Estimation Notes 3 (Problems 3.1 - 3.9) **pp.63 - 64**

Problem 3.1

(An example of a PDF that does NOT satisfy the regularity condition)

If x[n] for $n = 0, 1, \ldots, N-1$ are i.i.d according to $\mathcal{U}[0, \theta]$, show that the regularity condition does not hold or that

$$E\left[\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta}\right] \neq 0 \quad \text{for all } \theta > 0$$

Hence, the CRLB cannot be applied to this problem.

Answer:

1

Problem 3.2

In Example 3.1 assume that $w[0]$ has the PDF $p(w[0])$ which can now be arbitrary. Show that the CRLB for A is

$$var(\hat{A}) \geq \left[\int_{-\infty}^{\infty} \frac{(\frac{dp(u)}{du})^2}{p(u)}\right]^{-1}.$$

Evaluate this for the Laplacian PDF

$$p(w[0]) = \frac{1}{\sqrt{2}\sigma} \exp\big(-\frac{\sqrt{2}|w[0]|}{\sigma}\big)$$

and compare the result to the Gaussian case

Answer:

Since $x[0] = A + w[0]$, then $p_x(x[0]; A) = p_w(x[0] - A) = p(x[0] - A)$, then we have

$$I(A) = -E\left\{\frac{\partial^2 \ln p(x[0]-A)}{\partial^2 A}\right\} = E\left\{\left(\frac{\partial \ln p(x[0]-A)}{\partial A}\right)^2\right\}$$

because $\partial \ln x / \partial x = 1/x$, we have

$$\frac{\partial \ln p(x[0]-A)}{\partial A} = (-1)\frac{1}{p(x[0]-A)}\frac{\partial p(x[0]-A)}{\partial(x[0]-A)}$$

$$\Rightarrow E\left[\left(\frac{1}{p(x[0]-A)}\frac{\partial p(x[0]-A)}{\partial(x[0]-A)}\right)^2\right]$$

$$= \int_{-\infty}^{\infty}\left(\frac{\partial p(x[0]-A)}{\partial(x[0]-A)}\right)^2 \frac{1}{p^2(x[0]-A)}\underbrace{p_x(x[0];A)}_{=p(x[0]-A)}dx[0]$$

letting $u = x[0] - A$, then we have

$$I(A) = \int_{-\infty}^{\infty}\frac{(\frac{dp(u)}{du})^2}{p(u)}du$$

For $p(u) = \frac{1}{\sqrt{2}\sigma}\exp(-\sqrt{2}|u|/\sigma)$ we have

$$\frac{dp}{du} = -\frac{\sqrt{2}}{\sigma}\frac{1}{\sqrt{2}\sigma}\exp(-\sqrt{2}u/\sigma) \quad u > 0$$

$$I(A) = 2\int_0^{\infty}\frac{\frac{1}{\sigma^4}\exp(-2\sqrt{2}u/\sigma)}{\frac{1}{\sqrt{2}\sigma}\exp(-\sqrt{2}u/\sigma)}du$$

$$= \frac{2\sqrt{2}\sigma}{\sigma^4}\int_0^{\infty}\exp(-\sqrt{2}u/\sigma)du = \frac{2\sqrt{2}}{\sigma^4}\frac{\sqrt{2}\sigma}{2} = \frac{2}{\sigma^2}$$

$$\Rightarrow var(\hat{A}) \geq \frac{\sigma^2}{2}$$

the CRLB for Laplacian PDF is **half** of that for the Gaussian case. In fact, it can be known that the Gaussian PDF produces the **largest** CRLB.

Problem 3.3

(One of the variations on example 3.3)

The data $x[n] = Ar^n + w[n]$ for $n = 0, 1, \ldots, N-1$ are observed, where $w[n]$ is WGN with variance σ^2 and $r > 0$ is known. Find the CRLB for A. Show that an efficient estimator exists and find its variance. What happens to the variance as $N \to \infty$ for various values of r?

Answer:

We can know that

$$
\begin{aligned}
p(\mathbf{x}; A) &= \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - Ar^n)^2\Big] \\
&= \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\Big[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - Ar^n)^2\Big] \\
\Rightarrow \frac{\partial \ln p(\mathbf{x}; A)}{\partial A} &= \frac{\partial}{\partial A}\Big[-\ln((2\pi\sigma^2)^{\frac{N}{2}}) - \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - Ar^n)^2\Big] \\
&= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n] - Ar^n)r^n \\
\Rightarrow \frac{\partial^2 \ln p(\mathbf{x}; A)}{\partial A^2} &= -\frac{1}{\sigma^2}\sum_{n=0}^{N-1} r^{2n} \\
\Rightarrow -E\left\{\frac{\partial^2 \ln p(\mathbf{x}; A)}{\partial A^2}\right\} &= \frac{1}{\sigma^2}\sum_{n=0}^{N-1} r^{2n}
\end{aligned}
$$

or

$$
var(\hat{A}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}}
$$

To show that an efficient estimator, we have

$$
\frac{\partial \ln p(\mathbf{x}; A)}{\partial A} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n] - Ar^n)r^n
$$

$$= \frac{1}{\sigma^2}(\sum_{n=0}^{N-1} x[n]r^n - A\sum_{n=0}^{N-1} r^{2n})$$

$$= \underbrace{\frac{\sum_{n=0}^{N-1} r^{2n}}{\sigma^2}}_{I(A)} \Bigg(\underbrace{\frac{\sum_{n=0}^{N-1} x[n]r^n}{\sum_{n=0}^{N-1} r^{2n}}}_{\hat{\theta}} - A \Bigg)$$

$\hat{A}$ is efficient and $1/I(A)$ is the variance, therefore

$$var(\hat{A}) = \frac{\sigma^2}{\sum_{n=0}^{N-1} r^{2n}} \to \begin{cases} \sigma^2(1-r^2) & \text{if } 0 < r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \quad \text{as} \quad N \to \infty$$

Problem 3.4

(One of the variations on example 3.3)

If $x[n] = r^n + w[n]$ for $n = 0, 1, \dots, N-1$ are observed, where $w[n]$ is WGN with variance σ^2 and r is to be estimated, Find the CRLB. Does an efficient estimator exist and if so find its variance?

Answer:

$$p(\mathbf{x}; r) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\big[-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n] - r^n)^2\big]$$

$$\Rightarrow \frac{\partial \ln p(\mathbf{x}; r)}{\partial r} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n] - r^n)nr^{n-1}$$

it CANNOT put into form $I(r)(\hat{r} - r)$, so no efficient estimator

We have

$$\frac{\partial^2 \ln p(\mathbf{x}; r)}{\partial r^2} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\Bigg[x[n]n(n-1)r^{n-2} - n(2n-1)r^{2n-2}\Bigg]$$

since $E\big\{x[n]\big\} = r^n$,

$$\Rightarrow E\Bigg\{\frac{\partial^2 \ln p(\mathbf{x}; r)}{\partial r^2}\Bigg\} = -\frac{1}{\sigma^2}\sum_{n=0}^{N-1} n^2 r^{2n-2}$$

$$\Rightarrow var(\hat{r}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} n^2 r^{2n-2}}$$

Problem 3.5

(One of the variations on example 3.3)

If $x[n] = A + w[n]$ for $n = 0, 1, \ldots, N-1$ are observed and $\mathbf{w} = [w[0], w[1], \ldots, w[N-1]]^T \sim \mathcal{N}(\mathbf{0}, \mathbf{C})$, Find the CRLB for A. Does an efficient estimator exist and if so what is its variance?

Answer:

We can know that

Problem 3.6

For Example 2.3 compute the CRLB. Does it agree with the results given?

Answer:

From Example 2.3 we know that

$$x[0] \sim \mathcal{N}(\theta, 1)$$

$$x[1] \sim \begin{cases} \mathcal{N}(\theta, 1) & \text{if } \theta \geq 0 \\ \mathcal{N}(\theta, 2) & \text{if } \theta < 0 \end{cases}$$

then

$$p(\mathbf{x}; \theta) = \begin{cases} \frac{1}{2\pi} e^{-\frac{1}{2}[(x[0]-\theta)^2 + (x[1]-\theta)^2]} & \text{if } \theta \geq 0 \\ \frac{1}{2\pi\sqrt{2}} e^{-\frac{1}{2}[(x[0]-\theta)^2 + \frac{1}{2}(x[1]-\theta)^2]} & \text{if } \theta < 0 \end{cases}$$

For $\theta \geq 0$

$$\frac{\partial \ln p(\mathbf{x}; \theta)}{\partial \theta} = -\frac{1}{2}\big[2(x[0]-\theta)(-1) + 2(x[1]-\theta)(-1)\big] = (x[0]-\theta) + (x[1]-\theta)$$

$$\Rightarrow \frac{\partial^2 \ln p(\mathbf{x}; \theta)}{\partial \theta^2} = -2 \Rightarrow -E\big[\frac{\partial^2 \ln p(\mathbf{x}; \theta)}{\partial \theta^2}\big] = 2 \Rightarrow var(\hat{\theta}) \geq \frac{1}{2}$$

For $\theta < 0$

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = -\frac{1}{2}\big[-2(x[0]-\theta) + -(x[1]-\theta)\big] = (x[0]-\theta) + \frac{1}{2}(x[1]-\theta)$$

$$\Rightarrow \frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2} = -\frac{3}{2} \Rightarrow -E\big[\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\big] = \frac{3}{2} \Rightarrow var(\hat{\theta}) \geq \frac{2}{3}$$

Problem 3.7

Prove that in Example 3.4

$$\frac{1}{N}\sum_{n=0}^{N-1} \cos(4\pi f_0 n + 2\phi) \approx 0$$

What conditions on f_0 are required for this to hold? Hint: Note that

$$\sum_{n=0}^{N-1} \cos(\alpha n + \beta) = \mathrm{Re}\Big\{\sum_{n=0}^{N-1} \exp[j(\alpha n + \beta)]\Big\}$$

Answer:

Let $\alpha = 4\pi f_0$, $\beta = 2\phi$, then

$$\frac{1}{N}\sum_{n=0}^{N-1} \cos(\alpha n + \beta) = \frac{1}{N}\mathrm{Re}\Big\{\sum_{n=0}^{N-1} \exp(j(\alpha n + \beta))\Big\}$$

$$= \frac{1}{N}\mathrm{Re}\Big\{e^{j\beta}\underbrace{\sum_{n=0}^{N-1} e^{j\alpha n}}_{=\frac{1-e^{j\alpha N}}{1-e^{j\alpha}}}\Big\}$$

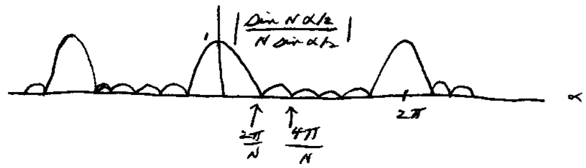
$$= \frac{1}{N}\mathrm{Re}\Big\{e^{j\beta}\cdot\frac{e^{j\alpha\frac{N}{2}}}{e^{j\alpha/2}}\cdot\frac{e^{-j\alpha\frac{N}{2}} - e^{j\alpha\frac{N}{2}}}{e^{-j\alpha/2} - e^{j\alpha/2}}\Big\} \qquad *$$

since $e^{jk} = \cos(k) + j\sin(k)$, then

$$* = \frac{1}{N}\cos[\alpha(\frac{N-1}{2}) + \beta]\cdot\frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$$= \cos[\alpha(\frac{N-1}{2}) + \beta]\cdot\frac{\sin\frac{N\alpha}{2}}{N\sin\frac{\alpha}{2}} = \cos[4\pi f_0(\frac{N-1}{2}) + \beta]\cdot\frac{\sin(N2\pi f_0)}{N\sin(2\pi f_0)}$$

therefore f_0 CANNOT be 0 or 1/2



Problem 3.8

Repeat the computation of the CRLB for Example 3.3 by using the alternative expression (3.12)

Answer:

1) the first method

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)$$

$$E\left\{(\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta})^2\right\} = var\left\{\frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)\right\} + E^2\left\{(\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta})\right\}$$

$$= \frac{1}{\sigma^4}N\sigma^2 + 0 = \frac{N}{\sigma^2} \Rightarrow var(\hat{A}) \geq \frac{\sigma^2}{N}$$

2) the second method

$$\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)$$

$$\Rightarrow (\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta})^2 = \frac{1}{\sigma^4}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left[(x[m]-A)(x[n]-A)\right]$$

$$\Rightarrow E\left\{(\frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta})^2\right\} = \frac{1}{\sigma^4}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}E\left\{(x[m]-A)(x[n]-A)\right\}$$

$$= \frac{N}{\sigma^2} \Rightarrow var(\hat{A}) \geq \frac{\sigma^2}{N}$$

Problem 3.9

Repeat the computation of the CRLB for Example 3.3 by using the alternative expression (3.12)

Answer:

1) the first method