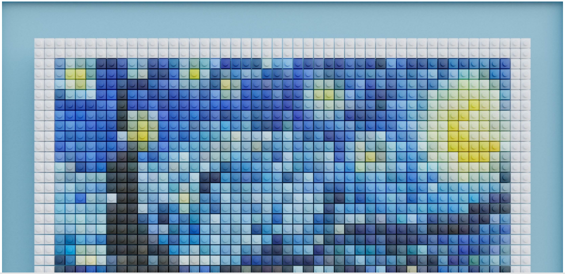
Introduction to Modern Controls

# Discretization of Continuous-time Transfer-function Models

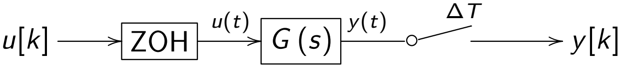


# Topic

1. Discretization of transfer-function models

# Overview

- consider the discrete-time controller implementation scheme

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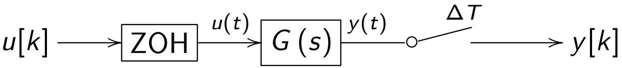


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- for this note, we use $[k]$ to distinguish DT signals from their CT counter parts

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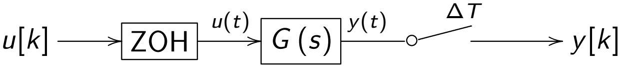


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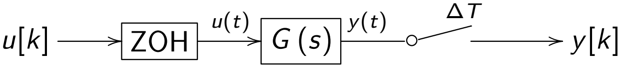


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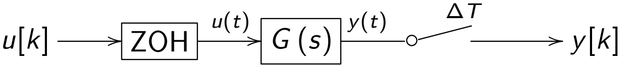


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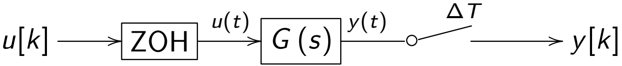


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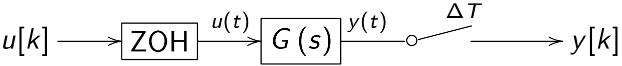


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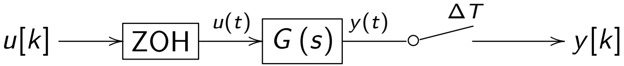


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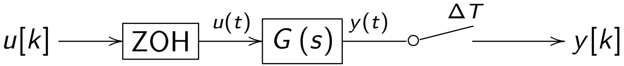
$$u(t) =$$

# Solution

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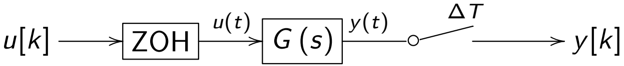
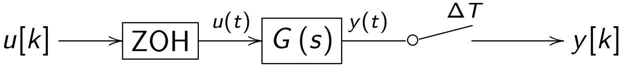



- $u[k]$ is a DT impulse $\Rightarrow$ after ZOH

$$u(t) = \begin{cases} 1, & 0 \leq t < \Delta T \\ 0, & \text{otherwise} \end{cases}$$




- $u[k]$ is a DT impulse $\Rightarrow$ after ZOH

$$u(t) = \begin{cases} 1, & 0 \leq t < \Delta T \\ 0, & \text{otherwise} \end{cases} = 1(t) - 1(t - \Delta T)$$

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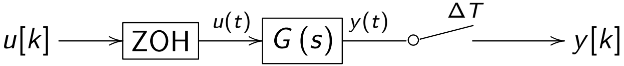


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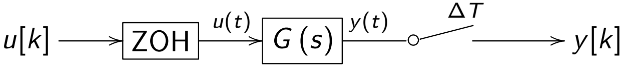


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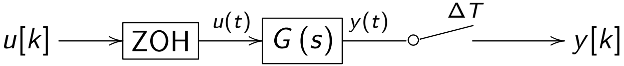
$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1 - e^{-s\Delta T}}{s}\right]$$

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$$y(t) = \mathcal{L}^{-1}\left[G(s)\frac{1 - e^{-s\Delta T}}{s}\right] = \mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right] - \mathcal{L}^{-1}\left[G(s)\frac{e^{-s\Delta T}}{s}\right]$$

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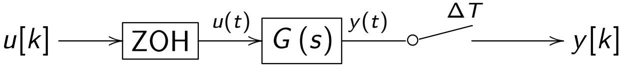


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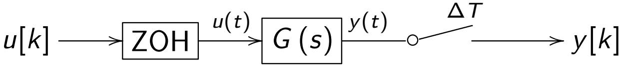


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$$\underbrace{Y(z)}_{=G(z)\times 1} = \mathcal{Z}\left\{ \underbrace{\overbrace{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]}^{\tilde{y}(t)}\Bigg|_{t=k\Delta T}}_{\triangleq \tilde{y}[k]} \right. \quad -$$

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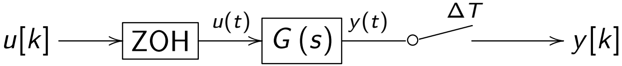


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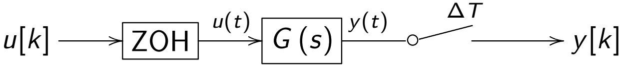


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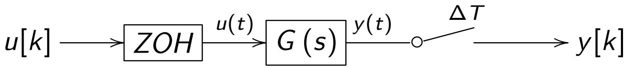

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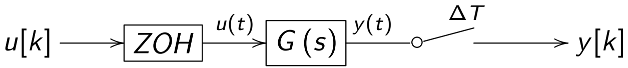


- $u[k] = \delta[k]$ $(U(z) = 1)$

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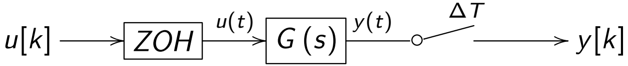


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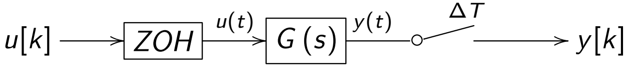


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- $\Rightarrow$the zero order hold equivalent of $G(s)$ is

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left[G(s)\frac{1}{s}\right]\Bigg|_{t=k\Delta T}\right\}$$

where $\Delta T$ is the sampling time

# Example: obtain the ZOH equivalent of

$$G(s) = \frac{a}{s + a}$$

- $\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$

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- sampling at $\Delta T$ gives $1[k] - e^{-ak\Delta T}1[k]$, whose Z transform is
$$\frac{z}{z-1} - \frac{z}{z-e^{-a\Delta T}} = \frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})}$$
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$$(1-z^{-1})\frac{z(1-e^{-a\Delta T})}{(z-1)(z-e^{-a\Delta T})} = \frac{1-e^{-a\Delta T}}{z-e^{-a\Delta T}}$$

# Matlab and Python commands

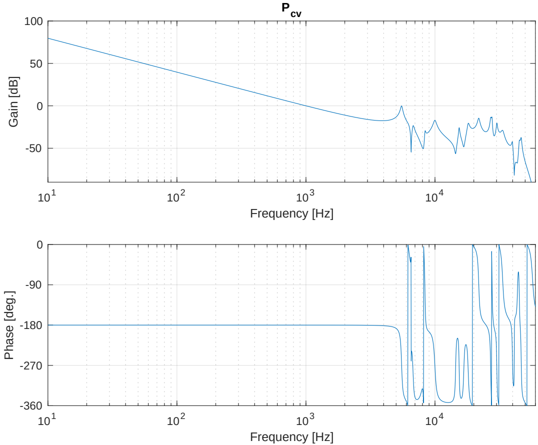
- *c2d(G,dt,'zoh')*
- e.g., $G(s) = \frac{1}{s^2}$ and $\Delta T = 1$

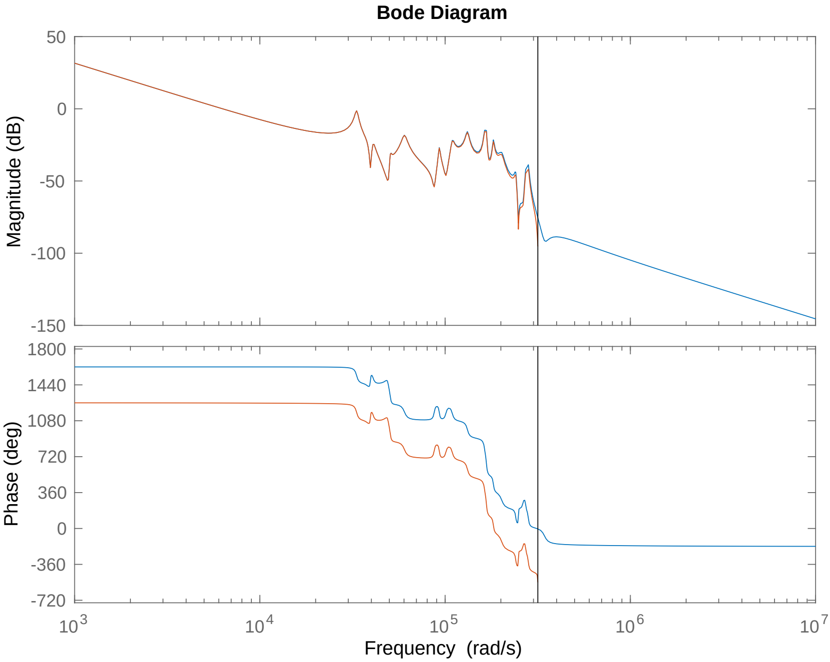
```
% matlab
dt = 1;
num = 1;
den = [1,0,0];
G = tf(num,den);
Gd = c2d(G,dt,'zoh');
```

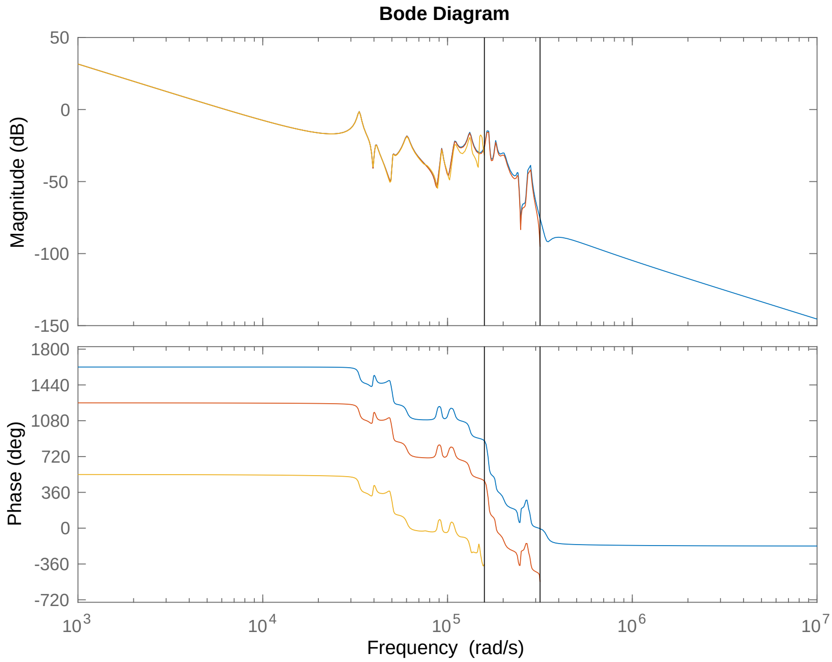
```
#Python
import control as ct
dt = 1
num = [1]
den = [1,0,0]
G = ct.tf(num,den)
Gd = ct.c2d(G,dt,'zoh')
print(Gd)
```

```
% MATLAB code to generate a single-stage HDD model
num_sector=420;                     % Number of sector
num_rpm=7200;                       % Number of RPM
Ts = 1/(num_rpm/60*num_sector);    % Sampling time

Kp_vcm=3.7976e+07;                  % VCM gain
omega_vcm=[0, 5300 ,6100 ,6500 ,8050 ,9600 ,14800 ,17400 ,21000 ,26000
↪   ,26600 ,29000 ,32200 ,38300 ,43300 ,44800]*2*pi;
kappa_vcm=[1, -1.0 ,+0.1 ,-0.1 ,0.04 ,-0.7 ,-0.2  ,-1.0  ,+3.0  ,-3.2
↪   ,2.1   ,-1.5  ,+2.0  ,-0.2  ,+0.3  ,-0.5 ];
zeta_vcm =[0, 0.02 ,0.04 ,0.02 ,0.01 ,0.03 ,0.01  ,0.02  ,0.02  ,0.012
↪   ,0.007 ,0.01  ,0.03  ,0.01  ,0.01  ,0.01 ];

Sys_Pc_vcm_c1=0;
for i=1:length(omega_vcm)
        Sys_Pc_vcm_c1=Sys_Pc_vcm_c1+tf([0,0,kappa_vcm(i)]*Kp_vcm,[1,
        ↪   2*zeta_vcm(i)*omega_vcm(i), (omega_vcm(i))^2]);
end

Sys_Pd_vcm_c1 = c2d(Sys_Pc_vcm_c1,Ts,'zoh')
Sys_Pd1_vcm_c1 = c2d(Sys_Pc_vcm_c1,Ts*2,'zoh')
Sys_Pd2_vcm_c1 = c2d(Sys_Pc_vcm_c1,Ts/2,'zoh')
```

```
figure, bodeplot(Sys_Pc_vcm_c1)
figure, bodeplot(Sys_Pc_vcm_c1,Sys_Pd2_vcm_c1)
figure, bodeplot(Sys_Pc_vcm_c1,Sys_Pd2_vcm_c1,Sys_Pd_vcm_c1)
figure,
↪ bodeplot(Sys_Pc_vcm_c1,Sys_Pd2_vcm_c1,Sys_Pd_vcm_c1,Sys_Pd1_vcm_c1)
```




Bode Diagram
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
1800
1440
1080
720
360
0
-360
-720
$10^3$
$10^4$
$10^5$
$10^6$
$10^7$
Frequency (rad/s)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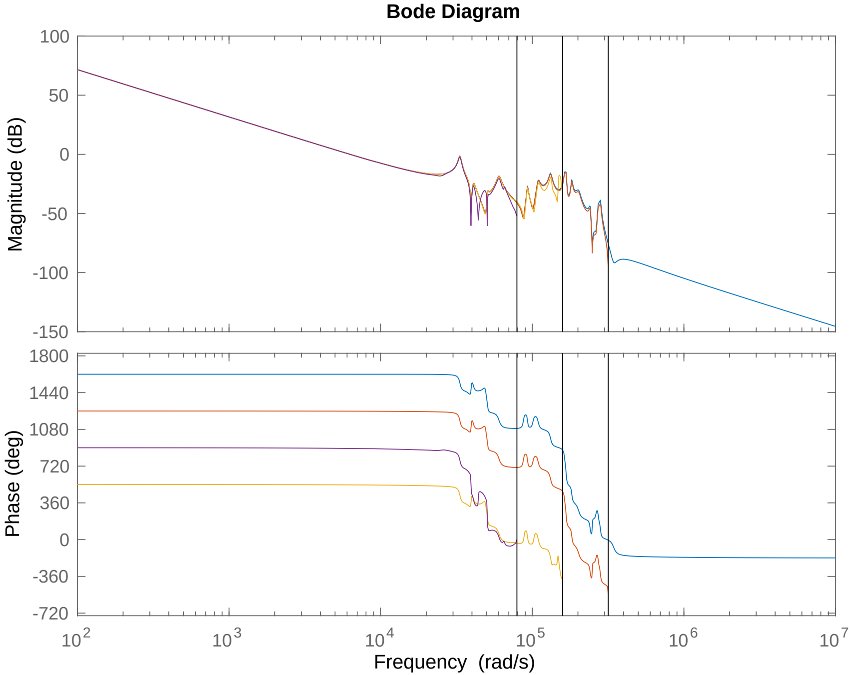
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-720
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$10^3$
$10^4$
$10^5$
$10^6$
$10^7$
Frequency (rad/s)

# Exercise

Find the zero order hold equivalent of $G(s) = e^{-Ls}$, $2\Delta T < L < 3\Delta T$, where $\Delta T$ is the sampling time.