Math Meet 1

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1 Prequisite Notes

The MCMA's Math Group project aims to be a long-term experimental case study in mathematical education, in particular, the development, effect, implication, cause, pros and cons, etc. of competition-oriented problem-solving. The project is led by students, taught by students, and experienced by students. This document focuses on the first meeting of Math Group 1, which is MCMA's first math group and the only existing group up to date. The group consists of four male 11th graders of RDFZ Class 1 with varying backgrounds. They were all taking the IB HL math curriculum. The "Summary" section provides detailed descriptions of the design of the entire activity. However, it should be noted that this is not meant to be a study on the teaching of mathematics; indeed, education covers many more aspects than just teaching. The details are provided so that the reader can have a holistic understanding of the entire process, from which they will form their own judgements on mathematics education. No final conclusion or judgement will be provided in this article. In addition, the observer is part of the math group, and will participate just as other members of the group unless otherwise specified. The observer serves as a role similar to both a member and discussion leader throughout these meets. The members of the group are labelled 1, 2, 3, 4, respectively.

2 Summary

Math Meet 1 was held on Sept. 15 from 1:40-2:20pm. As the first math meet of Math Group 1, no specific goal was intended other than an introductory experience in mathematical problem solving. It was also an oppertunity to obtain basic information about the members of the group. Combinatorical problems requiring minimal background knowledge were selected for obvious reasons. In total, four combinatorical questions were handed to the members of the math group. Problem (1) was indicated as *easy, elementary calculation problem.* Problem (2) was indicated as *excellent practice for mathematical induction.* Problem (3) was indicated as *more challenging, but standard problem.* Problem (4) was indicated as *additional challenge.* The four problems are shown below, in original form:
The problems were all selected from Chapter 1 of Pablo Soberon's book *Problem*

Problem 1.3 Show that if $n \geq k \geq r \geq s$, then

$$\binom{n}{k}\binom{k}{r}\binom{r}{s} = \binom{n}{s}\binom{n-s}{r-s}\binom{n-r}{k-r}.$$

Figure 1: (1)

Example 1.2.16 (IMO 2002) Let n be a positive integer and S the set of points (x, y) in the plane, where x and y are non-negative integers such that $x + y < n$. The points of S are colored in red and blue so that if (x, y) is red, then (x', y') is red as long as $x' \leq x$ and $y' \leq y$. Let A be the number of ways to choose n blue points such that all their x-coordinates are different and let B be the number of ways to choose n blue points such that all their y-coordinates are different. Prove that $A = B$.

Figure 2: (2)

Problem 1.14 (IMO 2006) We say that a diagonal of a regular polygon P of 2006 sides is a good segment if its extremities divide the boundary of P into two parts, each with an odd number of sides. The sides of P are also considered good segments. Suppose that P is divided into triangles using 2003 diagonals such that no two of them meet in the interior of P. Find the maximum number of isosceles triangles with two good segments as sides that can be in this triangulation.

Figure 3: (3)

Problem 1.9 (IMO 2000) A magician has cards numbered from 1 to 100, distributed in 3 boxes of different colors so that no box is empty. His trick consists in letting one person of the crowd choose two cards from different boxes without the magician watching. Then the person tells the magician the sum of the numbers on the two cards and he has to guess from which box no card was taken. In how many ways can the magician distribute the cards so that his trick always works?

Figure 4: (4)

Solving Methods in Olympiad Combinatorics. The problems were selected based on the below considerations:

1. Progressive level of difficulty
2. Low-level content knowledge
3. Major problem-solving aspects
4. Diversity of possible solutions
5. General quality
6. Approachable
7. Moderate level of mental exercise

Participants were given three days to complete the four problems. They were told that "Problems do not have to be completed, but thoughts should be prepared for discussion. You should record all ideas, motivations, intuitions, etc. which may have been used, all you've considered or tried, to the greatest possible extent." Of course, (1) was meant as an easy warm-up to review basic combinatorical definitions, though not directly related to the latter problems. In addition, participants were told to self-study the following basic combinatorical definitions and techniques, which were more than necessary to solve these problems:

1. Sets, operations on sets, special sets and set theory
2. Binomial expansion, Pascal's triangle, corresponding proofs
3. Elementary counting techniques, the counting of sets, related proofs
4. One-variable mathematical induction
5. Mathematical meaning of combination and arrangement

As a final side note, the observer did not look at any of the problem solutions or was given access to related information prior to the event, other then the information that (2) can be solved by induction.

3 Observations

Three participants attended the math meet, (1, 3, 4). Out of which, 2 of them (1, 4, includes the observer) attempted the standard problems. The discussion began with (2), as a general consensus was reached that (1) did not have mathematical value (in terms of problem-solving). The two members who attempted the standard problems knew the idea behind (2), but gave up on (3).

All participants were then encouraged to fully attempt all problems given; they were told that there was no new content to be learned, so if they don't attempt them completely, they'll never be able to solve them. The discussion then began smoothly, with insights being exchanged regarding (2), while the other participant was told to attempt the three problems. Apparently, the main difficulty presented in (2) was to present the proof in a mathematically correct manner. It was agreed that the problem could be solved through observation of the symmetry of configurations through $x = y$, despite being previously informed that it was an exercise on mathematical induction. Participant 1 stated that they shouldn't have been previously informed on the technique used in the official solution, as it would've restricted their own thinking. In the end, participant 4 proposed a mathematically correct proof by construction and proof of a bijection, afterwards proving the necessary conditions related to the arrangement of red and blue points stated in the problem. Participant 1 also managed to successfully construct the bijection immediately afterwards. At the end of the first discussion, participants were asked to explain in detail their train of thought during problem solving. Participant 1 explained that the symmetry was obvious to him by observation, and that intuition was helpful in solving such problems. Participant 4 explained that the key motivation came from the points being bounded by $y = -x$, and with contraints of various configurations being in the x and y axis, respectively. It was argued that intuition wasn't needed to observe the symmetry occuring in (2). Next, (3) was discussed. The main difference in (3) was that more steps were involved, and the information given was complex. Unfortunately, no one was able to find a complete solution for (3), despite that (3) was of a similar difficulty to (2). In fact, all participants made very little progress on (3), although participants found the optimal configuration of 1003 triangles. Participant 1 claimed that everything seemed very obvious but was not easy to prove. However, he quickly realized that the correctness of his configuration was not obvious at all. At first, participants 1 and 4 approached the problem for cases with less sides and tried to form more general observations, however, this quickly failed. However, it was evident that by the creation of good triangles, a polygon with least area P' would be formed within P, which has its vertices lying on the vertices of P, a property which becomes useful during the latter part of discussion. Then, participant 4 observed that by inclusion-exclusion of triangles, some useful properties may be found. Hence, the discussion shifted to the restrictions of a single good isosceles triangle. It was quickly noted by 4 that the two legs of the isosceles triangle must be good sides, and a simple proof by contradiction was devised. 1 also observed this, but was initially confused. Using this property, 1 and 4 desired to prove that the optimal configuration was one in which the good sides are sides of P. The most suitable way to prove this was seemingly using the inclusion-exclusion of the triangles, which implies the motivation that in any other situation, the triangle must occupy the space of some other triangles. This was rigorized by the counting of shared points, as every three points would have a one-one correspondence with each triangle. 4 noted that every pair of vertices of P corresponds to at most one good triangle under certain circumstances, but was not well rigorized.

1 rigorized the claim to a certain extent by noticing the even number of vertices of P, which implied that the base of a good triangle wouldn't be the base of another triangle, which 4 spent some effort in realizing. However, this still wasn't a sufficiently clear justification for 4's argument. Here, although not explicitly brought up in the discussion, the previous observations regarding P' were again used. By 1's observation, the base of each of the good triangles must be on one side of P'. Then, the result follows as the number of sides of P' should be optimized without overlapping with P.