

Massive Notes 3.2 pp.248 - 253

Key Points : MMSE Channel Estimation; Pilot Contamination

1. MMSE Channel Estimation

1. Recall (3.1) as follows

$$\mathbf{Y}_j^p = \underbrace{\sum_{k=1}^{K_j} \sqrt{p_{jk}} \mathbf{h}_{jk}^j \phi_{jk}^T}_{\text{Desired Pilots}} + \underbrace{\sum_{l=1, l \neq j}^L \sum_{i=1}^{K_l} \sqrt{p_{li}} \mathbf{h}_{li}^j \phi_{li}^T}_{\text{Inter-cell Pilots}} + \underbrace{\mathbf{N}_j^p}_{\text{Noise}} \quad (3.1)$$

We now derive an estimator of the channel response \mathbf{h}_{li}^j , based on the received pilot signal \mathbf{Y}_j^p in (3.1) and a pilot book with **mutually orthogonal** sequences.

2. The channel is a **realization of a random variable**, thus **Bayesian estimators** are desirable since they **take the statistical distributions of the variables into accounts**. Bayesian estimators require that the distributions are **known**. From (2.1) we know that $\mathbf{h}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{li}^j)$.

3. The **minimum mean-squared error (MMSE) estimator** of \mathbf{h}_{li}^j is the **vector** $\hat{\mathbf{h}}_{li}^j$ that **minimizes** the MSE

$$\mathbb{E}\{\|\mathbf{h}_{li}^j - \hat{\mathbf{h}}_{li}^j\|^2\}$$

4. **Theorem 3.1.** Using a pilot book with mutually orthogonal sequences, the MMSE estimate of the channel \mathbf{h}_{li}^j based on the observation \mathbf{Y}_j^p in (3.1)

$$\begin{aligned} \mathbf{Y}_j^p &= \underbrace{\sum_{l=1}^L \sum_{i=1}^{K_l} \sqrt{p_{li}} \mathbf{h}_{li}^j \phi_{li}^T}_{\text{Pilots}} + \underbrace{\mathbf{N}_j^p}_{\text{Noise}} \\ \Rightarrow \mathbf{y}_{jli}^p &= \mathbf{Y}_j^p \phi_{li}^* = \sqrt{p_{li}} \tau_p \mathbf{h}_{li}^j + \mathbf{n}_j^p \phi_{li}^* \end{aligned} \quad (3.1)$$

is

$$\hat{\mathbf{h}}_{li}^j = \sqrt{p_{li}} \mathbf{R}_{li}^j \Psi_{li}^j \mathbf{y}_{jli}^p \quad (3.9)$$

where

$$\Psi_{li}^j = \left(\sum_{(l', i') \in \mathcal{P}_{li}} p_{l'i'} \tau_p \mathbf{R}_{l'i'}^j + \sigma_{\text{UL}}^2 \mathbf{I}_{M_j} \right)^{-1} \quad (3.10)$$

and

$$\mathbf{h}_{lk}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{lk}^j) \quad (2.1)$$

5. The estimation error

$$\tilde{\mathbf{h}}_{li}^j = \mathbf{h}_{li}^j - \hat{\mathbf{h}}_{li}^j$$

has **Correlation Matrix** given by

$$\mathbf{C}_{li}^j = \mathbb{E}\{\tilde{\mathbf{h}}_{li}^j (\tilde{\mathbf{h}}_{li}^j)^H\} = \mathbf{R}_{li}^j - p_{li} \tau_p \mathbf{R}_{li}^j \Psi_{li}^j \mathbf{R}_{li}^j \quad (3.11)$$

recall that the estimation quality is represented by the MSE as follows

$$\mathbb{E}\{\|\mathbf{h}_{li}^j - \hat{\mathbf{h}}_{li}^j\|^2\} = \text{tr}(\mathbf{C}_{li}^j)$$

6. We define the **effective SNR** during pilot signaling from UE k in cell j to its serving BS j as

$$\text{SNR}_{jk}^p = \frac{p_{jk} \tau_p \beta_{jk}^j}{\sigma_{\text{UL}}^2} \quad (3.13)$$

where we recall that $\beta_{jk}^j = \frac{1}{M_j} \text{tr}(\mathbf{R}_{jk}^j)$ was defined in (2.2) as the **average channel gain** to the antennas in the BS array.

7. The **effective SNR** implies that the **Pilot Processing Gain** τ_p is included in the SNR. The processing gain is obtained from the fact that **the pilot sequence spans τ_p samples**. For example, if the pilot sequence are 10 samples long, then the effective SNR is 10 dB larger than the nominal SNR at a single sample.

2. Pilot Contamination

8. Recall that

$$\hat{\mathbf{h}}_{li}^j = \sqrt{p_{li}} \mathbf{R}_{li}^j \boldsymbol{\Psi}_{li}^j \mathbf{y}_{jli}^p \quad (3.9)$$

$$\tilde{\mathbf{h}}_{li}^j = \mathbf{h}_{li}^j - \hat{\mathbf{h}}_{li}^j$$

$$\mathbf{h}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{li}^j) \quad (2.1)$$

then we have

$$\hat{\mathbf{h}}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{li}^j - \mathbf{C}_{li}^j) \quad (3.14)$$

$$\tilde{\mathbf{h}}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{C}_{li}^j) \quad (3.15)$$

9. In practice, Theorem 3.1 is particularly important for **estimating the intra-cell channels**. However, also the **inter-cell channels** from any UE in the entire network to BS j can be estimated.

10. An important observation can be made by comparing the MMSE estimate in (3.9)

$$\hat{\mathbf{h}}_{li}^j = \sqrt{p_{li}} \mathbf{R}_{li}^j \boldsymbol{\Psi}_{li}^j \mathbf{y}_{jli}^p \quad (3.9)$$

of an **intra-cell channel** $\hat{\mathbf{h}}_{jk}^j$ with the estimate $\hat{\mathbf{h}}_{li}^j$ of a UE in **another cell that utilizes the same pilot sequence**, i.e.,

$$(l, i) \in \mathcal{P}_{jk} \quad \text{which implies} \quad \phi_{li} = \phi_{jk} \quad \text{and} \quad \mathcal{P}_{li} = \mathcal{P}_{jk}$$

in this case, we then have

$$\boldsymbol{\Psi}_{jk}^j = \boldsymbol{\Psi}_{li}^j = \left(\sum_{(l', i') \in \mathcal{P}_{li}} p_{l'i'} \tau_p \mathbf{R}_{l'i'}^j + \sigma_{\text{UL}}^2 \mathbf{I}_{M_j} \right)^{-1} \quad 3.10$$

and

$$\mathbf{y}_{jjk}^p = \mathbf{y}_{jli}^p = \mathbf{Y}_j^p \phi_{li}^* = \sqrt{p_{li}} \tau_p \mathbf{h}_{li}^j + \mathbf{n}_j^p \phi_{li}^*$$

thus the same matrix inverse is multiplied with the same processed received signal. It is only **the scalar** $\sqrt{p_{li}}$ and **the first matrix** \mathbf{R}_{li}^j in (3.9) that are different, as shown below

$$\hat{\mathbf{h}}_{li}^j = \underbrace{\sqrt{p_{li}}\mathbf{R}_{li}^j}_{\text{different from } \hat{\mathbf{h}}_{jk}^j} \underbrace{\Psi_{li}^j \mathbf{y}_{jli}^p}_{\text{the same as } \hat{\mathbf{h}}_{jk}^j} \quad (3.9)$$

$$\begin{aligned} \Rightarrow \frac{\hat{\mathbf{h}}_{li}^j}{\hat{\mathbf{h}}_{jk}^j} &= \frac{\sqrt{p_{li}}\mathbf{R}_{li}^j}{\sqrt{p_{jk}}\mathbf{R}_{jk}^j} \\ \Rightarrow \hat{\mathbf{h}}_{li}^j &= \frac{\sqrt{p_{li}}}{\sqrt{p_{jk}}} \mathbf{R}_{li}^j (\mathbf{R}_{jk}^j)^{-1} \hat{\mathbf{h}}_{jk}^j \end{aligned} \quad (3.16)$$

this implies that the two estimates are **strongly correlated**.

11. However, it is noteworthy that $\hat{\mathbf{h}}_{li}^j$ and $\hat{\mathbf{h}}_{jk}^j$ are **linearly independent** (i.e., non-parallel) since

$$\Rightarrow \hat{\mathbf{h}}_{li}^j = \frac{\sqrt{p_{li}}}{\sqrt{p_{jk}}} \underbrace{\mathbf{R}_{li}^j (\mathbf{R}_{jk}^j)^{-1}}_{\text{NOT a scalar}} \hat{\mathbf{h}}_{jk}^j \quad (3.16)$$

unless $\mathbf{R}_{li}^j (\mathbf{R}_{jk}^j)^{-1}$ is a scalar. In the special case of **Spatially Uncorrelated Channels** with

$$\begin{aligned} \mathbf{R}_{li}^j &= \beta_{li}^j \mathbf{I}_{M_j} \\ \mathbf{R}_{jk}^j &= \beta_{jk}^j \mathbf{I}_{M_j} \end{aligned}$$

the two channel estimates are **Parallel Vectors** that **only differ in scaling**. This is an **unwanted property** caused by the **inability** of BS j to separate UEs that have **transmitted the same pilot sequence** and have **the same spatial characteristics**.

12. As Figure 3.1, when two UEs transmit the same pilot sequence, their respective BSs receive a **superposition** of their signals—they **contaminate** each others' pilot transmissions. Since it is challenging for the BSs to separate the UEs, the estimates of their respective channels will be correlated.

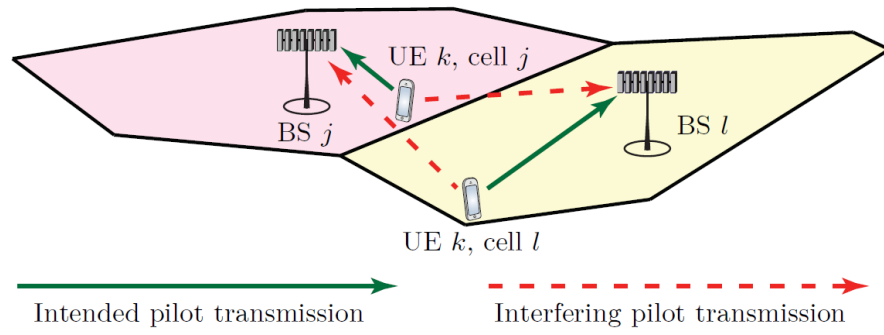


Figure 3.1 Pilot Contamination.