KU LEUVEN kulak

A definition of scale-independent communities & a flow-free vertex cut algorithm

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What is the subject?

We collect a lot of data...

We collect a lot of data...

- social networks

Image source: Paul Butler, facebook intern, 14 december 2010

We collect a lot of data...

- social networks
- street maps

Image source: ©2018 Google, Inst. Geogr. Nacional

We collect a lot of data...

- social networks
- street maps
- chemical interactions

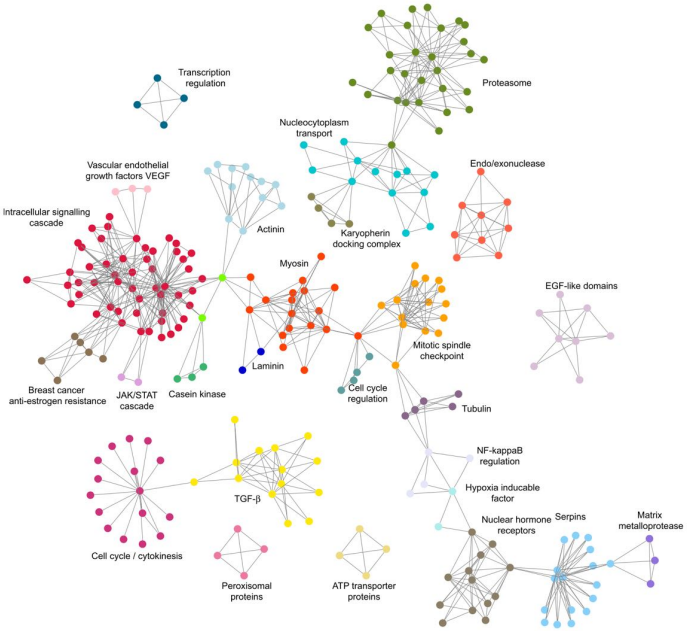

Image source: S. Fortunato, Community detection in graphs

We collect a lot of data...

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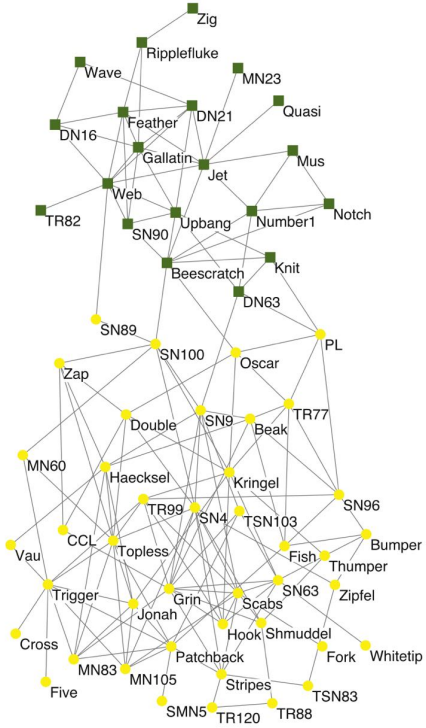
- animal behaviour



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We collect a lot of data...

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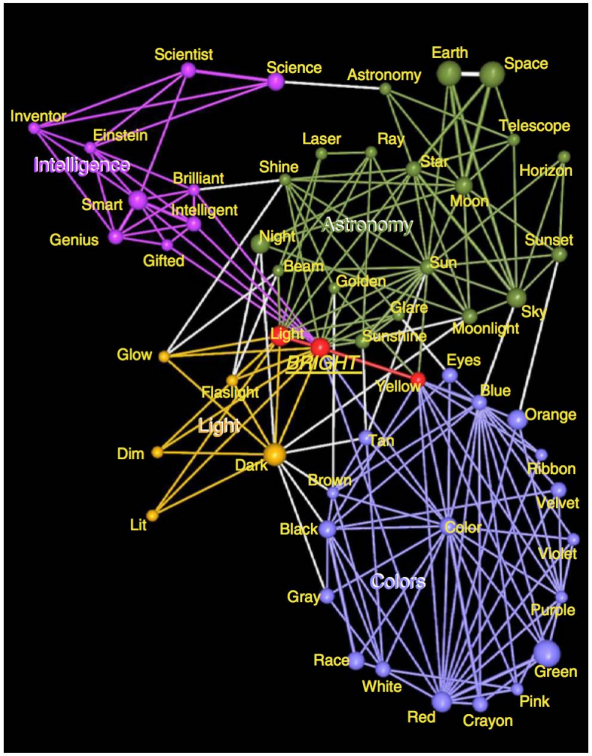
- chemical interactions
- animal behaviour
- word associations



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We collect a lot of data...

- social networks
- street maps
- chemical interactions
- animal behaviour
- word associations
- and many more ...

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... which we want to model.

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- social networks → social groups
- street maps
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- animal behaviour
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We collect a lot of data...

- social networks → social groups
- street maps → cities, villages
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- animal behaviour
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- and many more ...

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We collect a lot of data...

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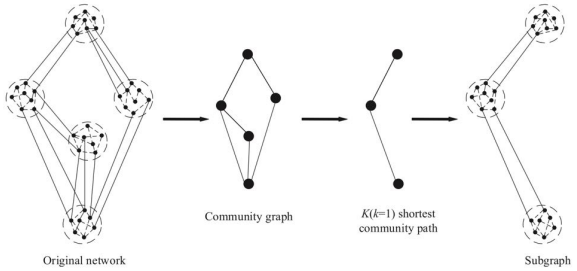

Image source: M. Gong et al., 2016

... which we want to model.

We collect a lot of data...

- social networks → social groups
- street maps → cities, villages
- chemical interactions → interacting molecules
- animal behaviour
- word associations
- and many more ...

... which we want to model.

We collect a lot of data...

- social networks → social groups
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- chemical interactions → interacting molecules
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- word associations
- and many more ...

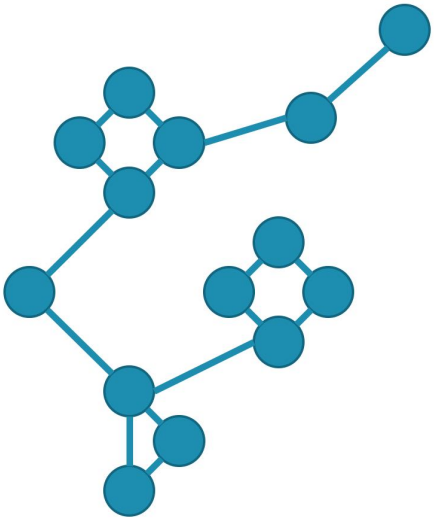
... which we want to model.

We collect a lot of data...

- social networks → social groups
- street maps → cities, villages
- chemical interactions → interacting molecules
- animal behaviour → communities
- word associations → concepts
- and many more ...

... which we want to model.

Problem of “community detection”



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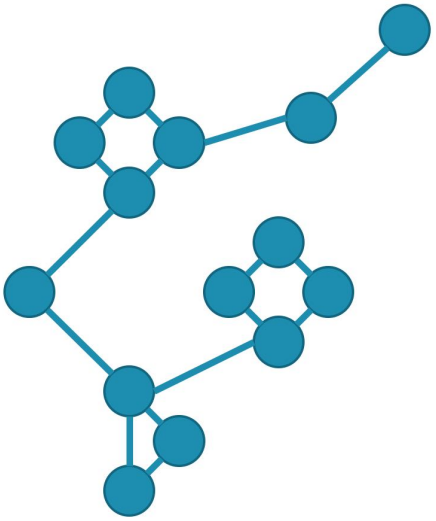
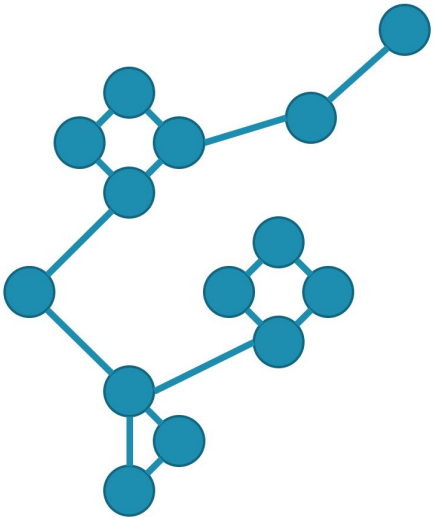


The data (input)

Problem of “community detection”



The data (input)

- graph = network = data

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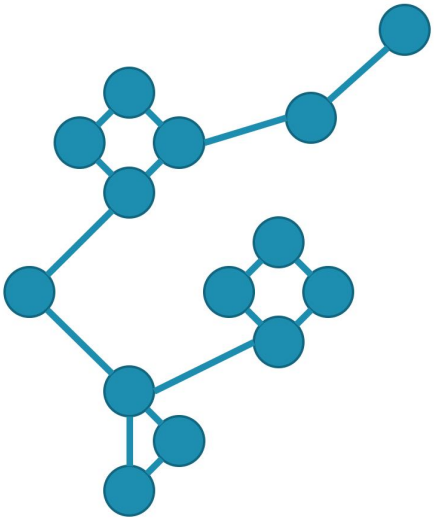

The data (input)

- graph = network = data
- vertex = node = object

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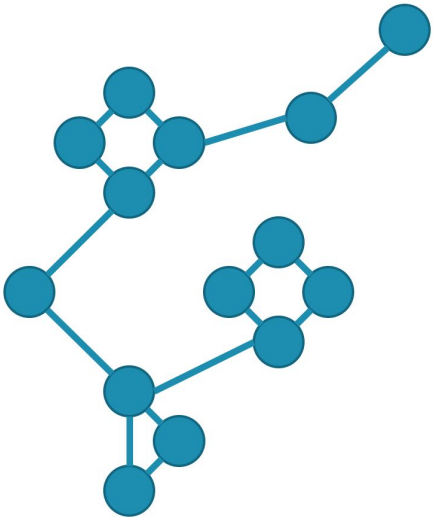

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- edge = arc = relation

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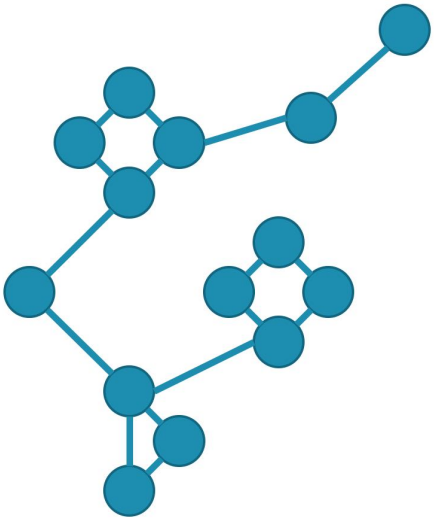

The data (input)

- graph = network = data
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- undirected

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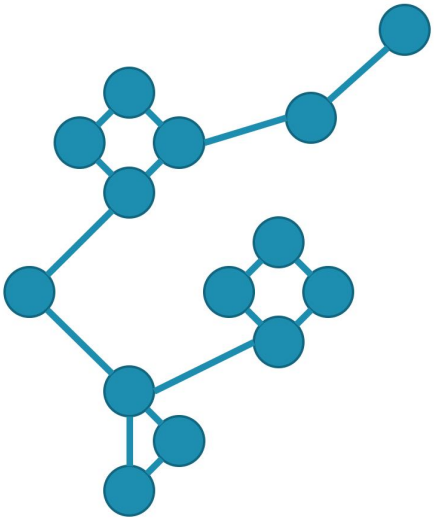

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- undirected
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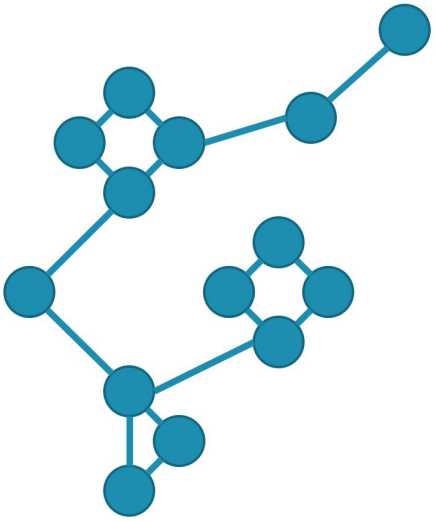

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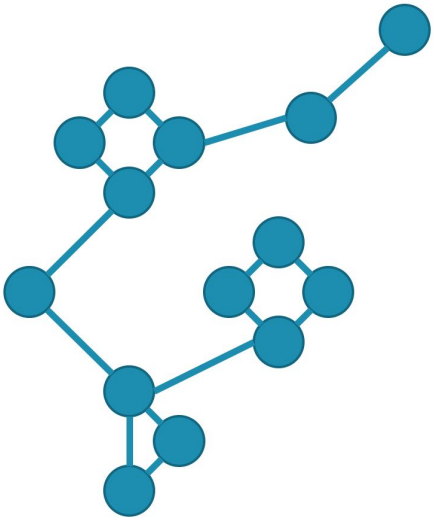

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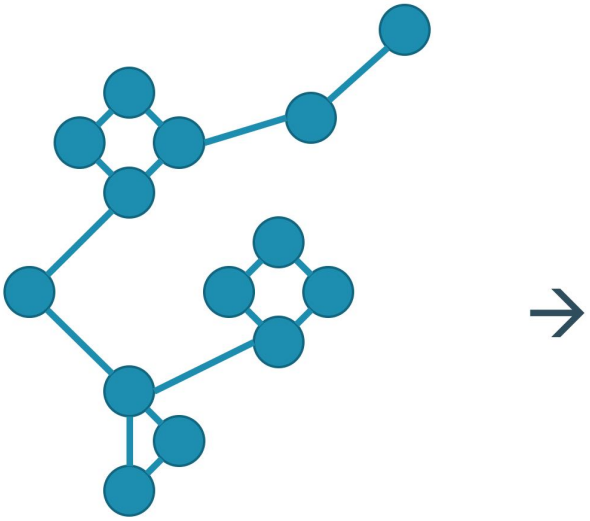

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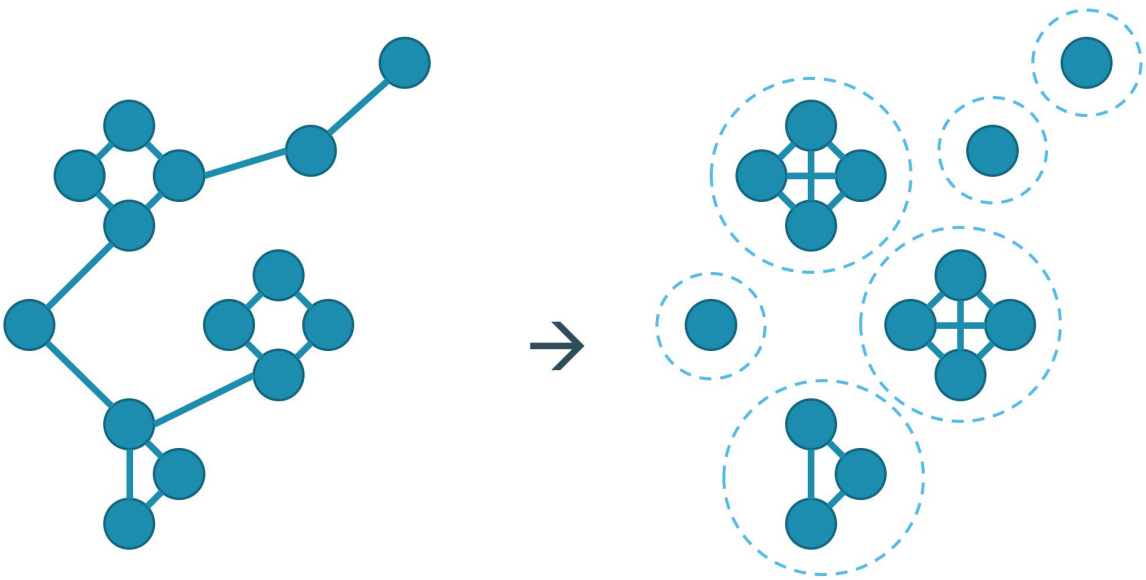

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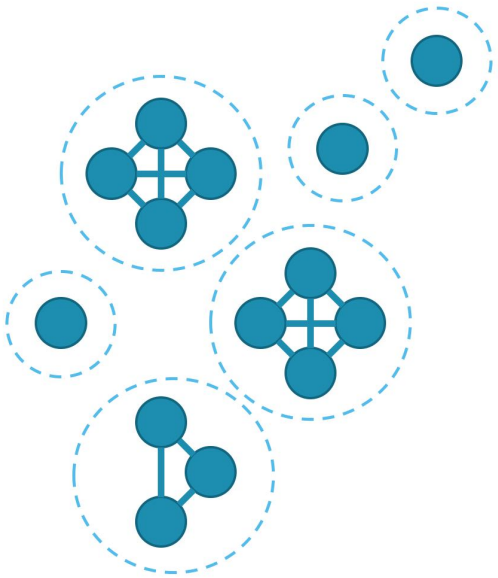
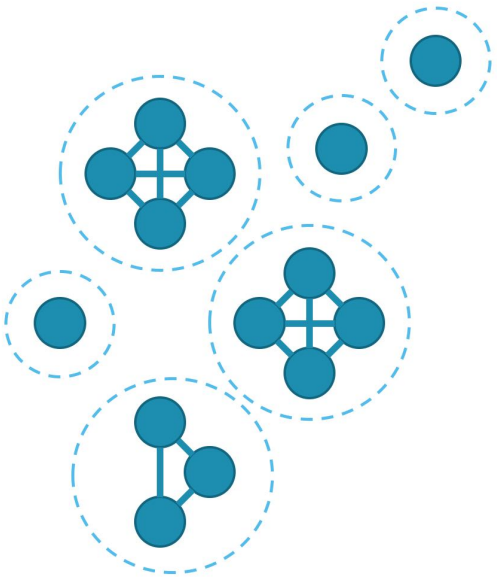


Problem of “community detection”

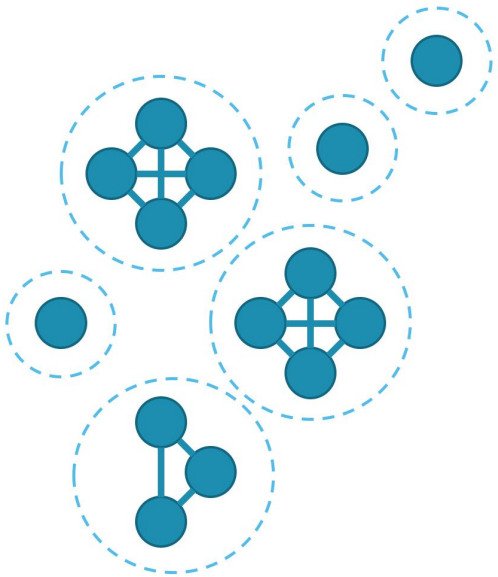
The result (output)



Problem of “community detection”

The result (output)

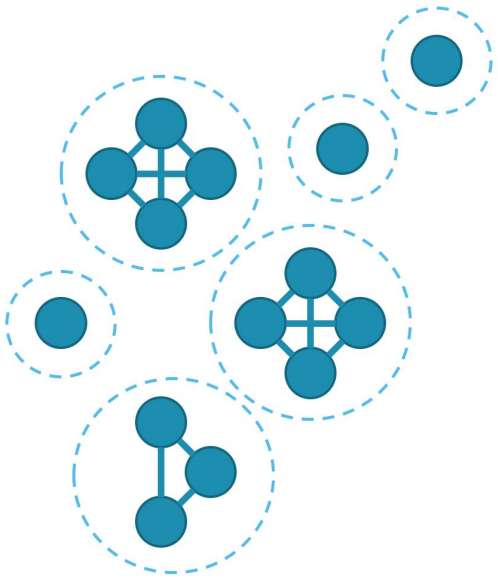
- “communities”



Problem of “community detection”

The result (output)

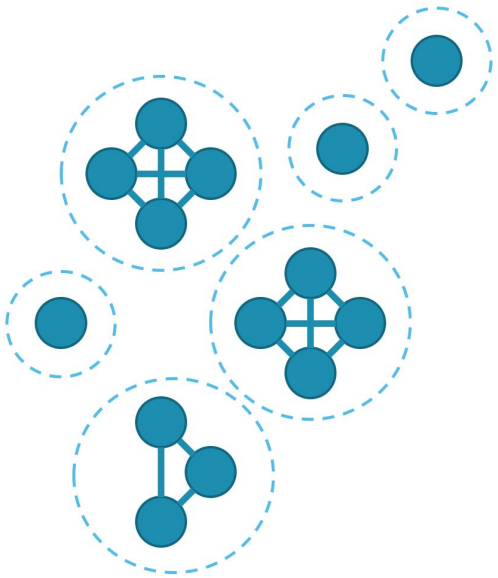
- “communities”
- equivalence relation



Problem of “community detection”

The result (output)

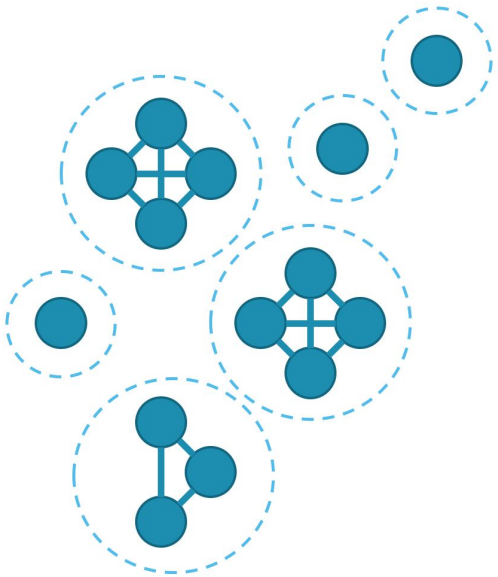
- “communities”
- equivalence relation
- sets of vertices



Problem of “community detection”

The result (output)

- “communities”
- equivalence relation
- sets of vertices
- possibly overlapping



Problem of “community detection”

The result (output)

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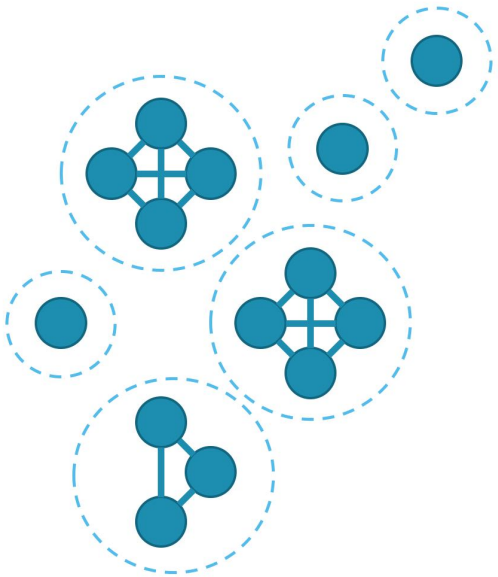

“strongly connected individuals with few outgoing connections”

Problem of “community detection”

The result (output)

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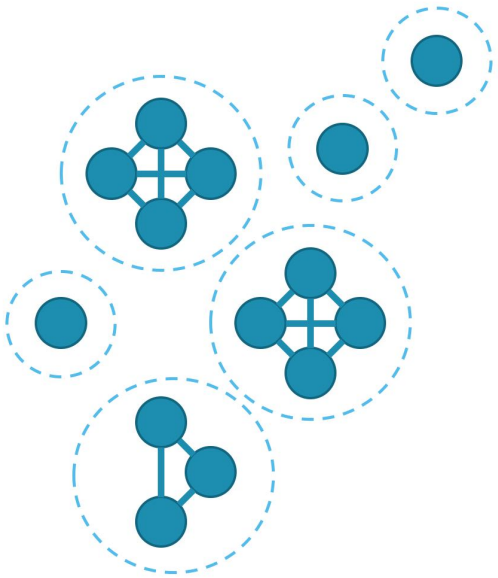
- equivalence relation
- sets of vertices
- possibly overlapping

“strongly connected individuals with few outgoing connections”



What are the research questions?

Research question 1

What is a good definition for a community such that it's properties correspond to the intuitive concept of "strongly connected individuals with few outgoing connections"?

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Research question 2

How does an algorithm for finding this definition scale with respect to the data size?

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Research question 1:

definition of community

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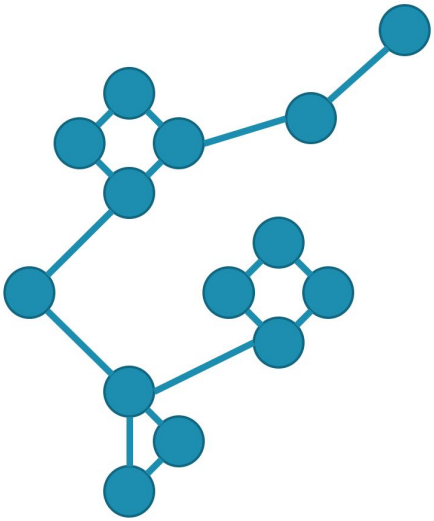
=

measurement of connectedness

definition of community

=

measurement of connectedness



definition of community

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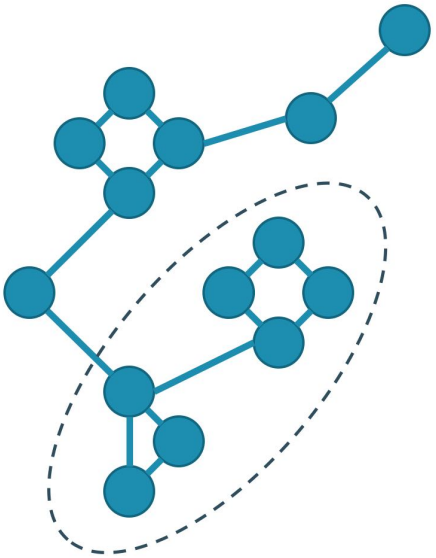
measurement of connectedness



definition of community

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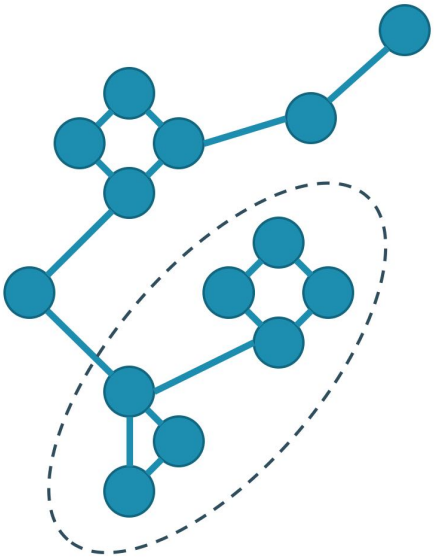

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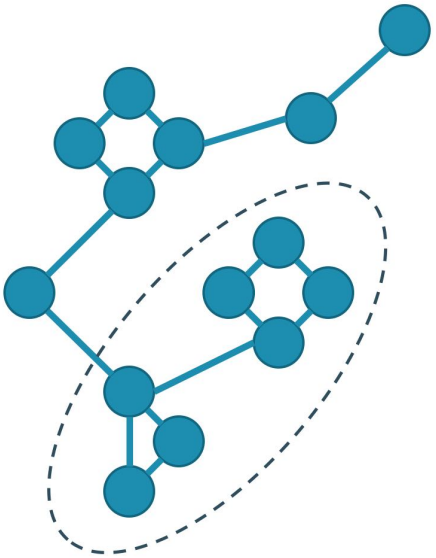


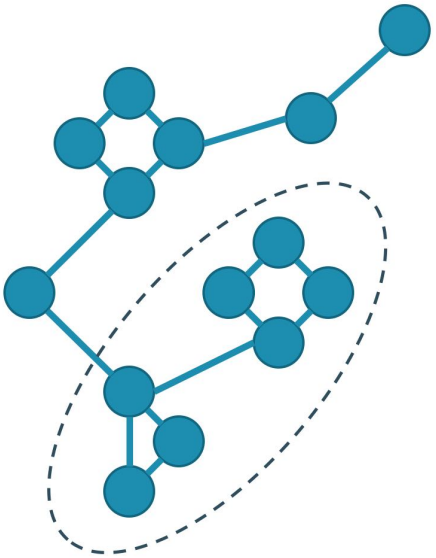
→

7
6
5
4
3

definition of community

=

measurement of connectedness



7
6
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The perfect community: clique

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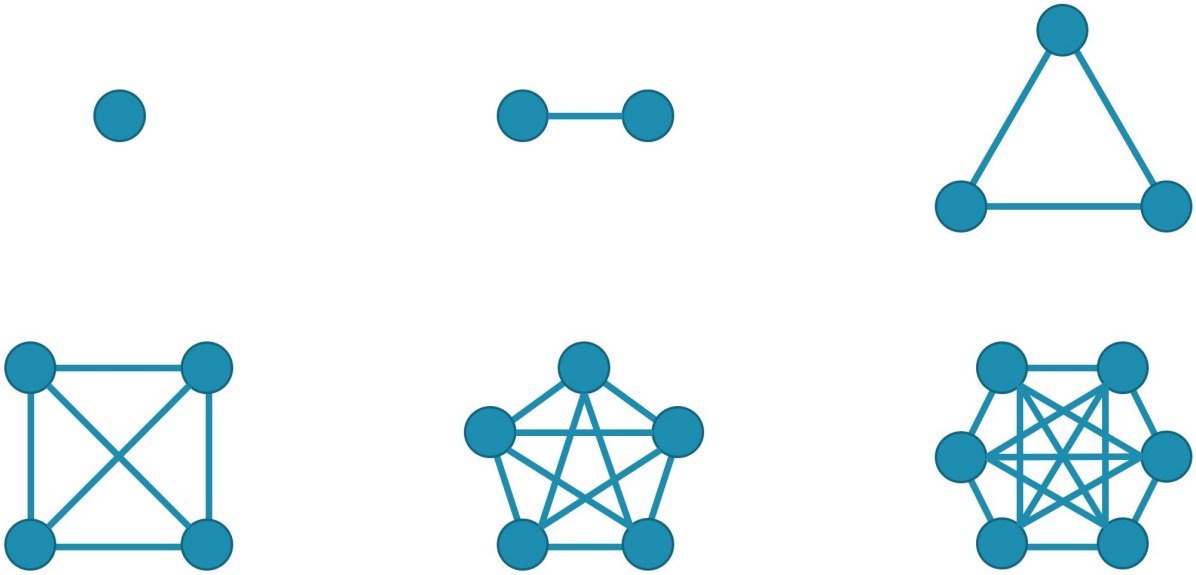
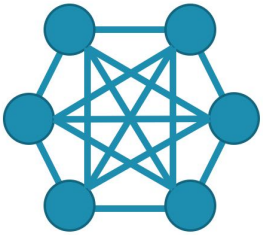


The perfect community: clique

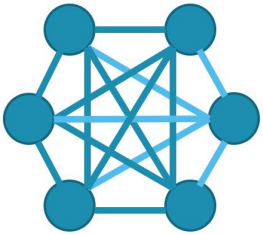
The perfect community: clique

Distance
Diameter
Domination
Degree
Edge connectivity
Vertex connectivity
Edge density



The perfect community: clique

- Distance
- Diameter
- Domination
- **Degree**
- Edge connectivity
- Vertex connectivity
- Edge density

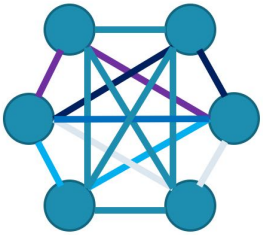


$$size - 1 = 5$$

The **degree** is
smallest number of neighbors of a vertex.

The perfect community: clique

- Distance
- Diameter
- Domination
- Degree
- **Edge connectivity**
- Vertex connectivity
- Edge density

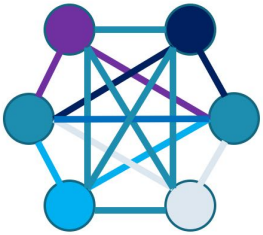


$size - 1 = 5$

The **edge connectivity** is
smallest number of edge-independent paths
between every pair of vertices.

The perfect community: clique

- Distance
- Diameter
- Domination
- Degree
- Edge connectivity
- **Vertex connectivity**
- Edge density

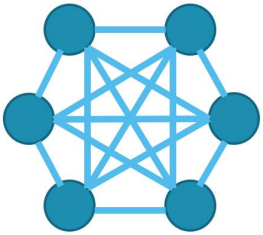


$size - 1 = 5$

The **vertex connectivity** is
smallest number of vertex-independent paths
between every pair of vertices.

The perfect community: clique

- Distance
- Diameter
- Domination
- Degree
- Edge connectivity
- Vertex connectivity
- **Edge density**



$$size * (size - 1) = 30$$

The **edge density** is number of edges.

The perfect community: clique

Distance
Diameter
Domination
Degree
Edge connectivity
Vertex connectivity
Edge density

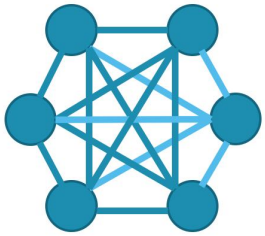
Clique relaxation framework

	ensure lower bound		relax upper bound	
	absolute	**relative**	**absolute**	**relative**
Distance	s-clique	def. 3	def. 8	def. 3
Diameter	s-club	def. 4	def. 9	def. 4
Domination	s-plex	def. 5	def. 10	def. 5
Degree	k-core	λ in (λ,γ)-quasi-clique	s-plex	λ in (λ,γ)-quasi-clique
Edge connectivity	def. 1	def. 6	def. 11	def. 6
Vertex connectivity	k-block	def. 7	s-bundle	def. 7
Edge density	def. 2	γ-quasi-clique	s-defective-clique	γ-quasi-clique

Property	Definition	Vertex connectivity k	
Distance	s-clique	$k \geq 0$ *	absolute lower bound
Diameter	s-club	$k \geq 1$ *	
Domination	s-plex	$k \geq n - 2x + 2$ *	
Degree	k-core	$k \geq 2x + 2 - n$ *	
Edge connectivity	def. 1	$k \geq 2x + 2 - n$	
Vertex connectivity	k-block	$k \geq x$ *	
Edge density	def. 2	$k \geq \lceil x\,n / 2 - (n-1)\,(n-2) / 2 \rceil$ *	
Distance	def. 3	$k \geq 0$ *	relative bound
Diameter	def. 4	$k \geq 1$ *	
Domination	def. 5	$k \geq n - 2x\,(n-1) + 2$ *	
Degree	λ in (λ,γ)-quasi-clique	$k \geq 2x\,(n-1) + 2 - n$ *	
Edge connectivity	def. 6	$k \geq 2x\,(n-1) + 2 - n$	
Vertex connectivity	def. 7	$k \geq x\,(n-1)$ *	
Edge density	γ-quasi-clique	$k \geq \lceil x\,n\,(n-1) / 2 - (n-1)\,(n-2) / 2 \rceil$ *	
Distance	def. 8	$k \geq 0$ *	absolute upper bound
Diameter	def. 9	$k \geq 1$ *	
Domination	def. 10	$k \geq n - 2\,(n-x) + 2$ *	
Degree	s-plex	$k \geq 2\,(n-x) + 2 - n$ *	
Edge connectivity	def. 11	$k \geq 2(n-x) + 2 - n$	
Vertex connectivity	s-bundle	$k \geq n - x$ *	
Edge density	s-defective-clique	$k \geq \lceil (n-x)\,n / 2 - (n-1)\,(n-2) / 2 \rceil$ *	
	clique (= fully connected)	$\boldsymbol{k = n - 1}$	

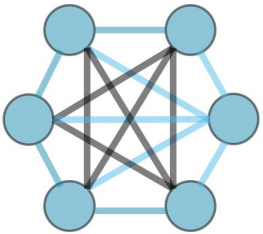
Whitney’s theorem

Whitney's theorem



Degree connectivity

Whitney's theorem



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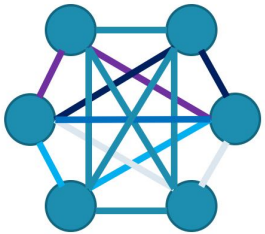
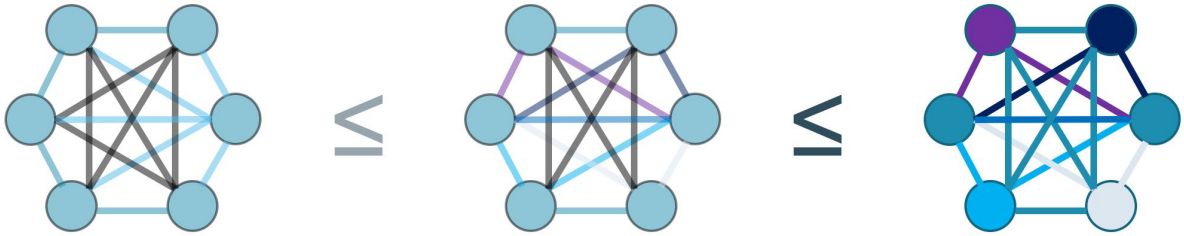


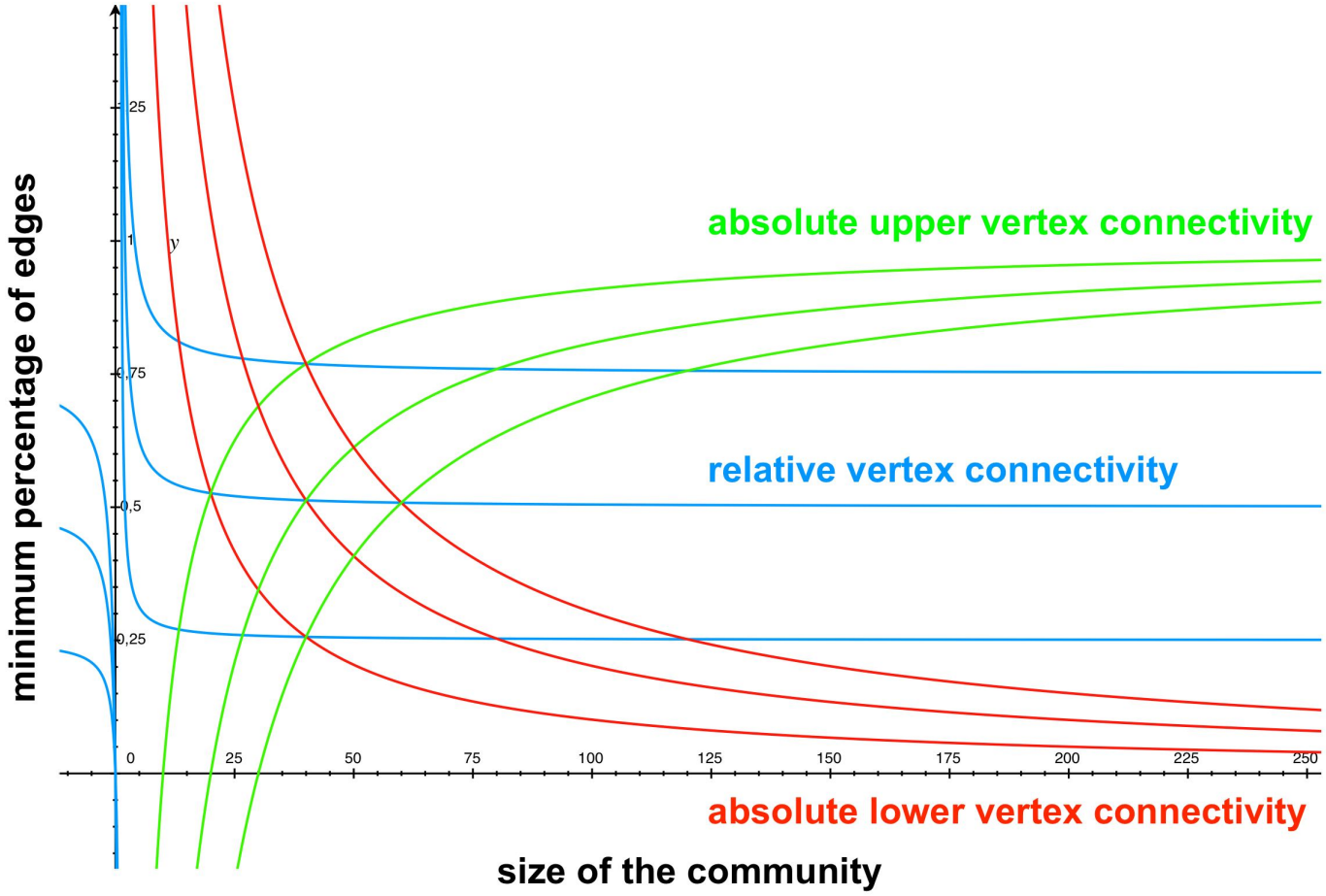
Degree connectivity

Edge connectivity

Whitney's theorem



preferred definition


minimum percentage of edges
absolute upper vertex connectivity
relative vertex connectivity
absolute lower vertex connectivity
size of the community
y
25
1
0,75
0,5
0,25
0
25
50
75
100
125
150
175
200
225
250

Guarantied properties

Distance	$dG(v1, v2) \leq \lfloor (n - 2) / x + 1 \rfloor$	**k-block**
Diameter	$diam(S) \leq \lfloor (n - 2) / x + 1 \rfloor$	
Domination	$\lvert D \rvert \leq n - x$	
Degree	$\delta(S) \geq x$	
Edge connectivity	$\lambda(S) \geq x$	
Vertex connectivity	$\kappa(S) \geq x$	
Edge density	$\rho(S) \geq x / (n - 1)$	
Distance	$dG(v1, v2) \leq \lfloor (n - 2) / (x (n - 1)) + 1 \rfloor$	**def. 7**
Diameter	$diam(S) \leq \lfloor (n - 2) / (x (n - 1)) + 1 \rfloor$	
Domination	$\lvert D \rvert \leq n - x (n - 1)$	
Degree	$\delta(S) \geq x (n - 1)$	
Edge connectivity	$\lambda(S) \geq x (n - 1)$	
Vertex connectivity	$\kappa(S) \geq x (n - 1)$	
Edge density	$\rho(S) \geq x$	
Distance	$dG(v1, v2) \leq \lfloor (n - 2) / (n - x) + 1 \rfloor$	**s-bundle**
Diameter	$diam(S) \leq \lfloor (n - 2) / (n - x) + 1 \rfloor$	
Domination	$\lvert D \rvert \leq x$	
Degree	$\delta(S) \geq n - x$	
Edge connectivity	$\lambda(S) \geq n - x$	
Vertex connectivity	$\kappa(S) \geq n - x$	
Edge density	$\rho(S) \geq (n - x) / (n - 1)$	

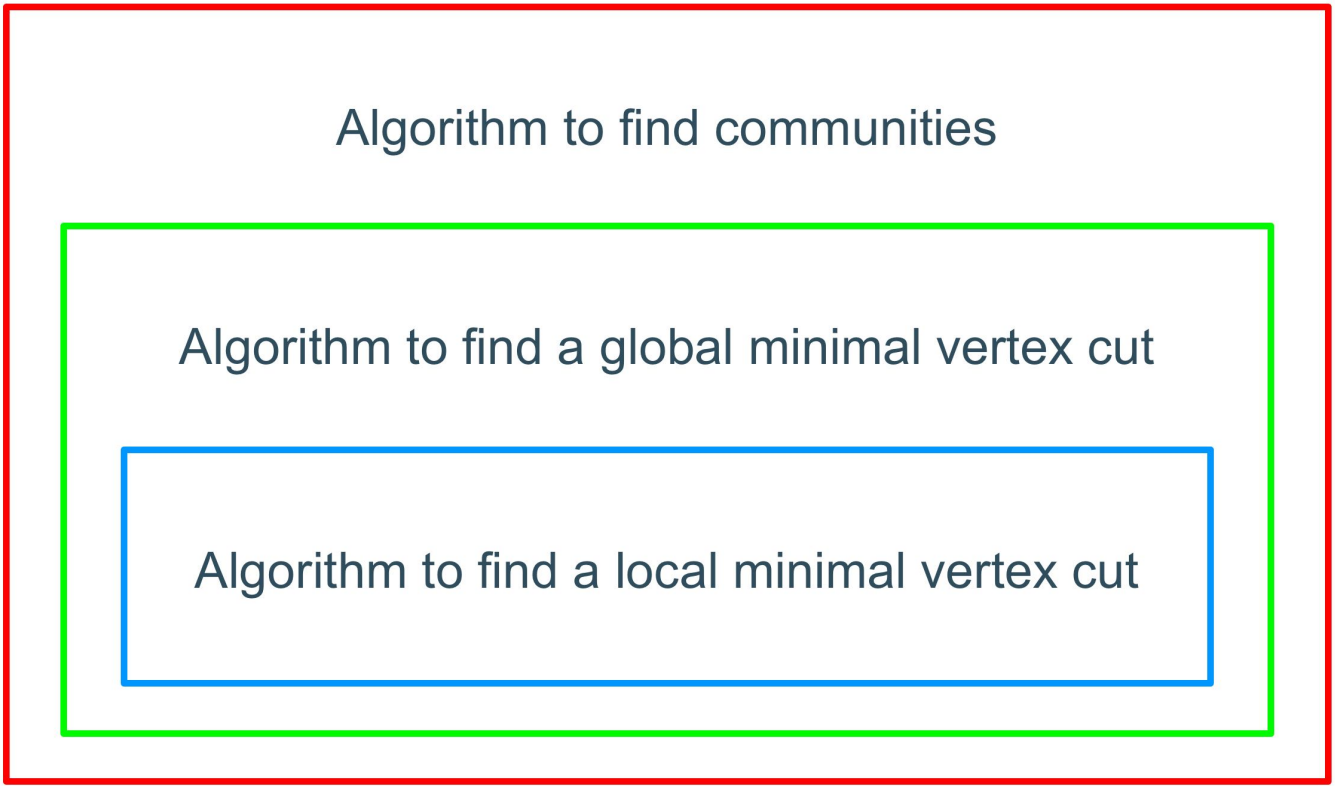
Research question 2:

a scalable algorithm

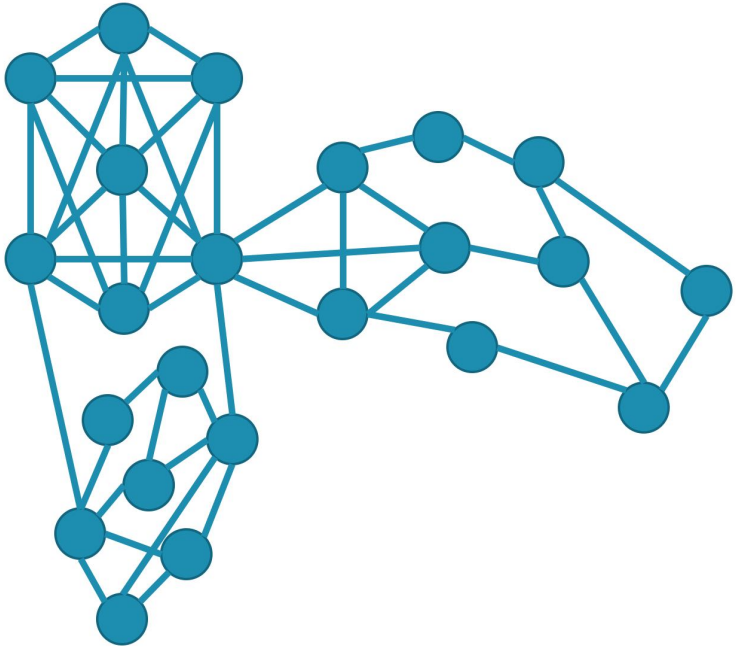
Algorithm to find communities

Algorithm to find communities

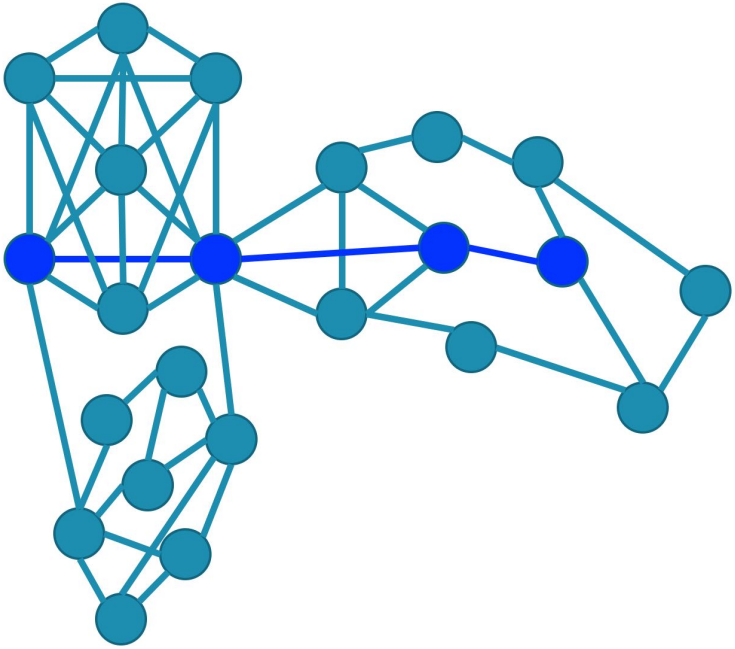
Algorithm to find a global minimal vertex cut


Algorithm to find communities
Algorithm to find a global minimal vertex cut
Algorithm to find a local minimal vertex cut

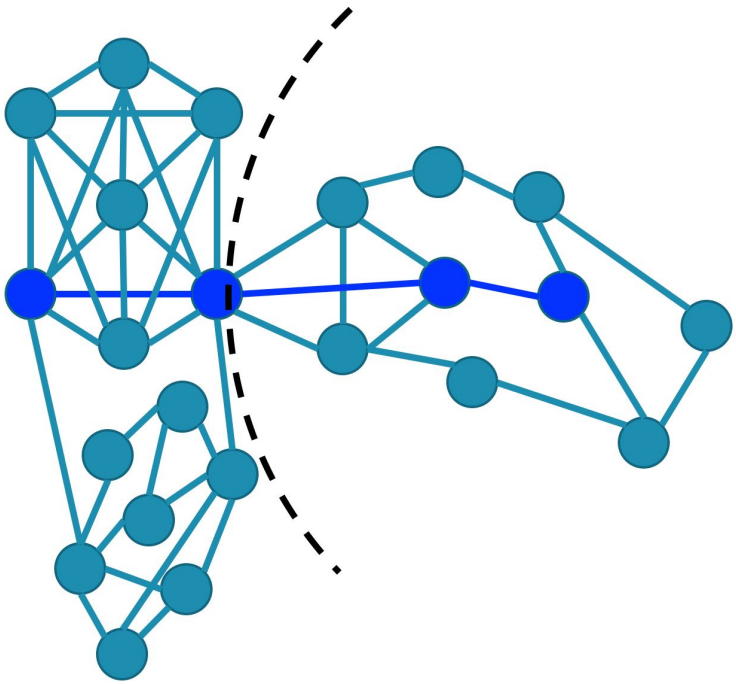
Local minimal flow-free vertex cut



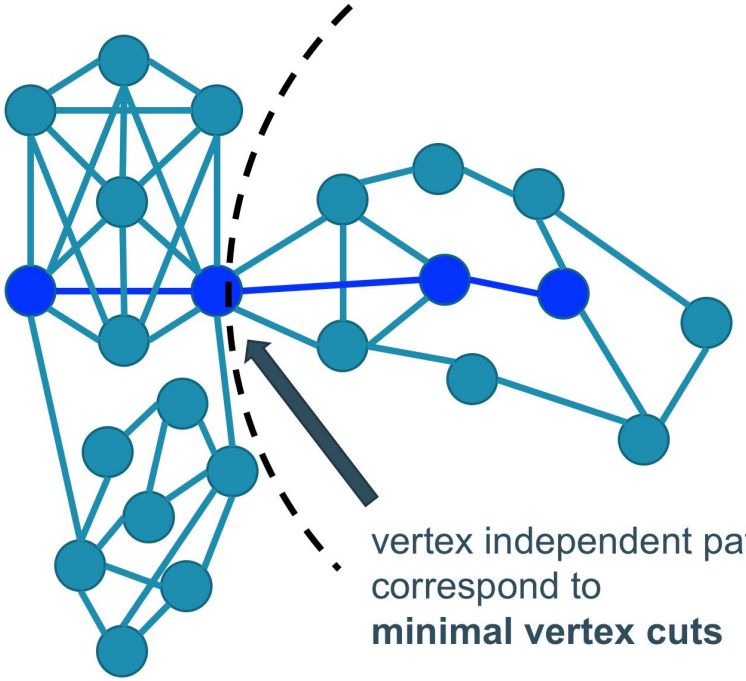
Local minimal flow-free vertex cut



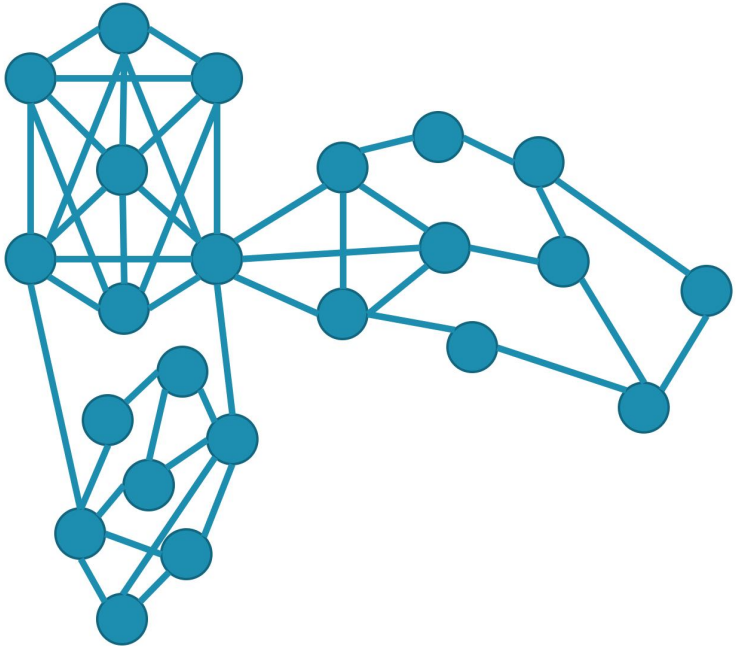
Local minimal flow-free vertex cut



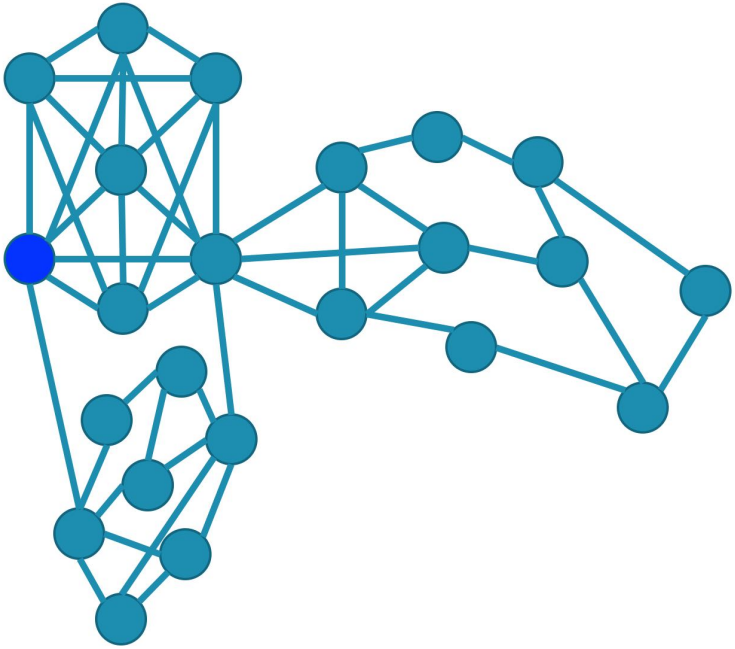
Local minimal flow-free vertex cut



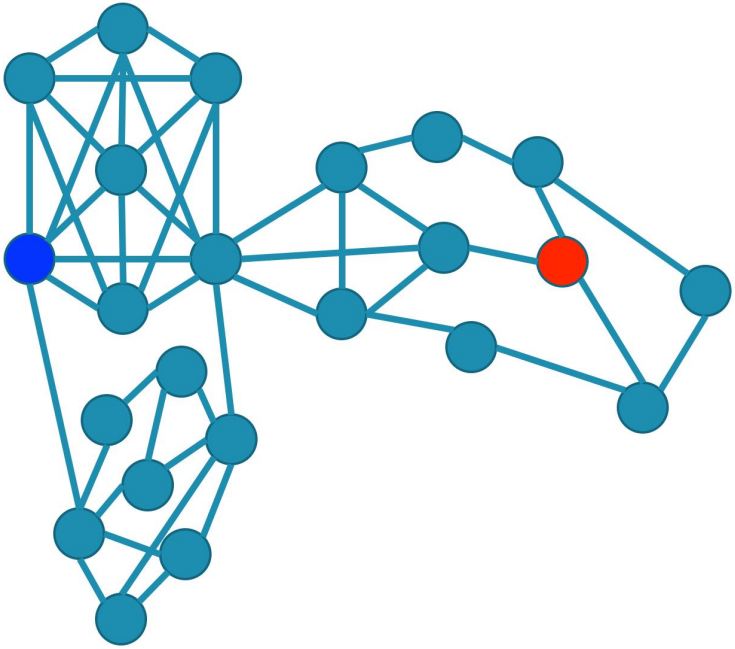
Local minimal flow-free vertex cut



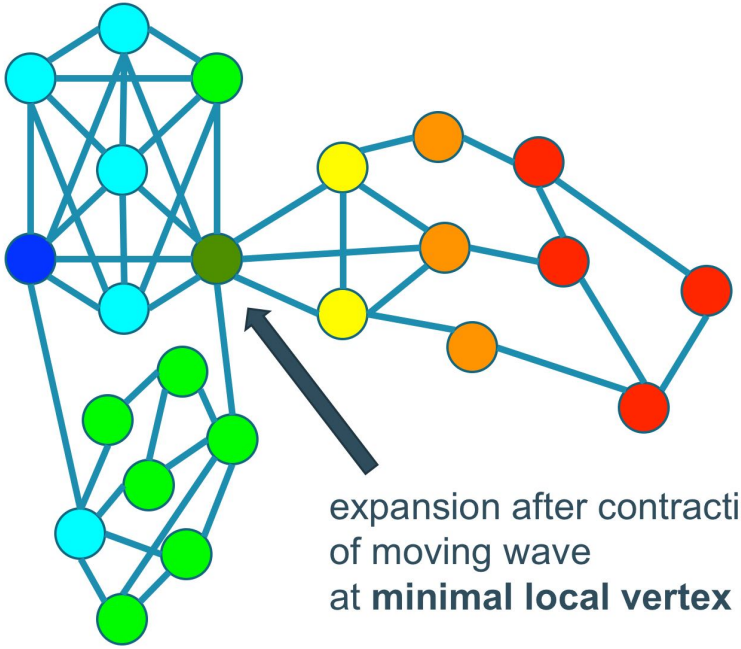
Local minimal flow-free vertex cut



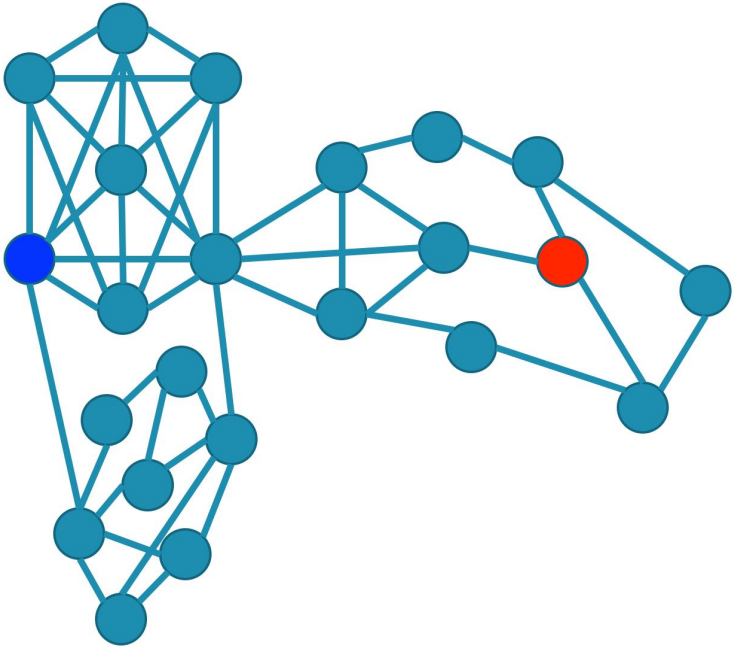
Local minimal flow-free vertex cut



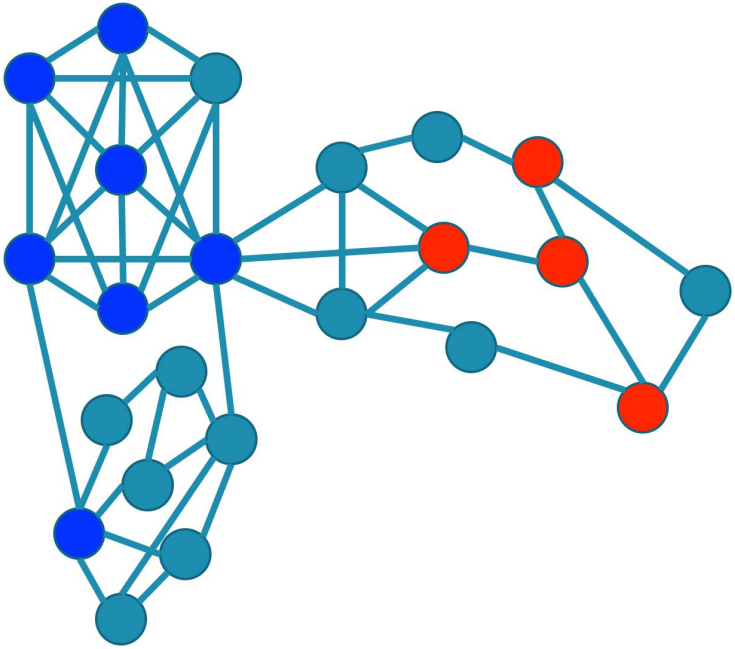
Local minimal flow-free vertex cut



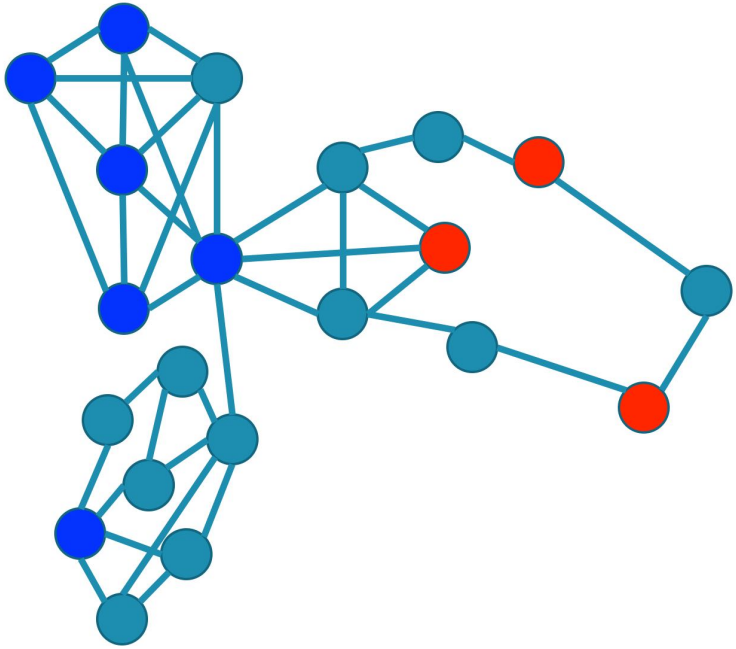
Local minimal flow-free vertex cut



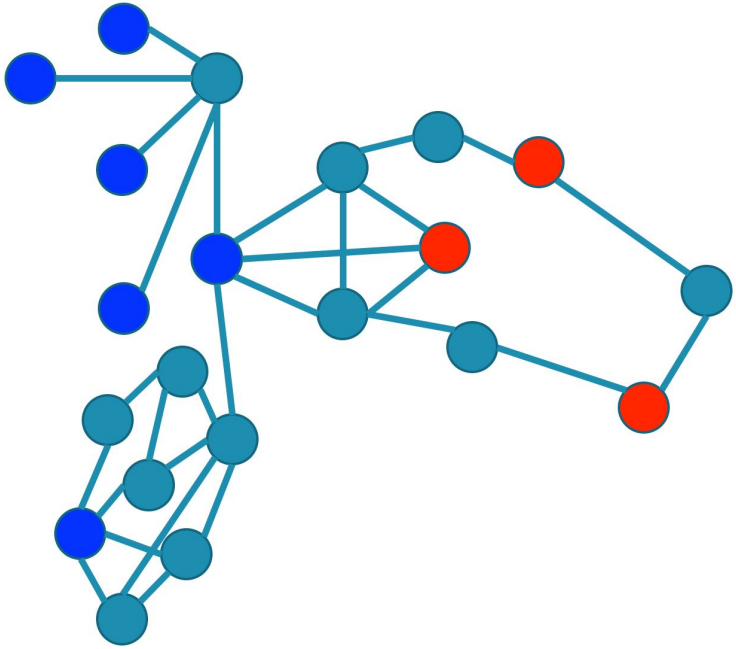
Local minimal flow-free vertex cut



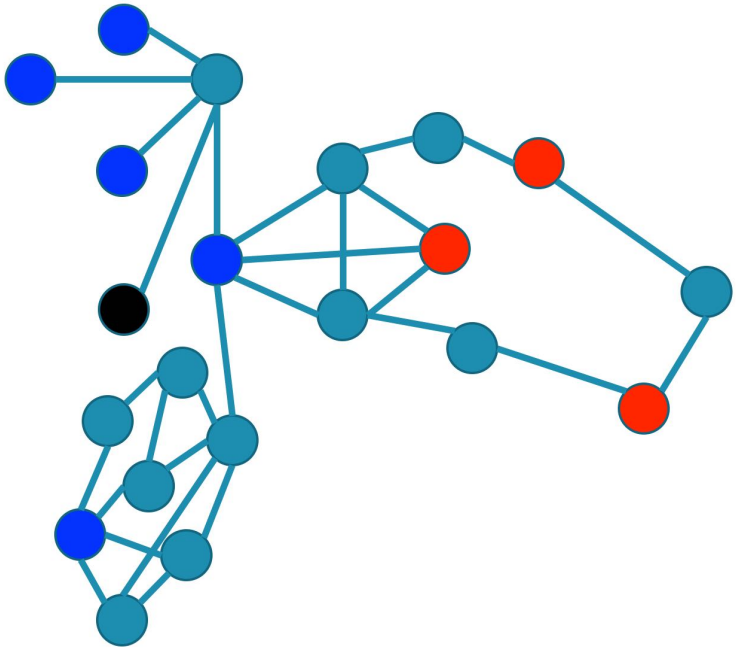
Local minimal flow-free vertex cut



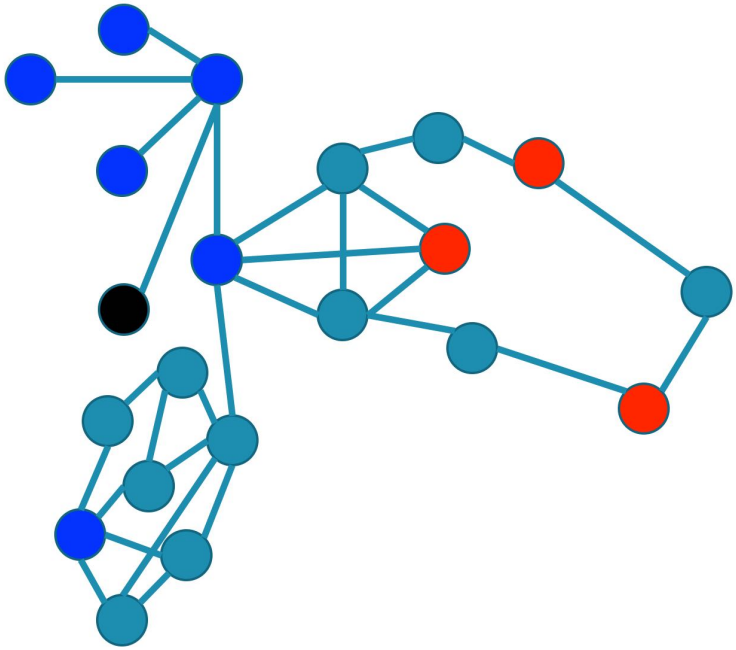
Local minimal flow-free vertex cut



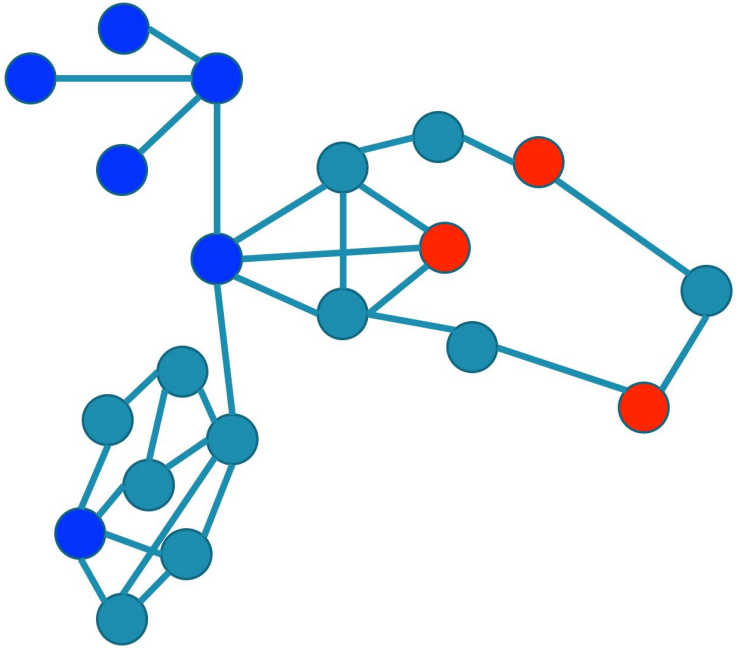
Local minimal flow-free vertex cut



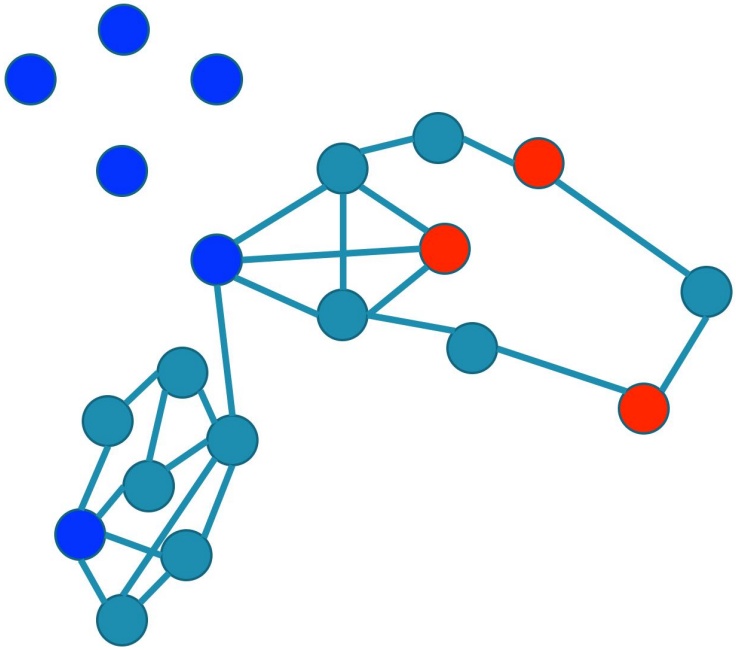
Local minimal flow-free vertex cut



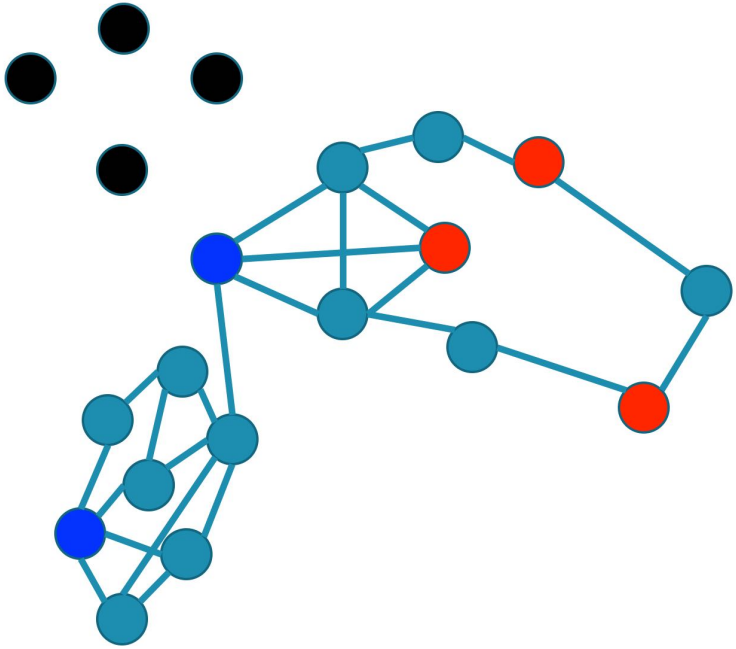
Local minimal flow-free vertex cut



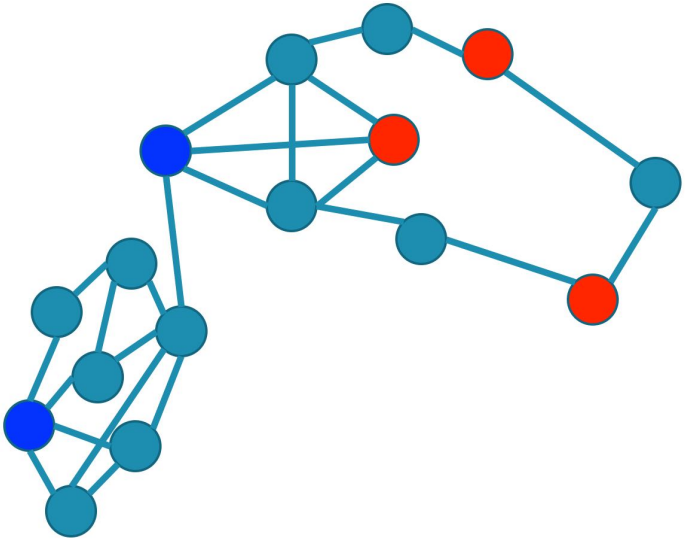
Local minimal flow-free vertex cut



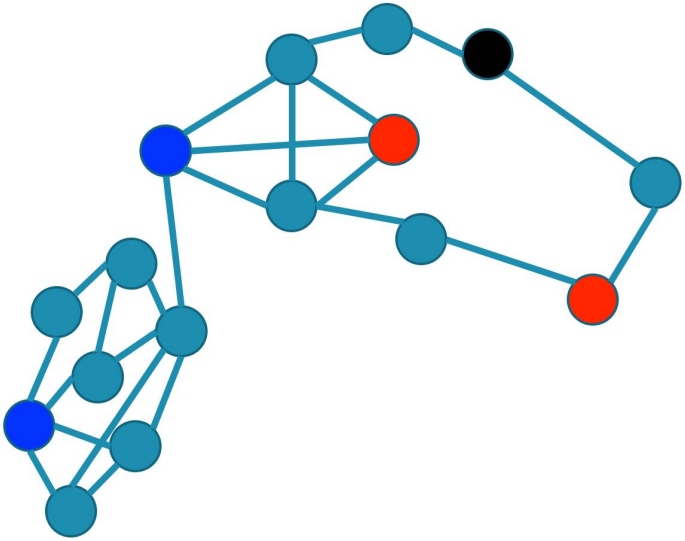
Local minimal flow-free vertex cut



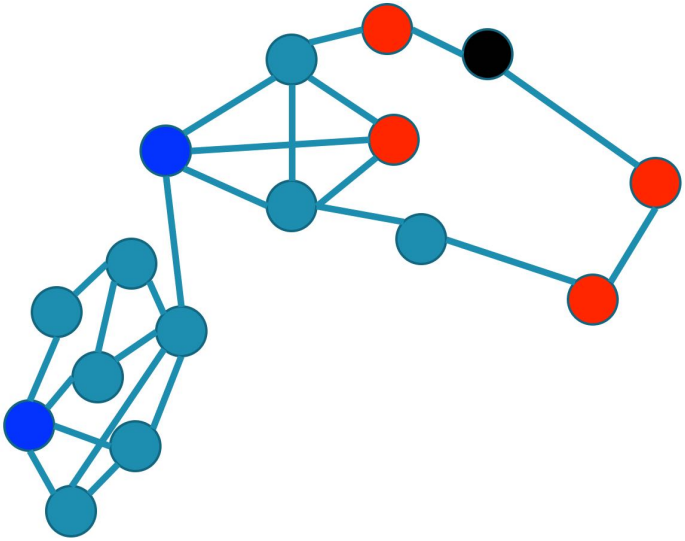
Local minimal flow-free vertex cut



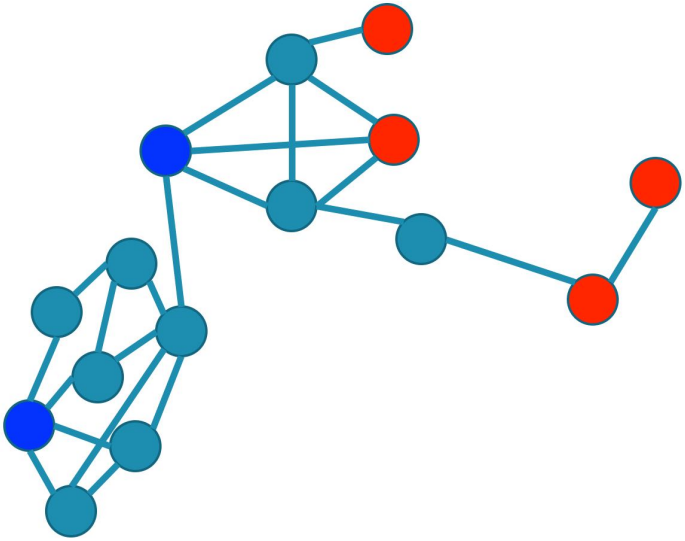
Local minimal flow-free vertex cut



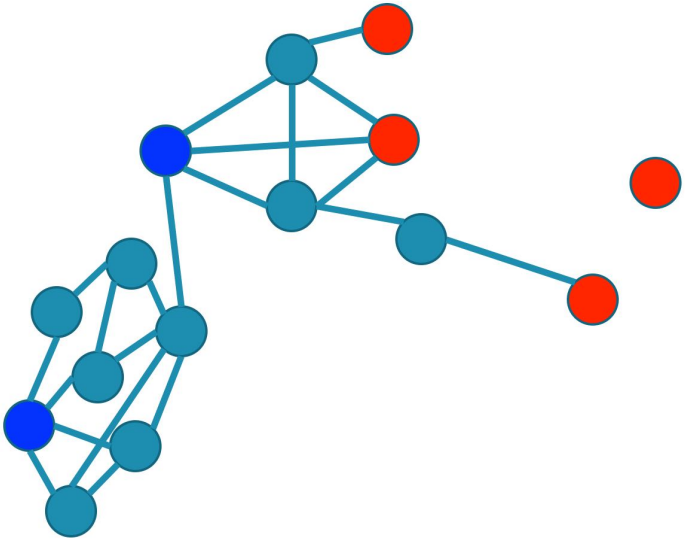
Local minimal flow-free vertex cut



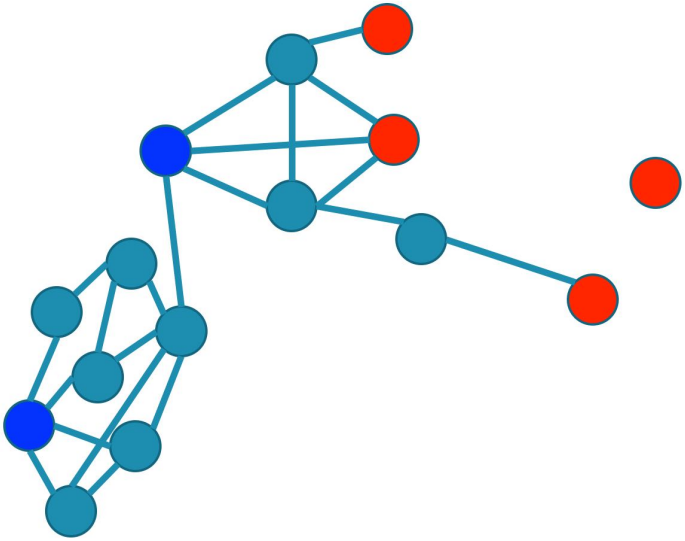
Local minimal flow-free vertex cut



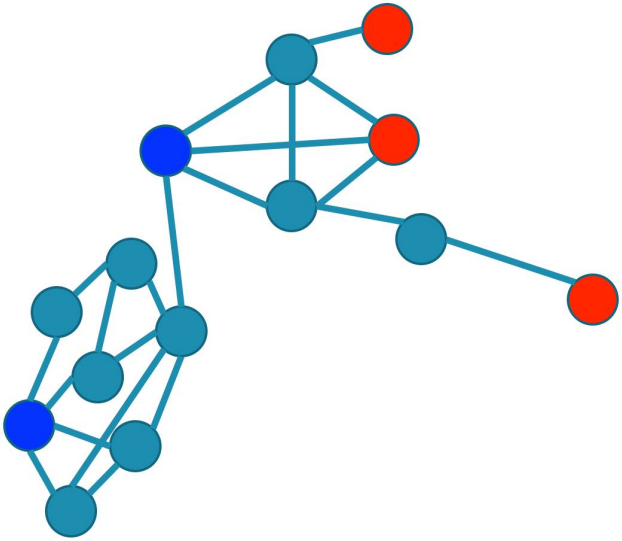
Local minimal flow-free vertex cut



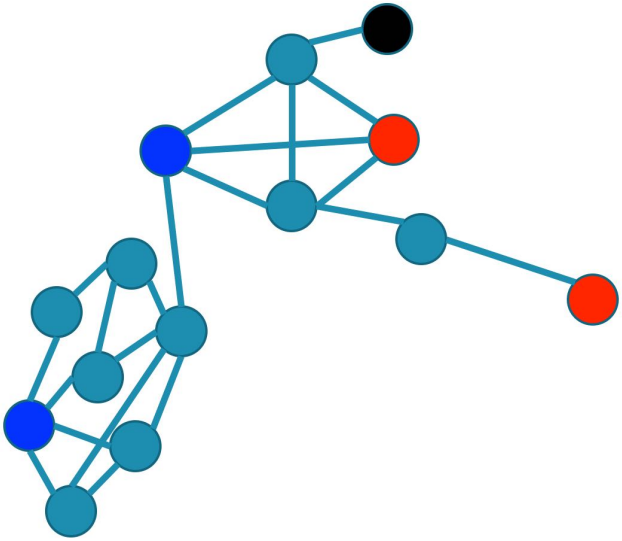
Local minimal flow-free vertex cut



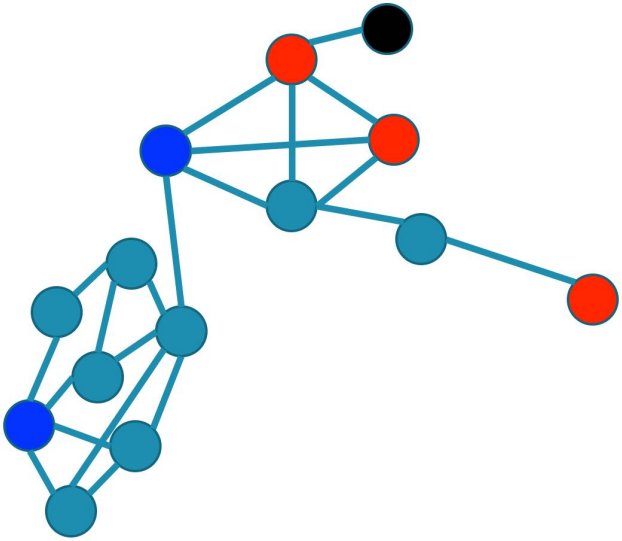
Local minimal flow-free vertex cut



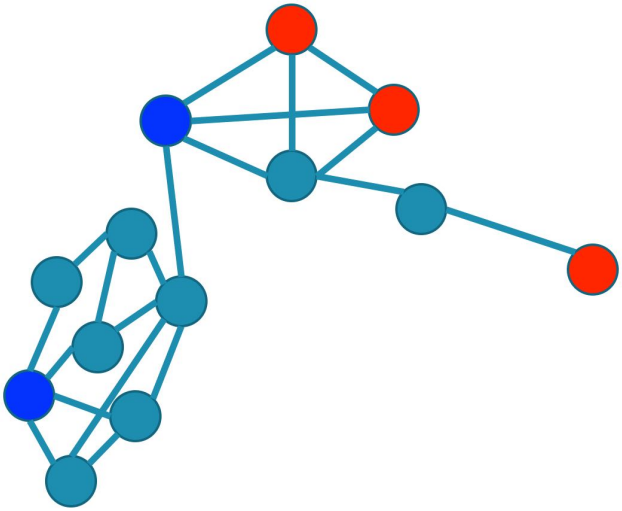
Local minimal flow-free vertex cut



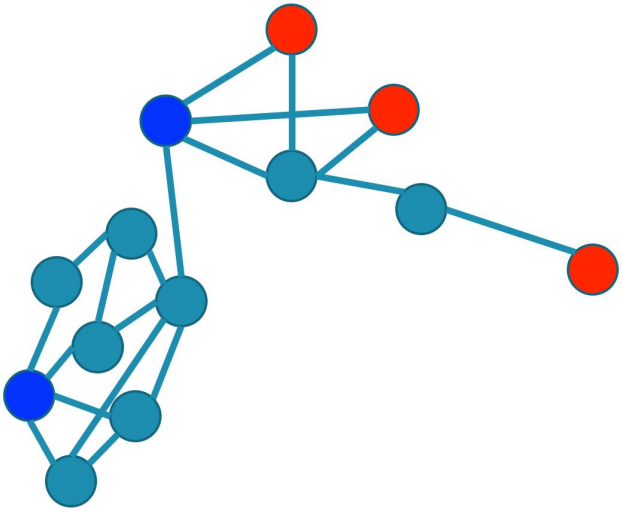
Local minimal flow-free vertex cut



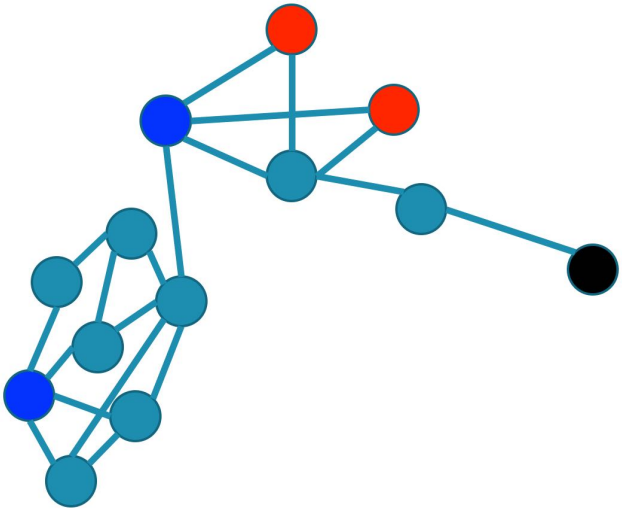
Local minimal flow-free vertex cut



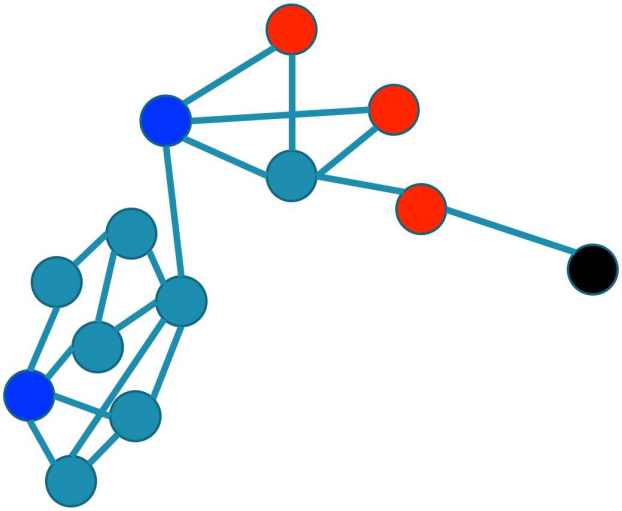
Local minimal flow-free vertex cut



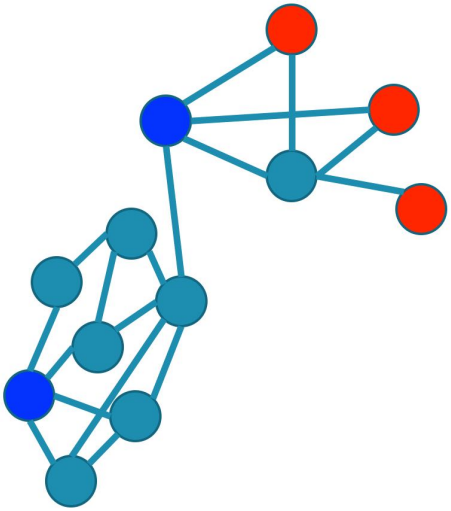
Local minimal flow-free vertex cut



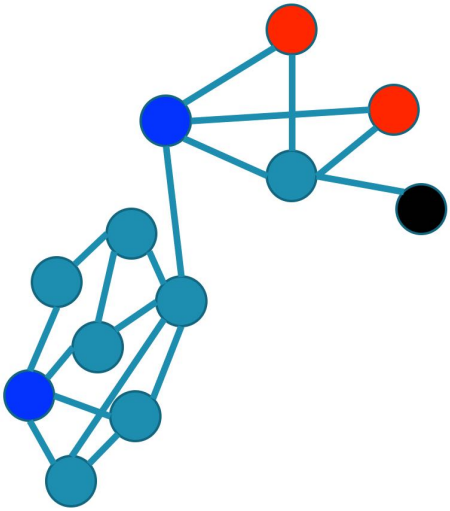
Local minimal flow-free vertex cut



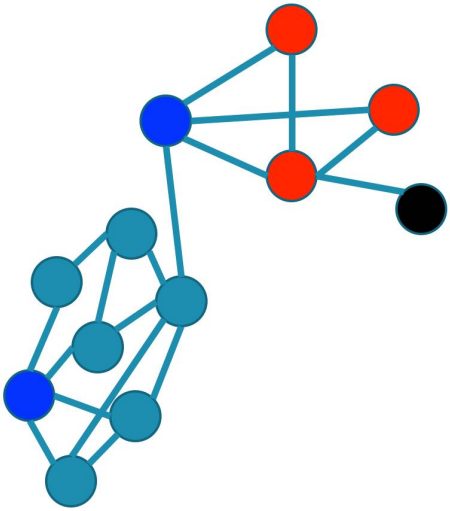
Local minimal flow-free vertex cut



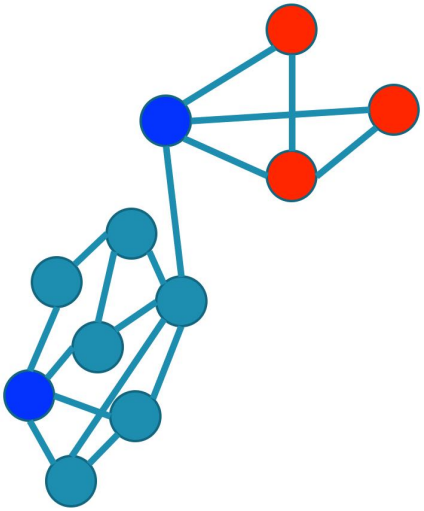
Local minimal flow-free vertex cut



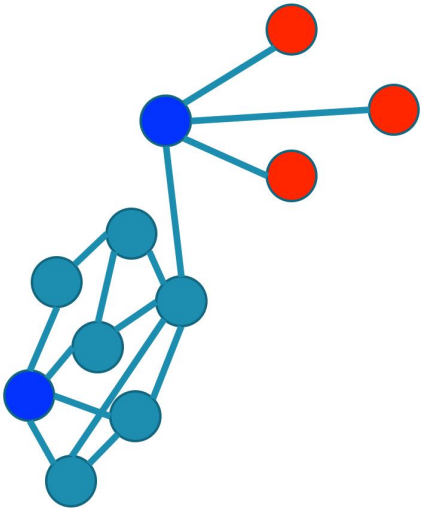
Local minimal flow-free vertex cut



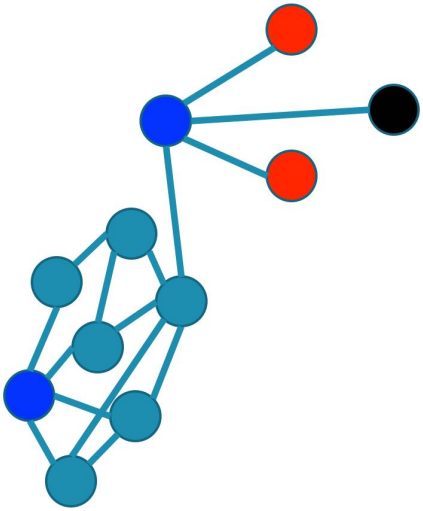
Local minimal flow-free vertex cut



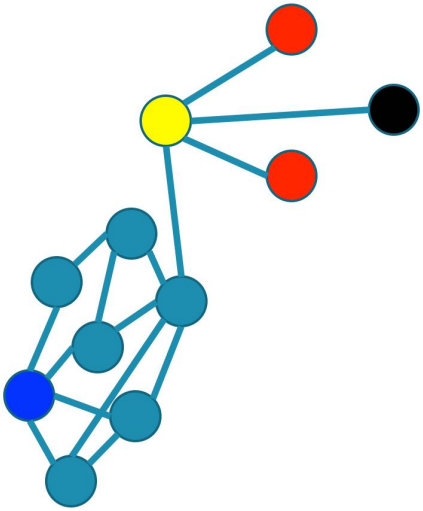
Local minimal flow-free vertex cut



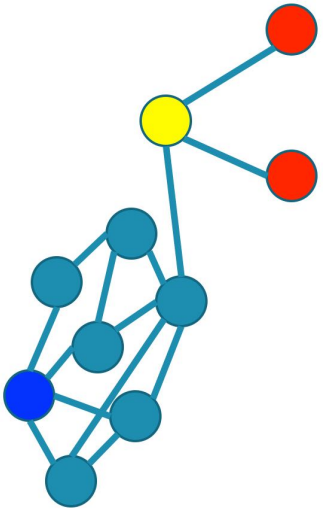
Local minimal flow-free vertex cut



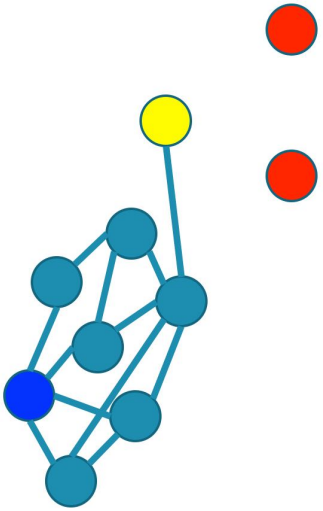
Local minimal flow-free vertex cut



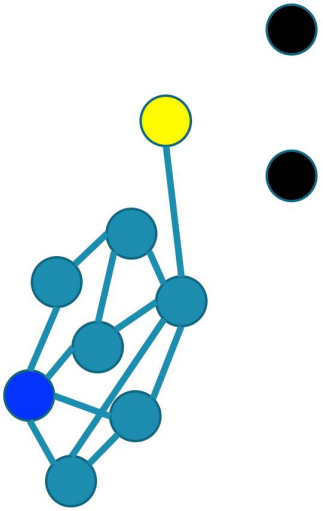
Local minimal flow-free vertex cut



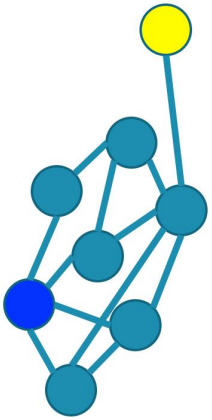
Local minimal flow-free vertex cut



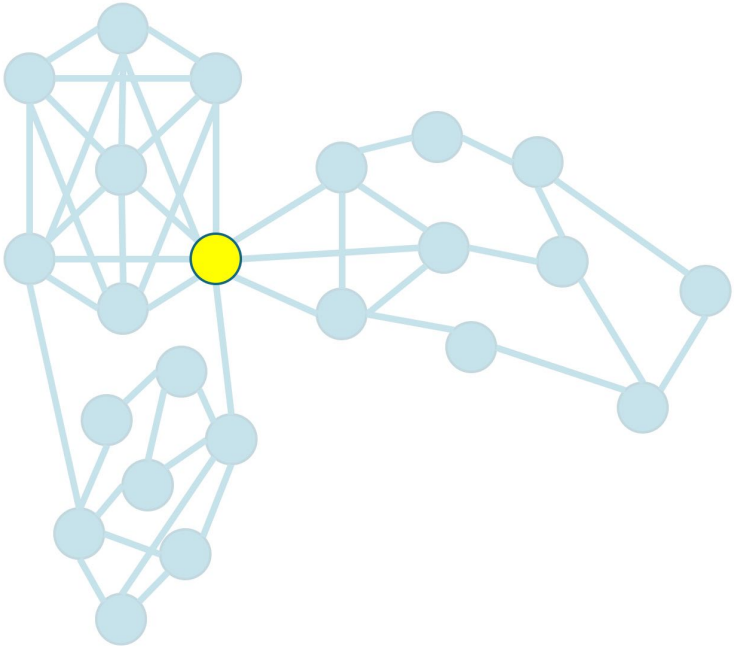
Local minimal flow-free vertex cut



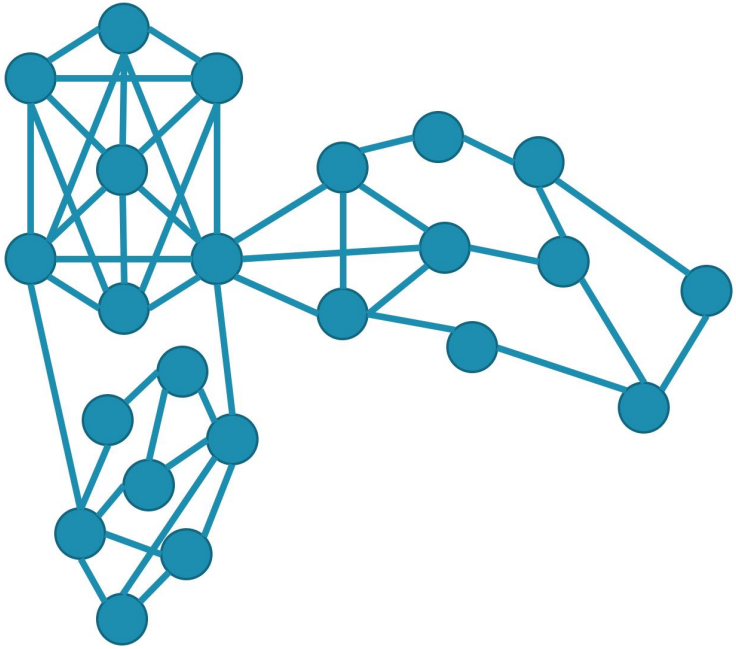
Local minimal flow-free vertex cut



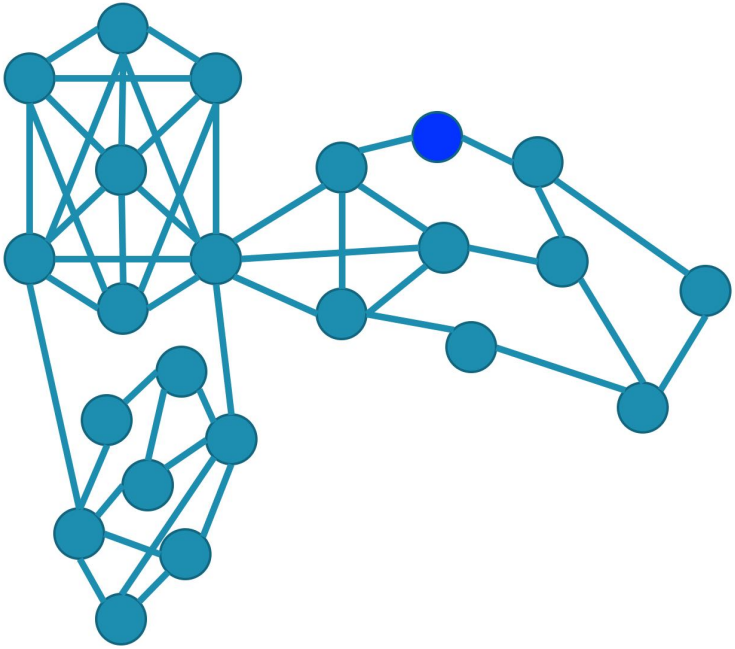
Local minimal flow-free vertex cut



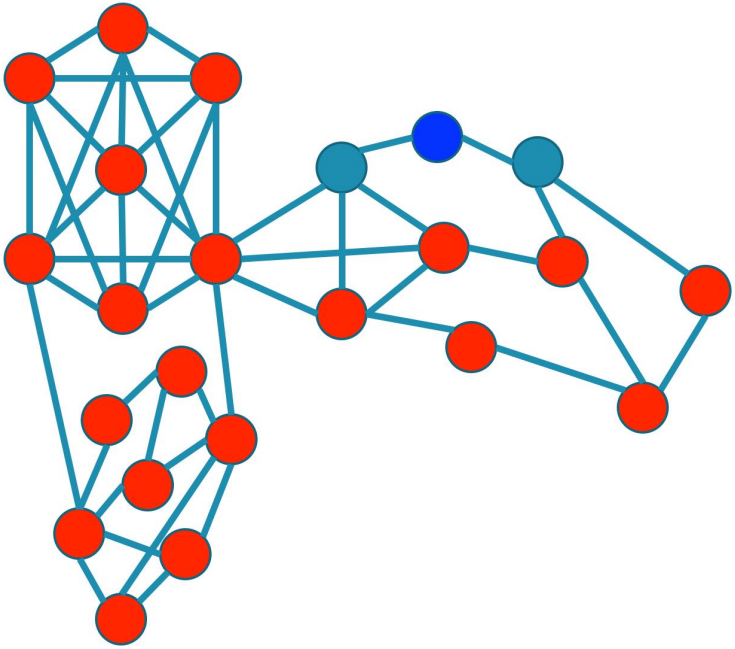
Global minimal vertex cut



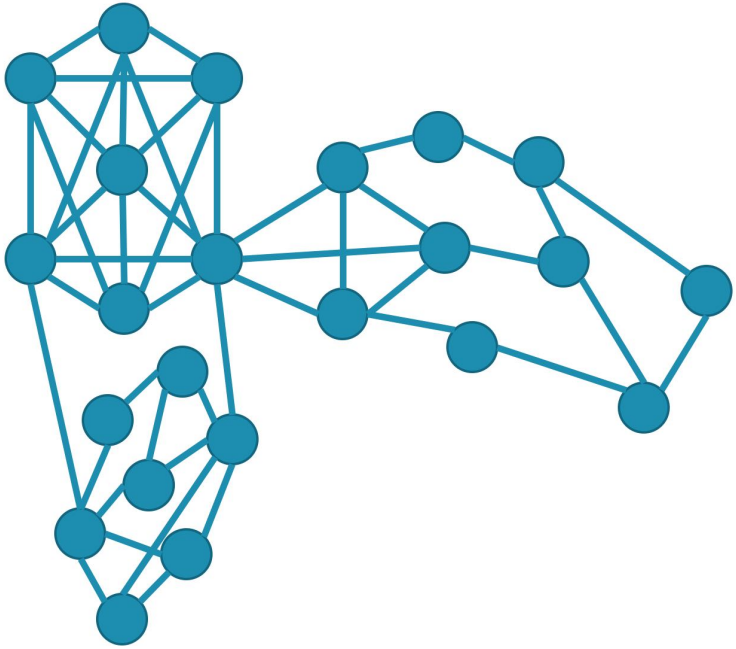
Global minimal vertex cut



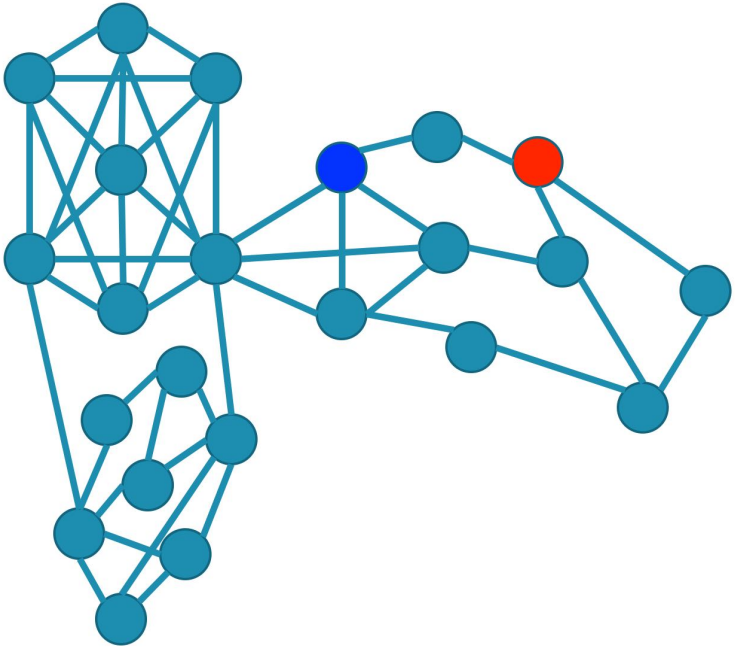
Global minimal vertex cut



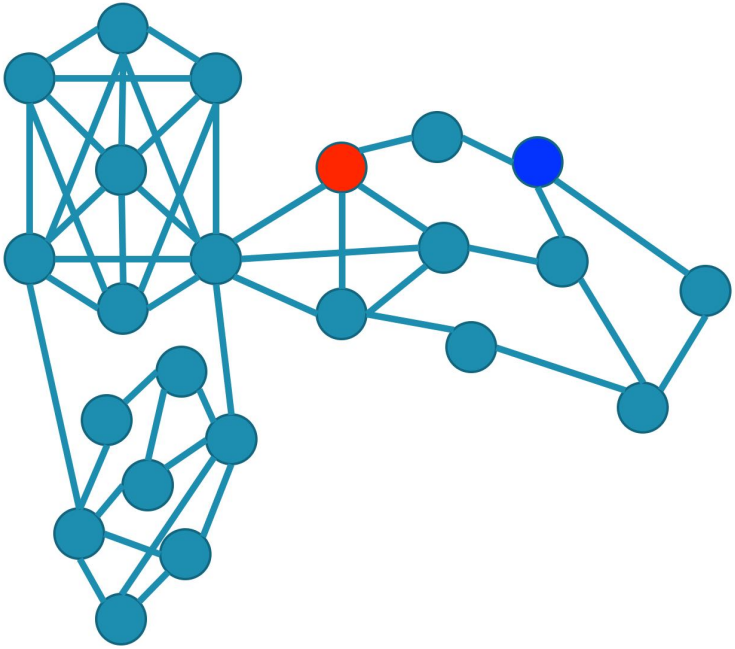
Global minimal vertex cut



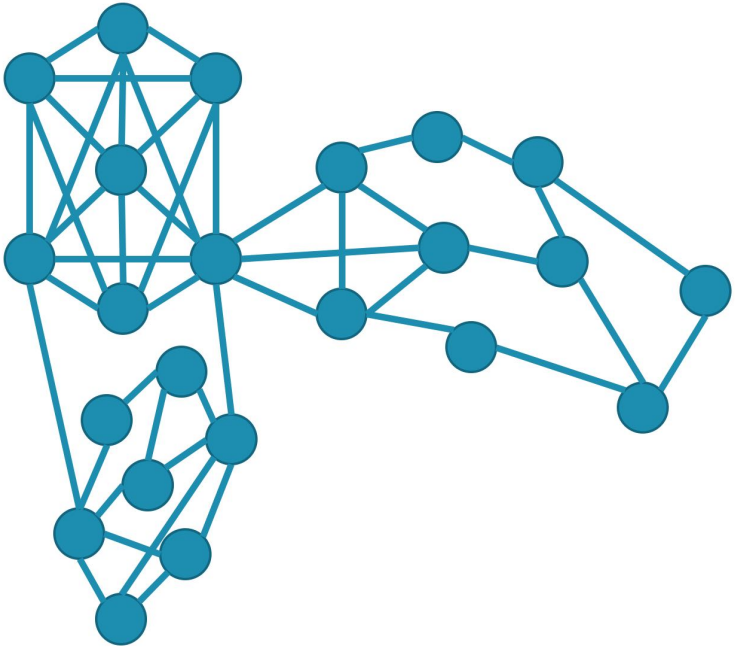
Global minimal vertex cut



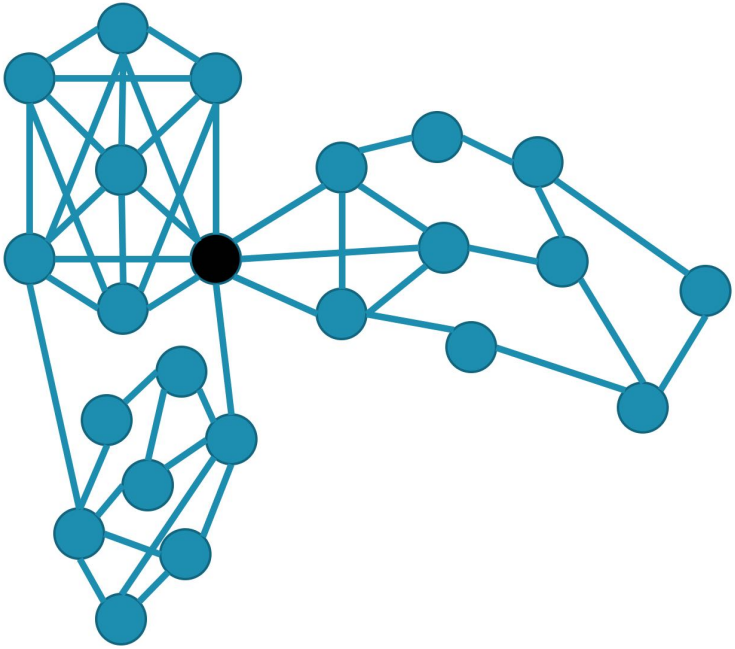
Global minimal vertex cut



Find communities (Moody & White)



Find communities (Moody & White)



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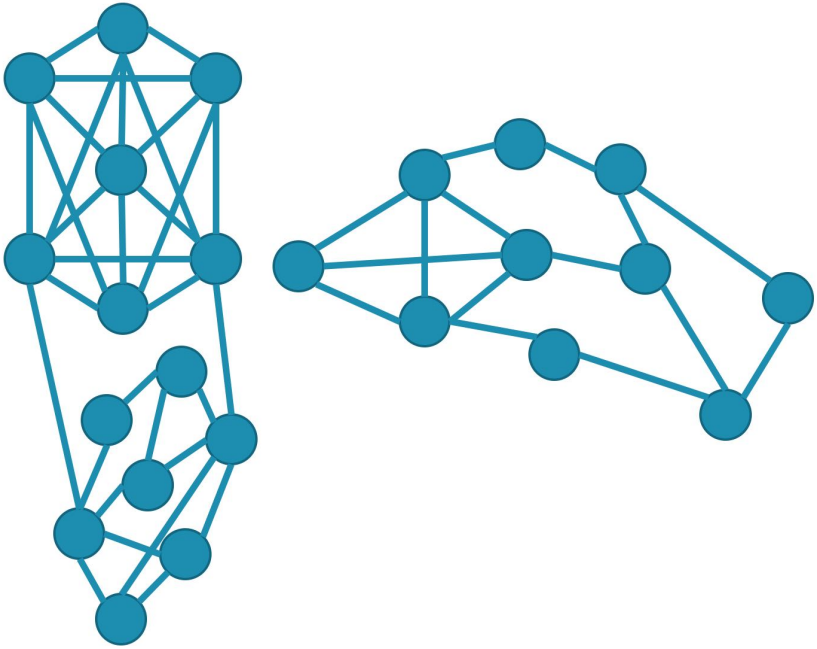
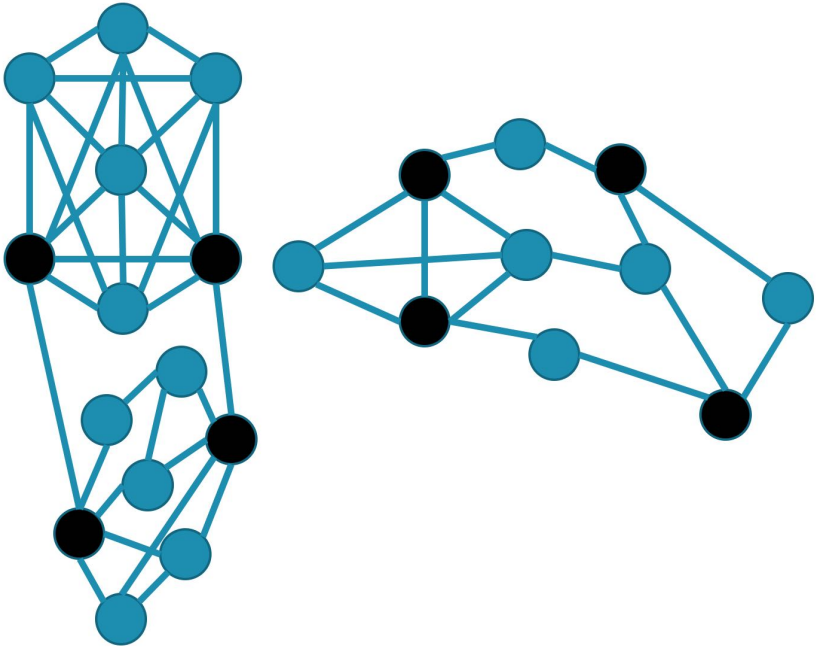
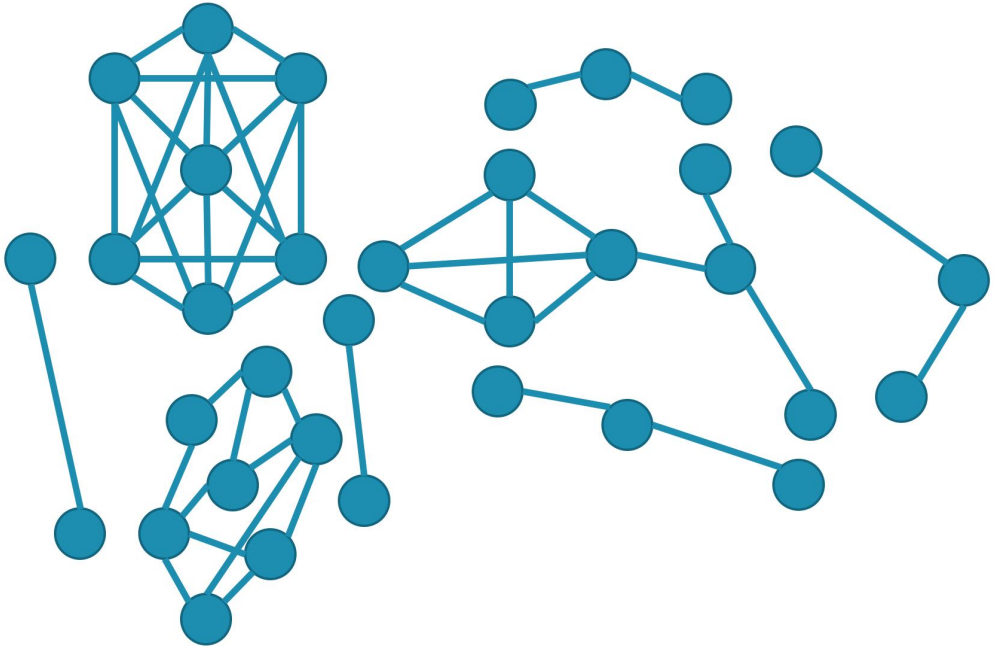


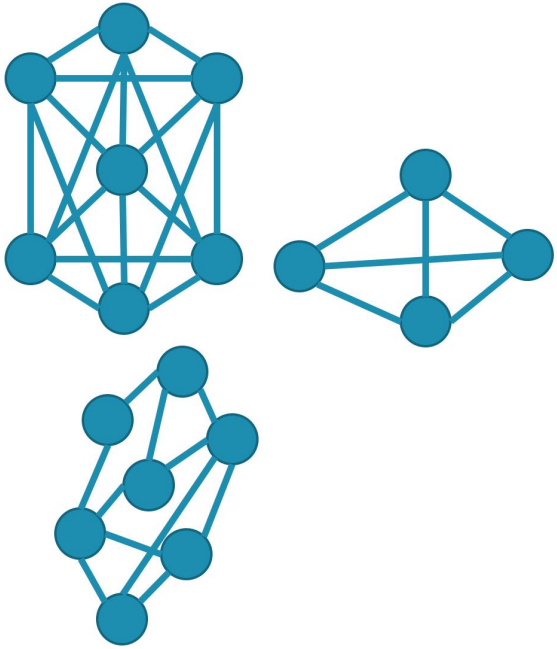
Find communities (Moody & White)



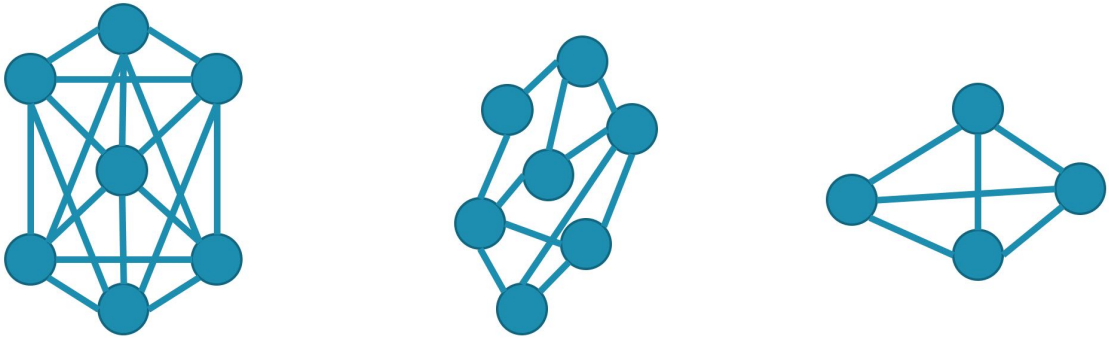
Find communities (Moody & White)

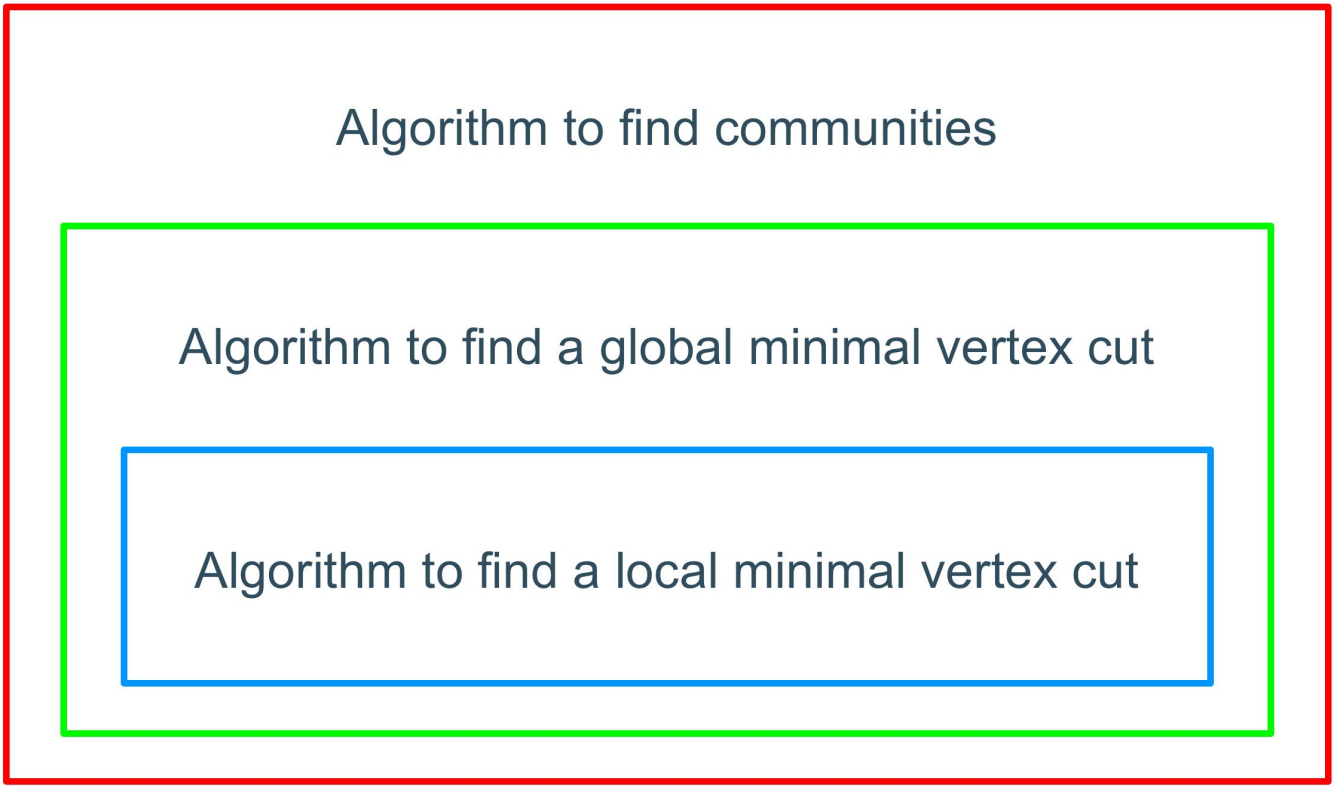


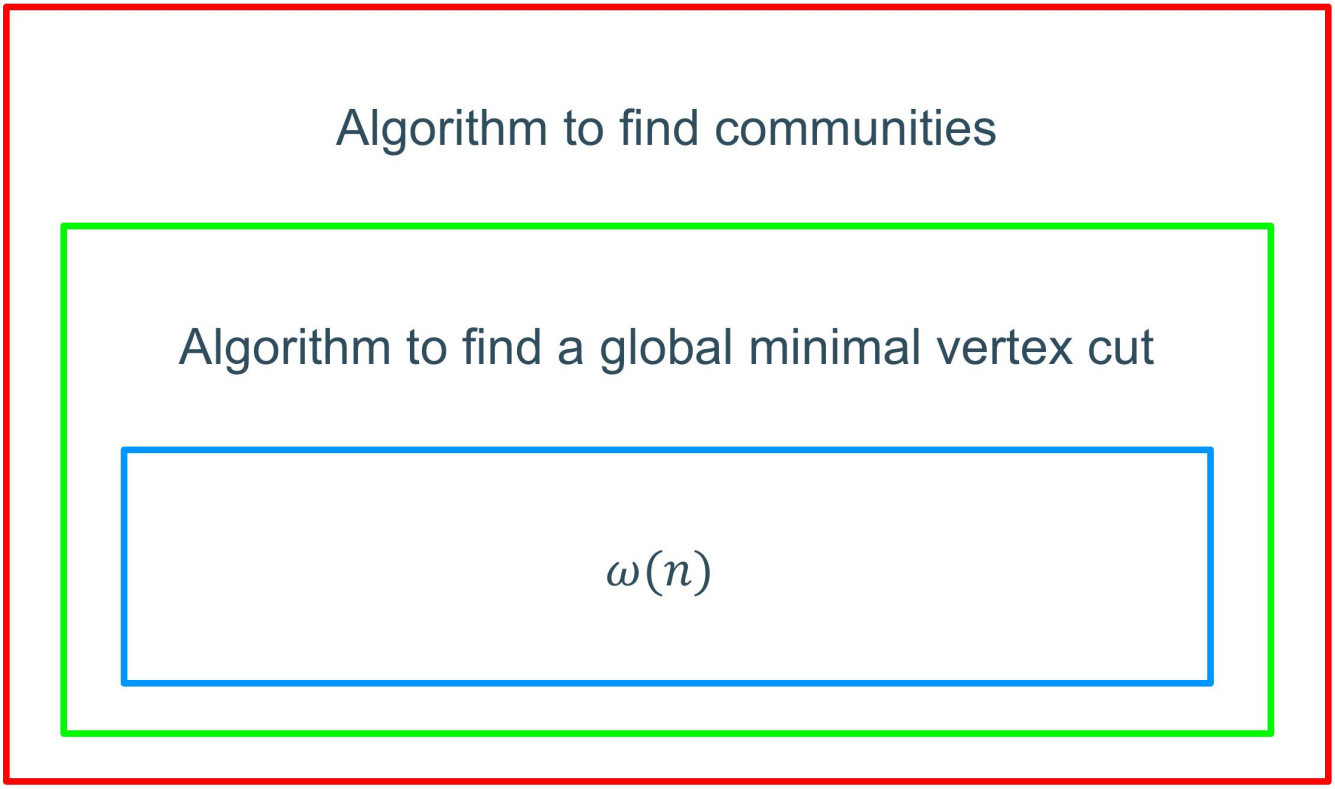
Find communities (Moody & White)

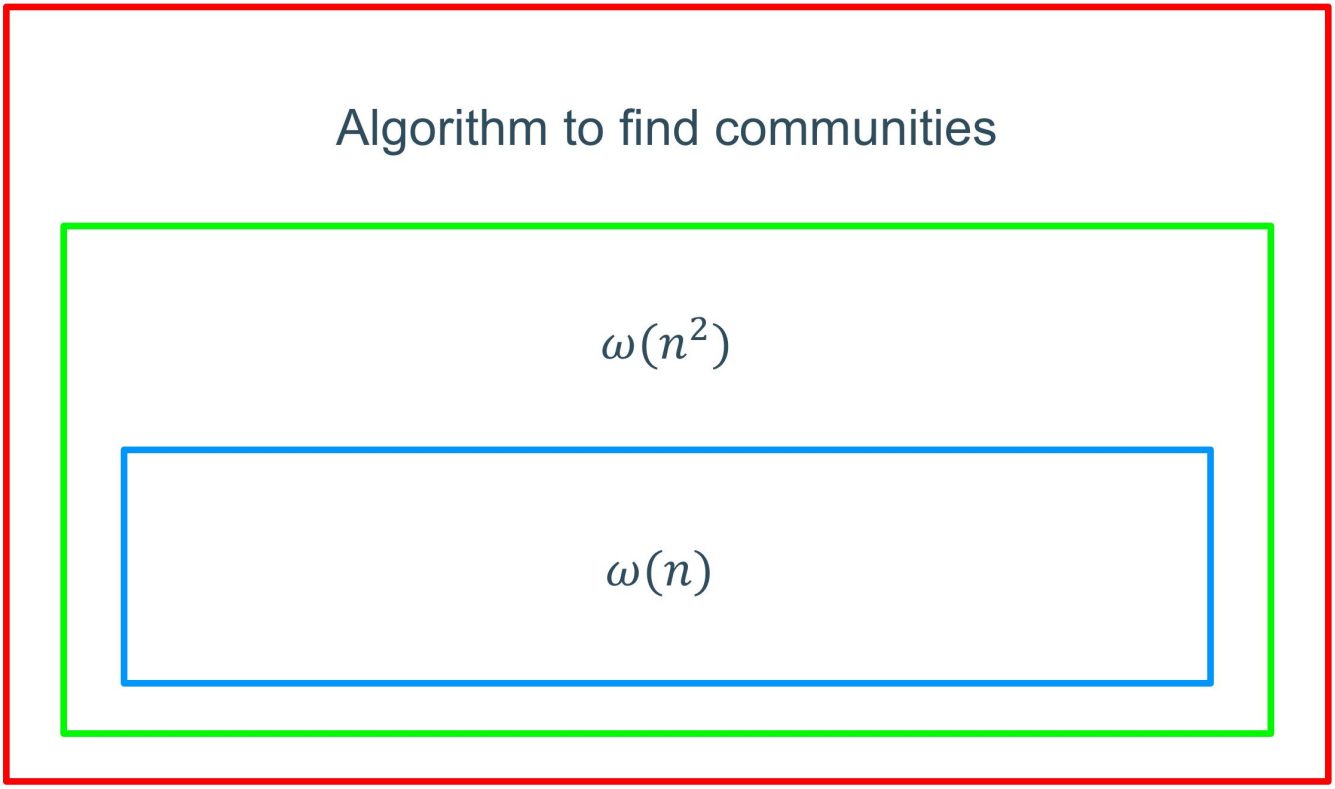


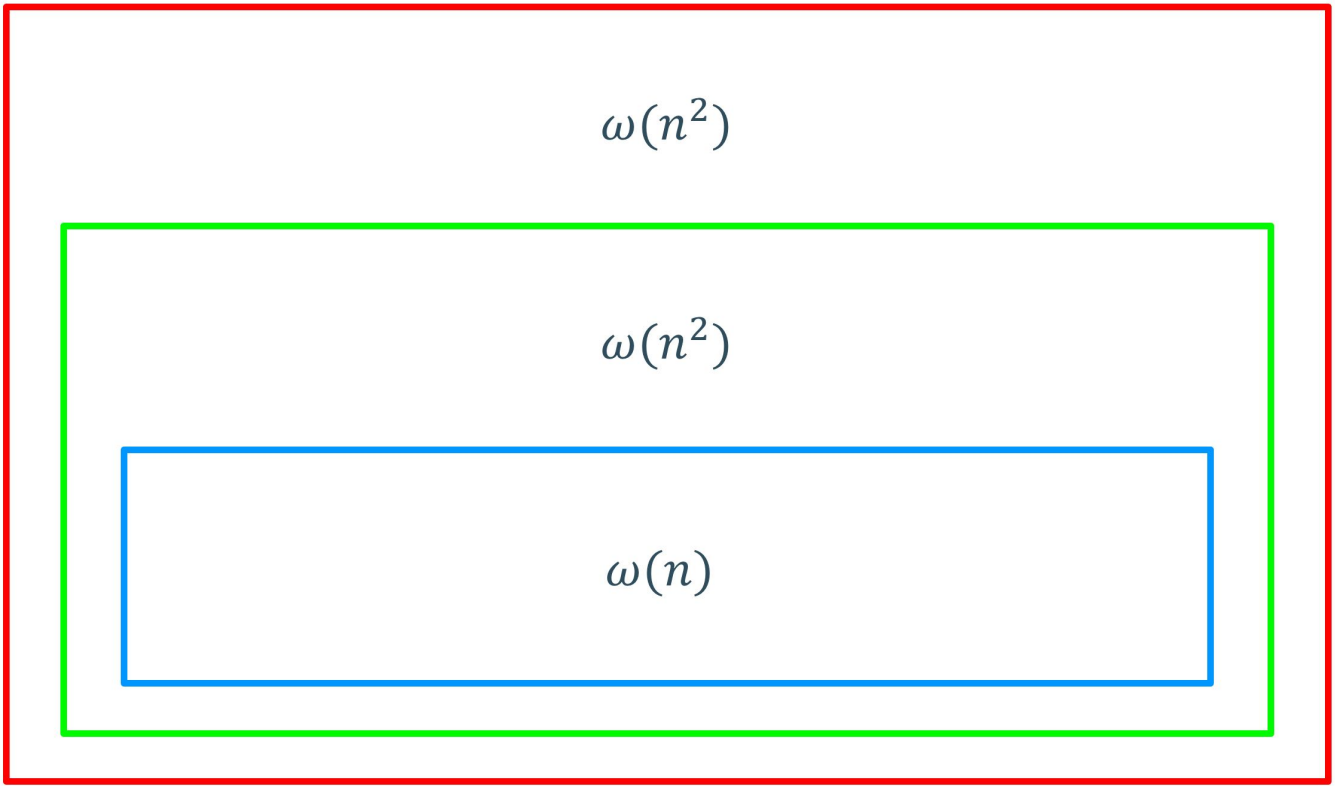
Find communities (Moody & White)




Algorithm to find communities
Algorithm to find a global minimal vertex cut
Algorithm to find a local minimal vertex cut


Algorithm to find communities
Algorithm to find a global minimal vertex cut
$\omega(n)$


Algorithm to find communities
$\omega(n^2)$
$\omega(n)$


$\omega(n^2)$
$\omega(n^2)$
$\omega(n)$

Thank you!

Any questions?

Source code at Github:
https://github.com/MCMXCVII/scale-independent-communities-flow-free-vertex-cut