Estimation Notes 1 (Problems 2.1 - 2.6) pp.23 - 24

Problem 2.1

The data $\{x[0], \ldots, x[N-1]\}$ are observed where the $x[n]$'s are independent and identically distributed (IID) as $\mathcal{N}(0, \sigma^2)$. We wish to estimate the variance σ^2 as

$$\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1} x^2[n]$$

Is this an unbiased estimator? Find the variance of $\hat{\sigma^2}$ and examine what happens as $N \to \infty$

Answer:

We can know that

$$E(\hat{\sigma^2}) = E\big[\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]\big] = E\big[(x[n]-0)^2\big] = \sigma^2 > 0$$

therefore, this estimator is unbiased. $x^2[n]$ are IID and the variance of $\hat{\sigma^2}$ is

$$var(\hat{\sigma^2}) = var(\frac{1}{N}\sum_{n=0}^{N-1} x^2[n]) = \frac{1}{N^2}var(\sum_{n=0}^{N-1} x^2[n])$$

$$= \frac{1}{N^2}N \cdot var(x^2[n]) = \frac{1}{N}\big[E(x^4[n]) - E^2(x[n])\big]$$

$$= 3\sigma^2 - \sigma^2 = 2\sigma^2$$

$$\Rightarrow var(\hat{\sigma^2}) = \frac{2\sigma^4}{N} \to 0 \quad \text{as} \quad N \to \infty$$

Hence, the PDF of $\hat{\theta^2}$ is close to the true value as $N \to \infty$

Note: $D(X+Y) = D(X) + D(Y)$, $D(X^2) = E(X^2) - E^2(X)$

Problem 2.2

Consider the data $\{x[0], \ldots, x[N-1]\}$, where each sample is distributed as $\mathcal{U}[0, \theta]$ and the samples are IID. Can you find an unbiased estimator for θ? The range of θ is $0 < \theta < \infty$.

Answer:

We

Problem 2.3

Prove that the PDF of $\hat{A}$ given in Example 2.1 is $\mathcal{N}(A, \sigma^2/N)$

Answer:

$\hat{A}$ is a linear function of independent random variables.

$$E(\hat{A}) = E(\frac{1}{N}\sum_{n=0}^{N-1} x[n]) = \frac{1}{N}\sum_{n=0}^{N-1} E(x[n]) = \frac{AN}{N} = A$$

$$var(\hat{A}) = var(\frac{1}{N}\sum_{n=0}^{N-1} x[n]) = \frac{1}{N^2} var(\sum_{n=0}^{N-1} x[n]) = \frac{1}{N^2} N \cdot var(x[n])$$

since the $x[n]$s' are i.i.d and thus uncorrelated, then

$$var(\hat{A}) = \frac{\sigma^2}{N}$$

Problem 2.4

The heart rate h of a patient is automatically recorded by a computer every 100 ms. In 1 s the measurements $\{\hat{h_1}, \ldots, \hat{h_{10}}\}$ are averaged to obtain $\hat{h}$. If $E(\hat{h_i}) = \alpha h$ for some constant α and $var(\hat{h_i}) = 1$ for each i, determine whether averaging improves the estimator if $\alpha = 1$ and $\alpha = 1/2$. Assume each measurement is uncorrelated.

Answer:

we have

$$\hat{h} = \frac{1}{10}\sum_{i=1}^{10} \hat{h}_i$$

and

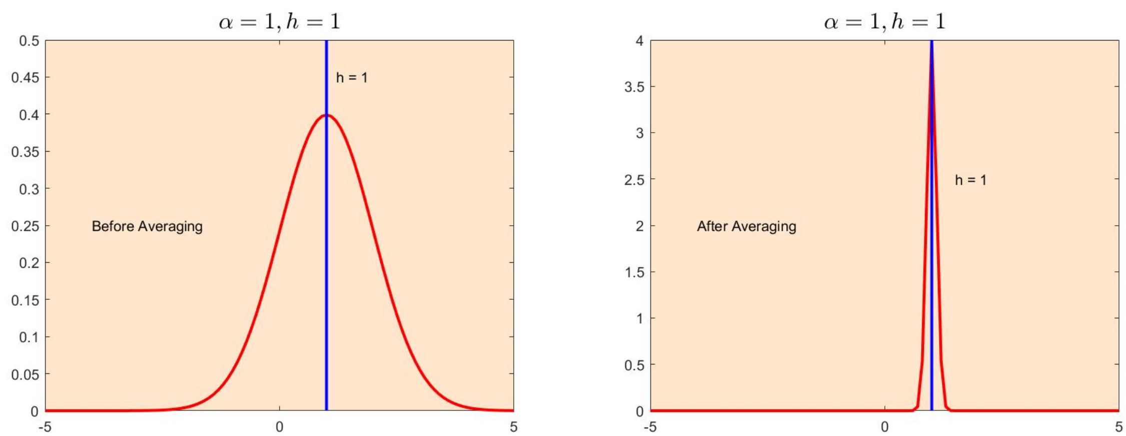
$$E(\hat{h}) = E(\frac{1}{10}\sum_{i=1}^{10}\hat{h}_i) = \frac{1}{10}\sum_{i=1}^{10}E(\hat{h}_i) = E(\hat{h}_i) = \alpha h$$

$$var(\hat{h}) = \frac{1}{100}var(\sum_{i=1}^{10}\hat{h}_i) = \frac{var(\hat{h}_i)}{10} = \frac{1}{10}$$

therefore, if $\alpha = 0.5$, $h = 1$ and before averaging, we use MATLAB to obtain the figure

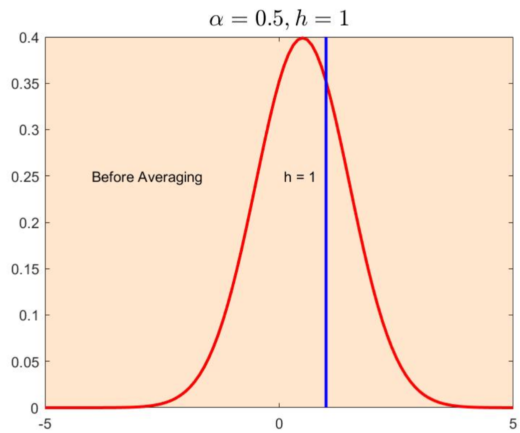
```
x = -5 : 0.1 : 5;
fun = normpdf(x, 0.5, 1);
plot(x, fun, '-r', 'linewidth', 2);
hold on
plot([1,1],[0, 4], '-b', 'linewidth', 2);
ylim([0, 0.4]);
set(gca, 'color',  [1, 0.9, 0.8]);
title('$\alpha = 0.5, h = 1$', 'interpreter','latex', 'FontSize', 16);
str = {'h = 1'};
text(0.1, 0.25, str);
str1 = {'Before Averaging'};
text(-4, 0.25, str1);
```

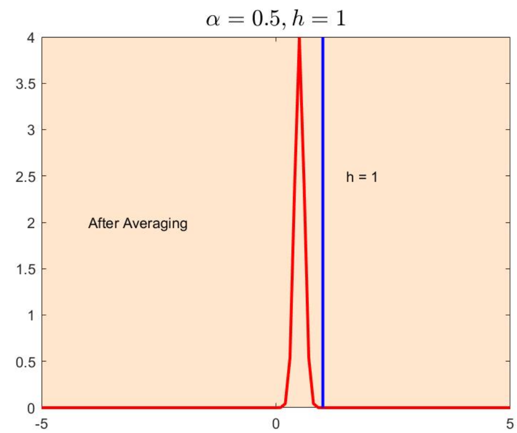
the figures we obtained is as follows



$h = 1$, before (variance = 1) and after (variance = 0.1) averaging.

similarly, when $\alpha = 0.5$, we can have





$h = 1$, before (variance $= 1$) and after (variance $= 0.1$) averaging.

In conclusion, if $\alpha = 0.5$, averaging decreases the probability of $\hat{h} = h$. But if $\alpha = 1$, averaging is benefital.

Problem 2.5

Two samples $\{x[0], x[1]\}$ are independently observed from a $\mathcal{N}(0, \sigma^2)$ distribution. The estimator

$$\hat{\sigma^2} = \frac{1}{2}(x^2[0] + x^2[1])$$

is unbiased. Find the PDF of $\hat{\sigma^2}$ to determine if is symmetric about σ^2

Answer:

We have $x[n] \sim \mathcal{N}(0, \sigma^2)$ or

$$\frac{x[n]}{\sigma} \sim \mathcal{N}(0, 1)$$

therefore

$$\left\{\frac{x[n]}{\sigma}\right\}^2 \sim \mathcal{X}_1^2$$

where $\mathcal{X}$ refers to **Chi-Square distribution** with **one degree of freedom**, and

$$y = \left\{\frac{x[0]}{\sigma}\right\}^2 + \left\{\frac{x[1]}{\sigma}\right\}^2 \sim \mathcal{X}_2^2$$

since the PDF of **Chi-Square distribution** is

$$p(y) = \begin{cases} \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} e^{-\frac{y}{2}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

in this case $n = 2$ and the Gamma function $\Gamma(2) = 1$, so

$$p(y) = \begin{cases} \frac{1}{2} e^{-\frac{y}{2}} & y > 0 \\ 0 & y < 0 \end{cases}$$

transforming, we have $\hat{\sigma^2} = \frac{\sigma^2}{2} y \Rightarrow y = \frac{2\hat{\sigma^2}}{\sigma^2}$, and let $\hat{\sigma^2}$ be the argument, so that

$$\begin{aligned} p(\hat{\sigma^2}) &= \frac{p_y(y(\hat{\sigma^2}))}{|d\hat{\sigma^2}/dy|} \\ &= \begin{cases} \frac{\frac{1}{2} e^{-\frac{1}{2}(2\hat{\sigma^2}/\sigma^2)}}{\sigma^2/2} & \hat{\sigma^2} > 0 \\ 0 & \hat{\sigma^2} < 0 \end{cases} \\ &= \begin{cases} \frac{1}{\sigma^2} e^{-\frac{\hat{\sigma^2}}{\sigma^2}} & \hat{\sigma^2} > 0 \\ 0 & \hat{\sigma^2} < 0 \end{cases} \end{aligned}$$

we use MATLAB to obtain the figure

```
>> syms x
fun = @(x) (1/2) * exp(-(x./2));
fplot(fun,[0, 20],'-*r','linewidth',2)
grid on
set(gca, 'color',  [1, 0.9, 0.8]);
title('$\sigma^2 = 2$','interpreter','latex', 'FontSize', 16);
```

assume $\sigma^2 = 2$, then the figure is as follows

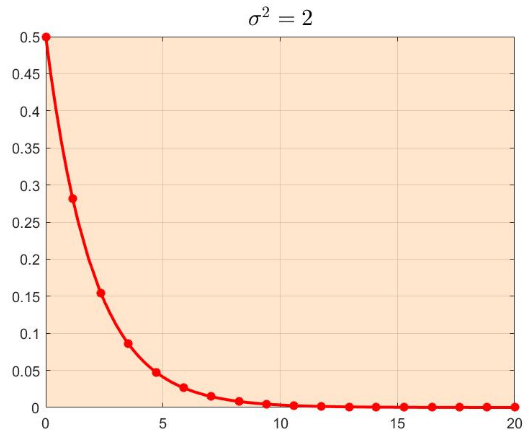


Figure 1. When $\sigma^2 = 2$.

Clearly, it is **NOT** symmetric

However

$$E(\hat{\sigma^2}) = \int_0^\infty \hat{\sigma^2} \frac{1}{\sigma^2} e^{-\frac{\hat{\sigma^2}}{\sigma^2}} d\hat{\sigma^2} = \sigma^2 \int_0^\infty u e^{-u} du = \sigma^2$$

In conclusion, $\hat{\sigma^2}$ is **Unbiased** but PDF is **NOT** symmetric about σ^2

Problem 2.6

For the problem described in Example 2.1 the more general estimator

$$\hat{A} = \sum_{n=0}^{N-1} a_n x[n]$$

is proposed. Find the a_n's so that the estimator is unbiased and the variance is minimized. Hint: Use Lagrangian multipliers with unbiasedness as the constraint equation.

Answer:

$$E(\hat{A}) = \sum_{n=0}^{N-1} a_n A = A \Rightarrow \sum_{n=0}^{N-1} a_n = 1$$

$$var(\hat{A}) = var\{\sum_{n=0}^{N-1} a_n x[n]\} = \sum_{n=0}^{N-1} a_n^2 var(x[n]) = \sum_{n=0}^{N-1} a_n^2 \sigma^2$$

Let

$$F = \sigma^2 \sum_{n=0}^{N-1} a_n^2 + \lambda(\sum_{n=0}^{N-1} a_n - 1)$$

then

$$\frac{\partial F}{\partial a_i} = 2\sigma^2 a_i + \lambda = 0, \quad i = 0, \ldots, N-1$$

$$\Rightarrow a_i = -\frac{\lambda}{2\sigma^2} \quad \text{for all } i$$