Introduction to Modern Controls

Relationship Between State-Space Models and Transfer Functions

Continuous-time LTI state-space description

$$u(t) \longrightarrow \boxed{\begin{array}{c}\text{System}\\ x_1, x_2, \ldots, x_n\end{array}} \longrightarrow y(t)$$

$$\begin{aligned}\frac{d}{dt}x(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Recap: LTI input/output description

$$u(t) \longrightarrow \boxed{\begin{array}{c}\text{System}\\ x_1,x_2,\ldots,x_n\end{array}} \longrightarrow y(t)$$

let $u(t) \in \mathbb{R}$ and $y(t) \in \mathbb{R}$, then

$$\boxed{\begin{aligned} y(t) &= (g \star u)(t) \\ &= \int_0^t g(t-\tau)u(\tau)d\tau \end{aligned}}$$

where $g(t)$ is the system's impulse response
Laplace domain:

$$\boxed{Y(s) = G(s)U(s)}$$

$Y(s) = \mathcal{L}\{y(t)\}, U(s) = \mathcal{L}\{u(t)\}, G(s) = \mathcal{L}\{g(t)\}$

From state space to transfer function

given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$\overset{\mathcal{L}}{\Rightarrow}$

$$\begin{aligned} sX(s) - x(0) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$

when $x(0) = 0$, we have

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \triangleq: G(s)}$$

–the transfer function between u and y

Analogously for discrete-time systems

for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$,

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

$\stackrel{\mathcal{Z}}{\Rightarrow}$

$$\begin{aligned} zX(z) - zx(0) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

when $x(0) = 0$, we have

$$\boxed{\frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D \triangleq: G(z)}$$

–the transfer function between u and y

From state space to transfer function: Observations

$$\frac{d}{dt}x(t) = A_{n\times n}x(t) + B_{n\times 1}u(t)$$
$$y(t) = C_{1\times n}x(t) + Du(t)$$

- dimensions:

$$G(s) = \underbrace{C}_{1\times n}\underbrace{(sI - A)^{-1}}_{n\times n}\underbrace{B}_{n\times 1} + D$$

$$\Sigma = \left[\begin{array}{c|c} A_{n\times n} & B_{n\times 1} \\ \hline C_{1\times n} & D_{1\times 1} \end{array}\right]$$

- uniqueness: $G(s)$ is unique given the state-space model

Matrix inverse

$$M^{-1} = \frac{1}{\det(M)} \mathrm{Adj}(M)$$

where $\mathrm{Adj}(M) = \{\text{Cofactor matrix of } M\}^T$

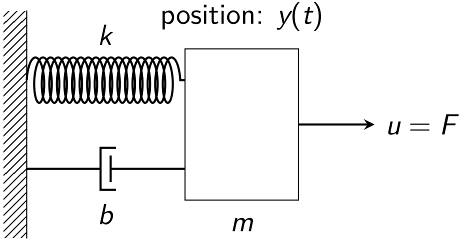
e.g.: $M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$, $\{\text{Cofactor matrix of } M\} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

where $c_{11} = \begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 24$, $c_{12} = -\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} = 5$, $c_{13} = \begin{vmatrix} 0 & 4 \\ 1 & 0 \end{vmatrix} = -4$,

$c_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 6 \end{vmatrix} = -12$, $c_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} = 3$, $c_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$,

$c_{31} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = -2$, $c_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} = -5$, $c_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} = 4$

Mass-spring-damper



$$
\frac{d}{dt}\underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t)
$$

$$
y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)}
$$

Mass-spring-damper

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ v(t) \end{bmatrix} \end{aligned}$$

$$\boxed{G(s) = C(sI - A)^{-1}B + D}$$

$\Rightarrow$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \right]^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

Mass-spring-damper

$$\left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}\right]^{-1} = \begin{bmatrix} s & -1 \\ \frac{k}{m} & s+\frac{b}{m} \end{bmatrix}^{-1}$$
$$= \frac{1}{s^2+\frac{b}{m}s+\frac{k}{m}} \begin{bmatrix} s+\frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

Mass-spring-damper

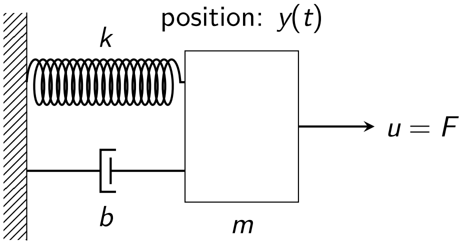
Putting the inverse in yields

$$\begin{aligned} G(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \\ &= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}{s^2 + \frac{b}{m}s + \frac{k}{m}} \end{aligned}$$

namely

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

Numerical example in MATLAB




```
m = 1; k = 2; b = 1;
A = [0 1; -k/m -b/m];
B = [0; 1/m];
C = [1 0];
D = 0;
sys = ss(A,B,C,D)
[num,den] = ss2tf(A,B,C,D);
sys_tf = tf(num,den)
figure, step(sys)
figure, step(sys_tf)
```

Numerical example in Python

```python
import control as co
import numpy as np
m = 1
k = 2
b = 1
A = np.array([[0,1],[-k/m,-b/m]])
B = np.array([[0], [1/m]])
C = np.array([1,0])
D = np.array([0])
sys = co.ss(A,B,C,D)
print(sys)
sys_tf = co.ss2tf(sys)
print(sys_tf)

print(co.poles(sys))
print(co.poles(sys_tf))
```

Exercise

Given the following state-space system parameters: $A = \begin{bmatrix} 0 & -6 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -6 & 0 & -3 \\ -2 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, obtain the transfer function $G(s)$.