

Linear Systems: Stability



1. Definitions in Lyapunov stability analysis
2. Stability of LTI systems: method of eigenvalue/pole locations
3. Lyapunov's approach to stability
 - Relevant tools
 - Lyapunov stability theorems
 - Instability theorem
 - Discrete-time case
4. Recap

Finite dimensional vector norms

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default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

Equilibrium state

For an n -th order unforced system

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- ▶ the condition must be satisfied by all $t \geq 0$
- ▶ if a system starts at equilibrium state, it stays there

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- ▶ origin $x_e = 0$ is always an equilibrium state
- ▶ when $A(t)$ is singular, multiple equilibrium states exist

Lyapunov's definition of stability

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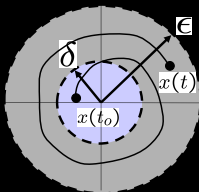


Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \forall t \geq t_0$.

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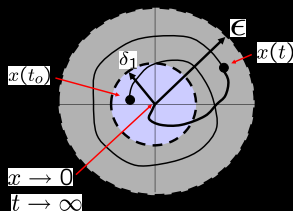


Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \rightarrow 0$.

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- ▶ $e^{\sigma t} \rightarrow 0$ if $\sigma < 0$; $e^{\lambda t} \rightarrow 0$ if $\lambda < 0$
- ▶ $\lambda^k \rightarrow 0$ if $|\lambda| < 1$; $r^k \rightarrow 0$ if $|r| = |\sqrt{\sigma^2 + \omega^2}| = |\lambda| < 1$

Stability of the origin for $\dot{x} = Ax$

stability at 0	$\lambda_i(A)$
unstable	$\text{Re}\{\lambda_i\} > 0$ for some λ_i or $\text{Re}\{\lambda_i\} \leq 0$ for all λ_i 's but for a repeated λ_m on the imaginary axis with multiplicity m , nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\text{Re}\{\lambda_i\} \leq 0$ for all λ_i 's and \forall repeated λ_m on the imaginary axis with multiplicity m , nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\text{Re}\{\lambda_i\} < 0 \forall \lambda_i$ (A is then called Hurwitz stable)

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- ▶ i.e., two repeated eigenvalues but needs a generalized eigenvector \Rightarrow Jordan form after similarity transform
- ▶ verify by checking $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$: t grows unbounded

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► verify by checking $e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Routh-Hurwitz criterion

- ▶ the Routh Test (by E.J. Routh, in 1877): a simple algebraic procedure to determine how many roots a given polynomial

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

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- ▶ popular if stability is the only concern and no details on eigenvalues (e.g., speed of response) are needed

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- ▶ simply apply the Routh Test to $A(s) = \det(sI - A)$
- ▶ recap: the poles of transfer function $G(s) = C(sI - A)^{-1}B + D$ come from $\det(sI - A)$ in computing the inverse $(sI - A)^{-1}$

The Routh Array

for $A(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$, construct

$$\begin{array}{c|cccc} s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \dots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \dots \\ s^{n-2} & q_{n-2} & q_{n-4} & q_{n-6} & \dots & \\ s^{n-3} & q_{n-3} & q_{n-5} & q_{n-7} & \dots & \\ \vdots & \vdots & \vdots & \vdots & & \\ s^1 & x_2 & x_0 & & & \\ s^0 & x_0 & & & & \end{array}$$

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 \vdots & \vdots & \vdots & \vdots & & \\
 s^1 & x_2 & x_0 & & & \\
 s^0 & x_0 & & & &
 \end{array}$$

- ▶ first two rows contain the coefficients of $A(s)$
- ▶ third row constructed from the previous two rows via

$$\begin{array}{c|ccc}
 \cdot & a & b & x & \cdot \\
 \cdot & c & d & y & \cdot \\
 \cdot & \frac{bc - ad}{c} & \frac{xc - ay}{c} & \cdot & \\
 \cdot & \cdot & \cdot & \cdot & \cdot
 \end{array}$$

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- All roots of $A(s)$ are on the left half s-plane if and only if all elements of the first column of the Routh array are positive.

The Routh Array

Example ($A(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$)

$$\begin{array}{c|ccc} s^4 & 2 & 3 & 10 \\ s^3 & 1 & 5 & 0 \\ s^2 & 3 - \frac{2 \times 5}{1} = -7 & 10 & 0 \\ s^1 & 5 - \frac{1 \times 10}{-7} & 0 & 0 \\ s^0 & 10 & 0 & 0 \end{array}$$

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- ▶ two sign changes in the first column
- ▶ unstable and two roots in the right half side of s-plane

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special cases:

- If the 1st element in any one row of Routh's array is zero, one can replace the zero with a small number ϵ and proceed further.

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- ▶ If the 1st element in any one row of Routh's array is zero, one can replace the zero with a small number ϵ and proceed further.
- ▶ There are other possible complications, which we will not pursue further. See, e.g. "Automatic Control Systems", by Kuo, 7th ed., pp. 339-340.

Stability of the origin for $x(k+1) = f(x(k), k)$

- stability analysis follows analogously for nonlinear time-varying discrete-time systems of the form

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- ▶ equilibrium point x_e :

$$f(x_e, k) = x_e, \quad \forall k$$

- ▶ without loss of generality, 0 is assumed an equilibrium point

Stability of the origin for $x(k+1) = Ax(k)$

stability at 0	$\lambda_i(A)$
unstable	$ \lambda_i > 1$ for some λ_i or $ \lambda_i \leq 1$ for all λ_i 's but for a repeated λ_m on the unit circle with multiplicity m , $\text{nullity}(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$ \lambda_i \leq 1$ for all λ_i 's but for any repeated λ_m on the unit circle with multiplicity m , $\text{nullity}(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$ \lambda_i < 1 \ \forall \lambda_i$ (such a matrix is called Schur stable)

Routh-Hurwitz criterion for DT LTI systems

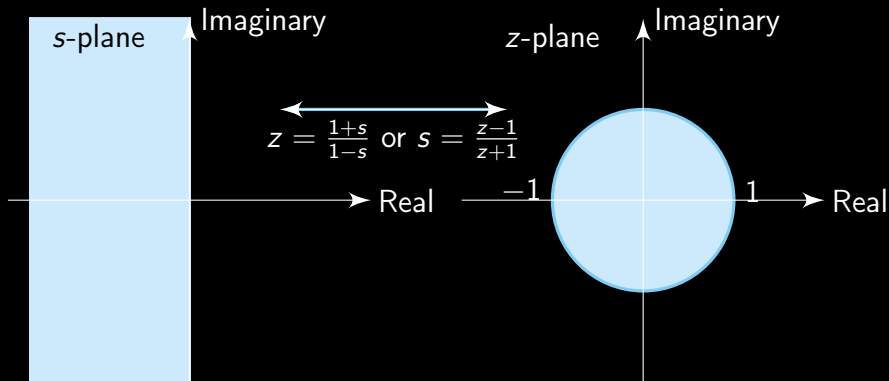
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- ▶ the stability domain $|\lambda_i| < 1$ is a unit disk
- ▶ Routh array validates stability in the left-half plane
- ▶ bilinear transformation maps the closed left half s -plane to the closed unit disk in z -plane



Routh-Hurwitz criterion for DT LTI systems

- ▶ Given $A(z) = z^n + a_1 z^{n-1} + \dots + a_n$, procedures of Routh-Hurwitz test:
 - ▶ apply bilinear transform
$$A(z)|_{z=\frac{1+s}{1-s}} = \left(\frac{1+s}{1-s}\right)^n + a_1 \left(\frac{1+s}{1-s}\right)^{n-1} + \dots + a_n =: \frac{A^*(s)}{(1-s)^n}$$
 - ▶ apply Routh test to
$$A^*(s) = a_n^* s^n + a_{n-1}^* s^{n-1} + \dots + a_0^* = A(z)|_{z=\frac{1+s}{1-s}} (1-s)^n$$

Routh-Hurwitz criterion for DT LTI systems

Example ($A(z) = z^3 + 0.8z^2 + 0.6z + 0.5$)

$$\begin{aligned} \blacktriangleright A^*(s) &= A(z)|_{z=\frac{1+s}{1-s}} (1-s)^3 = (1+s)^3 + 0.8(1+s)^2(1-s) + \\ &0.6(1+s)(1-s)^2 + 0.5(1-s)^3 = 0.3s^3 + 3.1s^2 + 1.7s + 2.9 \end{aligned}$$

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► Routh array

$$\begin{array}{c|cc} s^3 & 0.3 & 1.7 \\ s^2 & 3.1 & 2.9 \\ s & 1.7 - \frac{0.3 \times 2.9}{3.1} = 1.42 & 0 \\ s^0 & 2.9 & 0 \end{array}$$

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► all elements in first column are positive \Rightarrow roots of $A(z)$ are all in the unit circle

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Lyapunov's approach to stability

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- ▶ no need for explicit solutions to system responses

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- ▶ no need for explicit solutions to system responses
- ▶ an “energy” perspective
- ▶ fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \quad (x_1: \text{position}; x_2 : \text{velocity})$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2, \quad b > 0 \quad (\text{Newton's law})$$

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Stability from an energy viewpoint: Generalization

Consider unforced, time-varying, nonlinear systems

$$\begin{aligned}\dot{x}(t) &= f(x(t), t), \quad x(t_0) = x_0 \\ x(k+1) &= f(x(k), k), \quad x(k_0) = x_0\end{aligned}$$

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- ▶ main tool: matrix formulation, linear algebra, positive definite functions

Relevant tools

Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- general quadratic functions in matrix form

$$Q(x) = x^T P x, \quad P^T = P$$

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$$\text{general case: } P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

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namely, $a_j^T a_j = 1$ and $a_j^T a_m = 0 \ \forall j \neq m$.

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Also, $\bar{u}^T u \in \mathbb{R}$. Thus $\lambda = \frac{\bar{u}^T Au}{\bar{u}^T u}$ must also be a real number. □

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matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

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- ▶ λ_i 's: eigenvalues of A
- ▶ u_i : eigenvector associated to λ_i , normalized to have unity norms
- ▶ $U = [u_1, u_2, \dots, u_n]$ is orthogonal: $U^T U = U U^T = I$
- ▶ $\Lambda = \text{diagonal}(\lambda_1, \lambda_2, \dots, \lambda_n)$

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- ▶ if $A \in \mathbb{R}^{2 \times 2}$, then if you compute first λ_1 , λ_2 and u_1 , you won't need to go through the regular math to get u_2 , but can simply solve for a u_2 that is orthogonal to u_1 with $\|u_2\| = 1$.

Example: $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5 - \lambda & \sqrt{3} \\ \sqrt{3} & 7 - \lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$
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If $A = A^T \in \mathbb{R}^{n \times n}$, then the eigenvalues of A satisfy

$$\lambda_{\max} = \max_{x \in \mathbb{R}^n, x \neq 0} \frac{x^T A x}{\|x\|_2^2} \quad (2)$$

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Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	\mathbb{R}_+ axis
negative definite	negative	\mathbb{R}_- axis

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Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P = P^T$ and take any $v = [x, y]^T$, we have

$$\begin{aligned} v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\ &= x^2 + y^2 + (x - y)^2 \geq 0 \end{aligned}$$

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and the equality sign holds only when $x = y = 0$.

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which gives $x^T A x \in [\lambda_{\min} \|x\|_2^2, \lambda_{\max} \|x\|_2^2]$. Thus
 $x^T A x > 0, x \neq 0 \Leftrightarrow \lambda_{\min} > 0$. □

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Checking positive definiteness of a matrix.

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Definition

The leading principle minors of $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ are defined as

$$p_{11}, \det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}, \det P.$$

Relevant tools

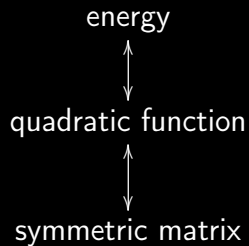
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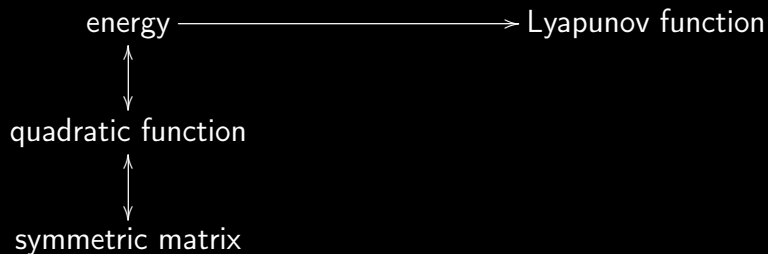
None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

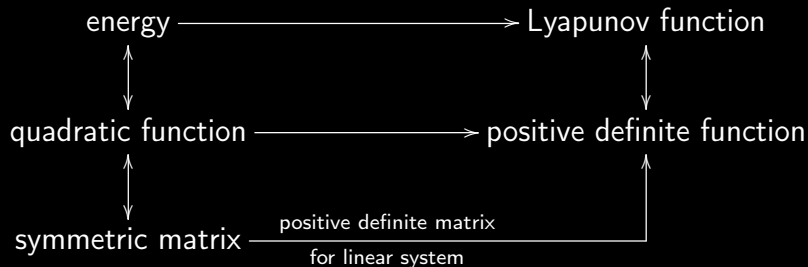
Recap



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Relevant tools

Definition (Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called to be PD, satisfying

- ▶ $W(x) > 0$ for all $x \neq 0$
- ▶ $W(0) = 0$
- ▶ $W(x) \rightarrow \infty$ as $|x| \rightarrow \infty$ uniformly in x

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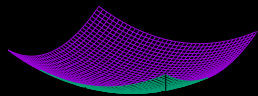
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A continuous time function $W : \mathbb{R}^n \rightarrow \mathbb{R}_+$, called to be PD, satisfying

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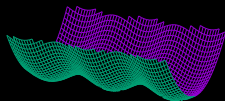
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Exercise

Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

1. $V(x) = x_1^4 + x_2^2 + x_3^4$

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1. Definitions in Lyapunov stability analysis
2. Stability of LTI systems: method of eigenvalue/pole locations
3. Lyapunov's approach to stability
 - Relevant tools
 - Lyapunov stability theorems
 - Instability theorem
 - Discrete-time case
4. Recap

Lyapunov stability theorems

- recall the spring mass damper example in matrix form

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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and its derivative is NSD:

$$\dot{\mathcal{E}}(t) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2} \right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 \quad (6)$$

$$\begin{aligned} &= k_1 x_1 x_2 + m x_2 \left(-\frac{k}{m} x_1 - \frac{b}{m} x_2 \right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2} \right] A x \quad (7) \\ &= -b x_2^2 \end{aligned}$$

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Lyapunov stability concept for linear systems

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► can stack the columns of $A^T P + PA$ and Q to yield

$$\begin{aligned} \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11}/ & a_{21}/ \\ a_{12}/ & a_{22}/ \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \\ \underbrace{\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}/ & a_{21}/ \\ a_{12}/ & a_{22}/ \end{bmatrix} \right\}}_{L_A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} &= - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \end{aligned}$$

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 - ▶ so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$
 - ▶ if $\lambda_i + \lambda_j \neq 0$, the operator is invertible

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- can simply write $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$ using the Kronecker

product notation $B \otimes C =$

$$\begin{bmatrix} b_{11}C & b_{12}C & \dots & b_{1n}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$$

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► e.g., $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\begin{aligned} L_A &= I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array} \right] \end{aligned}$$

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

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The eigenvalues of L_A are $-1, -1, -1 - \sqrt{3}, -1 + \sqrt{3}$, which are precisely $\lambda_1 + \lambda_1, \lambda_1 + \lambda_2, \lambda_2 + \lambda_1, \lambda_2 + \lambda_2$.

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$P \succ 0 \Rightarrow (\lambda_Q)_{\min} > 0$ and $(\lambda_P)_{\max} > 0$. Thus $\alpha > 0$; $V(t)$ decays exponentially to zero. $V(x) \succ 0 \Rightarrow V(x) = 0$ only at $x = 0$.

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For $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if for any symmetric positive definite matrix $Q \succ 0$, the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

Proof.

$$\text{"}\Rightarrow\text{"}: \frac{\dot{V}}{V} = -\frac{x^T Q x}{x^T P x} \leq -\underbrace{\frac{(\lambda_Q)_{\min}}{(\lambda_P)_{\max}}}_{\triangleq \alpha} \implies V(t) \leq e^{-\alpha t} V(0). \quad Q \succ 0 \text{ and}$$

$P \succ 0 \implies (\lambda_Q)_{\min} > 0$ and $(\lambda_P)_{\max} > 0$. Thus $\alpha > 0$; $V(t)$ decays exponentially to zero. $V(x) \succ 0 \implies V(x) = 0$ only at $x = 0$.

Therefore, $x \rightarrow 0$ as $t \rightarrow \infty$, regardless of the initial condition. \square

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$$\begin{aligned} \cancel{x^T(\infty)P} \overset{0}{x(\infty)} - x^T(0)Px(0) &= \int_0^\infty \frac{d}{dt} x^T(t)Px(t) dt = \int_0^\infty x^T(t) (A^TP + PA) x(t) dt \\ \Rightarrow x^T(0)Px(0) &= \int_0^\infty x^T(t)Qx(t) dt = \int_0^\infty x^T(0)e^{A^Tt}Qe^{At}x(0) dt \end{aligned}$$

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$$x^T(0)Px(0) = \int_0^\infty \|Ne^{At}x(0)\|^2 dt \geq 0$$

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“ \Leftarrow ”: if 0 of $\dot{x} = Ax$ is asymptotically stable, then all eigenvalues of A have negative real parts. For any Q , the Lyapunov equation has a unique solution P . Note $x(t) = e^{At}x_0 \rightarrow 0$ as $t \rightarrow \infty$. We have

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Thus $P \succ 0$. Furthermore

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$



Procedures of Lyapunov's direct method

1. Given A , select an arbitrary positive-definite symmetric matrix Q (e.g., I).
2. Find the solution matrix P to the Lyapunov equation
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 - ▶ if P is not positive-definite, then A has at least one eigenvalue with a positive real part and the origin is an unstable equilibrium.

Lyapunov stability theorems

Example

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. The Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_P + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = - \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_Q$$

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$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

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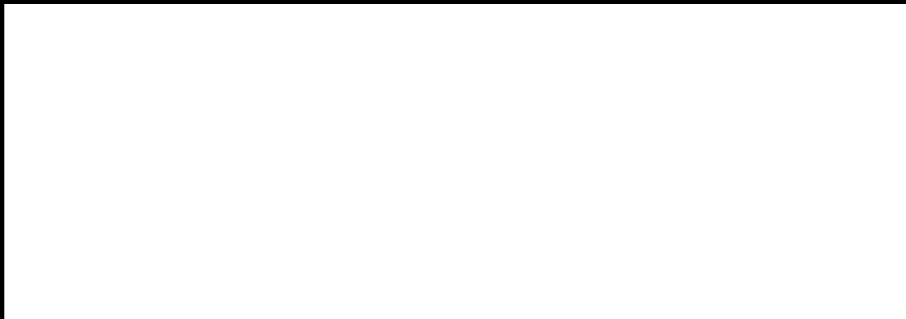
Leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$
 $\Rightarrow P \succ 0 \Rightarrow$ asymptotically stable

Lyapunov analysis with Matlab

$$\dot{x} = Ax, A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$


Lyapunov analysis with Python

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$$\begin{aligned} A^T P + P A &= -I \\ \Downarrow \\ \underbrace{N^T A^T N^{-T}}_{\tilde{A}^T} \underbrace{N^T P N}_{\tilde{P}} + \underbrace{N^T P N}_{\tilde{P}} \underbrace{N^{-1} A N}_{\tilde{A}} &= -N^T N \end{aligned}$$

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- $\tilde{A} = N^{-1} A N$ and A are similar matrices and have the same eigenvalues.
- $\tilde{P} = N^T P N$ and P have the same definiteness. If we can find a positive definite solution P then the \tilde{P} will also be positive definite. Vice versa.

Instability theorem

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Theorem

The equilibrium state 0 of $\dot{x} = f(x)$ is unstable if there exists a function $W(x)$ such that

- ▶ *$\dot{W}(x)$ is PD locally: $\dot{W}(x) > 0 \ \forall |x| < r$ for some r and $\dot{W}(0) = 0$*
- ▶ *$W(0) = 0$*
- ▶ *there exist states x arbitrarily close to the origin such that $W(x) > 0$*

Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x, \quad P = P^T \succ 0$$

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Asymptotic stability desires $\Delta V(x)$ to be negative.

DT Lyapunov stability theorem for linear systems

Theorem

For system $x(k+1) = Ax(k)$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if $\exists Q \succ 0$, such that the discrete-time Lyapunov equation

$$A^T P A - P = -Q$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

The DT Lyapunov Eq.

$$\boxed{A^T P A - P = -Q}$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur stable), comes from:

$$\begin{aligned} \cancel{V(x(\infty))} - V(x(0)) &= \sum_{k=0}^{\infty} x^T(k) [A^T P A - P] x(k) \\ &= - \sum_{k=0}^{\infty} x^T(0) (A^T)^k Q A^k x(0) \\ \Rightarrow P &= \sum_{k=0}^{\infty} (A^T)^k Q A^k \end{aligned}$$

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- can show that the DT Lyapunov operator $L_A = A^T P A - P$ is invertible if and only if $\forall i, j \ (\lambda_A)_i (\lambda_A)_j \neq 1$

DT Lyapunov analysis with MATLAB

Example

$$x(k+1) = Ax(k), \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

DT Lyapunov analysis with Python

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Recap

- ▶ Internal stability
 - ▶ Stability in the sense of Lyapunov: ε, δ conditions
 - ▶ Asymptotic stability
- ▶ Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
 - ▶ Based on the eigenvalues of A
 - ▶ Time response modes
 - ▶ Repeated eigenvalues on the imaginary axis
 - ▶ Routh's criterion
 - ▶ No need to solve the characteristic equation
 - ▶ Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)

Recap

► Lyapunov equations

Theorem: All eigenvalues of A have negative real parts iff for any given $Q \succ 0$, the Lyapunov equation

$$A^T P + PA = -Q$$

has a unique solution P and $P \succ 0$.

Given Q , the Lyapunov equation $A^T P + PA = -Q$ has a unique solution when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j .

Theorem: All eigenvalues of A are inside the unit circle iff for any given $Q \succ 0$, the Lyapunov equation

$$A^T P A - P = -Q$$

has a unique solution P and $P \succ 0$.

Given Q , the Lyapunov equation $A^T P A - P = -Q$ has a unique solution when $\lambda_{A,i} \lambda_{A,j} \neq 1$ for all i and j .

Recap

- ▶ P is positive definite if and only if any one of the following conditions holds:
 1. All the eigenvalues of P are positive.
 2. All the leading principle minors of P are positive.
 3. There exists a nonsingular matrix N such that $P = N^T N$.