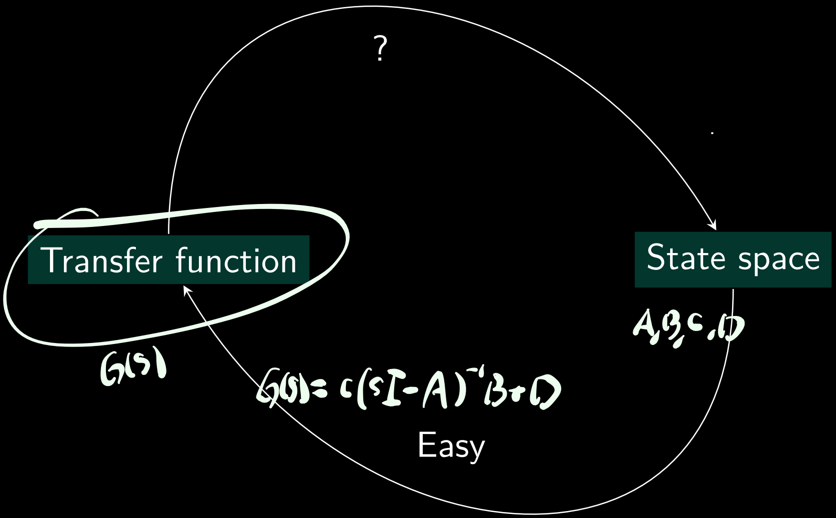
# Introduction to Modern Controls

# State-Space Realization Theory

# Topic


?
Transfer function
State space
G(s)
A, B, C, D
$G(s) = C(sI - A)^{-1}B + D$
Easy

# Goal

the realization problem:

$$\boxed{G(s) = \frac{B(s)}{A(s)}} \overset{?}{\Longrightarrow} \boxed{\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}$$

- existence and uniqueness: the same system can have infinite amount of state-space representations: e.g.

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \quad \begin{cases} \dot{x} &= Ax + \frac{1}{2}Bu \\ y &= 2Cx \end{cases}$$

1 2

$G(s)$

$C(sI-A)^{-1}B+D$

# Goal

the realization problem:

$$\boxed{G(s) = \frac{B(s)}{A(s)}} \overset{?}{\Longrightarrow} \boxed{\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}$$

- existence and uniqueness: the same system can have infinite amount of state-space representations: e.g.

$$\begin{cases} \dot{x} & = Ax + Bu \\ y & = Cx \end{cases} \qquad \begin{cases} \dot{x} & = Ax + \frac{1}{2}Bu \\ y & = 2Cx \end{cases}$$

- canonical realizations exist

# Goal

the realization problem:

$$\boxed{G(s) = \frac{B(s)}{A(s)}} \overset{?}{\Longrightarrow} \boxed{\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}$$

- existence and uniqueness: the same system can have infinite amount of state-space representations: e.g.

$$\begin{cases} \dot{x} & = Ax + Bu \\ y & = Cx \end{cases} \qquad \begin{cases} \dot{x} & = Ax + \frac{1}{2}Bu \\ y & = 2Cx \end{cases}$$

- canonical realizations exist
- relationship between different realizations?

# Goal

the realization problem:

$$\boxed{G(s) = \frac{B(s)}{A(s)}} \overset{?}{\Longrightarrow} \boxed{\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]}$$

- existence and uniqueness: the same system can have infinite amount of state-space representations: e.g.

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases} \qquad \begin{cases} \dot{x} &= Ax + \frac{1}{2}Bu \\ y &= 2Cx \end{cases}$$

- canonical realizations exist
- relationship between different realizations?
- unit problem:

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

# Topic

1 Introduction

2 CT controllable canonical form

ccf

3 CT observable canonical form

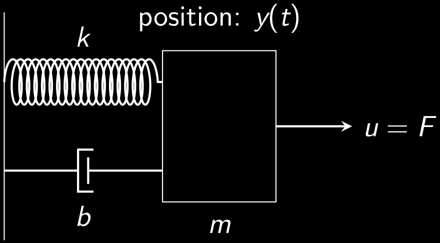
4 CT diagonal and Jordan canonical forms

5 Modified canonical form

6 DT state-space canonical forms

7 Similar realizations

# Recall




- $G(s) = \frac{1}{ms^2 + bs + k}$
- chose position $y(t)$ and velocity $\dot{y}(t)$ as state variables

$x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$

$\dfrac{1}{ms^2 + bs + k}$

$$u \longrightarrow \boxed{\dfrac{1}{s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow y$$

- choose similarly:

$$x_1 = y, \quad x_2 = \dot{x}_1 = \dot{y}, \quad x_3 = \dot{x}_2 = \ddot{y}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$sY(s)$

$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \longrightarrow y \Rightarrow (s^3+a_2s^2+a_1s+a_0)y = u$$

$$\Rightarrow a_0y + a_1\dot{y} + a_2\ddot{y} + \dddot{y} = u$$

- choose similarly:

$$x_1 = y,\ x_2 = \dot{x}_1 = \dot{y},\ x_3 = \dot{x}_2 = \ddot{y}$$

$\dot{x}_1 = \dot{y} = x_2$

$\dot{x}_3 = \ddot{x}_2 = \dddot{y}$

- $\Rightarrow$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \xrightarrow{\mathcal{L}} \begin{bmatrix} sX_1 - x_1(0) \\ \vdots \end{bmatrix}$$

$$\dot{x}_3 = \dddot{y} = -a_0y - a_1x_2 - a_2x_3 + u \Leftarrow \text{no } \frac{d}{dt}!$$

$$= -a_0x_1 - a_1x_2 - a_2x_3 + u$$

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow y$$

- choose $x_1$ such that

$$u \longrightarrow \boxed{\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}} \xrightarrow{x_1} \boxed{b_2 s^2 + b_1 s + b_0} \longrightarrow y$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ \ddot{x}_1 \end{bmatrix}$$

$$Y(s) = (b_2 s^2 + b_1 s + b_0) X_1(s)$$

$$\updownarrow$$

$$y(t) = b_2 \ddot{x}_1 + b_1 \dot{x}_1 + b_0 x_1$$

$$= b_2 x_3 + b_1 x_2 + b_0 x_1$$

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow y$$

- choose $x_1$ such that

$$u \longrightarrow \boxed{\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}} \xrightarrow{x_1} \boxed{b_2 s^2 + b_1 s + b_0} \longrightarrow y$$

- the first part

$$u \longrightarrow \boxed{\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}} \longrightarrow \underline{\tilde{y}(= x_1)}$$

is now familiar

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \xrightarrow{x_1} \boxed{b_2s^2+b_1s+b_0} \longrightarrow y$$

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \xrightarrow{x_1} \boxed{b_2s^2+b_1s+b_0} \longrightarrow y$$

$$X_1(s) = \frac{U(s)}{s^3+a_2s^2+a_1s+a_0} \Rightarrow \dddot{x}_1 + a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1 = u$$

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \xrightarrow{x_1} \boxed{b_2s^2+b_1s+b_0} \longrightarrow y$$

$$X_1(s) = \frac{U(s)}{s^3+a_2s^2+a_1s+a_0} \Rightarrow \dddot{x}_1 + a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1 = u$$

- let $x_2 = \dot{x}_1,\ x_3 = \dot{x}_2 \Rightarrow \dot{x}_3 = -a_2x_3 - a_1x_2 - a_0x_1 + u$

# Controllable canonical form (ccf)

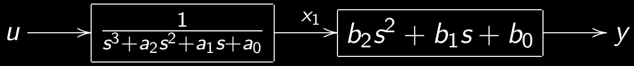
$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \xrightarrow{x_1} \boxed{b_2s^2+b_1s+b_0} \longrightarrow y$$

$$X_1(s) = \frac{U(s)}{s^3+a_2s^2+a_1s+a_0} \Rightarrow \dddot{x}_1 + a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1 = u$$

- let $x_2 = \dot{x}_1, \; x_3 = \dot{x}_2 \Rightarrow \dot{x}_3 = -a_2x_3 - a_1x_2 - a_0x_1 + u$
- $\Rightarrow$

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

# Controllable canonical form (ccf)




$$X_1(s) = \frac{U(s)}{s^3+a_2s^2+a_1s+a_0} \Rightarrow \dddot{x}_1 + a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0x_1 = u$$

- let $x_2 = \dot{x}_1,\ x_3 = \dot{x}_2 \Rightarrow \dot{x}_3 = -a_2x_3 - a_1x_2 - a_0x_1 + u$
- for the output:

$$Y(s) = \left(b_2s^2 + b_1s + b_0\right)X_1(s) \Rightarrow y = b_2\underbrace{\ddot{x}_1}_{x_3} + b_1\underbrace{\dot{x}_1}_{x_2} + b_0x_1$$

- $\Rightarrow$

$$y(t) = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}}_{C}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \underbrace{0}_{D} \cdot u$$

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \xrightarrow{x_1} \boxed{b_2s^2+b_1s+b_0} \longrightarrow y$$

- $x_2 = \dot{x}_1, \;\; x_3 = \dot{x}_2$
- $y = b_2 \underbrace{\ddot{x}_1}_{x_3} + b_1 \underbrace{\dot{x}_1}_{x_2} + b_0 x_1$

# Controllable canonical form (ccf)

$$u \longrightarrow \boxed{\frac{1}{s^3+a_2s^2+a_1s+a_0}} \xrightarrow{x_1} \boxed{b_2s^2+b_1s+b_0} \longrightarrow y$$

- $x_2 = \dot{x}_1, \;\; x_3 = \dot{x}_2$
- $y = b_2 \underbrace{\ddot{x}_1}_{x_3} + b_1 \underbrace{\dot{x}_1}_{x_2} + b_0 x_1$
- putting everything in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

# Block diagram realization of ccf

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

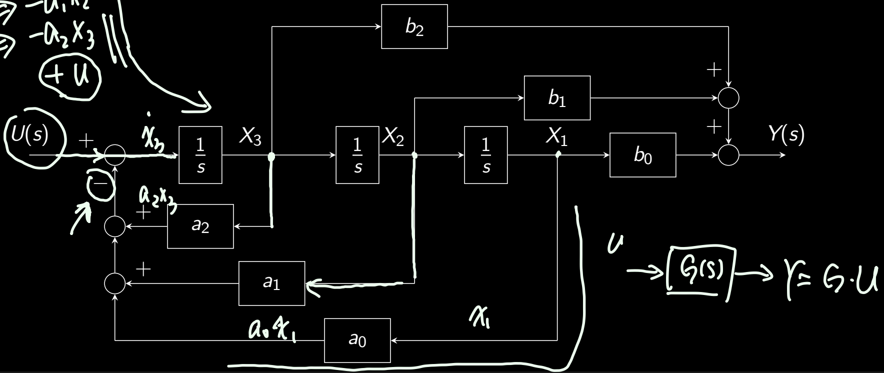
$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

# Block diagram realization of ccf

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$\dot{x}_3 = -a_0 x_1$
$-a_1 x_2$
$-a_2 x_3$
$+U$



# General ccf

general single-input single-output transfer function:

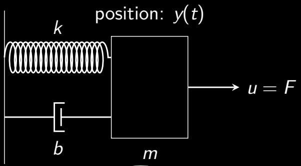
$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$$

- the following realizes $G(s)$

$$\Sigma_c = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} & d \end{array}\right]$$

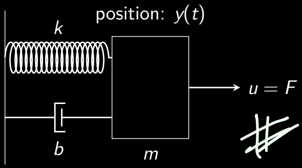
- this realization is called the *controllable canonical form*

# ccf example




$$\underbrace{\frac{d}{dt}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)}$$




$$\frac{1}{m}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} y(t) \\ v(t) \end{bmatrix}}_{x(t)}$$

$$G(s) = \frac{1}{ms^2+bs+k}$$

a slightly modified form of the ccf $\Rightarrow$

$$G(s) = \frac{1}{m}\frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} = \frac{1}{ms^2 + bs + k}$$

# Topic

# Observable canonical form (ocf)

$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s) \Rightarrow \boxed{\begin{aligned} &s^3 Y(s) + a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) \\ &= b_2 s^2 U(s) + b_1 s U(s) + b_0 U(s) \end{aligned}}$$

$$\Rightarrow Y(s) = -\frac{a_2}{s} Y(s) - \frac{a_1}{s^2} Y(s) - \frac{a_0}{s^3} Y(s) + \frac{b_2}{s} U(s) + \frac{b_1}{s^2} U(s) + \frac{b_0}{s^3} U(s)$$

$$\begin{aligned} s^3 Y(s) = {} & b_2 s^2 U(s) + b_1 s U(s) + b_0 U(s) \\ & - a_2 s^2 Y(s) - a_1 s Y(s) - a_0 Y(s) \end{aligned}$$

# Observable canonical form (ocf)

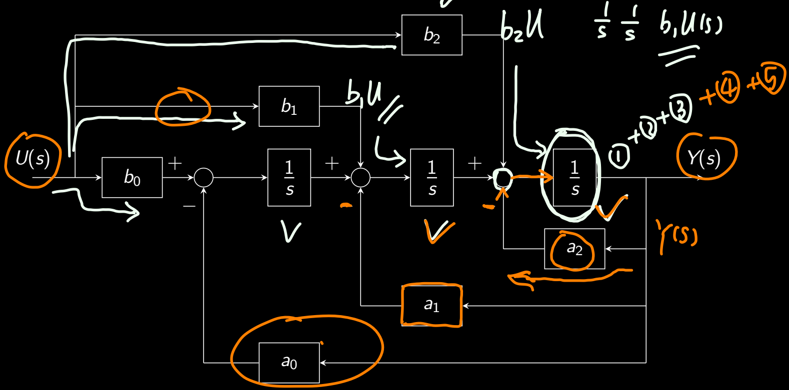
$$Y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

⑤ ⑥ ① ② ③

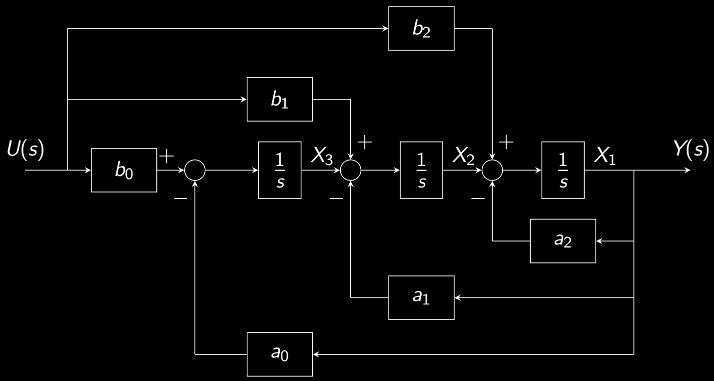
$$\Rightarrow Y(s) = -\frac{a_2}{s} Y(s) - \frac{\boxed{a_1}}{s^2} Y(s) - \frac{a_0}{s^3} Y(s) + \frac{b_2}{s} U(s) + \frac{b_1}{s^2} U(s) + \frac{b_0}{s^3} U(s)$$

④

in a block diagram, the above looks like



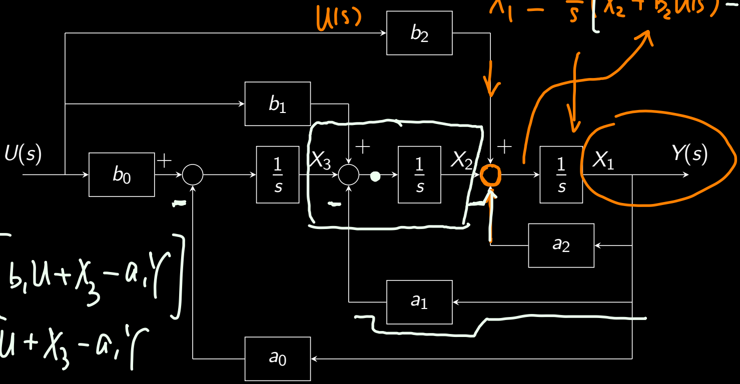
# Observable canonical form



# Observable canonical form

$$s\left\{ X_1 = \frac{1}{s}\left[X_2 + b_2 U - a_2 X_1\right]\right\}$$

$$X_1 = \frac{1}{s}\left[X_2 + b_2 U(s) - a_2 Y(s)\right]$$




$$X_2 = \frac{1}{s}\left[b_1 U + X_3 - a_1 Y\right]$$

$$sX_2 = b_1 U + X_3 - a_1 Y$$

here, the states are connected by

$$\begin{aligned}
Y(s) &= X_1(s) \\
sX_1(s) &= -a_2 X_1(s) + X_2(s) + b_2 U(s) \\
sX_2(s) &= -a_1 X_1(s) + X_3(s) + b_1 U(s) \\
sX_3(s) &= -a_0 X_1(s) + b_0 U(s)
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
y(t) &= x_1(t) \\
\dot{x}_1(t) &= -a_2 x_1(t) + x_2(t) + b_2 u(t) \\
\dot{x}_2(t) &= -a_1 x_1(t) + x_3(t) + b_1 u(t) \\
\dot{x}_3(t) &= -a_0 x_1(t) + b_0 u(t)
\end{aligned}$$

# Observable canonical form

$$
\begin{cases}
\dot{x}_1(t) &= -a_2 x_1(t) + x_2(t) + b_2 u(t) \\
\dot{x}_2(t) &= -a_1 x_1(t) + x_3(t) + b_1 u(t) \\
\dot{x}_3(t) &= -a_0 x_1(t) + b_0 u(t) \\
y(t) &= x_1(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\end{cases}
$$

# Observable canonical form

$\boxed{1}$ $\quad + \boxed{0} x_3$

$$
\begin{cases}
\dot{x}_1(t) & = -a_2 x_1(t) + x_2(t) + b_2 u(t) \\
\dot{x}_2(t) & = -a_1 x_1(t) + x_3(t) + b_1 u(t) \\
\dot{x}_3(t) & = -a_0 x_1(t) + b_0 u(t) \\
y(t) & = x_1(t)
\end{cases}
$$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$
\Rightarrow \dot{x}(t) = \underbrace{\begin{bmatrix} -a_2 & \boxed{1} & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}}_{A_o} x(t) + \underbrace{\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}}_{B_o} u(t)
$$

$$
y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C_o} x(t)
$$

this is called the *observable canonical form* realization of $G(s)$

# General ocf

general case for:

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$$

# General ocf

general case for:

$$G(s) = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} + d$$

*observable canonical form:*

$$\Sigma_o = \left[\begin{array}{c|c} A_o & B_o \\ \hline C_o & D_o \end{array}\right] = \left[\begin{array}{ccccc|c} -a_{n-1} & 1 & 0 & \cdots & 0 & b_{n-1} \\ -a_{n-2} & 0 & \ddots & \ddots & \vdots & b_{n-2} \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ -a_1 & 0 & \cdots & 0 & 1 & b_1 \\ -a_0 & 0 & \cdots & 0 & 0 & b_0 \\ \hline 1 & 0 & \cdots & 0 & 0 & d \end{array}\right]$$

# ocf in Python

```python
import control as ct
Gs  = ct.tf2ss([1,0,1],[1,2,10])
Gc, T = ct.canonical_form(Gs,'observable')

Gc.A
Gc.B
Gc.C
Gc.D
```

# ocf in Python

```python
import control as ct
Gs  = ct.tf2ss([1,0,1],[1,2,10])
Gc, T = ct.canonical_form(Gs,'observable')

Gc.A
Gc.B
Gc.C
Gc.D
```

ccf and ocf: no direct Matlab commands

# Topic

# Diagonal form

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

when the poles $p_1 \neq p_2 \neq p_3$, partial fractional expansion yields

# Diagonal form

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

when the poles $p_1 \neq p_2 \neq p_3$, partial fractional expansion yields

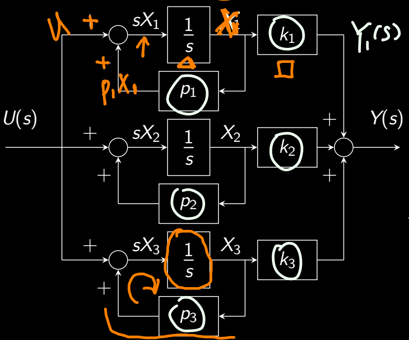
$$G(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \frac{k_3}{s - p_3}, \quad k_i = \lim_{p \to p_i} (s - p_i) \frac{B(s)}{A(s)}$$

# Diagonal form

$$G(s) = \frac{B(s)}{A(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

when the poles $p_1 \neq p_2 \neq p_3$, partial fractional expansion yields

$$G(s) = \frac{k_1}{s - p_1} + \frac{k_2}{s - p_2} + \frac{k_3}{s - p_3} \qquad k_i = \lim_{p \to p_i} (s - p_i) \frac{B(s)}{A(s)}$$




$Y_1(s) = \frac{k_1}{s - p_1} U(s)$

$s Y_1(s) - p_1 Y_1(s)$

$= k_1 U(s)$

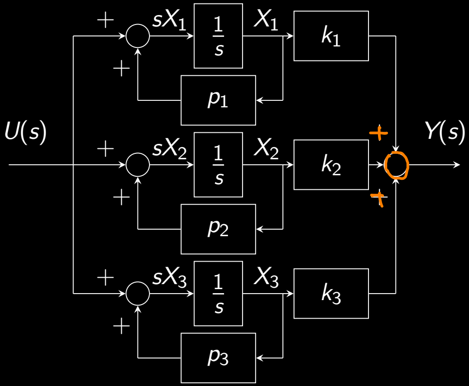
$Y_1 = k_1 X_1$

$\Rightarrow s k_1 X_1 - p_1 k_1 X_1 = k_1 U$

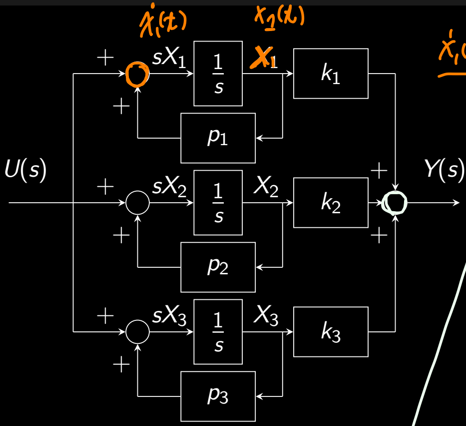
$\Rightarrow s X_1 - p_1 X_1 = U$

$\Rightarrow X_1 = \frac{1}{s} [p_1 X_1 + U]$

# Diagonal form



# Diagonal form




$\dot{x}_1(t) = p_1 x_1 + u$

$Y(s) = k_1 X_1(s) + k_2 X_2(s) + k_3 X(3)$

$\dot{x}_2(t) = p_2 x_2(t) + u$

$\dot{x}_3(t) = p_3 x_3 + u$

state-space realization:

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}, \quad D = 0$$

# Jordan form

if poles repeat, say,

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)(s - p_m)^2}, \; p_1 \neq p_m \in \mathbb{R}$$

then partial fraction expansion gives

# Jordan form

if poles repeat, say,

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{b_2 s^2 + b_1 s + b_0}{(s - p_1)(s - p_m)^2}, \; p_1 \neq p_m \in \mathbb{R}$$

then partial fraction expansion gives

$$G(s) = \frac{k_1}{s - p_1} + \frac{k_2}{(s - p_m)^2} + \frac{k_3}{s - p_m} \quad \text{w/} \; \begin{cases} k_1 & = \lim_{s \to p_1} G(s)(s - p_1) \\ k_2 & = \lim_{s \to p_m} G(s)(s - p_m)^2 \\ k_3 & = \lim_{s \to p_m} \frac{d}{ds} \left\{ G(s)(s - p_m)^2 \right\} \end{cases}$$

$\dot{x}_1$

$\dot{x}_2$

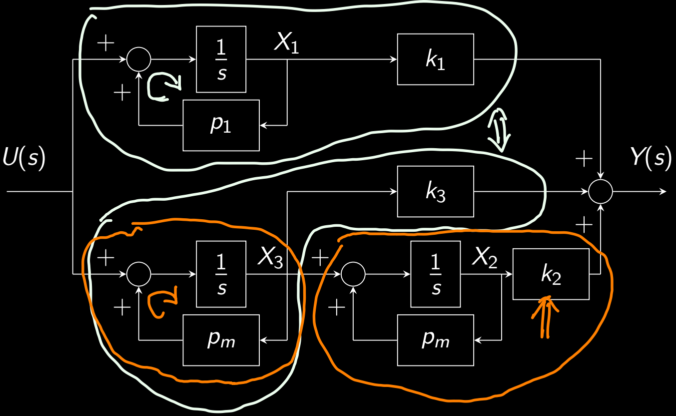
$\dot{x}_3$

# Jordan form

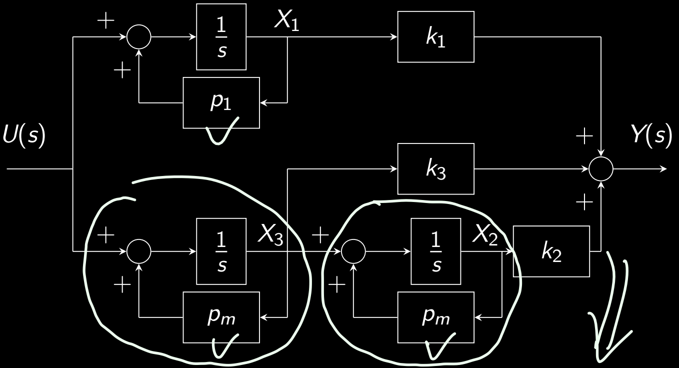
$$\frac{k_2}{(s-p_m)}\;\frac{1}{(s-p_m)}$$

$$G(s) = \frac{k_1}{s-p_1} + \frac{k_2}{(s-p_m)^2} + \frac{k_3}{s-p_m}$$

has the block diagram realization:



# Jordan form




state-space realization (called the Jordan canonical form):

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}, \quad D = 0$$

# Topic

# Modified canonical form

if the system has complex poles, say,

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{k_1}{s - p_1} + \frac{\alpha s + \beta}{(s-\sigma)^2 + \omega^2}$$

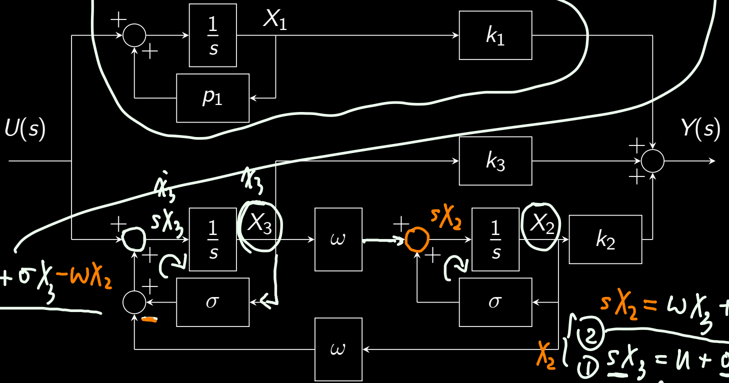
$\sigma \pm j\omega$

# Modified canonical form

if the system has complex poles, say,

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} = \frac{k_1}{s - p_1} + \boxed{\frac{\alpha s + \beta}{(s-\sigma)^2 + \omega^2}}$$

then




① $sX_3 = U + \sigma X_3 - \omega X_2$

$sX_2 = \omega X_3 + \sigma X_2$ ②

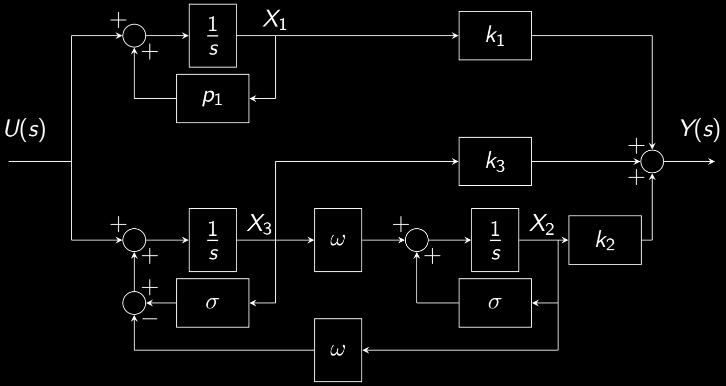
$X_2$ ① $sX_3 = U + \sigma X_3 - \omega X_2$

$\Rightarrow X_3 = \frac{1}{\omega}(sX_2 - \sigma X_2)$

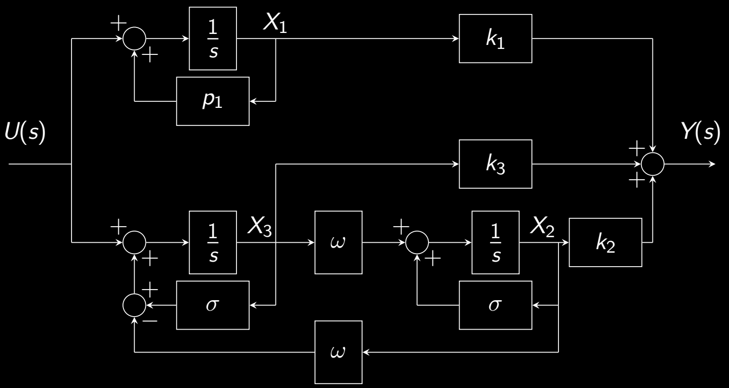
$\Rightarrow (s-\sigma)\frac{1}{\omega}(s-\sigma)X_2 = U - \omega X_2$

where $k_2 = (\beta + \alpha\sigma)/\omega$ and $k_3 = \alpha$

# Modified canonical form



# Modified canonical form

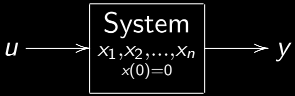



$\Rightarrow$ *modified Jordan form:*

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}, \quad D = 0$$

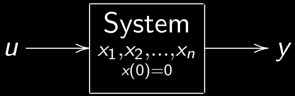
# Topic

# Continuous- and discrete-time state-space descriptions




$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

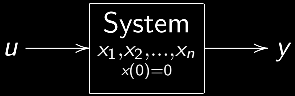
# Continuous- and discrete-time state-space descriptions




$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

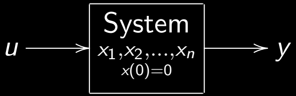
# Continuous- and discrete-time state-space descriptions




$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned}$$



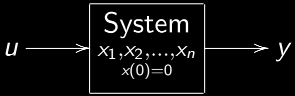

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$zX(z) = AX(z) + BU(z)$$
$$Y(z) = CX(z) + DU(z)$$

# Continuous- and discrete-time state-space descriptions




$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \qquad \begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

$$\begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s) \end{aligned} \qquad \begin{aligned} zX(z) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

- previous procedure applies to discrete-time systems
- replace $t$ with $k$, and $\dot{x}(t)$ with $x(k+1)$
- replace $s$ with $z$, and $\boxed{\frac{1}{s}}$ with $\boxed{z^{-1}}$ in block diagrams

# DT controllable canonical form

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- same transfer-function structure

# DT controllable canonical form

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- same transfer-function structure
- $\Rightarrow$ same $A$, $B$, $C$, $D$ matrices as those in CT

# DT controllable canonical form

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

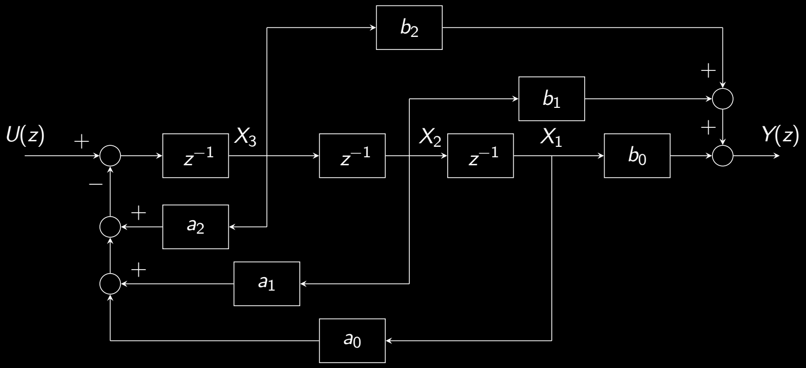
- same transfer-function structure
- $\Rightarrow$ same $A$, $B$, $C$, $D$ matrices as those in CT
- controllable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

# DT controllable canonical form

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$



# DT observable canonical form

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- observable canonical form:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

# DT diagonal form

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- diagonal form (distinct poles):

$$G(z) = \frac{k_1}{z - p_1} + \frac{k_2}{z - p_2} + \frac{k_3}{z - p_3}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

# DT Jordan form 1

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- Jordan form (2 repeated poles):

$$G(z) = \frac{k_1}{z - p_1} + \frac{k_2}{(z - p_m)^2} + \frac{k_3}{z - p_m}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_m & 1 \\ 0 & 0 & p_m \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

# DT Jordan form 2

$$G(z) = \frac{b_2 z^2 + b_1 z + b_0}{z^3 + a_2 z^2 + a_1 z + a_0}$$

- Jordan form (2 complex poles):

$$G(s) = \frac{k_1}{z - p_1} + \frac{\alpha z + \beta}{(z-\sigma)^2 + \omega^2}$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & \sigma & \omega \\ 0 & -\omega & \sigma \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

where $k_2 = (\beta + \alpha\sigma)/\omega, \; k_3 = \alpha.$

# Exercise

obtain the controllable canonical form:

- $G(z) = \frac{z^{-1} - z^{-3}}{1 + 2z^{-1} + z^{-2}}$

# Topic

# Relation between different realizations

- given one realization $\Sigma$ and a nonsingular $T \in \mathbb{R}^{n \times n}$

# Relation between different realizations

- given one realization $\Sigma$ and a nonsingular $T \in \mathbb{R}^{n \times n}$
- can define *new* states: $Tx^* = x$

# Relation between different realizations

- given one realization $\Sigma$ and a nonsingular $T \in \mathbb{R}^{n \times n}$
- can define *new* states: $Tx^* = x$
- then

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow \frac{d}{dt}\left(Tx^*(t)\right) = ATx^*(t) + Bu(t),$$

$$\Rightarrow \Sigma^* : \left\{ \begin{array}{rcl} \dot{x}^*(t) & = & T^{-1}ATx^*(t) + T^{-1}Bu(t) \\ y(t) & = & CTx^*(t) + Du(t) \end{array} \right.$$

# Relation between different realizations

- given one realization $\Sigma$ and a nonsingular $T \in \mathbb{R}^{n \times n}$
- can define *new* states: $Tx^* = x$
- then

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow \frac{d}{dt}\left(Tx^*(t)\right) = ATx^*(t) + Bu(t),$$

$$\Rightarrow \Sigma^* : \left\{ \begin{array}{rcl} \dot{x}^*(t) & = & T^{-1}ATx^*(t) + T^{-1}Bu(t) \\ y(t) & = & CTx^*(t) + Du(t) \end{array} \right.$$

- namely

$$\Sigma^* = \left[ \begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array} \right]$$

also realizes $G(s)$ and is said to be *similar* to $\Sigma$

# Relation between different realizations

verify that the following realize the same system

$$\Sigma = \left[\begin{array}{ccc|c} -a_2 & 1 & 0 & b_2 \\ -a_1 & 0 & 1 & b_1 \\ -a_0 & 0 & 0 & b_0 \\ \hline 1 & 0 & 0 & d \end{array}\right], \; \Sigma^* = \left[\begin{array}{ccc|c} 0 & 0 & -a_0 & b_0 \\ 1 & 0 & -a_1 & b_1 \\ 0 & 1 & -a_2 & b_2 \\ \hline 0 & 0 & 1 & d \end{array}\right]$$

# Recap