Estimation Notes 3.5 - 3.6 pp.35 - 39

Key Points : General CRLB for Signals in White Gaussian Noise; Transformation of Parameters

1. General CRLB for Signals in White Gaussian Noise

1. Assume that a **deterministic signal** with an **unknown** parameter θ is observed in WGN as

$$x[n] = s[n;\theta] + w[n] \quad n = 0, 1, \ldots, N-1$$

then the likelihood function is

$$p(\mathbf{x};\theta) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n] - s[n;\theta])^2 \right\}$$

$$\Rightarrow \frac{\partial \ln p(\mathbf{x};\theta)}{\partial \theta} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} (x[n] - s[n;\theta]) \frac{\partial s[n;\theta]}{\partial \theta}$$

$$\Rightarrow \frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2} = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left\{ \underbrace{(x[n] - s[n;\theta])}_{w[n]} \frac{\partial^2 s[n;\theta]}{\partial \theta^2} - \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2 \right\}$$

$$\Rightarrow E\left\{ \frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2} \right\} = -\frac{1}{\sigma^2} \sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2$$

$$\Rightarrow var(\hat{\theta}) \geq \frac{\sigma^2}{\sum_{n=0}^{N-1} \left(\frac{\partial s[n;\theta]}{\partial \theta}\right)^2} \tag{3.14}$$

2. Transformation of Parameters

2. If we want to estimate $\alpha = g(\theta)$, $g(\cdot)$ is a function of θ, then the CRLB is

$$var(\hat{\alpha}) \geq \frac{(\frac{\partial g}{\partial \theta})^2}{-E\left\{\frac{\partial^2 \ln p(\mathbf{x};\theta)}{\partial \theta^2}\right\}} \tag{3.16}$$

3. If $\alpha = g(\theta) = A^2$ and

$$var(\hat{A}^2) \geq \frac{(2A)^2}{N/\sigma^2} = \frac{4A^2\sigma^2}{N} \tag{3.17}$$

4. in Example 3.3 we know that the sample mean estimator $\bar{x} \sim \mathcal{N}(A, \sigma^2/N)$ was efficient for A, but it is noteworthy that $\bar{x}^2$ is not even an unbiased estimator, because

$$E(\bar{x}^2) = E^2(\bar{x}) + var(\bar{x}) = A^2 + \frac{\sigma^2}{N} \neq A^2 \tag{3.18}$$

so we can conclude that **the efficiency of an estimator is destroyed by a nonlinear transformation**.

5. Let's suppose that an efficient for θ exists and it given by $\hat{\theta}$. It is desired to estimate $g(\theta) = a\theta + b$. We choose $\widehat{g(\theta)} = g(\hat{\theta}) = a\hat{\theta} + b$ as our estimator of $g(\theta)$. Then,

$$E(\widehat{g(\theta)}) = E(g(\hat{\theta})) = E(a\hat{\theta} + b) = aE(\hat{\theta}) + b = g(\theta)$$

so that $\widehat{g(\theta)}$ is unbiased. The CRLB for $g(\theta)$ is

$$var(\widehat{g(\theta)}) \geq \frac{(\frac{\partial g}{\partial \theta})^2}{I(\theta)} = (\frac{\partial g}{\partial \theta})^2 var(\hat{\theta}) = a^2 var(\hat{\theta})$$

since $var(\widehat{g(\theta)}) = var(a\hat{\theta} + b) = a^2 var(\hat{\theta})$, so that the CRLB is achieved.

6. Although efficiency is preserved only over linear transformations, it is **approximately** maintained over nonlinear transformations **if the data record is large enough**.

7. Let's return to the previous example of estimating A^2 by $\bar{x}^2$. From (3.18) we know that $\bar{x}^2$ is **asymptotically** unbiased (i.e., unbiased as $N \to \infty$). since $\bar{x} \sim \mathcal{N}(A, \sigma^2/N)$, we have

$$var(\bar{x}^2) = E(\bar{x}^4) - E^2(\bar{x}^2)$$

by using the result that if $k \sim \mathcal{N}(\mu, \sigma^2)$ then $E(k^2) = \mu^2 + \sigma^2$ and $E(k^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$, and therefore $var(k^2) = E(k^4) - E^2(k^2) = 4\mu^2\sigma^2 + 2\sigma^4$.

So we have

$$var(\bar{x}^2) = E(\bar{x}^4) - E^2(\bar{x}^2) = \frac{4A^2\sigma^2}{N} + \frac{2\sigma^4}{N^2} \tag{3.19}$$

Thus, as $N \to \infty$, the variance approaches $4A^2\sigma^2/N$.

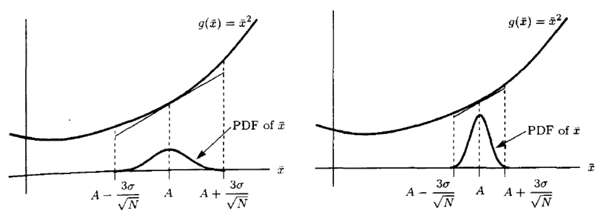

Figure 3.4 Statistical linearity of nonlinear transformations.

8. This situation occurs due to the **statistical linearity** of the transformation, as Figure 3.4 shows. As N increases, the PDF of $\bar{x}$ becomes **more concentrated** about the mean A. So the values of $\bar{x}$ that are observed lie in a small interval about $\bar{x} = A$. **Over this small interval the nonlinear transformation is approximately linear**.