Estimation Notes 3.8 - 3.9 pp.45 - 50

Key Points : Vector Parameter CRLB for Transformations; CRLB for the General Gaussian Case

1. Vector Parameter CRLB for Transformations

1. Assume that it is desired to estimate $\boldsymbol{\alpha} = \mathbf{g}(\boldsymbol{\theta})$, and $\mathbf{g}$ is an r-dimensional function. Then we have

$$\mathbf{C}_{\hat{\boldsymbol{\alpha}}} - \frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{g}(\boldsymbol{\theta})^{\mathrm{T}}}{\partial \boldsymbol{\theta}} \geq \mathbf{0} \tag{3.30}$$

where, as before, ≥ 0 is to be interpreted as **positive semidefinite**. Note that $\partial \mathbf{g}(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$ is the $r \times p$ **Jacobian Matrix** defined as

$$\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_p} \\ \frac{\partial g_2(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g_2(\boldsymbol{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_2(\boldsymbol{\theta})}{\partial \theta_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_r(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g_r(\boldsymbol{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_r(\boldsymbol{\theta})}{\partial \theta_p} \end{bmatrix}$$

2. Example 3.8 CRLB for Signal-to-Noise Ratio

Consider a DC level in WGN with A and σ^2 unknown. We wish to estimate $\alpha = A^2/\sigma^2$ which can be considered to be the SNR for a single sample. Here $\boldsymbol{\theta} = [A, \sigma^2]^{\mathrm{T}}$ and $g(\boldsymbol{\theta}) = \theta_1^2/\theta_2 = A^2/\sigma^2$. Then from Example 3.6 we have

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2\sigma^4} \end{bmatrix}$$

the Jacobian is

$$\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g(\boldsymbol{\theta})}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial g(\boldsymbol{\theta})}{\partial A} & \frac{\partial g(\boldsymbol{\theta})}{\partial \sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{2A}{\sigma^2} & -\frac{A^2}{\sigma^4} \end{bmatrix}$$

so that

$$\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \mathbf{I}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{g}(\boldsymbol{\theta})^{\mathrm{T}}}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{2A}{\sigma^2} & -\frac{A^2}{\sigma^4} \end{bmatrix} \begin{bmatrix} \frac{\sigma^2}{N} & 0 \\ 0 & \frac{2\sigma^4}{N} \end{bmatrix} \begin{bmatrix} \frac{2A}{\sigma^2} \\ -\frac{A^2}{\sigma^4} \end{bmatrix} = \frac{4\alpha + 2\alpha^2}{N}$$

finally we have

$$var(\hat{\alpha}) \geq \frac{4\alpha + 2\alpha^2}{N}$$

3. It is noteworthy that for vector parameter transformations, the linearity is maintained.

2. CRLB for the General Gaussian Case

4. In the case of **Gaussian Observations** we can derive the CRLB that that **generalizes** (3.14). Assume that both the mean and covariance of $\mathbf{x}$ may depend on $\boldsymbol{\theta}$

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}(\boldsymbol{\theta}), \mathbf{C}(\boldsymbol{\theta}))$$

then the **Fisher Information Matrix** is given by

$$[\mathbf{I}(\boldsymbol{\theta})]_{ij} = [\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i}]^{\mathrm{T}} \mathbf{C}^{-1}(\boldsymbol{\theta}) [\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_j}] + \frac{1}{2} \mathrm{tr}[\mathbf{C}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_i} \mathbf{C}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_j}] \tag{3.31}$$

where

$$\frac{\partial \boldsymbol{\mu}(\boldsymbol{\theta})}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [\boldsymbol{\mu}(\boldsymbol{\theta})]_1}{\partial \theta_i} \\ \frac{\partial [\boldsymbol{\mu}(\boldsymbol{\theta})]_2}{\partial \theta_i} \\ \vdots \\ \frac{\partial [\boldsymbol{\mu}(\boldsymbol{\theta})]_N}{\partial \theta_i} \end{bmatrix}$$

$$\frac{\partial \mathbf{C}(\boldsymbol{\theta})}{\partial \theta_i} = \begin{bmatrix} \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{11}}{\partial \theta_i} & \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{12}}{\partial \theta_i} & \cdots & \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{1N}}{\partial \theta_i} \\ \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{21}}{\partial \theta_i} & \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{22}}{\partial \theta_i} & \cdots & \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{2N}}{\partial \theta_i} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{N1}}{\partial \theta_i} & \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{N2}}{\partial \theta_i} & \cdots & \frac{\partial [\mathbf{C}(\boldsymbol{\theta})]_{NN}}{\partial \theta_i} \end{bmatrix}$$

5. For the scalar parameter case in which

$$\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}(\theta), \mathbf{C}(\theta))$$

this reduces to

$$I(\theta) = [\frac{\partial \boldsymbol{\mu}(\theta)}{\partial \theta}]^{\mathrm{T}} \mathbf{C}^{-1}(\theta) [\frac{\partial \boldsymbol{\mu}(\theta)}{\partial \theta}] + \frac{1}{2} \mathrm{tr}[(\mathbf{C}^{-1}(\theta) \frac{\partial \mathbf{C}(\theta)}{\partial \theta})^2] \tag{3.32}$$