Lyapunov Stability

Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

- a metric in vector space: a function that assigns a real-valued length to each vector in a vector space
- e.g., 2 (Euclidean) norm: $\|v\|_2 = \sqrt{v^T v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

Equilibrium state

For an *n*-th order unforced system

$$\dot{x} = f\left(x, t\right), \; x(t_0) = x_0$$

an equilibrium state/point x_e is one such that

$$\boxed{f\left(x_e, t\right) = 0, \;\; \forall t}$$

- the condition must be satisfied by all $t \geq 0$
- if a system starts at equilibrium state, it stays there

Equilibrium state of a linear system

For a linear system

$$\dot{x}(t) = A(t)x(t), \; x(t_0) = x_0$$

- origin $x_e = 0$ is always an equilibrium state
- when $A(t)$ is singular, multiple equilibrium states exist

Lyapunov's definition of stability

- The equilibrium state 0 of $\dot{x} = f(x, t)$ is *stable in the sense of Lyapunov (s.i.L)* if for all $\epsilon > 0$, and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $\|x(t)\|_2 < \epsilon$ for all $t \geq t_0$

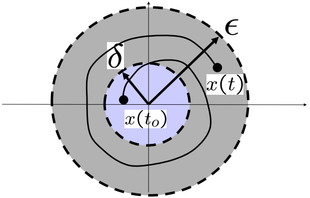


Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \; \forall t \geq t_0$.

Asymptotic stability

The equilibrium state 0 of $\dot{x} = f(x, t)$ is asymptotically stable if

- it is stable in the sense of Lyapunov, and
- for all $\epsilon > 0$ and t_0, there exists $\delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\|_2 < \delta$ gives $x(t) \rightarrow 0$

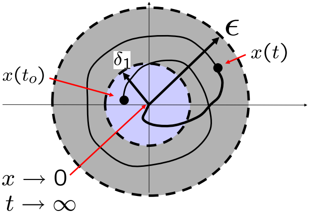


Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \rightarrow 0$.

Stability of LTI systems: method of eigenvalue/pole locations

the stability of the equilibrium point 0 for $\dot{x} = Ax$ or $x(k+1) = Ax(k)$ can be concluded immediately based on $\lambda(A)$:

- the response $e^{At}x(0)$ involves modes such as $e^{\lambda t}$, $te^{\lambda t}$, $e^{\sigma t}\cos\omega t$, $e^{\sigma t}\sin\omega t$
- the response $A^k x(0)$ involves modes such as λ^k, $k\lambda^{k-1}$, $r^k\cos k\theta$, $r^k\sin k\theta$
- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2 + \omega^2}\right| = |\lambda| < 1$

Lyapunov's approach to stability

The direct method of Lyapunov to stability problems:

- no need for explicit solutions to system responses
- an "energy" perspective
- fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \qquad\qquad (x_1\text{: position; } x_2\text{ : velocity})$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2,\ \ b > 0 \qquad\qquad \text{(Newton's law)}$$

- $\lambda(A)$'s are in the left-half s-plane$\Rightarrow$ asymptotically stable
- total energy

$$\mathcal{E}(t) = \text{potential energy} + \text{kinetic energy} = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$$

- energy dissipates / is dissipative:

$$\dot{\mathcal{E}}(t) = kx_1\dot{x}_1 + mx_2\dot{x}_2 = -bx_2^2 \leq 0$$

- $\dot{\mathcal{E}} = 0$ only when $x_2 = 0$. As $[x_1, x_2]^T = 0$ is the only equilibrium, the motion will not stop at $x_2 = 0$, $x_1 \neq 0$. Thus energy will keep decreasing toward 0 which is achieved at the origin.

Stability from an energy viewpoint: Generalization

Consider unforced, time-varying, nonlinear systems

$$\dot{x}(t) = f\left(x(t), t\right), \ x\left(t_0\right) = x_0$$
$$x\left(k+1\right) = f\left(x(k), k\right), \ x\left(k_0\right) = x_0$$

- assume the origin is an equilibrium state
- energy function $\Rightarrow$ Lyapunov function: a scalar function of x and t (or x and k)
- goal is to relate properties of the state through the Lyapunov function
- main tool: matrix formulation, linear algebra, positive definite functions

Relevant tools

Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- convenience of matrix formulation:

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + x_1x_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} \frac{k}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{m}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 + x_1x_2 + c = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{k}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{m}{2} & 0 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

- general quadratic functions in matrix form

$$Q(x) = x^T P x, \;\; P^T = P$$

Relevant tools

Symmetric matrices

- recall: a real square matrix A is
  - *symmetric* if $A = A^T$
  - *skew-symmetric* if $A = -A^T$
- examples:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

- *Any* real square matrix can be decomposed as the sum of a *symmetric* matrix and a *skew-symmetric* matrix:

$$\text{e.g.} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2.5 \\ 2.5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix}$$

$$\text{general case:} \quad P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

- a real square matrix $A \in \mathbb{R}^{n \times n}$ is *orthogonal* if $A^T A = AA^T = I$
- meaning that the columns of A form a orthonormal basis of $\mathbb{R}^n$

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & \dots & a_n \\ | & | & | & | \end{bmatrix}$$

$$A^T A = \begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & \vdots & \vdots \\ a_n^T a_1 & a_n^T a_2 & \dots & a_n^T a_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

namely, $a_j^T a_j = 1$ and $a_j^T a_m = 0 \ \forall j \neq m$.

Theorem

The eigenvalues of symmetric matrices are all real.

Proof: $\forall : A \in \mathbb{R}^{n\times n}$ with $A^T = A$.

Eigenvalue-eigenvector pair: $Au = \lambda u \Rightarrow \overline{u}^T A u = \lambda \overline{u}^T u$, where $\overline{u}$ is the complex conjugate of u. $\overline{u}^T A u$ is a real number, as

$$\begin{aligned}
\overline{\overline{u}^T A u} &= u^T \overline{A} \overline{u} \\
&= u^T A \overline{u} \quad \because A \in \mathbb{R}^{n\times n} \\
&= u^T A^T \overline{u} \quad \because A = A^T \\
&= \lambda u^T \overline{u} \quad \because (Au)^T = (\lambda u)^T \\
&= \lambda \overline{u}^T u \quad \because u^T \overline{u} \in \mathbb{R} \\
&= \overline{u}^T A u \quad \because Au = \lambda u
\end{aligned}$$

Also, $\overline{u}^T u \in \mathbb{R}$. Thus $\lambda = \frac{\overline{u}^T A u}{\overline{u}^T u}$ must also be a real number. $\square$

Example

- $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$: $\lambda = \pm 2$
- $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$: $\lambda = 1 \pm 2$

```
import numpy as np #larger-scale Python example
N = 100
P = np.random.randint(-200,200,size=(N,N))
P_symm = (P + P.T)/2
lambdas, _ = np.linalg.eig(P_symm)
print(lambdas)
```

Theorem

The eigenvalues of skew-symmetric matrices are all imaginary or zero.

- $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$: $\lambda = \pm 2j$

```
import numpy as np
N = 100
P = np.random.randint(-200,200,size=(N,N))
P_symm = (P - P.T)/2
lambdas, _ = np.linalg.eig(P_symm)
print(lambdas)
```

Theorem

All eigenvalues of an orthogonal matrix have a magnitude of 1.

- $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} : \lambda = 1 \pm 2j$

```
import numpy as np
from scipy.linalg import qr
n = 3
H = np.random.randn(n, n)
Q, _ = qr(H)
print (np.dot(Q,Q.T))
print (np.dot(Q.T,Q))
```

Important properties of symmetric matrices

Theorem

The eigenvalues of symmetric matrices are all real.

Theorem

The eigenvalues of skew-symmetric matrices are all imaginary or zero.

Theorem

All eigenvalues of an orthogonal matrix have a magnitude of 1.

matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

The spectral theorem for symmetric matrices

When $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, we can do diagonalization $A = U \Lambda U^{-1}$. When A is symmetric, things are even better:

Theorem (Symmetric eigenvalue decomposition (SED))

$\forall : A \in \mathbb{R}^{n \times n},\ A^T = A$, *there always exist* $\lambda_i \in \mathbb{R}$ *and* $u_i \in \mathbb{R}^n$, *s.t.*

$$A = \sum_{i=1}^{n} \lambda_i u_i u_i^T = U \Lambda U^T \tag{1}$$

- λ_i's: eigenvalues of A
- u_i: eigenvector associated to λ_i, normalized to have unity norms
- $U = [u_1, u_2, \cdots, u_n]$ is orthogonal: $U^T U = U U^T = I$
- $\Lambda = diagonal(\lambda_1, \lambda_2, \ldots, \lambda_n)$

Elements of proof for SED

Theorem

$\forall : A \in \mathbb{R}^{n\times n}$ with $A^T = A$, *then eigenvectors of* A*, associated with different eigenvalues, are* ***orthogonal****.*

Proof.

Let $Au_i = \lambda_i u_i$ and $Au_j = \lambda_j u_j$. Then $u_i^T A u_j = u_i^T \lambda_j u_j = \lambda_j u_i^T u_j$. Also, $u_i^T A u_j = u_i^T A^T u_j = (Au_i)^T u_j = \lambda_i u_i^T u_j$. So $\lambda_i u_i^T u_j = \lambda_j u_i^T u_j$. But $\lambda_i \neq \lambda_j$. It must be that $u_i^T u_j = 0$. □

SED now follows:

- If A has distinct eigenvalues, then $U = [u_1, u_2, \cdots, u_n]$ is orthogonal after normalizing all the eigenvectors to unity norm.
- If A has $r(< n)$ distinct eigenvalues, we can *choose* multiple orthogonal eigenvectors for the eigenvalues with none-unity multiplicities.

Rethinking symmetric matrices

With the spectral theorem, next time we see a symmetric matrix A, we immediately know that

- λ_i is real for all i
- associated with λ_i, we can always find a real eigenvector
- $\exists$ an orthonormal basis $\{u_i\}_{i=1}^n$, which consists of the eigenvectors
- if $A \in \mathbb{R}^{2\times 2}$, then if you compute first λ_1, λ_2 and u_1, you won't need to go through the regular math to get u_2, but can simply solve for a u_2 that is orthogonal to u_1 with $\|u_2\| = 1$.

Example: $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5-\lambda & \sqrt{3} \\ \sqrt{3} & 7-\lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$

$$\Rightarrow \lambda_1 = 4, \ \ \lambda_2 = 8$$

- first normalized eigenvector:

$$(A - \lambda_1 I)\, t_1 = 0 \Rightarrow \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix} t_1 = 0 \Rightarrow t_1 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

- A is symmetric $\Rightarrow$ eigenvectors are orthogonal to each other: *choose* $t_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$. No need to solve $(A - \lambda_2 I)\, t_2 = 0$!

Theorem (Eigenvalues of symmetric matrices)

If $A = A^T \in \mathbb{R}^{n \times n}$, then the eigenvalues of A satisfy

$$\lambda_{\max} = \max_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{2}$$

$$\lambda_{\min} = \min_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{3}$$

Proof.

Perform SED to get $A = \sum_{i=1}^n \lambda_i u_i u_i^T$ where $\{u_i\}_{i=1}^n$ spans $\mathbb{R}^n$. Then any vector $x \in \mathbb{R}^n$ can be decomposed as $x = \sum_{i=1}^n \alpha_i u_i$. Thus

$$\max_{x \neq 0} \frac{x^T A x}{\|x\|_2^2} = \max_{\alpha_i} \frac{\left(\sum_i \alpha_i u_i\right)^T \sum_i \lambda_i \alpha_i u_i}{\sum_i \alpha_i^2} = \max_{\alpha_i} \frac{\sum_i \lambda_i \alpha_i^2}{\sum_i \alpha_i^2} = \lambda_{\max}$$

$\square$

Positive definite matrices

- eigenvalues of symmetric matrices are real $\Rightarrow$ we can order the eigenvalues
- a symmetric matrix P is called positive-definite if all its eigenvalues are positive
- equivalently:

Definition (Positive Definite Matrices)

A symmetric matrix $P \in \mathbb{R}^{n \times n}$ is called **positive-definite**, written $P \succ 0$, if $x^T P x > 0$ for all $x \, (\neq 0) \in \mathbb{R}^n$.
P is called **positive-semidefinite,** written $P \succeq 0$, if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$

- $P \succ 0$ $(P \succeq 0) \Leftrightarrow P$ can be decomposed as $P = N^T N$ where N is nonsingular (singular)

Negative definite matrices

Definition

A symmetric matrix $Q \in \mathbb{R}^{n \times n}$ is called **negative-definite**, written $Q \prec 0$, if $-Q \succ 0$, i.e., $x^T Qx < 0$ for all $x\,(\neq 0) \in \mathbb{R}^n$.
Q is called **negative-semidefinite,** written $Q \preceq 0$, if $x^T Qx \leq 0$ for all $x \in \mathbb{R}^n$

Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	$\mathbb{R}_+$ axis
negative definite	negative	$\mathbb{R}_-$ axis

Caution

- positive-definite matrices can have negative entries:

Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P = P^T$ and take any $v = [x, y]^T$, we have

$$\begin{aligned} v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\ &= x^2 + y^2 + (x - y)^2 \geq 0 \end{aligned}$$

and the equality sign holds only when $x = y = 0$.

Caution

- conversely, matrices whose entries are all positive are not necessarily positive-definite:

Example

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is not positive-definite:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -2 < 0$$

Positive definite matrices

Theorem

For a symmetric matrix P, $P \succ 0$ if and only if all the eigenvalues of P are positive.

Proof.

Since P is symmetric, we have

$$\lambda_{\max}(P) = \max_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{4}$$

$$\lambda_{\min}(P) = \min_{x \in \mathbb{R}^n,\ x \neq 0} \frac{x^T A x}{\|x\|_2^2} \tag{5}$$

which gives $x^T A x \in [\lambda_{\min}\|x\|_2^2,\ \lambda_{\max}\|x\|_2^2]$. Thus $x^T A x > 0,\ x \neq 0 \Leftrightarrow \lambda_{\min} > 0$. □

Relevant tools

Checking positive definiteness of a matrix.

We often use the following necessary and sufficient conditions to check positive (semi-)definiteness:

- $P \succ 0$ ($P \succeq 0$) $\Leftrightarrow$ the leading principle minors defined below are positive (nonnegative)

Definition

The leading principle minors of $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ are defined as

p_{11}, $\det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\det P$.

Relevant tools

Checking positive definiteness of a matrix.

Example

None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Recap

$$\begin{array}{c}\text{energy}\\ \updownarrow \\ \text{quadratic function}\\ \updownarrow \\ \text{symmetric matrix}\end{array}$$

Recap

$$
\begin{tikzcd}
\text{energy} \arrow[rrr] \arrow[d, leftrightarrow] & & & \text{Lyapunov function} \\
\text{quadratic function} \arrow[d, leftrightarrow] & & & \\
\text{symmetric matrix} & & &
\end{tikzcd}
$$

Recap

$$\begin{tikzcd}
\text{energy} \arrow[r] \arrow[d, leftrightarrow] & \text{Lyapunov function} \arrow[d, leftrightarrow] \\
\text{quadratic function} \arrow[r] \arrow[d, leftrightarrow] & \text{positive definite function} \\
\text{symmetric matrix} \arrow[ur, "\text{positive definite matrix}", "\text{for linear system}"'] &
\end{tikzcd}$$

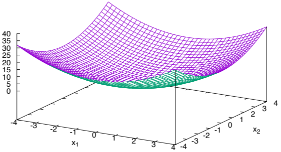
Relevant tools

Definition (Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$, called to be PD, satisfying

- $W(x) > 0$ for all $x \neq 0$
- $W(0) = 0$
- $W(x) \to \infty$ as $|x| \to \infty$ uniformly in x

In the 3D space, positive definite functions are "bowl-shaped", e.g., $W(x_1, x_2) = x_1^2 + x_2^2$.



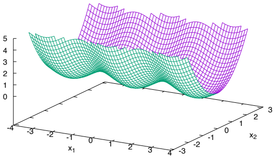
Relevant tools

Definition (Locally Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$, called to be LPD, satisfying

- $W(x) > 0$ for all $x \neq 0$ and $|x| < r$
- $W(0) = 0$

In the 3D space, locally positive definite functions are "bowl-shaped" locally, e.g., $W(x_1, x_2) = x_1^2 + \sin^2 x_2$ for $x_1 \in \mathbb{R}$ and $|x_2| < \pi$



Relevant tools

Exercise

Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

1. $V(x) = x_1^4 + x_2^2 + x_3^4$ (PD)
2. $V(x) = x_1^2 + x_2^2 + 3x_3^2 - x_3^4$ (LPD for $|x_3| < \sqrt{3}$)

Lyapunov stability theorems

- recall the spring mass damper example in matrix form

$$\frac{d}{dt}\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = A\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{array}\right]\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

- energy function is PD:
  $\mathcal{E}(t) =$ potential energy $+$ kinetic energy $= \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$
  and its derivative is NSD:

$$\dot{\mathcal{E}}(t) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right]\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 \tag{6}$$

$$= k_1 x_1 x_2 + m x_2 \left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right] Ax \tag{7}$$

$$= -bx_2^2$$

Theorem

The equilibrium point 0 of $\dot{x}(t) = f\left(x(t), t\right),\ x\left(t_0\right) = x_0$ is stable in the sense of Lyapunov if there exists a locally positive definite function $V(x, t)$ such that $\dot{V}\left(x, t\right) \leq 0$ for all $t \geq t_0$ and all x in a local region $x : |x| < r$ for some $r > 0$.

- such a $V(x, t)$ is called a Lyapunov function
- i.e., $V\left(x\right)$ is PD and $\dot{V}(x)$ is negative semidefinite in a local region $|x| < r$

Theorem

The equilibrium point 0 of $\dot{x}(t) = f\left(x(t), t\right),\ x\left(t_0\right) = x_0$ is locally asymptotically stable if there exists a Lyapunov function $V(x)$ such that $\dot{V}\left(x\right)$ is locally negative definite.

Theorem

The equilibrium point 0 *of* $\dot{x}(t) = f\left(x(t), t\right), \; x\left(t_0\right) = x_0$ *is globally asymptotically stable if there exists a Lyapunov function* $V(x)$ *such that* $V(x)$ *is positive definite and* $\dot{V}(x)$ *is negative definite.*

Lyapunov stability concept for linear systems

- for linear system $\dot{x} = Ax$, a good Lyapunov candidate is the quadratic function $V(x) = x^T P x$ where $P = P^T$ and $P \succ 0$
- the derivative along the state trajectory is then

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\ &= (Ax)^T P x + x^T P A x \\ &= x^T \left(A^T P + PA\right) x\end{aligned}$$

- such a $V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = Ax$ when $A^T P + PA \preceq 0$
- and the origin is stable in the sense of Lyapunov

Theorem (Lyapunov stability theorem for linear systems)

For $\dot{x} = Ax$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if for any symmetric positive definite matrix $Q \succ 0$, the Lyapunov equation

$$\boxed{A^T P + PA = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

Essense of the Lyapunov Eq.

Observations:

- $A^T P + PA$ is a linear operation on P: e.g.,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}, \quad P = \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix}$$

$$A^T \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} + \begin{bmatrix} | & | \\ p_1 & p_2 \\ | & | \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = - \begin{bmatrix} | & | \\ q_1 & q_2 \\ | & | \end{bmatrix}$$

$$\begin{aligned} A^T p_1 + a_{11} p_1 + a_{21} p_2 &= -q_1 \\ A^T p_2 + a_{12} p_1 + a_{22} p_2 &= -q_2 \end{aligned}$$

Essense of the Lyapunov Eq.

Observations: with now

$$A^T P + PA = Q \Leftrightarrow \begin{cases} A^T p_1 + a_{11} p_1 + a_{21} p_2 & = -q_1 \\ A^T p_2 + a_{12} p_1 + a_{22} p_2 & = -q_2 \end{cases}$$

- can stack the columns of $A^T P + PA$ and Q to yield

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\underbrace{\left\{ \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \right\}}_{L_A} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

The Lyapunov Eq.: Existence of solution

$$L_A(P) = A^T P + PA$$

- L_A is invertible if and only if $\lambda_i + \lambda_j \neq 0$ for all eigenvalues of A:
  - let $A^T u_i = \lambda_i u_i$ and $A^T u_j = \lambda_j u_j$
  - $L_A\left(u_i u_j^T\right) = u_i u_j^T A + A^T u_i u_j^T = u_i (\lambda_j u_j)^T + \lambda_i u_i u_j^T = (\lambda_i + \lambda_j) u_i u_j^T$
  - so $\lambda_i + \lambda_j$ is an eigenvalue of the operator $L_A(\cdot)$
  - if $\lambda_i + \lambda_j \neq 0$, the operator is invertible

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- can simply write $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$ using the Kronecker product notation $B \otimes C = \begin{bmatrix} b_{11}C & b_{12}C & \dots & b_{1n}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- e.g., $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$L_A = I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} -1-1 & -1 & -1 & 0 \\ 1 & 0-1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

$$L_A = I \otimes A^T + A^T \otimes I = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

The eigenvalues of L_A are -1, -1, $-1-\sqrt{3}$, $-1+\sqrt{3}$, which are precisely $\lambda_1+\lambda_1$, $\lambda_1+\lambda_2$, $\lambda_2+\lambda_1$, $\lambda_2+\lambda_2$.

```
import numpy as np
A = [[-1,1],[-1,0]]; I2=np.eye(2); AT=np.transpose(A)
L_A=np.kron(I2,AT)+np.kron(AT,I2)
eigLA,_=np.linalg.eig(L_A)
eigA,_=np.linalg.eig(A)
print(eigLA)
print(eigA)
```

Theorem (Lyapunov stability theorem for linear systems)

For $\dot{x} = Ax$ *with* $A \in \mathbb{R}^{n\times n}$*, the origin is asymptotically stable if and only if for any symmetric positive definite matrix* $Q \succ 0$*, the Lyapunov equation*

$$\boxed{A^T P + PA = -Q}$$

has a unique positive definite solution $P \succ 0$*,* $P^T = P$*.*

Proof.

"$\Rightarrow$": $\frac{\dot{V}}{V} = -\frac{x^T Q x}{x^T P x} \leq -\underbrace{\frac{(\lambda_Q)_{\min}}{(\lambda_P)_{\max}}}_{\triangleq \alpha} \Longrightarrow V(t) \leq e^{-\alpha t} V(0)$. $Q \succ 0$ and $P \succ 0 \Rightarrow (\lambda_Q)_{\min} > 0$ and $(\lambda_P)_{\max} > 0$. Thus $\alpha > 0$; $V(t)$ decays exponentially to zero. $V(x) \succ 0 \Rightarrow V(x) = 0$ only at $x = 0$. Therefore, $x \to 0$ as $t \to \infty$, regardless of the initial condition. □

Proof.

"$\Leftarrow$": if 0 of $\dot{x} = Ax$ is asymptotically stable, then all eigenvalues of A have negative real parts. For any Q, the Lyapunov equation has a unique solution P. Note $x(t) = e^{At}x_0 \to 0$ as $t \to \infty$. We have

$$\cancelto{0}{x^T(\infty)Px(\infty)} - x^T(0)Px(0) = \int_0^\infty \frac{d}{dt}x^T(t)Px(t)\,dt = \int_0^\infty x^T(t)\left(A^TP + PA\right)x(t)\,dt$$

$$\Rightarrow x^T(0)Px(0) = \int_0^\infty x^T(t)Qx(t)\,dt = \int_0^\infty x^T(0)e^{A^Tt}Qe^{At}x(0)\,dt$$

If $Q \succ 0$, there exists a nonsingular N matrix: $Q = N^TN$. Thus

$$x^T(0)Px(0) = \int_0^\infty \|Ne^{At}x(0)\|^2 dt \geq 0$$

$$x^T(0)Px(0) = 0 \text{ only if } x_0 = 0$$

Thus $P \succ 0$. Furthermore

$$\boxed{P = \int_0^\infty e^{A^Tt}Qe^{At}dt}$$

$\square$

Procedures of Lyapunov's direct method

1. Given A, select an arbitrary positive-definite symmetric matrix Q (e.g., I).
2. Find the solution matrix P to the Lyapunov equation $A^T P + PA = -Q$.
3. If a solution P cannot be found, the origin is not asymptotically stable.
4. If a solution is found:
   - if P is positive-definite, then A is Hurwitz stable and the origin is asymptotically stable;
   - if P is not positive-definite, then A has at least one eigenvalue with a positive real part and the origin is an unstable equilibrium.

Lyapunov stability theorems

Example

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. The Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}$$

We need

$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

Leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$
$\Rightarrow P \succ 0 \Rightarrow$asymptotically stable

Lyapunov analysis with Matlab

$$\dot{x} = Ax,\ A = \left[\begin{array}{cc} -1 & 1 \\ -1 & 0 \end{array}\right].$$

```
A = [-1,1;-1,0]
Q = eye(2)
P = lyap(A',Q)
w = eig(P)
```

Lyapunov analysis with Python

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

```
import control as ct
import numpy as np
A = np.array([[-1,1],[-1,0]])
Q = np.identity(2)
P = ct.lyap(A.transpose(),Q)
print(P)
w = np.linalg.eigvals(P)
print(f'eigenvalues of P: {w}')
```

It suffices to select $Q = I$

For linear systems we can let $Q = I$ and check whether the resulting P is positive definite. If it is, then we can assert the asymptotic stability:

- take any $Q \succ 0$. there exists $Q = N^T N$, where N is invertible, yielding

$$A^T P + PA = -I$$
$$\Updownarrow$$
$$\underbrace{N^T A^T N^{-T}}_{\tilde{A}^T} \underbrace{N^T P N}_{\tilde{P}} + \underbrace{N^T P N}_{\tilde{P}} \underbrace{N^{-1} A N}_{\tilde{A}} = -N^T N$$

  - $\tilde{A} = N^{-1}AN$ and A are similar matrices and have the same eigenvalues.
  - $\tilde{P} = N^T PN$ and P have the same definiteness. If we can find a positive definite solution P then the $\tilde{P}$ will also be positive definite. Vise versa.

Instability theorem

- for nonlinear systems, Lyapunov function can be nontrivial to find
- failure to find a Lyapunov function does not imply instability

Theorem

The equilibrium state 0 *of* $\dot{x} = f(x)$ *is unstable if there exists a function* $W(x)$ *such that*

- $\dot{W}(x)$ *is PD locally:* $\dot{W}(x) > 0 \; \forall \, |x| < r$ *for some* r *and* $\dot{W}(0) = 0$
- $W(0) = 0$
- *there exist states* x *arbitrarily close to the origin such that* $W(x) > 0$

Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x,\ \ P = P^T \succ 0$$

and compute $\Delta V(x)$ along the trajectory of the state

$$V(x(k+1)) - V(x(k)) = x^T(k) \underbrace{\left(A^T PA - P\right)}_{\triangleq -Q} x(k)$$

Asymptotic stability desires $\Delta V(x)$ to be negative.

DT Lyapunov stability theorem for linear systems

Theorem

For system $x(k+1) = Ax(k)$ *with* $A \in \mathbb{R}^{n\times n}$*, the origin is asymptotically stable if and only if* $\exists\ Q \succ 0$*, such that the discrete-time Lyapunov equation*

$$\boxed{A^T PA - P = -Q}$$

has a unique positive definite solution $P \succ 0$*,* $P^T = P$*.*

The DT Lyapunov Eq.

$$\boxed{A^T PA - P = -Q}$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur stable), comes from:

$$\begin{aligned}
\cancelto{0}{V(x(\infty))} - V(x(0)) &= \sum_{k=0}^{\infty} x^T(k)\left[A^T PA - P\right]x(k) \\
&= -\sum_{k=0}^{\infty} x^T(0)\left(A^T\right)^k QA^k x(0) \\
\Rightarrow P &= \sum_{k=0}^{\infty} \left(A^T\right)^k QA^k
\end{aligned}$$

- can show that the DT Lyapunov operator $L_A = A^T PA - P$ is invertible if and only if $\forall i, j \; (\lambda_A)_i (\lambda_A)_j \neq 1$

DT Lyapunov analysis with MATLAB

Example

$$x(k+1) = Ax(k), \;\; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
% MATLAB
A=[ 0 1 0; 0 0 1; 0.275 -0.225 -0.1]
Q = eye(3)
P = dlyap(A',Q) % check function definition in Matlab help
eig(P)
```

DT Lyapunov analysis with Python

Example

$$x(k+1) = Ax(k), \;\; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

```
#Python
import control as ct
import numpy as np
from numpy.linalg import eig
A = np.array([[0,1,0],[0,0,1],[0.275,-0.225,-0.1]])
Q = np.identity(3)
P = ct.dlyap(A.transpose(),Q)
w,v = eig(P)
print(w)
```

Recap

- Internal stability
  - Stability in the sense of Lyapunov: ε, δ conditions
  - Asymptotic stability
- Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
  - Based on the eigenvalues of A
    - Time response modes
    - Repeated eigenvalues on the imaginary axis
  - Routh's criterion
    - No need to solve the characteristic equation
    - Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)

Recap

- Lyapunov equations
  Theorem: All eigenvalues of A have negative real parts iff for any given $Q \succ 0$, the Lyapunov equation
  $$A^T P + PA = -Q$$
  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T P + PA = -Q$ has a unique solution when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j.
  Theorem: All eigenvalues of A are inside the unit circle iff for any given $Q \succ 0$, the Lyapunov equation
  $$A^T PA - P = -Q$$
  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T PA - P = -Q$ has a unique solution when $\lambda_{A,i}\lambda_{A,j} \neq 1$ for all i and j.

Recap

- P is positive definite if and only if any one of the following conditions holds:
  1. All the eigenvalues of P are positive.
  2. All the leading principle minors of P are positive.
  3. There exists a nonsingular matrix N such that $P = N^T N$.