Estimation Notes 2.3 - 2.6 pp.16 - 22

Key Points : Unbiased Estimator; Minimum Variance Criterion;Existence of the Minimum Variance Unbiased Estimator; Finding the MVU Estimator

2.3 Unbiased Estimator

1. For an estimator to be **unbiased** we mean that **on the average** the estimator will yield the true value of the unknown parameter.

2. **Unbiasedness** assert that **no matter what the true value of θ, our estimator will yield it on the average**. Mathematically, an estimator is **unbiased** if

$$E(\hat{\theta}) = \theta \quad a < \theta < b \tag{2.1}$$

where (a, b) denotes the range of possible values of θ

3. **Example 2.1 - Unbiased Estimator for DC Level in White Gaussian Noise**

Consider the observations

$$x[n] = A + w[n], \quad n = 0, 1, \ldots, N-1 \quad \text{and} \quad -\infty < A < \infty$$

where A is the parameter to be estimated and $w[n]$ is WGN. Then a reasonable estimator for the **average value** of $x[n]$ is

$$\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n] \tag{2.2}$$

and because of the **linearity properties** of the expectation operator $E(\hat{A}) = A$ for all A, the sample mean estimator (2.2) is **unbiased**.

4. In example 2.1, A can take on any value, and unbiased estimator tend to have **symmetric PDFs** centered about the true value of θ, although this is **NOT** necessary. We take Problem 2.5 as an example (please see the answer of Problem 2.5).

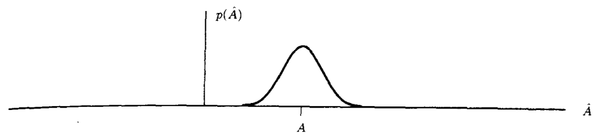

Figure 2.1 Probability density function for sample mean estimator.

5. For example 2.1 the PDF is shown in Figure 2.1 and we use Problem 2.3 to obtain $\mathcal{N}(A, \sigma^2/N)$. The answer of Problem 2.3 as follows

$\hat{A}$ is a linear function of independent random variables.

$$E(\hat{A}) = E(\frac{1}{N}\sum_{n=0}^{N-1} x[n]) = \frac{1}{N}\sum_{n=0}^{N-1} E(x[n]) = \frac{AN}{N} = A$$

$$var(\hat{A}) = var(\frac{1}{N}\sum_{n=0}^{N-1} x[n]) = \frac{1}{N^2}var(\sum_{n=0}^{N-1} x[n]) = \frac{1}{N^2}N \cdot var(x[n])$$

since the $x[n]$s' are i.i.d and thus uncorrelated, then

$$var(\hat{A}) = \frac{\sigma^2}{N}$$

6. The restriction that $E(\hat{\theta}) = \theta$ **for all** θ is an important one. Letting $\hat{\theta} = g(\mathbf{x})$, where $\mathbf{x} = \big[x[0], \ldots, x[N-1]\big]^T$, it asserts that

$$E(\hat{\theta}) = \int g(\mathbf{x})p(\mathbf{x};\theta)d\mathbf{x} = \theta \quad \text{for all } \theta \tag{2.3}$$

7. However, (2.3) may hold for some values of θ and **NOT** others.

8. **Example 2.2 - Biased Estimator for DC Level in White Noise**

Consider again Example 2.1 but with the **modified** sample mean estimator

$$\check{A} = \frac{1}{2N}\sum_{n=0}^{N-1} x[n]$$

then

$$E(\check{A}) = \frac{1}{2}A \begin{cases} = A & \text{if } A = 0 \\ \neq A & \text{if } A \neq 0 \end{cases}$$

We can see that only if $A = 0$, (2.3) is true. Therefore, $\hat{A}$ is a **Biased estimator**.

9. Unbiased estimator **DOES NOT** mean it is a good estimator. It only guarantees that **on the average it will attain the true value**.

10. **Biased estimators** are ones that are characterized by a **systematic error**, which presumably should not be present.

11. A persistent bias will always result in a poor estimator. Problem 2.4 will show the unbiased property has an important implication when several estimators are combined.

12. If we have lots of estimates of the same parameter, i.e., $\{\hat{\theta}_1, \dots, \hat{\theta}_n\}$, we may average them to get a better result

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i \tag{2.4}$$

from example 2.1 we know that $a < \theta < b$, and assume the estimators in (2.4) are all unbiased and uncorrelated with each other, then $E(\hat{\theta}) = \theta$ and $var(\hat{\theta}) = var(\hat{\theta}_i)/n$. Therefore, as more estimates are averaged (i.e., n becomes larger), the variance $var(\hat{\theta})$ will decrease. Ultimately, as $n \to \infty$, $\hat{\theta} \to \theta$.

13. if these estimators are **biased** or $E(\hat{\theta}_i) = \theta + b(\theta)$, then

$$E(\hat{\theta}) = \frac{1}{n}\sum_{n=1}^{n}E(\hat{\theta}_i) = \theta + b(\theta)$$

so no matter how many estimators are averaged, we **CANNOT** obtain the true value θ. In general, $b(\theta)$ is defined as the **Bias** of the estimator.

2.4 Minimum Variance Criterion

14. **Mean square error (MSE)** criterion is defined as follows and this measures **the average mean squared deviation of the estimator from the true value**.

$$mse(\hat{\theta}) = E\big[(\hat{\theta} - \theta)^2\big] \tag{2.5}$$

15. We rewrite the MSE as

$$\begin{aligned} mse(\hat{\theta}) &= E\Bigg\{\Big[\big(\hat{\theta} - E(\hat{\theta})\big) + \big(E(\hat{\theta}) - \theta\big)\Big]^2\Bigg\} \\ &= var(\hat{\theta}) + \big[E(\hat{\theta}) - \theta\big]^2 \\ &= var(\hat{\theta}) + b^2(\theta) \end{aligned} \tag{2.6}$$

which shows that **the MSE is composed of errors due to the variance of the estimator as well as the bias**.

16. Consider example 2.1, we now have a modified estimator as follows

$$\check{A} = a\frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

for some constant a. Since $E(\check{A}) = aA$ and $var(\check{A}) = a^2\sigma^2/N$, so

$$\begin{aligned} mse(\check{A}) &= \frac{a^2\sigma^2}{N} + \big[E(\hat{\theta}) - \theta\big]^2 \\ &= \frac{a^2\sigma^2}{N} + \big[aA - A\big]^2 \\ &= \frac{a^2\sigma^2}{N} + (a-1)^2A^2 \end{aligned}$$

and we can differentiating the MSE with respect to a and then obtain the optimum value

$$a_{\text{opt}} = \frac{A^2}{A^2 + \sigma^2/N}$$

17. However, we can see that a_{opt} is depends upon the unknown parameter A. Therefore, the estimator is **NOT** realizable.

18. An alternative approach is to **constrain the bias to be zero** and **find the estimator which minimizes the variance**. Such an estimator is termed the **Minimum Variance Unbiased (MVU) estimator**. The MSE of an unbiased estimator is just the variance.

$$mse(\check{A}) = \underbrace{var(\hat{\theta})}_{minimize} + \underbrace{b^2(\theta)}_{=0} = \underbrace{var(\hat{\theta})}_{minimize}$$

19. From problem 2.7 we know that minimizing the variance of an unbiased estimator also has the effect of concentrating the PDF of the estimation error, $\hat{\theta} - \theta$, about zero.

2.5 Existence of the Minimum Variance Unbiased Estimator

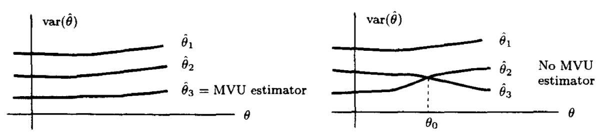

Figure 2.3 Possible dependence of estimator variance with θ.

20. So whether a MVU estimator exists, i.e., an unbiased estimator with minimum variance for all θ, is a question that is worth noting. From figure 2.3, we know that in the figure 2.3a, $\hat{\theta_3}$ is the MVU estimator and sometimes referred to as the **Uniformly Minimum Variance Unbiased** estimator. But in figure 2.3b, there is no MVU estimator. In conclusion, **the MVU estimator does not always exist**.

21. From problem 2.11, **it is also possible that there may not exist even a single unbiased estimator**.

2.6 Finding the MVU Estimator

22. Even if a MVU estimator exists, we may not be able to find it. We have three possible approaches. They are

1) Determine the **Cramer-Rao lower bound (CRLB)** and check to see if some estimator satisfies it

2) Apply the **Rao-Blackwell-Lehmann-Scheffe (RBLS)** theorem

3) Further restrict the class of estimators to be **not only unbiased but also linear**. Then, find the minimum variance estimator within this **restricted class**. (END)