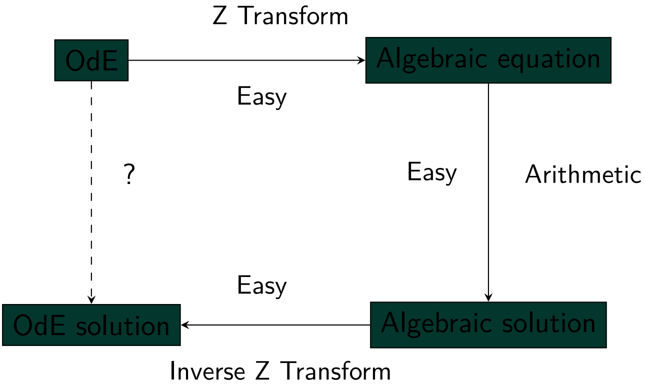
Introduction to Modern Controls

Z transform

The Z transform approach to Ordinary difference Equations (OdEs)



- analogous to Laplace transform for continuous-time signals

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$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots$$

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  where $z \in \mathbb{C}$
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- the series $1 + \gamma + \gamma^2 + \ldots$ converges to $\frac{1}{1-\gamma}$ for $|\gamma| < 1$ [region of convergence (ROC)]
- (also, recall that $\sum_{k=0}^{N} \gamma^k = \frac{1-\gamma^{N+1}}{1-\gamma}$ if $\gamma \neq 1$)

Example: geometric sequence $\{a^k\}_{k=0}^{\infty}$

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- $x(k) = a^k$
- $\mathcal{Z}\{a^k\} = \sum_{k=0}^{\infty} a^k z^{-k} = \boxed{\dfrac{1}{1-az^{-1}}} = \frac{z}{z-a}$

Example: step sequence (discrete-time unit step function)

$$\boxed{\mathcal{Z}\{a^k\} = \frac{1}{1 - az^{-1}}}$$

- $1(k) = \begin{cases} 1, & \forall k = 1, 2, \ldots \\ 0, & \forall k = \ldots, -1, 0 \end{cases}$

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- $1(k) = \begin{cases} 1, & \forall k = 1, 2, \ldots \\ 0, & \forall k = \ldots, -1, 0 \end{cases}$
- $\mathcal{Z}\{1(k)\} = \mathcal{Z}\{a^k\}\big|_{a=1} = \boxed{\dfrac{1}{1 - z^{-1}}} = \frac{z}{z-1}$

Example: discrete-time impulse

- $\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$

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- $\mathcal{Z}\{\delta(k)\} = 1$

Exercise: $\cos(\omega_0 k)$

$f(k)$	$F(z)$	ROC
$\delta(k)$	1	All z
$a^k 1(k)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^k 1(-k-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$ka^k 1(k)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-ka^k 1(-k-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\cos(\omega_0 k)$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
$\sin(\omega_0 k)$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\|z\| > 1$
$a^k\cos(\omega_0 k)$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$
$a^k\sin(\omega_0 k)$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\|z\| > \|a\|$

Properties of Z transform: time shift

- let $\mathcal{Z}\{x(k)\} = X(z)$ and $x(k) = 0 \; \forall k < 0$

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- one-step delay:

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\begin{aligned}
\mathcal{Z}\{x(k-1)\} &= \sum_{k=0}^{\infty} x(k-1)z^{-k} = \sum_{k=1}^{\infty} x(k-1)z^{-k} + x(-1) \\
&= \sum_{k=1}^{\infty} x(k-1)z^{-(k-1)}z^{-1} + x(-1) \\
&= z^{-1}X(z) + \cancel{x(-1)} = \boxed{z^{-1}X(z)}
\end{aligned}
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- analogously, $\underline{\mathcal{Z}\{x(k+1)\}} = \sum_{k=0}^{\infty} x(k+1)z^{-k} \boxed{= zX(z) - zx(0)}$

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- analogously, $\underline{\mathcal{Z}\{x(k+1)\}} = \sum_{k=0}^{\infty} x(k+1)z^{-k} \boxed{= zX(z) - zx(0)}$
- thus, if $x(k+1) = Ax(k) + Bu(k)$ and $x(0) = 0$,

$$zX(z) = AX(z) + BU(z) \Rightarrow X(z) = (zI - A)^{-1}BU(z)$$

provided that $(zI - A)$ is invertible

Solving difference equations

Solve the difference equation

$$y(k) + 3y(k-1) + 2y(k-2) = u(k-2)$$

where $y(-2) = y(-1) = 0$ and $u(k) = 1(k)$.

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- $\Rightarrow \boxed{Y(z) = \frac{1}{z^2 + 3z + 2}U(z)}$

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- $\Rightarrow Y(z) = \frac{z}{(z-1)(z+2)(z+1)} = \frac{1}{6}\frac{z}{z-1} + \frac{1}{3}\frac{z}{z+2} - \frac{1}{2}\frac{z}{z+1}$ (careful with the partial fraction expansion)

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- inverse Z transform then gives $y(k) = \frac{1}{6}1(k) + \frac{1}{3}(-2)^k - \frac{1}{2}(-1)^k,\ k \geq 0$

From difference equation to transfer functions

- general discrete-time OdE:

$$y(k)+a_{n-1}y(k-1)+\cdots+a_0 y(k-n) = b_m u(k+m-n)+\cdots+b_0 u(k-n)$$

where $y(k) = 0 \ \forall k < 0$

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- applying Z transform to the OdE yields

  $$\left(z^n + a_{n-1}z^{n-1} + \cdots + a_0\right) Y(z) = \left(b_mz^m + b_{m-1}z^{m-1} + \cdots + b_0\right) U(z)$$

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- hence

$$Y(z) = \underbrace{\frac{b_m z^m + b_{m-1}z^{m-1} \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}}_{G_{yu}(z)\text{: discrete-time transfer function}} U(z)$$

DC gain of discrete-time transfer functions

- general discrete-time OdE and transfer function:

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- assuming constant input and convergent output, then at steady state,
  - $y(k) = y(k-1) = \cdots = y(k-n) \triangleq y_{ss}$ and
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  - $y_{ss} + a_{n-1}y_{ss} + \cdots + a_0y_{ss} = b_mu_{ss} + \cdots + b_0u_{ss}$

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  - $y_{ss}+a_{n-1}y_{ss}+\cdots+a_0y_{ss}=b_mu_{ss}+\cdots+b_0u_{ss}$
- thus,

$$\underline{\text{DC gain of } G_{yu}(z)} = \frac{b_m+b_{m-1}+\cdots+b_0}{1+a_{n-1}+\cdots+a_0} = \underline{G_{yu}(z)\big|_{z=1}}$$

Transfer functions in two domains

$$y(k) + a_{n-1}y(k-1) + \cdots + a_0 y(k-n) = b_m u(k+m-n) + \cdots + b_0 u(k-n)$$

$$\iff G_{yu}(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

v.s.

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_0 u(t)$$

$$\iff G_{yu}(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

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Properties	$G_{yu}(s)$	$G_{yu}(z)$
poles and zeros	roots of $A(s)$ and $B(s)$	roots of $A(z)$ and $B(z)$
causality condition	$n \geq m$	$n \geq m$
DC gain / steady-state response to unit step	$G_{yu}(0)$	$G_{yu}(1)$

Coding a discrete-time transfer function

```
num = [0.09952, -0.08144];
den = [1, -1.792, 0.8187];
Ts = 0.1;
sys_tf = tf(num,den,Ts)
poles = pole(sys_tf);
zeros = zero(sys_tf);
disp(['System Poles = ',num2str(poles')])
disp(['System Zeros = ',num2str(zeros')])

[yout, T] = step(sys_tf);
figure, stairs(T, yout)
figure, impulse(sys_tf)

u1 = 2*ones(length(T),1);
u2 = sin(T);
figure, lsim(sys_tf,u1,T)
figure, lsim(sys_tf,u2,T)
```

```python
import control as co
import matplotlib.pyplot as plt
import numpy as np
Ts = 0.1 # sampling time
num = [0.09952, -0.08144] # Numerator co-efficients
den = [1, -1.792, 0.8187] # Denominator co-efficients
sys_tf = co.tf(num,den, Ts)
print(sys_tf)

poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout = co.step_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.step(T,np.append(0,yout[0:-1]))
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()
```

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poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout_i = co.impulse_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.step(T,np.append(0,yout_i[0:-1]))
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()
```

Additional useful properties of Z transform

- time shifting (assuming $x(k) = 0$ if $k < 0$):

$$\mathcal{Z}\left\{x(k - n_d)\right\} = z^{-n_d} X(z)$$

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- differentiation: $\mathcal{Z}\left\{k x\left(k\right)\right\} = -z \frac{dX(z)}{dz}$

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- Z-domain scaling: $\mathcal{Z}\left\{a^k x\left(k\right)\right\} = X\left(a^{-1} z\right)$
- differentiation: $\mathcal{Z}\left\{k x\left(k\right)\right\} = -z \frac{dX(z)}{dz}$
- time reversal: $\mathcal{Z}\left\{x\left(-k\right)\right\} = X\left(z^{-1}\right)$

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- convolution: let $f(k) * g(k) \triangleq \sum_{j=0}^{k} f(k-j)\, g(j)$, then

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