

Lyapunov Stability



1. Definitions in Lyapunov stability analysis

2. Lyapunov's approach to stability

- Relevant tools

- Lyapunov stability theorems

- Instability theorem

- Discrete-time case

3. Recap

Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

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default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

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- ▶ the condition must be satisfied by all $t \geq 0$
- ▶ if a system starts at equilibrium state, it stays there

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- ▶ origin $x_e = 0$ is always an equilibrium state
- ▶ when $A(t)$ is singular, multiple equilibrium states exist

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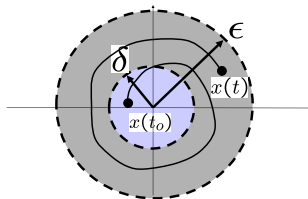


Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \forall t \geq t_0$.

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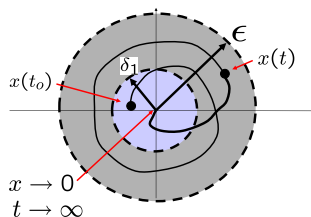


Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \rightarrow 0$.

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- ▶ $e^{\sigma t} \rightarrow 0$ if $\sigma < 0$; $e^{\lambda t} \rightarrow 0$ if $\lambda < 0$
- ▶ $\lambda^k \rightarrow 0$ if $|\lambda| < 1$; $r^k \rightarrow 0$ if $|r| = |\sqrt{\sigma^2 + \omega^2}| = |\lambda| < 1$

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- ▶ no need for explicit solutions to system responses
- ▶ an “energy” perspective
- ▶ fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \quad (x_1: \text{position}; x_2 : \text{velocity})$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2, \quad b > 0 \quad (\text{Newton's law})$$

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Stability from an energy viewpoint: Generalization

Consider unforced, time-varying, nonlinear systems

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- ▶ goal is to relate properties of the state through the Lyapunov function
- ▶ main tool: matrix formulation, linear algebra, positive definite functions

Relevant tools

Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- convenience of matrix formulation:

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- general quadratic functions in matrix form

$$Q(x) = x^T P x, \quad P^T = P$$

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$$\text{e.g. } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2.5 \\ 2.5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$$

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$$\text{general case: } P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

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$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | & | \end{bmatrix}$$

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$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

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namely, $a_j^T a_j = 1$ and $a_j^T a_m = 0 \ \forall j \neq m$.

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The eigenvalues of symmetric matrices are all real.

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import numpy as np #larger-scale Python example
N = 100
P = np.random.randint(-200,200,size=(N,N))
P_symm = (P + P.T)/2
lambdas, _ = np.linalg.eig(P_symm)
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import numpy as np
from scipy.linalg import qr
n = 3
H = np.random.randn(n, n)
Q, _ = qr(H)
print (np.dot(Q,Q.T))
print (np.dot(Q.T,Q))
```


Important properties of symmetric matrices

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matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

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Theorem (Symmetric eigenvalue decomposition (SED))

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- ▶ $\Lambda = \text{diagonal}(\lambda_1, \lambda_2, \dots, \lambda_n)$

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- ▶ If A has distinct eigenvalues, then $U = [u_1, u_2, \dots, u_n]$ is orthogonal after normalizing all the eigenvectors to unity norm.
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Example: $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5 - \lambda & \sqrt{3} \\ \sqrt{3} & 7 - \lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$
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If $A = A^T \in \mathbb{R}^{n \times n}$, then the eigenvalues of A satisfy

$$\lambda_{\max} = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T A x}{\|x\|_2^2} \quad (2)$$

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Positive definite matrices

- ▶ eigenvalues of symmetric matrices are real \Rightarrow we can order the eigenvalues

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- ▶ $P \succ 0$ ($P \succeq 0$) $\Leftrightarrow P$ can be decomposed as $P = N^T N$ where N is nonsingular (singular)

Negative definite matrices

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A symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is called **negative-definite**, written $Q \prec 0$, if $-Q \succ 0$, i.e., $x^T Q x < 0$ for all $x (\neq 0) \in \mathbb{R}^n$.

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Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	\mathbb{R}_+ axis
negative definite	negative	\mathbb{R}_- axis

Caution

- ▶ positive-definite matrices can have negative entries:

Cautious

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Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P = P^T$ and take any $v = [x, y]^T$, we have

$$\begin{aligned} v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\ &= x^2 + y^2 + (x - y)^2 \geq 0 \end{aligned}$$

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and the equality sign holds only when $x = y = 0$.

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$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 < 0$$

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For a symmetric matrix P , $P \succ 0$ if and only if all the eigenvalues of P are positive.

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Proof.

Since P is symmetric, we have

$$\lambda_{\max}(P) = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T A x}{\|x\|_2^2} \quad (4)$$

$$\lambda_{\min}(P) = \min_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T A x}{\|x\|_2^2} \quad (5)$$

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which gives $x^T A x \in [\lambda_{\min} \|x\|_2^2, \lambda_{\max} \|x\|_2^2]$. Thus
 $x^T A x > 0, x \neq 0 \Leftrightarrow \lambda_{\min} > 0$. □

Relevant tools

Checking positive definiteness of a matrix.

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Definition

The leading principle minors of $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ are defined as

$$p_{11}, \det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \det P.$$

Relevant tools

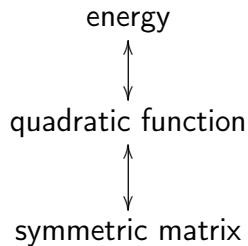
Checking positive definiteness of a matrix.

Example

None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

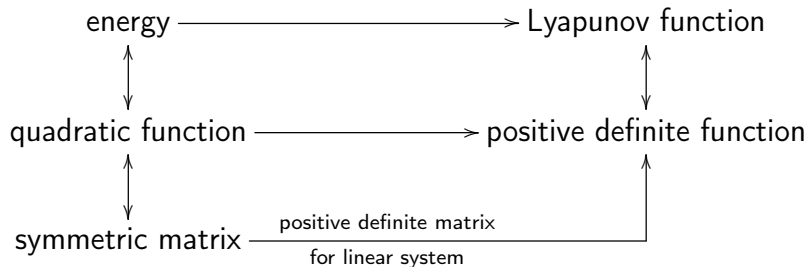
Recap



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Definition (Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called to be PD, satisfying

- ▶ $W(x) > 0$ for all $x \neq 0$
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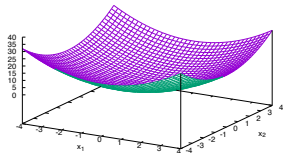
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In the 3D space, positive definite functions are “bowl-shaped”, e.g., $W(x_1, x_2) = x_1^2 + x_2^2$.



Definition (Locally Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called to be LPD, satisfying

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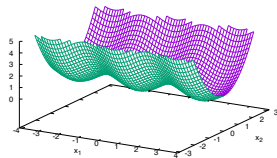
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In the 3D space, locally positive definite functions are “bowl-shaped” locally, e.g., $W(x_1, x_2) = x_1^2 + \sin^2 x_2$ for $x_1 \in \mathbb{R}$ and $|x_2| < \pi$



Exercise

Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

1. $V(x) = x_1^4 + x_2^2 + x_3^4$

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1. Definitions in Lyapunov stability analysis

2. Lyapunov's approach to stability

- Relevant tools

- Lyapunov stability theorems

- Instability theorem

- Discrete-time case

3. Recap

Lyapunov stability theorems

- recall the spring mass damper example in matrix form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$\mathcal{E}(t) = \text{potential energy} + \text{kinetic energy} = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

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$\mathcal{E}(t)$ = potential energy + kinetic energy = $\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$
and its derivative is NSD:

$$\dot{\mathcal{E}}(t) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2} \right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 \quad (6)$$

$$\begin{aligned} &= k_1 x_1 x_2 + m x_2 \left(-\frac{k}{m} x_1 - \frac{b}{m} x_2 \right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2} \right] A x \quad (7) \\ &= -b x_2^2 \end{aligned}$$

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Lyapunov stability concept for linear systems

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$$A^T \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} + \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

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► can stack the columns of $A^T P + PA$ and Q to yield

$$\begin{aligned} \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11}/ & a_{21}/ \\ a_{12}/ & a_{22}/ \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ \underbrace{\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}/ & a_{21}/ \\ a_{12}/ & a_{22}/ \end{bmatrix} \right\}}_{L_A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \end{aligned}$$

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 - ▶ so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$

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- ▶ L_A is invertible if and only if $\lambda_i + \lambda_j \neq 0$ for all eigenvalues of A :
 - ▶ let $A^T u_i = \lambda_i u_i$ and $A^T u_j = \lambda_j u_j$
 - ▶ $L_A(u_i u_j^T) = u_i u_j^T A + A^T u_i u_j^T = u_i (\lambda_j u_j)^T + \lambda_i u_i u_j^T = (\lambda_i + \lambda_j) u_i u_j^T$
 - ▶ so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$
 - ▶ if $\lambda_i + \lambda_j \neq 0$, the operator is invertible

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

► can simply write $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$ using the Kronecker

product notation $B \otimes C =$

$$\begin{bmatrix} b_{11}C & b_{12}C & \dots & b_{1n}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$$

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► e.g., $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{aligned} L_A &= I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] \end{aligned}$$

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

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The eigenvalues of L_A are $-1, -1, -1 - \sqrt{3}, -1 + \sqrt{3}$, which are precisely $\lambda_1 + \lambda_1, \lambda_1 + \lambda_2, \lambda_2 + \lambda_1, \lambda_2 + \lambda_2$.

```
import numpy as np
A = [[-1,1],[-1,0]]; I2=np.eye(2); AT=np.transpose(A)
L_A=np.kron(I2,AT)+np.kron(AT,I2)
eigLA,_=np.linalg.eig(L_A)
eigA,_=np.linalg.eig(A)
print(eigLA)
print(eigA)
```

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has a unique positive definite solution $P \succ 0$, $P^T = P$.

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Therefore, $x \rightarrow 0$ as $t \rightarrow \infty$, regardless of the initial condition. □

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$$\begin{aligned} \cancel{x^T(\infty)Px(\infty)} - x^T(0)Px(0) &= \int_0^\infty \frac{d}{dt} x^T(t)Px(t) dt = \int_0^\infty x^T(t) (A^TP + PA) x(t) dt \\ &\Rightarrow x^T(0)Px(0) = \int_0^\infty x^T(t) Qx(t) dt = \int_0^\infty x^T(0) e^{A^T t} Q e^{At} x(0) dt \end{aligned}$$

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If $Q \succ 0$, there exists a nonsingular N matrix: $Q = N^T N$. Thus

$$x^T(0)Px(0) = \int_0^\infty \|Ne^{At}x(0)\|^2 dt \geq 0$$

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Thus $P \succ 0$. Furthermore

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$



Procedures of Lyapunov's direct method

1. Given A , select an arbitrary positive-definite symmetric matrix Q (e.g., I).
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 - ▶ if P is positive-definite, then A is Hurwitz stable and the origin is asymptotically stable;
 - ▶ if P is not positive-definite, then A has at least one eigenvalue with a positive real part and the origin is an unstable equilibrium.

Lyapunov stability theorems

Example

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. The Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_P + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = - \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_Q$$

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$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

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Leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$
 $\Rightarrow P \succ 0 \Rightarrow$ asymptotically stable

Lyapunov analysis with Matlab

$$\dot{x} = Ax, A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

```
A = [-1,1;-1,0]
Q = eye(2)
P = lyap(A',Q)
w = eig(P)
```

Lyapunov analysis with Python

$$\dot{x} = Ax, A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

```
import control as ct
import numpy as np
A = np.array([[ -1, 1], [-1, 0]])
Q = np.identity(2)
P = ct.lyap(A.transpose(), Q)
print(P)
w = np.linalg.eigvals(P)
print(f'eigenvalues of P: {w}')
```

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- $\tilde{A} = N^{-1} A N$ and A are similar matrices and have the same eigenvalues.
- $\tilde{P} = N^T P N$ and P have the same definiteness. If we can find a positive definite solution P then the \tilde{P} will also be positive definite. Vice versa.

Instability theorem

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Theorem

The equilibrium state 0 of $\dot{x} = f(x)$ is unstable if there exists a function $W(x)$ such that

- ▶ $\dot{W}(x)$ is PD locally: $\dot{W}(x) > 0 \quad \forall |x| < r$ for some r and $\dot{W}(0) = 0$
- ▶ $W(0) = 0$
- ▶ *there exist states x arbitrarily close to the origin such that $W(x) > 0$*

Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x, \quad P = P^T \succ 0$$

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$$V(x(k+1)) - V(x(k)) = x^T(k) \underbrace{(A^T P A - P)}_{\triangleq -Q} x(k)$$

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Asymptotic stability desires $\Delta V(x)$ to be negative.

DT Lyapunov stability theorem for linear systems

Theorem

For system $x(k+1) = Ax(k)$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if $\exists Q \succ 0$, such that the discrete-time Lyapunov equation

$$A^T P A - P = -Q$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

The DT Lyapunov Eq.

$$A^T P A - P = -Q$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur stable), comes from:

$$\begin{aligned} \cancel{V(x(\infty))} - V(x(0)) &= \sum_{k=0}^{\infty} x^T(k) [A^T P A - P] x(k) \\ &= - \sum_{k=0}^{\infty} x^T(0) (A^T)^k Q A^k x(0) \\ \Rightarrow P &= \sum_{k=0}^{\infty} (A^T)^k Q A^k \end{aligned}$$

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- can show that the DT Lyapunov operator $L_A = A^T P A - P$ is invertible if and only if $\forall i, j \ (\lambda_A)_i (\lambda_A)_j \neq 1$

DT Lyapunov analysis with MATLAB

Example

$$x(k+1) = Ax(k), \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
% MATLAB
A=[ 0 1 0; 0 0 1; 0.275 -0.225 -0.1]
Q = eye(3)
P = dlyap(A',Q) % check function definition in Matlab help
eig(P)
```

DT Lyapunov analysis with Python

Example

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```
#Python
import control as ct
import numpy as np
from numpy.linalg import eig
A = np.array([[0,1,0],[0,0,1],[0.275,-0.225,-0.1]])
Q = np.identity(3)
P = ct.dlyap(A.transpose(),Q)
w,v = eig(P)
print(w)
```

Recap

- ▶ Internal stability
 - ▶ Stability in the sense of Lyapunov: ε , δ conditions
 - ▶ Asymptotic stability
- ▶ Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
 - ▶ Based on the eigenvalues of A
 - ▶ Time response modes
 - ▶ Repeated eigenvalues on the imaginary axis
 - ▶ Routh's criterion
 - ▶ No need to solve the characteristic equation
 - ▶ Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)

Recap

► Lyapunov equations

Theorem: All eigenvalues of A have negative real parts iff for any given $Q \succ 0$, the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique solution P and $P \succ 0$.

Given Q , the Lyapunov equation $A^T P + PA = -Q$ has a unique solution when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j .

Theorem: All eigenvalues of A are inside the unit circle iff for any given $Q \succ 0$, the Lyapunov equation

$$A^T P A - P = -Q$$

has a unique solution P and $P \succ 0$.

Given Q , the Lyapunov equation $A^T P A - P = -Q$ has a unique solution when $\lambda_{A,i} \lambda_{A,j} \neq 1$ for all i and j .

Recap

- ▶ P is positive definite if and only if any one of the following conditions holds:
 1. All the eigenvalues of P are positive.
 2. All the leading principle minors of P are positive.
 3. There exists a nonsingular matrix N such that $P = N^T N$.