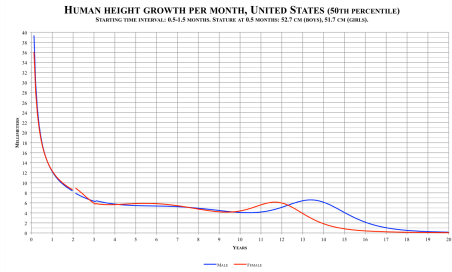


Introduction to Modern Controls

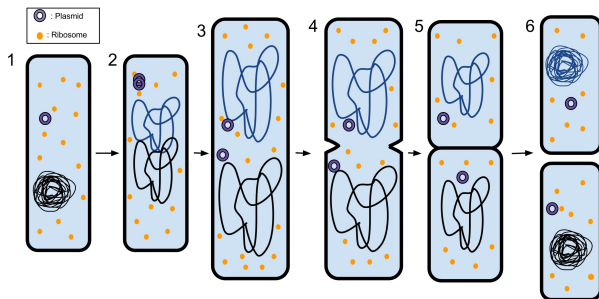
Solution of LTI State-Space Equations



- 1 Introduction
- 2 Continuous-time state-space solution
- 3 Discrete-time state-space solution
- 4 Explicit computation of the state transition matrix e^{At}
- 5 Explicit Computation of the State Transition Matrix A^k
- 6 Transition Matrix via Inverse Transformation

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Population dynamics

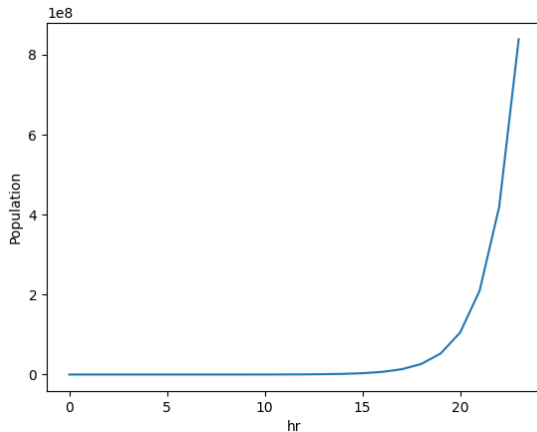
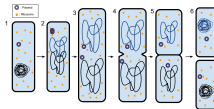


prokaryotic fission

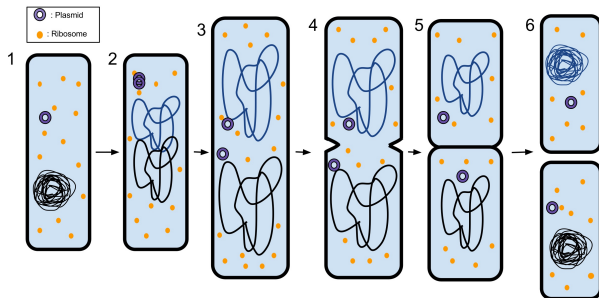
- ~1 hour / division with infinite resource

$$100 \xrightarrow{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} 800 \xrightarrow{1\text{hr}} \dots$$

Population dynamics



Population dynamics



prokaryotic fission

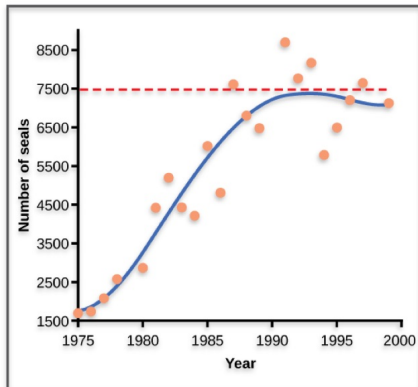
- ~1 hour / division with infinite resource

$$100 \xrightarrow{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} 800 \xrightarrow{1\text{hr}} \dots$$

- after 1 day:

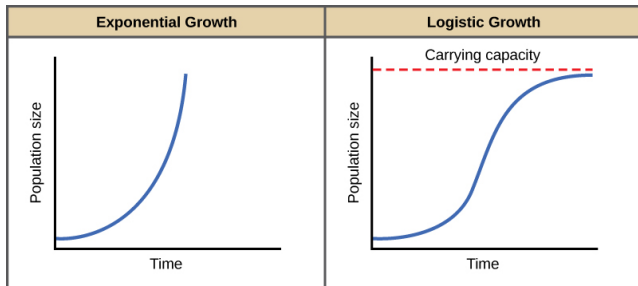
$$100 \xrightarrow[\frac{\Delta N}{N}=1]{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} \dots \longrightarrow 100 \times 2^{24} = 1.7\text{B!}$$

Population dynamics



Environmental limits to population growth: Figure 1, by OpenStax College, Biology, CC BY 4.0.

The exponential function and population dynamics



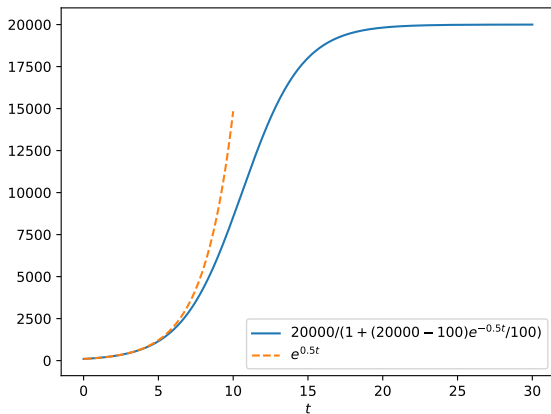
- more general population dynamics (w/ infinite resources)

$$\frac{dN}{dt} = \overbrace{(\text{birth rate} - \text{death rate})}^r N \Rightarrow N(t) = e^{rt} N(0)$$

- logistic growth (w/ limited resources in reality)

$$\frac{dN}{dt} = r \frac{K - N}{K} N \Rightarrow N(t) = \frac{KN_0 e^{rt}}{(K - N_0) + N_0 e^{rt}} = \frac{K}{1 + \frac{K - N_0}{N_0} e^{-rt}}$$

The exponential function and the logistic S curve: example



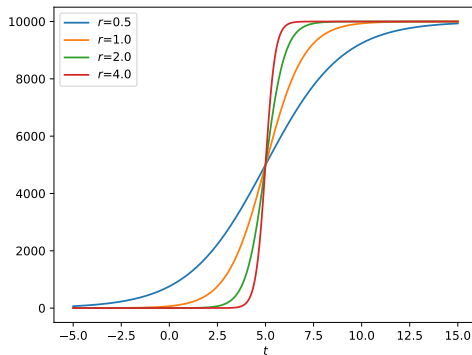
The logistic S curve

$$\frac{K}{1 + \frac{K - N_0}{N_0} e^{-rt}}$$

can also be written as

$$\frac{K}{1 + e^{-r(t-t_o)}}$$

- K : final value
- r : logistic growth rate
- t_o : midpoint



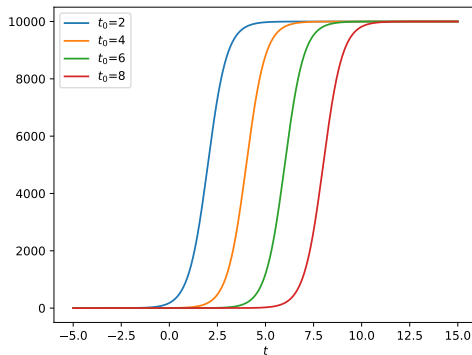
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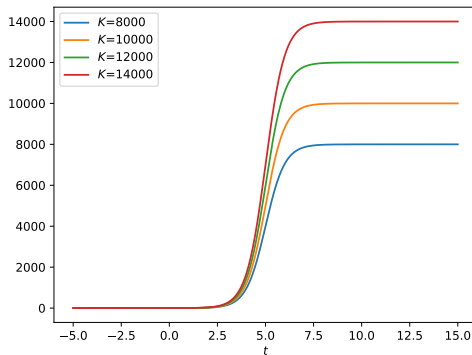
The logistic S curve

$$\frac{K}{1 + \frac{K - N_0}{N_0} e^{-rt}}$$

can also be written as

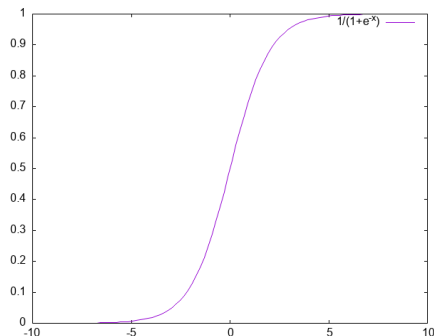
$$\frac{K}{1 + e^{-r(t-t_o)}}$$

- K : final value
- r : logistic growth rate
- t_o : midpoint



The logistic function in deep learning

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$



- transforms the input variables into a probability value between 0 and 1
- represents the likelihood of the dependent variable being 1 or 0

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General LTI continuous-time state equation

$$\frac{dx}{dt} = Ax + Bu$$

$$\Sigma = \left[\begin{array}{c|c} A_{n \times n} & B_{n \times m} \\ \hline C_{n_y \times n} & D_{n_y \times m} \end{array} \right]$$

- to solve the vector equation $\dot{x} = Ax + Bu$, we start with the scalar case when $x, a, b, u \in \mathbb{R}$.

The solution to $\dot{x} = ax + bu$

- fundamental property of exponential functions

$$\frac{d}{dt}e^{at} = ae^{at}, \quad \frac{d}{dt}e^{-at} = -ae^{-at}$$

- $\dot{x}(t) = ax(t) + bu(t), \quad a \neq 0 \quad \because e^{-at} \neq 0 \implies e^{-at}\dot{x}(t) - e^{-at}ax(t) = e^{-at}bu(t)$
- namely,

$$\frac{d}{dt} \{e^{-at}x(t)\} = e^{-at}bu(t) \Leftrightarrow d \{e^{-at}x(t)\} = e^{-at}bu(t) dt$$

$$\implies \boxed{e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^t e^{-a\tau}bu(\tau) d\tau}$$

The solution to $\dot{x} = ax + bu$

$$e^{-at}x(t) = e^{-at_0}x(t_0) + \int_{t_0}^t e^{-a\tau} bu(\tau) d\tau$$

when $t_0 = 0$, we have

$$x(t) = \underbrace{e^{at}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{a(t-\tau)} bu(\tau) d\tau}_{\text{forced response}}$$

About e

- $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$
 - ▶ Taylor expansion

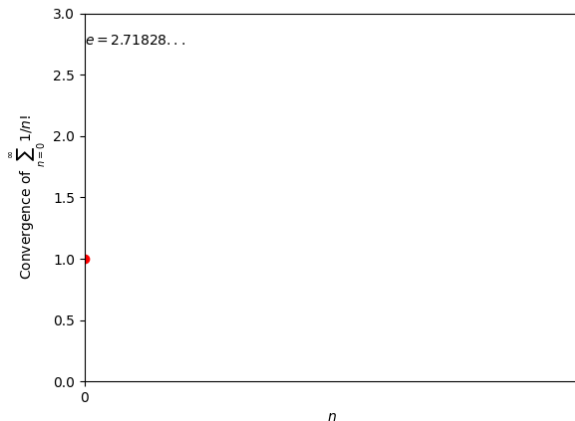
$$e^x = 1 + \frac{x}{1!} + \frac{1}{2!}(x)^2 + \dots + \frac{1}{n!}(x)^n + \dots$$

- ▶ letting $x = 1$ gives $e = \sum_{n=0}^{\infty} \frac{1}{n!}$
- Python demonstration:

```
import math
math.e
for ii in range(10):
    print(sum(1/math.factorial(k) for k in range(ii)))
```

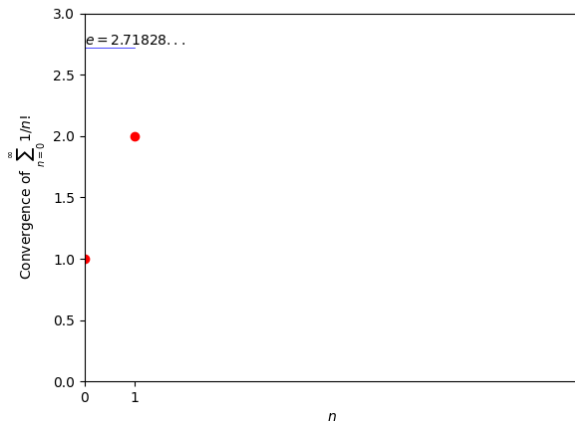
About e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$



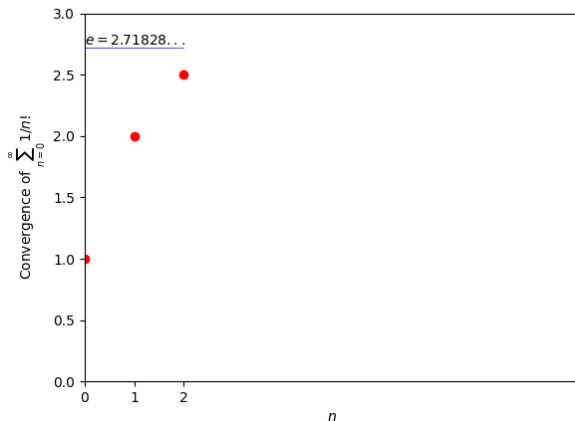
About e

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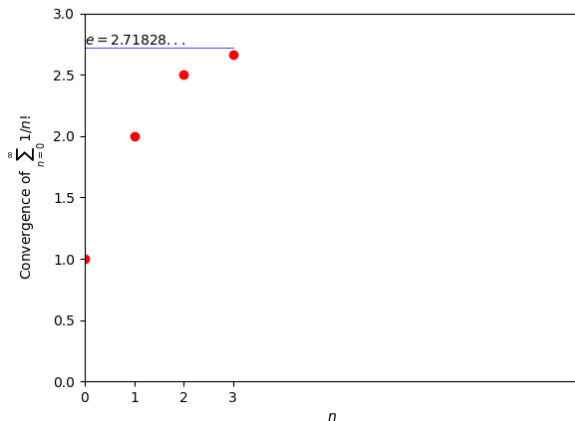
About e

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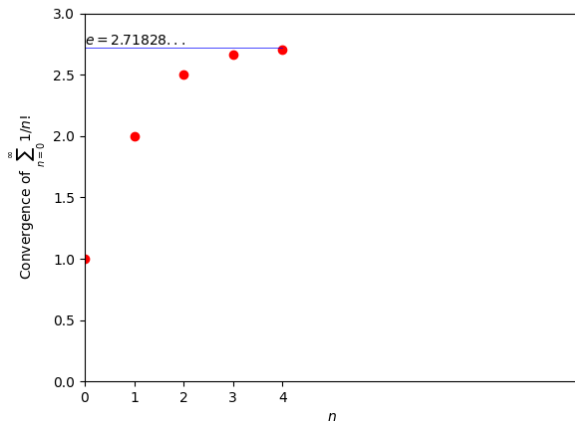
About e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$



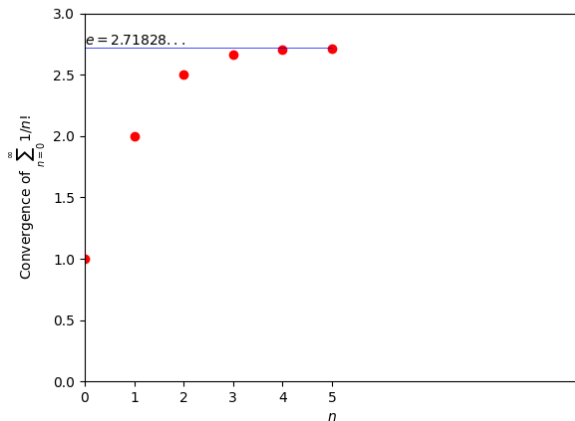
About e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$



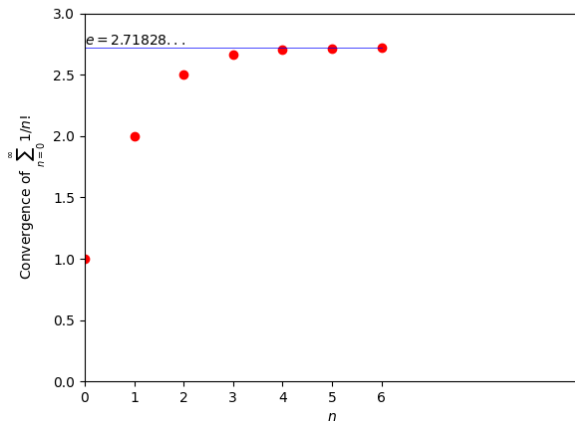
About e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$



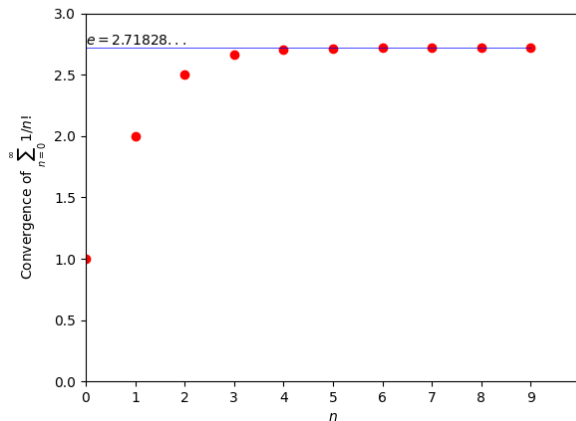
About e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$



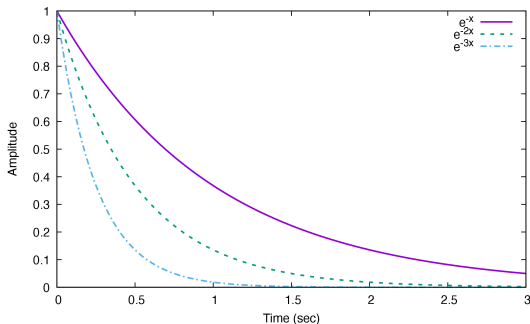
About e

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71828 \dots$$



The solution to $\dot{x} = ax + bu$

Solution concepts of $e^{at}x(0)$



$e = 2.71828 \dots$

$e^{-1} \approx 37\%$,

$e^{-2} \approx 14\%$,

$e^{-3} \approx 5\%$,

$e^{-4} \approx 2\%$

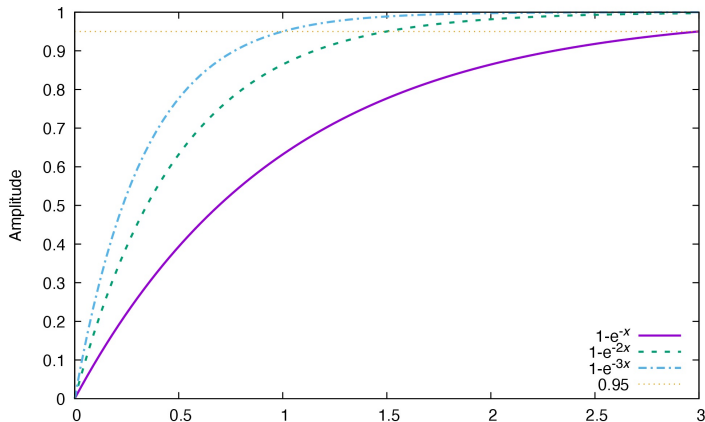
time constant $\tau \triangleq \frac{1}{|a|}$ when
 $a < 0$: after 3τ , $e^{at}x(0)$, the
transient has approximately
converged

The solution to $\dot{x} = ax + bu$

Unit step response

when $a < 0$ and $u(t) = 1(t)$ (the step function), the solution is

$$x(t) = \frac{b}{|a|}(1 - e^{at})$$



The solution to n^{th} -order LTI systems

- general state-space equation

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases} \quad x(t_0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

- solution

$$x(t) = \underbrace{e^{A(t-t_0)}x_0}_{\text{free response}} + \underbrace{\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau}_{\text{forced response}}$$

$$y(t) = Ce^{A(t-t_0)}x_0 + C \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- in both the free and the forced responses, computing e^{At} is key
- $e^{A(t-t_0)}$: called the transition matrix

The state transition matrix e^{At}

scalar case with $a \in \mathbb{R}$: Taylor expansion gives

$$e^{at} = 1 + at + \frac{1}{2}(at)^2 + \cdots + \frac{1}{n!}(at)^n + \dots$$

the transition scalar $\Phi(t, t_0) = e^{a(t-t_0)}$ satisfies

$$\Phi(t, t) = 1 \quad (\text{transition to itself})$$

$$\Phi(t_3, t_2)\Phi(t_2, t_1) = \Phi(t_3, t_1) \quad (\text{consecutive transition})$$

$$\Phi(t_2, t_1) = \Phi^{-1}(t_1, t_2) \quad (\text{reverse transition})$$

The state transition matrix e^{At}

matrix case with $A \in \mathbb{R}^{n \times n}$:

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots$$

- as I_n and A^i are matrices of dimension $n \times n$, e^{At} must $\in \mathbb{R}^{n \times n}$
- the transition matrix $\Phi(t, t_0) = e^{A(t-t_0)}$ satisfies

$$\begin{aligned} e^{A0} &= I_n & \Phi(t, t) &= I_n \\ e^{At_1}e^{At_2} &= e^{A(t_1+t_2)} & \Phi(t_3, t_2)\Phi(t_2, t_1) &= \Phi(t_3, t_1) \\ e^{-At} &= \left[e^{At}\right]^{-1} & \Phi(t_2, t_1) &= \Phi^{-1}(t_1, t_2) \end{aligned}$$

- note, however, that $e^{At}e^{Bt} = e^{(A+B)t}$ if and only if $AB = BA$ (check by using Taylor expansion)

Computing e^{At} when A is diagonal or in Jordan form

convenient when A is a diagonal or Jordan matrix

the case with a diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$:

- $A^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \dots, A^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$

- all matrices on the right side of

$$e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots$$

are easy to compute

Computing a structured e^{At} via Taylor expansion

the case with a diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$:

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{n!}A^nt^n + \dots \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 t & 0 & 0 \\ 0 & \lambda_2 t & 0 \\ 0 & 0 & \lambda_3 t \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\lambda_1^2 t^2 & 0 & 0 \\ 0 & \frac{1}{2}\lambda_2^2 t^2 & 0 \\ 0 & 0 & \frac{1}{2}\lambda_3^2 t^2 \end{bmatrix} + \dots \\ &= \begin{bmatrix} 1 + \lambda_1 t + \frac{1}{2}\lambda_1^2 t^2 + \dots & 0 & 0 \\ 0 & 1 + \lambda_2 t + \frac{1}{2}\lambda_2^2 t^2 + \dots & 0 \\ 0 & 0 & 1 + \lambda_3 t + \frac{1}{2}\lambda_3^2 t^2 + \dots \end{bmatrix} \\ &= \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{bmatrix} \end{aligned}$$

Computing a structured e^{At} via Taylor expansion

the case with a Jordan matrix $A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$:

- decompose $A = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_N \Rightarrow e^{At} = e^{(\lambda I_3 t + Nt)}$
- also, $(\lambda I_3 t)(Nt) = \lambda N t^2 = (Nt)(\lambda I_3 t)$ and hence $e^{(\lambda I_3 t + Nt)} = e^{\lambda I_3 t} e^{Nt}$
- thus

$$\underline{e^{At} = e^{(\lambda I_3 t + Nt)} = e^{\lambda I_3 t} e^{Nt} \because e^{\lambda I_3 t} = e^{\lambda t I} e^{\lambda t} e^{Nt}}$$

Computing a structured e^{At} via Taylor expansion

$$\underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_N, \quad e^{At} = e^{\lambda t} e^{Nt}$$

- N is *nilpotent*¹: $N^3 = N^4 = \dots = 0I_3$, yielding

$$e^{Nt} = I_3 + Nt + \frac{1}{2}N^2t^2 + \frac{1}{3!}N^3t^3 + \dots = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

- thus

$$e^{At} = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

¹“nil” \sim zero; “potent” \sim taking powers.

Computing a structured e^{At} via Taylor expansion

Mass moving on a straight line with zero friction and no external force

$x(t) = e^{At}x(0)$ where

$$e^{At} = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t + \underbrace{\frac{1}{2!} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}} t^2 + \dots = \underline{\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}}.$$

Computing low-order e^{At} via column solutions

an intuition of the matrix entries in e^{At} : consider:

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0$$

$$\begin{aligned} x(t) = e^{At}x(0) &= \left[\underbrace{\text{1st column}}_{a_1(t)} \mid \underbrace{\text{2nd column}}_{a_2(t)} \right] \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= a_1(t)x_1(0) + a_2(t)x_2(0) \end{aligned} \quad (1)$$

observation

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x(t) = a_1(t)$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow x(t) = a_2(t)$$

Computing low-order e^{At} via column solutions

$$\dot{x} = Ax = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x, \quad x(0) = x_0$$

hence, we can obtain e^{At} from:

- write out $\begin{matrix} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_2(t) \end{matrix} \Rightarrow \begin{matrix} x_1(t) = e^{0t}x_1(0) + \int_0^t e^{0(t-\tau)}x_2(\tau)d\tau \\ x_2(t) = e^{-t}x_2(0) \end{matrix}$
- let $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\begin{matrix} x_1(t) \equiv 1 \\ x_2(t) \equiv 0 \end{matrix}$, namely $x(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- let $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $x_2(t) = e^{-t}$ and $x_1(t) = 1 - e^{-t}$, or more compactly, $x(t) = \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$
- using (1), write out directly $e^{At} = \begin{bmatrix} 1 & 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$

Computing low-order e^{At} via column solutions

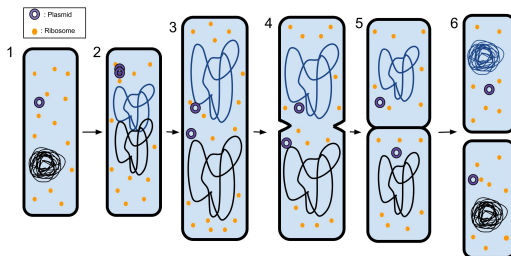
Compute e^{At} where

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

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Recall: population dynamics



prokaryotic fission

- ~1 hour / division with infinite resource
- after 1 day:

$$100 \xrightarrow[\frac{\Delta N}{N}=1]{1\text{hr}} 200 \xrightarrow{1\text{hr}} 400 \xrightarrow{1\text{hr}} \dots \longrightarrow 100 \times 2^{24} = 1.7\text{B!}$$

- or: $N(k+1) = 2N(k) \Rightarrow N(k) = 2^k N(0)$

Solution to discrete-time state equation

discrete-time system:

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0,$$

iteration of the state-space equation gives:

$$x(k) = A^{k-k_0}x(k_0) + \begin{bmatrix} A^{k-k_0-1}B, A^{k-k_0-2}B, \dots, B \end{bmatrix} \begin{bmatrix} u(k_0) \\ u(k_0+1) \\ \vdots \\ u(k-1) \end{bmatrix}$$

$$\Leftrightarrow x(k) = \underbrace{A^{k-k_0}x(k_0)}_{\text{free response}} + \underbrace{\sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j)}_{\text{forced response}}$$

Solution to discrete-time state equation

$$x(k) = \underbrace{A^{k-k_0}x(k_0)}_{\text{free response}} + \underbrace{\sum_{j=k_0}^{k-1} A^{k-1-j}Bu(j)}_{\text{forced response}}$$

$\Phi(k,j) = A^{k-j}$: the transition matrix:

$$\Phi(k,k) = 1$$

$$\Phi(k_3, k_2)\Phi(k_2, k_1) = \Phi(k_3, k_1) \quad k_3 \geq k_2 \geq k_1$$

$$\Phi(k_2, k_1) = \Phi^{-1}(k_1, k_2) \quad \text{if and only if } A \text{ is nonsingular}$$

The state transition matrix A^k

similar to the continuous-time case, when A is a diagonal or Jordan matrix, A^k is easy

- diagonal matrix $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} : A^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$

Computing a structured A^k via Taylor expansion

- Jordan canonical form

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{\lambda I_3} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_N :$$

$$A^k = (\lambda I_3 + N)^k \\ = (\lambda I_3)^k + k(\lambda I_3)^{k-1} N + \underbrace{\binom{k}{2} (\lambda I_3)^{k-2} N^2}_{2 \text{ combination}} + \underbrace{\binom{k}{3} (\lambda I_3)^{k-3} N^3 + \dots}_{N^3=N^4=\dots=0I_3}$$

$$= \begin{bmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^k \end{bmatrix} + k\lambda^{k-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{k(k-1)}{2} \lambda^{k-2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!} k(k-1) \lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

Computing a structured A^k via Taylor expansion

Recall that $\binom{k}{3} = \frac{1}{3!}k(k-1)(k-2)$. Show

$$A = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$
$$\Rightarrow A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} & \frac{1}{3!}k(k-1)(k-2)\lambda^{k-3} \\ 0 & \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} \\ 0 & 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & 0 & \lambda^k \end{bmatrix}$$

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Explicit computation of a general e^{At}

- why another method: general matrices may not be diagonal or Jordan
- approach: transform a general matrix to a diagonal or Jordan form, via similarity transformation

Computing e^{At} via similarity transformation

principle concept:

- given

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

- find a nonsingular $T \in \mathbb{R}^{n \times n}$ such that a coordinate transformation defined by $x(t) = Tx^*(t)$ yields

$$\begin{aligned} \frac{d}{dt} (Tx^*(t)) &= ATx^*(t) + Bu(t) \\ \frac{d}{dt} x^*(t) &= \underbrace{T^{-1}AT}_{\triangleq \Lambda: \text{diagonal or Jordan}} x^*(t) + \underbrace{T^{-1}B}_{B^*} u(t) \end{aligned}$$

$$x^*(0) = T^{-1}x_0$$

Computing e^{At} via similarity transformation

- when $u(t) = 0$

$$\dot{x}(t) = Ax(t) \xrightarrow{x=Tx^*} \frac{d}{dt}x^*(t) = \underbrace{T^{-1}AT}_{\triangleq \Lambda: \text{diagonal or Jordan}} x^*(t)$$

- now $x^*(t)$ can be solved easily: e.g., if $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then

$$x^*(t) = e^{\Lambda t} x^*(0) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} x_1^*(0) \\ e^{\lambda_2 t} x_2^*(0) \end{bmatrix}$$

- $x(t) = Tx^*(t)$ then yields

$$x(t) = Te^{\Lambda t} x^*(0) = Te^{\Lambda t} T^{-1} x_0$$

- on the other hand, $x(t) = e^{At} x_0 \Rightarrow$

$$\boxed{e^{At} = Te^{\Lambda t} T^{-1}}$$

Similarity transformation

- existence of solutions: T comes from the theory of eigenvalues and eigenvectors in linear algebra
- if A and $B \in \mathbb{C}^{n \times n}$ are similar: $A = TBT^{-1}$, $T \in \mathbb{C}^{n \times n}$, then
 - ▶ their A^n and B^n are also similar: e.g.,

$$A^2 = TBT^{-1}TBT^{-1} = TB^2T^{-1}$$

- ▶ their exponential matrices are also similar

$$e^{At} = Te^{Bt}T^{-1}$$

as

$$\begin{aligned}Te^{Bt}T^{-1} &= T(I_n + Bt + \frac{1}{2}B^2t^2 + \dots)T^{-1} \\&= TI_nT^{-1} + TBtT^{-1} + \frac{1}{2}TB^2t^2T^{-1} + \dots \\&= I + At + \frac{1}{2}A^2t^2 + \dots = e^{At}\end{aligned}$$

Similarity transformation

- for $A \in \mathbb{R}^{n \times n}$, an eigenvalue $\lambda \in \mathcal{C}$ of A is the solution to the characteristic equation

$$\boxed{\det(A - \lambda I) = 0} \quad (2)$$

- the corresponding eigenvectors are the nonzero solutions to

$$At = \lambda t \Leftrightarrow (A - \lambda I) t = 0 \quad (3)$$

Similarity transformation

The case with distinct eigenvalues (diagonalization)

recall: when $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues such that

$$Ax_1 = \lambda_1 x_1$$

$$\vdots$$

$$Ax_n = \lambda_n x_n$$

or equivalently

$$A \underbrace{[x_1, x_2, \dots, x_n]}_{\triangleq T} = [x_1, x_2, \dots, x_n] \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}}_{\Lambda}$$

$[x_1, x_2, \dots, x_n]$ is square and invertible. Hence

$$A = T\Lambda T^{-1}, \Lambda = T^{-1}AT$$

Similarity transform: diagonalization

Physical interpretations

- diagonalized system:

$$x^*(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1^*(0) \\ x_2^*(0) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} x_1^*(0) \\ e^{\lambda_2 t} x_2^*(0) \end{bmatrix}$$

- $x(t) = T x^*(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$ then decomposes the state trajectory into two modes parallel to the two eigenvectors.

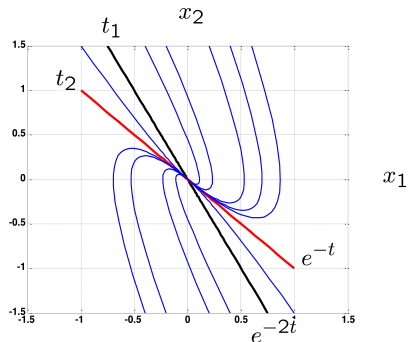
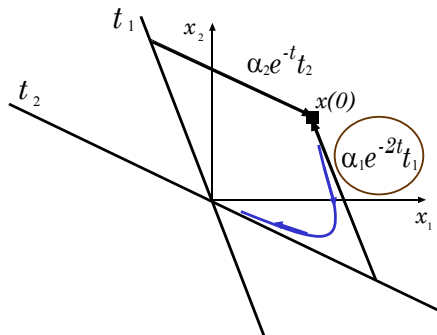
Similarity transform: diagonalization

Physical interpretations

- if $x(0)$ is aligned with one eigenvector, say, t_1 , then $x_2^*(0) = 0$ and $x(t) = e^{\lambda_1 t} x_1^*(0) t_1 + e^{\lambda_2 t} x_2^*(0) t_2$ dictates that $x(t)$ will stay in the direction of t_1
- i.e., if the state initiates along the direction of one eigenvector, then the free response will stay in that direction without “making turns”
- if $\lambda_1 < 0$, then $x(t)$ will move towards the origin of the state space; if $\lambda_1 = 0$, $x(t)$ will stay at the initial point; and if positive, $x(t)$ will move away from the origin along t_1
- furthermore, the magnitude of λ_1 determines the speed of response

Similarity transform: diagonalization

Physical interpretations



Similarity transformation

The case with complex eigenvalues

consider the undamped spring-mass system

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \det(A - \lambda I) = \lambda^2 + 1 = 0 \Rightarrow \lambda_{1,2} = \pm j.$$

the eigenvectors are

$$\lambda_1 = j: (A - jI)t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$\lambda_2 = -j: (A + jI)t_2 = 0 \Rightarrow t_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix} \quad (\text{complex conjugate of } t_1)$$

hence

$$T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

Similarity transformation

The case with complex eigenvalues

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\lambda_{1,2} = \pm j$
- $T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \quad T^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$
- we have

$$e^{At} = T e^{\Lambda t} T^{-1} = T \begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix} T^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Similarity transformation

The case with complex eigenvalues

for a general $A \in \mathbb{R}^{2 \times 2}$ with complex eigenvalues $\sigma \pm j\omega$, by using $T = [t_R, t_I]$, where t_R and t_I are the real and the imaginary parts of t_1 , an eigenvector associated with $\lambda_1 = \sigma + j\omega$, $x = Tx^*$ transforms $\dot{x} = Ax$ to

$$\dot{x}^*(t) = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} x^*(t)$$

and

$$e^{\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t} = \begin{bmatrix} e^{\sigma t} \cos \omega t & e^{\sigma t} \sin \omega t \\ -e^{\sigma t} \sin \omega t & e^{\sigma t} \cos \omega t \end{bmatrix}$$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

consider $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$: two repeated eigenvalues $\lambda(A) = 1$, and

$$(A - \lambda I) t_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- No other linearly independent eigenvectors exist. What next?
- A is already very similar to the Jordan form. Try instead

$$A \begin{bmatrix} t_1 & t_2 \end{bmatrix} = \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

which requires $At_2 = t_1 + \lambda t_2$, i.e.,

$$(A - \lambda I) t_2 = t_1 \Leftrightarrow \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} t_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow t_2 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$$

t_2 is linearly independent from $t_1 \Rightarrow t_1$ and t_2 span \mathbb{R}^2 . (t_2 is called a generalized eigenvector.)

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

for general 3×3 matrices with $\det(\lambda I - A) = (\lambda - \lambda_m)^3$, i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_m$, we look for T such that

$$A = TJT^{-1}$$

where J has three canonical forms:

$$\begin{aligned} i), & \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}, \quad iii), \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix} \\ ii), & \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix} \end{aligned}$$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$i), A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

this happens

- when A has three linearly independent eigenvectors, i.e., $(A - \lambda_m I)t = 0$ yields t_1 , t_2 , and t_3 that span \mathbb{R}^3
- mathematically: when nullity $(A - \lambda_m I) = 3$, namely, $\text{rank}(A - \lambda_m I) = 3 - \text{nullity}(A - \lambda_m I) = 0$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$ii), A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

- this happens when $(A - \lambda_m I)t = 0$ yields two linearly independent solutions, i.e., when nullity $(A - \lambda_m I) = 2$
- we then have, e.g.,

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$\Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, \lambda_m t_3] = [At_1, At_2, At_3]$$

- t_1 and t_3 are the directly computed eigenvectors
- for t_2 , the second column of the above gives $(A - \lambda_m I) t_2 = t_1$

Similarity transformation

The case with repeated eigenvalues via generalized eigenvectors

$$\text{iii), } A = TJT^{-1}, J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

- this is for the case when $(A - \lambda_m I)t = 0$ yields only one linearly independent solution, i.e., when $\text{nullity}(A - \lambda_m I) = 1$
- We then have

$$A[t_1, t_2, t_3] = [t_1, t_2, t_3] \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 1 \\ 0 & 0 & \lambda_m \end{bmatrix}$$

$$\Leftrightarrow [\lambda_m t_1, t_1 + \lambda_m t_2, t_2 + \lambda_m t_3] = [At_1, At_2, At_3]$$

yielding

$$(A - \lambda_m I) t_1 = 0$$

$$(A - \lambda_m I) t_2 = t_1, \quad (t_2 : \text{generalized eigenvector})$$

Example

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \det(A - \lambda I) = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0, J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- two repeated eigenvalues with $\text{rank}(A - 0I) = 1 \Rightarrow$ only one linearly independent eigenvector: $(A - 0I) t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- generalized eigenvector: $(A - 0I) t_2 = t_1 \Rightarrow t_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- coordinate transform matrix:

$$T = [t_1, t_2] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$e^{At} = T e^{Jt} T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & te^{0t} \\ 0 & e^{0t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$$

Example

$$A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \det(A - \lambda I) = \lambda^2 \Rightarrow \lambda_1 = \lambda_2 = 0.$$

observation:

- $\lambda_1 = 0$, $t_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ implies that if $x_1(0) = x_2(0)$ then the response is characterized by $e^{0t} = 1$
- i.e., $x_1(t) = x_1(0) = x_2(0) = x_2(t)$. This makes sense because $\dot{x}_1 = -x_1 + x_2$ from the state equation

Exercise

Obtain the eigenvectors of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \quad (\lambda_1 = 5, \lambda_2 = \lambda_3 = -3).$$

Generalized eigenvectors

Physical interpretation

when $\dot{x} = Ax$, $A = TJT^{-1}$ with $J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$, we have

$$\begin{aligned} x(t) &= e^{At}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} T^{-1}x(0) \\ &= T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} \cancel{T^{-1}} \overset{!}{T} x^*(0) \end{aligned}$$

- if the initial condition is in the direction of t_1 , i.e., $x^*(0) = [x_1^*(0), 0, 0]^T$ and $x_1^*(0) \neq 0$, the above equation yields $x(t) = x_1^*(0)t_1 e^{\lambda_m t}$

Generalized eigenvectors

Physical interpretation

when $\dot{x} = Ax$, $A = TJT^{-1}$ with $J = \begin{bmatrix} \lambda_m & 1 & 0 \\ 0 & \lambda_m & 0 \\ 0 & 0 & \lambda_m \end{bmatrix}$, we have

$$\begin{aligned} x(t) &= e^{At}x(0) = T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} T^{-1}x(0) \\ &= T \begin{bmatrix} e^{\lambda_m t} & te^{\lambda_m t} & 0 \\ 0 & e^{\lambda_m t} & 0 \\ 0 & 0 & e^{\lambda_m t} \end{bmatrix} \cancel{T^{-1}} T x^*(0) \end{aligned}$$

- if $x(0)$ starts in the direction of t_2 , i.e., $x^*(0) = [0, x_2^*(0), 0]^T$, then $x(t) = x_2^*(0)(t_1 te^{\lambda_m t} + t_2 e^{\lambda_m t})$. In this case, the response does not remain in the direction of t_2 but is confined in the subspace spanned by t_1 and t_2

Example

Obtain eigenvalues of J and e^{Jt} by inspection:

$$J = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

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Explicit computation of A^k

everything in getting the similarity transform applies to the DT case:

$$A^k = T\Lambda^k T^{-1} \text{ or } A^k = TJ^k T^{-1}$$

J	J^k
$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$	$\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{1}{2!}k(k-1)\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$
$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$	$\begin{bmatrix} \lambda^k & k\lambda^{k-1} & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$
$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$	$r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$ $r = \sqrt{\sigma^2 + \omega^2}$ $\theta = \tan^{-1} \frac{\omega}{\sigma}$

Example

Write down J^k for $J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ and

$$J = \begin{bmatrix} -10 & 1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -100 & 1 \\ 0 & 0 & 0 & -1 & -100 \end{bmatrix}.$$

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Transition matrix via inverse transformation

	Continuous-time system
state eq.	$\dot{x}(t) = Ax(t) + Bu(t), x(0) = x_0$
solution	$x(t) = \underbrace{e^{At}x(0)}_{\text{free response}} + \underbrace{\int_0^t e^{A(t-\tau)} Bu(\tau) d\tau}_{\text{forced response}}$
transition matrix	e^{At}

On the other hand, from Laplace transform:

$$\dot{x}(t) = Ax(t) + Bu(t) \Rightarrow X(s) = \underbrace{(sI - A)^{-1} x(0)}_{\text{free response}} + \underbrace{(sI - A)^{-1} BU(s)}_{\text{forced response}}$$

Comparing $x(t)$ and $X(s)$ gives

$$e^{At} = \mathcal{L}^{-1} \{ (sI - A)^{-1} \}$$

Example

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1} \left[\begin{bmatrix} s - \sigma & -\omega \\ \omega & s - \sigma \end{bmatrix}^{-1} \right] \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{(s - \sigma)^2 + \omega^2} \begin{bmatrix} s - \sigma & \omega \\ -\omega & s - \sigma \end{bmatrix} \right\} \\ &= e^{\sigma t} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \end{aligned}$$

Transition matrix via inverse transformation (DT case)

	Discrete-time system
state eq.	$x(k+1) = Ax(k) + Bu(k), x(0) = x_0$
solution	$x(k) = \underbrace{A^k x(0)}_{\text{free response}} + \underbrace{\sum_{j=0}^{(k-1)} A^{(k-1-j)} Bu(j)}_{\text{forced response}}$
transition matrix	transition matrix A^k

On the other hand, from Z transform:

$$X(z) = (zI - A)^{-1} zx(0) + (zI - A)^{-1} BU(s)$$

Hence

$$A^k = \mathcal{Z}^{-1} \{ (zI - A)^{-1} z \}$$

Example

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

$$\begin{aligned} A^k &= \mathcal{Z}^{-1} \left\{ z \begin{bmatrix} z - \sigma & -\omega \\ \omega & z - \sigma \end{bmatrix}^{-1} \right\} \\ &= \mathcal{Z}^{-1} \left\{ \frac{z}{(z - \sigma)^2 + \omega^2} \begin{bmatrix} z - \sigma & \omega \\ -\omega & z - \sigma \end{bmatrix} \right\} \\ &= \mathcal{Z}^{-1} \left\{ \frac{z}{z^2 - 2r \cos \theta z + r^2} \begin{bmatrix} z - r \cos \theta & r \sin \theta \\ -r \sin \theta & z - r \cos \theta \end{bmatrix} \right\} \\ &\quad , \quad r = \sqrt{\sigma^2 + \omega^2}, \quad \theta = \tan^{-1} \frac{\omega}{\sigma} \\ &= r^k \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix} \end{aligned}$$

Example

Consider $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.5 \end{bmatrix}$. We have

$$\begin{aligned} & (zI - A)^{-1} z \\ &= \begin{bmatrix} \frac{z(z-0.5)}{(z-0.8)(z-0.4)} & \frac{0.3z}{(z-0.8)(z-0.4)} \\ \frac{0.1z}{(z-0.8)(z-0.4)} & \frac{z(z-0.7)}{(z-0.8)(z-0.4)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{0.75z}{z-0.8} + \frac{0.25z}{z-0.4} & \frac{0.75z}{z-0.8} - \frac{0.75z}{z-0.4} \\ \frac{0.25z}{z-0.8} - \frac{0.25z}{z-0.4} & \frac{0.25z}{z-0.8} + \frac{0.75z}{z-0.4} \end{bmatrix} \\ \Rightarrow A^k &= \begin{bmatrix} 0.75(0.8)^k + 0.25(0.4)^k & 0.75(0.8)^k - 0.75(0.4)^k \\ 0.25(0.8)^k - 0.25(0.4)^k & 0.25(0.8)^k + 0.75(0.4)^k \end{bmatrix} \end{aligned}$$