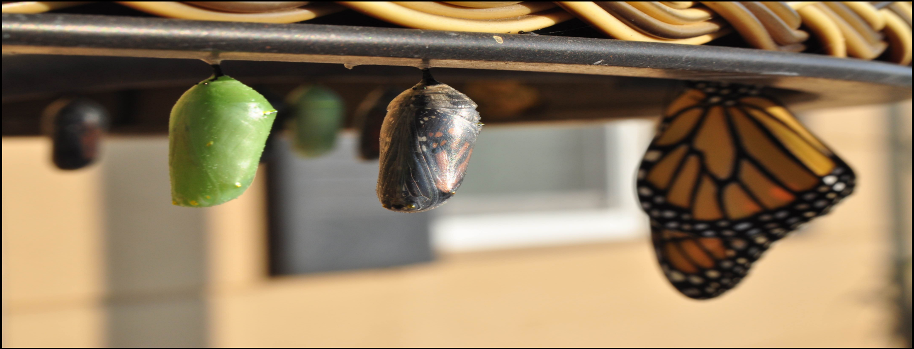
Introduction to Modern Controls

Z transform



The Z transform approach to Ordinary difference Equations (OdEs)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{OdE} \arrow[r, "\text{Z Transform}", "\text{Easy}"'] \arrow[d, dashed, "?"] & \text{Algebraic equation} \arrow[d, "\text{Easy}"', "\text{Arithmetic}"] \\
\text{OdE solution} & \text{Algebraic solution} \arrow[l, "\text{Easy}"', "\text{Inverse Z Transform}"]
\end{tikzcd}$$

- analogous to Laplace transform for continuous-time signals

Definition

- let $x(k)$ be a real discrete-time sequence that is zero if $k < 0$
- the (one-sided) Z transform of $x(k)$ is

$$X(z) \triangleq \mathcal{Z}\{x(k)\} = \sum_{k=0}^{\infty} x(k) z^{-k}$$
$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots$$

  where $z \in \mathbb{C}$
- a linear operator: $\mathcal{Z}\{\alpha f(k) + \beta g(k)\} = \alpha \mathcal{Z}\{f(k)\} + \beta \mathcal{Z}\{g(k)\}$
- the series $1 + \gamma + \gamma^2 + \ldots$ converges to $\frac{1}{1-\gamma}$ for $|\gamma| < 1$ [region of convergence (ROC)]
- (also, recall that $\sum_{k=0}^{N} \gamma^k = \frac{1-\gamma^{N+1}}{1-\gamma}$ if $\gamma \neq 1$)

Example: geometric sequence $\{a^k\}_{k=0}^{\infty}$

$$\boxed{\sum_{k=0}^{\infty} \gamma^k = \frac{1}{1-\gamma}}$$

- $x(k) = a^k$
- $\mathcal{Z}\{a^k\} = \sum_{k=0}^{\infty} a^k z^{-k} = \boxed{\dfrac{1}{1-az^{-1}}} = \frac{z}{z-a}$

Example: step sequence (discrete-time unit step function)

$$\boxed{\mathcal{Z}\{a^k\} = \frac{1}{1 - az^{-1}}}$$

- $1(k) = \begin{cases} 1, & \forall k = 1, 2, \ldots \\ 0, & \forall k = \ldots, -1, 0 \end{cases}$
- $\mathcal{Z}\{1(k)\} = \mathcal{Z}\{a^k\}\big|_{a=1} = \boxed{\frac{1}{1 - z^{-1}}} = \frac{z}{z-1}$

Example: discrete-time impulse

- $\delta(k) = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$
- $\mathcal{Z}\{\delta(k)\} = 1$

Exercise: $\cos(\omega_0 k)$

$f(k)$	$F(z)$	ROC
$\delta(k)$	1	All z
$a^k 1(k)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^k 1(-k-1)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$ka^k 1(k)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-ka^k 1(-k-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 k)$	$\frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 k)$	$\frac{z^{-1}\sin(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}}$	$\lvert z\rvert > 1$
$a^k\cos(\omega_0 k)$	$\frac{1-az^{-1}\cos(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^k\sin(\omega_0 k)$	$\frac{az^{-1}\sin(\omega_0)}{1-2az^{-1}\cos(\omega_0)+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Properties of Z transform: time shift

- let $\mathcal{Z}\{x(k)\} = X(z)$ and $x(k) = 0 \ \forall k < 0$
- one-step delay:

$$
\begin{aligned}
\mathcal{Z}\{x(k-1)\} = \sum_{k=0}^{\infty} x(k-1)z^{-k} &= \sum_{k=1}^{\infty} x(k-1)z^{-k} + x(-1) \\
&= \sum_{k=1}^{\infty} x(k-1)z^{-(k-1)}z^{-1} + x(-1) \\
&= z^{-1}X(z) + \cancel{x(-1)} = \boxed{z^{-1}X(z)}
\end{aligned}
$$

- analogously, $\underline{\mathcal{Z}\{x(k+1)\}} = \sum_{k=0}^{\infty} x(k+1)z^{-k} \boxed{= zX(z) - zx(0)}$
- thus, if $x(k+1) = Ax(k) + Bu(k)$ and $x(0) = 0$,

$$zX(z) = AX(z) + BU(z) \Rightarrow X(z) = (zI - A)^{-1}BU(z)$$

provided that $(zI - A)$ is invertible

Solving difference equations

Solve the difference equation

$$y(k) + 3y(k-1) + 2y(k-2) = u(k-2)$$

where $y(-2) = y(-1) = 0$ and $u(k) = 1(k)$.

- $\mathcal{Z}\{y(k-1)\} = z^{-1}\mathcal{Z}\{y(k)\} = z^{-1}Y(z)$
- $\mathcal{Z}\{y(k-2)\} = z^{-1}\mathcal{Z}\{y(k-1)\} = z^{-2}Y(z)$
- $\mathcal{Z}\{u(k-2)\} = z^{-2}U(z)$
- $\Rightarrow (1 + 3z^{-1} + 2z^{-2})Y(z) = z^{-2}U(z)$
- $\Rightarrow \boxed{Y(z) = \frac{1}{z^2 + 3z + 2}U(z)}$

Solving difference equations

Solve the difference equation

$$y(k) + 3y(k-1) + 2y(k-2) = u(k-2)$$

where $y(-2) = y(-1) = 0$ and $u(k) = 1(k)$.

- $\boxed{Y(z) = \dfrac{1}{z^2+3z+2}U(z) = \dfrac{1}{(z+2)(z+1)}U(z)}$
- $u(k) = 1(k) \Rightarrow U(z) = 1/(1-z^{-1})$
- $\Rightarrow Y(z) = \frac{z}{(z-1)(z+2)(z+1)} = \frac{1}{6}\frac{z}{z-1} + \frac{1}{3}\frac{z}{z+2} - \frac{1}{2}\frac{z}{z+1}$ (careful with the partial fraction expansion)
- inverse Z transform then gives $y(k) = \frac{1}{6}1(k) + \frac{1}{3}(-2)^k - \frac{1}{2}(-1)^k,\ k \geq 0$

From difference equation to transfer functions

- general discrete-time OdE:

$$y(k)+a_{n-1}y(k-1)+\cdots+a_0 y(k-n) = b_m u(k+m-n)+\cdots+b_0 u(k-n)$$

where $y(k) = 0 \; \forall k < 0$

- applying Z transform to the OdE yields

$$\left(z^n + a_{n-1}z^{n-1} + \cdots + a_0\right) Y(z) = \left(b_m z^m + b_{m-1}z^{m-1} + \cdots + b_0\right) U(z)$$

- hence

$$Y(z) = \underbrace{\frac{b_m z^m + b_{m-1}z^{m-1} \cdots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0}}_{G_{yu}(z)\text{: discrete-time transfer function}} U(z)$$

DC gain of discrete-time transfer functions

- general discrete-time OdE and transfer function:

$$y(k)+a_{n-1}y(k-1)+\cdots+a_0y(k-n)=b_mu(k+m-n)+\cdots+b_0u(k-n)$$

$$Y(z)=\underbrace{\frac{b_mz^m+b_{m-1}z^{m-1}\cdots+b_1z+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0}}_{G_{yu}(z)\text{: discrete-time transfer function}}U(z)$$

- assuming constant input and convergent output, then at steady state,
  - $y(k)=y(k-1)=\cdots=y(k-n)\triangleq y_{ss}$ and
    $u(k+m-n)=u(k+m-n-1)=\cdots=u(k-n)\triangleq u_{ss}$
  - $y_{ss}+a_{n-1}y_{ss}+\cdots+a_0y_{ss}=b_mu_{ss}+\cdots+b_0u_{ss}$
- thus,

$$\underline{\text{DC gain of }G_{yu}(z)}=\frac{b_m+b_{m-1}+\cdots+b_0}{1+a_{n-1}+\cdots+a_0}=\underline{G_{yu}(z)\big|_{z=1}}$$

Transfer functions in two domains

$$y(k) + a_{n-1}y(k-1) + \cdots + a_0 y(k-n) = b_m u(k+m-n) + \cdots + b_0 u(k-n)$$
$$\iff G_{yu}(z) = \frac{B(z)}{A(z)} = \frac{b_m z^m + b_{m-1} z^{m-1} \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

v.s.

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_0 u(t)$$
$$\iff G_{yu}(s) = \frac{B(s)}{A(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Properties	$G_{yu}(s)$	$G_{yu}(z)$
poles and zeros	roots of $A(s)$ and $B(s)$	roots of $A(z)$ and $B(z)$
causality condition	$n \geq m$	$n \geq m$
DC gain / steady-state response to unit step	$G_{yu}(0)$	$G_{yu}(1)$

Coding a discrete-time transfer function

```
num = [0.09952, -0.08144];
den = [1, -1.792, 0.8187];
Ts = 0.1;
sys_tf = tf(num,den,Ts)
poles = pole(sys_tf);
zeros = zero(sys_tf);
disp(['System Poles = ',num2str(poles')])
disp(['System Zeros = ',num2str(zeros')])

[yout, T] = step(sys_tf);
figure, stairs(T, yout)
figure, impulse(sys_tf)

u1 = 2*ones(length(T),1);
u2 = sin(T);
figure, lsim(sys_tf,u1,T)
figure, lsim(sys_tf,u2,T)
```

```python
import control as co
import matplotlib.pyplot as plt
import numpy as np
Ts = 0.1 # sampling time
num = [0.09952, -0.08144] # Numerator co-efficients
den = [1, -1.792, 0.8187] # Denominator co-efficients
sys_tf = co.tf(num,den, Ts)
print(sys_tf)

poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout = co.step_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.step(T,np.append(0,yout[0:-1]))
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()
```

```python
import control as co
import matplotlib.pyplot as plt
import numpy as np
Ts = 0.1 # sampling time
num = [0.09952, -0.08144] # Numerator co-efficients
den = [1, -1.792, 0.8187] # Denominator co-efficients
sys_tf = co.tf(num,den, Ts)
print(sys_tf)

poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout_i = co.impulse_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.step(T,np.append(0,yout_i[0:-1]))
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()
```

Additional useful properties of Z transform

- time shifting (assuming $x(k) = 0$ if $k < 0$):

$$\mathcal{Z}\left\{x\left(k-n_d\right)\right\} = z^{-n_d} X\left(z\right)$$

- Z-domain scaling: $\mathcal{Z}\left\{a^k x\left(k\right)\right\} = X\left(a^{-1} z\right)$
- differentiation: $\mathcal{Z}\left\{kx\left(k\right)\right\} = -z\frac{dX(z)}{dz}$
- time reversal: $\mathcal{Z}\left\{x\left(-k\right)\right\} = X\left(z^{-1}\right)$
- convolution: let $f(k) * g(k) \triangleq \sum_{j=0}^{k} f\left(k-j\right) g\left(j\right)$, then

$$\mathcal{Z}\left\{f(k) * g(k)\right\} = F\left(z\right) G\left(z\right)$$

- initial value theorem: $f\left(0\right) = \lim_{z\to\infty} F\left(z\right)$
- final value theorem: $\lim_{k\to\infty} f\left(k\right) = \lim_{z\to 1}\left(z-1\right) F\left(z\right)$, if $\lim_{k\to\infty} f(k)$ exists and is finite

Mortgage payment

- image you borrow \$100,000 (e.g., for a mortgage)
- annual percent rate: $APR = 4.0\%$
- plan to pay off in 30 years with fixed monthly payments
- interest computed monthly
- what is your monthly payment?

Mortgage payment

- borrow \$100,000 $\Rightarrow$ initial debt $y(0) = 100,000$
- $APR = 4.0\% \Rightarrow MPR = \frac{4.0\%}{12} = 0.0033$
- pay off in 30 years ($N = 30 \times 12 = 360$ months) $\Rightarrow y(N) = 0$
- debt at month $k+1$:
  $$y(k+1) = \underbrace{(1 + MPR)}_{a} y(k) - \underbrace{b}_{\text{monthly payment}} 1(k)$$
- $\Rightarrow Y(z) = \frac{z}{z-a} y(0) - \frac{1}{z-a} \frac{b}{1-z^{-1}}$
  $\Rightarrow Y(z) = \frac{1}{1-az^{-1}} y(0) + \frac{b}{1-a} \left(\frac{1}{1-az^{-1}} - \frac{1}{1-z^{-1}} \right)$
- $\Rightarrow y(k) = a^k y(0) + \frac{b}{1-a} \left(a^k - 1\right)$
- need $y(N) = 0 \Rightarrow a^N y(0) = -\frac{b}{1-a} \left(a^N - 1\right)$
- $\Rightarrow b = \frac{a^N y(0)(a-1)}{a^N - 1} = \$477.42$