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Publisher's Editorial

Good Ideas, Tireless Efforts

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After the conclusion of this year's Mathematical Contest in Modeling®, I and others deeply involved in the Contest received the following from Contest Director Frank Giordano:

Dear Friends of the MCM,

It has been my pleasure to be the Contest Director of the MCM for the past 20 contests. At the end of this year's contest, I am delighted that the contest directorship will be in the able and enthusiastic hands of Bill Fox.

I would like to thank all of you for your continued support of the MCM over the years: beginning with Ben Fusaro, the founder of the contest, Sol Garfunkel and COMAP for sponsoring the contest, and founding fathers Marvin Keener and Maynard Thompson who have been at the Final Judging since the beginning. I would also like to thank Courtney Coleman and Bob Borrelli for hosting the Final Judging at Claremont for many years. We remember fondly Mike Moody for his support, friendship, and warm hospitality. And who can forget Bob's wonderful wine tastings! More recently, we thank Gordon McCormick and the Naval Postgraduate School (NPS) for hosting and sponsoring us and facilitating our work.

I sincerely appreciate the counsel, wisdom, and leadership of Marvin Keener and Maynard Thompson, who have served as Head Judges each of the last 20 years. They helped make my job both easy and enjoyable. Their leadership during all phases of the contest has been spectacular.

I taught at West Point with Pat Driscoll and Bill Fox. Drawing on that experience, they knew I would need help, and they willingly gave

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it as Associate Directors. Additionally, as the number of contestants began to increase, Pat headed up the Regional Judging at West Point while Bill did the same at NPS with both sites accomplishing huge amounts of work in a short period of time, always with good cheer. Pat and Bill would often stay late and come early at the Final Judging as well to solve software problems and anything else that needed to be done. Their professional and dedicated service has allowed the contest to expand and take on new challenges without degrading the high standards of the contest. Thank you!

I would especially like to thank our faculty advisors for organizing, training, and motivating their teams. Having been a faculty advisor on the first several contests, I know that the job requires devotion but is a very rewarding experience for the dedicated educator. Additionally, we entrust the faculty advisor with the enormous responsibility to ensure compliance with the contest rules. Thank you for establishing the spirit of fair competition that has characterized our contest. Over the years, I have met or spoken with many of you. I appreciate your camaraderie and feedback that has allowed us to continually examine and improve our rules and procedures. Additionally, the feedback I have gotten from past and present contestants is high testimony to the difference that you have made in their lives. A heartfelt thanks to all of you!

A major decision by COMAP was to allow unlimited participation by the schools and provide optional feedback to the students and faculty advisors. The number of contestants increased rapidly from a couple hundred to about 3,600 this year. COMAP felt that if students would devote 96 hours to solving an interesting problem, we should be able to find folks willing to judge their work and provide useful feedback if the teams asked for it. Participation and feedback are certainly noble goals that we wanted to attain without any loss in quality. This challenge was met due to the hard work at our triage sites, including at Appalachian State University under the leadership of Bill Bauldry, the National Security Agency (NSA) led by Peter Anspach, Carroll College led by Marie Vanisko, West Point led by Pat Driscoll, and NPS led by Bill Fox. I give my heartfelt thanks to each of the judges who volunteered their time at those sites. Also at Carroll, Steve Harper deserves special recognition for designing and constantly improving the contest software that allows us to enter and process large amounts of information quickly and assign stratified judging assignments based upon the history of each paper. With all the elements in place, we found that we could converge to the very top papers, the Finalists as we now call them, without loss of quality. Problem C has become the independent Interdisciplinary Contest in Modeling under the wise and able leadership of Chris Arney, who has provided interesting problems featuring a variety of associated disciplines. Thanks

to all of you at the various sites for allowing the contest to become increasingly an international event with 18 countries participating this year.

Of course, the contest can only be as good as the contest problems it provides. We have been blessed with extremely creative and prolific authors including Jerry Griggs, Mike Tortorella, Kelly Black, Paul Campbell, Joe Malkevitch, Veena Mendiratta, Danny Solow, Doug Faires, and Yves Nievergelt, to name a few. And we remember fondly Don Miller for authoring problems, and cheerfully participating in the judging. Paul Campbell, Pat Driscoll, Bill Fox, and Kelly Black have been invaluable in editing and accomplishing background research on candidate problems.

The folks at NSA deserve special recognition, beginning with Gene Berg who saw the potential of the contest and helped sponsor the contest in a relationship that began in the early 1990s and continues to this day. Additionally, Peter Anspach organized a triage stage at NSA that has been instrumental in allowing the growth of the contest. Jim Case has been invaluable in providing his experience and expertise to help train the team at NSA. Thank you!

Those of us who are seriously interested in education are fortunate to have lived during the COMAP era. I can think of no other organization that has contributed as much as COMAP has to education at all levels under the wise leadership of Sol Garfunkel. I especially thank Sol Garfunkel and Laurie Aragon for allowing us to have fun on all the COMAP projects—each project has been a very worthwhile and enjoyable experience. Gary Feldman has handled our administration with a caring hand. John Tomicek has been a workhorse, amazingly available at all hours of the day, and facilitating our work in so many creative ways—thanks, COMAP!

The future of MCM is bright, as Sol plans to increase U.S. participation, the number of countries participating, and the number of students from current and new countries.

I would especially like to thank each of you for your hard work, dedication, and pleasant demeanor throughout the past 20 years. Like my good friend Ben, I intend to support the MCM as long as I can make a contribution. I hope that we as a team have achieved the vision that Ben, Sol, and the other founding fathers had for the contest. Thanks to all of you, both the ride and the destination have been both pleasant and rewarding!

Sincerely,
    Frank Giordano

Frank Giordano.

Responses

I have always known that the success COMAP has enjoyed over the years is due to the tireless efforts and the kindness of people dedicated to the improvement of mathematics education. Frank Giordano is living proof of that fact. I honestly believe that the success of MCM and its growth into a mature, internationally respected contest could not and would not have happened without Frank. There are no words to express my gratitude. It has been my honor to have worked with Frank over these years. It is my joy to know him as a friend.

Thanks, Frank—for everything

Sol Garfunkel
Publisher, *The UMAP Journal*

We in the mathematics community owe Frank Giordano a debt that we cannot begin to repay. If it were just MCM, HiMCM, ILAPs, the legion of quality textbooks on modeling and applications, the quality graduates of the USMA Mathematical Sciences department that have peopled universities and schools nationwide, . . . , but it is even more. Frank taught us, in the words of an ancient Chinese motto, "It is better to light one candle than to curse the darkness."

Frank has done this for each of us and for our profession over and over by suggesting ideas that grew to fruition in greater change than one could have imagined. He did it structurally in issues that came back as tangible programs (like MCM), in course ideas that permeated curricular change, and in personal touches that challenged each of us to see the possibilities. . . .

Thank you, Frank!

John Dossey
Illinois State University

On behalf of all of the MCM Friends that didn't get a chance to thank you this weekend and the national mathematics and operations research communities—THANKS for all you have done for MCM. You have shown us that MCM is special because of the people involved and we have seen that it is successful because of a special person—Frank Giordano.

Best wishes to you (with all that free time you now have). Best of luck to Bill in filling those big shoes.

Chris Arney
U.S. Military Academy

I want to start by saying the sign of a good leader is the ability to adapt to the changing times and circumstances. During Frank's tenure at the helm, we have seen MCM go from a contest primarily administered via postal mail to a contest on the verge of going paperless. We have also seen the numbers grow from 314 teams in 1994 to 2,741 teams in this current year.

Frank has made my unenviable task of reading and sorting thousands of emails and papers on a yearly basis manageable. I also want to thank Frank for being open and understanding to new and innovative ideas for administrating the contest; you will be dearly missed.

Thanks for making my job/life easier.

John Tomicek
COMAP

Thanks to you for all the hard work you've put in as MCM head for the last 20. I think it's a meritorious—no, make that Outstanding—service to the mathematics community.

Peter Anspach
National Security Agency

About the Correspondents

Brigadier General (retired) Frank R. Giordano graduated from the U.S. Military Academy at West Point in 1964. He taught there for 21 years, including seven years as Professor and Head of the Department of Mathematical Sciences. He currently is Professor of Defense Analysis and Operations Research at the Naval Postgraduate School, Monterey, CA.

Solomon Garfunkel picked up on Ben Fusaro's idea for an "applied Putnam contest," secured funding for it, and has overseen the growth of the resulting MCM/ICM over the past 27 years.

John Dossey has been president of the National Council of Teachers of Mathematics, chair of the Conference Board of the Mathematical Sciences,

and chair of the College Board's Mathematical Sciences Advisory Committee. He is now Distinguished Professor Emeritus at Illinois State University.

Chris Arney has been co-director for many years of COMAPs Interdisciplinary Contest in Modeling® (ICM).

John Tomicek has administered the MCM for the past several years.

Peter Anspach organizes a triage judging session for the MCM.

Editor's Note

Errata to Genetic Inversion Module

Vol. 31, No. 3 (2010)
BioMath Module: Genetic Inversion
(with Module page number in parentheses)

p. 227 (9), vertical label in left margin: Homework 2 $\rightarrow$ Homework 1

p. 228 (10), vertical label in right margin: Activity 1 $\rightarrow$ Activity

p. 229 (11), vertical label in left margin: Homework 1 $\rightarrow$ Homework 2

p. 231 (13), vertical label in left margin: Homework 1 $\rightarrow$ Homework 3

p. 231 (13), middle paragraph beginning "A strip..." should read:

A strip with *exactly one element and appearing at the beginning or end* of the sequence is considered an **increasing** strip. All other strips with *exactly one* element are considered **decreasing** strips.

p. 232 (14), vertical label in right margin: Homework 1 $\rightarrow$ Homework 3

p. 232 (14), Exercise 2b should read:

b. Invert the strip (or group of adjacent strips) that results in x and $x-1$ being adjacent; note: sometimes you will invert a subsequence ending with x and other times you will invert a subsequence ending with $x-1$.

p. 233 (15), vertical label in left margin: Homework 1 $\rightarrow$ Homework 3

MCM Modeling Forum

Results of the 2011 Mathematical Contest in Modeling

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Monterey, CA 93943–5000
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Introduction

A total of 2,775 teams of undergraduates from hundreds of institutions and departments in 17 countries spent a weekend in February working on applied mathematics problems in the 27th Mathematical Contest in Modeling (MCM)®.

The 2011 MCM began at 8:00 P.M. EST on Thursday, February 10, and ended at 8:00 P.M. EST on Monday, February 14. During that time, teams of up to three undergraduates researched, modeled, and submitted a solution to one of two open-ended modeling problems. Students registered, obtained contest materials, downloaded the problem and data, and entered completion data through COMAP's MCM Website. After a weekend of hard work, solution papers were sent to COMAP on Monday. Two of the top papers appear in this issue of *The UMAP Journal*, together with commentaries.

In addition to this special issue of *The UMAP Journal*, COMAP has made available a special supplementary *2011 MCM-ICM CD-ROM* containing the press releases for the two contests, the results, the problems, unabridged versions of the Outstanding papers, and judges' commentaries. Information about ordering is at `http://www.comap.com/product/cdrom/index.html` or at (800) 772–6627.

Results and winning papers from the first 26 contests were published in special issues of *Mathematical Modeling* (1985–1987) and *The UMAP Journal* (1985–2010). The 1994 volume of *Tools for Teaching*, commemorating the tenth anniversary of the contest, contains the 20 problems used in the first 10 years of the contest and a winning paper for each year. That volume and the special

MCM issues of the *Journal* for the last few years are available from COMAP. The 1994 volume is also available on COMAP's special *Modeling Resource* CD-ROM. Also available is *The MCM at 21* CD-ROM, which contains the 20 problems from the second 10 years of the contest, a winning paper from each year, and advice from advisors of Outstanding teams. These CD-ROMs can be ordered from COMAP at `http://www.comap.com/product/cdrom/index.html`.

This year, the two MCM problems represented significant challenges:

- Problem A, "Snowboard Course," asked teams to design a snowboard course for maximum "vertical air," as well as consider other requirements and trade-offs.

- Problem B, "Repeater Coordination," asked teams to determine the number of VHF radio repeaters needed to amplify and retransmit signals from mobile units, for 1,000 and for 10,000 users in a 40-mile-radius flat area.

COMAP also sponsors:

- The MCM/ICM Media Contest (see p. 108).

- The Interdisciplinary Contest in Modeling (ICM)®, which runs concurrently with MCM and next year will offer a modeling problem involving network science. Results of this year's ICM are on the COMAP Website at `http://www.comap.com/undergraduate/contests`. The contest report, an Outstanding paper, and commentaries appear in this issue.

- The High School Mathematical Contest in Modeling (HiMCM)®, which offers high school students a modeling opportunity similar to the MCM. Further details are at `http://www.comap.com/highschool/contests`.

2011 MCM Statistics

- 2,775 teams participated (with 735 more in the ICM)

- 13 high school teams (<0.5%)

- 347 U.S. teams (12%)

- 2,428 foreign teams (88%), from Canada, China, Finland, Germany, Indonesia, Ireland, Malaysia, Mexico, Pakistan, Scotland, Singapore, South Africa, South Korea, Spain, Taiwan, and the United Kingdom

- 8 Outstanding Winners (<0.5%)

- 23 Finalist Winners (1%)

- 354 Meritorious Winners (13%)

- 842 Honorable Mentions (30%)

- 1,545 Successful Participants (55%)

Problem A: Snowboard Course

Determine the shape of a snowboard course (currently known as a "half-pipe") to maximize the production of "vertical air" by a skilled snowboarder. "Vertical air" is the maximum vertical distance above the edge of the halfpipe. Tailor the shape to optimize other possible requirements, such as maximum twist in the air. What tradeoffs may be required to develop a "practical" course?

Problem B: Repeater Coordination

The VHF radio spectrum involves line-of-sight transmission and reception. This limitation can be overcome by "repeaters," which pick up weak signals, amplify them, and retransmit them on a different frequency. Thus, using a repeater, low-power users (such as mobile stations) can communicate with one another in situations where direct user-to-user contact would not be possible. However, repeaters can interfere with one another unless they are far enough apart or transmit on sufficiently separated frequencies. In addition to geographical separation, the "continuous tone-coded squelch system" (CTCSS), sometimes nicknamed "private line" (PL) technology, can be used to mitigate interference problems. This system associates to each repeater a separate sub-audible tone that is transmitted by all users who wish to communicate through that repeater. The repeater responds only to received signals with its specific PL tone. With this system, two nearby repeaters can share the same frequency pair (for receive and transmit); so more repeaters (and hence more users) can be accommodated in a particular area.

For a circular flat area of radius 40 miles, determine the minimum number of repeaters necessary to accommodate 1,000 simultaneous users. Assume that the spectrum available is 145 to 148 MHz, the transmitter frequency in a repeater is either 600 kHz above or 600 kHz below the receiver frequency, and there are 54 different PL tones available.

How does your solution change if there are 10,000 users?

Discuss the case where there might be defects in line-of-sight propagation caused by mountainous areas.

The Results

The solution papers were coded at COMAP headquarters so that names and affiliations of the authors would be unknown to the judges. Each paper was then read preliminarily by two "triage" judges at either Appalachian State University (Snowboard Course Problem) or at the National Security Agency (Repeater Coordination Problem) or at Carroll College (Repeater Coordination

Problem). At the triage stage, the summary and overall organization are the basis for judging a paper. If the judges' scores diverged for a paper, the judges conferred; if they still did not agree, a third judge evaluated the paper.

Additional Regional Judging sites were created at the U.S. Military Academy, the Naval Postgraduate School, and Carroll College to support the growing number of contest submissions.

Final judging took place at the Naval Postgraduate School, Monterey, CA. The judges classified the papers as follows:

	Outstanding	Finalist	Meritorious	Honorable Mention	Successful Participation	Total
Snowboard Course Problem	4	9	157	300	820	1290
Repeater Coordination Problem	4	14	197	542	725	1482
	8	23	354	842	1545	2732

We list here the 8 teams that the judges designated as Outstanding; the list of all participating schools, advisors, and results is at the COMAP Website.

Outstanding Teams

Institution and Advisor   **Team Members**

Snowboard Course Problem

"Designing a Half-Pipe for Advanced Snurfers"
Eastern Oregon University   Alex Macavoy
La Grande, OR   Jadon Herron
Anthony A. Tovar   Rachel Burton

"Higher in the Air: Design of a Snowboard Course"
Peking University   Yingfei Wang
Beijing, China   Chu Wang
Zijing Dong   Binghong Han

"A Half-Blood Half-Pipe, A Perfect Performance"
Tsinghua University   Enhao Gong
Beijing, China   Xiaoyun Wang
Jimin Zhang   Rongsha Li

"Designing the Optimal Snowboard Half-Pipe"
University of Western Ontario   Zhe Chen
London, ON, Canada   Markus Sturm
Allan B. MacIsaac   Simon Xu

Repeater Coordination Problem

"Clustering on a Network"
Harvey Mudd College   Louis Ryan
Claremont, CA   Dylan Marriner
Susan E. Martonosi   Daniel Furlong

"VHF Repeater Placement"
Rensselaer Polytechnic Institute   Emily P. Meissen
Troy, NY   Joseph H. Gibney
Peter Roland Kramer   Yonatan Naamad

"Fewest Repeaters for a Circular Area:
    Iterative Extremal Optimization Based on
    Voronoi Diagrams"
University of Electronic Science and Technology   Wengqiang Wang
Chengdu, Sichuan, China   Zimo Yang
Tao Zhou   Yue Cao

"Optimizing VHF Repeater Coordination Using
    Cluster Analysis"
Virginia Polytechnic Institute and State University   John W. Frey
Blacksburg, VA   Patrick O'Neil
John F. Rossi   Evan Menchini

Awards and Contributions

Each participating MCM advisor and team member received a certificate signed by the Contest Director and the appropriate Head Judge.

INFORMS, the Institute for Operations Research and the Management Sciences, recognized the teams from University of Western Ontario (Snowboard Course Problem) and University of Electronic Science and Technology (Repeater Coordination Problem) as INFORMS Outstanding teams and provided the following recognition:

- a letter of congratulations from the current president of INFORMS to each team member and to the faculty advisor;

- a check in the amount of $300 to each team member;

- a bronze plaque for display at the team's institution, commemorating team members' achievement;

- individual certificates for team members and faculty advisor as a personal commemoration of this achievement; and

- a one-year student membership in INFORMS for each team member, which includes their choice of a professional journal plus the *OR/MS Today* periodical and the INFORMS newsletter.

The Society for Industrial and Applied Mathematics (SIAM) designated one Outstanding team from each problem as a SIAM Winner. The teams were from Tsinghua University (Snowboard Course Problem) and Harvey Mudd College (Repeater Coordination Problem). Each of the team members was awarded a $300 cash prize, and the teams received partial expenses to present their results in a special Minisymposium at the SIAM Annual Meeting in Vancouver, BC, Canada in July. Their schools were given a framed hand-lettered certificate in gold leaf.

The Mathematical Association of America (MAA) designated one Outstanding North American team from each problem as an MAA Winner. The teams were from Eastern Oregon University (Snowboard Course Problem) and Virginia Polytechnic Institute and State University (Repeater Coordination Problem). With partial travel support from the MAA, the teams presented their solution at a special session of the MAA Mathfest in Lexington, KY in August. Each team member was presented a certificate by an official of the MAA Committee on Undergraduate Student Activities and Chapters.

Ben Fusaro Award

One Meritorious or Outstanding paper was selected for each problem for the Ben Fusaro Award, named for the Founding Director of the MCM and awarded for the seventh time this year. It recognizes an especially creative approach; details concerning the award, its judging, and Ben Fusaro are in Vol. 25 (3) (2004): 195–196. The Ben Fusaro Award Winners were the teams from University of Western Ontario (Snowboard Course Problem) and University of Electronic Science and Technology (Repeater Coordination Problem). A commentary on the latter appears in this issue.

Judging

Director
Frank R. Giordano, Naval Postgraduate School, Monterey, CA

Associate Directors
Patrick J. Driscoll, U.S. Military Academy, West Point, NY
William P. Fox, Dept. of Defense Analysis, Naval Postgraduate School, Monterey, CA

Snowboard Course Problem

Head Judge
Marvin S. Keener, Executive Vice-President, Oklahoma State University,
    Stillwater, OK
Associate Judges
William C. Bauldry, Chair, Dept. of Mathematical Sciences,
    Appalachian State University, Boone, NC (Head Triage Judge)
Kelly Black, Mathematics Dept., Union College, Schenectady, NY
Patrick J. Driscoll, Dept. of Systems Engineering, U.S. Military Academy,
    West Point, NY (INFORMS Judge)
Ben Fusaro, Dept. of Mathematics, Florida State University, Tallahassee, FL
    (SIAM Judge)
Jerry Griggs, Mathematics Dept., University of South Carolina,
    Columbia, SC
Mario Juncosa, RAND Corporation, Santa Monica, CA (retired)
Michael Tortorella, Dept. of Industrial and Systems Engineering,
    Rutgers University, Piscataway, NJ (Problem Author)
Richard Douglas West, Francis Marion University, Florence, SC

Regional Judging Session at the U.S. Military Academy
Head Judge
Patrick J. Driscoll, Dept. of Systems Engineering
Associate Judges
Tim Elkins, Dan McCarthy, and Kenny McDonald,
    Dept. of Systems Engineering
Steve Horton, Dept. of Mathematical Sciences
    —all from the United States Military Academy at West Point, NY

Triage Session at Appalachian State University
Head Triage Judge
William C. Bauldry, Chair, Dept. of Mathematical Sciences
Associate Judges
Jeffry Hirst, Rene Salinas, Tracie McLemore Salinas,Katie Mawhinney,
    Greg Rhoads, and Kevin Shirley
    —all from the Dept. of Mathematical Sciences, Appalachian State
        University, Boone, NC

Repeater Coordination Problem

Head Judge
Maynard Thompson, Mathematics Dept., University of Indiana,
    Bloomington, IN
Associate Judges
Robert Burks, Operations Research Dept., Naval Postgraduate School,
    Monterey, CA

Jim Case (SIAM Judge)
J. Douglas Faires, Youngstown State University, Youngstown, OH
Steve Horton, Dept. of Mathematical Sciences, U.S. Military Academy,
    West Point, NY (MAA Judge)
Michael Jaye, Dept. of Defense Analysis, Naval Postgraduate School,
    Monterey, CA
Veena Mendiratta, Lucent Technologies, Naperville, IL
Greg Mislick, Naval Postgraduate School, Monterey, CA
David H. Olwell, Naval Postgraduate School, Monterey, CA
Kathleen M. Shannon, Dept. of Mathematics and Computer Science,
    Salisbury University, Salisbury, MD (MAA Judge)
Dan Solow, Case Western Reserve University, Cleveland, OH
    (INFORMS Judge)
Marie Vanisko, Dept. of Mathematics, Engineering, and Computer Science,
    Carroll College, Helena, MT (Ben Fusaro Award Judge)

Regional Judging Session at the Naval Postgraduate School
Head Judges
William P. Fox, Dept. of Defense Analysis
Frank R. Giordano, Dept. of Defense Analysis
Associate Judges
Michael Jaye, Dept. of Defense Analysis
Robert Burks, Greg Mislick, and Scott Nestler, Operations Research Dept.
David H. Olwell, Dept. of Systems Engineering
    —all from the Naval Postgraduate School, Monterey, CA

Triage Session at Carroll College
Head Judge
Marie Vanisko, Dept. of Mathematics, Engineering, and Computer Science
Associate Judge
Terry Mullen, Dept. of Mathematics, Engineering, and Computer Science
    —both from Carroll College, Helena, MT

Triage Session at the National Security Agency
Head Triage Judge
Peter Anspach, National Security Agency (NSA), Ft. Meade, MD
Associate Judges
Jim Case
Other judges from inside and outside NSA, who wish not to be named.

Sources of the Problems

Both the Snowboard Course Problem and the Repeater Coordination Problem were contributed by Michael Tortorella (Rutgers University).

Acknowledgments

Cautions

To the reader of research journals:
Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each paper here is the result of undergraduates working on a problem over a weekend. Editing (and usually substantial cutting) has taken place; minor errors have been corrected, wording altered for clarity or economy, and style adjusted to that of *The UMAP Journal*. The student authors have proofed the results. Please peruse these students' efforts in that context.

To the potential MCM advisor:
It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

COMAP's Mathematical Contest in Modeling and Interdisciplinary Contest in Modeling are the only international modeling contests in which students work in teams. Centering its educational philosophy on mathematical modeling, COMAP serves the educational community as well as the world of work by preparing students to become better-informed and better-prepared citizens.

Editor's Note

The complete roster of participating teams and results has become too long to reproduce in the printed copy of the *Journal*. It can now be found at the COMAP Website, in separate files for each problem:

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http://www.comap.com/undergraduate/contests/mcm/contests/
    2011/results/2011-Problem-A.pdf
http://www.comap.com/undergraduate/contests/mcm/contests/
    2011/results/2011-Problem-B.pdf
```

Media Contest

This year, COMAP organized the first MCM/ICM Media Contest.

Over the years, contest teams have increasingly taken to various forms of documentation of their activities over the grueling 96 hours—frequently in video, slide, or presentation form. This material has been produced to provide comic relief and let off steam, as well as to provide some memories days, weeks, and years after the contest. We *love* it, and we want to encourage teams (outside help is allowed) to create media pieces and share them with us and the MCM/ICM community.

The media contest is *completely separate* from MCM and ICM. No matter how creative and inventive the media presentation, it has *no* effect on the judging of the team's paper for MCM or ICM. We do not want work on the media project to detract or distract from work on the contest problems in any way. This is a separate competition, one that we hope is fun for all.

Further information about the contest is at

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http://www.comap.com/undergraduate/contests/mcm/media.html.
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Results of the 2011 Media Contest are at

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http://www.comap.com/undergraduate/contests/mcm/contests/
    2011/results/media/media.html:
```
Outstanding: Dalian Maritime University (Yuqing Shi, Hao Yin, Jingyu Qi)
Finalists:

Beijing Institute of Technology (Tengfei Yu, Yongbo Chen, Keyu Wu)
North Carolina School of Science and Mathematics
    (Christy Vaughn, Matt Jordan, Kevin Valakuzhy)

A Half-Blood Half-Pipe,
A Perfect Performance

Enhao Gong
Xiaoyun Wang
Rongsha Li
Tsinghua University
Beijing, China

Advisor: Jimin Zhang

Abstract

Our basic model has two parts: to find a half-pipe shape that can maximize vertical air, and to adapt the shape to maximize the possible total angle of rotation. In an extended model, we analyze the snowboarder's effect on vertical air and on rotation. Finally, we discuss the feasibility and the tradeoffs of building a practical course.

The major assumption is that resistance includes the friction of snow plus air drag, with the former proportional to the normal force. We find air drag negligible.

We first obtain and solve a differential equation for energy lost to friction and drag based on force analysis and energy conservation. We calculate vertical air by analyzing projectile motion. We then calculate the angular momentum before the flight and discuss factors influencing it. In an extended model, we take the snowboarder's influence into account.

We compare analytical and numerical results with reality, using default parameters; we validate that our method is correct and robust. We analyze the effects on vertical air of width, height, and gradient angle of the half-pipe. We find that a wider, steeper course with proper depth and the path of a skilled snowboarder are best for vertical air. Using a genetic algorithm, we globally optimize the course shape to provide either the greatest vertical air or maximal potential rotation; there is a tradeoff. Implementing a hybrid scoring system as the objective function, we optimize the course shape to a "half-blood" shape that would provide the eclectically best snowboard performance.

Background

A half-pipe is the venue for extreme sports such as snowboarding and skateboarding. It usually consists of two concave ramps (including a transition and a vert), topped by copings and decks, facing each other across a transition as shown in **Figure 1**. Half-pipe snowboarding has been a part of the Winter Olympics since 2002; the riders take two runs, performing tricks such as straight airs, grabs, spins, flips, and inverted rotations.

Figure 1. End-on schematic view of a half-pipe. (Source: Wikimedia Commons; created by Dennis Dowling.)

We find no analysis of the "best" shape for a half-pipe. However, usually it is 100–150 m long, 17–19.5 m wide, and 5.4–6.5 m from floor to crown, with slope angle 16–18.5° [Postins n.d.]. In addition, the Fédération Internationale de Ski (FIS) recommends that the width, height, transition, and the bottom flat be 15 m, 3.5 m, 5 m, and 5 m, respectively [2003, 36].

Half-pipe snowboarding is currently judged using subjective measures. Still, there is strong community perception that air time and degree of rotation play a major role in competition success [Harding et al. 2008a; Harding et al. 2008b]. According to Harding et al. [2008b] and Harding and James [2010], who have attempted to introduce objective analysis into the scoring, air time and total rotation are the two most critical evaluation criteria.

Terminology and Definitions

Cycle: The start of a cycle is when the snowboarder reaches the edge of the half-pipe after a flight, and the end of a cycle is the next start.

Flight: the part of the movement when the snowboarder is airborne.

Flight distance (S_f): displacement along the z direction during the flight.

Flight time (t_f): duration of the flight.

Cycle distance (S_c): displacement along z direction during a cycle.

Assumptions

- The cross section of the half-pipe is a convex curve that is smooth (has second-order derivative) everywhere except the endpoints.

- The snowboarder crouches during the performance until standing up to gain speed right at the edge of the half-pipe before the flight.

- We neglect the rotational kinetic energy of the snowboarder before considering the twist performance.

- The friction of the snow is proportional to the normal force of the snow exerted on the snowboarder but has nothing to do with velocity (that is, the angle between the direction of the snowboard and the snowboarder's velocity is constant).

- Air drag is proportional to the square of speed.

- The snowboarder's body is perpendicular to the tangential surface of the half-pipe during movement on the half-pipe.

- The force exerted on the board can be considered as acting at its center.

- We neglect the influence of natural factors such as uneven sunshine (which may result from an east or west orientation), altitude, etc.

Basic Model

Model Overview

A cycle can be divided into two parts: movement on the half-pipe, and the airborne performance.

For the first, we focus on the conversion and conservation of energy. The loss of mechanical energy E_{lost} due to the resistance of snow and air is the key. We derive a differential equation for it. We cannot neglect the snowboarder's increasing the mechanical energy by stretching the body (standing up) and doing work against the centrifugal force.

To derive an expression for vertical air, we apply Newton's Second Law. If we neglect air drag during the flight (we later show that it is indeed negligible), we can calculate vertical air, duration of the flight, flight distance, gravitational potential decrease, etc.

Next, we discuss the airborne rotation of the snowboarder. Since the shape of the half-pipe directly influences the initial angular momentum of the snowboarder, and the angular momentum cannot change during the flight, the relationship between the half-pipe shape and the initial angular momentum is the key to our discussion. After deriving an expression for the initial angular momentum, we can find the optimal shape of the half-pipe.
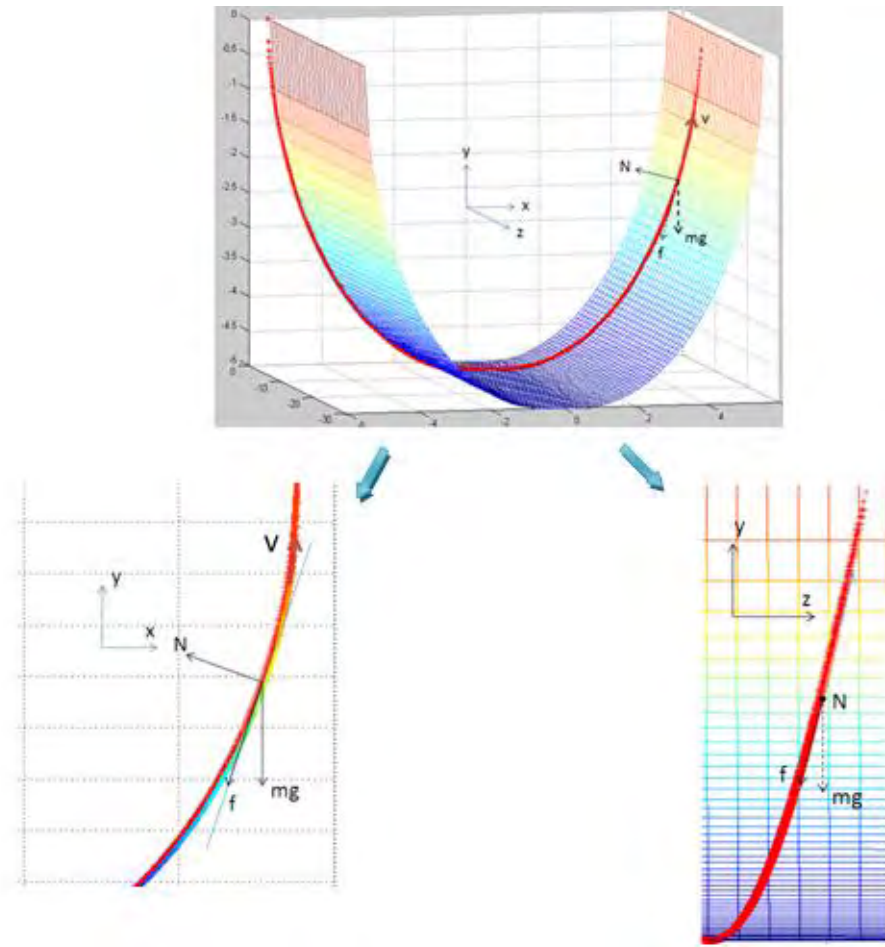
Figure 2. Force analysis.

The Model

Vertical Air

Step 1. Force Analysis: The top part of **Figure 2** shows the definition of the coordinate variables: x is the free variable, while y and z are functions of x. The relationship between y and x depends on the shape of the half-pipe, while the relationship between z and x depends on the path chosen by the snowboarder.

Three forces act on the snowboarder: gravity (mg), normal force (N), and resistance (f).

Resistance can be represented as

$$f = \alpha N + \beta(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \alpha N + \beta(1 + y'^2 + z'^2)\dot{x}^2.$$

For the normal force, only the part of centripetal acceleration that is parallel

Parameter	Meaning
x, y, z	Coordinate variables
$\dot{x}, \dot{y}, \dot{z}$	Velocities
y', y''	$\partial y/\partial x$, $\partial^2 y/\partial x^2$
z'	$\partial z/\partial x$
s	Length of the path
E_0	Initial mechanical energy at the beginning of a cycle
E_{leave}	Kinetic energy right before the flight
E_{reach}	Kinetic energy at the end of the flight
E_{lost}	Mechanical energy lost due to friction of the snow and air drag
N	Normal force of the snow exerted on the snowboarder
W_{human}	Work done by the snowboarder at the edge of the half-pipe when (s)he stands up
W_G	Decrease in gravitational potential during the flight
f	Friction of the snow plus air drag
m	Mass of the snowboarder
α	Friction coefficient between the snow and snowboard
β	Drag coefficient of air
θ	Angle between z-axis and the horizontal plane
Δh	Rise of the mass point of the snowboarder when (s)he stands up from a crouching position
ρ	Radius of curvature at a point on the cross section of the half-pipe
$x_t, y_t, z_t, y_t', z_t', \dot{y}_t, \dot{z}_t, y_t''$	Values right before the flight
H_f	Vertical air

to the direction of N needs to be considered:

$$\rho = \frac{\left(1 + y'^2\right)^{3/2}}{y''},$$

$$N = \frac{\dot{x}^2 + \dot{y}^2}{\rho} m + \frac{mg \cos\theta}{\sqrt{1 + y'^2}} = \left(y'' \dot{x}^2 + g\cos\theta\right) \frac{m}{\sqrt{1 + y'^2}}.$$

Path length unit can be represented as

$$ds = \sqrt{1 + \dot{y}^2 + \dot{z}^2} \, dx.$$

Step 2. Energy Conservation: According to the Energy Conservation Principle, we have

$$\tfrac{1}{2}m \left(1 + y'^2 + z'^2\right) \dot{x}^2 = E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta).$$

Then we have

$$\dot{x}^2 = \frac{2}{m\left(1 + y'^2 + z'^2\right)} \left[E_0 - E_{\text{lost}} - mg(y\cos\theta - z\sin\theta)\right].$$

Step 3. E_{lost}:

$$E_{\text{lost}} = \int_{-x_0}^{x} f \cdot ds$$

$$= \int_{-x_0}^{x} \left[\alpha N \sqrt{1 + y'^2 + z'^2} + \beta \left(1 + y'^2 + z'^2\right)^{3/2} \cdot \dot{\tau}^2 \right] d\tau$$

$$= \int_{-x_0}^{x} \left\{ \left(\alpha \frac{my''}{\sqrt{(1 + y'^2)(1 + y'^2 + z'^2)}} + \beta \right) \times \right.$$

$$\frac{2}{m} \left[E_0 - E_{\text{lost}} - mg(y \cos \theta - z \sin \theta) \right]$$

$$\left. + \alpha mg \cos \theta \frac{\sqrt{1 + y'^2 + z'^2}}{\sqrt{1 + y'^2}} \right\} d\tau.$$

The integral has a variable upper limit. Differentiating both sides and solving the resulting first-order linear ordinary differential equation, we get an expression for E_{lost}, which we would like to minimize. However, since the relationship between y and x is unknown, as is that between z and x, the expression is a functional. The expression and the calculation are too complicated, so we use a numerical method to solve the problem.

Step 4. W_{human}: When the snowboarder stands up, (s)he does work overcoming the centrifugal force. At high speed, the centrifugal force is huge, so W_{human} is considerable and cannot be neglected. The work done by the snowboarder is

$$W_{\text{human}} = \frac{\dot{t}_t^2}{\rho} m \cdot \Delta h = \frac{y_t'^2 y_t''}{\left(1 + y_t'\right)^{3/2}} m \cdot \Delta h.$$

Step 5. Vertical Air: At the edge of the half-pipe before the flight, we have $\dot{x} = 0$. From the Energy Conservation Principle, we get

$$\tfrac{1}{2} m \left(\dot{y}_t^2 + \dot{z}_t^2 \right) = E_0 - E_{\text{lost}}(x_t) = mg \cdot z_t \sin \theta + \frac{y_t'^2 y_t''}{\left(1 + y_t'\right)^{3/2}} m \cdot \Delta h.$$

Since $\dfrac{\dot{y}_t}{\dot{z}_t} = \dfrac{y_t'}{z_t'}$, we have $\dot{z}_t = \dfrac{z_t'}{y_t'} \dot{y}_t$ and

$$\dot{y}_t^2 = \frac{2}{m \left[1 + \left(\frac{z_t'}{y_t'} \right)^2 \right]} \left(E_0 - E_{\text{lost}}(x_t) + mg \cdot z_t \sin \theta + \frac{y_t'^2 y_t''}{\left(1 + y_t'\right)^{3/2}} m \cdot \Delta h \right).$$

If we neglect air drag during the flight, **Figure 3** shows that vertical air is

$$H_f = \frac{1}{\cos\theta}\frac{\dot{y}_t^2}{2g\cos\theta} = \frac{E_0 - E_{\text{lost}}(x_t) + mg\cdot z_t\sin\theta + \dfrac{y_t'^2 y_t''}{(1+y_t')^{3/2}}m\cdot\Delta h}{mg\cdot\cos^2\theta\left[1+\left(\frac{z_t'}{y_t'}\right)^2\right]}.$$
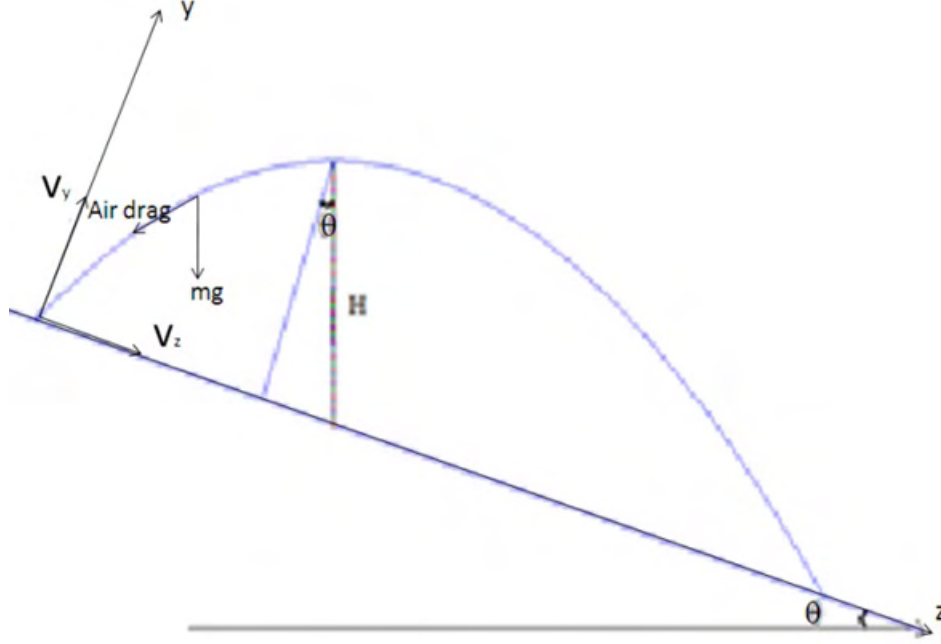
Figure 3. The path and force analysis of the flight.

Step 6. Flight Distance: To compare the initial kinetic energy of two adjacent cycles, we must calculate the decrease of the gravitational potential at the beginning and at the end of the flight. The duration of the flight is

$$t = \frac{2\dot{y}_t}{g\cos\theta}.$$

The flight distance is $S_f = \dot{z}_t t + \frac{1}{2}g\sin\theta\cdot t^2$.
The decrease of gravitational potential is $W_G = mg\cdot S_f\sin\theta$.

Finally, we give an equation to describe the energy conversion and conservation relationship at the beginning and the end of a cycle:

$$E_0 = E_{\text{lost}} + mg\cdot z_t\sin\theta + W_G + W_{\text{human}} + E_{\text{reach}}.$$

From this, we get

$$E_{\text{reach}} = E_0 - E_{\text{lost}} - mg\cdot z_t\sin\theta - W_G - W_{\text{human}}.$$

Rotation

[EDITOR'S NOTE: The authors regard the snowboarder as a stick and use considerations of conservation of angular momentum (in the absence of air drag) to explain various tricks. We omit the details.]

Numerical Computation

We determine values for some parameters.

The coefficient α of kinetic friction between snow and snowboard is generally 0.03–0.2 [Yan et al. 2009], while Chen et al. [1992] determined it to be 0.0312. Thus, we generally assume $\alpha = 0.03$

The air drag coefficient β is about 0.15 [Yan 2006].

Since the mass of Olympic Champion Shaun White is 63 kg, we assume that the mass of a snowboarder is typically 60 kg. Moreover, according to ZAUGG AG EGGIWIL [2008], the drop-in ramp height should be at least 5.5 m and the distance from ramp to pipe should be at least 9 m. Since the slope angle is about 18°, the potential energy is $mg(5.5 + 9\cos 18° \approx 8267$ J.

Since $\alpha < 0.2$ (for which E_0 would be 6613 J), we conservatively assume the initial mechanical energy at the beginning of a cycle (E_0) to be 7000 J.

We need to define $y(x)$ and $z(x)$, the shape of the ramp cross-section and the path that the snowboarder chooses. As we assumed before, $y(x)$ is a smooth convex curve, which is symmetric in reality. There is a horizontal flat connecting two parts of the ramp and each part consists of a smooth transition part and possibly vertical part.

As **Figure 4** shows, with constant friction the energy loss is minimized if z is proportional to $\tau(x)$, the length of the projection curvature of the three-dimensional curve $\vec{p}(x) = \big((x, y(x), z(x)\big)$ on the xy-plane. Thus, we define the coordinate along the z-axis as $z = z\big(\tau(x)\big) \approx k\tau(x)$. Practically, we control $z(x)$ using the location of the point right before the flight.

Given the relationships between $y(x)$ and x and between $z(x)$ and x, we use numerical approximation to solve the differential equation for E_{lost}. **Figure 4** shows the output figure from numerical simulation.

To validate our results, we compared analytical results and numerical results for the dimensions

$$\text{width} = 15 \text{ m}, \quad \text{flat} = 5 \text{ m}, \quad \text{depth} = 3.5 \text{ m}, \quad z_t = 10 \text{ m},$$

with conditions

$$\theta = 18°, \quad m = 60 \text{ kg}, \quad g = 9.8 \text{ m/s}^2, \quad \beta = 0.15 \text{ kg/m},$$

and transition part standard elliptic.

Analytical and numerical results differed negligibly, yielding the results of **Table 2**. The resulting maximum vertical air of about 7 m matches Shaun White's best performance and is consistent with common performances (12–20 ft for men, 6–15 ft for women).

Sensitivity

We modified parameter values by 10%, and the results (**Table 3**) support that the numerical simulation is robust.
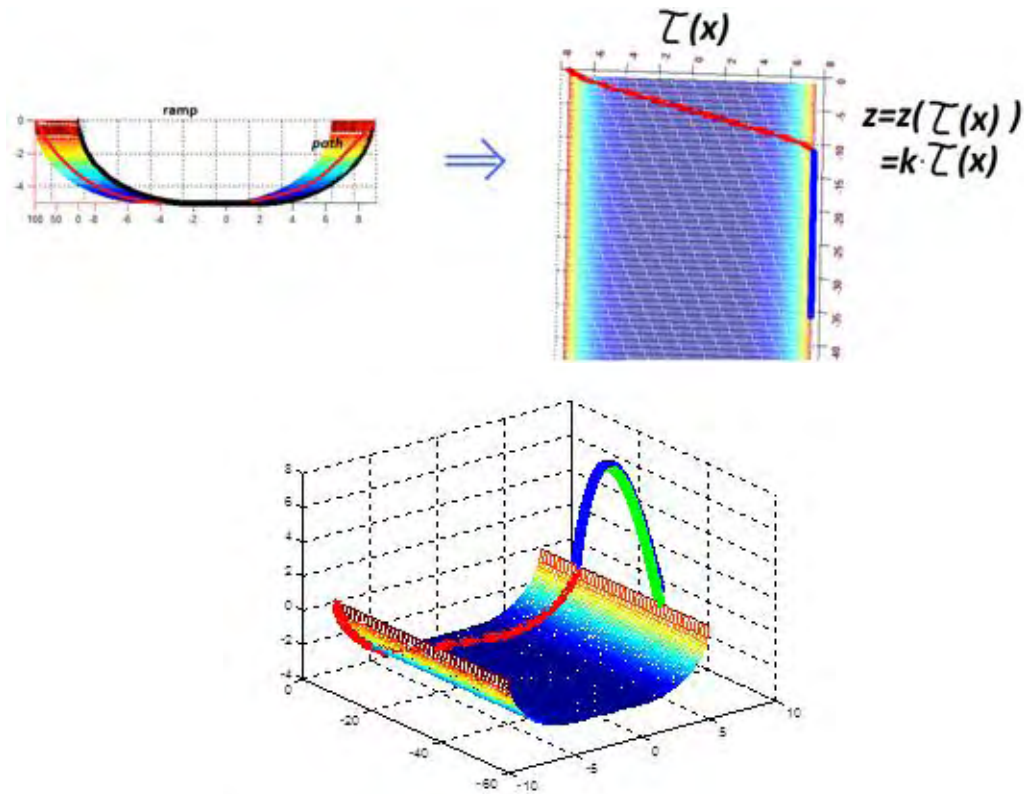
Figure 4. Path of the snowboarder.

Table 2.

Results.

Condition	$\dot{y}$ (m/s)	Vertical air (m)	Flight distance (m)	Duration of flight (s)
No air resistance	11.2	7.12	25.5	2.41
Air resistance	11.2	7.00	24.5	2.38

Table 3.

Sensitivity of vertical air to $\pm 10\%$ change in parameter values.

Parameter	Percentage change in vertical air for	
	+10% in parameter	−10% change in parameter
Width	+5.5%	−6.1%
Depth	+2.5%	−2.6%
Flat	+0.9%	−0.9%
Z_t (Flight point)	+8.2%	−8.1%

Extended Model

[EDITOR'S NOTE: We omit the authors' extended model, which takes into account the snowboarder's actions' effect on $\dot{z}_t$ and total degree of rotation.]

Solutions to the Requirements

Question 1: Snowboard Course for Maximal Vertical Air

Maximum vertical air is determined by the parameters width W, depth H, flat B, flight point z_t, and transition shape ($y(x)$). Using an elliptical transition shape and standard values for the other parameters, we find:

- The wider the ramp, the faster the snowboarder can speed up before the flight and consequently the greater vertical air is.

- A higher ramp can result in higher speed and greater vertical air. But when the ramp is higher than 15 m, the speed and the vertical air decrease with height. Commonly, the height of ramp is around 3–6 m.

- Longer flat provides greater vertical air. But the accompanying decreasing value of E_{leave} means that the potential maximum vertical air is decreased; the actual vertical air is affected by the direction in which the snowboarder flies.

- The steeper the venue is (described by θ), the faster snowboarders can slide and the higher they can fly.

- The path of the snowboarder (z_t) plays a significant part in vertical air. A steeper path provides higher speed, and a greater chance to fly farther down the pipe, but at the expense of fewer tricks. A shallower path ($z_t < 5$ m) means cutting straight across and straight up the wall, but it provides less momentum and lower speed. The snowboarder's skill in choosing a path can help optimize vertical air in some tricks.

Global Optimal

We examine a multidimensional space of parameter values for an elliptical transition:

- width from 13 m to 18 m,
- height from 2 m to 6 m,
- flat length from 4 m to 6 m,
- z_t from 0 m to 15 m, and
- θ from 15° to 20°.

Using the kinetic energy before the flight (E_{leave}) and vertical air as criteria, we find that the best values of the parameters for vertical air (when the snowboarder does not change direction right before the flight) are:

width = 18 m, height = 3 m, flat = 4 m, $z_t = 6$ m, $\theta = 20°$.

The shape of the transition plays a significant role, since the curve directly determines the energy loss caused by friction along the path. To find the optimal transition shape, we applied a genetic algorithm, generating random convex curves and using third-order controlling splines. Vertical air was the fitness function, and the genes were the position of the control points of the B-splines. **Figure 5** shows the result, which matches previous work claiming that the transition should be an ellipse [Fédération de Ski Internationale 2003].
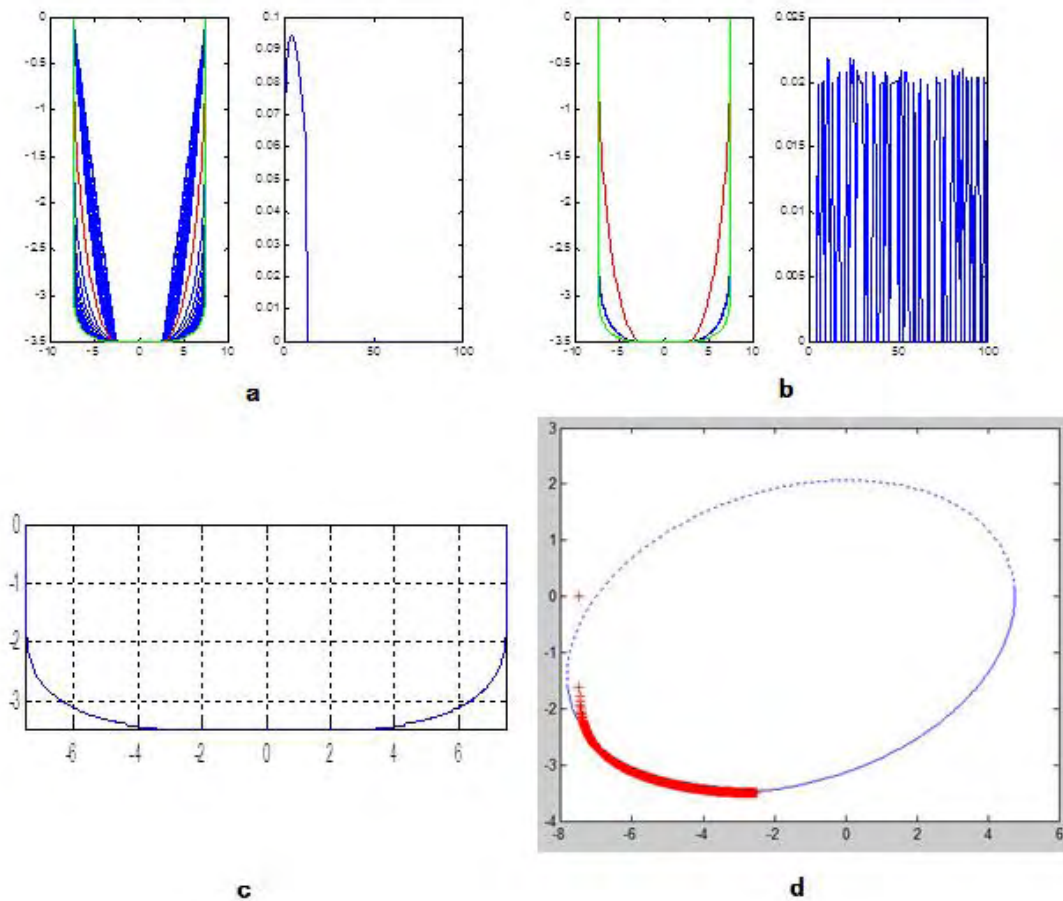
Figure 5. Result of optimizing transition shape using a genetic algorithm.
(a) Initial generation of curves (blue) and their fitness values.
(b) Final generation of curves (blue) and their fitness values.
(c) Shape of the optimized course.
(d) Fitting the optimized course with a second-order curve, which turns out to be a segment of an ellipse.

Question 2: Tailoring the Shape to Other Requirements

[EDITOR'S NOTE: Here the authors apply their genetic algorithm technique to tailor the shape of the course to optimize separately total duration of flight, initial angular momentum in the z-direction, and total degree of rotation.]

   We also modified the objective function and implemented a genetic algorithm for the tradeoff between vertical air and total degree of rotation. As Harding et al. [2008c] have shown, the overall performance of half-pipe snowboard can be treated as a complex combination of vertical air, average air time (AAT), and average degree of rotation (ADR). (As we have derived, air time (flight time) is almost proportion to the square root of vertical air in the motion of a projectile, so they can be regarded as a single criterion). They put forward an equation to predict the score of a snowboarder:

$$\text{Predicted score} = 11.424(\text{AAT}) + 0.013(\text{ADR}) - 2.223.$$

They justify this function empirically.

   We set the objective function for the genetic algorithm to be the predicted score. The resulting optimal shape of the course can be fitted by a combination of the two ellipses that fit optimal curves for vertical air and total degree of rotation. Judging by the predicted score, a snowboarder could get a score greater than 46 on this course.

Question 3: Tradeoffs for a Practical Course

Radius of curvature: The radius of curvature of the half-pipe cross-section cannot be too small, or the snow may fall off. Since the snowboard is more than 1 m long, too small a radius of curvature is dangerous.

Flat bottom: Since the 1980s, half-pipes have had extended flat ground (the flat bottom) added between the quarter-pipes; the original-style half-pipes have become deprecated. The flat ground gives the athlete time to regain balance after landing and more time to prepare for the next trick.

   Moreover, according to our numerical computation, when a half-pipe has a wider flat bottom, then E_{reach} and E_{leave} decrease, whereas the duration of flight and vertical air increase. The change in number of cycles is negligible.

E_{reach}**:** E_{reach} is the final kinetic energy of a cycle, as well as the E_0 of the next cycle. Considering that the snowboarder experiences more than one cycle, it is not wise to chase a higher jump at the cost of less E_{reach}.

   According to our results, the wider the course, the larger the vertical air. However, a wider course leads to lower final kinetic energy (E_{reach}), which is unwelcome. The situation is similar with the height of the half-pipe: As long as the height is less than the width, a higher course results in larger vertical air but less E_{reach}. Flat also has the same effect.

   Therefore, the half-pipe should not be too wide, too high, or too flat.

Number of cycles: In the Olympics, a snowboarder must perform 5 to 8 acrobatic tricks (hence 5 to 8 cycles) along the half-pipe's 110 m extent.

From our computation, both vertical air and E_{reach} become larger for larger θ. Nonetheless, since S_f (the displacement along the z direction during the flight) increases as θ increases, a snowboarder would find it difficult to perform enough acrobatic tricks on a very steep half-pipe.

Hence, θ should not be too large.

Strengths and Weaknesses

- The model describes the motion in detail, with coordination among the many physical quantities.

- The numerical computations are precise.

- The results generated by numerical computation agree with empirical data, lending support to the model.

- The model takes the subjective influence of snowboarders into account.

- We establish an objective function to compare different course shapes.

- We optimize the course locally (to learn the individual impact of the parameters) as well as globally (to shed light on the design of a half-pipe), and obtain numerical solutions.

- The model does not provide an analytic solution for the optimal course.

- The model does not take into account detailed mechanical characteristics and on-snow performance of snowboards.

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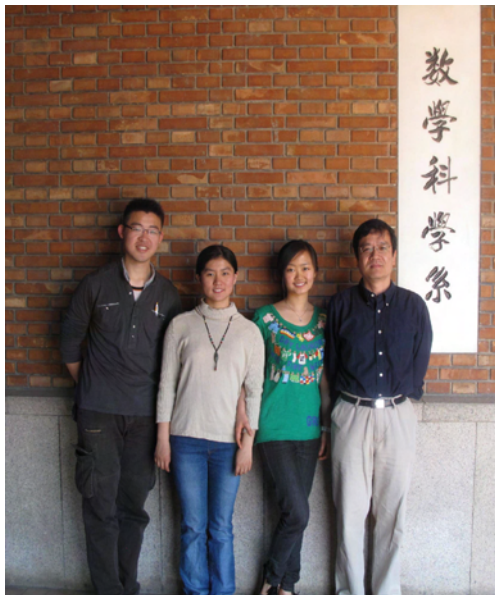
Enhao Gong, Rongsha Li, and Xiaoyun Wang, with advisor Jimin Zhang.
(The sign says "Department of Mathematics.")

Judges' Commentary:
The Outstanding Snowboard Course Papers

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Overview

The focus of the problem was design of a snowboard course that allows snowboarders to achieve the largest possible vertical jump. The problem also required that teams identify issues associated with athletes performing other tricks and identify potential tradeoffs for other considerations.

We provide an overview of a few select observations of some of the judges. Students are required to put together a well-formed report on a complex topic in only a few days. Every year there are inconsistencies and errors in even the best reports, and the judges always struggle to find ways to balance the positive and negative aspects of each team's submission.

The problem examined this year is no exception; this problem is even more complex than usual. The Outstanding papers represent remarkable work by talented teams. Careful reading of the reports can reveal specific errors, but it is important to recognize the limitations of the event and examine the report as a whole.

This overview is divided into four parts:

- a broad overview of the judging process,

- an overview of the models and their derivation that were submitted by many teams,

- issues of examining the sensitivity of the resulting mathematical models, and

- an overview of how some student teams presented their overall results.

The Judging Process

The judging process proceeds in three sets of rounds.

- **The triage process.** Every paper is read several times by different people. The goal is to determine which papers should be given more careful attention and could possibly achieve a higher rating.

   The amount of time available per paper during the triage round is limited. The main concern is whether or not a team has answered the question. The importance of the summary is amplified for these initial readings. A paper that provides a good overview of the entire paper, is written well, and provides a good overview within each section has a stronger probability of being passed on to the later rounds.

- **Screening rounds.** The judges are given more time to read each paper. In the triage round, papers perceived to be good tend to be given the benefit of doubt and be passed on; in the screening round, this is still true, but the goal begins to shift from removing papers that are not likely to achieve a higher ranking to trying to identify good papers that require more careful reading.

   During the screening rounds, the judges spend more time examining the mathematical model. Papers that provide a clear description of the model and offer substantial analysis of it tend to receive higher marks. The judges can begin to spend more time and focus on the whole submission. There is a higher expectation that the analysis, results, and writing be more consistent.

- **Final rounds.** The judges are given an increased amount of time to focus on the teams' submissions. During this set of rounds a judge may spend between half an hour to a full hour reading a single paper. During these rounds, the complete focus is on identifying the best papers. The judges focus on particular details and are able to make detailed comparisons between papers.

At the end of the final rounds, there are typically 12 to 16, and each remaining paper is given a rating of Finalist. Time is allotted so that each paper is read by every judge. At the end of the reading time, the judges assemble, and together they discuss each paper in order. The judges then make the final decision about which papers receive a rating of Outstanding. After deciding which teams receive Outstanding, the members of each of the sponsoring societies assemble in smaller groups to decide which paper should receive their award.

Modeling

This competition requires students to examine a nontrivial problem and identify a potential solution in a short amount of time. The problem this

year required that the teams put together a nontrivial physical model and then apply mathematical tools in the analysis of it.

This overview of the modeling issues is broken up into two parts. First the physics of the various approaches is examined. Then the mathematics associated with the various approaches is examined. The majority of the teams used one of two approaches to the physics. The types of analysis cannot be easily divided with respect to the approach that the teams took to derive the physical model.

Physics

One of the difficulties in this problem is that it required the teams to model nontrivial dynamics. The first task required the teams to describe the physical situation and describe the terms found within the complex equations describing the physical situation.

Overall, the teams tended to take one of two approaches to develop a model, centering on use of either:

- the work-energy relationship, or

- Newton's Second Law.

Each team then had to translate the approach into a system of equations. For this second issue, the teams made use of a wide variety of techniques.

Deriving the Physical Situation

The first task for the teams was to describe the physical setup of a snowboarding "half-pipe." The International Olympic Committee has specific restrictions on the design of a half-pipe, and the majority of teams tried to stay consistent with the Olympic specifications.

The majority of teams broke down the construction of the half-pipe into a small number of parts. For example, a common construction included a middle flat part down the center of the half-pipe, round corners at the ends, and a flat lip along the top of the sides. Describing those restrictions can be difficult, and in this case most teams made use of a diagram that greatly simplified the task of translating those restrictions for the reader.

Once the parameters associated with the half-pipe were defined a coordinate system also had to be defined. Different teams used different coordinate systems, and there is no one obvious coordinate system to use. Teams that clearly indicated the coordinate system and showed it in a diagram had an immediate advantage when it came to describing the derivation of their model.

The parameters for the snowboarder had to be defined. An immediate discriminator for a paper in this regard was whether or not a free body diagram of the snowboarder was included. Teams that included one made it much easier for the judges to understand the resulting model.

Physical Principles

The teams whose primary approach made use of the relationship between work and energy faced a number of difficulties. The first is that the relationship between work and energy is a scalar relationship, and it does not easily lend itself to determining the height in a multivariate setting.

The teams also had to determine the total work done on a person while traveling on a snowboard, which then required that they model the system using Newton's Second Law. Upon successfully modeling the motion of a rider, the team then had to decide which forces were relevant to calculating the work integral for the friction forces and then approximate the integral. By itself, calculating the work integral was a difficult task to accomplish.

Teams that focused solely on the differential equations derived from Newton's Second Law had fewer complications. Even using a the free body diagram, the team still needed to do a correct derivation of the differential equations. It also required determining how to represent the forces in the different parts of the half-pipe including the straight section, the corners, the upper lip, and moving through the air.

Some papers used different parametrizations for the different sections, which caused a number of difficulties. Also, a common mistake found in even the best papers was to use mv^2/r to represent the magnitude of the radial force in the round corners. This is only true for constant radial velocity, which is not the case in this situation.

Mathematical Models of the Physical Principles

Once a team decided which physical principle to use and which terms were most important, the team had to formulate a system of equations. The entries that tended to receive the most positive attention used systems of differential equations. Given the complex paths different teams made use of different ways to express these equations and divided them into the various situations in different ways.

For example, some teams broke up the equations in terms of the location of the snowboarder in the half-pipe. Also, teams parametrized in terms of time, position, or other quantities. Because of the structure of the course and the multivariate nature of the problem, it was important for a team to describe carefully the parametrization and what equation was used for different portions of the half-pipe.

Bringing all of the physical principles together, keeping them consistent for the whole of the path within the half-pipe, translating the motion correctly into a system of equations, and then implementing the model in a consistent way was an extremely difficult task. Every team's entry included errors, and some of those errors were basic problems dealing with details such as the multivariate chain rule, numerical approximation, or assumptions about the values of physical terms.

The judges made every effort to try to balance the difficulty of the prob-

lem and the short time allotted to the teams with the desire to have a clear, correct solution. This not possible in the best of situations, and the judges had a difficult task in comparing entries to decide which team put together a better solution. In the end, it was a matter of judgment, and the work of the teams that more clearly discussed how they were able to arrive at a conclusion and justify their work made a more positive impression.

Sensitivity

The exploration of the sensitivity of the models tends to mark a significant difference between the top tier of the submissions and the rest of the entries. The judges expect that the best papers will include some indication of which parameters are most important and are the most sensitive in terms of what happens to the predictions in the presence of small changes in their values or what happens under slightly different assumptions. This year, the physical situation offered a rich set of options to explore the sensitivity of the resulting models.

The goal is to determine what happens to the snowboarder's performance for small changes in one or more parameters. The impact in terms of both the height of the jumps and safety for the snowboarders are important questions to address through the sensitivity analysis.

The exploration can take many forms. The most straightforward approach is to examine small changes in the results when different individual parameters are changed. For example, a team might examine what happens when the width of the half-pipe is changed by some small amount.

The sensitivity of different parameters is *always* an important aspect to the development of a mathematical model. *Every year*, the judges look closely at this aspect of the problem; *every year*, very few teams explore this aspect of the problem. A simple way for a team to have their submission stand out from the other submissions is to include a coherent exploration of the sensitivity of the mathematical model.

Discussion of Results

The majority of the teams used one of a few standard approaches. The differences between the entries were the combination of techniques used and how extensively the model was analyzed. The three things that make an entry stand out and receive positive attention from the judges are the following:

- the combination of techniques to assemble a mathematical model,

- the analysis of the model, and

- the writing and presentation of the model and results.

Advice

The first impression that a team can make on a judge comes from how the material is presented. To make a positive impression a team must provide a coherent structure to their document. The summary must be coherent and include an overview of the problem, an overview of the paper, and the team's specific results. The document itself should follow some basic rules and maintain a consistent presentation throughout the paper.

There are some simple rules for any entry:

- The nomenclature adopted by a team should be clearly described. (Keep in mind that different teams use different terms and variable names, which can make it difficult for a judge to compare different papers.)

- Every graph, table, or plot should be clearly described in the text, and the teams should explicitly explain what to look for in it and why it is relevant to the paper.

- Every equation should be numbered and proper punctuation employed to integrate the equations within the text.

- A picture can make complicated ideas much easier to understand.

- A free body diagram and a clear picture that shows the coordinate system can make it much easier for a judge to determine what a team was able to accomplish.

- When a plot is used, the axes should be clearly labeled and the units stated.

- Just having a table of contents at the beginning of the document can make it much easier for a judge reading a paper in the early rounds.

- Finally, team members should know the difference between a citation and a reference. The references are the sources listed at the end of the document and are a vital part of a paper. Citations are the indications within the text that help the reader decide which references are associated with specific ideas. A vast number of entries include a list of references but do not include citations within the text. Simply including consistent citations is an easy way to make a team's entry stand apart from the other entries.

Conclusions

The problem this year was difficult. Determining the important parameters and designing a half-pipe for snowboarders is a challenge that required the teams to bring together complex physics principles and use a wide array of mathematical topics. Every team was unable to avoid some basic

pitfalls, but most of the submissions reflected the teams' overall desire to complete an excellent submission for this event.

The majority of teams made use of similar physical principles, but the different ways that those principles were applied and translated into a mathematical model made the difference between submissions. The judges were aware that this is a difficult problem, and the teams had a limited time to explore the topic. Despite these difficulties, the teams were able to bring together a high level of talent and desire that resulted in an impressive collection of entries.

In the end, the difference between the papers judged to be the top entries came down to the analysis of the subsequent models and the way in which the teams conveyed their results.

About the Author

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Fewest Repeaters for a Circular Area: Iterative Extremal Optimization Based on Voronoi Diagrams

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Abstract

We propose a two-tiered network in which lower-power users communicate with one another through repeaters, which amplify signals and retransmit them, have limited capacity, and may interfere with one another if their transmitter frequencies are close and they share the same private-line tone.

Our objective is the fewest repeaters so that either every user is covered by at least one repeater or else every user can communicate with any other user anywhere in the considered area.

Motivated by cellular networks, we give a naïve solution where the number of repeaters and their positions can be obtained analytically. In a circular area with radius 40 miles, 12 repeaters can accommodate 1,000 simultaneous users.

We further propose an iterative refinement algorithm consisting of three fundamental modules that draw the Voronoi diagram, determine the centers of the circumscribed circles of the Voronoi regions, and escape the local optimum by using extremal optimization. The algorithm obtains a solution with 11 repeaters, which we prove to be the absolute minimum. For 10,000 users, it uses 104 repeaters, better than the naïve solution's 108.

We further discuss how to assign frequencies and private-line tones (based on maximum and minimum spanning tree techniques), accommodating simultaneous users, the fluctuation of user density in reality, how the landscape can affect repeaters' locations, and the strengths and weaknesses of the model and the algorithms.

Introduction

Amplify-and-forward relay networks are very helpful for long-distance communication. Through relay nodes, low-power users can communicate with one another in situations where direct user-to-user communication would not be possible. For example, amateur radio "hams" communicate through relay nodes (repeaters), and in wireless sensor networks relay nodes help information dissemination [Akyildiz et al. 2002].

In many such networks, the nodes are homogeneous, and a node can simultaneously play the roles of source node, sink node, and relay node; a typical example is a cellphone. In other scenarios, the relay nodes usually do not initiate communications but only help communications between other nodes. For example, in amateur radio, repeaters can be considered relay nodes, and hams usually carry their own radios (transceivers); the functionalities of repeaters and radios are different, as are their power specifications.

Pan et al. [2003] proposed a two-tiered relay network model, where base stations are considered low-power users and some interapplication nodes play the role of relay nodes. However, they did not address the issue of covering all users. Gupta and Younis considered fault-tolerant [2003a] and traffic load balance [2003b] problems in a two-tiered relay network model but did not address the placement of relay nodes. Tang et al. [2006] proposed two algorithms for placing the fewest relay nodes.

Those works differ from ours because in a more general scenario the capacity of a repeater and interference among nearby repeaters should be taken into account. We propose a model in which each repeater can simultaneously manage at most C users and two nearby repeaters interfere with each other if their transmitter frequencies are close and they share the same private-line tone. Our objective is the fewest repeaters that can satisfy the users' communication requirement. We consider two such requirements:

- **Weak requirement:** Every user is covered by at least one repeater; this is equivalent to a circle-covering problem.

- **Strong requirement:** Every user can communicate with any other user.

The circle-covering problem is NP-hard [Fowler et al. 1981] and is usually very time-consuming even for a small number of circles. For example, Nurmela and Östergård [2000] proposed a simulated annealing algorithm to obtain near-optimal solutions to cover the unit square with up to 30 equal circles; their algorithm has to run more than 2 weeks for 27 circles.

Motivated by cellular networks, we give a naïve solution in which the number of repeaters and their positions can be obtained analytically. In numerical simulation, this naïve solution performs unexpectedly well. For a circular area with radius 40 miles, 12 repeaters accommodate 1,000 simultaneous users; for 10,000 users, 108 suffice. We propose an iterative refinement algorithm that draws the Voronoi diagram, determine the centers of

the circumscribed circles of the Voronoi regions, and escapes the local optimum by using extremal optimization. For 1,000 users, this method has 11 repeaters, which we prove is optimal. For 10,000 users, it has 104 repeaters.

Problem Description

Given a circular flat area Γ of radius Φ, we are to determine the fewest radio repeaters to accommodate N users. A repeater is a combination receiver/transmitter that picks up weak signals, amplifies them, and retransmits them on a different frequency. The three parameters characterizing a repeater are receiver frequency f_r, transmitter frequency f_t, and private-line (PL) tone n_{PL}. A repeater responds only to signals on its receiver frequency that contain its PL tone and retransmits with the same PL tone. Both f_r and f_t are in the range [145 MHz, 148 MHz], we have $|f_r - f_t| = 0.6$ MHz, and there are $N_{\mathrm{PL}} = 54$ PL tones available.

The maximal communication distance r from a user to a repeater is the same for every user, and it is considerably smaller than the communication radius R of a repeater. Every user should be covered by at least one repeater.

The primary problem is to determine the minimum number of repeaters to satisfy the communication requirement.

Model

We use a two-tiered directed network $D\{V_u, V_r, E_{ur}, E_{rr}\}$, where $V_u = \{u_1, u_2, ..., u_N\}$ and $V_r = \{r_1, r_2, ..., r_M\}$ denote the sets of users and repeaters, and E_{ur} and E_{rr} are the sets of directed links from users to repeaters and between repeaters. A user u_i is identified by a location $\big(x(u_i), y(u_i)\big)$ in the plane, and a repeater r_j is identified by its location, receiver frequency, transmitter frequency and PL tone as $\big(x(r_j), y(r_j), f_r(r_j), f_t(r_j), n_{PL}(r_j)\big)$. The frequencies of a repeater r_j satisfy $f_r(r_j), f_t(r_j) \in [145\,\mathrm{MHz}, 148\,\mathrm{MHz}]$ and $|f_r(r_j - f_r)| = 0.6$ Mhz. Since the considered area is circular with radius Φ, the location of a user or a repeater satisfies $x^2 + y^2 \leq \Phi^2$, where $\Phi = 40$ mi. A directed link from u_i to r_j exists if $|u_i - r_j| \leq r$. A directed link from r_j to r_k exists (i.e., $(r_j, r_k \in E_{rr})$) if $|r_j - r_k \leq R$, the transmitter frequency of r_j equals the receiver frequency of r_k (i.e., $f_t(r_j) = f_r(r_k)$), and they share the same PL tone (i.e., $n_{PL}(r_j) = n_{PL}(r_k)$). Clearly, we need to know the locations of users and repeaters.

A network D is a solution if the following three conditions $(\Omega_1, \Omega_2, \Omega_3)$ are all satisfied.

- Ω_1 - **Capacity**. For simplicity, we assume that users are uniformly distributed and each communicates with the nearest repeater. Each repeater can manage at most C users at the same time. For repeater r_j, there is a

connected area $S_V(r_j)$, the *Voronoi region* of r_j, such that for every point inside r_j is the nearest repeater and for every point outside, r_j is definitely not the nearest repeater. The number of users inside the Voronoi region of a repeater must be no more than its capacity C.

- Ω_2 - **Interference avoidance**. If two repeaters share the same PL tone and are less than $2R$ apart, the difference between their transmitter frequencies must be no less than the threshold $f_c = 0.6$ MHz.

- Ω_3 - **Connectivity**. Every user is covered by at least one repeater. That is, for every u_i, $\exists r_j$ such that $(u_r, r_j) \in E_{ur}$.

Our goal is a solution with the minimum number of repeaters M. Although every user is covered by at least one repeater, the user's signals may not reach the desired position in Γ, since the coverage of a repeater is also limited and the solution does not guarantee a multi-hop path through several repeaters to reach the desired position. Ignoring the small area that can be reached directly by a user without the help of a repeater, the reachable area $S_r(u_i)$ of user u_i is the area in Γ that can be covered by at least one reachable repeater of u_i (each repeater covers a circle with radius R).

The set of reachable repeaters $R_r(u_i)$ for u_i consists of:

- repeaters directly reachable by u_i (i.e., the repeaters located within the circle with radius r and centered at u_i), and

- repeaters reachable through links in E_{rr} from the directly-reachable repeaters.

Figure 1 illustrates a simple example where r_2 can be directly reached by u_1, and r_4 and r_5 can be further reached starting from r_2. Two nearby repeaters, r_2 and r_3, may not have a link between them, since they may not match in frequency or PL tone. The reachable repeaters of u_1 are r_2, r_4, and r_5, and the reachable area of u_1 is the union of their coverage areas. The following condition must be satisfied to guarantee every user can in principle reach any position of the considered area through multi-hop repeaters.
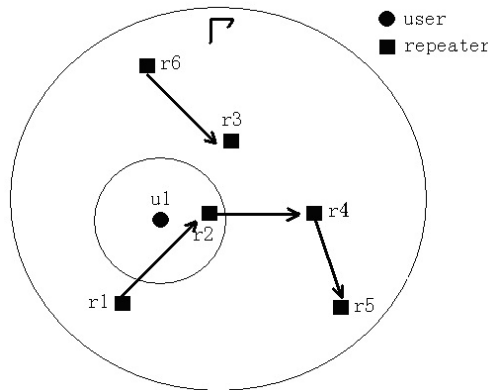
Figure 1. An illustration of repeaters reachable from u_1. The circle centered at u_1 has radius r.

- Ω_4 - **Global reachability**. The reachable area of every user is all of Γ.

A network D is a *strong solution* if $(\Omega_1, \Omega_2, \Omega_4)$ are all satisfied. When $R \geq 2\Phi$, any solution is a strong solution.

To find a strong solution is much more difficult than to find a solution, and the two tasks are equivalent only if $R \geq 2\Phi$.

Analysis

We calculate the communication ranges for repeaters and users, as well as the repeater's capacity, using Shannon's information theory. Taking into consideration mobility of users, we show that continuous approximation of the distribution of users' locations is necessary to address the problem. We present a naïve solution with repeaters arranged in a cellular network.

Communication Radius

We assume that there is no interference from fog, rivers, hills, buildings, sunspots, etc.

Let $P_{r,\text{out}}$ be the power of the signal transmitted by a repeater. Its average power P in a unit area at a distance D from the repeater is

$$P = \frac{P_{r,\text{out}}}{4\pi d^2}.$$

According to antenna theory [Balanis 2005], the effective receiving area of an antenna is $\lambda^2/4\pi$, where λ is the wavelength of the signal. So the receiving power of the signal is

$$P' = \frac{P_{r,\text{out}}}{4\pi d^2} \times \frac{\lambda^2}{4\pi}.$$

Replacing λ by c/f, where c is the velocity of light and f is the frequency of the signal, we obtain

$$P' = P_{r,\text{out}} \left(\frac{c}{4\pi df} \right)^2.$$

In terms of Shannon's information theory, the loss L_s is

$$L_s = 10 \log_{10} \left(\frac{P_{r,\text{out}}}{P'} \right) = 92.4 + 20 \log_{10} d + 20 \log_{10} f,$$

where L_s is in dB, d is in km, and f is in GHz. The actual power of the received signal $P_{r,\text{in}}$ is

$$P_{r,\text{in}} = P_{r,\text{out}} + (G_{\text{out}} + G_{\text{in}}) - (L_{f,\text{out}} + L_{f,\text{in}}) - (L_{b,\text{out}} + L_{b,\text{in}}) - L_s.$$

From one repeater (transmitter) to another repeater (receiver), the equations that hold, and the conventional values, are

$$L_{f,\text{out}} = L_{f,\text{in}} = 20 \text{ db} \quad \text{(the loss of the feed system)},$$
$$L_{b,\text{out}} = L_{b,\text{in}} = 1 \text{ db} \quad \text{(other loss of the system)},$$
$$G_{\text{out}} = G_{\text{in}} = 39 \text{ db} \quad \text{(the gain of the antenna)}.$$

For this problem, the frequency of signals is about 146.5 MHz (the midpoint of the available spectrum 145–148 MHz), and thus

$$d = 10^{\frac{10 \log_{10}\left(\frac{P_{r,\text{out}}}{P_{r,\text{in}}}\right) - 37.2328}{20}}. \tag{1}$$

The effective radiated power of most repeaters is $P_{r,\text{out}} = 100$ W [Utah VHF Society 2011], and normally a repeater can receive a signal with power no less than $1\mu W$ (i.e., $P_{r,in} \geq 1\mu W$). According to **(1)**, the communication radius of a repeater is $R \approx 85.5$ mi. Analogously, the average working power for a user (according to several wireless devices) is $P_{u,\text{out}} = 3.2W$ and $P_{u,\text{in}} \geq 1\mu W$, resulting in a communication radius $r \approx 15.28$ miles.

Repeater's Capacity

We calculate the capacity C of a repeater. Ignoring background noise and the interference, we assume that signals from one repeater do not affect others. A mainstream method to estimate the capacity of information over a noisy channel, according to Shannon's theory, is

$$\phi = B \log_2(1 + \text{SNR}) \tag{2}$$

where ϕ is the information bit rate (dB), SNR is the signal-to-noise ratio (dimensionless), and B is the total bandwidth (Hz).

The transmitter frequency in a repeater is an exact value rather than in a broad band. We use the equation

$$\frac{E_b}{N_0} = \frac{G_{\text{out}}}{V(C-1)(1 + I_{\text{other}}/I_{\text{self}})},$$

where E_b/N_0 is the level that ensures operation of bit-performance at the level required for digital voice transmission, G is the gain of the antenna, V is the gain of voice, I_{other} is the interference from other repeaters, and I_{self} is the interference of a repeater with itself. The SNR can be regarded as the ratio of effective information in the total received signal:

$$\text{SNR} = \frac{P_{ur}}{(C-1)P_{ur}} = \frac{1}{C-1},$$

where P_{ur} is the power of the signal from a single user as received by the repeater (measured in watts). Normally, $G = 39$ dB $= 7966.40$ W and $V = 0.4$ dB $= 1.07$ W (in the calculation, the units must be watts). We ignore interference from other repeaters: We set $I_{\text{other}}/I_{\text{self}} = 0$. Setting $E_b/E_0 = 18$ dB $= 63.01$ W (the bit energy-to-noise density ratio always ranges from 5 to 30 dB; we set it at the midpoint 18 dB), we get the capacity of a repeater as

$$C = 1 + \frac{G_{\text{out}}}{V(1 + I_{\text{other}}/I_{\text{self}}E_b/N_0)} \approx 119.$$

Continuous Approximation

Because users are mobile, the (fixed) repeaters must cover all of the considered area Γ. A solution for one distribution of users may not be a solution for another distribution. Therefore, a practical solution should not depend on a specific distribution.

Consequently, we use a continuous uniform distribution instead of a discrete uniform distribution of users. The user density is $\rho = N/\pi\Phi^2$. Omitting the bow-irrelevant Ω_2, the other three constraints changed to

- Ω_1^* - **Capacity**. Every repeater r_j, which has $S_V(r_j)$ as the area of its Voronoi region, must satisfy $\rho S_V(r_j) \leq C$.

- Ω_2^* - **Connectivity**. Every point is covered by a repeater.

- Ω_3^* - **Global reachability**. The reachable area of every point (considering that at every point, there can be user) is equal to the considered area Γ.

In the frequency range [145 MHz, 148 MHz] with $f_c = 0.6$ MHz, in a PL tone, if $R \geq 2\Phi$, there are at most 6 different repeater transmitter frequencies without interference (145.0 MHz, 145.6 MHz, 146.2 MHz,..., 148.0 MHz).

Let repeater i have receiver frequency $f_{i,1}$ and transmitter frequency $f_{i,2}$. With more than 6 repeaters, there must be at least one pair i and j with "inverse frequencies," i.e., $f_{i,1} - f_{i,2} = f_{j,2} - f_{j,1}$. Repeater i may amplify signals and send them to repeater j, and repeater j may amplify those signals and send them back to repeater i, and so on. To avoid this problem, we put only noninteracting repeaters in a PL tone group; the maximum size of such a group is 5. So when $R \geq \Phi$, with $N_{\text{PL}}=54$ different PL tones, the maximum number of repeaters without any interference is $54 \times 6 = 324$, and without any interactions is $54 \times 5 = 270$. Therefore, if the required number of small circles is no more than 324, we do not need to consider interference avoidance but just set repeaters not to interfere with one another; if the required number is no more than 270, we can make sure there will not be interactions between repeaters.

According to the connectivity constraint, the (integer) number of re-

peaters M should satisfy

$$M \geq \left\lceil \frac{\pi \Phi^2}{\pi r^2} \right\rceil = \left\lceil \frac{\Phi^2}{r^2} \right\rceil \approx \left\lceil \frac{40^2}{15.28^2} \right\rceil \approx \lceil 6.9 \rceil = 7. \tag{3}$$

According to the capacity constraint, M should satisfy

$$M \geq \left\lceil \frac{N}{C} \right\rceil = \left\lceil \frac{N}{118} \right\rceil = \left\lceil \frac{1000}{118} \right\rceil \approx \lceil 8.5 \rceil = 9. \tag{4}$$

For $N = 10,000$, we need $M \geq 85$.

naïve Solution

We use a cellular network of equal-size regular hexagons centered at repeaters. The pattern is the *Voronoi diagram* of the repeater sites.

We first consider an inverse problem: to determine the largest circle that can be covered by a number of such hexagons with edge length 1. **Table 1** gives the results up to 13 regular hexagons, calculated by hand.

Table 1.

Radius of the largest circle coverable by a network of M regular hexagons of side 1.

M	1	2	3	4	5	6	7	8	9	10	11	12	13
Radius	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{7}}{2}$	$\frac{7}{5}$	$\sqrt{3}$	2	2	2	$\frac{\sqrt{19}}{2}$	$\frac{\sqrt{19}}{2}$	$\sqrt{7}$	$\sqrt{7}$

We show how to obtain a solution for $R \geq 2\Phi$ and $N = 1,000$ by using **Table 1**. The user density is $\rho = N/\pi\Phi^2 \approx 0.1989$. To make sure that each point is covered by at least one repeater, the edge length r_h of the regular hexagon should not exceed the communication range r. In addition, according to the capacity constraint, r_h should satisfy

$$\frac{3\sqrt{3}}{2} r_h^2 \rho \leq C. \tag{5}$$

These two conditions determine the longest possible edge length; to cover as large circle as possible, we always use the longest r_h. For our case, $r = 15.28$ mi, and according to **(5)**, r_h should be 15.28 mi. The circle to be covered has radius $\Phi = 40$ mi. Since

$$\frac{\sqrt{19}}{2} \approx 2.18 < \frac{\Phi}{r_h} \approx \frac{40}{15.28} \approx 2.62 < \sqrt{7} \approx 2.65,$$

according to **Table 1**, 12 repeaters are sufficient (but 11 won't do) (**Figure 2**). (We consider only $R \geq 2\Phi$, for which frequencies and PL tones are easily arranged; in our example, we use 3 PL tones.) The algorithm for the naïve solution is to find the longest allowable r_h and search **Table 2** (extended
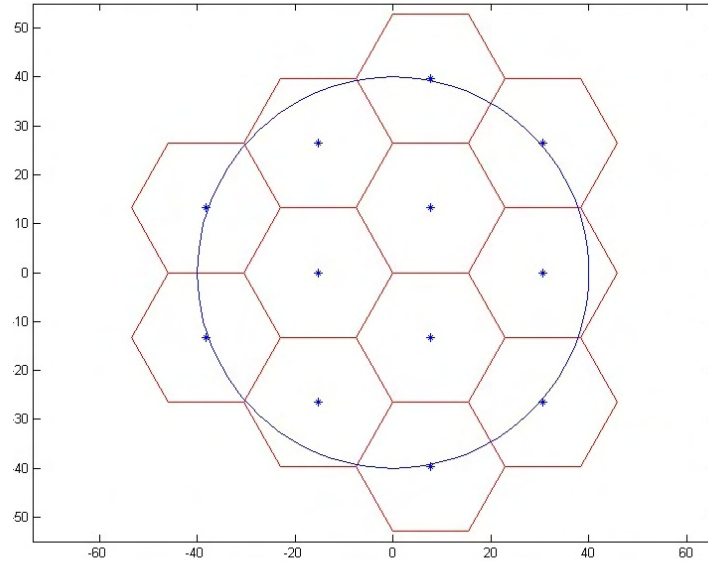
Figure 2. The solution with 12 repeaters arranged in a cellular network.

as necessary). However, extending the table is not easy (at least by hand). We get analytical results, for up to 121 circles, for two particular cases: the covered circle's center is at the center of the central hexagon, or it is at the intersection of three more central hexagons (**Table 3**).

Table 3.
Analytical results for two particular cases:
Left: The covered circle is centered at the center of the central hexagon.
Right: The covered circle is centered at the intersection of the three central hexagons.

Center of a circle		Intersection of three circles	
Cells	Radius of largest coverable circle	Cells	Radius of largest coverable circle
1	$\sqrt{3}/2$	3	1
7	2	6	$\sqrt{3}$
13	$3\sqrt{3}/2$	12	$\sqrt{7}$
19	$\sqrt{13}$	18	3
31	$5\sqrt{3}/2$	27	4
37	5	36	$\sqrt{21}$
55	$7\sqrt{3}/2$	48	$\sqrt{43}$
61	$\sqrt{43}$	60	6
85	$9\sqrt{3}/2$	75	7
91	8	90	$\sqrt{57} \approx 7.5$
121	$\sqrt{91} \approx 9.5$	108	9

Applying this approach to the case $N = 10{,}000$, we have $\rho = N/\pi\Phi^2 \approx 1.989$ and $r_h = 4.8$ miles. Since $8 < \Phi/r_h < 9$, according to **Table 3**, 108 repeaters are sufficient. Since this number is smaller than 324, it is easy to arrange them when $R \geq 2\Phi$. [EDITOR'S NOTE: We omit the authors' figure.]

Algorithms

We present algorithms

- to adaptively place the repeaters so as to solve the circle-covering problem with the fewest circles; and

- to assign receiver frequencies, transmitter frequencies, and PL tones.

Algorithm for Repeater Locations

1. Randomly place M_0 (the lower bound from **(3)** and **(4)**) repeaters.

2. Determine the Voronoi region of each repeater [Aurenhammer 1991].

3. Determine the circumscribed circle of each Voronoi region [Megiddo 1983].

4. Calculate the coordinates of the center of each circumscribed circle.

5. Calculate the distance from each repeater's current location to the center of its circumscribed circle. Sum up the distances over all repeaters and compare the sum to a threshold ζ. If the sum is less, the current locations are considered to be converged; otherwise, move each repeater to the center of its circumscribed circle and return to Step 2.

6. Check if the number of users in each Voronoi region is less than the repeater's capacity C, and if the radius of the circumscribed circle is less than the communication radius of users. If so, stop the algorithm and output the current solution. Otherwise, go to Step 7.

7. If the number of extremal optimization operations is less than a threshold T_c, pick up the repeater with the smallest Voronoi region, move it to a random position and go to Step 2; and advance the count of extremal optimization operations. Otherwise, add one more repeater, randomize the positions of all repeaters, and go back to Step 2.

Extremal optimization is a method to escape a local optimum by changing the individual with the least fitness (here, the repeater with the smallest Voronoi region). This idea comes from the Bak and Sneppen [1993], who describe the punctuated equilibrium in evolution caused by the annihilation of the least-fit species. In our simulation, $\xi = 0.01$ and $T_c = 100$.

Algorithm for Assignments

This algorithm does not change locations of repeaters as obtained from the first algorithm nor add new repeaters. It tries to maximize the reachable area of users just by rearranging the frequencies and PL tones of repeaters.
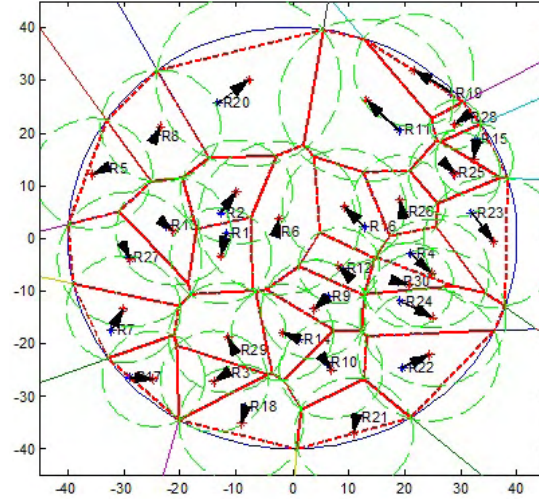
Figure 3. Example of application of the algorithm for repeater placement. The labeled plus signs are the original locations of the repeaters, and the black asterisks are their final locations; arrows show the movements. Voronoi regions are outlined by red (thick) line segments, and circumscribed circles are drawn in green (dashed) arcs (with red asterisks for their centers).
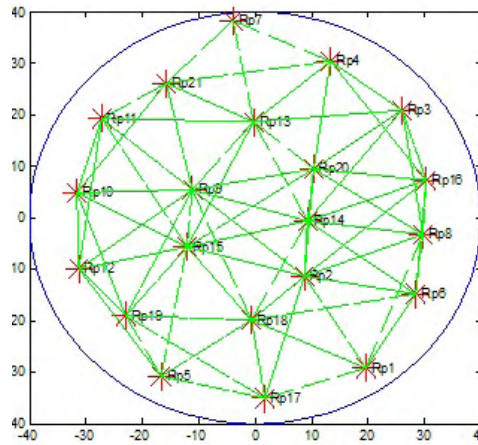
Figure 4. The red asterisks represent repeaters, and a green line connects two repeaters if they can communicate (that is, are less than R apart).
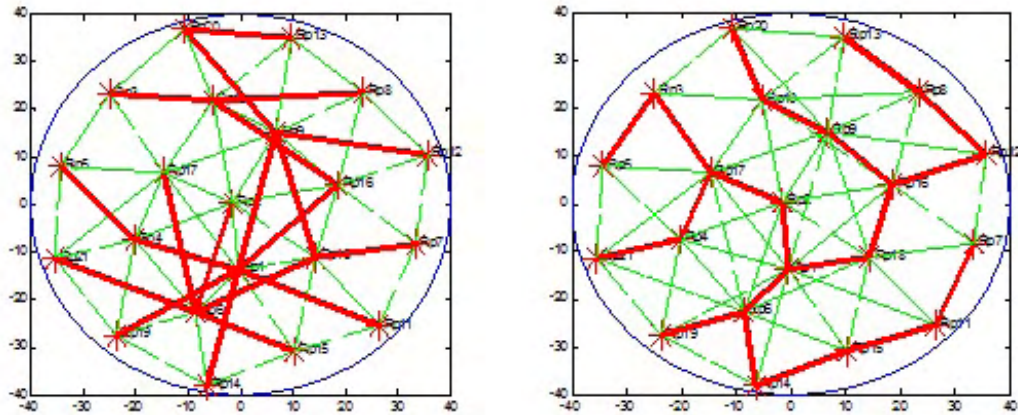
Figure 5. Maximum spanning tree (left) and the minimum spanning tree (right). The red (thick) lines are the edges in the spanning trees and the green (thin) lines are the other edges in the graph.
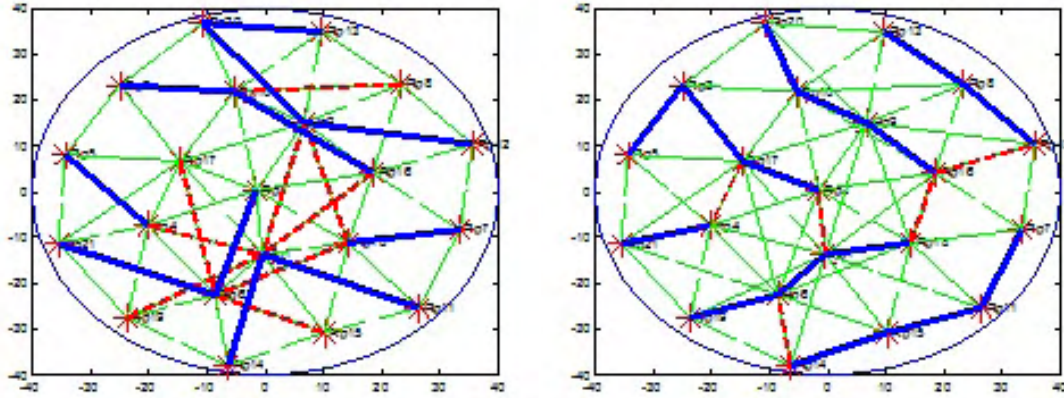
Figure 6. Resulting routings after the removal of edges corresponding to the spanning trees in **Figure 5**. The red (dashed) lines are the removed edges, while the blue (thick solid) lines are the preserved ones. Repeaters connected by a blue line share the same tone.

1. Construct a graph G to represent the relation that two repeaters can transmit to each other: If the distance between them is less than R, add an edge between them.

2. Find the minimum spanning tree (mST) T or the maximum spanning tree (MST) T' of G. From the illustration in **Figure 4**, we see that edges in the minimum spanning tree will not intersect. If we build the transmission paths along the mST, the signals received by repeaters are fewer than for the maximum spanning tree. However, since the distance between two adjacent repeaters along the MST tree is the shortest, the received signals will be much stronger. So the assignment based on the mST is suitable for communication in a local area. In contrast, most edges in the MST intersect with one another, the number of signals in many areas is very large, and signals can cover larger areas. However, this situation increases the chance of interference. Depending on purpose, one can choose either spanning tree to continue the algorithm.

3. Remove edges from the tree. For any node i with degree $k > 3$, delete $k - 2$ edges. Then the node i will be apart from $k - 2$ connected components. Let the size of component j be SC_j, then the method of removing edges can be presented as follows: Find $k - 2$ edges to remove in order to minimize $\sum_{a,b} |\text{SC}_a - \text{SC}_b|$. The results can be found in **Figure 5**.

4. After Step 3, we may have several signal routes that do not connect. Assign a different PL tone to each route. Then for the repeaters in each route, assign transmitting frequency and receiving frequency. Make sure that the transmitting frequency of a repeater is the receiving frequency of the repeater's neighbors in the same route. (See **Figure 6**.)
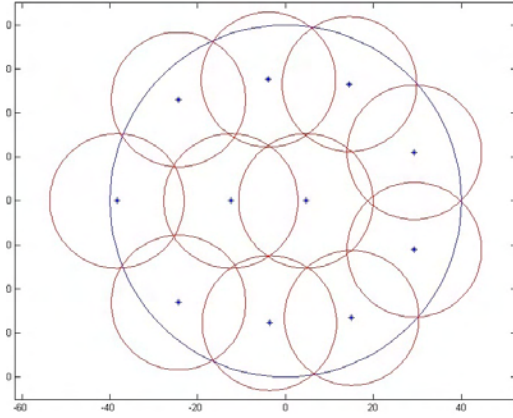
Figure 7. Solution with 11 repeaters obtained by our algorithm.
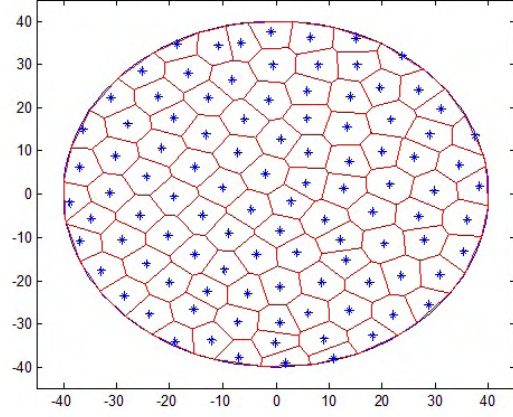
Figure 8. Solution with 104 repeaters obtained by our algorithm.

Simulation

N = 1,000, R = 85.45 mi

Figure 7 shows a solution with 11 repeaters obtained by our algorithm. The maximal Voronoi area is 560.56, the user density is 0.1989, and thus the largest capacity demand is 112, smaller than the repeater's capacity $C = 119$. Compared with the naïve solution, fewer repeaters are required and sizes of the Voronoi regions are more homogeneous.

We prove that 11 repeaters is optimal: 10 repeaters with radius no more than 15.28 miles cannot cover a circle with radius 40 miles.

Lemma [Toth 2005]. *Let $r(n)$ be the maximum radius of a circular disc that can be covered by n closed unit circles, then*

$$r(n) = 1 + 2\cos\left(\frac{2\pi}{n-1}\right)$$

for $n = 8$, $n = 9$, and $n = 10$.

According to the Lemma,

$$r(10) = 1 + 2\cos\left(\frac{2\pi}{9}\right) \approx 2.53 < \frac{40}{15.28} \approx 2.62,$$

so coverage by 10 circles is not possible.

N = 10,000, R = 85.45 mi

Figure 8 shows a solution with 104 repeaters obtained by our algorithm; 21 PL tones are used to guarantee that every pair of repeaters will not interact each other.
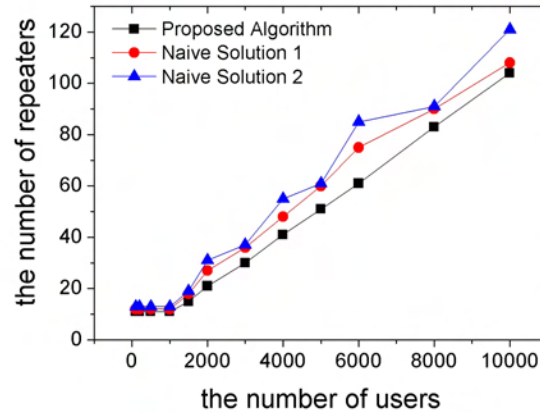
Figure 9. Comparison of fewest required repeaters obtained by our algorithm vs. the naïve solution. The largest coverable circle in the naïve Solution 1 is centered at the center of the central hexagon, while the largest coverable circle in naïve Solution 2 is centered at the intersection of the three more central hexagons.

N =1,000, R = 40 mi

The repeaters' locations are the same as earlier, but the frequencies and PL tones are different.

To test the effectiveness of the secondary algorithm, we randomly pick 100,000 ordered pairs of points (u, v) inside the considered area and see in how many pairs u can send to v. The answer is 90,708. So, given a user, the probability that the system can satisfy this user's requirement to communicate with any other user at random is about 91%.

N = 100,000, R = 40 mi

It seems that when the required number of repeaters increases, the reachable area of a user increases. For example, in this case, for the solution found by our algorithm, the corresponding probability is 97%.

Sensitivity Analysis

Sensitivity of Parameters

We discuss to what extent the results depend on the parameters. **Figure 9** displays the fewest repeaters required vs. the number of users and shows that our algorithm is better than the naïve solution. There is a transition point at about $N = 1,000$, which approximately satisfies the equation

$$\frac{N}{\pi \Phi^2} = \frac{C}{\pi r^2}.$$
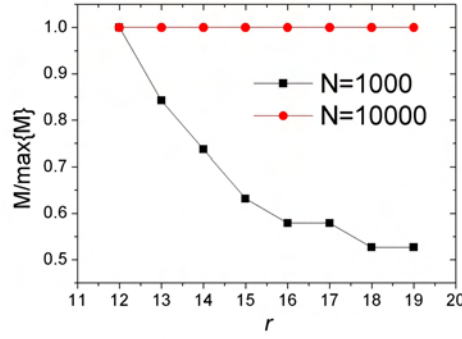
Figure 10. Comparison between the fewest required repeaters obtained by our algorithm vs. the naïve solution. The black points correspond to the case of $N = 10,000$ and the red points to $N = 10,000$. The two curves have been normalized.

Up to the transition point, the number of repeaters mainly depends on the communication range r, while beyond it is capacity that becomes the bottleneck determining the fewest required repeaters. Since the capacity constraint plays a major role when there are many users, it is not a surprise that the number of repeaters grows linearly with the number of users.

Figure 10 reports the relation between the fewest required repeaters and the user's communication range r. Each curve has been normalized by dividing by its respective largest value. In the case $N = 10,000$, the number of repeaters never changes with r, again indicating that the capacity limitation determines the result, while when $N = 1,000$, the number of repeaters decreases with increase of the user's communication range.

User Density Fluctuation

In the analysis of our model, the user density is constant, behaving like a real variable. However, in reality, the number of users can only be an integer. For the discrete case where users are distributed uniformly, each user belongs to Voronoi area S_ν with probability $S_\nu/\pi\Phi^2$, and thus the number X of users in this area obeys a Bernoulli distribution

$$P(X) = \binom{N}{X} \left(\frac{S_\nu}{\pi\Phi^2}\right)^X \left(1 - \frac{S_\nu}{\pi\Phi^2}\right)^{N-X},$$

whose expectation and standard deviation are

$$E(X) = \frac{NS_\nu}{\pi\Phi^2} = \rho S_\nu, \quad \sigma(X) = \sqrt{\frac{NS_\nu}{\pi\Phi^2}\left(1 - \frac{S_\nu}{\pi\Phi^2}\right)} = \sqrt{\rho S_\nu}\sqrt{1 - \frac{S_\nu}{\pi\Phi^2}}.$$

When the total number N of users is small, capacity is not a big problem; and when N is big (corresponding to high user density), the area S_ν should

be small due to the capacity limitation. Therefore, the standard deviation is approximately the square root of the expected number of users in the Voronoi region. We are interested in the case when the number of users in S_ν approaches the capacity limitation. For example, in this problem, we have $C = 119$, so the standard deviation is about 11. If we set up a tolerance of two standard deviations, the tolerant capacity C' should satisfy

$$C' + 2\sqrt{C'} = C,$$

leading to $C' = 99$.

Effects of Landscape: Mountainous Areas

[EDITOR'S NOTE: We must omit the team's discussion of this point.]

Conclusion and Discussion

We propose a two-tiered network model, where lower-power users communicate with one another through repeaters, taking into account in our model capacity constraints and interference.

We give a naïve solution in which the number of repeaters and their positions can be obtained analytically. We further develop an algorithm based on Voronoi diagrams, which outperforms the naïve solution. For 1,000 users, the algorithm proposes 11 repeaters, which we prove to be optimal. For the 10,000 users, the algorithm obtains a solution with 104 repeaters.

Moreover, we offer an algorithm, based on maximum and minimum spanning trees, to assign frequencies and private-line tones. This algorithm does not introduce any new repeaters yet can broaden the reachable areas of users.

Compared with the related model for sensor wireless networks and mobile communication networks, our model is more general. Our algorithm is effective and efficient: It runs much faster than the simulated annealing approach [Nurmela and Östergård 2000] and is better able escape the local optimum than another iterative refinement algorithm [Das et al. 2006]. Our algorithm does not require any specific geographical features of the considered area, while many efficient circle-covering algorithms work well only for squares.

There are also two considerable weaknesses in our work:

- We have not developed an algorithm that satisfies the constraint of global reachability without adding too many repeaters.

- Our model does not take into account heterogeneity of users or repeaters, or wave reflection and refraction by the atmosphere.

Appendix

Summary of Assumptions

A1 Users are uniformly distributed in the considered area, and in a more strong assumption, we consider the number of users as a real variable and the user density in the considered area is a constant.

A2 Users prefer to communicate with the nearest repeaters.

A3 Consider two repeaters sharing the same PL tone, with the difference between their transmitter frequencies less than a threshold $f_c = 0.6\,\text{MHz}$. If the distance between them is less than $2R$, they will interfere with each other.

A4 In the considered circular area, wireless signals can fade freely; there are no other sources of interference such as fogs, rivers, hills, buildings, activities of the Sun, and so forth, so that the fading of signals is due only to the distance involved.

A5 There is no background noise in this system.

A6 Repeaters don't have noisy impact on others.

A7 Functionalities and specifications of users' radios are the same (i.e., homogeneous users' radios). Functionalities and specifications of radio repeaters are the same (i.e., homogeneous repeaters).

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Judges' Commentary:
The Outstanding Repeater
Coordination Papers

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Overview

This year's problem dealt with finding the number of repeaters needed to create a VHS network to cover a circular region of radius 40 miles and simultaneously serve first 1,000, then 10,000, users. Naturally, there is quite a bit of literature available related to this topic.

Approaches

The approaches used could be broken down into two categories. Some papers focused first on covering the area, others on covering the population.

Covering the Area

There is much to be said for the simplicity and directness of the method of covering the area first. The most common approach was to tile the region with hexagons inscribed within circles of radius equal to the distance that a user's signal will reach effectively. Some papers shifted their hexagonal lattice back and forth to capture the minimum number of hexagons needed to cover the 40-mile circle. Good papers then generated simulated populations, generally uniformly distributed, to check if the number of repeaters was adequate for the usage load. Most then added more repeaters

for the 10,000-user case. The better papers tested their results against non-uniformly distributed populations, either following some other distribution or concentrated in groups or towns. Some of the population generation that we saw was quite creative and demonstrative of good modeling.

Covering the Population

Many of the papers simulated user populations first, then attempted to cover all (or a high percentage of) the users with a minimal number of repeaters. Of course, if a population distribution is simulated and then covered with K repeaters, in general additional argument is needed before one can conclude that K repeaters will work for any such distribution. Most of the papers used repeated simulations as their argument. There were some interesting approaches used to cover the populations minimally, including greedy and genetic algorithms. One of the more creative papers assumed that although the goal was to cover 1,000 or 10,000 simultaneous users, there were in fact, more users than that; and that team's algorithm was designed to capture just the required number of users. Although this was a simplifying assumption that dramatically changed the problem, and judges felt that the uncovered users might not appreciate this approach, we could find nothing in the problem statement to preclude it; and the paper in question stated and justified the assumption.

The most disappointing feature of these papers, which were in general creative and presented interesting modeling, was that although their approaches clearly relied on advance knowledge of the locations of all the users in the region, virtually none of the papers highlighted this fact, either in the assumptions or in the weaknesses of their models. Although (as one might expect) this approach generally (especially in the 1,000-user case) required fewer repeaters than the area-first approach, almost without exception teams that took this approach failed to indicate that such an approach requires collecting and entering great amounts of data that may not even be available in a real-world application. While this fact does not necessarily negate the validity of the model or its results, the papers should have clearly stated the assumption that these locations must be known for the model to be useful and should also have mentioned this requirement as a disadvantage. Almost no teams made the reader aware of this critical fact.

Generally, the judges in the final stages, referring to flaws in papers, call a flaw that keeps a very good paper from being outstanding a "fatal flaw"; and our discussions and deliberations frequently come down to arguing whether or not a discovered flaw should be considered "fatal." Some felt that requiring knowledge of users' locations, while neither including such knowledge in the assumptions nor acknowledging the need as a weakness, should be a "fatal flaw"; but eventually the desire to have some Outstanding papers outweighed those feelings.

Determining the Required Spacing for the Repeaters

There was quite a bit of disagreement in the ranges used for the repeaters and the users. Papers generally correctly assumed that the range of the repeaters would be greater that the range of the users' equipment, making the latter the determining factor. But we saw ranges for repeaters going from about 3 miles to 100 miles. It is possible, using the radius of the Earth (and assuming that the Earth is perfectly spherical) to compute the "line of sight" distance to the horizon as a function of the height of the repeater. Some papers found this relationship, either in the literature or by computing it themselves. Others made reference to online sources giving the ranges for repeaters. Given the time constraints and the fact that this is a modeling contest, not a contest to distinguish engineering prowess, we did not use the range value as a discriminator, even though we suspected that some of the sources referenced may not have actually referred to VHS repeaters.

Use of Sources

In a contest of this nature, it is expected that participants will rely on sources; but it is also expected that the participants will cite and evaluate those sources. Many papers used graphics that—since we saw them in a number of papers—must have come from some online source, but they failed to specifically credit the source for the graphic. Also, many used models that they found in the literature, such as the Hata Model. This is appropriate; however, if you choose a model from the literature, then you should explain why you choose that equation to use, what assumptions led you to that equation, and what value-added you gave it as you adapted it to the given problem. It is also important that if you use equations from the literature, that you adapt the notation to match what you use in the rest of the paper, and that you clearly explain any notation that you use.

Mountainous Terrain

Most papers that considered mountainous terrain spent some time dealing with line-of-sight issues relating to the terrain. A few simulated some mountainous terrain or found some sample elevation maps and indicated what changes would be necessary in repeater placement for these samples. Some papers discussed changes to the population distribution caused by the terrain. The judges acknowledged that it would have been unreasonable to expect models that would independently deal with any terrain, but we looked for papers that indicated how one would approach uneven ground.

General Modeling Principles

One of the things teams needed to do for this problem—and which many neglected—was to decide consciously which portions of their model should be deterministic and which should be stochastic.

Assumptions were also important factors. When you make assumptions, you need to justify them—not simply state them. You should not include assumptions that are unnecessary for your model or have nothing to do with it. But even with the assumptions that you do need, you should indicate how sensitive your results are to those assumptions. It is OK to justify an assumption by indicating that it was necessary for your model, even though in reality it may not hold (for example, in this problem the assumption that population is uniformly distributed might fall into this category); but in that case, it is essential to discuss how the results depend on that assumption.

It is important not to make assumptions that defeat the purpose of the problem. Some papers assumed that repeaters were connected by wires. That was not at all in keeping with the statement of the problem, and it eventually eliminated some otherwise well-written papers.

Sensitivity Analysis, Error Analysis, and Model Testing

An important area that turned out to be one of the major discriminators at the end was testing and sensitivity analysis. How does the number of repeaters change if your population is distributed in a different fashion? If you used normal distributions, for example, how do your results change given an $x\%$ change in the assumed standard deviation? What if the range of a repeater is less than assumed? The better papers also tested their results, some by comparison with the actual distribution of users and repeaters in various locations, and others by simulations of one sort or another.

Finally, always do a commonsense check. If you are running out of time, and your commonsense check fails, you should at least acknowledge that. We had some beautifully written papers that had results where you needed on the order of 2,000 repeaters for 1,000 users. One could argue that such might be possible if the area coverage was what was driving the need. But when the same paper then required 15,000+ repeaters for 10,000 users, the reality check certainly failed. This was a "fatal flaw"! Always ask if the results "make sense" logically.

Executive Summary

Every year, we seem to reiterate the importance of a good executive summary. We continue to threaten not to read beyond a poor summary; and while we have yet to live up to that threat, it is certainly the case that the summary sets the expectation of the reader for the rest of the paper. The summary should

- be the last thing written,

- stand alone,

- make sense, and

- be satisfying, even if the reader has not read the problem and never intends to read the paper.

The results, a description of the model, any key assumptions, and recommendations should be clearly included. Important strengths and weaknesses should be highlighted. It takes some skill to write a good executive summary, but it is a skill that will take you far. Out in that "real world," you frequently need to boil down months of work into a well-crafted executive summary for the decision makers. Your MCM summary should be good practice. Look at the Outstanding paper printed in this issue, which exemplifies what we look for in a good summary. That paper consistently, through all rounds of judging, received the highest marks.

Writing and Organization

Even a brilliant team will not go far if the members cannot convey their work effectively. A few tips for the writing:

- Even when you divide up tasks such as sections to write, have your best writer do a final edit. Do this *after* you have run the grammar and spellchecker, and then run them one more time after the final edit.

- If you try some additional models and abandon them, not using them in the final analysis, put them in an appendix rather that in the body of the paper, where they distract the reader.

- Keep in mind that judges have very limited time to read your paper. The salient points need to be easy to find. If your paper is long, it may be that although many judges have looked at it, no single judge has had time to read the whole thing.

- Avoid unnecessary repetition, use good section headings, and offset/display important parts.

- Label graphics in such a way that a reader flipping through your paper will see what they represent.

- Have conclusions at the end of each section, and make sure that results are easy to find.

Conclusion

This problem led to a variety of solution techniques and approaches. It allowed for a great deal of creativity, and in the end the creativity in the solution was one of the primary factors for bringing papers recognition. Mathematical modeling is an art, and in the long run it will be the kind of creativity we see in these papers that will help solve the problems facing the world. We commend all the participants for developing these crucial skills. We are proud of your accomplishments and the drive that led you to devote your time and energy to this endeavor.

About the Author

Kathleen Shannon is Professor of Mathematics at Salisbury University and former chair of the Department of Mathematics and Computer Science. She earned her bachelor's degree at the College of the Holy Cross with a double major in Mathematics and Physics and her Ph.D. in Applied Mathematics from Brown University in the mid 1980s under the direction of Philip J. Davis. Since then, she has been primarily interested in undergraduate mathematics education and mathematical modeling. She has been involved since 1990 with the MCM as, at different times, a team advisor, a triage judge, and a final judge (sometimes as an MAA or SIAM judge). She has also been a co-Principal Investigator on two National Science Foundation Grants for the PascGalois Project (`http://www.PascGalois.org`) on visualizing abstract mathematics.

Judges' Commentary:
The Fusaro Award for the Repeater Coordination Problem

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Introduction

MCM Founding Director Fusaro attributes the competition's popularity in part to the challenge of working on practical problems. "Students generally like a challenge and probably are attracted by the opportunity, for perhaps the first time in their mathematical lives, to work as a team on a realistic applied problem," he says. The most important aspect of the MCM is the impact it has on its participants and, as Fusaro puts it, "the confidence that this experience engenders." The Ben Fusaro Award for the 2011 discrete problem went to a team from the University of Electronic Science and Technology (UES&T), Web Sciences Center, in Chengdu, Sichuan, China. This solution paper was in the top group, the Outstanding papers. Characteristics it exemplified were:

- Presented a high-quality application of the complete modeling process.

- Demonstrated noteworthy originality and creativity in the modeling effort to solve the problem as given.

- Written clearly and concisely, making it a pleasure to read.

The Problem

This year's problem dealt with finding the number of repeaters needed to create a VHS network to cover a circular region of radius 40 miles and

simultaneously serve first 1,000 users, then 10,000 users. The approaches that different teams took could be broken down into two categories. Some papers focused first on covering the area, others on covering the population. This team did both. Students found many publications related to this topic. While to receive an Outstanding or Meritorious designation it was important for a team to review the literature, teams had to address all the issues raised and come up with a solution that demonstrated their own creativity.

The University of Electronic Science and Technology Paper

One-Page Summary Sheet

This team did an outstanding job with its executive summary. Although it was a bit long with very small print, in one page they motivated the reader and provided the reader with a precise summary of what they had accomplished. It gave an overview of everything from the assumptions, to the modeling and how it was done, to the testing of their models, and finally, to the analysis of the accuracy of their results and limitations of their models. It was well written and among the best examples of what an executive summary should be. The team's executive summary was followed by a one-page abstract. Typically, an executive summary contains more information (and often more sensitive information) than the abstract does.

Assumptions

As was the case with many teams, this team began with the assumption that the distribution of users in the area was uniform. Other teams considered a variety of other distributions as well, but this team did not. The second assumption stated the conditions under which repeaters would interfere with each other and the third was that the wireless signals can fade freely. Of critical importance, the team showed how their assumptions were used in the development of their model.

The Model and Methods

The team proposed a two-tiered network model consisting of user nodes and repeater nodes. As many teams did, they covered the circle with a sufficient number of hexagons to yield transmission among users based on the maximum communication distances for each type of node. However, they also used Voronoi diagrams to optimize communication with the least

number of repeaters. Spanning trees were used to assign frequencies in the desired ranges and private line (PL) tones.

Testing Their Models

After determining the communication distances for their users and repeaters, along with the maximum capacity for a repeater, the UES&T team computed lower bounds for the number of repeaters that would be needed for each population size. They then developed an algorithm to place the users and repeaters within the designated circle of radius 40 miles and subdivided the area using Voronoi diagrams. They refined their algorithm to make certain that in the Voronoi regions, no repeater had users beyond its threshold capacity. Then they tested their models—not with just one size region, but with many, for user numbers of 1,000 and 10,000. By analyzing their results, they were able to comment on the sensitivity and robustness of their models. This was something that very few papers were able to do, and it is a very important step in the modeling process.

Recognizing Limitations of the Model

Recognizing the limitations of a model is an important last step in the completion of the modeling process. The students recognized that their algorithms would have to be modified if the terrain were not flat or if the users were distributed differently.

References and Bibliography

The list of references used was fairly thorough, but specific documentation of where those references were used was not always clear. Precise documentation of references used should have been included throughout the paper.

Conclusion

The careful exposition in the development of the mathematical models made this paper one that the judges felt was worthy of the Outstanding designation. The team members are to be congratulated on their analysis, their clarity, and on using the mathematics they knew to create and justify their own model for the Repeater Coordination Problem.

About the Author

Marie Vanisko is a Mathematics Professor Emerita from Carroll College in Helena, Montana, where she taught for more than 30 years. She was also a Visiting Professor at the U.S. Military Academy at West Point and taught for five years at California State University Stanislaus. She chairs the Board of Directors at the Montana Learning Center at Canyon Ferry and serves on the Engineering Advisory Board at Carroll College. She has been a judge for both the MCM and HiMCM.

ICM Modeling Forum

Results of the 2011 Interdisciplinary Contest in Modeling

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Introduction

A total of 735 teams from four countries spent a weekend in February working in the 13th Interdisciplinary Contest in Modeling (ICM)®. This year's contest began on Thursday, Feb. 10, and ended on Monday, Feb. 14. During that time, teams of up to three undergraduate or high school students researched, modeled, analyzed, solved, wrote, and submitted their solutions to an openended interdisciplinary modeling problem involving electric vehicles. After the weekend of challenging and productive work, the solution papers were sent to COMAP for judging. One of the top six papers judged as Outstanding appears in this issue of *The UMAP Journal*.

COMAP's Interdisciplinary Contest in Modeling (ICM), along with it sibling, the Mathematical Contest in Modeling (MCM)®, involves students working in teams to model and analyze an open problem. Centering its educational philosophy on mathematical modeling, COMAP supports the use of mathematical tools to explore real-world problems. It serves society by developing students as problem solvers in order to become better informed and prepared as citizens, contributors, consumers, workers, and community leaders. The ICM and MCM are examples of COMAP's efforts in working towards its goals.

The problem required teams to understand the complexity of energy and transportation and to model that complexity in the effects of the electric vehicle on the future of energy production and transportation. In order to accomplish their tasks, the students had to consider many difficult and complex disci-

plinary and interdisciplinary issues. The problem also included the requirements of the ICM to use thorough analysis, research, creativity, and effective communication. All members of the 735 competing teams are to be congratulated for their excellent work and dedication to modeling and problem solving.

Next year, we will shift the ICM theme for the contest problem to *network science*. Teams preparing for the 2012 contest should consider reviewing interdisciplinary topics in the area of networks and assemble teams accordingly.

The Problem Statement:
The Electric Vehicle Problem

How environmentally and economically sound are electric vehicles? Is their widespread use feasible and practical? Here are some issues to consider, but, of course, there are many more, and you will not be able to consider all the issues in your model(s):

- Would the widespread use of electric vehicles actually save fossil fuels or would we merely be trading one use of fossil fuel for another given that electricity is currently mostly produced by burning fossil fuels? What conditions would need to be put in place to maximize the savings through use of electric vehicles?

- Consider how much the amount of electricity generated by alternatives such as wind and solar would need to climb during the twenty-first century to make the widespread use of electric vehicles feasible and environmentally beneficial. Assess whether or not the needed growth of these alternate sources of electricity is likely and possible.

- Would charging batteries at off-peak times be beneficial and increase the feasibility of widespread use of electric vehicles? How quickly would batteries need to charge to maximize the efficiency and practicality of electric vehicles? How would progress in these areas change the equation regarding the environmental savings and practicality of widespread use of electric vehicles?

- What method of basic transportation is most efficient? Does the efficiency of a method dependent on the nation or region where it is used?

- Pollution caused directly by electric vehicles is low, but are there hidden sources of pollutants associated with electric vehicles? Gasoline and diesel fuel burned in internal combustion engines for transportation account for nitrites of oxygen and vehicle-born monoxide and carbon dioxide pollution; but are these byproducts something that we really should worry about? What are the short- and the long-term effects of these substances on the climate and our health?

- How would the pollution caused by the need to dispose of increasing numbers of large batteries affect the comparison between the environmental effects of electric vehicles versus the effects of fossil fuel-burning vehicles?

- You also should consider economic and human issues such as the convenience of electric vehicles. Can batteries be recharged or replaced fast enough to meet most transportation needs or would their ranges be limited? Would electric vehicles have only a limited role in transportation, good only for short hauls (commuters or light vehicles on short trips) or could they practically be used for heavier and longer-range transportation and shipping? Should governments give subsidies to developers of electric vehicle technologies and if so, why, how much, and in what form?

Requirements:

- Model the environmental, social, economic, and health impacts of the widespread use of electric vehicles and detail the key factors that governments and vehicle manufacturers should consider when determining if and how to support the development and use of electric vehicles. What data do you have to validate your model(s)?

- Use your model(s) to estimate how much oil (fossil fuels) the world would save by widely using electric vehicles.

- Provide a model of the amount and type of electricity generation that would be needed to support your recommendations regarding the amount and type of electric vehicle use that will produce the largest number of benefits to the environment, society, business, and individuals.

- Write a *20-page* report (not including the summary sheet) to present your model and your analysis of the key issues associated with the electric vehicle and electricity generation. Be sure to include the roles that governments should play to ensure safe, efficient, effective transportation. Discuss whether the introduction of widespread use of electric vehicles is a worthwhile endeavor and an important part of an overall strategy to address global energy needs in the face of dwindling fossil fuel supplies.

References:

Getting reliable global data on controversial issues like this one can be difficult. As a start on global energy information, we provide this link:

```
http://www.bp.com/liveassets/bp_internet/globalbp/globalbp_
uk_english/reports_and_publications/statistical_energy_
review_2008/STAGING/local_assets2010_downloadsstatistical_
review_of_world_energy_full_report_2010.pdf
```
.

A concise summary of energy generation and usage in the U.S. is at:

```
http://www.eia.doe.gov/aer/pecss_diagram.html.
```

More global data in spreadsheet form are found here:

```
http://www.eia.doe.gov/iea/.
```

The Results

The 735 solution papers were coded at COMAP headquarters so that names and affiliations of the authors were unknown to the judges. Each paper was then read preliminarily by triage judges at the U.S. Military Academy at West Point, NY. At the triage stage, the summary, the model description, and overall organization are the primary elements in judging a paper. Final judging by a team of modelers, analysts, and subject-matter experts took place in late March. The judges classified the 735 submitted papers as follows:

	Outstanding	Finalist	Meritorious	Honorable Mention	Successful Participant	Total
Electric Car	6	5	146	292	286	735

Outstanding Teams

Institution and Advisor	Team Members

"Electric Cars as a Widespread Means of
    Transportation"
Humboldt State University   Zachary Stanko
Arcata, CA   Brenda Howell
Brad Finney   Rick Bailey

"Putting the Spark Back in the Electric Car"
North Carolina School of Science and Mathematics   Christy J. Vaughn
Durham, NC   Matt G. Jordan
Daniel J. Teague   Kevin E. Valakuzhy

"Can Electric Vehicles Be Widely Used?"
Northwestern Polytechnical University   YuBing Zhang
Xi'an, China   FengJiang Li
Huayong Xiao   PengCheng Li

"Can Electric Vehicles Be a Shining Star of Tomorrow?"
South China University of Technology   Weiyang Liu
Guangzhou, China   Jianhan Mei
Weijian Ding   Weikai Wang

"What Will the Electric Vehicle Bring to the World?"
Southeast University   Chenxi Zhai
Nanjing, China   Xinru Zheng
Zhiqiang Zhang   Yuchong Huo

"An Analysis of the Future Development of
  Electric Vehicles"

Zhejiang University	Qiqin Dai
Hangzhou, China	Yi Li
An Zhang	Huangyu Ding

Awards and Contributions

Each participating ICM advisor and team member received a certificate signed by the Contest Directors and the Head Judge. Additional awards were presented to the team from Zhejiang University by the Institute for Operations Research and the Management Sciences (INFORMS).

Judging

Contest Director
Chris Arney, Dept. of Mathematical Sciences, U.S. Military Academy,
    West Point, NY

Associate Directors
Joseph Myers, Computing Sciences Division, Army Research Office,
    Research Triangle Park, NC
Rodney Sturdivant, Dept. of Mathematical Sciences,
    U.S. Military Academy, West Point, NY

Judges
Rachel DeCoste, Dept. of Mathematics and Computer Science,
    Wheaton College, Norton, MA

Tina Hartley, Dept. of Mathematical Sciences, U.S. Military Academy,
    West Point, NY
John Kobza, Dept. of Industrial Engineering, Texas Tech University,
    Lubbock, TX
Kathleen Snook, COMAP Consultant, Bedford, MA
Tan Yongji, Dept. of Mathematics, Fudan University, Shanghai, China

Triage Judges

Chris Arney, Kristin Arney, Gabe Costa, Michelle Craddock,
Chris Eastburg, Aaron Elliott, Kingsley Fink, Ben Gatzke, Andy Glen,
Tina Hartley, Alex Heidenberg, Tim Hudson, James Jones,
Bill Kaczynski, Phil LaCasse, Mike Landin, Craig Lennon, Chris Marks,
Donovan Phillips, Bill Pulleyblank, Jeremy Riehl, Elizabeth Russell,

Libby Schott, Mick Smith, Csilla Szabo, Josh Thibeault, Eric Thornburg, Chris Weld, JoAnna Crixell Whitener, Brian Winkel, and Shaw Yoshitani.

—all of Dept. of Mathematical Sciences, U.S. Military Academy, West Point, NY; and

Amanda Beecher, Dept. of Mathematics, Ramapo College of New Jersey, Mahwah, NJ.

Acknowledgments

Cautions

To the reader of research journals:

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each of the team papers here is the result of undergraduates working on a problem over a weekend. Editing (and usually substantial cutting) has taken place; minor errors have been corrected, wording altered for clarity or economy, and style adjusted to that of *The UMAP Journal.* The student authors have proofed the results. Please peruse these students' efforts in that context.

To the potential ICM advisor:

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

Electric Cars as a Widespread Means of Transportation

Rick Bailey
Brenda Howell
Zachary Stanko

Humboldt State University
Arcata, CA

Advisor: Brad Finney

Summary

We adapt a a Lotka-Volterra ecological competition model to describe the car (and light truck) market. We assume that gasoline-powered internal combustion engine vehicles (ICE), plug-in hybrid vehicles (PHEV), and battery-electric vehicles (BEVs) perform like organisms competing for a shared but limited resource. For organisms, this resource might be a food supply; in the car market, manufacturers compete for consumer dollars. The equations describe the rates of change of three dependent variables, the populations of cars of each type. The model parameters describe growth rates, interspecies competition, and carrying capacities, which indirectly relate to consumer preferences, economic conditions, government influences, and improvements in automotive technologies. Variables and parameters used in the model are listed in **Table 1**.

We assume that intrinsic growth rates are constant, but refinements to the model could describe them as functions of time, market forces, or stochastic variables. We assume that the carrying capacity grows at 1%, consistent with the human population growth rate of the U.S. [The World Bank Group 2011]. We lump together model parameters, with deterministic variables reflecting all aspects influencing consumer choice.

We investigate five scenarios of changes in conditions affecting the auto market. A base scenario uses current yearly growth rates and current populations; others investigate effects of high oil prices, increased battery performance, government investment, and high electricity rates.

We compare the present value of two car models currently available, to examine the competitiveness of these cars. Without current government subsidies, the Nissan Leaf has lower present value than the Honda Civic and so is at a disadvantage against the Civic. The Leaf would be competitive without subsidies with a linear rise in gas prices to \$5 /gallon, increased BEV efficiency (in kWh/mile driven), and higher resale values.

Table 1.

Symbol table.

Variable	Description	Value/Units
	Growth Model	
$G(t)$	number of ICE vehicles	-
$E(t)$	number of BEVs	-
$G(t)$	number of PHEVs	-
r_G, r_E, r_H	intrinsic growth rates	$1.52 \, \text{yr}^{-1}$
K_G, K_E, K_H	maximum populations	300×10^6
β_G	effect of ICE on BEV	0.7
γ_G	effect of ICE on PHEV	1.2
α_E	effect of BEV on ICE	1.2
γ_E	effect of BEV on PHEV	0.8
α_H	effect of PHEV on ICE	1.2
β_H	effect of PHEV on BEV	1.0
	Economic Model	
C_0	initial capital investment	US \$
L	resale value at end of ownership	US \$
B_t	net economic benefit in year t	US \$
R_t	repair/maintenance cost in year t	US \$
n	vehicle ownership life	yr
I	nominal interest rate	%
k	compounding periods per year	12
i	effective annual interest rate	%

The primary weakness of our model is lack of data for calibration and testing.

All scenarios predict an eventual shift to BEVs from ICEs, but the timing depends on growth rates and the values of the competitiveness parameters. With a significant initial investment to increase the numbers of BEVs, the equilibrium point between ICEs and BEVs occurs in 2030, compared to a base case of 2035. If coal is expensive and battery prices high, this point does not occur until 2043; if oil prices rise rapidly, increasing the competitiveness coefficients of BEVs, this point occurs in 2028. When all factors combine–oil prices rise, battery technology improves, and electricity remains inexpensive—then BEVs, ICEs, and PHEVs would be present in equal numbers in 2027.

Introduction

We adapt an ecological competition model to describe the population dynamics among ICE cars, BEVs, and PHEVs. We interpret model results using present-day figures for ownership costs, tailpipe and power-plant emissions, electrical generation capacity, and oil consumption. This analysis allows a rough estimate of the greenhouse gas (GHG) emissions, extra electricity demand due to electric vehicle use, and specific measures for manufacturers and governments to speed electric vehicle adoption.

Problem Formulation

Because most electricity generated in the U.S. comes from coal, BEVs do not represent a wholesale movement away from fossil fuels. The environmental impact of coal-produced electricity varies by location, depending on infrastructure and emissions regulations. Price fluctuations for oil and coal are unavoidable and difficult to predict, varying both regionally and temporally. Additionally, initial costs of new technology are high; but with increased production, prices can be expected to decrease. Government support in the early stages of development of new technologies is critical but cannot be undertaken without research into the viability and marketability of the technology.

Electric cars have limitations: short ranges and little infrastructure for recharging in public locations. Both factors can be expected to improve as more electric cars are sold. Charging time is also a consideration; a full charge takes approximately five hours [Perujo and Cuiffo 2010], and rapid charging at public stations would place huge demands on the electrical grid. Alternatively, a battery exchange program would allow charging at a slower rate and off peak; however, higher initial capital would be required to store, maintain, and stockpile batteries. Incentive to buy BEVs would depend strongly on such infrastructure already be in place or planned.

Model Goals

We develop a model to estimate market penetration of electric cars under varying conditions, including the prices and availability of oil and coal, investment in infrastructure, and the rate of development of battery technology. We use the Lotka-Volterra equations describing interspecies competition to simulate the competition between standard ICEs, BEVs, and PHEVs, and we use the resulting estimates of market share to calculate changes in CO_2 production, oil consumption, and electricity demand. We also perform a separate but related economic analysis comparing ownership costs of the Nissan Leaf and the Honda Civic SI over an eight-year life.

Methodology

Mathematical Model

The Lotka-Volterra equations express an ecological approach to competition modeling [Wolfram Demonstrations Project 2010], which considers the effects of multiple species competing for the same resource. The interaction between the numbers of gasoline ($G(t)$), electric ($E(t)$), and hybrid ($H(t)$) cars can be represented by a Lotka-Volterra system of ordinary differential equations **(1)–(3)**, with initial values determined from current data for registered cars [North American Transportation Statistics Database 2011].

$$\frac{dG}{dt} = r_G G \left[1 - \frac{G + \alpha_E E + \alpha_H H}{K_G} \right], \tag{1}$$

$$\frac{dE}{dt} = r_E E \left[1 - \frac{E + \beta_G G + \beta_H H}{K_E} \right], \tag{2}$$

$$\frac{dH}{dt} = r_H H \left[1 - \frac{H + \gamma_E E + \gamma_G G}{K_H} \right], \tag{3}$$

$$G(0) = 246 \times 10^6, \qquad E(0) = 0.01 \times 10^6, \qquad H(0) = 0.1 \times 10^6.$$

We solve the system via the Runge-Kutta-Fehlberg method, using coefficients derived by Cash-Karp. This method gives efficient solutions without introducing excessive round-off error [Chapra and Canale 2002].

Limited data make reliable calibration of the model impossible. The model is therefore best used for running various scenarios and comparing outcomes.

Economic Effects

The cost of fuel (gasoline and electricity) can be represented in the growth model via the competition parameters. If gasoline prices rise, electric cars become more competitive; if the cost of electricity rises, gasoline cars become more competitive.

A larger investment in charging or battery swap stations, charging ports in public parking facilities, and research to improve technology should result in a greater influx of BEVs.

Battery technology has been improving rapidly. Prices, life cycles, range, and efficiencies of batteries will dictate the strength of the BEV population.

Present Value Model

The economic viability of the BEV and the PHEV can be quantified in terms of ownership cost over the life of a vehicle. We convert forecasts of gasoline prices and electricity rates to cost per year based on driving an average of 11,720 mi/yr [U.S. Energy Information Agency 2010]. We add average annual maintenance and repair costs (R_t) of \$500/year [Automotive.com 2011]. With data for the Nissan Leaf and the Honda Civic SI from Callaway [2011] and Penn [2011], we compare the cost of purchasing and operating each car for eight years [Willis and Finney 2004]. The model can easily be adjusted for changes in interest rate, resource costs, maintenance costs, driving distance, battery technology, and resale value:

$$\text{Net Ownership Cost } (n) = -C_0 + \frac{L}{(1+i)^n} + \sum_{t=1}^{n} \frac{B_t - R_t}{(1+i)^t},$$

where

- Net Ownership Cost(n) is the present value of the ownership cost over an n-year horizon,

- C_0 is the initial cost,

- L is the is the resale value after n years,

- $B(t)$ and $R(t)$ are the benefits and the repair/maintenance costs,

- I is the nominal inflation rate, and the effective annual inflation rate i is

$$i = \left(1 + \tfrac{I}{k}\right)^k - 1.$$

(This calculation does not take into account the cost of buying the car on credit and paying for it over a number of years.)

Effects on the Environment

The potential power demand of 100,000 electric cars charging simultaneously would be 440 GW [Perujo and Cuiffo 2010]. If charging coincided with peak demand, additional capacity would be needed. At 236 MW per typical coal-fired plant [U.S. Energy Information Agency 2011a], these 100,000 cars would require the equivalent of 1,865 more power plants.

Effects on Human Health

In the model, potential health risks are determined from the amounts of tailpipe emissions and emissions from coal-fired power plants. Health risks per ICE have been estimated to cost \$103 per vehicle annually from particulate matter [Guo et al. 2010], by assigning monetary values to premature deaths and increased illness rates.

Results

As represented in **(1)–(3)**, the behavior of G, E, and H is mostly controlled by the value of the interspecific competition coefficients. We matched model parameters to the best available data for growth rates of PHEVs, BEVs and ICEs, their relative competitiveness, and the number of registered cars. To describe the competition between BEVs and ICEs, the model assumes that the BEV has a competitive edge vs. the ICE. The PHEV is assumed to have an equal competitiveness factor.

Economics of Current BEVs vs. ICEs

A 2003 Honda Civic SI sold originally for an average of $16,680, according to HowStuffWorks.com [2011]; and a good-condition trade-in value after eight years and 100,000 miles is about $4,000 [Kelley Blue Book 2011]. This is an average resale value of 24%, without factoring in the time value of money. Multiplying by the compound interest factor $(1 + i)^n$, with a nominal annual interest rate $I = 2\%$ compounded monthly (so $i = I/12$ and $n = 96$ months), the value in current (2011) dollars of the $16,680 in 2003 is $19,571, which yields a resale fraction of $4,000/$19,571 \approx 20\%$. Assuming a similar resale fraction, an eight-year-old 100,000-mile 2011 Honda Civic SI will have a resale value of $22,405 \times 20\% = \$4,551$. With these values, the ownership cost of the 2011 Civic SI over eight years is PV = $33,487.

Lack of long-term data on electric cars make estimating resale value for the Nissan Leaf difficult. If the maintenance costs and resale values turn out to be equivalent to the Civic, an initial comparison can be made on the cost of fuel and capital costs. For the 2011 Leaf, with a MSRP of $32,780, an incentive tax credit of $7,500 [Penn 2011], and an eight-year 20% resale value of $6,556, we get an ownership cost of $27,803, which is $5,684 less than the Civic. **Table 2** presents a sensitivity analysis.

Table 2.

Sensitivity of the cost advantage of the Leaf over the Civic to changes in resale value, efficiency, maintenance costs, and annual interest rate.

Parameter	Change	Change in cost advantage of Leaf over Civic
Leaf resale value	$\pm 5\%$	$\mp 24\%$
Civic resale value	$+5\%$	-16%
Civic resale value	-5%	$+18\%$
Maintenance cost of Leaf	$\pm 20\%$	$\mp 13\%$
Maintenance cost of Civic	$\pm 20\%$	$\pm 13\%$
$/kWh	$\pm 5\%$	$\mp 2\%$
kWh/km	-5%	$+ 4\%$
$/gal	$\pm 5\%$	$\pm 10\%$
Interest rate	$2\% \rightarrow 5\%$	-26%

Population Model Results

To compare future scenarios, we use the time t when $G(t) = E(t)$. Because the populations of cars span several orders of magnitude, we use a semi-log scale is to display the results.

Base Scenario

As a base case, we presume that the BEV and PHEV are more competitive than the ICE, based on projected increase in battery technology combined with rising oil prices. Further evidence of the BEV's competitive advantage is given by the economic analysis above of the Nissan Leaf vs. the Honda Civic SI. For the base case, the model parameters are those in **Table 1**. Results are given in **Figure 1**; G and E are equal in 2035.
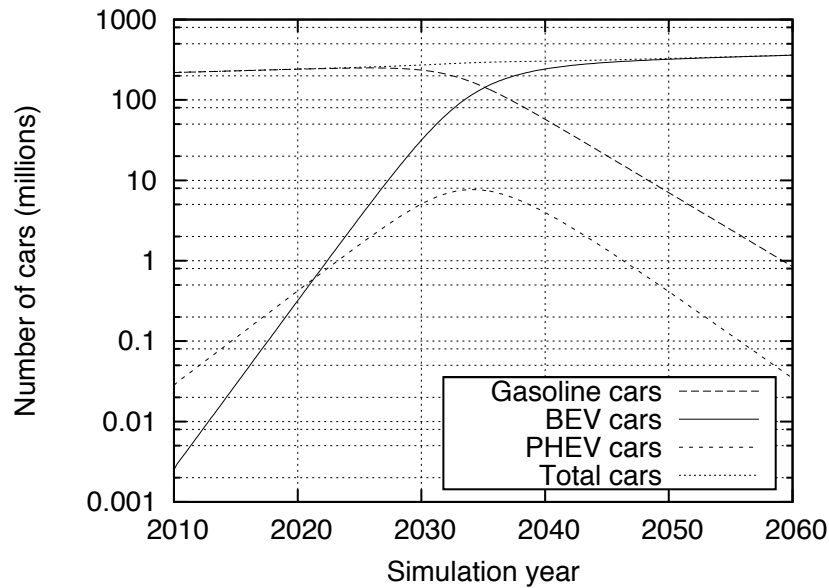
Figure 1. Base scenario. The BEV enters the market with a competitive advantage over the PHEVs and ICE cars, but low initial market penetration hinders spread of BEVs.

Scenario 1: Heavy Investment in Electric Cars

Additional investment will permit a larger initial BEV and PHEV population. This investment is assumed consist of measures that:

- Convert ICE factories to produce BEVs and PHEVs.

- Encourage consumers to purchase BEVs and PHEVs instead of ICEs.

- Support infrastructure development such as electrical grid improvements and charging stations.

- Promote research into improved battery capacity and reduced battery cost.

- Incentivize zero-carbon electricity generation.

We assume that these measures effect a 10-fold increase in initial BEVs and PHEVs populations in 2010. This change would move the ICE and BEV equilibrium point forward 10 years with respect to the base scenario: $G = E$ in 2025.

Scenario 2: Electricity Becomes More Expensive

If electricity were to become much more expensive, growth of BEVs would be hindered. PHEVs would still hold a market share, but battery performance would dictate to what degree.

For this analysis, we assume that battery technology improves slowly, while the price of oil continues to climb. PHEVs would enjoy a decided market advantage over BEVs until oil prices rise too high. The increased cost of driving a PHEV then causes a slide in competitiveness against the purely electric BEV, despite high electric costs.

To accomplish this change in the model, we decreased the intrinsic growth rate r_E of the BEV 20%, from 1.52 to 1.2, to simulate the reluctance of customers to purchase a car with a high per-mile cost.

Eventually, the BEV becomes competitive and overtakes ICEs. However, the timing of the equivalence point is retarded with respect to the base case: $G = E$ in 2042.

Scenario 3: Oil Prices Rise

This scenario assumes that oil prices rise and stay high. The increased cost to operate the ICE is reflected in decreased competitiveness factors against the BEV and PHEV: α_E and α_B are both increased from 1.2 to 1.3, while β_H is decreased from 1.0 to 0.6 and γ_G is decreased from 1.2 to to 0.7. These measures simulate the penalties that gasoline-burning cars face against a purely electric BEV. The equivalence point would occur in 2028, only three years after the equivalence point in the base scenario.

Scenario 4: Best Case for BEVs

For a best-case scenario for BEVs, we combine Scenarios 1 and 3 with advances in battery technology. This case would produce an equilibrium point in 2018 (**Figure 2**).

Emissions

Widespread adoption of the electric car would not reduce emissions of all pollutants. While electric cars do not have tailpipe emissions, particulate matter from coal burning could be 10 times as much as from ICEs [Argonne National Laboratories 2011]. Moreover, emission rates of some pollutants appeared to be driven by economic growth irrespective of car choice.
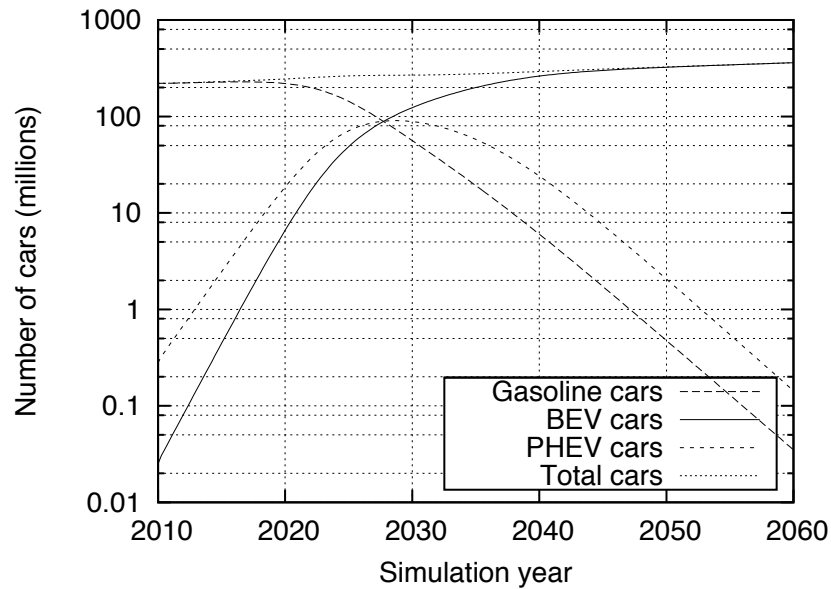
Figure 2. Scenario 4. BEVs and PHEVs enter the market with a competitive advantage over ICEs, an advantage heightened by early infrastructure investment, cheap battery technology, and high oil prices. Both PHEVs and BEVs are adopted early, but the BEV ultimately dominates the market due to reliance on non-petroleum energy sources.

Airborne Pollutants

While widespread use of the electric car would be expected to reduce emissions in cities, the overall effect on emissions would be mixed. Using data from the GREET model [Argonne National Laboratories 2011] and assuming the power generation mix of the present-day U.S., total emissions would change over the modeled period as shown in **Table 3**.

Table 3.
Total change in emissions 2010–2060 in the base scenario.

CH_4	GHGs	CO	PM_{10}	SO_x	N_2O	VOCs	NO_x	$PM_{2.5}$
+39%	24%	−87%	+940%	+860%	−64%	−84%	+94%	+580%

We also used data from Argonne National Laboratories [2011] to predict emissions per mile for each type of car. Although the model predicts a 10-fold increase in $PM_{2.5}$ and PM_{10} if electric cars become widespread, the model was not designed to describe the spatial distribution of particulate pollution. Higher concentration of particulate pollution emitted from power plants in areas with low population density might have a less costly effect than the relatively low particulate pollution density emitted at ICE tailpipes in densely populated areas. The scrubbers, and other emissions equipment added to new generation equipment might cause the overall emissions distribution from electricity generation to decline over time. The estimates in **Table 3** do not take such possible changes into account.

CO_2

Using information from the Environmental Protection Agency [2011], we analyzed the results from each scenario to predict CO_2 emissions attributable to cars. As shown in **Figure 3**, the greatest reduction in CO_2 production corresponds to bringing the more energy-efficient PHEVs and BEVs to market dominance the fastest ("best case"). Initial and trailing rising trends reflect assumptions about the growth rate of the cars. Because the CO_2 emissions of BEVs and PHEVs occur at power-generation facilities rather than at the vehicle tailpipe, a growing population of BEVs and PHEVs would still cause higher emissions. However, because their energy efficiency is higher than that of ICEs, substantial carbon savings could be realized by switching to electric cars.
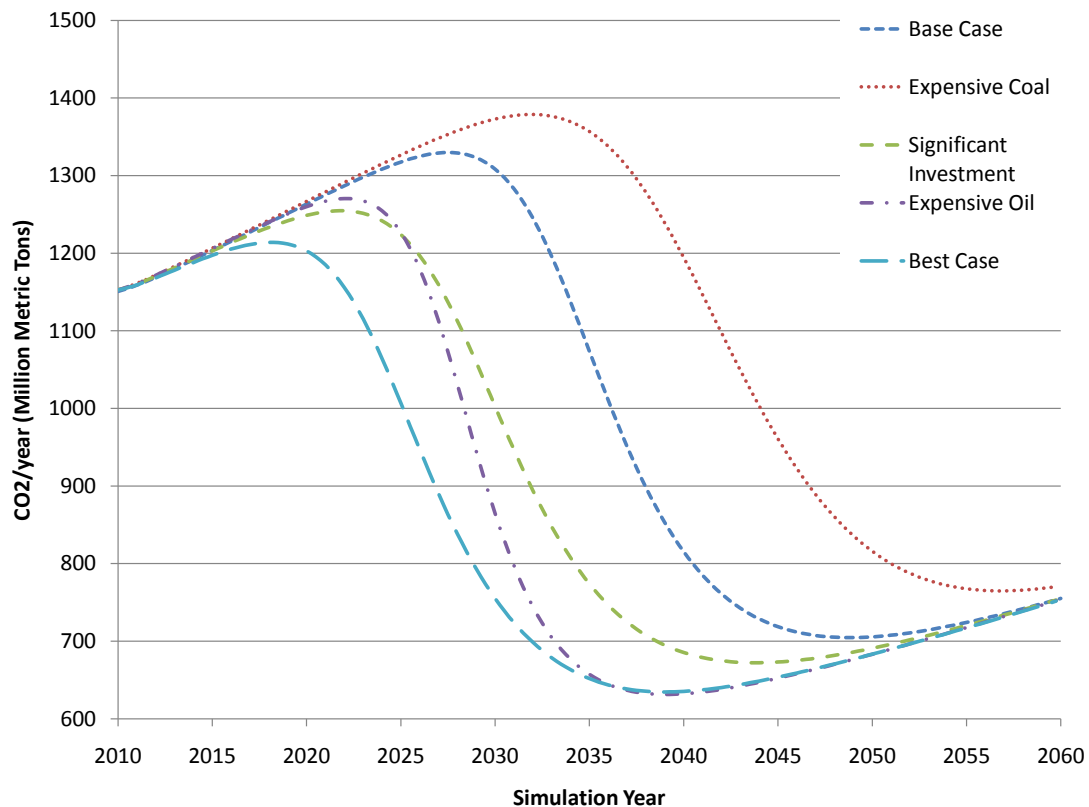
Figure 3. CO_2 production drops as the BEV begins to dominate the market. Advancing the competitiveness of the BEV and PHEV leads to the greatest reductions in CO_2.

Electricity Demand

The U.S. consumed 4 billion MWh of electricity in 2009 [U.S. Energy Information Agency 2011a]. All scenarios predict additional electricity demand for cars of 1 billion MWh per year by 2060.

Discussion

Model Parameters

The parameters for competitiveness (α, β, and γ), as well as the initial conditions and the carrying capacities values (the K parameters) will vary with region of the country.

The values used in the base case for the growth rates for the types of vehicles (r_G, r_E, and r_H) are based on current trends, and they are assumed constant throughout the modeling period. In reality, they would depend on supply and demand and could change dramatically over time.

Demand

The demand for BEVs will be restricted by range and infrastructure. Current buyers would likely be urban commuters, who may also own another vehicle for longer trips. Improvements in infrastructure allowing charging or battery exchange at public locations would increase the appeal to individuals who need just one vehicle for all purposes.

Power Supply

The demand on electricity could potentially be unmanageable. The model predicts increased annual electricity demands of more than 1 billion MWh over 50 years regardless of scenario, reflecting an almost complete switch to electric cars. In the "best-case" scenario, half of this increased demand would occur in the next 20 years. To avoid an electricity shortage, additional generation and conservation will be necessary. Off-peak charging would reduce required maximum capacity but not overall demand. Further improvements could increase battery energy density while decreasing battery mass, which in turn would lead to increased efficiency (in kWh/mi).

Generating more electricity from coal-fired power plants to increase capacity would have environmental consequences beyond reduced ICE emissions. However, estimates by Perujo and Cuiffo [2010] indicate the BEV's superior energy efficiency would still offset production of electricity from high-carbon sources. Furthermore, predictions indicate that the proportion of electricity from coal will steadily decrease [U.S. Energy Information Agency 2010], which in turn could decrease transportation-related CO_2 emissions by an even greater margin than just by the switch to electric cars.

Health

Healthcare costs have been estimated at $103 per vehicle driven [Guo et al. 2010], based on tailpipe emissions. BEVs have zero tailpipe emissions but increase emissions from power generation, relocating emissions from urban areas to rural power plants.

Environment and Pollution

The model scenarios (**Figure 3**) predict peak CO_2 production from 1,200 to 1,400 million metric tonnes per year from cars. The amount produced depends on the number of ICEs replaced by BEVs. Increasing BEVs without developing renewable energy sources may only slow the rate of climate change.

Oil Consumption

According to the model, the decline of oil consumption depends strongly on the competitiveness factors α_E and α_H.

Economics

Table 2 provides useful information on the economic feasibility of electric cars. The largest effect is seen from the resale value for the Nissan Leaf An important factor is the current $7,500 tax credit for buying the Leaf. To overcome this incentive and make the Nissan Leaf and Honda Civic SI equal in overall cost, the resale value of the Leaf would need to be 26%, vs. 20% for the Civic.

Scenario 2 considers a rise in electricity costs along with rising oil prices. Even with electricity rates increasing 10% from the predicted rates, the Leaf still costs $5,500 less than the Civic. If the tax credit is removed, only a 33% increase in BEV efficiency coupled with a 10% decrease in electricity and a 10% increase in gas prices would make the two cars cost the same.

Scenario 3 assumes the status quo in BEV technology but incorporates higher oil prices. If gasoline were to become $5/gal in eight years and increase linearly, the Leaf would be $8,438 cheaper than the Civic, more than enough to overcome the need for a tax credit.

Scenario 4 combines rising oil prices with improved technology and initial investment. Without a tax credit, the consumer would save $3,085 with a Leaf over a Civic.

Of course, a change in the interest rate used in the calculations would affect these numbers.

Government Support

According to the model, early subsidies that increase the number of electric cars would have a large effect. Government support for research and development in battery efficiency, vehicle range, and cost could improve market penetration as well as decrease electricity demand per vehicle.

Conclusions

To compete with the ICE, the BEV needs battery technology improvements and infrastructure investment. The savings in CO_2 and oil become greater the faster the electric car comes to dominate the market.

Subsidies are currently needed to make the Nissan Leaf competitive against the Honda Civic SI; but if gasoline hits \$5/gal in 2019, the cost of owning and operating a Leaf would be thousands less than for a Civic.

The model suffers from considerable uncertainty, and data are not available to calibrate it.

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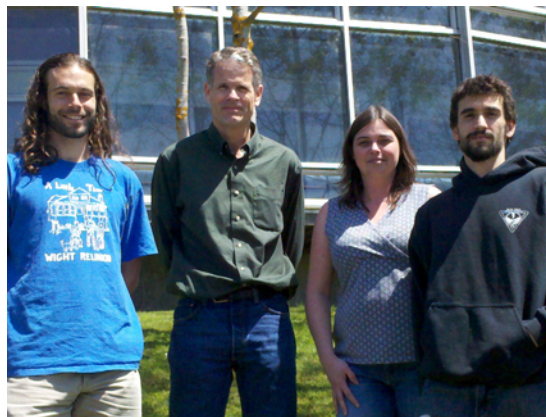
Rick Bailey, Brad Finney (advisor), Brenda Howell, and Zachary Stanko.

Judges' Commentary:
The Outstanding Electric Car Papers

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Introduction

The Interdisciplinary Contest in Modeling (ICM) is an opportunity for teams of students to tackle challenging real-world problems that require a wide breadth of understanding in multiple academic subjects. The difficult nature of the problems and the short time limit require effective communication and coordination of effort among team members. In one sense the real problem is how to best use each team member's skills and talents. Teams that have solved this problem usually submit solutions that rise to the final rounds of judging.

The Problem

Increasing prosperity in the developing countries is tied to an increasing demand in these rapidly growing economies for both energy and automobiles. The difficulty of meeting the increasing demand for oil and the potential environmental impact of increasing numbers of fossil-fuel vehicles are global challenges. This year's problem addressed the effect of transitioning

from vehicles powered by fossil fuels to electric vehicles. The many aspects involved made this an especially appropriate topic for modeling. The main tasks expected of the students were to:

- model the impacts of the widespread use of electric vehicles,

- estimate how much oil the world would save by widely using electric vehicles,

- estimate the amount and type of electricity generation needed, and

- discuss the role governments should play to ensure safe, efficient, effective transportation.

Overall, the judges were impressed both by the strength of many of the submissions of individual teams, and by the variety of approaches that students used to address the questions that were posed by the ICM problem.

Judges' Criteria

To ensure that individual judges assessed submissions on the same criteria, a rubric was developed. The framework used to evaluate submissions is described below.

Executive Summary

It was important that students succinctly and clearly explained the highlights of their submissions. The executive summary should contain a brief description of the modeling approach and the bottom-line results. The remaining report provides a more detailed statement of the contents of the executive summary. One mark of an outstanding paper was a well-connected and concise description of the approach used, the results obtained, and any recommendations.

Modeling

Multiple models were needed to determine the amount of oil saved and how the electricity would be generated to replace this energy. The assumptions needed for these models and the development of these models were important to evaluating the quality of the solutions that were submitted. The better submissions explicitly discussed why key assumptions were made and discussed how these affected the model development. The stronger submissions presented these discussions as a balanced mix of mathematics and English rather than as a series of equations and parameter values without explanation.

Science

The conversion from fossil fuel to electric vehicles involves many scientific and technological issues related to the different methods of producing electricity, how this energy is efficiently transmitted and stored, and how it can be effectively used to power vehicles. In addition, all these areas will experience significant technological improvements in the future. Understanding these issues and trends was very important in creating models with meaningful output.

Data/Validity/Sensitivity

Once the model has been created, the choice of input data and checks on the accuracy and robustness of the solution help to build confidence in the problem approach. Sensitivity analysis to determine the relative rates of change can often be more meaningful than specific output values.

Strengths/Weaknesses

A discussion of the strengths and weaknesses of the developed models is where students truly demonstrate their understanding of what they have created. The ability of a team to make useful recommendations fades quickly if they do not understand the limitations of their assumptions or the implications of their modeling methodology. A simple model that a team can understand and explain is better than a complicated equation pulled out of context from the literature.

Communication/Visuals/Charts

Although mathematics is a precise language used in science and engineering, it is not widely understood outside these disciplines. To clearly explain solutions, teams must use multiple modes of expression including diagrams and graphs, and, in the case of this competition, English. A solution that could not be understood did not progress to the final rounds of judging.

Recommendations

Teams were specifically asked to discuss "the roles that governments should play" and whether the use of electric vehicles is "an important part of an overall strategy to address global energy needs." The ability of teams to evaluate the results of their analysis and make recommendations was important in identifying strong submissions.

Discussion of the Outstanding Papers

Many teams used differential equations, often with simulations, to model the growth of electric vehicles and associated economic impacts. The Analytic Hierarchy Process (AHP) was a common method for addressing the electricity generation mix to most benefit the environment, business, society and individuals. Some chose to model using a few representative vehicles, while others worked at the macro level. As a result, the submissions this year were diverse and interesting to read.

Submissions that did not reach final judging generally suffered from one or both of two shortcomings:

- Some lacked any real mathematical modeling to support their conclusions and recommendations.

- Others had sophisticated and potentially sound models but either failed to clearly present the models or failed to connect them to the science and use them in making recommendations.

In general, poor communication was the biggest discriminator in determining which papers reached the final judging stage. Although the Outstanding papers included different aspects of the basic issues in their approaches, they all addressed the problem in a comprehensive way. The papers were well-written and presented clear recommendations. In several, a unique or innovative approach distinguished them from the rest of the finalists. Others were noteworthy for either the thoroughness of their modeling or exceptionally well communicated results.

- The Northwestern Polytechnical University submission titled "Can Electric Vehicles Be Widely Used?" presented a series of models that flowed from predicting the future number of vehicles, predicting the proportion of different types of vehicles, then mapping this to demand for electricity. An optimization model was then used to determine the best mix of types of electricity production. These results were then used to determine future trends in CO_2 production and oil savings. Weak submissions often show poor transitions as pieces of the report that were done by different individuals or groups come together. This team's report has a smooth progression of model development. They also apply these models to three countries, showing that future trends depend on political and economic contexts.

- The "What Will the Electric Vehicle Bring to the World?" paper by Southeast University uses a Bass diffusion model to predict the increase in the number of electric vehicles sold. They use a neural network model to predict overall vehicle demand, and then combine the results to use in predicting changes in oil consumption and CO_2 emission. The paper stood out for this unique approach, in which the situation was modeled

as optimization of an "ant colony." Health and environmental effects are considered in their optimization model for electricity generation.

- The paper from the South China University of Technology titled "Can Electric Vehicles Be a Shining Star of Tomorrow?" begins with the development of a life-cycle cost model. The model is used to identify the factors that are most important for electric vehicles to increase their share of the vehicle market. A diffusion model is later used to predict the transition from conventional vehicles to electric vehicles. An Analytic Hierarchy Process (AHP) model is used to determine the appropriate proportion of different types of electricity generation. The team also analyzes the sources and patterns of dispersion of pollutants associated with conventional and electric vehicles. Although somewhat compartmentalized, this paper is well written and considers a broader set of issues through the use of the life-cycle cost and pollutant dispersion models.

- The Humboldt State University submission titled "Electric Cars as a Widespread Means of Transportation" models the transition from fossil fuel vehicles to electric vehicles using a competition model represented as a system of ordinary differential equations. Team members examine a number of scenarios as part of their sensitivity analysis. The graphs showing their analysis of oil and electricity consumption under five different cases are very well done, and the paper was among the best in terms of communicating results from the models.

- The "Putting the Spark Back in the Electric Car" paper by the North Carolina School of Science and Mathematics uses a polynomial model to predict future demand for vehicles and coupled differential equations to represent the transition from fossil fuel to electric vehicles. An optimization model determines the amount of electricity generated by each power generation method. Finally, these models are joined together into a model that uses different amounts of government incentive funding to initially incentivize the transition from fossil fuel to electric vehicles. The result allows governments to minimize the total cost of the transition. The team's cellular automata approach was another of the unique methods that distinguished some of the Outstanding papers.

- The paper from Zhejiang University titled "An Analysis of the Future Development of Electric Vehicles" begins with a model of the interactions among oil prices, tax rates, and demand for different types of vehicles. A large portion of the paper focuses on the team's optimization model, which goes beyond considering only different types of electricity generation. Their model considers all forms of energy sources and allocates them among different energy uses. The objective function minimizes the total environmental cost. They analyze several versions of the model to gain insight into the problem and make recommendations.

Conclusion

There were a large number of strong submissions this year, as evidenced by the number of Outstanding papers. This can make judging difficult. However, it is gratifying to see so many students with the ability to combine mathematics, science, and effective communication skills in order to understand a problem and recommend solutions. We look forward to next year's competition, which will involve network science.

Recommendations for Future Participants

- **Answer the problem.** Weak papers do not address a significant part of the problem (e.g., electricity generation of government recommendations). Outstanding teams cover all the bases and then go beyond.

- **Time management is critical.** Every year, a large number of teams do an outstanding job on one aspect of the problem, then "run out of gas" and are unable to complete their solution. Outstanding teams have a plan and adjust as needed to submit a complete solution.

- **Coordinate your plan.** It is pretty obvious in many weak papers how the work was spilt between group members, then pieced together into the final report. For example, the output from one model doesn't match the input for the next model or a section appears in the paper that does not fit with the rest of the report. The more your team can coordinate the efforts of its members, the stronger your final submission will be.

- **The model is not the solution.** Weak teams create a model, then stop. Outstanding teams use their models to understand the problem and recommend a solution.

- **Explain what you are doing and why.** Weak teams tend to use lots of equations and few words. Problem approaches appear out of nowhere. Outstanding teams explain what they are doing and why.

About the Authors

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