

Linear Systems

Observer and Observer State Feedback



Introduction

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- ▶ the state estimation problem
 - ▶ deterministic case: observer design
 - ▶ stochastic case: the most frequent option is Kalman filter

Open-loop observer

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with a best guess of initial estimate $\hat{x}(0) \stackrel{\text{e.g.}}{=} 0$.

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- ▶ error dynamics: $e = x - \hat{x}$:

$$\dot{e}(t) = Ae(t), \quad e(k+1) = Ae(k), \quad e(0) = x_0 - \hat{x}(0)$$

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- ▶ if A is not Hurwitz/Schur stable, the error diverges

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- ▶ sensitive to input disturbances
 - ▶ if A is not Hurwitz/Schur stable, the error diverges
- ▶ open-loop observers look simple but do not work in practice

Luenberger (closed-loop) observer concept

- ▶ given system dynamics

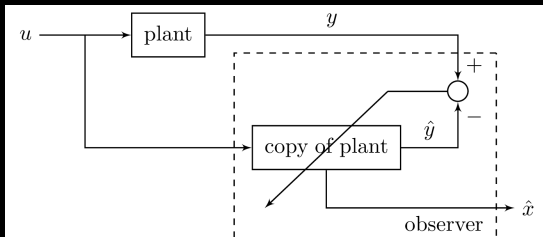
$$\begin{aligned}\dot{x} &= Ax + Bu, \quad x(0) = x_0, \quad A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times r} \\ y &= Cx, \quad y \in \mathbb{R}^{m \times n}\end{aligned}$$

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- ▶ in contrast to open-loop observers, the Luenberger observer adds correction based on output differences



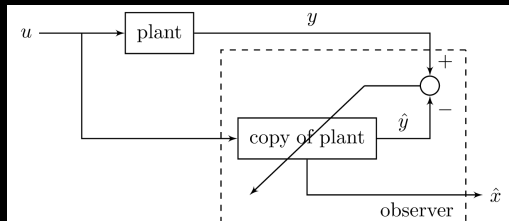
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plant:

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observer concept



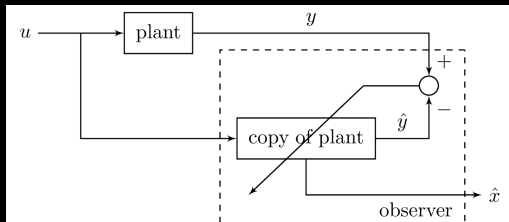
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► observer realization:

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) = A\hat{x} + Bu + L(y - C\hat{x}), \quad \hat{x}(0) = 0 \\ &= (A - LC)\hat{x} + Ly + Bu\end{aligned}$$

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$$\dot{e} = Ae - LCe = (A - LC)e, \quad e(0) = x(0)$$

- ▶ if all eigenvalues of $A - LC$ are on the left half plane, then the error dynamics can be made asymptotically stable

Luenberger (closed-loop) observer

Theorem

If (A, C) is an observable pair, then all the eigenvalues of $A - LC$ can be arbitrarily assigned

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- ▶ we show the SISO case when A and C are in observable canonical form

Luenberger (closed-loop) observer

Theorem

If (A, C) is an observable pair, then all the eigenvalues of $A - LC$ can be arbitrarily assigned, provided that they are symmetric with respect to the real axis of the complex plane.

- we show the SISO case when A and C are in observable canonical form (if not, a similarity transform can help out):

$$A = \begin{bmatrix} -\alpha_{n-1} & 1 & 0 & \dots \\ \vdots & 0 & \ddots & \ddots \\ -\alpha_1 & \vdots & \ddots & 1 \\ -\alpha_0 & 0 & \dots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \beta_{n-1} \\ \vdots \\ \beta_1 \\ \beta_0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \end{bmatrix}, \quad D = d$$
$$\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1s + \alpha_0$$

Observer eigenvalue placement: o.c.f.

- Luenberger observer with correction:

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- Goal: place eigenvalues of the observer at locations $\bar{p}_1, \dots, \bar{p}_n$:

$$\begin{aligned}\det(sI - (A - LC)) &= (s - \bar{p}_1)(s - \bar{p}_2) \cdots (s - \bar{p}_n) \\ &= s^n + \bar{\gamma}_{n-1}s^{n-1} + \cdots + \bar{\gamma}_1s + \bar{\gamma}_0\end{aligned}$$

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- Let $L = [l_0, l_1, \dots, l_{n-1}]^T$. The unique structures of A and C give

$$LC = \begin{bmatrix} l_0 \\ \vdots \\ l_{n-2} \\ l_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} l_0 & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ l_{n-2} & \ddots & \ddots & 0 \\ l_{n-1} & 0 & \dots & 0 \end{bmatrix}$$

$$A - LC = \begin{bmatrix} -\alpha_{n-1} - l_0 & 1 & 0 & \dots & 0 \\ -\alpha_{n-2} - l_1 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ -\alpha_1 - l_{n-2} & \vdots & \ddots & 0 & 1 \\ -\alpha_0 - l_{n-1} & 0 & \dots & 0 & 0 \end{bmatrix}$$

Observer eigenvalue placement: o.c.f.

- A and $A - LC$ have the same structure:

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- Recall: $\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \dots + \alpha_1s + \alpha_0$.
- Thus

$$\det(sI - (A - LC)) = s^n + \underbrace{(\alpha_{n-1} + l_0)}_{\text{target: } \bar{\gamma}_{n-1}} s^{n-1} + \dots + \underbrace{(\alpha_0 + l_{n-1})}_{\text{target: } \bar{\gamma}_0}$$

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- Hence

$$\begin{aligned} l_0 &= \bar{\gamma}_{n-1} - \alpha_{n-1} \\ &\vdots \\ l_{n-1} &= \bar{\gamma}_0 - \alpha_0 \end{aligned}$$

General observer eigenvalue placement

- What if (A, B, C, D) is not in the observable canonical form?

General observer eigenvalue placement

- ▶ What if (A, B, C, D) is not in the observable canonical form?
- ▶ We can transform it to o.c.f. via a similarity transform:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad x = R^{-1}x_{ob} \quad \Longrightarrow \quad \begin{cases} \dot{x}_{ob} = \underbrace{RAR^{-1}}_{A_o} x_{ob} + \underbrace{RB}_{B_o} u \\ y = C_o x_{ob} = CR^{-1}x_{ob} \end{cases}$$

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- use previous formulas to design \tilde{L} in:

$$\dot{\hat{x}}_{ob} = \left(A_o - \tilde{L}C_o \right) \hat{x}_{ob} + \tilde{L}y + B_o u \quad (\text{analysis form})$$

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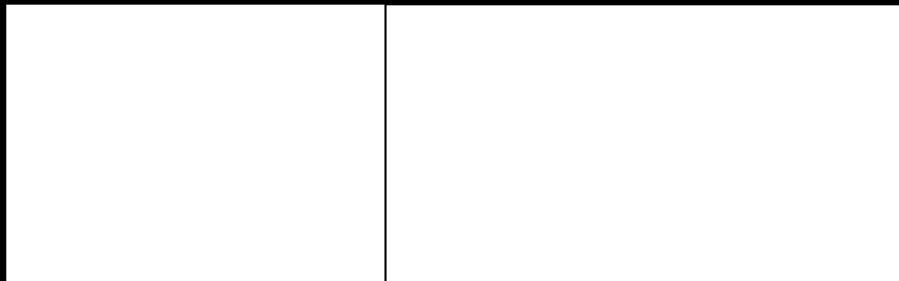
$$R\dot{\hat{x}} = \left(RAR^{-1} - \tilde{L}CR^{-1} \right) R\hat{x} + \tilde{L}y + RBu$$

$$\Rightarrow \dot{\hat{x}} = \left(A - \overbrace{R^{-1}\tilde{L}C}^L \right) \hat{x} + Ly + Bu \quad (\text{implementation form})$$

General observer eigenvalue placement

- **Powerful fact:** if system $\Sigma = (A, B, C, D)$ is observable, then we can arbitrarily place the observer eigenvalues.

Observer design in MATLAB and Python



Motor control example

Example 12.2.1 (Motor Control) To see an example performance of the observer design, we apply the algorithm to a motor system and program Equation 12.10. Here, the three states are the current in the motor electronics, the angular position and angular velocity of the motor. Only the angular position is directly measured on the output side. We compute first the plant eigenvalues, then check observability and place the observer eigenvalues.

```
% observer/motorobs.m
% State observer design for motion control in MATLAB
%% Continuous-time system model
% motor parameters
L = 1e-3; R = 1; J = 5e-5; B = 1e-4; K = 0.1;

% state-space model
A = [-R/L, 0, -K/L; 0, 0, 1; K/J, 0, -B/J];
B = [1/L; 0; 0];
C = [0, 1, 0];
D = [0];

% check original eigenvalues
eig(A)

%% Observer design
% check observability
O = obsv(A,C);
rank(O)

% desired poles for the observer
pole_des = [-500+250j, -500-250j, -1000];

% design observer by placing poles of A-LC
Lt = place(A',C',pole_des);
L = Lt.'

% check poles of estimator-error dynamics
est_poles = eig(A - L*C)
```

```
%% Simulation
% define augmented system to run the simulation
Aaug = [A, zeros(3,3); L*C, A-L*C];
Baug = [B; 0];
Caug = [C, zeros(1,3)];
Daug = 0;
sys = ss(Aaug,Baug,Caug,Daug);

% define initial conditions
x0 = [10, 2, 10]'; xhat0 = [0, 0, 0]'; X0 = [x0; xhat0];

% define simulink parameters
Tend = 0.03; % simulation end time
amplitude = 10; % sin wave input amplitude
initpha = 0; % initial phase
freq = 600; % sin wave freq (rad/s)
t = 0:1e-4:Tend;

u = amplitude*sin(freq*t+initpha);
```

```
x0 = np.array([10, 2, 10]); xhat0 = np.array([0, 0, 0]); X0 =
    np.array([x0, xhat0]).reshape((6, 1))

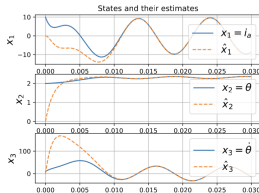
Tend = 0.03; amplitude = 10; initpha = 0; freq = 600
t = np.arange(0, Tend, 1e-4)

u = amplitude * np.sin(freq * t + initpha)

[V, T, X] = ct.lsim(sys, u, t, X0)

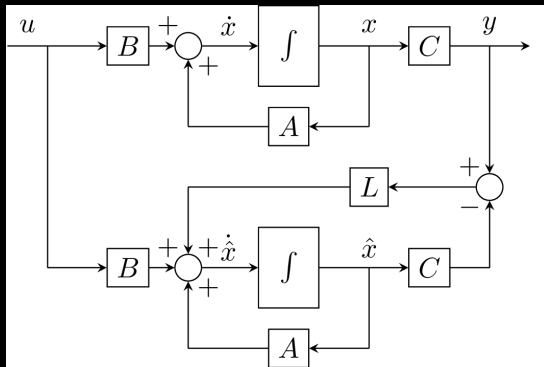
plt.figure()
plt.subplot(3, 1, 1)
plt.plot(t, X[:, 0], t, X[:, 3], '--', linewidth=1.5)
plt.xlabel('time (sec)')
plt.legend(['$x_1 = i_a$', '$\hat{x}_1$'], fontsize=16)
plt.grid()
plt.ylabel('$x_1$')
plt.title('States and their estimates')
plt.subplot(3, 1, 2)
plt.plot(t, X[:, 1], t, X[:, 4], '--', linewidth=1.5)
plt.xlabel('time (sec)')
plt.legend(['$x_2 = \theta$', '$\hat{x}_2$'], fontsize=16)
plt.grid()
plt.ylabel('$x_2$')
plt.subplot(3, 1, 3)
plt.plot(t, X[:, 2], t, X[:, 5], '--', linewidth=1.5)
plt.xlabel('time (sec)')
plt.legend(['$x_3 = \dot{\theta}$', '$\hat{x}_3$'], fontsize=16)
plt.grid()
plt.ylabel('$x_3$')
plt.show()
```

From the generated result below, we see that despite the initial error between the true states and the estimated states, the estimation errors quickly converge to zero for all the three states after about 0.01 second. Try modify the observer eigenvalues and see how they affect the convergence.



Luenberger observer summary

- ▶ observer dynamics: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$, $\hat{x}(0) = 0$
- ▶ block diagram



Luenberger observer summary

► system dynamics

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► augmented system

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u$$

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- to see the distribution of eigenvalues, note the error dynamics $\dot{e} = (A - LC)e \Rightarrow$

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\Rightarrow eigenvalues are separated into: $\lambda(A)$ and observer eigenvalues

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\Rightarrow eigenvalues are separated into: $\lambda(A)$ and observer eigenvalues

- underlying similarity transform: $\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$

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Discrete-time observers: Introduction

- ▶ full state feedback is usually not available
- ▶ often observers are implemented in the discrete-time domain
- ▶ the discrete-time observer design
 - ▶ basic form: analogous to the continuous-time Luenberger observer
 - ▶ predict and correct form:
 - ▶ direct DT design
 - ▶ leverages discrete-time signal properties

Discrete-time full state observer

- ▶ standard discrete-time observer:

$$x(k+1) = Ax(k) + Bu(k)$$

$$\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$$

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Discrete-time full state observer

- ▶ standard discrete-time observer:

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- ▶ **Powerful fact:** the error dynamics can be arbitrarily assigned if the system is observable.

DT full state observer with predictor

- motivation: $\hat{x}(k+1) = A\hat{x}(k) + Bu(k) + L(y(k) - C\hat{x}(k))$
doesn't use most recent measurement $y(k+1) = Cx(k+1)$

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- ▶ discrete-time observer **with predictor**:

predictor: $\hat{x}(k+1|k) = A\hat{x}(k|k) + Bu(k)$

corrector: $\hat{x}(k+1|k+1) = \hat{x}(k+1|k) + L(y(k+1) - C\hat{x}(k+1|k))$

- ▶ $\hat{x}(k|k)$: estimate of $x(k)$ based on measurements up to time k
- ▶ $\hat{x}(k|k-1)$: estimate based on measurements up to time $k-1$
- ▶ $e(k) \triangleq x(k) - \hat{x}(k|k)$: estimation error

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 - ▶ $e(k) \triangleq x(k) - \hat{x}(k|k)$: estimation error
- ▶ error dynamics

$$\hat{x}(k+1|k+1) = (I - LC)\hat{x}(k+1|k) + Ly(k+1)$$

$$= (I - LC)A\hat{x}(k|k) + (I - LC)Bu(k) + Ly(k+1)$$

$$\Rightarrow e(k+1) = x(k+1) - Ly(k+1) - (I - LC)A\hat{x}(k|k) - (I - LC)Bu(k)$$

$$= (A - LCA)e(k)$$

DT full state observer with predictor

$$e(k+1) = \left(A - L \underbrace{CA}_{\tilde{C}} \right) e(k), \quad e(0) = (I - LC) x_0$$

- ▶ the error dynamics can be arbitrarily assigned if the pair $(A, \tilde{C}) = (A, CA)$ is observable

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- ▶ observability matrix

$$\tilde{Q}_d = \begin{bmatrix} \tilde{C} \\ \tilde{C}A \\ \vdots \\ \tilde{C}A^{n-1} \end{bmatrix} = \overbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}^{Q_d} A$$

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- ▶ if A is invertible, then \tilde{Q}_d has the same rank as Q_d
- ▶ (A, \tilde{C}) is observable if (A, C) is observable and A is nonsingular (guaranteed if discretized from a CT system)

Example

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} u(k),$$

$y(k) = x_1(k)$. Place all eigenvalues of an **observer with predictor** at the origin.

$$\begin{aligned} A - LCA &= \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} -a_2 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (l_1 - 1)a_2 & 1 - l_1 & 0 \\ l_2 a_2 - a_1 & -l_2 & 1 \\ l_3 a_2 - a_0 & -l_3 & 0 \end{bmatrix} \end{aligned}$$

$\det(A - LCA - \lambda I) = ((l_1 - 1)a_2 - \lambda)(l_2 + \lambda)\lambda + (1 - l_1)(l_3 a_a - a_0) + l_3((l_1 - 1)a_2 - \lambda) + \lambda(1 - l_1)(l_2 a_2 - a_1)$
roots must be all 0 $\Rightarrow l_1 = 1, l_2 = l_3 = 0$.

1. Concepts
2. Continuous-time Luenberger observer
3. Discrete-time observers
 - DT full state observer
 - DT full state observer with predictor
4. Observer state feedback

Observer state feedback

given system dynamics:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

- ▶ state feedback control: arbitrary eigenvalue assignment if system controllable

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- ▶ state feedback control: arbitrary eigenvalue assignment if system controllable
- ▶ observer design: arbitrary observer eigenvalue assignment for state estimation if system observable
- ▶ when full states are not available, what's the performance if we combine both?

$$u = -K\hat{x} + v$$

Closed-loop dynamics

- full closed-loop system

$$\dot{x} = Ax + Bu$$

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$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

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$$\Rightarrow \frac{d}{dt} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BK \\ LC & A - LC - BK \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} v$$

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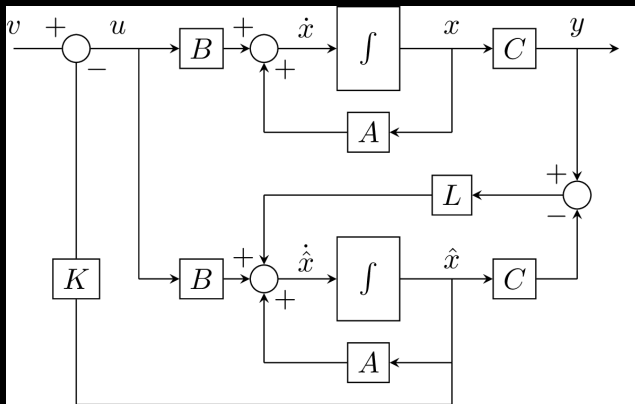
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- using again similarity transform $\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$ gives

$$\frac{d}{dt} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

Block diagram

► $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), u = -K\hat{x} + v$



The separation theorem

- ▶ closed-loop dynamics

$$\frac{d}{dt} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} v$$

- ▶ **powerful result: separation theorem:** closed-loop eigenvalues consist of
 - ▶ eigenvalues of $A - BK$ from the state feedback control design
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- ▶ if system is controllable and observable, we can arbitrarily assign the closed-loop eigenvalues
- ▶ rule of thumb: assign observer dynamics to be faster than state-feedback dynamics