Introduction to Modern Controls

State-Space Dynamic System Models

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- how much information from the past is needed?

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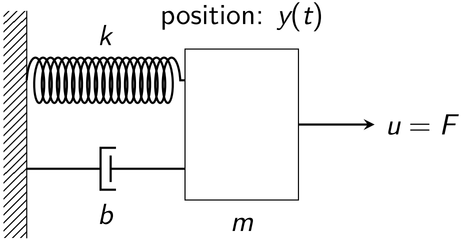
- loosely speaking:
  - the “aggregated effect of past inputs”
  - the necessary “memory” that the dynamic system keeps at each time instance

Example



- to predict the future motion, we need to know

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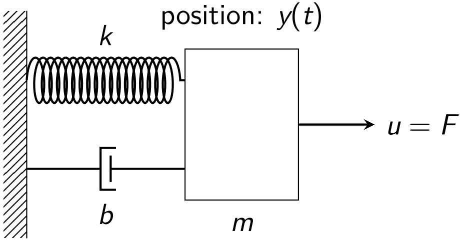

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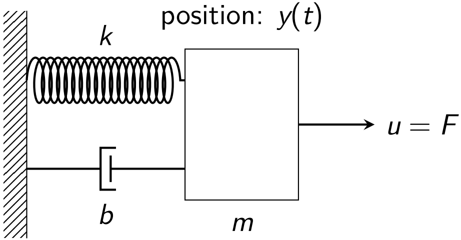

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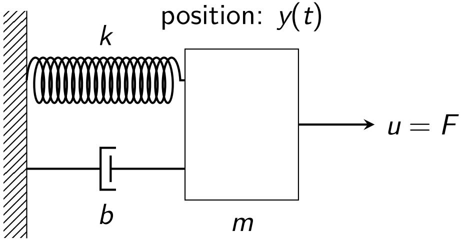
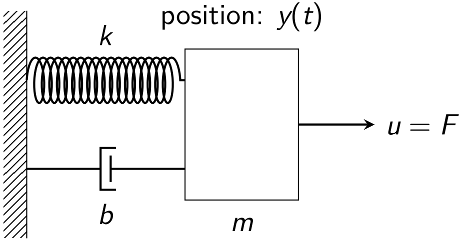


- to predict the future motion, we need to know
  - *current* position and velocity
  - *future* force
- $\Rightarrow$ states: position and velocity

The order of a dynamic system



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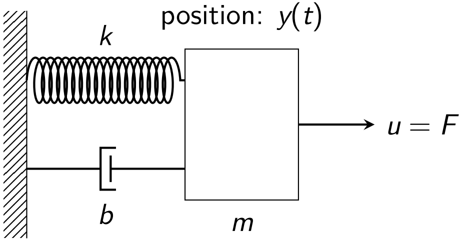

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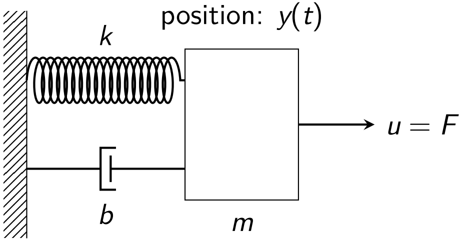

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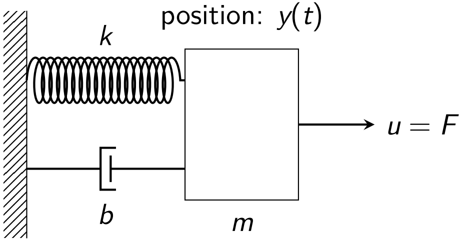

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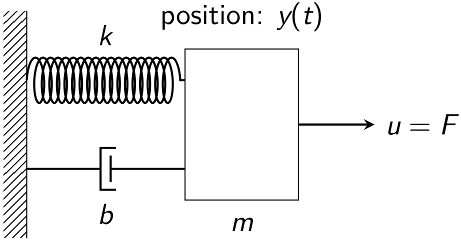

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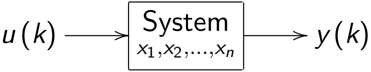
  - however, its order n is fixed
  - i.e. you need not more than n but not less than n state variables

States of a discrete-time system

consider a discrete-time dynamic system:



- the state at any instance k_o is the minimum set of variables,

$$x_1(k_o), x_2(k_o), \cdots, x_n(k_o)$$

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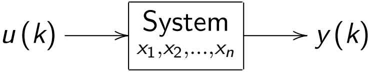

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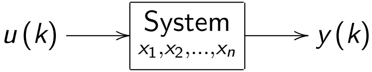
- loosely speaking, $x_1(k_o), x_2(k_o), \cdots, x_n(k_o)$ defines the system's memory

Discrete-time state-space description



general case

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\begin{aligned}
x(k+1) &= f(x(k), u(k), k) \\
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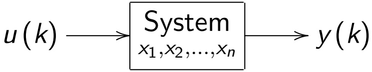

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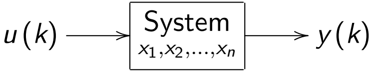

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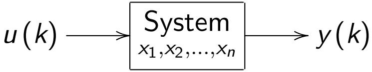

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$$u(k) \longrightarrow \boxed{\begin{array}{c}\text{System}\\ x_1, x_2, \ldots, x_n\end{array}} \longrightarrow y(k)$$

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linear time-invariant (LTI) case

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

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- $\Sigma(A, B, C, D)$ denotes a state-space realization
- also written as $\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Continuous-time state-space description

$$u(t) \longrightarrow \boxed{\begin{array}{c}\text{System}\\ x_1, x_2, \ldots, x_n\end{array}} \longrightarrow y(t)$$

general case

$$\begin{aligned}\frac{dx(t)}{dt} &= f(x(t), u(t), t)\\ y(t) &= h(x(t), u(t), t)\end{aligned}$$

LTI case

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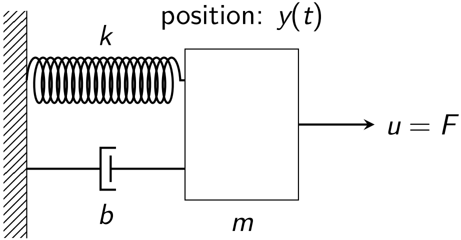
Example: mass-spring-damper



$$x(t) = \begin{bmatrix} \overbrace{y(t)}^{\text{mass position}} \\ \underbrace{v(t)}_{\text{mass velocity}} \end{bmatrix} \in \mathbb{R}^2$$

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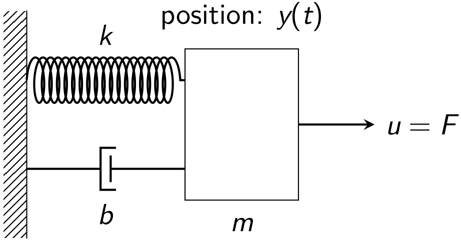

$$
\begin{aligned}
\frac{d}{dt}\underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} &= \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t) \\
y(t) &= \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)}
\end{aligned}
$$

Coding a continuous-time state-space system in MATLAB

```
A = [0,1;-3,-2];
B = [0;1];
C = [2,1];
D = 0;
sys_ss = ss(A,B,C,D)

[yout, T] = step(sys_ss);
figure, plot(T, yout)
```

Coding a continuous-time state-space system in Python

```python
import control as co
import matplotlib.pyplot as plt
import numpy as np
A = np.array([[0,1],[-3,-2]])
B = np.array([[0],[1]])
C = np.array([2,1])
D = np.array([0])

sys_ss = co.ss(A,B,C,D)
print(sys_ss)

T,yout = co.step_response(sys_ss)

plt.figure(1,figsize = (6,4))
plt.plot(T,yout)
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()
```