Lyapunov Stability

1. Definitions in Lyapunov stability analysis

2. Lyapunov's approach to stability
    - Relevant tools
    - Lyapunov stability theorems
    - Instability theorem
    - Discrete-time case

3. Recap

Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

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default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

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- the condition must be satisfied by all $t \geq 0$
- if a system starts at equilibrium state, it stays there

Equilibrium state of a linear system

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- when $A(t)$ is singular, multiple equilibrium states exist

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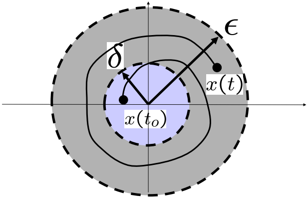

Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \ \forall t \geq t_0$.

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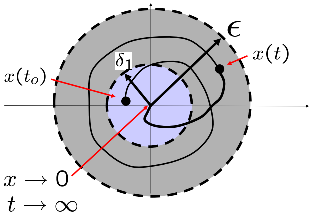

Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \rightarrow 0$.

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- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2 + \omega^2}\right| = |\lambda| < 1$

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- no need for explicit solutions to system responses
- an "energy" perspective
- fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \qquad\qquad (x_1\text{: position; } x_2 \text{ : velocity})$$

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- goal is to relate properties of the state through the Lyapunov function
- main tool: matrix formulation, linear algebra, positive definite functions

Relevant tools

Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- general quadratic functions in matrix form

$$Q(x) = x^T P x, \; P^T = P$$

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$$\text{general case: } P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

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namely, $a_j^T a_j = 1$ and $a_j^T a_m = 0 \ \forall j \neq m$.

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$$\begin{aligned}
\overline{\overline{u}^T Au} &= u^T \overline{A}\overline{u} \\
&= u^T A\overline{u} \quad \because A \in \mathbb{R}^{n\times n} \\
&= u^T A^T \overline{u} \quad \because A = A^T \\
&= \lambda u^T \overline{u} \quad \because (Au)^T = (\lambda u)^T \\
&= \lambda \overline{u}^T u \quad \because u^T \overline{u} \in \mathbb{R} \\
&= \overline{u}^T Au \quad \because Au = \lambda u
\end{aligned}$$

Theorem

The eigenvalues of symmetric matrices are all real.

Proof: $\forall : A \in \mathbb{R}^{n \times n}$ with $A^T = A$.

Eigenvalue-eigenvector pair: $Au = \lambda u \Rightarrow \overline{u}^T A u = \lambda \overline{u}^T u$, where $\overline{u}$ is the complex conjugate of u. $\overline{u}^T A u$ is a real number, as

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Also, $\overline{u}^T u \in \mathbb{R}$.

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Also, $\overline{u}^T u \in \mathbb{R}$. Thus $\lambda = \frac{\overline{u}^T Au}{\overline{u}^T u}$ must also be a real number. $\square$

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```
import numpy as np #larger-scale Python example
N = 100
P = np.random.randint(-200,200,size=(N,N))
P_symm = (P + P.T)/2
lambdas, _ = np.linalg.eig(P_symm)
print(lambdas)
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```
import numpy as np
from scipy.linalg import qr
n = 3
H = np.random.randn(n, n)
Q, _ = qr(H)
print (np.dot(Q,Q.T))
print (np.dot(Q.T,Q))
```

Important properties of symmetric matrices

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matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

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Theorem (Symmetric eigenvalue decomposition (SED))

$\forall : A \in \mathbb{R}^{n \times n},\ A^T = A$, there always exist $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$, s.t.

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- If A has distinct eigenvalues, then $U = [u_1, u_2, \cdots, u_n]$ is orthogonal after normalizing all the eigenvectors to unity norm.
- If A has $r(< n)$ distinct eigenvalues, we can *choose* multiple orthogonal eigenvectors for the eigenvalues with none-unity multiplicities.

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Example: $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5-\lambda & \sqrt{3} \\ \sqrt{3} & 7-\lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$

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If $A = A^T \in \mathbb{R}^{n \times n}$, then the eigenvalues of A satisfy

$$\lambda_{\max} = \max_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{2}$$

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- $P \succ 0$ ($P \succeq 0$) $\Leftrightarrow$ P can be decomposed as $P = N^T N$ where N is nonsingular (singular)

Negative definite matrices

Definition

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Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	$\mathbb{R}_+$ axis
negative definite	negative	$\mathbb{R}_-$ axis

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Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P = P^T$ and take any $v = [x, y]^T$, we have

$$\begin{aligned} v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\ &= x^2 + y^2 + (x - y)^2 \geq 0 \end{aligned}$$

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and the equality sign holds only when $x = y = 0$.

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$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 < 0$$

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Theorem

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Proof.

Since P is symmetric, we have

$$\lambda_{\max}(P) = \max_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{4}$$

$$\lambda_{\min}(P) = \min_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{5}$$

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which gives $x^T A x \in [\lambda_{\min} \|x\|_2^2,\ \lambda_{\max} \|x\|_2^2]$. Thus $x^T A x > 0,\ x \neq 0 \Leftrightarrow \lambda_{\min} > 0$. $\square$

Relevant tools

Checking positive definiteness of a matrix.

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Definition

The leading principle minors of $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ are defined as p_{11}, $\det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\det P$.

Example

None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Recap

$$
\begin{array}{c}
\text{energy} \\
\updownarrow \\
\text{quadratic function} \\
\updownarrow \\
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\end{array}
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Recap

$$\begin{tikzcd}
\text{energy} \arrow[r] \arrow[d, leftrightarrow] & \text{Lyapunov function} \\
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Definition (Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$, called to be PD, satisfying

- $W(x) > 0$ for all $x \neq 0$
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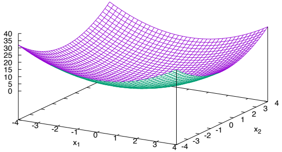

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Definition (Locally Positive Definite Functions)

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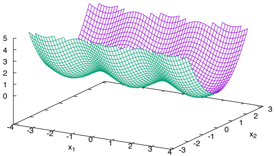

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Exercise

Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

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2. $V(x) = x_1^2 + x_2^2 + 3x_3^2 - x_3^4$ (LPD for $|x_3| < \sqrt{3}$)

Lyapunov stability theorems

- recall the spring mass damper example in matrix form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- energy function is PD:
$\mathcal{E}(t)$ = potential energy + kinetic energy $= \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$
and its derivative is NSD:

$$\dot{\mathcal{E}}(t) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right]\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 \tag{6}$$

$$= k_1 x_1 x_2 + m x_2\left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right] Ax \tag{7}$$

$$= -bx_2^2$$

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$$A^T \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} + \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

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$$A^T P + PA = Q \Leftrightarrow \begin{cases} A^T p_1 + a_{11} p_1 + a_{21} p_2 & = -q_1 \\ A^T p_2 + a_{12} p_1 + a_{22} p_2 & = -q_2 \end{cases}$$

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- can stack the columns of $A^T P + PA$ and Q to yield

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

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  - so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$

The Lyapunov Eq.: Existence of solution

$$L_A(P) = A^T P + PA$$

- L_A is invertible if and only if $\lambda_i + \lambda_j \neq 0$ for all eigenvalues of A:
  - let $A^T u_i = \lambda_i u_i$ and $A^T u_j = \lambda_j u_j$
  - $L_A\left(u_i u_j^T\right) = u_i u_j^T A + A^T u_i u_j^T = u_i(\lambda_j u_j)^T + \lambda_i u_i u_j^T = (\lambda_i + \lambda_j)\, u_i u_j^T$
  - so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$
  - if $\lambda_i + \lambda_j \neq 0$, the operator is invertible

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- can simply write $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$ using the Kronecker product notation $B \otimes C = \begin{bmatrix} b_{11}C & b_{11}C & \dots & b_{11}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$

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- e.g., $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$L_A = I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} -1-1 & -1 & -1 & 0 \\ 1 & 0-1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

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The eigenvalues of L_A are -1, -1, $-1-\sqrt{3}$, $-1+\sqrt{3}$, which are precisely $\lambda_1+\lambda_1$, $\lambda_1+\lambda_2$, $\lambda_2+\lambda_1$, $\lambda_2+\lambda_2$.

```
import numpy as np
A = [[-1,1],[-1,0]]; I2=np.eye(2); AT=np.transpose(A)
L_A=np.kron(I2,AT)+np.kron(AT,I2)
eigLA,_=np.linalg.eig(L_A)
eigA,_=np.linalg.eig(A)
print(eigLA)
print(eigA)
```

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For $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if

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$$\xcancel{x^T(\infty)Px(\infty)}^{\,0} - x^T(0)Px(0) = \int_0^\infty \frac{d}{dt} x^T(t)Px(t)\,dt = \int_0^\infty x^T(t)\left(A^TP + PA\right)x(t)\,dt$$

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If $Q \succ 0$, there exists a nonsingular N matrix: $Q = N^TN$. Thus

$$x^T(0)Px(0) = \int_0^\infty \|Ne^{At}x(0)\|^2 dt \geq 0$$

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Thus $P \succ 0$. Furthermore

$$\boxed{P = \int_0^\infty e^{A^Tt}Qe^{At}dt}$$

□

Procedures of Lyapunov's direct method

1. Given A, select an arbitrary positive-definite symmetric matrix Q (e.g., I).
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   - if P is not positive-definite, then A has at least one eigenvalue with a positive real part and the origin is an unstable equilibrium.

Lyapunov stability theorems

Example

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. The Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}$$

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We need

$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

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Leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$
$\Rightarrow P \succ 0 \Rightarrow$asymptotically stable

Lyapunov analysis with Matlab

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

```
A = [-1,1;-1,0]
Q = eye(2)
P = lyap(A',Q)
w = eig(P)
```

Lyapunov analysis with Python

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

```
import control as ct
import numpy as np
A = np.array([[-1,1],[-1,0]])
Q = np.identity(2)
P = ct.lyap(A.transpose(),Q)
print(P)
w = np.linalg.eigvals(P)
print(f'eigenvalues of P: {w}')
```

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$$A^T P + PA = -I$$
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$$\underbrace{N^T A^T N^{-T}}_{\tilde{A}^T} \underbrace{N^T P N}_{\tilde{P}} + \underbrace{N^T P N}_{\tilde{P}} \underbrace{N^{-1} A N}_{\tilde{A}} = -N^T N$$

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  - $\tilde{A} = N^{-1}AN$ and A are similar matrices and have the same eigenvalues.
  - $\tilde{P} = N^T P N$ and P have the same definiteness. If we can find a positive definite solution P then the $\tilde{P}$ will also be positive definite. Vise versa.

Instability theorem

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Theorem

The equilibrium state 0 *of* $\dot{x} = f(x)$ *is unstable if there exists a function* $W(x)$ *such that*

- $\dot{W}(x)$ *is PD locally:* $\dot{W}(x) > 0 \; \forall \, |x| < r$ *for some* r *and* $\dot{W}(0) = 0$
- $W(0) = 0$
- *there exist states* x *arbitrarily close to the origin such that* $W(x) > 0$

Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x,\ \ P = P^T \succ 0$$

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and compute $\Delta V(x)$ along the trajectory of the state

$$V(x(k+1)) - V(x(k)) = x^T(k) \underbrace{\left(A^T PA - P\right)}_{\triangleq -Q} x(k)$$

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Asymptotic stability desires $\Delta V(x)$ to be negative.

DT Lyapunov stability theorem for linear systems

Theorem

For system $x(k+1) = Ax(k)$ *with* $A \in \mathbb{R}^{n \times n}$, *the origin is asymptotically stable if and only if* $\exists\ Q \succ 0$, *such that the discrete-time Lyapunov equation*

$$\boxed{A^T P A - P = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

The DT Lyapunov Eq.

$$\boxed{A^T P A - P = -Q}$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur stable), comes from:

$$\begin{aligned}
\cancelto{0}{V\left(x\left(\infty\right)\right)} - V\left(x\left(0\right)\right) &= \sum_{k=0}^{\infty} x^T\left(k\right)\left[A^T P A - P\right] x\left(k\right) \\
&= -\sum_{k=0}^{\infty} x^T\left(0\right)\left(A^T\right)^k Q A^k x\left(0\right) \\
\Rightarrow P &= \sum_{k=0}^{\infty} \left(A^T\right)^k Q A^k
\end{aligned}$$

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$$\cancelto{0}{V\left(x\left(\infty\right)\right)} - V\left(x\left(0\right)\right) = \sum_{k=0}^{\infty} x^T\left(k\right)\left[A^T PA - P\right]x\left(k\right)$$

$$= -\sum_{k=0}^{\infty} x^T\left(0\right)\left(A^T\right)^k QA^k x\left(0\right)$$

$$\Rightarrow P = \sum_{k=0}^{\infty}\left(A^T\right)^k QA^k$$

- can show that the DT Lyapunov operator $L_A = A^T PA - P$ is invertible if and only if $\forall i, j \ (\lambda_A)_i (\lambda_A)_j \neq 1$

DT Lyapunov analysis with MATLAB

Example

$$x(k+1) = Ax(k), \;\; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
% MATLAB
A=[ 0 1 0; 0 0 1; 0.275 -0.225 -0.1]
Q = eye(3)
P = dlyap(A',Q) % check function definition in Matlab help
eig(P)
```

DT Lyapunov analysis with Python

Example

$$x(k+1) = Ax(k),\ \ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
#Python
import control as ct
import numpy as np
from numpy.linalg import eig
A = np.array([[0,1,0],[0,0,1],[0.275,-0.225,-0.1]])
Q = np.identity(3)
P = ct.dlyap(A.transpose(),Q)
w,v = eig(P)
print(w)
```

Recap

- Internal stability
  - Stability in the sense of Lyapunov: ε, δ conditions
  - Asymptotic stability
- Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
  - Based on the eigenvalues of A
    - Time response modes
    - Repeated eigenvalues on the imaginary axis
  - Routh's criterion
    - No need to solve the characteristic equation
    - Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)

Recap

- Lyapunov equations
  Theorem: All eigenvalues of A have negative real parts iff for any given $Q \succ 0$, the Lyapunov equation
  $$A^T P + PA = -Q$$
  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T P + PA = -Q$ has a unique solution when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j.
  Theorem: All eigenvalues of A are inside the unit circle iff for any given $Q \succ 0$, the Lyapunov equation
  $$A^T PA - P = -Q$$
  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T PA - P = -Q$ has a unique solution when $\lambda_{A,i}\lambda_{A,j} \neq 1$ for all i and j.

Recap

- P is positive definite if and only if any one of the following conditions holds:
    1. All the eigenvalues of P are positive.
    2. All the leading principle minors of P are positive.
    3. There exists a nonsingular matrix N such that $P = N^T N$.