Massive Notes 3.2 pp.248 - 253

Key Points : MMSE Channel Estimation; Pilot Contamination

1. MMSE Channel Estimation

1. Recall (3.1) as follows

$$\mathbf{Y}_j^p = \underbrace{\sum_{k=1}^{K_j} \sqrt{p_{jk}} \mathbf{h}_{jk}^j \phi_{jk}^{\mathrm{T}}}_{\text{Desired Pilots}} + \underbrace{\sum_{l=1, l \neq j}^{L} \sum_{i=1}^{K_l} \sqrt{p_{li}} \mathbf{h}_{li}^j \phi_{li}^{\mathrm{T}}}_{\text{Inter-cell Pilots}} + \underbrace{\mathbf{N}_j^p}_{\text{Noise}} \tag{3.1}$$

We now derive an estimator of the channel response $\mathbf{h}_{li}^j$, based on the received pilot signal $\mathbf{Y}_j^p$ in (3.1) and a pilot book with **mutually orthogonal** sequences.

2. The channel is a **realization of a random variable**, thus **Bayesian estimators** are desirable since they **take the statistical distributions of the variables into accounts**. Bayesian estimators require that the distributions are **known**. From (2.1) we know that $\mathbf{h}_{li}^j \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{li}^j)$.

3. The **minimum mean-squared error (MMSE) estimator** of $\mathbf{h}_{li}^j$ is the **vector** $\hat{\mathbf{h}}_{li}^j$ that **minimizes** the MSE

$$\mathbb{E}\{\|\mathbf{h}_{li}^j - \hat{\mathbf{h}}_{li}^j\|^2\}$$

4. **Theorem 3.1**. Using a pilot book with mutually orthogonal sequences, the MMSE estimate of the channel $\mathbf{h}_{li}^j$ based on the observation $\mathbf{Y}_j^p$ in (3.1)

$$\mathbf{Y}_j^p = \underbrace{\sum_{l=1}^{L} \sum_{i=1}^{K_l} \sqrt{p_{li}} \mathbf{h}_{li}^j \phi_{li}^{\mathrm{T}}}_{\text{Pilots}} + \underbrace{\mathbf{N}_j^p}_{\text{Noise}} \tag{3.1}$$

$$\Rightarrow \mathbf{y}_{jli}^p = \mathbf{Y}_j^p \phi_{li}^* = \sqrt{p_{li}} \tau_p \mathbf{h}_{li}^j + \mathbf{n}_j^p \phi_{li}^*$$

is

$$\hat{\mathbf{h}}_{li}^{j} = \sqrt{p_{li}} \mathbf{R}_{li}^{j} \mathbf{\Psi}_{li}^{j} \mathbf{y}_{jli}^{p} \tag{3.9}$$

where

$$\mathbf{\Psi}_{li}^{j} = \left(\sum_{(l',i') \in \mathcal{P}_{li}} p_{l'i'} \tau_p \mathbf{R}_{l'i'}^{j} + \sigma_{\mathrm{UL}}^2 \mathbf{I}_{M_j} \right)^{-1} \tag{3.10}$$

and

$$\mathbf{h}_{lk}^{j} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{lk}^{j}) \tag{2.1}$$

5. The **estimation error**

$$\tilde{\mathbf{h}}_{li}^{j} = \mathbf{h}_{li}^{j} - \hat{\mathbf{h}}_{li}^{j}$$

has **Correlation Matrix** given by

$$\mathbf{C}_{li}^{j} = \mathbb{E}\{\tilde{\mathbf{h}}_{li}^{j} (\tilde{\mathbf{h}}_{li}^{j})^{\mathrm{H}}\} = \mathbf{R}_{li}^{j} - p_{li} \tau_p \mathbf{R}_{li}^{j} \mathbf{\Psi}_{li}^{j} \mathbf{R}_{li}^{j} \tag{3.11}$$

recall that the estimation quality is represented by the MSE as follows

$$\mathrm{MSE}_{li}^{j} = \mathbb{E}\{\|\mathbf{h}_{li}^{j} - \hat{\mathbf{h}}_{li}^{j}\|^2\} = \mathrm{tr}(\mathbf{C}_{li}^{j})$$

to compare the estimation quality obtained with different estimation schemes in different scenarios, the **normalized MSE (NMSE)** defined as

$$\mathrm{NMSE}_{li}^{j} = \frac{\mathrm{tr}(\mathbf{C}_{li}^{j})}{\mathrm{tr}(\mathbf{R}_{li}^{j})} \tag{3.20}$$

6. We define the **effective SNR** during pilot signaling from UE k in cell j to its serving BS j as

$$\mathrm{SNR}_{jk}^{p} = \frac{p_{jk} \tau_p \beta_{jk}^{j}}{\sigma_{\mathrm{UL}}^2} \tag{3.13}$$

where we recall that $\beta_{jk}^{j} = \frac{1}{M_j} \mathrm{tr}(\mathbf{R}_{jk}^{j})$ was defined in (2.2) as the **average channel gain** to the antennas in the BS array.

7. The **effective SNR** implies that the **Pilot Processing Gain** τ_p is included in the SNR. The processing gain is obtained from the fact that **the pilot sequence spans** τ_p **samples**. For example, if the pilot sequence are 10 samples long, then the effective SNR is 10 dB larger than the nominal SNR at a single sample.

2. Pilot Contamination

8. Recall that

$$\hat{\mathbf{h}}_{li}^{j} = \sqrt{p_{li}}\mathbf{R}_{li}^{j}\mathbf{\Psi}_{li}^{j}\mathbf{y}_{jli}^{p} \tag{3.9}$$

$$\tilde{\mathbf{h}}_{li}^{j} = \mathbf{h}_{li}^{j} - \hat{\mathbf{h}}_{li}^{j}$$

$$\mathbf{h}_{li}^{j} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{li}^{j}) \tag{2.1}$$

then we have

$$\hat{\mathbf{h}}_{li}^{j} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{R}_{li}^{j} - \mathbf{C}_{li}^{j}) \tag{3.14}$$

$$\tilde{\mathbf{h}}_{li}^{j} \sim \mathcal{N}_{\mathbb{C}}(\mathbf{0}_{M_j}, \mathbf{C}_{li}^{j}) \tag{3.15}$$

9. In practice, Theorem 3.1 is particularly important for **estimating the intra-cell channels**. However, also the **inter-cell channels** from any UE in the entire network to BS j can be estimated.

10. An important observation can be made by comparing the MMSE estimate in (3.9)

$$\hat{\mathbf{h}}_{li}^{j} = \sqrt{p_{li}}\mathbf{R}_{li}^{j}\mathbf{\Psi}_{li}^{j}\mathbf{y}_{jli}^{p} \tag{3.9}$$

of an **intra-cell channel** $\hat{\mathbf{h}}_{jk}^{j}$ with the estimate $\hat{\mathbf{h}}_{li}^{j}$ of a UE in **another cell that utilizes the same pilot sequence**, i.e.,

$$(l, i) \in \mathcal{P}_{jk} \quad \text{which implies} \quad \phi_{li} = \phi_{jk} \quad \text{and} \quad \mathcal{P}_{li} = \mathcal{P}_{jk}$$

in this case, we then have

$$\mathbf{\Psi}_{jk}^{j} = \mathbf{\Psi}_{li}^{j} = \left(\sum_{(l',i') \in \mathcal{P}_{li}} p_{l'i'}\tau_p \mathbf{R}_{l'i'}^{j} + \sigma_{\mathrm{UL}}^{2}\mathbf{I}_{M_j} \right)^{-1} \qquad 3.10$$

and

$$\mathbf{y}_{jjk}^{p} = \mathbf{y}_{jli}^{p} = \mathbf{Y}_{j}^{p}\phi_{li}^{*} = \sqrt{p_{li}}\tau_{p}\mathbf{h}_{li}^{j} + \mathbf{n}_{j}^{p}\phi_{li}^{*}$$

thus the same matrix inverse is multiplied with the same processed received signal. It is only **the scalar** $\sqrt{p_{li}}$ and **the first matrix** $\mathbf{R}_{li}^{j}$ in (3.9) that are different, as shown below

$$\hat{\mathbf{h}}_{li}^{j} = \underbrace{\sqrt{p_{li}}\mathbf{R}_{li}^{j}}_{\text{different from } \hat{\mathbf{h}}_{jk}^{j}} \overbrace{\mathbf{\Psi}_{li}^{j}\mathbf{y}_{jli}^{p}}^{\text{the same as } \hat{\mathbf{h}}_{jk}^{j}} \tag{3.9}$$

$$\Rightarrow \frac{\hat{\mathbf{h}}_{li}^{j}}{\hat{\mathbf{h}}_{jk}^{j}} = \frac{\sqrt{p_{li}}\mathbf{R}_{li}^{j}}{\sqrt{p_{jk}}\mathbf{R}_{jk}^{j}}$$

$$\Rightarrow \hat{\mathbf{h}}_{li}^{j} = \frac{\sqrt{p_{li}}}{\sqrt{p_{jk}}}\mathbf{R}_{li}^{j}(\mathbf{R}_{jk}^{j})^{-1}\hat{\mathbf{h}}_{jk}^{j} \tag{3.16}$$

this implies that the two estimates are **strongly correlated**.

11. However, it is noteworthy that $\hat{\mathbf{h}}_{li}^{j}$ and $\hat{\mathbf{h}}_{jk}^{j}$ are **linearly independent** (i.e., non-parallel) since

$$\Rightarrow \hat{\mathbf{h}}_{li}^{j} = \frac{\sqrt{p_{li}}}{\sqrt{p_{jk}}}\underbrace{\mathbf{R}_{li}^{j}(\mathbf{R}_{jk}^{j})^{-1}}_{\text{NOT a scalar}}\hat{\mathbf{h}}_{jk}^{j} \tag{3.16}$$

unless $\mathbf{R}_{li}^{j}(\mathbf{R}_{jk}^{j})^{-1}$ is a scalar. In the special case of **Spatially Uncorrelated Channels** with

$$\mathbf{R}_{li}^{j} = \beta_{li}^{j}\mathbf{I}_{M_j}$$
$$\mathbf{R}_{jk}^{j} = \beta_{jk}^{j}\mathbf{I}_{M_j}$$

the two channel estimates are **Parallel Vectors** that **only differ in scaling**. This is an **unwanted property** caused by the **inability** of BS j to separate UEs that have **transmitted the same pilot sequence** and have **the same spatial characteristics**.

12. As Figure 3.1, when two UEs transmit the same pilot sequence, their respective BSs receive a **superposition** of their signals—they **contaminate** each others' pilot transmissions. Since it is challenging for the BSs to separate the UEs, the estimates of their respective channels will be correlated.

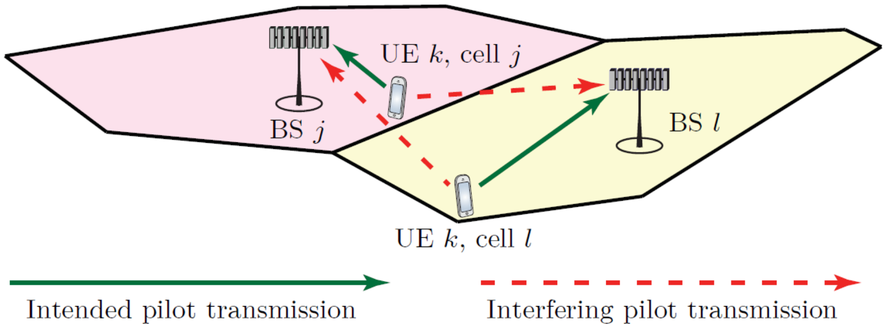

Figure 3.1 Pilot Contamination.