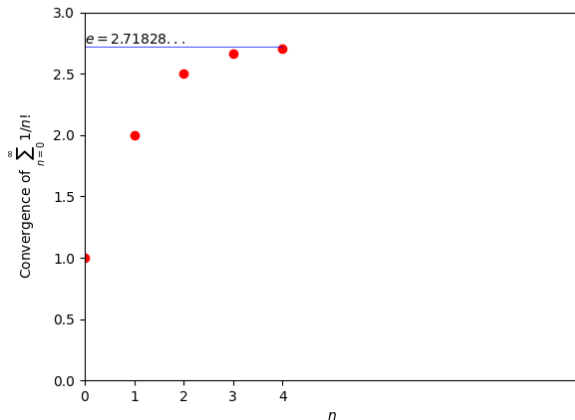
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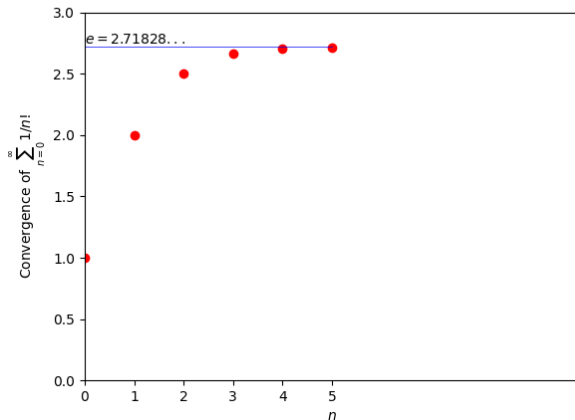
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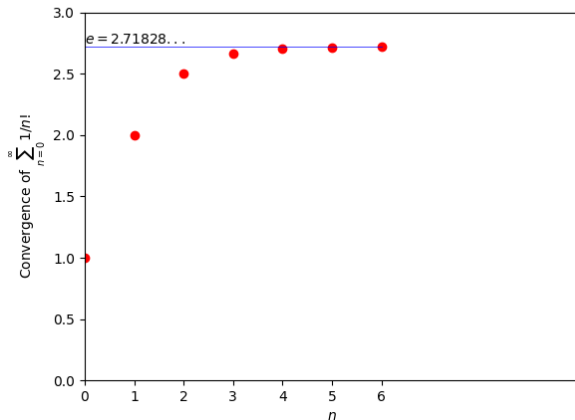
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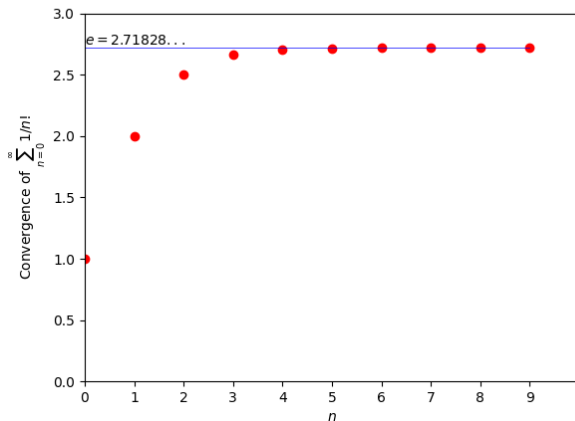
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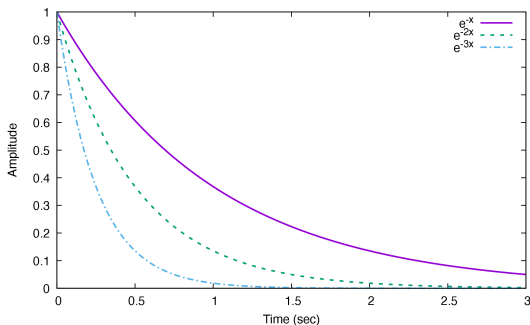
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# The solution to $\dot{x} = ax + bu$

Solution concepts of  $e^{at}x(0)$



$e = 2.71828 \dots$

$e^{-1} \approx 37\%$ ,

$e^{-2} \approx 14\%$ ,

$e^{-3} \approx 5\%$ ,

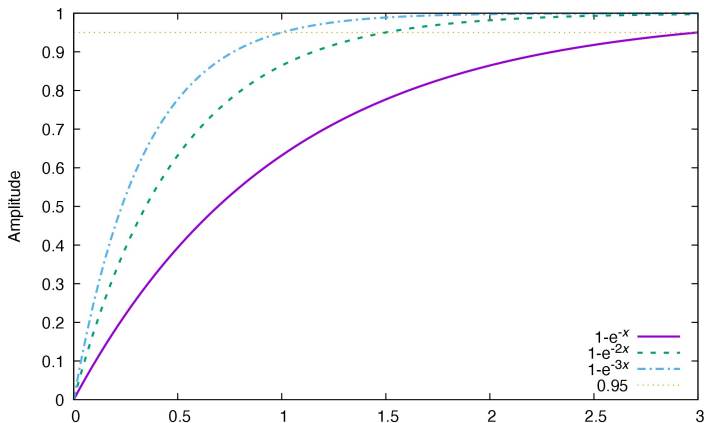
$e^{-4} \approx 2\%$

time constant  $\tau \triangleq \frac{1}{|a|}$  when  
 $a < 0$ : after  $3\tau$ ,  $e^{at}x(0)$ , the  
transient has approximately  
converged

# The solution to $\dot{x} = ax + bu$

## Unit step response

when  $a < 0$  and  $u(t) = 1(t)$  (the step function), the solution is  
$$x(t) = \frac{b}{|a|}(1 - e^{at})$$



# The solution to $n^{\text{th}}$ -order LTI systems

- general state-space equation

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

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- solution

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{free response}} + \underbrace{\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$$

$$y(t) = Ce^{A(t-t_0)}x_0 + C \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

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- in both the free and the forced responses, computing  $e^{At}$  is key
- $e^{A(t-t_0)}$ : called the transition matrix

# The state transition matrix $e^{At}$

scalar case with  $a \in \mathbb{R}$ : Taylor expansion gives

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \cdots$$

the transition scalar  $\Phi(t, t_0) = e^{a(t-t_0)}$  satisfies

$$\Phi(t, t) = 1 \quad (\text{transition to itself})$$

$$\Phi(t_3, t_2)\Phi(t_2, t_1) = \Phi(t_3, t_1) \quad (\text{consecutive transition})$$

$$\Phi(t_2, t_1) = \Phi^{-1}(t_1, t_2) \quad (\text{reverse transition})$$



# The state transition matrix $e^{At}$

matrix case with  $A \in \mathbb{R}^{n \times n}$ :

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots$$

- as  $I_n$  and  $A^i$  are matrices of dimension  $n \times n$ ,  $e^{At}$  must  $\in \mathbb{R}^{n \times n}$

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$$\begin{aligned} e^{A0} &= I_n & \Phi(t, t) &= I_n \\ e^{At_1} e^{At_2} &= e^{A(t_1+t_2)} & \Phi(t_3, t_2)\Phi(t_2, t_1) &= \Phi(t_3, t_1) \\ e^{-At} &= \left[ e^{At} \right]^{-1} & \Phi(t_2, t_1) &= \Phi^{-1}(t_1, t_2) \end{aligned}$$

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- note, however, that  $e^{At}e^{Bt} = e^{(A+B)t}$  if and only if  $AB = BA$  (check by using Taylor expansion)

# Computing $e^{At}$ when $A$ is diagonal or in Jordan form

convenient when  $A$  is a diagonal or Jordan matrix

the case with a diagonal matrix  $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ :

$$\bullet A^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \dots, A^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

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- all matrices on the right side of

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots$$

are easy to compute

# Computing a structured $e^{At}$ via Taylor expansion

the case with a diagonal matrix  $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ :

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1^2 t^2 & 0 & 0 \\ 0 & \frac{1}{2}\lambda_2^2 t^2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda_3^2 t^2 \end{bmatrix} + \dots \\ &= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2}\lambda_1^2 t^2 + \dots & 0 & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2}\lambda_2^2 t^2 + \dots & 0 \\ 0 & 0 & 1 + \lambda_3 t + \frac{1}{2}\lambda_3^2 t^2 + \dots \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} \end{aligned}$$

# Computing a structured $e^{At}$ via Taylor expansion

the case with a Jordan matrix  $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$ :

- decompose  $A = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_N \Rightarrow e^{At} = e^{(\lambda I_3 t + Nt)}$

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- thus

$$\underline{e^{At} = e^{(\lambda I_3 t + Nt)} = e^{\lambda I_3 t} e^{Nt} \because e^{\lambda I_3 t} = e^{\lambda t I} e^{\lambda t} e^{Nt}}$$

# Computing a structured $e^{At}$ via Taylor expansion

$$\underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_N, \quad e^{At} = e^{\lambda t} e^{Nt}$$

- $N$  is *nilpotent*<sup>1</sup>:  $N^3 = N^4 = \dots = 0_{I_3}$ , yielding

$$e^{Nt} = I_3 + Nt + \frac{1}{2}N^2t^2 + \frac{1}{3!}N^3t^3 + \dots = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

*(Note: In the original image, arrows point from the terms  $\frac{1}{3!}N^3t^3$  and the ellipsis to 0, indicating they are zero.)*

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<sup>1</sup>“nil”  $\sim$  zero; “potent”  $\sim$  taking powers.

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- thus

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

---

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# Computing a structured $e^{At}$ via Taylor expansion

Mass moving on a straight line with zero friction and no external force

$x(t) = e^{At}x(0)$  where

$$e^{At} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \underbrace{\frac{1}{2!} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} t^2 + \dots = \underline{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}.$$

# Computing low-order $e^{At}$ via column solutions

an intuition of the matrix entries in  $e^{At}$ : consider:

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0$$

$$\begin{aligned} x(t) = e^{At}x(0) &= \left[ \underbrace{\text{1st column}}_{a_1(t)} \mid \underbrace{\text{2nd column}}_{a_2(t)} \right] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= a_1(t)x_1(0) + a_2(t)x_2(0) \end{aligned} \quad (1)$$

observation

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x(t) = a_1(t)$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x(t) = a_2(t)$$

# Computing low-order $e^{At}$ via column solutions

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0$$

hence, we can obtain  $e^{At}$  from:

• write out  $\begin{matrix} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_2(t) \end{matrix} \Rightarrow \begin{matrix} x_1(t) = e^{0t}x_1(0) + \int_0^t e^{0(t-\tau)}x_2(\tau)d\tau \\ x_2(t) = e^{-t}x_2(0) \end{matrix}$

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- let  $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ , then  $\begin{matrix} x_1(t) \equiv 1 \\ x_2(t) \equiv 0 \end{matrix}$ , namely  $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

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- let  $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , then  $x_2(t) = e^{-t}$  and  $x_1(t) = 1 - e^{-t}$ , or more compactly,  $x(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$



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- using (1), write out directly  $e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$

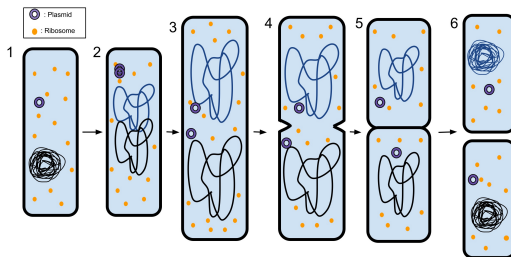
# Computing low-order $e^{At}$ via column solutions

Compute  $e^{At}$  where

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

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- 2 Continuous-time state-space solution
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- 5 Explicit Computation of the State Transition Matrix  $A^k$
- 6 Transition Matrix via Inverse Transformation

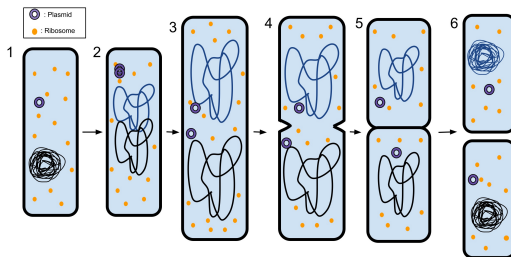
# Recall: population dynamics



prokaryotic fission

- ~1 hour / division with infinite resource

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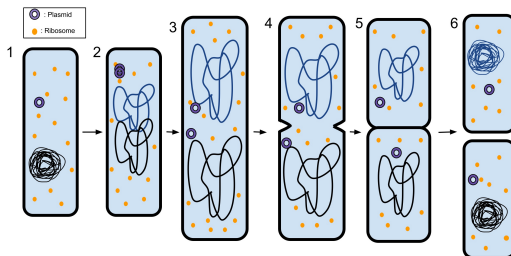


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- ~1 hour / division with infinite resource
- after 1 day:

$$100 \xrightarrow[\frac{\Delta N}{N}=1]{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} \dots \longrightarrow 100 \times 2^{24} = 1.7\text{B!}$$

# Recall: population dynamics



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- or:  $N(k+1) = 2N(k) \Rightarrow N(k) = 2^k N(0)$

# Solution to discrete-time state equation

discrete-time system:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0,$$

iteration of the state-space equation gives:

$$x(k) = A^{k-k_0}x(k_0) + \begin{bmatrix} A^{k-k_0-1}B, A^{k-k_0-2}B, \dots, B \end{bmatrix} \begin{bmatrix} u(k_0) \\ u(k_0+1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$\Leftrightarrow x(k) = \underbrace{A^{k-k_0}x(k_0)}_{\text{free response}} + \underbrace{\sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j)}_{\text{forced response}}$$

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$\Phi(k, j) = A^{k-j}$ : the transition matrix:

$$\Phi(k, k) = 1$$

$$\Phi(k_3, k_2) \Phi(k_2, k_1) = \Phi(k_3, k_1) \quad k_3 \geq k_2 \geq k_1$$

$$\Phi(k_2, k_1) = \Phi^{-1}(k_1, k_2) \quad \text{if and only if } A \text{ is nonsingular}$$



# The state transition matrix $A^k$

similar to the continuous-time case, when  $A$  is a diagonal or Jordan matrix,  $A^k$  is easy

- diagonal matrix  $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} : A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$

# Computing a structured $A^k$ via Taylor expansion

- Jordan canonical form

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_N :$$

$$\begin{aligned} A^k &= (\lambda I_3 + N)^k \\ &= (\lambda I_3)^k + k (\lambda I_3)^{k-1} N + \underbrace{\binom{k}{2} (\lambda I_3)^{k-2} N^2}_{\text{2 combination}} + \underbrace{\binom{k}{3} (\lambda I_3)^{k-3} N^3 + \dots}_{N^3=N^4=\dots=0I_3} \end{aligned}$$

$$\begin{aligned} &= \begin{bmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^k \end{bmatrix} + k\lambda^{k-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{k(k-1)}{2} \lambda^{k-2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \underline{\begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!} k(k-1) \lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}} \end{aligned}$$

# Computing a structured $A^k$ via Taylor expansion

Recall that  $\binom{k}{3} = \frac{1}{3!}k(k-1)(k-2)$ . Show

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$
$$\Rightarrow A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} & \frac{1}{3!}k(k-1)(k-2)\lambda^{k-3} \\ 0 & \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} \\ 0 & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & 0 & \lambda^k \end{bmatrix}$$

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# Explicit computation of a general $e^{At}$

- why another method: general matrices may not be diagonal or Jordan

# Explicit computation of a general $e^{At}$

- why another method: general matrices may not be diagonal or Jordan
- approach: transform a general matrix to a diagonal or Jordan form, via similarity transformation

# Computing $e^{At}$ via similarity transformation

principle concept:

- given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

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$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

- find a nonsingular  $T \in \mathbb{R}^{n \times n}$  such that a coordinate transformation defined by  $x(t) = Tx^*(t)$  yields

$$\frac{d}{dt} (Tx^*(t)) = ATx^*(t) + Bu(t)$$



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$$\frac{d}{dt} x^*(t) = \underbrace{T^{-1}AT}_{\triangleq \Lambda: \text{diagonal or Jordan}} x^*(t) + \underbrace{T^{-1}B}_{B^*} u(t)$$

$$x^*(0) = T^{-1}x_0$$

# Computing $e^{At}$ via similarity transformation

- when  $u(t) = 0$

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- now  $x^*(t)$  can be solved easily: e.g., if  $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ , then

$$x^*(t) = e^{\Lambda t} x^*(0) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} x_1^*(0) \\ e^{\lambda_2 t} x_2^*(0) \end{bmatrix}$$

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- on the other hand,  $x(t) = e^{At} x_0 \Rightarrow$

$$\boxed{e^{At} = Te^{\Lambda t} T^{-1}}$$

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- ▶ their exponential matrices are also similar

$$e^{At} = Te^{Bt}T^{-1}$$

as

$$\begin{aligned} Te^{Bt}T^{-1} &= T(I_n + Bt + \frac{1}{2}B^2t^2 + \dots)T^{-1} \\ &= TI_nT^{-1} + TBtT^{-1} + \frac{1}{2}TB^2t^2T^{-1} + \dots \\ &= I + At + \frac{1}{2}A^2t^2 + \dots = e^{At} \end{aligned}$$

# Similarity transformation

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- the corresponding eigenvectors are the nonzero solutions to

$$At = \lambda t \Leftrightarrow (A - \lambda I)t = 0 \quad (3)$$

# Similarity transformation

The case with distinct eigenvalues (diagonalization)

recall: when  $A \in \mathbb{R}^{n \times n}$  has  $n$  distinct eigenvalues such that

$$Ax_1 = \lambda_1 x_1$$

$$\vdots$$

$$Ax_n = \lambda_n x_n$$

or equivalently

$$A \underbrace{[x_1, x_2, \dots, x_n]}_{\triangleq T} = [x_1, x_2, \dots, x_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$$

$[x_1, x_2, \dots, x_n]$  is square and invertible. Hence

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# Similarity transform: diagonalization

## Physical interpretations

- diagonalized system:

$$x^*(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} x_1^*(0) \\ e^{\lambda_2 t} x_2^*(0) \end{bmatrix}$$

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- $x(t) = Tx^*(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$  then decomposes the state trajectory into two modes parallel to the two eigenvectors.

# Similarity transform: diagonalization

## Physical interpretations

- if  $x(0)$  is aligned with one eigenvector, say,  $t_1$ , then  $x_2^*(0) = 0$  and  $x(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$  dictates that  $x(t)$  will stay in the direction of  $t_1$

# Similarity transform: diagonalization

## Physical interpretations

- if  $x(0)$  is aligned with one eigenvector, say,  $t_1$ , then  $x_2^*(0) = 0$  and  $x(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$  dictates that  $x(t)$  will stay in the direction of  $t_1$
- i.e., if the state initiates along the direction of one eigenvector, then the free response will stay in that direction without “making turns”

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- if  $\lambda_1 < 0$ , then  $x(t)$  will move towards the origin of the state space; if  $\lambda_1 = 0$ ,  $x(t)$  will stay at the initial point; and if positive,  $x(t)$  will move away from the origin along  $t_1$

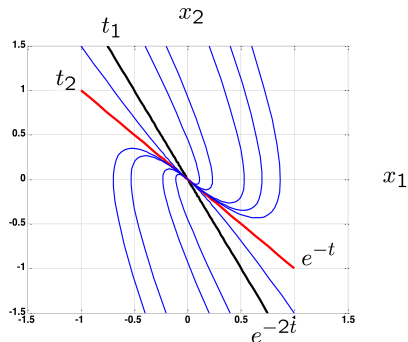
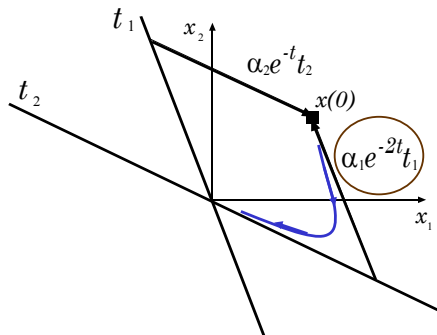
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- furthermore, the magnitude of  $\lambda_1$  determines the speed of response

# Similarity transform: diagonalization

## Physical interpretations



# Similarity transformation

## The case with complex eigenvalues

consider the undamped spring-mass system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \det(A - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm j.$$

the eigenvectors are

$$\lambda_1 = j: (A - jI)t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$\lambda_2 = -j: (A + jI)t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \quad (\text{complex conjugate of } t_1)$$

hence

$$T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$



# Similarity transformation

The case with complex eigenvalues

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- $\lambda_{1,2} = \pm j$
- $T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$
- we have

$$e^{At} = T e^{\Lambda t} T^{-1} = T \begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix} T^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

# Similarity transformation

## The case with complex eigenvalues

for a general  $A \in \mathbb{R}^{2 \times 2}$  with complex eigenvalues  $\sigma \pm j\omega$ , by using  $T = [t_R, t_I]$ , where  $t_R$  and  $t_I$  are the real and the imaginary parts of  $t_1$ , an eigenvector associated with  $\lambda_1 = \sigma + j\omega$ ,  $x = Tx^*$  transforms  $\dot{x} = Ax$  to

$$\dot{x}^*(t) = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} x^*(t)$$

and

$$e^{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t} = \begin{bmatrix} e^{\sigma t} \cos \omega t & e^{\sigma t} \sin \omega t \\ -e^{\sigma t} \sin \omega t & e^{\sigma t} \cos \omega t \end{bmatrix}$$

# Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

consider  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ : two repeated eigenvalues  $\lambda(A) = 1$ , and

$$(A - \lambda I) t_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- No other linearly independent eigenvectors exist. What next?
- $A$  is already very similar to the Jordan form. Try instead

$$A \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

which requires  $At_2 = t_1 + \lambda t_2$ , i.e.,

$$(A - \lambda I) t_2 = t_1 \Leftrightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow t_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

$t_2$  is linearly independent from  $t_1 \Rightarrow t_1$  and  $t_2$  span  $\mathbb{R}^2$ . ( $t_2$  is called a generalized eigenvector.)

# Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

for general  $3 \times 3$  matrices with  $\det(\lambda I - A) = (\lambda - \lambda_m)^3$ , i.e.,  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_m$ , we look for  $T$  such that

$$A = TJT^{-1}$$

where  $J$  has three canonical forms:

$$\begin{aligned} i), & \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}, \quad iii), \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix} \\ ii), & \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix} \end{aligned}$$

# Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$i), A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

# Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$i), A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

this happens



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$$i), A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

this happens

- when  $A$  has three linearly independent eigenvectors, i.e.,  $(A - \lambda_m I)t = 0$  yields  $t_1$ ,  $t_2$ , and  $t_3$  that span  $\mathbb{R}^3$

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this happens

- when  $A$  has three linearly independent eigenvectors, i.e.,  $(A - \lambda_m I)t = 0$  yields  $t_1, t_2$ , and  $t_3$  that span  $\mathbb{R}^3$
- mathematically: when nullity  $(A - \lambda_m I) = 3$ , namely,  $\text{rank}(A - \lambda_m I) = 3 - \text{nullity}(A - \lambda_m I) = 0$

# Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$ii), A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

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- this happens when  $(A - \lambda_m I)t = 0$  yields two linearly independent solutions, i.e., when  $\text{nullity}(A - \lambda_m I) = 2$

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- this happens when  $(A - \lambda_m I)t = 0$  yields two linearly independent solutions, i.e., when  $\text{nullity}(A - \lambda_m I) = 2$
- we then have, e.g.,

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$
$$\Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, \lambda_m t_3] = [At_1, At_2, At_3]$$

# Similarity transformation

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- $t_1$  and  $t_3$  are the directly computed eigenvectors

# Similarity transformation

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- $t_1$  and  $t_3$  are the directly computed eigenvectors
- for  $t_2$ , the second column of the above gives  $(A - \lambda_m I)t_2 = t_1$

# Similarity transformation

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$$\text{iii), } A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

- this is for the case when  $(A - \lambda_m I)t = 0$  yields only one linearly independent solution, i.e., when  $\text{nullity}(A - \lambda_m I) = 1$

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- We then have

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$\Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, t_2 + \lambda_m t_3] = [At_1, At_2, At_3]$$

yielding

$$(A - \lambda_m I) t_1 = 0$$

$$(A - \lambda_m I) t_2 = t_1, (t_2 : \text{generalized eigenvector})$$

# Example

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \det(A - \lambda I) = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0, J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- two repeated eigenvalues with  $\text{rank}(A - 0I) = 1 \Rightarrow$  only one linearly independent eigenvector:  $(A - 0I) t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

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- two repeated eigenvalues with  $\text{rank}(A - 0I) = 1 \Rightarrow$  only one linearly independent eigenvector:  $(A - 0I) t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- generalized eigenvector:  $(A - 0I) t_2 = t_1 \Rightarrow t_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- coordinate transform matrix:

$$T = [t_1, t_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = T e^{Jt} T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & te^{0t} \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$$

# Example

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \det(A - \lambda I) = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

observation:

- $\lambda_1 = 0, t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  implies that if  $x_1(0) = x_2(0)$  then the response is characterized by  $e^{0t} = 1$

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- i.e.,  $x_1(t) = x_1(0) = x_2(0) = x_2(t)$ . This makes sense because  $\dot{x}_1 = -x_1 + x_2$  from the state equation

# Exercise

Obtain the eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \quad (\lambda_1 = 5, \lambda_2 = \lambda_3 = -3).$$



# Generalized eigenvectors

## Physical interpretation

when  $\dot{x} = Ax$ ,  $A = TJT^{-1}$  with  $J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$ , we have

$$\begin{aligned} x(t) &= e^{At}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} T^{-1}x(0) \\ &= T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} \cancel{T^{-1}} \overset{!}{T} x^*(0) \end{aligned}$$

- if the initial condition is in the direction of  $t_1$ , i.e.,  $x^*(0) = [x_1^*(0), 0, 0]^T$  and  $x_1^*(0) \neq 0$ , the above equation yields  $x(t) = x_1^*(0)t_1 e^{\lambda_m t}$

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- if  $x(0)$  starts in the direction of  $t_2$ , i.e.,  $x^*(0) = [0, x_2^*(0), 0]^T$ , then  $x(t) = x_2^*(0)(t_1 te^{\lambda_m t} + t_2 e^{\lambda_m t})$ . In this case, the response does not remain in the direction of  $t_2$  but is confined in the subspace spanned by  $t_1$  and  $t_2$

## Example

Obtain eigenvalues of  $J$  and  $e^{Jt}$  by inspection:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

- 1 Introduction
- 2 Continuous-time state-space solution
- 3 Discrete-time state-space solution
- 4 Explicit computation of the state transition matrix  $e^{At}$
- 5 Explicit Computation of the State Transition Matrix  $A^k$**
- 6 Transition Matrix via Inverse Transformation

# Explicit computation of $A^k$

everything in getting the similarity transform applies to the DT case:

$$A^k = T \Lambda^k T^{-1} \text{ or } A^k = T J^k T^{-1}$$

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$J$	$J^k$
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$
$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$	$r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$ $r = \sqrt{\sigma^2 + \omega^2}$ $\theta = \tan^{-1} \frac{\omega}{\sigma}$

---

# Example

Write down  $J^k$  for  $J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$  and

$$J = \begin{bmatrix} -10 & 1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -100 & 1 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix}.$$

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# Transition matrix via inverse transformation

---

	Continuous-time system
state eq.	$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$
solution	$x(t) = \underbrace{e^{At}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$
transition matrix	$e^{At}$

---

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On the other hand, from Laplace transform:

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow X(s) = \underbrace{(sI - A)^{-1}x(0)}_{\text{free response}} + \underbrace{(sI - A)^{-1}BU(s)}_{\text{forced response}}$$

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Comparing  $x(t)$  and  $X(s)$  gives

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

## Example

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1} \left[ \begin{bmatrix} s - \sigma & -\omega \\ \omega & s - \sigma \end{bmatrix}^{-1} \right] \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{(s - \sigma)^2 + \omega^2} \begin{bmatrix} s - \sigma & \omega \\ -\omega & s - \sigma \end{bmatrix} \right\} \\ &= e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \end{aligned}$$

# Transition matrix via inverse transformation (DT case)

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Discrete-time system

state eq.  $x(k+1) = Ax(k) + Bu(k), x(0) = x_0$

solution 
$$x(k) = \underbrace{A^k x(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)} Bu(j)}_{\text{forced response}}$$

transition matrix    transition matrix  $A^k$

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# Transition matrix via inverse transformation (DT case)

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	Discrete-time system
state eq.	$x(k+1) = Ax(k) + Bu(k), x(0) = x_0$
solution	$x(k) = \underbrace{A^k x(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)} Bu(j)}_{\text{forced response}}$
transition matrix	transition matrix $A^k$

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On the other hand, from Z transform:

$$X(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} BU(s)$$

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On the other hand, from Z transform:

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Hence

$$A^k = \mathcal{Z}^{-1} \{ (zI - A)^{-1} z \}$$

## Example

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\begin{aligned} A^k &= \mathcal{Z}^{-1} \left\{ z \begin{bmatrix} z - \sigma & -\omega \\ \omega & z - \sigma \end{bmatrix}^{-1} \right\} \\ &= \mathcal{Z}^{-1} \left\{ \frac{z}{(z - \sigma)^2 + \omega^2} \begin{bmatrix} z - \sigma & \omega \\ -\omega & z - \sigma \end{bmatrix} \right\} \\ &= \mathcal{Z}^{-1} \left\{ \frac{z}{z^2 - 2r \cos \theta z + r^2} \begin{bmatrix} z - r \cos \theta & r \sin \theta \\ -r \sin \theta & z - r \cos \theta \end{bmatrix} \right\} \\ &\quad , \quad r = \sqrt{\sigma^2 + \omega^2}, \quad \theta = \tan^{-1} \frac{\omega}{\sigma} \\ &= r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \end{aligned}$$



## Example

Consider  $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.5 \end{bmatrix}$ . We have

$$\begin{aligned} & (zI - A)^{-1} z \\ &= \begin{bmatrix} \frac{z(z-0.5)}{(z-0.8)(z-0.4)} & \frac{0.3z}{(z-0.8)(z-0.4)} \\ \frac{0.1z}{(z-0.8)(z-0.4)} & \frac{z(z-0.7)}{(z-0.8)(z-0.4)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{0.75z}{z-0.8} + \frac{0.25z}{z-0.4} & \frac{0.75z}{z-0.8} - \frac{0.75z}{z-0.4} \\ \frac{0.25z}{z-0.8} - \frac{0.25z}{z-0.4} & \frac{0.25z}{z-0.8} + \frac{0.75z}{z-0.4} \end{bmatrix} \\ \Rightarrow A^k &= \begin{bmatrix} 0.75(0.8)^k + 0.25(0.4)^k & 0.75(0.8)^k - 0.75(0.4)^k \\ 0.25(0.8)^k - 0.25(0.4)^k & 0.25(0.8)^k + 0.75(0.4)^k \end{bmatrix} \end{aligned}$$