Linear Systems: Stability

1. Definitions in Lyapunov stability analysis

2. Stability of LTI systems: method of eigenvalue/pole locations

3. Lyapunov's approach to stability
   - Relevant tools
   - Lyapunov stability theorems
   - Instability theorem
   - Discrete-time case

4. Recap

Finite dimensional vector norms

Let $v \in \mathbb{R}^n$. A norm is:

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default in this set of notes: $\|\cdot\| = \|\cdot\|_2$

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- the condition must be satisfied by all $t \geq 0$
- if a system starts at equilibrium state, it stays there

Equilibrium state of a linear system

For a linear system

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- origin $x_e = 0$ is always an equilibrium state
- when $A(t)$ is singular, multiple equilibrium states exist

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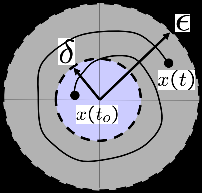

Figure: Stable s.i.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon \; \forall t \geq t_0$.

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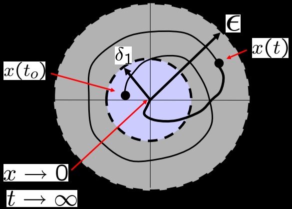

Figure: Asymptotically stable i.s.L: $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| \to 0$.

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- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2 + \omega^2}\right| = |\lambda| < 1$

Stability of the origin for $\dot{x} = Ax$

stability at 0	$\lambda_i(A)$
unstable	$\mathrm{Re}\{\lambda_i\} > 0$ for some λ_i or $\mathrm{Re}\{\lambda_i\} \leq 0$ for all λ_i's but for a repeated λ_m on the imaginary axis with multiplicity m, nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\mathrm{Re}\{\lambda_i\} \leq 0$ for all λ_i's and $\forall$ repeated λ_m on the imaginary axis with multiplicity m, nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\mathrm{Re}\{\lambda_i\} < 0 \ \forall \lambda_i$ (A is then called Hurwitz stable)

Example (Unstable moving mass)

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- verify by checking $e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$: t grows unbounded

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Routh-Hurwitz criterion

- the Routh Test (by E.J. Routh, in 1877): a simple algebraic procedure to determine how many roots a given polynomial

$$A(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

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- German mathematician Adolf Hurwitz independently proposed in 1895 to approach the problem from a matrix perspective
- popular if stability is the only concern and no details on eigenvalues (e.g., speed of response) are needed

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- recap: the poles of transfer function $G(s) = C(sI - A)^{-1}B + D$ come from $\det(sI - A)$ in computing the inverse $(sI - A)^{-1}$

The Routh Array

for $A(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$, construct

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & q_{n-2} & q_{n-4} & q_{n-6} & \cdots & \\
s^{n-3} & q_{n-3} & q_{n-5} & q_{n-7} & \cdots & \\
\vdots & \vdots & \vdots & \vdots & & \\
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- first two rows contain the coefficients of $A(s)$
- third row constructed from the previous two rows via

$$
\begin{array}{c|cccc}
\cdot & a & b & x & \cdot \\
\cdot & c & d & y & \cdot \\
\cdot & \dfrac{bc-ad}{c} & \dfrac{xc-ay}{c} & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

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\end{array}$$

- **All roots of $A(s)$ are on the left half s-plane if and only if all elements of the first column of the Routh array are positive.**

The Routh Array

Example $(A(s) = 2s^4 + s^3 + 3s^2 + 5s + 10)$

$$
\begin{array}{c|ccc}
s^4 & 2 & 3 & 10 \\
s^3 & 1 & 5 & 0 \\
s^2 & 3 - \frac{2\times 5}{1} = -7 & 10 & 0 \\
s^1 & 5 - \frac{1\times 10}{-7} & 0 & 0 \\
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- two sign changes in the first column
- unstable and two roots in the right half side of s-plane

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- If the 1st element in any one row of Routh's array is zero, one can replace the zero with a small number ϵ and proceed further.
- There are other possible complications, which we will not pursue further. See, e.g. "Automatic Control Systems", by Kuo, 7th ed., pp. 339-340.

Stability of the origin for $x(k+1) = f(x(k), k)$

- stability analysis follows analogously for nonlinear time-varying discrete-time systems of the form

$$x(k+1) = f(x(k), k), \quad x(k_0) = x_0$$

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- equilibrium point x_e:

$$f(x_e, k) = x_e, \;\; \forall k$$

- without loss of generality, 0 is assumed an equilibrium point

Stability of the origin for $x(k+1) = Ax(k)$

stability at 0	$\lambda_i(A)$
unstable	$\lvert\lambda_i\rvert > 1$ for some λ_i or $\lvert\lambda_i\rvert \leq 1$ for all λ_i's but for a repeated λ_m on the unit circle with multiplicity m, nullity $(A - \lambda_m I) < m$ (Jordan form)
stable i.s.L	$\lvert\lambda_i\rvert \leq 1$ for all λ_i's but for any repeated λ_m on the unit circle with multiplicity m, nullity $(A - \lambda_m I) = m$ (diagonal form)
asymptotically stable	$\lvert\lambda_i\rvert < 1$ $\forall \lambda_i$ (such a matrix is called Schur stable)

Routh-Hurwitz criterion for DT LTI systems

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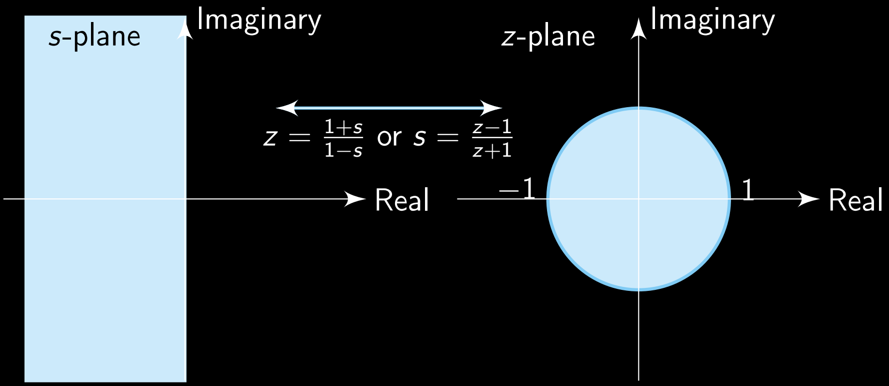
- the stability domain $|\lambda_i| < 1$ is a unit disk
- Routh array validates stability in the left-half plane
- bilinear transformation maps the closed left half *s*-plane to the closed unit disk in *z*-plane



Routh-Hurwitz criterion for DT LTI systems

- Given $A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$, procedures of Routh-Hurwitz test:
  - apply bilinear transform
    $A(z)|_{z=\frac{1+s}{1-s}} = \left(\frac{1+s}{1-s}\right)^n + a_1 \left(\frac{1+s}{1-s}\right)^{n-1} + \cdots + a_n =: \frac{A^*(s)}{(1-s)^n}$
  - apply Routh test to
    $A^*(s) = a_n^* s^n + a_{n-1}^* s^{n-1} + \cdots + a_0^* = A(z)|_{z=\frac{1+s}{1-s}} (1-s)^n$

Routh-Hurwitz criterion for DT LTI systems

Example $\left(A(z) = z^3 + 0.8z^2 + 0.6z + 0.5\right)$

- $A^*(s) = A(z)\big|_{z=\frac{1+s}{1-s}} (1-s)^3 = (1+s)^3 + 0.8\,(1+s)^2\,(1-s) + 0.6\,(1+s)\,(1-s)^2 + 0.5\,(1-s)^3 = 0.3s^3 + 3.1s^2 + 1.7s + 2.9$

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- Routh array

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s^3 & 0.3 & 1.7 \\
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- all elements in first column are positive $\Rightarrow$ roots of $A(z)$ are all in the unit circle

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- the response $A^k x(0)$ involves modes such as λ^k, $k\lambda^{k-1}$, $r^k\cos k\theta$, $r^k\sin k\theta$
- $e^{\sigma t} \to 0$ if $\sigma < 0$; $e^{\lambda t} \to 0$ if $\lambda < 0$
- $\lambda^k \to 0$ if $|\lambda| < 1$; $r^k \to 0$ if $|r| = \left|\sqrt{\sigma^2 + \omega^2}\right| = |\lambda| < 1$

Lyapunov's approach to stability

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- an "energy" perspective
- fit for general dynamic systems (linear/nonlinear, time-invariant/time-varying)

Stability from an energy viewpoint: Example

Consider spring-mass-damper systems:

$$\dot{x}_1 = x_2 \qquad (x_1\text{: position; } x_2 : \text{ velocity})$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2,\ \ b > 0 \qquad \text{(Newton's law)}$$

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Stability from an energy viewpoint: Generalization

Consider unforced, time-varying, nonlinear systems

$$\dot{x}(t) = f\left(x(t), t\right), \ x\left(t_0\right) = x_0$$
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- main tool: matrix formulation, linear algebra, positive definite functions

Relevant tools

Quadratic functions

- intrinsic in energy-like analysis, e.g.

$$\frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- general quadratic functions in matrix form

$$Q(x) = x^T P x, \;\; P^T = P$$

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- recall: a real square matrix A is
  - *symmetric* if $A = A^T$
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$$\text{general case: } P = \frac{P + P^T}{2} + \frac{P - P^T}{2}$$

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namely, $a_j^T a_j = 1$ and $a_j^T a_m = 0 \ \forall j \neq m$.

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Also, $\overline{u}^T u \in \mathbb{R}$. Thus $\lambda = \frac{\overline{u}^T Au}{\overline{u}^T u}$ must also be a real number. $\square$

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matrix structure	analogy in complex plane
symmetric	real line
skew-symmetric	imaginary line
orthogonal	unit circle

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Theorem (Symmetric eigenvalue decomposition (SED))

$\forall : A \in \mathbb{R}^{n \times n},\; A^T = A$, there always exist $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$, s.t.

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SED now follows:

- If A has distinct eigenvalues, then $U = [u_1, u_2, \cdots, u_n]$ is orthogonal after normalizing all the eigenvectors to unity norm.
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- if $A \in \mathbb{R}^{2\times 2}$, then if you compute first λ_1, λ_2 and u_1, you won't need to go through the regular math to get u_2, but can simply solve for a u_2 that is orthogonal to u_1 with $\|u_2\| = 1$.

Example: $A = \begin{bmatrix} 5 & \sqrt{3} \\ \sqrt{3} & 7 \end{bmatrix}$

Computing the eigenvalues gives

$$\det \begin{bmatrix} 5-\lambda & \sqrt{3} \\ \sqrt{3} & 7-\lambda \end{bmatrix} = 35 - 12\lambda + \lambda^2 - 3 = (\lambda - 4)(\lambda - 8) = 0$$

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If $A = A^T \in \mathbb{R}^{n \times n}$*, then the eigenvalues of* A *satisfy*

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Updated matrix analogies

matrix structure	eigenvalues	analogy in complex plane
symmetric	real	real axis
skew-symmetric	on imaginary axis	imaginary axis
orthogonal	magnitude 1	unit circle
positive definite	positive	$\mathbb{R}_+$ axis
negative definite	negative	$\mathbb{R}_-$ axis

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Example

$P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive-definite, as $P = P^T$ and take any $v = [x, y]^T$, we have

$$\begin{aligned} v^T P v &= \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 2y^2 - 2xy \\ &= x^2 + y^2 + (x-y)^2 \geq 0 \end{aligned}$$

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and the equality sign holds only when $x = y = 0$.

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which gives $x^T A x \in [\lambda_{\min}\|x\|_2^2,\ \lambda_{\max}\|x\|_2^2]$. Thus $x^T A x > 0,\ x \neq 0 \Leftrightarrow \lambda_{\min} > 0$. □

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Checking positive definiteness of a matrix.

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Definition

The leading principle minors of $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$ are defined as

p_{11}, $\det \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, $\det P$.

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None of the following matrices are positive definite:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Recap

$$\begin{array}{c}\text{energy}\\ \updownarrow \\ \text{quadratic function}\\ \updownarrow \\ \text{symmetric matrix}\end{array}$$

Recap

$$\begin{tikzcd}
\text{energy} \arrow[rr] \arrow[d, leftrightarrow] & & \text{Lyapunov function} \\
\text{quadratic function} \arrow[d, leftrightarrow] & & \\
\text{symmetric matrix} & &
\end{tikzcd}$$

Recap

$$\begin{tikzcd}
\text{energy} \arrow[r] \arrow[d, leftrightarrow] & \text{Lyapunov function} \arrow[d, leftrightarrow] \\
\text{quadratic function} \arrow[r] \arrow[d, leftrightarrow] & \text{positive definite function} \\
\text{symmetric matrix} \arrow[ur, "\text{positive definite matrix}", "\text{for linear system}"'] &
\end{tikzcd}$$

Relevant tools

Definition (Positive Definite Functions)

A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$, called to be PD, satisfying

- $W(x) > 0$ for all $x \neq 0$
- $W(0) = 0$
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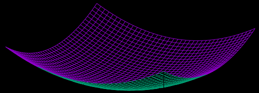

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A continuous time function $W : \mathbb{R}^n \to \mathbb{R}_+$, called to be LPD, satisfying

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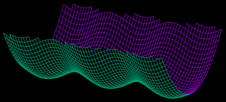
In the 3D space, locally positive definite functions are "bowl-shaped" locally, e.g., $W(x_1, x_2) = x_1^2 + \sin^2 x_2$ for $x_1 \in \mathbb{R}$ and $|x_2| < \pi$



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Exercise

Let $x = [x_1, x_2, x_3]^T$. Check the positive definiteness of the following functions

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1. Definitions in Lyapunov stability analysis

2. Stability of LTI systems: method of eigenvalue/pole locations

3. Lyapunov's approach to stability
    - Relevant tools
    - Lyapunov stability theorems
    - Instability theorem
    - Discrete-time case

4. Recap

Lyapunov stability theorems

- recall the spring mass damper example in matrix form

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and its derivative is NSD:

$$\begin{aligned} \dot{\mathcal{E}}(t) &= \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right]\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = k_1 x_1 \dot{x}_1 + m x_2 \dot{x}_2 && (6) \\ &= k_1 x_1 x_2 + m x_2\left(-\frac{k}{m}x_1 - \frac{b}{m}x_2\right) = \left[\frac{\partial \mathcal{E}}{\partial x_1}, \frac{\partial \mathcal{E}}{\partial x_2}\right] A x && (7) \\ &= -b x_2^2 \end{aligned}$$

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- can stack the columns of $A^T P + PA$ and Q to yield

$$\begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} a_{11} I & a_{21} I \\ a_{12} I & a_{22} I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = - \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

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  - if $\lambda_i + \lambda_j \neq 0$, the operator is invertible

The Lyapunov operator: eigenvalues

$$L_A = \begin{bmatrix} A^T & 0 \\ 0 & A^T \end{bmatrix} + \begin{bmatrix} a_{11}I & a_{21}I \\ a_{12}I & a_{22}I \end{bmatrix}$$

- can simply write $L_A = \underbrace{I \otimes A^T + A^T \otimes I}_{\text{mirror symmetric}}$ using the Kronecker product notation $B \otimes C = \begin{bmatrix} b_{11}C & b_{12}C & \dots & b_{1n}C \\ b_{21}C & b_{22}C & \dots & b_{2n}C \\ \vdots & \vdots & \dots & \vdots \\ b_{m1}C & b_{m2}C & \dots & b_{mn}C \end{bmatrix}$

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- e.g., $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$L_A = I \otimes A^T + A^T \otimes I = \begin{bmatrix} A^T + a_{11}I & a_{21}I \\ a_{12}I & A^T + a_{22}I \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc} -1-1 & -1 & -1 & 0 \\ 1 & 0-1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} -2 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ \hline 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Example: $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $\lambda_{1,2} = -0.5 \pm i\sqrt{3}/2$

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The eigenvalues of L_A are -1, -1, $-1-\sqrt{3}$, $-1+\sqrt{3}$, which are precisely $\lambda_1 + \lambda_1$, $\lambda_1 + \lambda_2$, $\lambda_2 + \lambda_1$, $\lambda_2 + \lambda_2$.

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$$\boxed{P = \int_0^\infty e^{A^Tt}Qe^{At}dt}$$

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Lyapunov stability theorems

Example

$\dot{x} = Ax$, $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$. The Lyapunov equation is

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}^T \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q}$$

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$$\begin{cases} -2p_{11} - 2p_{12} = -1 \\ -p_{12} - p_{22} + p_{11} = 0 \\ 2p_{12} = -1 \end{cases} \Rightarrow \begin{cases} p_{11} = 1 \\ p_{22} = 3/2 \\ p_{12} = -1/2 \end{cases}$$

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Leading principle minors: $p_{11} > 0$, $p_{11}p_{22} - p_{12}^2 > 0$
$\Rightarrow P \succ 0 \Rightarrow$asymptotically stable

Lyapunov analysis with Matlab

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Lyapunov analysis with Python

$$\dot{x} = Ax,\ A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}.$$

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  - $\tilde{A} = N^{-1} AN$ and A are similar matrices and have the same eigenvalues.
  - $\tilde{P} = N^T PN$ and P have the same definiteness. If we can find a positive definite solution P then the $\tilde{P}$ will also be positive definite. Vise versa.

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Theorem

The equilibrium state 0 *of* $\dot{x} = f(x)$ *is unstable if there exists a function* $W(x)$ *such that*

- $\dot{W}(x)$ *is PD locally:* $\dot{W}(x) > 0\ \forall\, |x| < r$ *for some* r *and* $\dot{W}(0) = 0$
- $W(0) = 0$
- *there exist states* x *arbitrarily close to the origin such that* $W(x) > 0$

Discrete-time case: key concept of Lyapunov

For the discrete-time system

$$x(k+1) = Ax(k)$$

we consider a quadratic Lyapunov function candidate

$$V(x) = x^T P x, \;\; P = P^T \succ 0$$

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Asymptotic stability desires $\Delta V(x)$ to be negative.

DT Lyapunov stability theorem for linear systems

Theorem

For system $x(k+1) = Ax(k)$ with $A \in \mathbb{R}^{n \times n}$, the origin is asymptotically stable if and only if $\exists\ Q \succ 0$, such that the discrete-time Lyapunov equation

$$\boxed{A^T PA - P = -Q}$$

has a unique positive definite solution $P \succ 0$, $P^T = P$.

The DT Lyapunov Eq.

$$\boxed{A^T P A - P = -Q}$$

- Solution to the DT Lyapunov equation, when asymptotic stability holds (A is Schur stable), comes from:

$$\cancelto{0}{V(x(\infty))} - V(x(0)) = \sum_{k=0}^{\infty} x^T(k)\left[A^T P A - P\right]x(k)$$

$$= -\sum_{k=0}^{\infty} x^T(0)\left(A^T\right)^k Q A^k x(0)$$

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- can show that the DT Lyapunov operator $L_A = A^T PA - P$ is invertible if and only if $\forall i, j\ (\lambda_A)_i (\lambda_A)_j \neq 1$

DT Lyapunov analysis with MATLAB

Example

$$x(k+1) = Ax(k), \;\; A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.275 & -0.225 & -0.1 \end{bmatrix}$$

DT Lyapunov analysis with Python

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Recap

- Internal stability
  - Stability in the sense of Lyapunov: ε, δ conditions
  - Asymptotic stability
- Stability analysis of linear time invariant systems ($\dot{x} = Ax$ or $x(k+1) = Ax(k)$)
  - Based on the eigenvalues of A
    - Time response modes
    - Repeated eigenvalues on the imaginary axis
  - Routh's criterion
    - No need to solve the characteristic equation
    - Discrete time case: bilinear transform ($z = \frac{1+s}{1-s}$)

Recap

- Lyapunov equations
  Theorem: All eigenvalues of A have negative real parts iff for any given $Q \succ 0$, the Lyapunov equation
  $$A^T P + PA = -Q$$
  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T P + PA = -Q$ has a unique solution when $\lambda_{A,i} + \lambda_{A,j} \neq 0$ for all i and j.
  Theorem: All eigenvalues of A are inside the unit circle iff for any given $Q \succ 0$, the Lyapunov equation
  $$A^T PA - P = -Q$$
  has a unique solution P and $P \succ 0$.
  Given Q, the Lyapunov equation $A^T PA - P = -Q$ has a unique solution when $\lambda_{A,i}\lambda_{A,j} \neq 1$ for all i and j.

Recap

- P is positive definite if and only if any one of the following conditions holds:
  1. All the eigenvalues of P are positive.
  2. All the leading principle minors of P are positive.
  3. There exists a nonsingular matrix N such that $P = N^T N$.