Linear Systems
State Feedback Control

Motivation

- At the center of designing control systems is the idea of feedback.
- In such transfer-function approaches as lead-lag and root locus methods, the primal goal is to achieve a proper map of closed-loop poles with output feedback.

Key questions:

- How much freedom do we have for state-space systems?
- Are there fundamental system properties that yield higher achievable performance?
- How to implement the design algorithms?

1. Goal and realization of state feedback

2. Closed-loop eigenvalue placement by state feedback

Goal

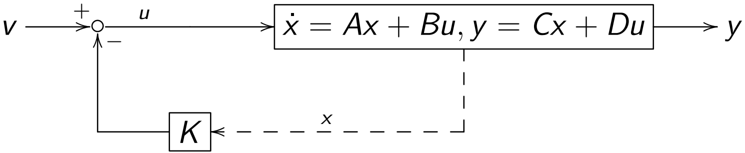
Consider an n-dimensional state-space system

$$\Sigma : \begin{cases} \dot{x}(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{cases} \quad x(t_0) = x_0$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, and $y \in \mathbb{R}^m$.

- Denominators of the transfer function $G(s) = C(sI - A)^{-1} B + D$ come from the characteristic polynomial $\det(sI - A)$ that arises when computing the inverse $(sI - A)^{-1}$.
- We shall investigate the use of feedback to alter the qualitative behavior of the system by changing the eigenvalues of the closed-loop "A" matrix.

Realization



Consider the *state-feedback law*

$$u = -Kx + v \tag{1}$$

- v: new input which we will deal with later
- $K \in \mathbb{R}^{m \times n}$: n-number of states, m-number of inputs
- closed-loop system:

$$\Sigma_{cl} : \begin{cases} \dot{x}(t) &= (A - BK)\,x(t) + Bv(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \qquad x(t_0) = x_0 \tag{2}$$

- key closed-loop property: eigenvalues of $A - BK$.
- How freely can we place the eigenvalues of $A_{cl} = A - BK$?

1. Goal and realization of state feedback

2. Closed-loop eigenvalue placement by state feedback

Eigenvalue placement by state feedback

Fact

If $\Sigma = (A, B, C, D)$ is in controllable canonical form, we can completely change all the eigenvalues of $A - BK$ by choice of state-feedback gain matrix K.

- Problem setup: single-input single-output system in c.c.f.

$$H(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} + d, \quad \Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \dots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ -\alpha_0 & \dots & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_0 & \beta_1 & \dots & \dots & \beta_{n-1} \end{bmatrix}, \; D = d$$

$$\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 \tag{3}$$

Eigenvalue placement by state feedback: c.c.f.

- Goal: achieve desired closed-loop eigenvalue locations $p_1, \cdots, p_n$, i.e.

$$\det\left(sI - (A - BK)\right) = (s - p_1)(s - p_2)\cdots(s - p_n) \tag{4}$$
$$= s^n + \gamma_{n-1}s^{n-1} + \cdots + \gamma_1 s + \gamma_0 \tag{5}$$

- Let $K = [k_0, k_1, \ldots, k_{n-1}]$. The structured A and B give

$$BK = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} [k_0, k_1, \ldots, k_{n-1}] = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \vdots \\ \vdots & \ldots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 0 \\ k_0 & \ldots & \ldots & k_{n-2} & k_{n-1} \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ldots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -\alpha_0 - k_0 & \ldots & \ldots & -\alpha_{n-2} - k_{n-2} & -\alpha_{n-1} - k_{n-1} \end{bmatrix}$$

Eigenvalue placement by state feedback: c.c.f.

- A and $A - BK$ have the same structure
- the only difference is the last row:

matrix	last row
A	$\begin{bmatrix} -\alpha_0 & \dots & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{bmatrix}$
$A - BK$	$\begin{bmatrix} -\alpha_0 - k_0 & \dots & \dots & -\alpha_{n-2} - k_{n-2} & -\alpha_{n-1} - k_{n-1} \end{bmatrix}$

- recall (3): $\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0$.
- thus

$$\det(sI - (A - BK)) = s^n + \underbrace{(\alpha_{n-1} + k_{n-1})}_{\text{target: } \gamma_{n-1}} s^{n-1} + \cdots + \underbrace{(\alpha_0 + k_0)}_{\text{target: } \gamma_0}$$

- hence

$$k_0 = \gamma_0 - \alpha_0$$
$$\vdots$$
$$k_{n-1} = \gamma_{n-1} - \alpha_{n-1}$$

Eigenvalue placement by state feedback: c.c.f.

Eigenvalue-placement Algorithm

1. determine desired eigenvalue locations $p_1, \cdots, p_n$
2. calculate desired closed-loop characteristic polynomial
   $$(s - p_1)(s - p_2) \cdots (s - p_n) = s^n + \gamma_{n-1} s^{n-1} + \cdots + \gamma_1 s + \gamma_0$$
3. calculate open-loop characteristic polynomial
   $$\det(sI - A) = s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0$$
4. define the matrices:
   $K = [\gamma_0 - \alpha_0, \ldots, \gamma_{n-1} - \alpha_{n-1}]$

Powerful result: if the system is in controllable canonical form, we can arbitrarily place the closed-loop eigenvalues by state feedback!

General eigenvalue placement by state feedback

- What if the given state-space realization $\Sigma = (A, B, C, D)$ is not in the required form?
- We can then transform it to c.c.f. via a similarity transformation (See lecture on controllability and observability).
- **Powerful fact**: if system $\Sigma = (A, B, C, D)$ is controllable, then we can arbitrarily place the closed-loop eigenvalues via state feedback.

Stabilization

- if a single-input system is uncontrollable, arbitrary closed-loop eigenvalue plaement is not available
- Kalman decomposition gives

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_c \\ \bar{x}_{uc}\end{bmatrix} = \begin{bmatrix} \overbrace{\bar{A}_c}^{\text{controllable part}} & \bar{A}_{12} \\ 0 & \underbrace{\bar{A}_{uc}}_{\text{uncontrollable part}} \end{bmatrix}\begin{bmatrix}\bar{x}_c \\ \bar{x}_{uc}\end{bmatrix} + \begin{bmatrix}\bar{B}_c \\ 0\end{bmatrix} u$$

applying controll law

$$u = -\left[\bar{K}_c, \bar{K}_{uc}\right]\begin{bmatrix}\bar{x}_c \\ \bar{x}_{uc}\end{bmatrix} + v$$

gives

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_c \\ \bar{x}_{uc}\end{bmatrix} = \begin{bmatrix}\bar{A}_c - \bar{B}_c\bar{K}_c & \bar{A}_{12} - \bar{B}_c\bar{K}_{uc} \\ 0 & \bar{A}_{uc}\end{bmatrix}\begin{bmatrix}\bar{x}_c \\ \bar{x}_{uc}\end{bmatrix} + \begin{bmatrix}\bar{B}_c \\ 0\end{bmatrix} v$$

Stabilization cont'd

- closed-loop dynamics

$$\frac{d}{dt}\begin{bmatrix}\bar{x}_c\\ \bar{x}_{uc}\end{bmatrix} = \underbrace{\begin{bmatrix}\bar{A}_c - \bar{B}_c\bar{K}_c & \bar{A}_{12} - \bar{B}_c\bar{K}_{uc}\\ 0 & \bar{A}_{uc}\end{bmatrix}}_{\bar{A}_{cl}}\begin{bmatrix}\bar{x}_c\\ \bar{x}_{uc}\end{bmatrix} + \begin{bmatrix}\bar{B}_c\\ 0\end{bmatrix} v$$

- closed-loop eigenvalues come from

$$\det\left(\bar{A}_{cl} - \lambda I\right) = \underbrace{\det \overbrace{\left(\left(\bar{A}_c - \bar{B}_c\bar{K}_c\right) - \lambda I\right)}^{\text{eigenvalues can be arbitrarily placed}}}_{\text{from the controllable subsystem}} \cdot \underbrace{\det\left(\bar{A}_{uc} - \lambda I\right)}_{\text{uncontrollable eigenvalues}}$$

- $\Rightarrow$: single-input systems are stabilizable if and only if the uncontrollable portion of the system does not have any unstable eigenvalue.

Discrete-time case

- the eigenvalue assignment of discrete-time systems is analogous:
  - system dynamics:

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k)$$

  - controller: $u(k) = -Kx(k) + v(k)$
  - closed-loop dynamics:

$$x(k+1) = Ax(k) - BKx(k) + Bv(k) = (A - BK)x(k) + Bv(k)$$

- arbitrary closed-loop eigenvalue assignment if system is controllable

The case with output feedback

- if the full state is not measurable, state feedback control is not feasible
- consider output feedback

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \\ u &= -Fy + v \end{cases} \Rightarrow \dot{x} = Ax - BFy + Bv = (A - BFC)\,x + Bv$$

- $A - BFC$ not as structured as $A - BK$ (exercise: write out the case for SISO systems)
- arbitrary closed-loop eigenvalue assignment not feasible

The case with output feedback

Example

Controllable mass-spring-damper system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$\overset{u^* \triangleq \frac{u}{m}}{=} \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u^*$$

- arbitrary closed-loop eigenvalue assignment if $u^* = -k_1 x_1 - k_2 x_2$, namely $U^*(s) = -k_1 X_1(s) - k_2 X_2(s) = -\left(k_1 + k_2 s\right) X_1(s) \Rightarrow$ a proportional plus derivative (PD) control law
- if with only proportional control, $u^* = -k_1 x_1$, arbitrary closed-loop eigenvalue assignment is not possible