Estimation Notes 1 (Problems) pp.14

Problem 1.1

In a radar system an estimator of round trip delay τ_0 has the PDF $\hat{\tau}_0 \sim \mathcal{N}(\tau_0, \sigma^2_{\hat{\tau}_0})$, where τ_0 is the true value. If the range is to be estimated, propose an estimator $\hat{R}$ and find its PDF. Next determine the standard deviation $\sigma_{\hat{\tau}_0}$ so that 99% of the time the range estimate will be within 100 m of the true value. Use $c = 3 \times 10^8$m/s for the speed of electromagnetic propagation.

Answer:

1) From the radar theory we can know that

$$R = \frac{C\tau_0}{2} \tag{1}$$

therefore

$$\hat{R} = \frac{C\hat{\tau}_0}{2} \tag{2}$$

since $\hat{\tau}_0 \sim \mathcal{N}(\tau_0, \sigma^2_{\hat{\tau}_0})$, we then have

$$\hat{R} \sim \mathcal{N}\big[\frac{C\tau_0}{2}, (\frac{C}{2}\sigma_{\hat{\sigma}_0})^2\big] = \mathcal{N}\big(\frac{C\tau_0}{2}, \frac{C^2}{4}\sigma^2_{\hat{\sigma}_0}\big) \tag{3}$$

2) In order to within 100 m, we must have

$$Pr\big\{|\hat{R} - \frac{C\tau_0}{2}| < 100\big\} = 0.99 \tag{4}$$

because $(\hat{R} - \frac{C\tau_0}{2}) \sim \mathcal{N}(0, \frac{\sigma^2}{4}\sigma^2_{\hat{\sigma}_0})$, then

$$\frac{\hat{R} - \frac{C\tau_0}{2}}{\frac{C}{2}\sigma_{\hat{\sigma}_0}} \sim \mathcal{N}(0,1) \tag{5}$$

from (4) and (5), we can obtain that

$$Pr\{|\underbrace{\frac{\hat{R}-\frac{C\tau_0}{2}}{\frac{C}{2}\sigma_{\hat{\sigma}_0}}}_{\mathcal{N}(0,1)}|<\frac{100}{\frac{C}{2}\sigma_{\hat{\sigma}_0}}\}=0.99 \tag{6}$$

using MATLAB we then have

```
>> normpdf(0.99, 0, 1)

ans =

    0.2444
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so

$$\frac{100}{\frac{C}{2}\sigma_{\hat{\sigma}_0}}=0.2444 \Rightarrow \hat{\sigma}_0=2.7278\times 10^{-6}m \tag{7}$$

Problem 1.2

An unknown parameter θ influences the outcome of an experiment which is modeled by the random variable x. The PDF of x is

$$p(x;\theta)=\frac{1}{\sqrt{2\pi}}\exp\big[-\frac{1}{2}(x-\theta)^2\big]$$

a series of experiments is performed, and x is found to always be in the interval $[97,103]$. As a result, the investigator concludes that θ must have been 100. Is this assertion correct?

Answer:

No, in fact θ could have been any value. If θ were indeed 100, then

$$p(x;\theta)=\frac{1}{\sqrt{2\pi}}\exp\big[-\frac{1}{2}(x-100)^2\big] \tag{8}$$

and the probability of x always in the interval $[97,103]$ is 0.999. Hence, this assertion is **likely** to be correct. However, we **CANNOT** be certain, since if $\theta=0.99$, then the probability of x always in the interval $[97,103]$ is

$$\int_{97}^{103}\frac{1}{\sqrt{2\pi}}\exp\big[-\frac{1}{2}(x-99)^2\big]dx=\int_{-2}^{4}\frac{1}{\sqrt{2\pi}}\exp\big[-\frac{1}{2}u^2\big]du\approx 0.977 \tag{9}$$

then, $\theta = 99$ is also an possible choice.

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>>  fun = @(x) (1/sqrt(2*pi)) * exp(-0.5*(x.^2));
>> integral(fun, -2, 4)

ans =

    0.9772
```

Problem 1.3

Let $x = \theta + w$, where w is a random variable with PDF $p_w(w)$. If θ is a deterministic parameter, find the PDF of x in terms of p_w and denote it by $p(x;\theta)$. Next assume that θ is a random variable independent of w and find the conditional PDF $p(x|\theta)$. Finally, do not assume that θ and w are independent and determine $p(x|\theta)$. What can you say about $p(x;\theta)$ versus $p(x|\theta)$?

Answer:

1) since $x = \theta + w$, then $w = x - \theta$, we have

$$p(x;\theta) = p_w(x-\theta) \tag{10}$$

2) Assume θ a random variable independent of w

$$p(x|\theta) = \frac{p_{x\theta}(x,\theta)}{p(\theta)} = \frac{p_{w\theta}(x-\theta,\theta)}{p(\theta)} = \frac{p_w(x-\theta)p(\theta)}{p(\theta)} = p_w(x-\theta) \tag{11}$$

which is the same as before.

3) If w and θ are not independent, then

$$p(x|\theta) = \frac{p_{w|\theta}(x-\theta)p(\theta)}{p(\theta)} = p_{w|\theta}(x-\theta) \tag{12}$$

which will be different than $p_w(x-\theta)$

Problem 1.4

It is desired to estimate the value of a DC level A in WGN or

$$x[n] = A + w[n] \quad n = 0, 1, \ldots, N-1$$

where $w[n]$ is zero mean and uncorrelated, and each sample has variance $\sigma^2 = 1$, Consider the two estimators

$$\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

$$\check{A} = \frac{1}{N+2}\left(2x[0] + \sum_{n=1}^{N-2} x[n] + 2x[N-1]\right)$$

which one is better? Does it depend on the value of A?

Answer:

1) We firstly compare the average value of these two estimators

$$E(\hat{A}) = E\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right) = \frac{1}{N}\sum_{n=0}^{N-1} E(x[n]) = A \tag{13}$$

$$\begin{aligned} E(\check{A}) &= E\left\{\frac{1}{N+2}\left(2x[0] + \sum_{n=1}^{N-2} x[n] + 2x[N-1]\right)\right\} \\ &= \frac{1}{N+2}\left\{E(2x[0]) + \sum_{n=1}^{N-2} E(x[n]) + E(2x[N-1])\right\} \\ &= \frac{1}{N+2}\left\{2A + (N-2)A + 2A\right\} = A \end{aligned} \tag{14}$$

2) Then we compare the variance of these two estimators

$$var(\hat{A}) = var\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right) = \frac{1}{N^2}\sum_{n=0}^{N-1} var(x[n]) = \frac{1}{N^2}N\sigma^2 = \frac{\sigma^2}{N} \tag{15}$$

$$\begin{aligned} var(\check{A}) &= var\left\{\frac{1}{N+2}\left(2x[0] + \sum_{n=1}^{N-2} x[n] + 2x[N-1]\right)\right\} \\ &= (\frac{1}{N+2})^2 var\left\{\left(2x[0] + \sum_{n=1}^{N-2} x[n] + 2x[N-1]\right)\right\} \end{aligned}$$

$$= (\frac{1}{N+2})^2(4\sigma^2 + \sum_{n=1}^{N-2}\sigma^2 + 4\sigma^2) = \frac{N+6}{(N+2)^2} \quad \text{when} \quad \sigma^2 = 1 \tag{16}$$

therefore, we have

$$var(\check{A}) - var(\hat{A}) = \frac{2N-4}{N(N+2)^2} > 0 \quad \text{for} \quad N > 2 \tag{17}$$

In conclusion, $var(\check{A})$ and $var(\hat{A})$ have the same average value, but when $N > 2$, the variance of $var(\hat{A})$ is smaller. Besides, the value of A **DOES NOT** affect the estimator.

Problem 1.5

For the same data set as Problem 1.4 the following estimator is proposed

$$\hat{A} = \begin{cases} x[0] & \frac{A^2}{\sigma^2} = A^2 > 1000 \\ \frac{1}{N}\sum_{n=0}^{N-1} x[n] & \frac{A^2}{\sigma^2} = A^2 \leq 1000 \end{cases}$$

the rationale for this estimator is that for a high enough SNR or A^2/σ^2, we do not need to reduce the effect of noise by averaging and hence can avoid the added computation. Comment on this approach.

Answer:

$\hat{A}$ is not an estimator since in order to implement it we have to know SNR firstly. In other words, we have to obtain A.