2023 FALL MCOMT

General Mathematics (Precalculus)

Exam paper and solutions written by volunteers of the Secondary Science Society of RDFZ

Revised and Proofread by the MCIME task group of MCMA

International Paper (English version)

DO NOT DISCLOSE

1. Calculate $i^{2022} + 2i^{2021} + 3i^{2020} + \cdots + 2021i^2 + 2022i$.

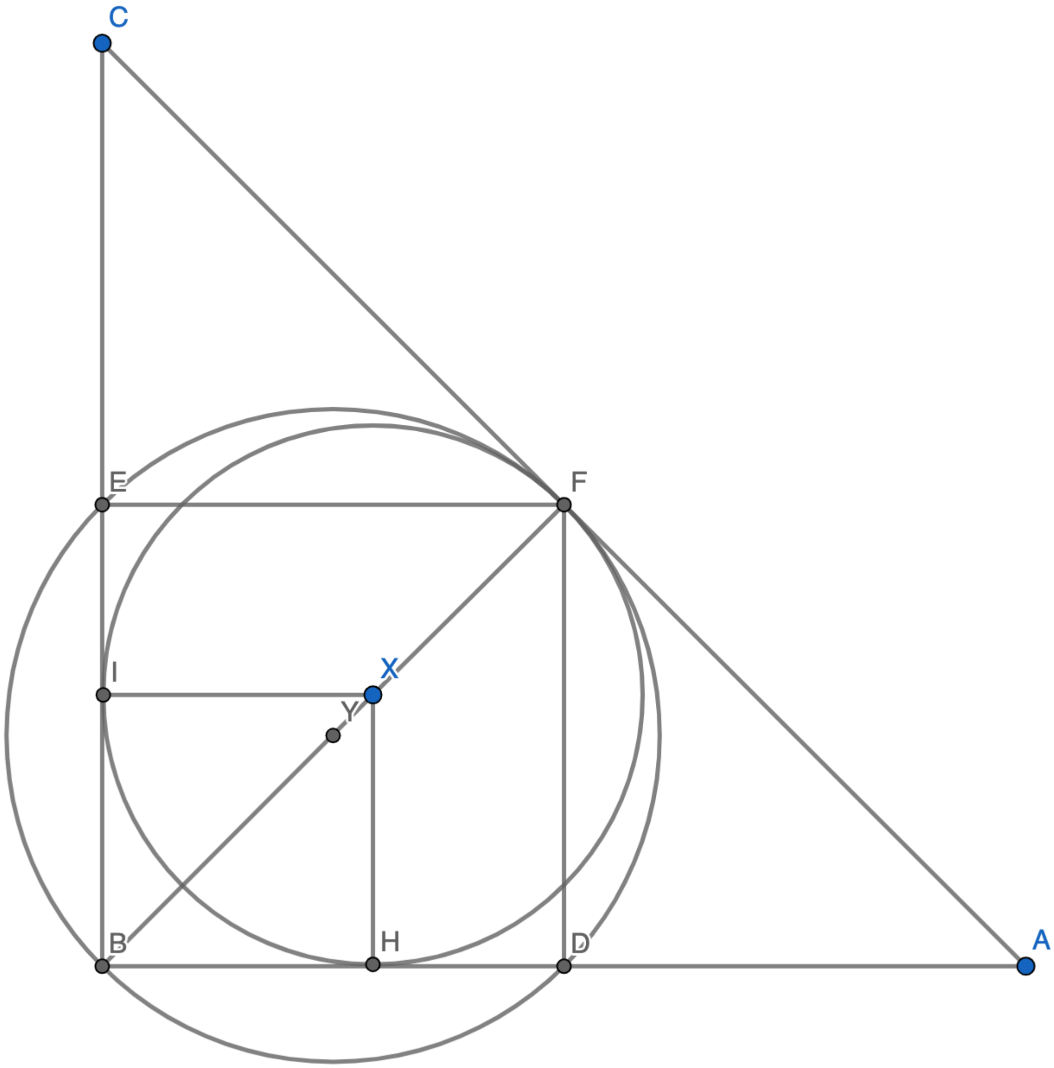
   Solution: $i^{4k} = 1, i^{4k+2} = -1, i^{4k+1} = i, i^{4k+3} = -i$
   Original equation $= (3 + 7 + 11 + \cdots + 2019)(1) + (1 + 5 + 9 + \cdots + 2021)(-1) + (2 + 6 + 10 + \cdots + 2022)(i) + (4 + 8 + 12 + \cdots + 2020)(-i)$
   $= (3 - 1 + 7 - 5 + 11 - 9 + \cdots + 2019 - 2017) - 2021 + i\big((2 - 4 + 6 - 8 + 10 - 12 + \cdots + 2018 - 2020) + 2022\big)$
   $= \left[\left(\frac{2019-3}{4} + 1\right)(2) - 2021\right] + i[\left(\frac{2018-2}{4} + 1\right)(-2) + 2022]$
   $= (1010 - 2021) + i(2022 - 1010)$
   $= 1012i - 1011.$

2. Given relation $f\colon A \to B, a \in A, b \in B$ and there are 32 distinct ordered pairs (a, b), calculate $\min(|A| + |B|)$.

   Solution: Essentially, we want the sum of the number of elements in set A and set B to be as small as possible. Obviously, this is satisfied when the number of ordered pairs $(a, b) = |A||B|$. Thus, $32 = |A||B|$. In order for $|A| + |B|$ to be as small as possible, we need $\big||A| - |B|\big|$ to be as small as possible. Furthermore, because $|A|, |B|$ are factors of 32, $\min(|A| + |B|) = 8 + 4 = 12$.

3. Have isosceles right triangle ABC with $AB = BC = \sqrt{2}$. Let circle X be tangent to all sides of the triangle. Let D, E, F be midpoints of AB, BC, and AC, respectively. Construct circle Y tangent to AC at point F and passes through points D, E. Calculate distance XY.

Solution:



Obviously, the hypotenuse has length AC=2. Furthermore, circle X is the incircle of triangle ABC. Thus,

$$[\mathrm{ABC}] = \frac{r_X\left(2\sqrt{2}+2\right)}{2}$$
$$1 = \left(\sqrt{2}+1\right)r_X$$
$$r_X = \sqrt{2}-1$$

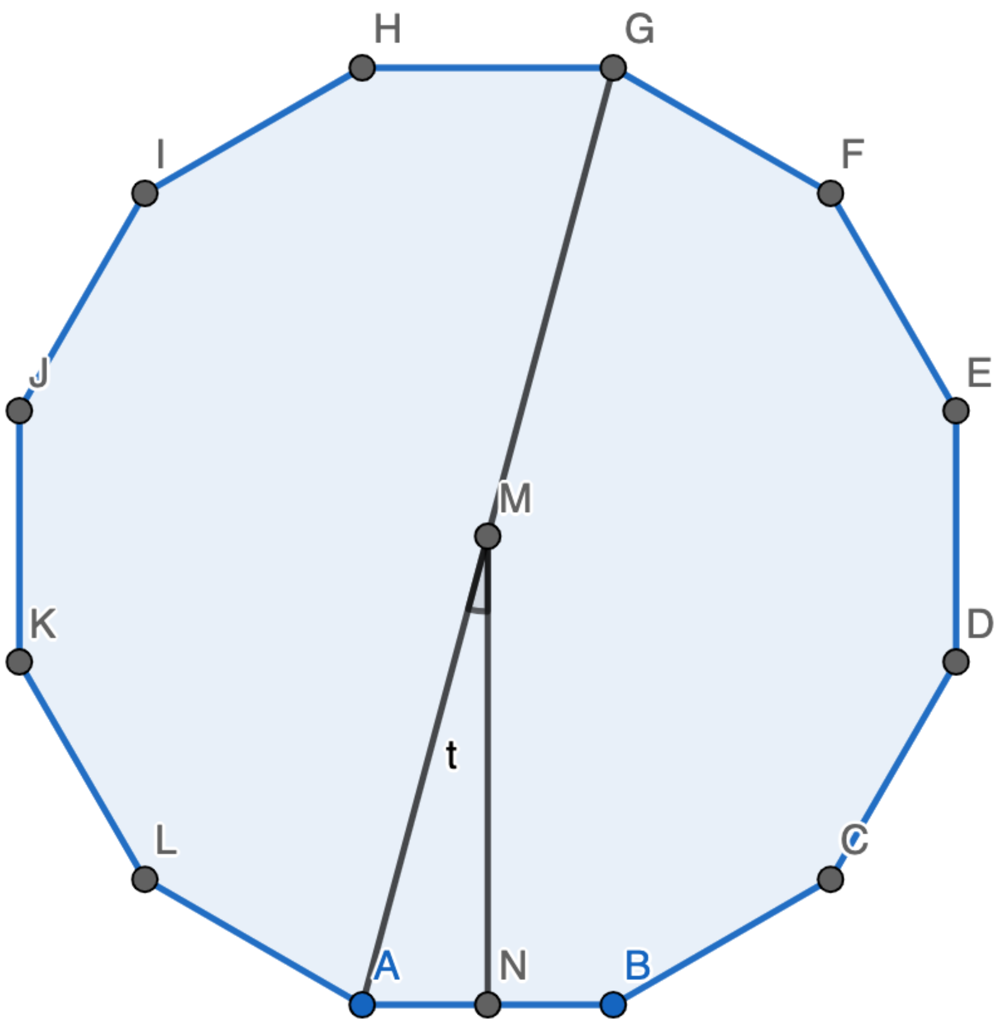
Furthermore, it is easy to see that EF=FD and that the two segments are perpendicular. Note that EFDB is a square. Because circle Y passes through three vertices of the square, it must also pass through the fourth vertex at point B. Finally, the length of segment XY is $FY - FX = \frac{3}{2} - \sqrt{2}$.

4. Calculate the exact area (in simplest form) of a regular dodecagon (12-sided polygon), assuming that the greatest possible distance between two vertices on the figure is 2022 units.

Solution: Using the area equation for a regular polygon:

$$A_{dodecagon} = \frac{(apothem\ length)(length\ of\ one\ side)(number\ of\ sides)}{2}$$

The apothem length is equivalent to $\frac{(length\ of\ one\ side)}{2\tan\frac{180}{12}}$.



Note $\angle t = \left(\frac{360}{24}\right)° = 15°$, thus the length of one side of the dodecagon is

$$s = 2(\sin(15°) * 1011)$$

From which the length of the apothem of the dodecagon is

$$\frac{2022\sin(15°)}{2\tan(15°)} = 1011\cos(15°)$$

Thus

$$A_{dodecagon} = \frac{(1011\cos(15°))(2022\sin(15°))(12)}{2} = 12(1011^2\tan(15°))$$

Note that $12 * 1011^2 = 12265452$ and

$$\tan(15°) = \tan\left(\frac{30}{2}°\right) = \frac{1-\cos(30°)}{\sin(30°)} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$$

Finally,

$$A_{dodecagon} = 24530904 - 12265452\sqrt{3}$$

5. Given that $\log_{10} 3 = 0.4771$, find the number of digits in 3^{2022}.

   Solution: Note, that due to the special characteristics of $\log_{10} x$, if $\log_{10} x < 1$, then x has one digit (because $x < 10$), if $1 \le \log_{10} x < 2$, then x has two digits, and so on. Now note that

   $$\log_{10} 3^{2022} = 2022(\log_{10} 3) = 964.6962$$

   Because $964 \le 964.6962 < 965$, 3^{2022} has 965 digits.

6. If $f(x) = \log_x 2022$,

   $$-f\left(\frac{1}{2}\right)f\left(\frac{2}{3}\right)f\left(\frac{3}{4}\right)\dots f\left(\frac{2022}{2023}\right) = \left(f(4b_1)\right)(\log_{2022} b_1)\prod_{i=1}^{2022}\frac{a}{\log_{2022} b_i},$$

   Solve for

   $$b_1 + a.$$

   Solution:

   Original expression $= -\log_{\frac{1}{2}} 2022 * \log_{\frac{2}{3}} 2022 * \log_{\frac{3}{4}} 2022 * \dots * \log_{\frac{2022}{2023}} 2022$

   Furthermore,

$$-\left(\log_{\frac{1}{2}} 2022\right)\left(\log_{\frac{2}{3}} 2022\right)\dots\left(\log_{\frac{2022}{2023}} 2022\right)$$

$$= (\log_2 2022)\left(\frac{1}{\frac{1}{\log_2 2022}-\frac{1}{\log_3 2022}}\right)\left(\frac{1}{\frac{1}{\log_3 2022}-\frac{1}{\log_4 2022}}\right)\dots\left(\frac{1}{\frac{1}{\log_{2022} 2022}-\frac{1}{\log_{2023} 2022}}\right)$$

$$= \left(\frac{1}{\log_{2022} 2}\right)\left(\frac{1}{\log_{2022}\frac{2}{3}}\right)\left(\frac{1}{\log_{2022}\frac{3}{4}}\right)\dots\left(\frac{1}{\log_{2022}\frac{2022}{2023}}\right)$$

Thus,

$$
\begin{aligned}
f(4b_1)(\log_{2022} b_1) \prod_{i=1}^{2022} \frac{a}{\log_{2022} b_i}
&= \frac{1}{\log_{2022} 2} \prod_{i=2}^{2022} \frac{1}{\log_{2022} \frac{i}{i+1}} \\
&= \frac{\log_{2022} \frac{1}{2}}{\log_{2022} 2} \prod_{i=1}^{2022} \frac{1}{\log_{2022} \frac{i}{i+1}} \\
&= (\log_2 2022)(\log_{2022} \frac{1}{2}) \prod_{i=1}^{2022} \frac{1}{\log_{2022} \frac{i}{i+1}} \\
&\therefore b_1 = \frac{1}{2}, a = 1, b_1 + a = \frac{3}{2}.
\end{aligned}
$$