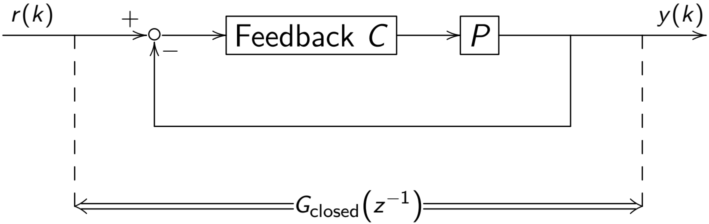
Feedforward Control
Zero Phase Error Tracking

Big picture
Stable pole-zero cancellation
Phase error
Zero phase error tracking

Big picture

why are we learning this:

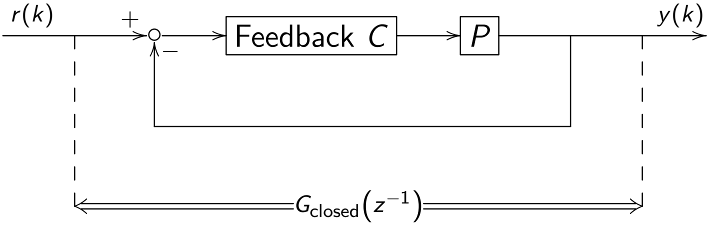


- two basic control problems: tracking (the reference) and regulation (against disturbances)
- feedback control has performance limitations
- For tracking $r(k)$, ideally we want

$$G_{\text{closed}}\left(z^{-1}\right) = 1$$

which is **not attainable** by feedback. We thus need **feedforward** control.

Big picture



- notation:

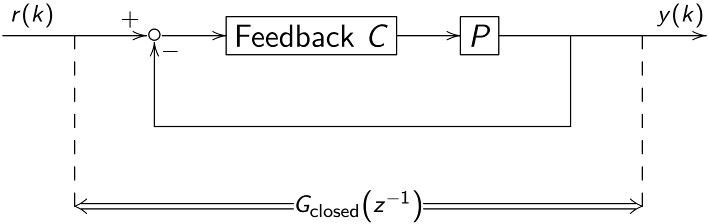
$$G_{\text{closed}}\left(z^{-1}\right) = \frac{z^{-d}B_c\left(z^{-1}\right)}{A_c\left(z^{-1}\right)}$$

where

$$B_c\left(z^{-1}\right) = b_{c0} + b_{c1}z^{-1} + \cdots + b_{cm}z^{-m},\ b_{co} \neq 0$$
$$A_c\left(z^{-1}\right) = 1 + a_{c1}z^{-1} + \cdots + a_{cn}z^{-n}$$

- z^{-1}: one-step delay operator. $z^{-1}r(k) = r(k-1)$

Big picture



one naive approach: to let $y(k)$ track $y_d(k)$, we can do

$$r(k) = G_{\text{closed}}^{-1}\left(z^{-1}\right) y_d(k) = \frac{z^d A_c\left(z^{-1}\right)}{B_c\left(z^{-1}\right)} y_d(k) = \frac{A_c\left(z^{-1}\right)}{B_c\left(z^{-1}\right)} y_d(k+d) \quad (1)$$

- **causality**: (1) requires knowledge of $y_d(k)$ for at least d steps ahead (usually not an issue)
- **stability**: poles of $G_{\text{closed}}^{-1}\left(z^{-1}\right)$, i.e., zeros of $G_{\text{closed}}\left(z^{-1}\right)$, must be all stable (usually an issue)
- **robustness**: the model $G_{\text{closed}}\left(z^{-1}\right)$ needs to be accurate

The cancellable parts in $G_{\text{closed}}\left(z^{-1}\right)$

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

- $G_{\text{closed}}(z^{-1})$ is always stable $\Rightarrow$ $A_c\left(z^{-1}\right)$ can be fully canceled
- $B_c\left(z^{-1}\right)$ may contain *uncancellable parts* (zeros on or outside the unit circle)
- partition $G_{\text{closed}}(z^{-1})$ as

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})} = \frac{z^{-d}\overbrace{B_c^{+}(z^{-1})}^{\text{cancellable}}\ \overbrace{B_c^{-}(z^{-1})}^{\text{uncancellable}}}{A_c(z^{-1})} \tag{2}$$

Stable pole-zero cancellation

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

feedforward via stable pole-zero cancellation:

$$G_{\text{spz}}\left(z^{-1}\right) = \frac{z^{d}A_c\left(z^{-1}\right)}{B_c^{+}\left(z^{-1}\right)}\frac{1}{B_c^{-}(1)} \tag{3}$$

where $B_c^{-}(1) = B_c^{-}\left(z^{-1}\right)\Big|_{z^{-1}=1}$

- $B_c^{-}(1)$ makes the overall DC gain from $y_d(k)$ to $y(k)$ to be one:

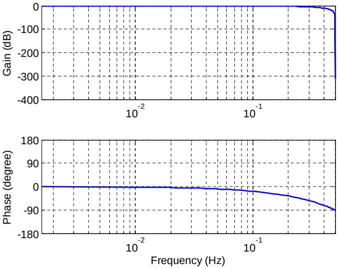
$$G_{y_d \to y}\left(z^{-1}\right) = G_{\text{spz}}\left(z^{-1}\right)G_{\text{closed}}\left(z^{-1}\right) = \frac{B_c^{-}(z^{-1})}{B_c^{-}(1)}$$

- example: $B_c^{-}\left(z^{-1}\right) = 1 + z^{-1}$, $B_c^{-}(1) = 2$, then

$$G_{y_d \to y}\left(z^{-1}\right) = \frac{1+z^{-1}}{2}\text{: a moving-average low-pass filter}$$

Stable pole-zero cancellation

properties of $G_{y_d\to y}\left(z^{-1}\right)=\frac{1+z^{-1}}{2}$:



- there is always a **phase error** in tracking
- example: if $y_d(k)=\alpha k$ (a ramp signal)

$$y(k)=G_{y_d\to y}\left(z^{-1}\right)y_d(k)=\alpha k-\frac{\alpha}{2}$$

which is always delayed by a factor of $\alpha/2$

Zero Phase Error Tracking (ZPET)

$$y_d(k) \longrightarrow \boxed{\text{Feedforward}} \longrightarrow \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c^+(z^{-1})B_c^-(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

Zero Phase Error Tracking (ZPET): extend (3) by adding a $B_c^-(z)$ part

$$\boxed{G_{\text{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2}} \qquad (4)$$

where $B_c^-(z) = b_{c0}^- + b_{c1}^- z + \cdots + b_{cs}^- z^s$ if
$B_c^-(z^{-1}) = b_{c0}^- + b_{c1}^- z^{-1} + \cdots + b_{cs}^- z^{-s}$

- overall dynamics between $y(k)$ and $y_d(k)$:

$$G_{y_d \to y}\left(z^{-1}\right) = G_{\text{closed}}\left(z^{-1}\right) G_{\text{ZPET}}\left(z^{-1}\right) = \frac{B_c^-(z)B_c^-(z^{-1})}{\left[B_c^-(1)\right]^2} \qquad (5)$$

Zero Phase Error Tracking (ZPET)

understanding (5):

- the frequency response always has **zero phase error**:

$$B_c^-\left(e^{j\omega}\right)=\overline{B_c^-\left(e^{-j\omega}\right)}\text{ (a complex conjugate pair)}$$

- $B_c^-(1)^2$ normalizes $Gy_d \to y$ to have unity DC gain:

$$G_{y_d\to y}\left(e^{-j\omega}\right)\Big|_{\omega=0}=\frac{B_c^-(e^{j\omega})\big|_{\omega=0}\,B_c^-(e^{-j\omega})\big|_{\omega=0}}{\left[B_c^-(1)\right]^2}=\cancelto{1}{\frac{\left[B_c^-(1)\right]^2}{\left[B_c^-(1)\right]^2}}$$

- example: $B_c^-\left(z^{-1}\right)=1+z^{-1}$, then

$$G_{y_d\to y}\left(z^{-1}\right)=\frac{(1+z)\left(1+z^{-1}\right)}{4}$$

- if $y_d(k)=\alpha k$, then $y(k)=\alpha k$!
- fact: ZPET provides perfect tracking to step and ramp signals at steady state (see ME 233 course reader)

Zero Phase Error Tracking (ZPET)

Example: $B_c^{-}\left(z^{-1}\right) = 1 + 2z^{-1}$

$$G_{y_d \to y}\left(z^{-1}\right) = \frac{(1+2z)(1+2z^{-1})}{9} = \frac{2z+5+2z^{-1}}{9}$$

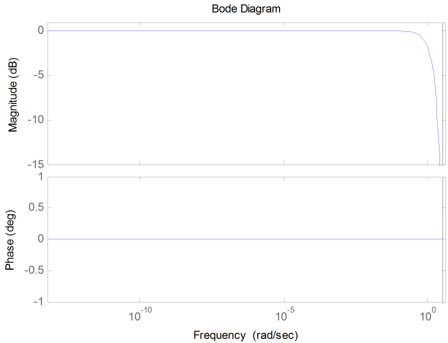


Figure: Bode Plot of $\frac{2z+5+2z^{-1}}{9}$

Implementation

$$y_d(k) \longrightarrow \boxed{G_{\text{ZPET}}\left(z^{-1}\right)} \xrightarrow{r(k)} \boxed{G_{\text{closed}}(z^{-1}) = \frac{z^{-d}B_c^+(z^{-1})B_c^-(z^{-1})}{A_c(z^{-1})}} \longrightarrow y(k)$$

$$r(k) = \left[\frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})}\frac{B_c^-(z)}{B_c^-(1)^2}\right] y_d(k)$$

- z^d is not causal $\Rightarrow$ do instead

$$r(k) = \left[\frac{A_c(z^{-1})}{B_c^+(z^{-1})}\frac{B_c^-(z)}{B_c^-(1)^2}\right] y_d(k+d)$$

- $B_c^-(z) = b_{c0}^- + b_{c1}^- z + \cdots + b_{cs}^- z^s$ is also not causal $\Rightarrow$ do instead

$$r(k) = \left[\frac{A_c(z^{-1})}{B_c^+(z^{-1})}\frac{z^{-s}B_c^-(z)}{B_c^-(1)^2}\right] y_d(k+d+s)$$

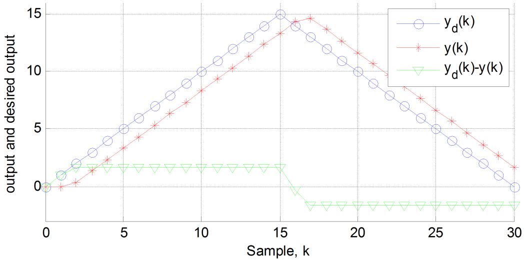
- at time k, requires $y_d(k+d+s)$: $d+s$ steps preview of the desired output

Implementation

Example:

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}(1+2z^{-1})}{3}$$

- without feedforward control:

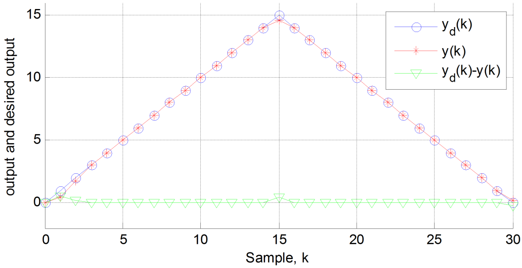


Implementation

Example:

$$G_{\text{closed}}(z^{-1}) = \frac{z^{-1}(1+2z^{-1})}{3}$$

- with ZPET feedforward:



Implementation

ZPET extensions:

- standard form:

$$G_{\mathrm{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2}$$

- extended bandwidth (ref: B. Haack and M. Tomizuka, "The effect of adding zeros to feedforward controllers," *ASME J. Dyn. Syst. Meas. Control*, vol. 113, pp. 6-10, 1991):

$$G_{\mathrm{ZPET}}(z^{-1}) = \frac{z^d A_c(z^{-1})}{B_c^+(z^{-1})} \frac{B_c^-(z)}{B_c^-(1)^2} \frac{\left(z^{-1} - b\right)(z - b)}{(1 - b)^2},\ 0 < b < 1$$

- remark: $\left(z^{-1} - b\right)(z - b) / (1 - b)^2$, $0 < b < 1$ is a high-pass filter with unity DC gain