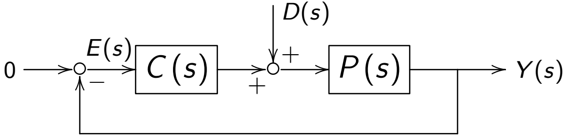
Internal Model Principle and Repetitive Control

Big picture

review of integral control in PID design

example:



where

$$P(s) = \frac{1}{ms+b},\ \ C(s) = k_p + k_i\frac{1}{s} + k_d s,\ \ k_p, k_i, k_d > 0$$

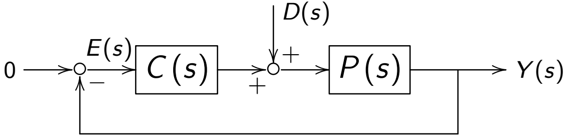
- the integral action in PID control perfectly rejects (asymptotically) constant disturbances ($D(s) = d_o/s$):

$$E(s) = \frac{-P(s)}{1+P(s)C(s)}D(s) = \frac{-d_o}{(m+k_d)s^2 + (k_p+b)s + k_i}$$

$$\Rightarrow e(t) \to 0$$

Big picture

review of integral control in PID design



the "structure" of the reference/disturbance is built into the integral controller:

- controller:

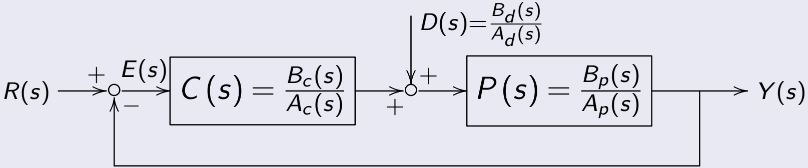
$$C(s) = k_p + k_i \frac{1}{s} + k_d s = \boxed{\frac{1}{s}} \left(k_d s^2 + k_p s + k_i \right)$$

- constant disturbance:

$$d(t) = d_o \Leftrightarrow \mathscr{L}\{d(t)\} = \boxed{\frac{1}{s}} d_o$$

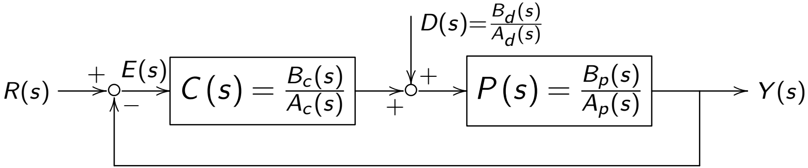
General case: internal model principle (IMP)

Theorem (Internal Model Principle)



Assume $B_p(s) = 0$ *and* $A_d(s) = 0$ *do not have common roots.*
If the closed loop is asymptotically stable,
and $A_c(s)$ *can be factorized as* $A_c(s) = A_d(s)\, A_c^{'}(s)$,
then the disturbance is asymptotically rejected.

General case: internal model principle (IMP)



Proof: The steady-state error response to the disturbance is

$$E(s) = \frac{-P(s)}{1 + P(s)C(s)} D(s) = \frac{-B_p(s)A_c(s)}{A_p(s)A_c(s) + B_p(s)B_c(s)} \frac{B_d(s)}{A_d(s)}$$

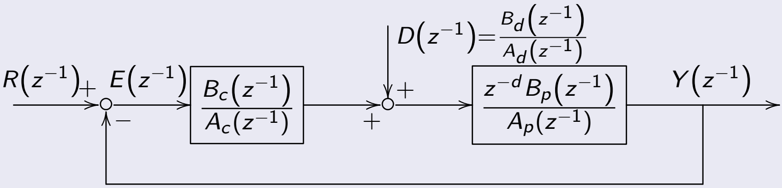
$$= \frac{-B_p(s)A_c^{'}(s)B_d(s)}{A_p(s)A_c(s) + B_p(s)B_c(s)}$$

where all roots of $A_p(s)A_c(s) + B_p(s)B_c(s) = 0$ are on the left half plane. Hence $e(t) \to 0$

Internal model principle

discrete-time case:

Theorem (Discrete-time IMP)

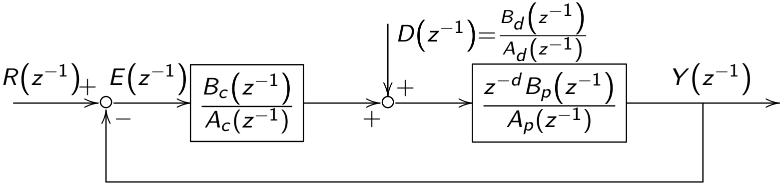


Assume $B_p(z^{-1}) = 0$ and $A_d(z^{-1}) = 0$ do not have common zeros.
If the closed loop is asymptotically stable,
and $A_c(z^{-1})$ can be factorized as $A_c(z^{-1}) = A_d(z^{-1})A_c^{'}(z^{-1})$,
then the disturbance is asymptotically rejected.

Proof: analogous to the continuous-time case.

Internal model principle

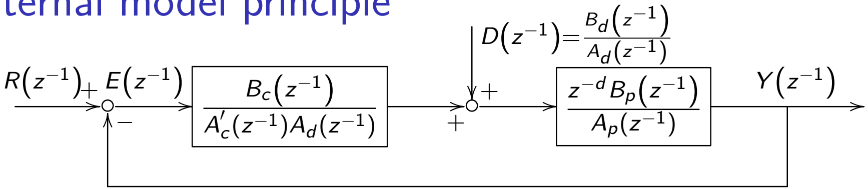
the disturbance structure:



example disturbance structures:

$d(k)$	$A_d(z^{-1})$
constant d_o	$1 - z^{-1}$
$\cos(\omega_0 k)$ and $\sin(\omega_0 k)$	$1 - 2z^{-1}\cos(\omega_0) + z^{-2}$
shifted ramp signal $d(k) = \alpha k + \beta$	$1 - 2z^{-1} + z^{-2}$
periodic: $d(k) = d(k-N)$	$1 - z^{-N}$

Internal model principle



observations:

- the controller contains a "counter disturbance" generator
- high-gain control: the open-loop frequency response

$$P\left(e^{-j\omega}\right)C\left(e^{-j\omega}\right) = \frac{e^{-dj\omega}B_p\left(e^{-j\omega}\right)B_c\left(e^{-j\omega}\right)}{A_p\left(e^{-j\omega}\right)A_c'\left(e^{-j\omega}\right)A_d\left(e^{-j\omega}\right)}$$

  is large at frequencies where $A_d(e^{-j\omega}) = 0$
- $D\left(z^{-1}\right) = B_d\left(z^{-1}\right)/A_d\left(z^{-1}\right)$ means $d\left(k\right)$ is the impulse response of $B_d\left(z^{-1}\right)/A_d\left(z^{-1}\right)$:

$$\delta(k)\left(\mathscr{Z}\{\delta(k)\}=1\right) \longrightarrow \boxed{\frac{B_d(z^{-1})}{A_d(z^{-1})}} \longrightarrow d(k)\left(\mathscr{Z}\{d(k)\} = D\left(z^{-1}\right)\right)$$

Outline

1. Big Picture
   review of integral control in PID design

2. Internal Model Principle
   theorems
   typical disturbance structures

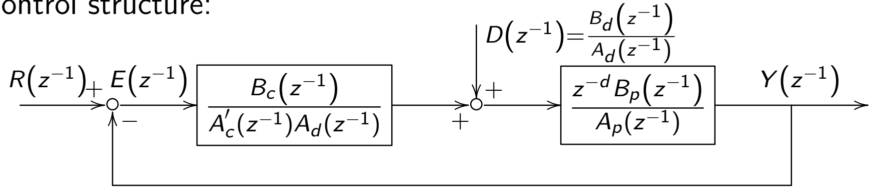
3. Repetitive Control
   use of internal model principle
   design by pole placement
   design by stable pole-zero cancellation

Repetitive control

Repetitive control focus on attenuating periodic disturbances with

$$A_d\left(z^{-1}\right) = 1 - z^{-N}$$

Control structure:



It remains to design $B_c\left(z^{-1}\right)$ and $A_c'\left(z^{-1}\right)$. We discuss two methods:

- **pole placement**
- (partial) cancellation of plant dynamics: **prototype repetitive control**

1, Pole placement: prerequisite

Theorem

Consider $G(z) = \frac{\beta(z)}{\alpha(z)} = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n}$. $\alpha(z)$ *and* $\beta(z)$ *are coprime (no common roots) iff* S *(Sylvester matrix) is nonsingular:*

$$S = \begin{bmatrix} 1 & 0 & \dots & 0 & \beta_1 & 0 & \dots & \dots & 0 \\ \alpha_1 & 1 & \ddots & \vdots & \beta_2 & \beta_1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \alpha_1 & 1 & \beta_{n-1} & & \ddots & \ddots & 0 \\ \alpha_{n-1} & & & \alpha_1 & \beta_n & \ddots & & \ddots & \beta_1 \\ \alpha_n & \ddots & & \vdots & 0 & \beta_n & \ddots & & \beta_2 \\ 0 & \alpha_n & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \alpha_{n-1} & \vdots & & \ddots & \beta_n & \beta_{n-1} \\ 0 & \dots & 0 & \alpha_n & 0 & \dots & \dots & 0 & \beta_n \end{bmatrix}_{(2n-1)\times(2n-1)}$$

1, Pole placement: prerequisite

Example:

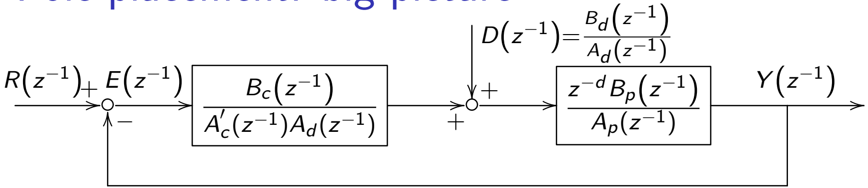
$$G(z) = \frac{\beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n} = \frac{z^{n-1} + \alpha_1 z^{n-2} + \cdots + \alpha_{n-1}}{z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1} z + 0}$$

i.e.

$$\begin{aligned} \beta_1 &= 1 \\ \beta_i &= \alpha_{i-1} \ \forall i \geq 2 \\ \alpha_n &= 0 \end{aligned}$$

$\alpha(z)$ and $\beta(z)$ are not coprime, and S is clearly singular.

1, Pole placement: big picture

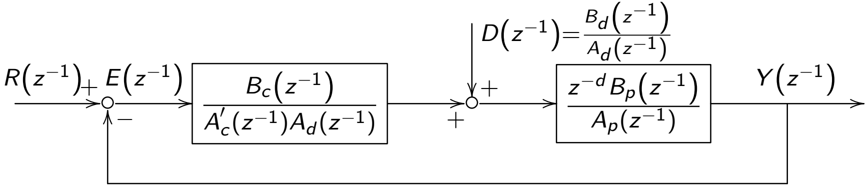


Disturbance model: $A_d\left(z^{-1}\right) = 1 - z^{-N}$
Pole placement assigns the closed-loop characteristic equation:

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right) + A_p\left(z^{-1}\right)A_c'\left(z^{-1}\right)A_d\left(z^{-1}\right) = \underbrace{1 + \eta_1 z^{-1} + \eta_2 z^{-2} + \cdots + \eta_q z^{-q}}_{\eta(z^{-1})}$$

which is in the structure of a *Diophantine equation*.
Design procedure: specify the desired closed-loop dynamics $\eta\left(z^{-1}\right)$; match coefficients of z^{-i} on both sides to get $B_c\left(z^{-1}\right)$ and $A_c'\left(z^{-1}\right)$.

1, Pole placement: big picture



Diophantine equation in Pole placement:

$$z^{-d}B_p\left(z^{-1}\right)B_c\left(z^{-1}\right) + A_p\left(z^{-1}\right)A_c^{'}\left(z^{-1}\right)A_d\left(z^{-1}\right)$$
$$= \underbrace{1 + \eta_1 z^{-1} + \eta_2 z^{-2} + \cdots + \eta_q z^{-q}}_{\eta(z^{-1})}$$

Questions:

- what are the constraints for choosing $\eta\left(z^{-1}\right)$?
- how to guarantee unique solution in Diophantine equation?

Design and analysis procedure

General procedure of control design:

- Problem definition
- Control design for solution (current stage)
- Prove stability
- Prove stability robustness
- Case study or implementation results

1, Pole placement: details

Theorem (Diophantine equation)

Given
$$\eta\left(z^{-1}\right)=1+\eta_1 z^{-1}+\eta_2 z^{-2}+\cdots+\eta_q z^{-q}$$
$$\alpha\left(z^{-1}\right)=1+\alpha_1 z^{-1}+\cdots+\alpha_n z^{-n}$$
$$\beta\left(z^{-1}\right)=\beta_1 z^{-1}+\beta_2 z^{-2}+\cdots+\beta_n z^{-n}$$

The Diophantine equation
$$\alpha\left(z^{-1}\right)\sigma\left(z^{-1}\right)+\beta\left(z^{-1}\right)\gamma\left(z^{-1}\right)=\eta\left(z^{-1}\right)$$

can be solved uniquely for $\sigma\left(z^{-1}\right)$ *and* $\gamma\left(z^{-1}\right)$
$$\sigma\left(z^{-1}\right)=1+\sigma_1 z^{-1}+\cdots+\sigma_{n-1} z^{-(n-1)}$$
$$\gamma\left(z^{-1}\right)=\gamma_0+\gamma_1 z^{-1}+\cdots+\gamma_{n-1} z^{-(n-1)}$$

if the numerators of $\alpha\left(z^{-1}\right)$ *and* $\beta\left(z^{-1}\right)$ *are coprime and* $\deg\left(\eta\left(z^{-1}\right)\right)=q\leq 2n-1$

1, Pole placement: details

Proof of Diophantine equation Theorem (key ideas):

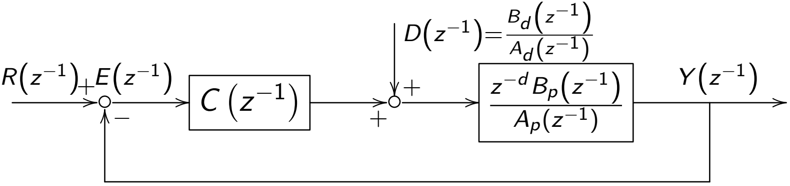
$$\alpha\left(z^{-1}\right)\underbrace{\sigma\left(z^{-1}\right)}_{\text{unknown}}+\beta\left(z^{-1}\right)\underbrace{\gamma\left(z^{-1}\right)}_{\text{unknown}}=\eta\left(z^{-1}\right)$$

Matching the coefficients of z^{-i} gives (see one numerical example in course reader)

$$S\begin{bmatrix}\sigma_1\\ \sigma_2\\ \vdots\\ \sigma_{n-1}\\ \gamma_0\\ \vdots\\ \gamma_{n-1}\end{bmatrix}+\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\\ 0\\ \vdots\\ 0\end{bmatrix}=\begin{bmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_{n-1}\\ \eta_n\\ \vdots\\ \eta_{2n-1}\end{bmatrix}$$

The coprime condition assures S is invertible. $\deg\eta\left(z^{-1}\right)\leq 2n-1$ assures the proper dimension on the right hand side of the equality.

2, Prototype repetitive control: simple case



$$A_d\left(z^{-1}\right) = 1 - z^{-N}$$

If all poles and zeros of the plant are stable, then prototype repetitive control uses

$$C\left(z^{-1}\right) = \frac{k_r z^{-N+d} A_p\left(z^{-1}\right)}{\left(1 - z^{-N}\right) B_p\left(z^{-1}\right)}$$

Theorem (Prototype repetitive control)

Under the assumptions above, the closed-loop system is asymptotically stable for $0 < k_r < 2$

2, Prototype repetitive control: stability

Proof of Theorem on prototype repetitive control:
From

$$1+\frac{k_r z^{-N+d}A_p\left(z^{-1}\right)}{\left(1-z^{-N}\right)B_p\left(z^{-1}\right)}\frac{z^{-d}B_p\left(z^{-1}\right)}{A_p\left(z^{-1}\right)}=0$$

the closed-loop characteristic equation is

$$A_p\left(z^{-1}\right)B_p\left(z^{-1}\right)\left[1-\left(1-k_r\right)z^{-N}\right]=0$$

- roots of $A_p\left(z^{-1}\right)B_p\left(z^{-1}\right)=0$ are all stable
- roots of $1-\left(1-k_r\right)z^{-N}=0$ are

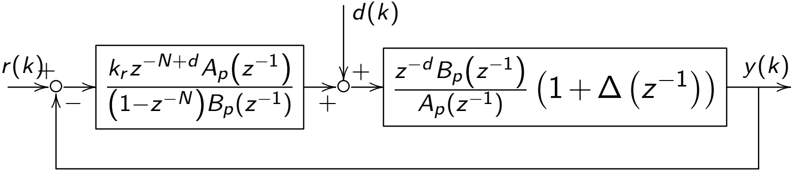
$$\left|1-k_r\right|^{\frac{1}{N}}e^{j\frac{2\pi i}{N}},\ i=0,\pm1,\dots,\ \text{for } 0<k_r\leq 1$$

$$\left|1-k_r\right|^{\frac{1}{N}}e^{j\left(\frac{2\pi i}{N}+\frac{\pi}{N}\right)},\ i=0,\pm1,\dots,\ \text{for } 1<k_r$$

which are all inside the unit circle

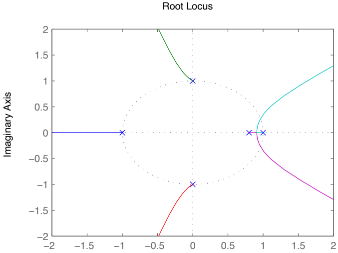
2, Prototype repetitive control: stability robustness

Consider the case with plant uncertainty



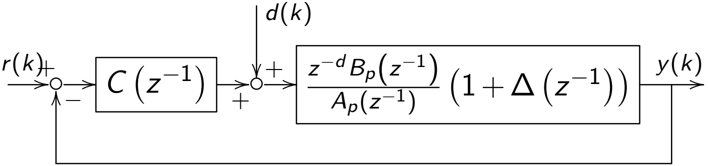
N open-loop poles on the unit circle

Root locus example: $N=4$, $1+\Delta\left(z^{-1}\right)=q/(z-p)$



$\forall\ k_r > 0$, the closed-loop system is now stable!

2, Prototype repetitive control: stability robustness



To make the controller robust to plant uncertainties, do instead

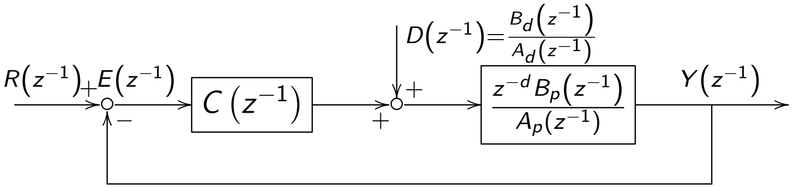
$$C\left(z^{-1}\right)=\frac{k_r q(z,z^{-1})z^{-N+d}A_p\left(z^{-1}\right)}{\left(1-q(z,z^{-1})z^{-N}\right)B_p\left(z^{-1}\right)} \tag{1}$$

$q(z,z^{-1})$: low-pass filter. e.g. zero-phase low pass $\dfrac{\alpha_1 z^{-1}+\alpha_0+\alpha_1 z}{\alpha_0+2\alpha_1}$

which shifts the marginary stable open-loop poles to be inside the unit circle:

$$A_p\left(z^{-1}\right)B_p\left(z^{-1}\right)\left[1-\left(1-k_r\right)q(z,z^{-1})z^{-N}\right]=0$$

2, Prototype repetitive control: extension



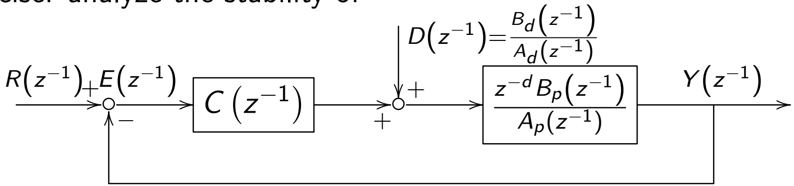
If poles of the plant are stable but **NOT all zeros are stable**, let $B_p(z^{-1}) = B_p^-(z^{-1})B_p^+(z^{-1})$ [$B_p^-(z^{-1})$—the uncancellable part] and

$$C\left(z^{-1}\right) = \frac{k_r z^{-N+\mu} A_p(z^{-1}) B_p^-(z) z^{-\mu}}{(1 - z^{-N}) B_p^+(z^{-1}) z^{-d} b}, \; b > \max_{\omega \in [0,\pi]} |B_p^-(e^{j\omega})|^2 \quad (2)$$

Similar as before, can show that the closed-loop system is stable (in-class exercise).

2, Prototype repetitive control: extension

Exercise: analyze the stability of



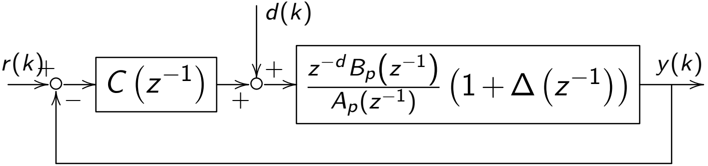
$$C\left(z^{-1}\right) = \frac{k_r z^{-N+\mu} A_p(z^{-1}) B_p^-(z) z^{-\mu}}{(1-z^{-N}) B_p^+(z^{-1}) z^{-d} b},\ b > \max_{\omega\in[0,\pi]} |B_p^-(e^{j\omega})|^2 \quad (3)$$

Key steps: $\left|\frac{B_p^-\left(e^{j\omega}\right)B_p^-\left(e^{-j\omega}\right)}{b}\right| < 1$; $\left|\frac{k_r B_p^-\left(e^{j\omega}\right)B_p^-\left(e^{-j\omega}\right)}{b} - 1\right| < 1$; all roots from

$$z^{-N}\left[\frac{k_r B_p^-(z) B_p^-\left(z^{-1}\right)}{b} - 1\right] + 1 = 0$$

are inside the unit circle.

2, Prototype repetitive control: extension



Robust version in the presence of plant uncertainties:

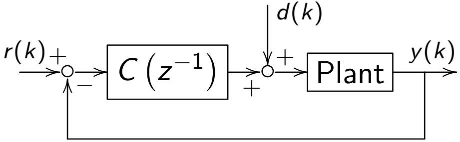
$$C\left(z^{-1}\right)=\frac{k_r z^{-N+\mu}q(z,z^{-1})A_p(z^{-1})B_p^{-}(z)z^{-\mu}}{(1-q(z,z^{-1})z^{-N})B_p^{+}(z^{-1})z^{-d}b} \tag{4}$$

where

$q(z,z^{-1})$: low-pass filter. e.g. zero-phase low pass $\dfrac{\alpha_1 z^{-1}+\alpha_0+\alpha_1 z}{\alpha_0+2\alpha_1}$

and μ is the order of $B_p^{-}(z)$

Example



disturbance period: $N = 10$; nominal plant:

$$\frac{z^{-d}B_p\left(z^{-1}\right)}{A_p\left(z^{-1}\right)} = \frac{z^{-1}}{\left(1-0.8z^{-1}\right)\left(1-0.7z^{-1}\right)}$$

$$C\left(z^{-1}\right) = k_r \frac{\left(1-0.8z^{-1}\right)\left(1-0.7z^{-1}\right)q(z,z^{-1})z^{-10}}{z^{-1}\left(1-q(z,z^{-1})z^{-10}\right)}$$

Additional reading

- X. Chen and M. Tomizuka, "An Enhanced Repetitive Control Algorithm using the Structure of Disturbance Observer," in *Proceedings of 2012 IEEE/ASME International Conference on Advanced Intelligent Mechatronics*, Taiwan, Jul. 11-14, 2012, pp. 490-495
- X. Chen and M. Tomizuka, "New Repetitive Control with Improved Steady-state Performance and Accelerated Transient," *IEEE Transactions on Control Systems Technology*, vol. 22, no. 2, pp. 664-675 (12 pages), Mar. 2014

Summary

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   theorems
   typical disturbance structures

3. Repetitive Control
   use of internal model principle
   design by pole placement
   design by stable pole-zero cancellation