Introduction to Modern Controls

Inverse Laplace transform

Common Laplace transform pairs

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$	e^{-at}	$\dfrac{1}{s+a}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$	t	$\dfrac{1}{s^2}$
$tx(t)$	$-\dfrac{dX(s)}{ds}$	t^2	$\dfrac{2}{s^3}$
$\frac{x(t)}{t}$	$\int_s^\infty X(s)\,ds$	te^{-at}	$\dfrac{1}{(s+a)^2}$
$\delta(t)$	1	$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
$1(t)$	$\dfrac{1}{s}$	$e^{-at}\cos(\omega t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

Overview of inverse Laplace transform: modularity and decomposition

- goal: to break a large Laplace transform into small blocks, so that we can use elemental examples of Laplace transfer functions:

$$G(s) = \frac{B(s)}{A(s)} = \frac{B_1(s)}{A_1(s)} + \frac{B_2(s)}{A_2(s)} + \dots$$

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- we will use examples to demonstrate strategies for common fractional expansions

Real and distinct roots in $A(s)$

example 1

$$G(s) = \frac{B(s)}{A(s)} = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$

residues:

- $K_1 = \lim_{s\to 0} sG(s) = 1$

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- $K_3 = \lim_{s\to -8}(s+8)G(s) = 1$

Coding partial fraction expansions

```
% MATLAB
syms s
G = 32/s/(s+4)/(s+8)
partfrac(G)
```

```
# Python
import sympy
s = sympy.symbols('s')
G = 32/s/(s+4)/(s+8)
print(sympy.apart(G))
```

```
1/(s + 8) - 2/(s + 4) + 1/s
```

Real and repeated roots in $A(s)$

example 2

$$G(s) = \frac{2}{(s+1)(s+2)^2} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{(s+2)^2}$$

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- $K_1 = \lim_{s \to -1} (s+1) G(s) = 2$

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- $K_1 = \lim_{s\to -1}(s+1) G(s) = 2$
- for K_2, we hit both sides with $(s+2)^2$ then differentiate once w.r.t. s, to get

$$K_2 = \lim_{s\to -2} \frac{d}{ds}(s+2)^2 G(s) = -2$$

Coding partial fraction expansions

```
# Python
import sympy
s = sympy.symbols('s')
G = 2/(s+1)/(s+2)**2
print(sympy.apart(G))
```

```
-2/(s + 2) - 2/(s + 2)**2 + 2/(s + 1)
```

Solution of a first-order ODE

example 1: Let $a > 0, b > 0, y(0) = y_0 \in \mathbb{R}$, obtain the solution to the ODE:

$$\dot{y}(t) = -ay(t) + b1(t)$$

where $1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$

- Laplace transform: $\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0)$

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- $\Rightarrow$ solution in Laplace domain:

$$Y(s) = \frac{1}{s+a}y(0) + \frac{b}{s(s+a)} = \frac{1}{s+a}y(0) + \frac{b}{a}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

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- apply inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \ldots$

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- solution:

$$y(t) = e^{-at}y(0) + \frac{b}{a}(1(t) - e^{-at})$$

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observations:

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- using final value theorem,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \frac{b}{a}$$

Solution of a first-order ODE

example 2: Let $a > 0, b > 0, y(0) = y_0 \in \mathbb{R}$, obtain the solution to the ODE:

$$\dot{y}(t) = -ay(t) + b\delta(t)$$

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- Q: what's the initial value from initial value theorem? what does the impulse do to the initial condition?

Connecting two domains

- n-th order differential equation:

$$\frac{d^n y}{dt^n}+a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}+\cdots+a_1\dot{y}+a_0 y = b_m\frac{d^m u}{dt^m}+b_{m-1}\frac{d^{m-1}u}{dt^{m-1}}+\cdots+b_1\dot{u}+b_o u$$

where $y(0) = 0, \frac{dy}{dt}|_{t=0} = 0, \ldots, \frac{d^{n-1}y}{dt^{n-1}}|_{t=0} = 0$

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- applying Laplace transform yields

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_0)Y(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)U(s)$$

$$\Rightarrow Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}U(s)$$

Transfer functions

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

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  - *proper* if $n \geq m$
  - *strictly proper* if $n > m$
- examples: $G_1(s) = K$, $G_2(s) = \frac{k}{s+a}$

Coding transfer functions in Python

```python
import control as co
import matplotlib.pyplot as plt
import numpy as np
num = [1,2] # Numerator co-efficients
den = [1,2,3] # Denominator co-efficients
sys_tf = co.tf(num,den)
print(sys_tf)
poles = co.pole(sys_tf)
zeros = co.zero(sys_tf)
print('\nSystem Poles = ', poles, '\nSystem Zeros = ', zeros)

T,yout = co.step_response(sys_tf)
plt.figure(1,figsize = (6,4))
plt.plot(T,yout)
plt.grid(True)
plt.ylabel("y")
plt.xlabel("Time (sec)")
plt.show()
```

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T,yout = co.step_response(sys_tf)
u1 = np.full((1,len(T)),2) # Create an array of 2's
u2 = np.sin(T)
T,yout_u1 = co.forced_response(sys_tf,T,u1) # Response to input 1
T,yout_u2 = co.forced_response(sys_tf,T,u2) # Response to input 2
```

The DC gain

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

- DC gain: the ratio of a stable system's output to its input after all transients have decayed

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- DC gain: the ratio of a stable system's output to its input after all transients have decayed
- can use the Final Value Theorem to find the DC gain:

$$\underline{\text{DC gain of } G(s)} = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sG(s)\frac{1}{s} = \underline{\lim_{s \to 0} G(s)}$$

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- example: find the DC gain of $G_1(s) = K$ and $G_2(s) = \frac{k}{s+a}$. Try (i) solve the ODE and (ii) the Final Value Theorem

The DC gain in Matlab and Python

```
% MATLAB
s = tf('s');
G = (2*s+3)/(4*s^2+3*s+1);
dcgain(G)
```

```
# Python
import control as co
s = co.tf('s')
G = (2*s+3)/(4*s**2+3*s+1);
print(co.dcgain(G))
```

The DC gain in Matlab and Python

find the DC gain of the system corresponding to $Y_2(s) = \frac{3}{s-2}$

```
# Python
import control as co
H = co.tf([0, 3],[1, -2])
print(co.dcgain(H))
T, yout = co.step_response(H)
print(yout)
```

- exercise: verify the result in Matlab

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- exercise: verify the result in Matlab
- is the result correct?