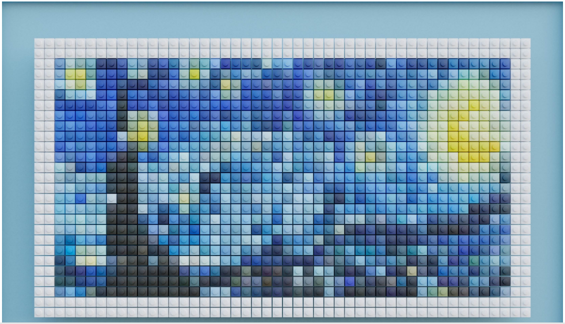
Introduction to Modern Controls

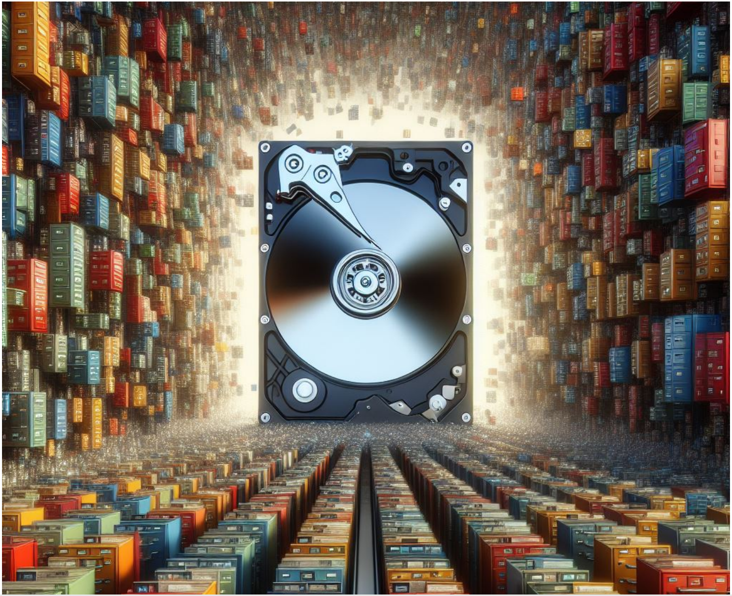
Discretization of State-Space System Models



Topic

1. Discretization of state-space models

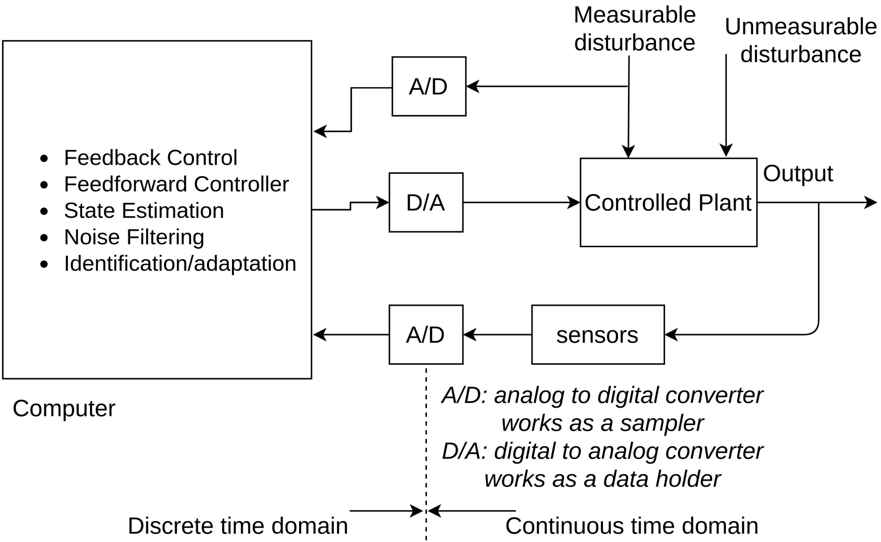
1TB vs 1,300 filing cabinets of paper



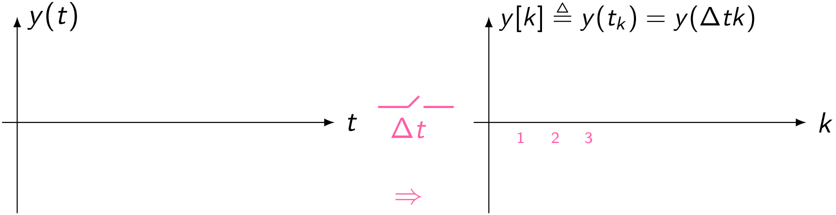
Inherent sampling in practice

$$\Delta t = \frac{1}{(\text{rpm}/60) \times \text{sector number}}$$

Practical control systems

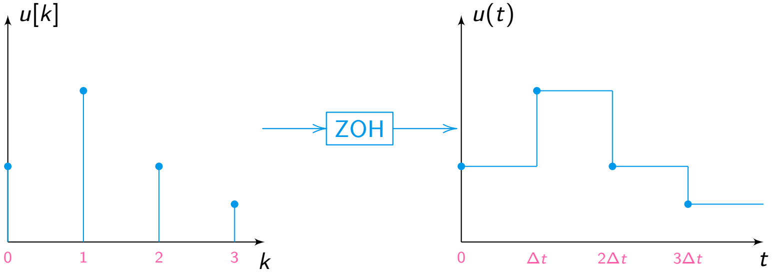


- sampler: converts a time function into a discrete sequence,

e.g., 

Signal holding

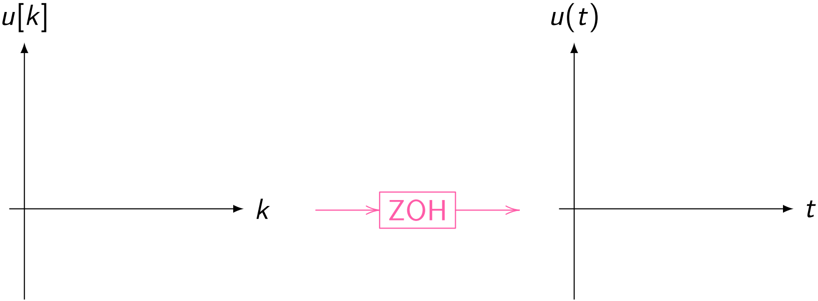
- Zero-order Hold (ZOH): converts a sequence into a "stair-case" time function, e.g.,



- $u(t) = u[k]$ for $t \in [k\Delta t, (k+1)\Delta t)$

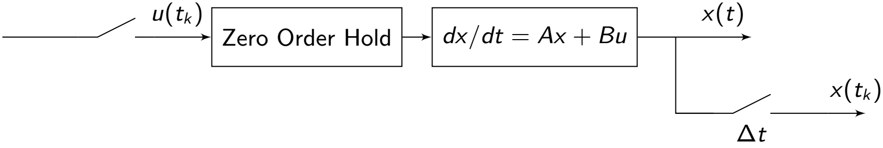
Signal holding

- more faithful presentation with fast sampling



Problem definition

continuous-time system preceded by a ZOH:



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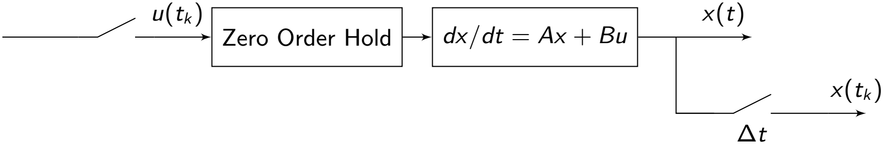

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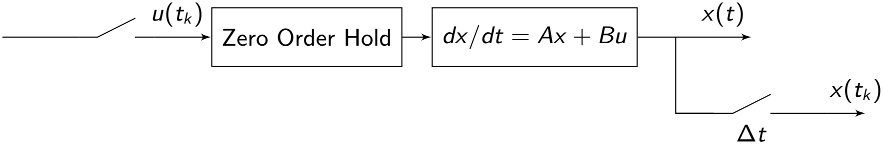

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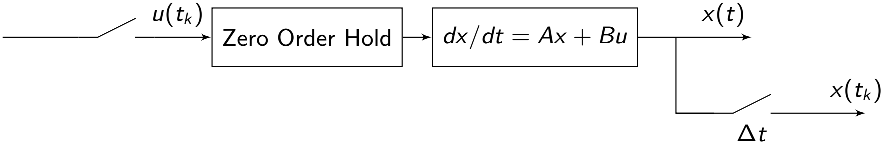

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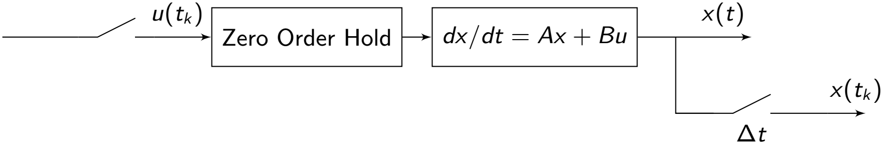

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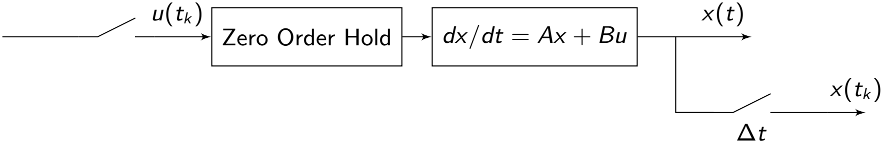

- $u(t_k)$: discrete-time input
- $x(t)$: continuous-time output
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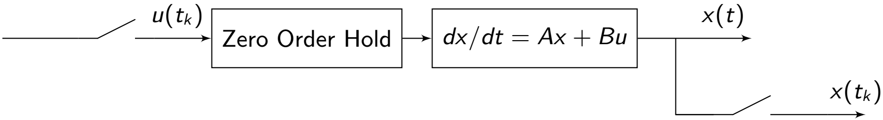
- Δt: sampling time
- goal: to obtain the model between $u(t_k)$ and $x(t_k)$

Solution



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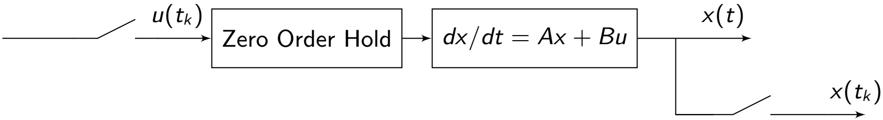

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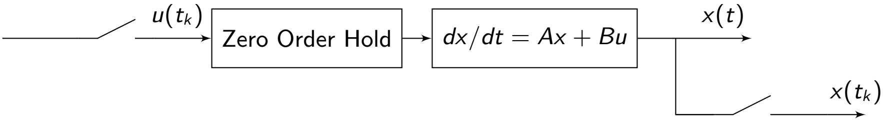

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$$x(t_{k+1}) = e^{A(t_{k+1}-t_k)}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau_o)}Bu(\tau_o)d\tau_o$$

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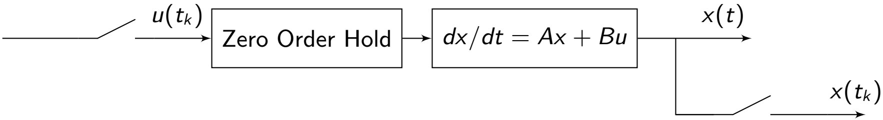

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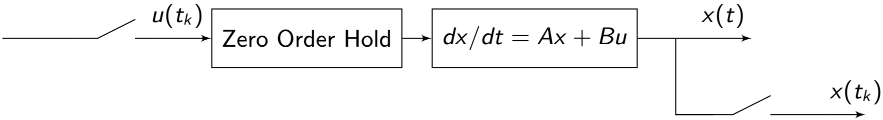

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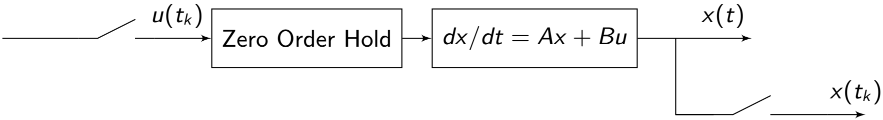

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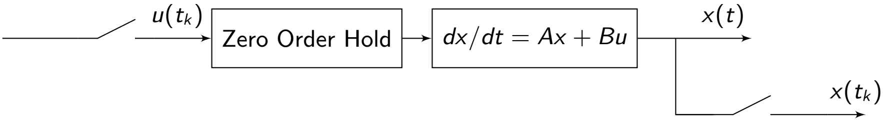

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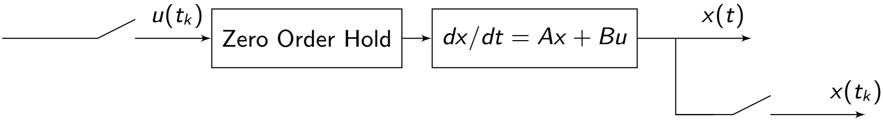

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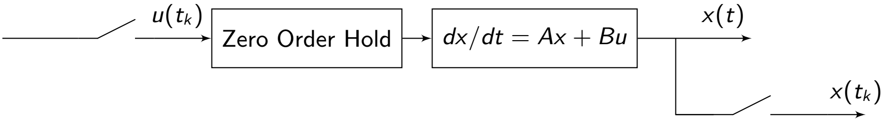

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$$\boxed{x[k+1] = A_d x[k] + B_d u[k],\ A_d = e^{A\Delta t},\ B_d = \int_0^{\Delta t} e^{A\tau}Bd\tau}$$

Mapping of eigenvalues

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- e^{At} has the same eigenvalues as $e^{\Lambda t}$
- $\Rightarrow$ eigenvalues of $A_d = e^{A\Delta t}$ are $e^{\lambda_i \Delta t}$'s where λ_i is an eigenvalue of A

Example

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t)$$

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$$C_d = C$$

Numerical example in Python

```python
import control
import numpy
m = 1
dt = 0.1
A = [[0, 1], [0, 0]]
B = [[0], [1]]
C = [[1/m, 0]]
D = 0

G_s = control.ss(A, B, C, D)
G_z = control.c2d(G_s, dt, 'zoh')
print(G_z.A)

# eigenvalues of continuous-time system
eigA, eigvecA = numpy.linalg.eig(A)
print(eigA)
# eigenvalues of discretized system
eigAd, eigvecAd = numpy.linalg.eig(G_z.A)
print(eigAd)
```

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eigA, eigvecA = numpy.linalg.eig(A)
print(eigA)
```

```
[99.8+2000.j 99.8-2000.j]
```