Estimation Notes 1 (Problems 2.7 - 2.11) pp.24 - 25

Problem 2.7

Two unbiased estimators are proposed whose variances satisfy $var(\hat{\theta}) < var(\check{\theta})$. If both estimators are Gaussian, prove that

$$\Pr\Big\{|\hat{\theta}-\theta| > \epsilon\Big\} < \Pr\Big\{|\check{\theta}-\theta| > \epsilon\Big\}$$

for any $\epsilon > 0$. This says that the estimator with less variance is to be preferred since its PDF is more concentrated about the true value.

Answer:

Since $\hat{\theta} \sim \mathcal{N}(\theta, \sigma^2)$, then

$$\frac{\hat{\theta}-\theta}{\sigma} \sim \mathcal{N}(0,1)$$

$$\Rightarrow \frac{\hat{\theta}-\theta}{\sqrt{var(\hat{\theta})}} \sim \mathcal{N}(0,1)$$

$$\Rightarrow \Pr\Big\{|\hat{\theta}-\theta| > \epsilon\Big\} = \Pr\Bigg\{\Bigg|\frac{\hat{\theta}-\theta}{\sqrt{var(\hat{\theta})}}\Bigg| > \frac{\epsilon}{\sqrt{var(\hat{\theta})}}\Bigg\}$$

Let $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}t^2)dt$ be the CDF for $\mathcal{N}(0,1)$, and $\Phi(x) = \Pr(X \leq x)$. There are two possible situations that satisfied $|\hat{\theta}-\theta| > \epsilon$ when $\epsilon > 0$. The first case is $\hat{\theta}-\theta > \epsilon$ and the second is $\hat{\theta}-\theta < -\epsilon$. We consider $\hat{\theta}-\theta < -\epsilon$ firstly. So we have

$$\Pr\Big\{\hat{\theta}-\theta < -\epsilon\Big\} = \Phi\Bigg(\frac{-\epsilon}{\sqrt{var(\hat{\theta})}}\Bigg)$$

and $\hat{\theta}-\theta > \epsilon$ is the same. Therefore, we have

$$\Pr\left\{|\hat{\theta}-\theta|>\epsilon\right\}=2\Phi\left(\frac{-\epsilon}{\sqrt{var(\hat{\theta})}}\right)$$

since $\Phi(x)$ is monotonically increasing and $var(\hat{\theta})<var(\check{\theta})$, then

$$2\Phi\left(\frac{-\epsilon}{\sqrt{var(\hat{\theta})}}\right)<2\Phi\left(\frac{-\epsilon}{\sqrt{var(\check{\theta})}}\right)$$

finally we have

$$\Pr\left\{|\hat{\theta}-\theta|>\epsilon\right\}<\Pr\left\{|\check{\theta}-\theta|>\epsilon\right\}$$

Problem 2.8

For the problem described in Example 2.1 show that as $N\to\infty$, $\hat{A}\to A$ by using the result of Problem 2.3. To do so prove that

$$\lim_{N\to\infty}\Pr\left\{|\hat{A}-A|>\epsilon\right\}=0$$

any $\epsilon>0$. In this case the estimator $\hat{A}$ is said to be **consistent**. Investigate what happens if the alternative estimator

$$\check{A}=\frac{1}{2N}\sum_{n=0}^{N-1}x[n]$$

is used instead.

Answer:

From problem 2.3, $\hat{A}\sim\mathcal{N}(A,\sigma^2/N)$, and then

$$\Pr\left\{|\hat{A}-A|>\epsilon\right\}=\Pr\left\{|\frac{\hat{A}-A}{\sqrt{\sigma^2/N}}|>\frac{\epsilon}{\sqrt{\sigma^2/N}}\right\}=2\Phi(\frac{-\epsilon}{\sqrt{\sigma^2/N}}\Bigg)\to 0$$

and

$$\frac{-\epsilon}{\sqrt{\sigma^2/N}}\to-\infty\quad\text{as}\quad N\to\infty$$

If $\check{A} = \frac{1}{2N}\sum_{n=0}^{N-1} x[n]$ is used, then

$$\check{A} \sim \mathcal{N}(\frac{A}{2}, \frac{\sigma^2}{4N})$$

as $N \to \infty$, the variance $\to \infty$, and the PDF approaches

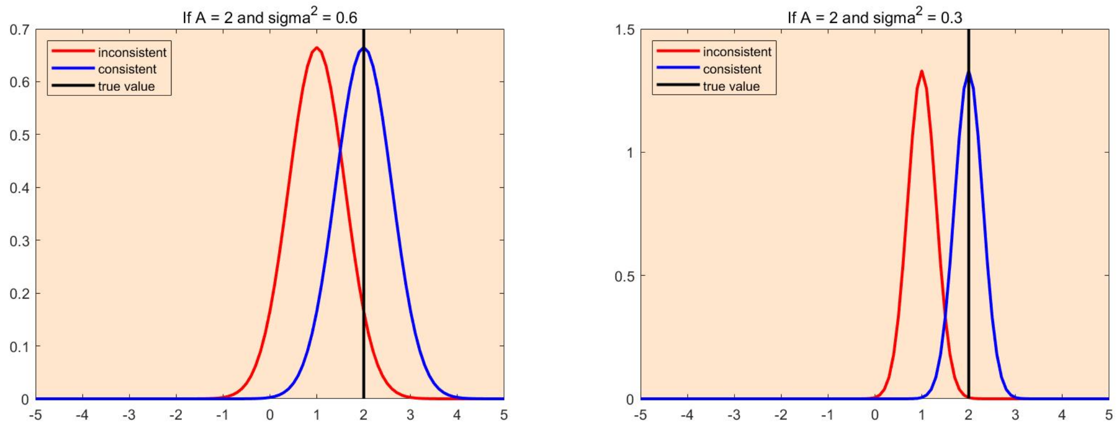

Figure 2.8

thus

$$\Pr\Big\{|\check{A} - A| > \epsilon\Big\} = 1 \quad \text{for any } \epsilon > 0$$

In conclusion, $\hat{A}$ is **consistent** while $\check{A}$ is **inconsistent**.

Problem 2.9

This problem illustrates what happens to an unbiased estimator when it undergoes a nonlinear transformation. In Example 2.1, if we choose to estimate the unknown parameter $\theta = A^2$ by

$$\hat{\theta} = \left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right)^2,$$

can we say that the estimator is unbiased? What happens as $N \to \infty$?

Answer:

We can know that

$$\hat{\theta} = (\hat{A})^2$$

where $\hat{A} \sim \mathcal{N}(A, \sigma^2/N)$, so

$$E(\hat{\theta}) = E[\hat{A}^2] = var(\hat{A}) + E^2[\hat{A}] = \theta + \frac{\sigma^2}{N} \neq \theta$$

therefore, θ is **biased** but as $N \to \infty$, it is **unbiased**.

Problem 2.10

In Example 2.1 assume now that in addition to A, the value of σ^2 is also unknown. We wish to estimate the vector parameter

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} A \\ \sigma^2 \end{bmatrix}$$

Is the estimator

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{A} \\ \hat{\sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{N}\sum_{n=0}^{N-1} x[n] \\ \frac{1}{N-1}\sum_{n=0}^{N-1} \left(x[n] - \hat{A}\right)^2 \end{bmatrix}$$

unbiased?

Answer:

We can know that

$$E(\hat{A}) = A$$

$$E(\sigma^2) = \frac{1}{N-1}\sum_{n=0}^{N-1} E\big\{(x[n] - \hat{A})^2\big\}$$

since $E\big\{(x[n] - \hat{A})^2\big\}$ can be simplified as follows

$$E\Big\{(x[n]-\hat{A})^2\Big\} = E\Big\{(x[n]-\frac{1}{N}\sum_{m=0}^{N-1}x[m])^2\Big\}$$

$$= E\Big\{\big[x[n](1-\frac{1}{N})-\frac{1}{N}\sum_{m=0,m\neq n}^{N-1}x[m]\big]^2\Big\}$$

$$= E\Big\{x^2[n](\frac{N-1}{N})^2 - 2x[n]\frac{N-1}{N}\frac{1}{N}\sum_{m=0,m\neq n}^{N-1}x[m] + \frac{1}{N^2}(\sum_{m=0,m\neq n}^{N-1}x[m])^2\Big\}$$

$$= (\frac{N-1}{N})^2E\Big\{x^2[n]\Big\} - \frac{2(N-1)}{N^2}E\Big\{x[n]\cdot\sum_{m=0,m\neq n}^{N-1}x[m]\Big\} + \frac{1}{N^2}E\Big\{(\sum_{m=0,m\neq n}^{N-1}x[m])^2\Big\}$$

$$= (\frac{N-1}{N})^2(\sigma^2+A^2) - \frac{2(N-1)}{N^2}E\Big\{x[n]\Big\}E\Big\{\sum_{m=0,m\neq n}^{N-1}x[m]\Big\} + \frac{1}{N^2}var(\sum_{m=0,m\neq n}^{N-1}x[m])$$

$$+\frac{1}{N^2}E\Big\{\sum_{m=0,m\neq n}^{N-1}x[m]\Big\}^2$$

$$= (\frac{N-1}{N})^2(\sigma^2+A^2) - \frac{2(N-1)}{N^2}A(N-1)A + \frac{1}{N^2}(N-1)\sigma^2 + \frac{1}{N^2}\big[(N-1)A\big]^2$$

$$= \sigma^2\frac{N-1}{N}$$

therefore

$$E(\sigma^2) = \frac{1}{N-1}\sum_{n=0}^{N-1}E\big\{(x[n]-\hat{A})^2\big\} = \frac{1}{N-1}\sum_{n=0}^{N-1}\sigma^2\frac{N-1}{N} = \sigma^2$$

this estimator is **unbiased**

Problem 2.11

This problem

Answer:

We can know that