Discretization and Implementation of Continuous-time Design

Big picture
Discrete-time frequency response
Discretization of continuous-time design
Aliasing and anti-aliasing

Big picture

why are we learning this:

- nowadays controllers are implemented in discrete-time domain
- implementation media: digital signal processor, field-programmable gate array (FPGA), etc
- either: controller is designed in continuous-time domain and implemented digitally
- or: controller is designed directly in discrete-time domain

Frequency response of LTI SISO digital systems

$$a\sin(\omega T_s k) \longrightarrow \boxed{G(z)} \longrightarrow b\sin(\omega T_s k+\phi) \text{ at steady state}$$

- sampling time: T_s
- $\phi\left(e^{j\omega T_s}\right)$: phase difference between the output and the input
- $M\left(e^{j\omega T_s}\right) = b/a$: magnitude difference

continuous-time frequency response:

$$G(j\omega) = G(s)|_{s=j\omega} = |G(j\omega)|e^{j\angle G(j\omega)}$$

discrete-time frequency response:

$$G\left(e^{j\omega T_s}\right) = G(z)|_{z=e^{j\omega T_s}} = \left|G\left(e^{j\omega T_s}\right)\right|e^{j\angle G\left(e^{j\omega T_s}\right)}$$
$$= M\left(e^{j\omega T_s}\right)e^{j\phi\left(e^{j\omega T_s}\right)}$$

Sampling

sufficient samples must be collected (i.e., fast enough sampling frequency) to recover the frequency of a continuous-time sinusoidal signal (with frequency ω in rad/sec)

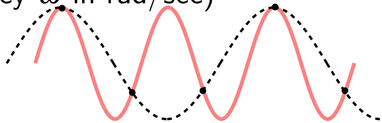


Figure: Sampling example (source: Wikipedia.org)

- the sampling frequency $= \frac{2\pi}{T_s}$
- Shannon's sampling theorem: the Nyquist frequency ($\triangleq \frac{\pi}{T_s}$) must satisfy

$$-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}$$

Approximation of continuous-time controllers

bilinear transform

formula:

$$\boxed{s = \frac{2}{T_s}\frac{z-1}{z+1} \qquad z = \frac{1+\frac{T_s}{2}s}{1-\frac{T_s}{2}s}} \tag{1}$$

intuition:

$$z = \mathrm{e}^{sT_s} = \frac{\mathrm{e}^{sT_s/2}}{\mathrm{e}^{-sT_s/2}} \approx \frac{1+\frac{T_s}{2}s}{1-\frac{T_s}{2}s}$$

implementation: start with $G(s)$, obtain the discrete implementation

$$G_d(z) = G(s)\big|_{s=\frac{2}{T_s}\frac{z-1}{z+1}} \tag{2}$$

Bilinear transformation maps the closed left half s-plane to the closed unit ball in z-plane

Stability reservation: $G(s)$ stable $\Longleftrightarrow$ $G_d(z)$ stable

Approximation of continuous-time controllers

history

Bilinear transform is also known as Tustin transform.

Arnold Tustin (16 July 1899 – 9 January 1994):

- British engineer, Professor at University of Birmingham and at Imperial College London
- served in the Royal Engineers in World War I
- worked a lot on electrical machines

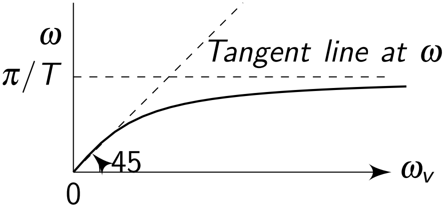
Approximation of continuous-time controllers

frequency mismatch in bilinear transform

$$\left.\frac{2}{T_s}\frac{z-1}{z+1}\right|_{z=e^{j\omega T_s}}=\frac{2}{T_s}\frac{e^{j\omega T_S/2}\left(e^{j\omega T_S/2}-e^{-j\omega T_S/2}\right)}{e^{j\omega T_S/2}\left(e^{j\omega T_S/2}+e^{-j\omega T_S/2}\right)}=j\overbrace{\frac{2}{T_s}\tan\left(\frac{\omega T_s}{2}\right)}^{\omega_v}$$

$G(s)|_{s=j\omega}$ is the true frequency response at ω; yet bilinear implementation gives,

$$G_d\left(e^{j\omega T_s}\right)=G(s)|_{s=j\omega_v}\neq G(s)|_{s=j\omega}$$



Approximation of continuous-time controllers

bilinear transform with prewarping

goal: extend bilinear transformation such that

$$G_d(z)\big|_{z=e^{j\omega T_s}} = G(s)\big|_{s=j\omega}$$

at a particular frequency ω_p

solution:

$$\boxed{s = p\frac{z-1}{z+1}, \qquad z = \frac{1+\frac{1}{p}s}{1-\frac{1}{p}s}, \qquad p = \frac{\omega_p}{\tan\left(\frac{\omega_p T_s}{2}\right)}}$$

which gives

$$G_d(z) = G(s)\big|_{s=\frac{\omega_p}{\tan\left(\frac{\omega_p T}{2}\right)}\frac{z-1}{z+1}}$$

and

$$\left.\frac{\omega_p}{\tan\left(\frac{\omega_p T_s}{2}\right)}\frac{z-1}{z+1}\right|_{z=e^{j\omega_p T_s}} = j\frac{\omega_p}{\cancel{\tan\left(\frac{\omega_p T_s}{2}\right)}}\cancel{\tan\left(\frac{\omega_p T_s}{2}\right)}$$

Approximation of continuous-time controllers

bilinear transform with prewarping

choosing a prewarping frequency ω_p:

- must be below the Nyquist frequency:

$$0 < \omega_p < \frac{\pi}{T_s}$$

- standard bilinear transform corresponds to the case where $\omega_p = 0$
- the best choice of ω_p depends on the important features in control design

example choices of ω_p:

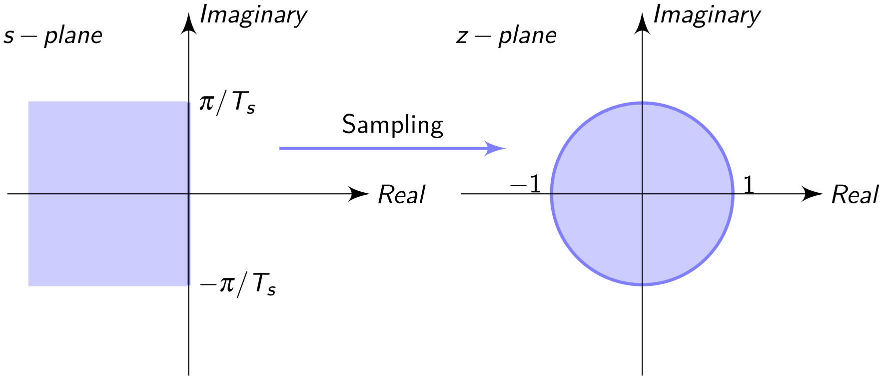
- at the cross-over frequency (which helps preserve phase margin)
- at the frequency of a critical notch for compensating system resonances

Sampling and aliasing

sampling maps the continuous-time frequency

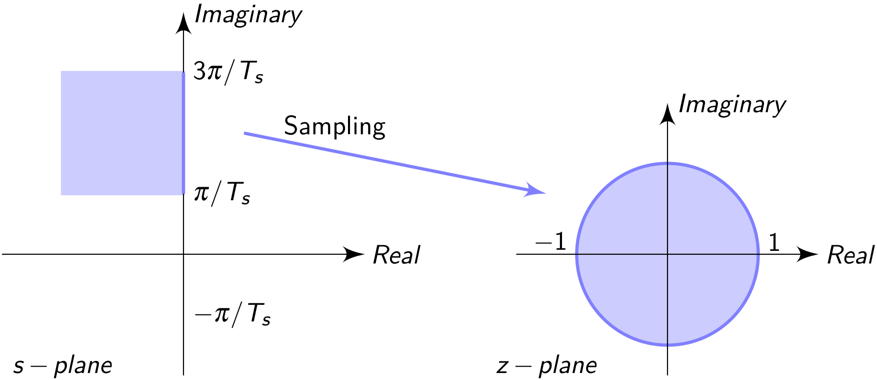
$$-\frac{\pi}{T_s} < \omega < \frac{\pi}{T_s}$$

onto the unit circle



Sampling and aliasing

sampling also maps the continuous-time frequencies $\frac{\pi}{T_s} < \omega < 3\frac{\pi}{T_s}$, $3\frac{\pi}{T_s} < \omega < 5\frac{\pi}{T_s}$, etc, onto the unit circle



Sampling and aliasing

Example (Sampling and Aliasing)

$T_s = 1/60$ sec (Nyquist frequency 30 Hz).

a continuous-time 10-Hz signal [10 Hz$\leftrightarrow 2\pi \times 10$ rad/sec $\in (-\pi/T_s, \pi/T_s)$]

$$y_1(t) = \sin(2\pi \times 10t)$$

is sampled to

$$y_1(k) = \sin\left(2\pi \times \frac{10}{60}k\right) = \sin\left(2\pi \times \frac{1}{6}k\right)$$

a 70-Hz signal [$2\pi \times 70$ rad/sec $\in (\pi/T_s, 3\pi/T_s)$]

$$y_2(t) = \sin(2\pi \times 70t)$$

is sampled to

$$y_2(k) = \sin\left(2\pi \times \frac{70}{60}k\right) = \sin\left(2\pi \times \frac{1}{6}k\right) \equiv y_1(k)!$$

Anti-aliasing

need to avoid the negative influence of *aliasing* beyond the Nyquist frequencies

- sample faster: make π/T_s large; the sampling frequency should be high enough for good control design
- anti-aliasing: perform a low-pass filter to filter out the signals $|\omega| > \pi/T_s$

Summary

1. Big picture
2. Discrete-time frequency response
3. Approximation of continuous-time controllers
4. Sampling and aliasing

Sampling example

- continuous-time signal

$$y(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases}, \quad a > 0$$

$$\mathscr{L}\{y(t)\} = \frac{1}{s+a}$$

- discrete-time sampled signal

$$y(k) = \begin{cases} e^{-aT_s k}, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$$\mathscr{Z}\{y(k)\} = \frac{1}{1 - z^{-1} e^{-aT_s}}$$

- sampling maps the continuous-time pole $s_i = -a$ to the discrete-time pole $z_i = e^{-aT_s}$, via the mapping

$$z_i = e^{s_i T_s}$$