On the Dynamics of a Magnetic Monopole in the Presence of an Electromagnetic Field: A Case-Study of an Ideal Hyperbolic Penning Trap

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BY

VICTOR AFIGBO

SUPERVISOR

PROF. MATTHEW HEDMAN

University of Idaho, Moscow Campus

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Abstract

Magnetic Monopoles are interesting theoretical objects that permit a more symmetric formulation of Maxwell's equations. I examined the dynamics of a sample magnetic monopole within the electric and magnetic fields of an ideal hyperbolic Penning Trap for point charges. The Lorenz force equation for a given magnetic charge in cylindrical coordinates, like that of its electric counterpart, was used to decompose the prospective motion of the sample magnetic monopole of a given mass to determine its likely axis of stability. The theoretical analysis showed that the motion of the magnetic monopole exhibited a more stable trajectory about the axis of the Penning trap's constant magnetic field.

1.0 INTRODUCTION

1.1 History

The subject of magnetic monopole research holds a strategic position in the scheme of physics, and generally how we interpret fundamental particles in particle physics research. These theoretical objects possess magnetic charges, just like their electric counterparts that are characterized by the possession of electric charges, either positive or negative electric charges.

Pierre Curie, in 1894, highlighted the fact that there might be possible existence of entities with magnetic charges, in symphony with the electrically charged entities; an idea that was very radical, compared to those of members of the scientific society as at that time[1].

Dirac simply argued that the quantization of electric charge is a direct consequence of the existence of a magnetic monopole, leaving basically all electric charges with integer multiples of a fundamental unit [3][11]. Buttressing arguments were put further by Polyakov [7] and 't Hooft [8], that the existence of monopoles follows from general ideas that are linked to the unification of the fundamental interactions in nature [11].

Particle theorists believe that the strong and electroweak gauge interactions, comprising three apparently independent gauge coupling constants, achieve unification at extremely short distances, becoming a single gauge interaction with a singular gauge coupling constant [9][10][11]. Polyakov and 't Hooft showed that magnetic monopoles are the missing parts to such "grand unified" theory of particle physics [11].

Sequel to the development of quantum mechanics and Dirac's demonstration of the consistency of magnetic monopoles with quantum electrodynamics[11], the field of monopole research seemed to be dominated by quantum mechanical descriptions, sometime way abstract to understand the basic nature of such theoretical entities. Still, a classical model for studying the properties of magnetic monopoles in the presence of given electromagnetic configurations is necessary [2]. This not only supports our basic understanding of fundamental particles, but it also paves the way for us to apply first-principle electromagnetic theory, towards knowing more about the behavior of fundamental entities, such as magnetic monopoles, their assessable extrinsic properties, as well as their predictable dynamics in given electromagnetic field configurations[11].

This research paper aims to supplement, classically, the already existing theory about magnetic monopoles, especially as it applies to their dynamic nature within the confines of an electromagnetic field, in this case the ideal hyperbolic Penning trap.

2.0 THEORETICAL BACKGROUND

2.1 What are Magnetic Monopoles?

Basically, magnets are known for their attached "north-south" polar configurations. Those poles are known to exist in pairs, with like poles repelling while unlike poles are attracted to each other. What if the fields are separated by some mechanism, and they continue their individual existence in time and space? This brings us closer to the concept of magnetic monopoles. Magnetic monopoles can be visualized as theoretical objects with segregated poles, for example, north pole far apart from a south pole or a south pole far apart from a north pole. Entities with such configuration are simply referred to as "magnetic monopoles". In comparation with the features of electric charges, magnetic monopoles are said to possess net "magnetic charge," which could either be a positive magnetic charge or a negative magnetic charge. Also, given the existing knowledge about magnetic monopoles, it is worthy of note to state that these entities are not responsible for the physical phenomenon of magnetism, as in the case of bar magnets and other forms of electromagnets [4].

Furthermore, magnetic monopoles possess magnetic charge that comes in quanta of size $\frac{\hbar c}{2e}$, as first derived by Dirac in 1931 [3]. Such quantization of magnetic charge is synonymous with that of electric charge.

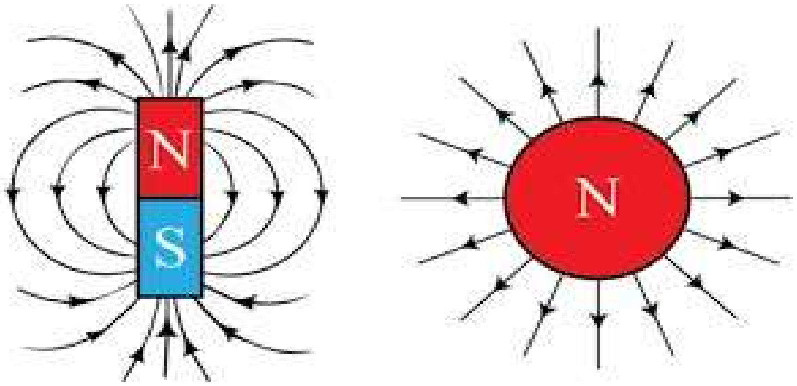

Figure 1: A magnet and a magnetic monopole [19].

2.2 Highlights of Maxwell's Equations and Magnetic Monopoles

Classical electrodynamics has been elegantly and explicitly put forward by the Maxwellian equations given below [4]:

$$\nabla \cdot \mathrm{E} = \frac{\rho_e}{\varepsilon_o}$$

$$\nabla \cdot \mathrm{B} = 0$$

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla \times \mathrm{B} = \frac{1}{c^2}\frac{\partial \mathrm{E}}{\partial t} + \mu_o j_e$$

$$\mathrm{F} = q_e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Given Maxwell's equations, we can say that an electric charge has an electric charge density which depicts the nature of the divergence of the E-field produced by such electric charge. Since a magnetic charge has similar features with its electric counterpart, there is a possibility that such magnetic charge would give non-zero divergence to the B-field produced. Similarly, a current of magnetic monopoles would give a circulating E-field along that current, in like manner as a current of electric charges which gives a circulating B-filed along that current [5]. Again, the Lorentz force equation on an electric charge has two components, the electric force on the electric charge as well as the magnetic charge on the electric force, which is dependent on the velocity of the electric charge. The theoretical existence of a magnetic monopole depicts that such Lorentz force equation for a magnetic charge exists in symmetric form to that of its electric counterpart. This brings about the dire need for a more symmetric Maxwell's equation of electromagnetism.

Considering magnetic monopoles, Maxwell's equations take a particularly symmetric form given below [4][6]:

$$\nabla \cdot \mathrm{E} = \frac{\rho_e}{\varepsilon_o}$$

$$\nabla \cdot \mathrm{B} = \mu_o \rho_m$$

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t} - \mu_o j_m$$

$$\nabla \times \mathrm{B} = \frac{1}{c^2}\frac{\partial \mathrm{E}}{\partial t} + \mu_o j_e$$

$$\mathrm{F} = q_e\left(\vec{E} + \vec{v} \times \vec{B}\right) + q_m\left(\vec{B} - \vec{v} \times \frac{\vec{E}}{c^2}\right)$$

As described previously, the presence of a magnetic charge density guarantees a non-zero divergence to the B-filed, in like manner with the electric charge to the E-field. Similarly, a current of magnetic charge produces a circulating E-field, in symphony with the electric charge and its circulating B-field. Lastly, Lorentz force equation has a component for a hypothetical magnetic charge, as seen in the symmetric Maxwell's equation of electromagnetism. The Maxwell's equations listed above were in Ampere-meter convention.

2.3 Some Detection Methods for Magnetic Monopole

There are two main groups of detection methods for magnetic monopoles:

2.3.1 Detection of pre-existing magnetic monopoles

These group of detectors are responsible for detecting pre-existing magnetic monopoles [4]. An explanation of their detection techniques is given below:

2.3.1.1 Superconducting induction devices

These devices work based on the principle of electromagnetic interaction between the magnetic charge and the macroscopic quantum state of the superconducting ring involved in the detector.

Theoretically, when a magnetic monopole propagates through the cross-sectional area of a coil of wire, a net current is induced in the coil, with respect to Faraday's law of electromagnetic induction. This phenomenon is used as a technique for testing the presence of magnetic monopoles [4]. If the resistance of the wire is finite, there is a dissipation of induced current, converting its energy heat. To avoid such loss in energy, a superconducting loop is utilized. Here, the induced current flows for a long while, unlike the first instance. In principle, a highly sensitive "superconducting quantum interference device" (SQUID) can help to achieve the right result for detection of a single magnetic monopole, owing to its current-preserving feature [4].

Consider a propagating magnetic monopole which induces an electromotive force, and corresponding current (Δi), in a coil of N turns and inductance, L.

$$\Delta i = \frac{4\pi N n g_D}{L} = 2\Delta i_0$$

Δi_0 is the change in current corresponding to one-unit change in the flux quantum of superconductivity

Practically, the parameters can be outlined as follows:

$\Delta i \simeq 10^{-9}$ A, L $\simeq$ few μH, energy $\simeq 4 \times 10^{-1}$ erg [12]

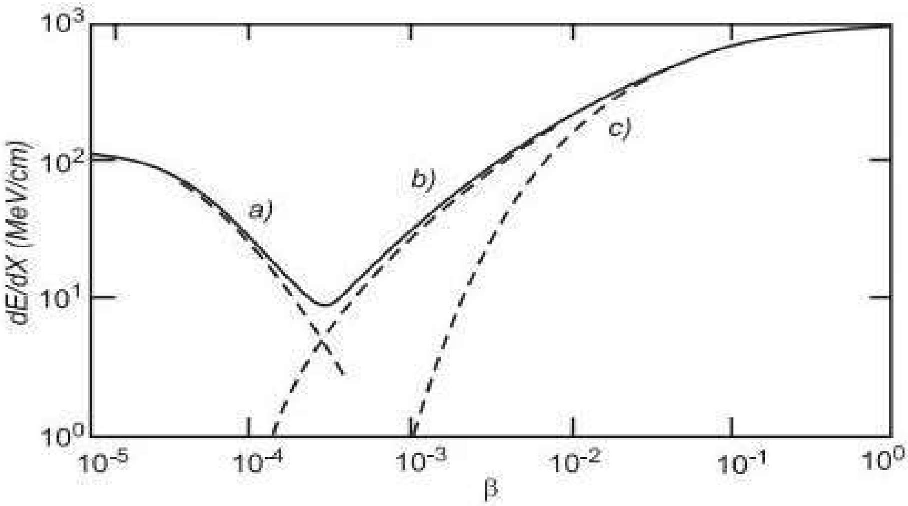

Figure 2: Energy losses in MeV/cm, of g = g_D magnetic monopoles in liquid hydrogen as a function of β. Key: (a) Corresponds to elastic monopole-hydrogen atom scattering; (b) Corresponds to interactions with level crossings; (c) Description of the ionization energy loss. $g_D = \frac{\hbar c}{2e}$ = Dirac charge [12].

2.3.1.2 Scintillation counters

Scintillation counters are well known for their ability to detect tracks of ionizing particles, as they pass through each layer of these counters, and into the scintillator material. As they interact with the scintillator material, atoms are excited along a track; the tracks for these charged particles are simply their trajectories along the given region of confinement [4]. Using gamma rays (uncharged), as an example, their energy is converted to an energetic electron via either the photoelectric effect, Compton scattering or pair production [4]. Similarly, with multiple layers of scintillators or gaseous detectors, a "signal" for a monopole could be detected by recording the place and the time of passage in each layer of the scintillator material. Here, figure 3 [13] shows the divergent polynomial curves for g = n*g_D monopole with n variable from 1-9. At $\beta \sim 10^{-4}$, the light signal is at its maximum when compared to that of a relativistic muon [12]. For $\beta > 0.1$, there is an observable increase in light yield since many delta rays are produced at this point. An estimated value for $\beta \simeq 10^{-4}$ was gotten by Ficenec et al. [14], for n-p elastic scattering in liquid scintillators which is applicable to slow protons [12].

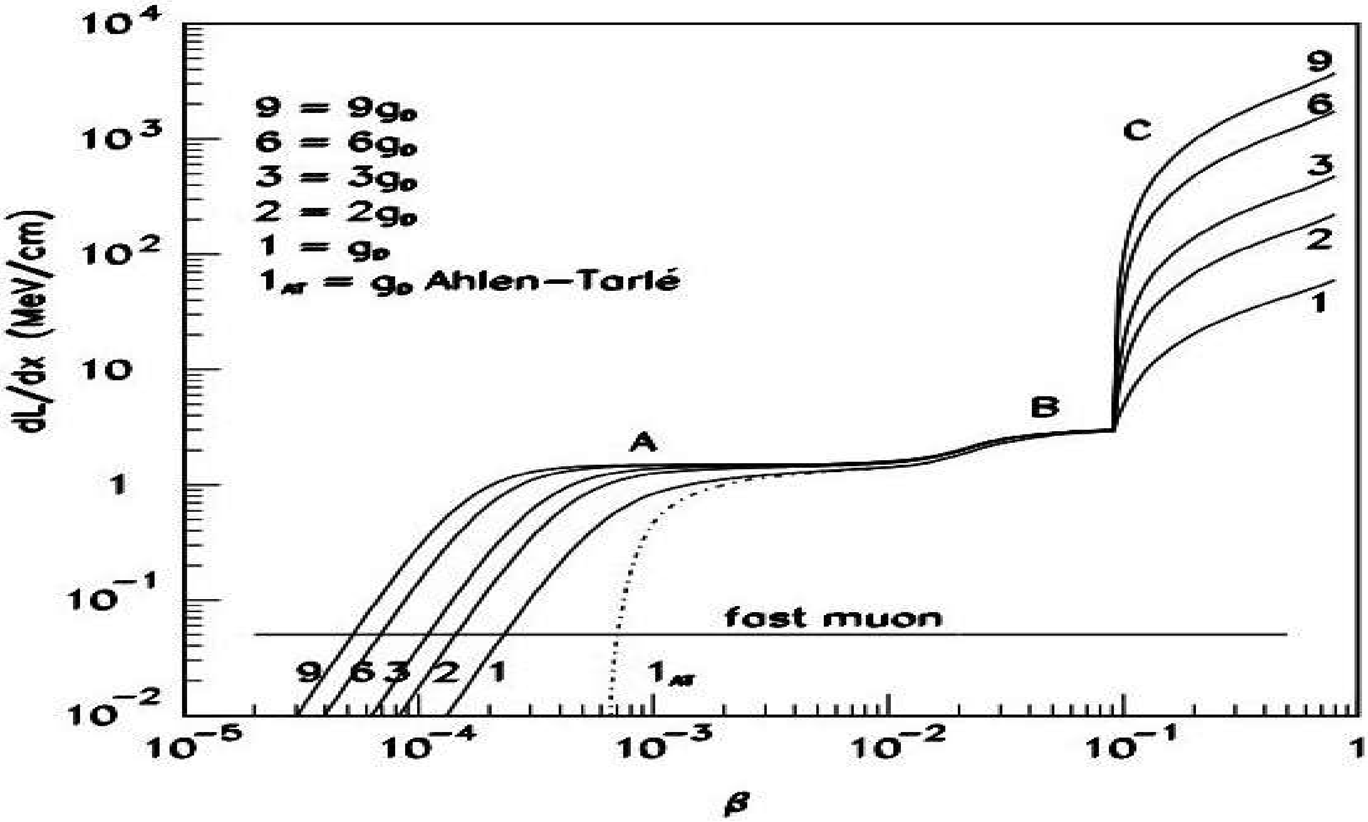

Figure 3: Light yield of magnetic monopoles in the plastic scintillator NE110(ρ = 1.032 g/cm^3) and in the MACRO liquid scintillator (ρ = 0.86 g/cm^3), versus β for g = n*g_D magnetic charge with n = 1-9 [13]

2.3.2 Creation and detection of new magnetic monopoles

Owing to their multi-purpose use, high energy particle colliders have been utilized in the process to create magnetic monopoles, in pairs, but the basic challenge is on how to proffer solutions to lower bounds on the mass of magnetic monopoles, for collider-based searches for these theoretical objects. Conservation of energy allows only magnetic monopoles with masses less than half of the center of mass energy of the colliding particles to be produced. As a result, the theoretical analysis of the creation of magnetic monopoles in high energy particle collisions is limited. This is due to their large magnetic charge, which does not converge with known computational techniques. The only predictable parameter for the search for magnetic monopoles is the existence of defined upper bounds on the probability of cross section of pair production, as a function of energy. As such, we shall explore the common detection methods under this sub-heading[4].

2.3.2.1 Monopole and Exotics Detector at the Large Hadron Collider (MoEDAL) detectors

The MoEDAL experiment is used for direct search for magnetic monopoles and other stably massive fundamental particles that are highly ionizing. It is designed to detect a track left behind by a moving magnetic monopole as it is slowed down and trapped in an aluminum detector. To confine such particle, the aluminum bars can trap sufficiently slowly moving magnetic monopoles, and thereafter the bars can then be analyzed by passing them through a very sensitive magnetometer for measuring highly delicate magnetic fields, based on the principle of superconducting loops. An example of such superconducting loop is the **superconducting quantum interference device** (SQUID).

2.3.2.2 A Toroidal Large Hadron Collider ApparatuS (ATLAS)

So far, this detection technique with ATLAS is characterized by a precise cross section limits for magnetic monopoles of 1 and 2 Dirac charges, which are produced through Drell-Yan pair production mechanism. It also involves searches for these particles based on theories that classify them as long lived and predominantly ionizing. In 2019 the search for magnetic monopoles in the ATLAS detector documented its results from data collected from the LHC Run 2 collisions at center of mass energy of 13 TeV, which at 34.4 fb^{-1} is the largest dataset analyzed to date [15]. The technique involved was to measure signal patterns in data for large energy deposits, which are produced by magnetic monopoles in the ATLAS particle detector [15]. Theoretically, the observed energy deposits are proportional to the magnetic charge squared. The field of magnetic monopole research has been an interesting field, even though a magnetic monopole has not yet been detected. It will be interesting to consider how a magnetic monopole behaves within a region under the influence of an electromagnetic field, for reasons of the fact that it makes Maxwell's equations symmetric, besides other reasons stated previously.

2.4 A Case-study of Detection Method with an ideal hyperbolic Penning trap

An ideal hyperbolic Penning trap is a device that is used in creating an isolated and stable environment required for particle confinement. Here, the particles do not interact with the containing walls or the residual gaseous environment . This enables the aforesaid device to trap, efficiently, exotic particles like, positrons, antiprotons, and highly charged ions. Only static electromagnetic fields are used for the configuration of the Penning trap. It does not offer unwanted perturbations to particles within its confinement[16]. So far, the Penning trap has been extraordinarily useful in predicting and gauging the essential physical properties of the aforesaid fundamental entities like, cyclotron and spin resonances, monitoring the temperature of electrons in a Penning trap, and laser cooling demonstration experiment, to name a few. It is worthy of note to state that much is yet to be known about the possible motion of a hypothetical magnetic monopole, confined within an ideal hyperbolic Penning trap. We can theoretically configure a Penning trap to sustain and measure definite extrinsic properties of our proposed magnetic monopole.

2.5 Geometric Description of an ideal hyperbolic Penning trap

The ideal hyperbolic Penning trap comprises the following configuration

Three electrodes

The upper and lower end cap electrodes, in addition to the ring electrode, form hyperboloids of revolution which generate pure quadrupolar potential within the Penning trap, through the DC potential applied between endcaps and ring. This enables ions to be trapped in the axial direction, thereby creating an unstable radial motion for the trapped ion[16].

Constant Magnetic Field

Strong B field applied along the axis provides radial confinement for the trapped ion, forcing it into loops of cyclotron motion[16].

Ultra-high Vacuum

Again, the Penning trap creates a stable environment that helps the trapped ion to avoid collisions, which could lead to ion loss due to instability of radial motion[16].

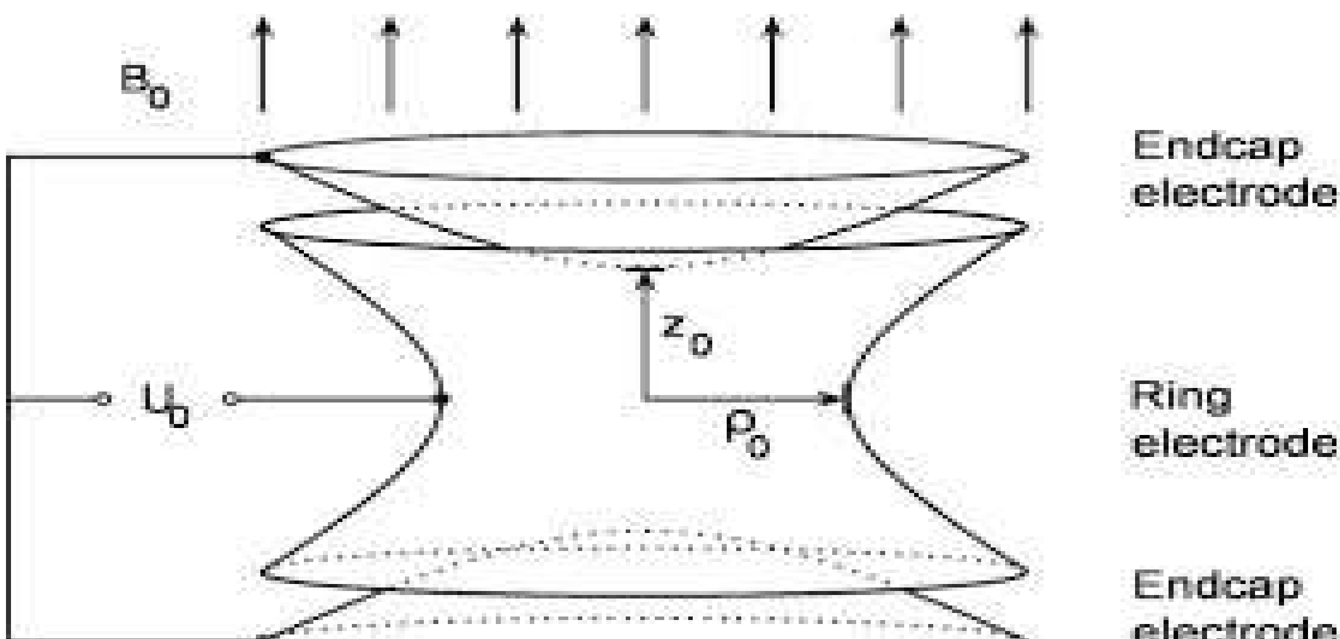

Figure 4: Diagram of an ideal hyperbolic Penning trap[20]

2.6 Motion of a Confined Ion in an ideal hyperbolic Penning trap

Considering the combined configuration of the Penning trap, its hyperbolic end cap and ring electrodes form a 3-dimensional quadrupolar electric field. Superposition of the 3-dimensional quadrupolar electric potential and a homogenous magnetic field creates the right environment for particle confinement. When an ion is confined within the Penning trap, it is subject to three forms of motion outlined below.

Axial motion

This motion is characterized by a sinusoidal movement about the direction of the magnetic field lines of the Penning trap, and between the endcaps. This is labelled as (a) in the figure below[16].

Modified cyclotron motion

This type of motion simply involves orbiting around a magnetic field line. It is also labelled as (9b)[16].

Magnetron motion

It involves a slow orbital movement around trap center, as a result of the influence of the electric and magnetic forces on the ion within the confines of the Penning trap. It is also a form of unstable motion of the ion, within the Penning trap[16].

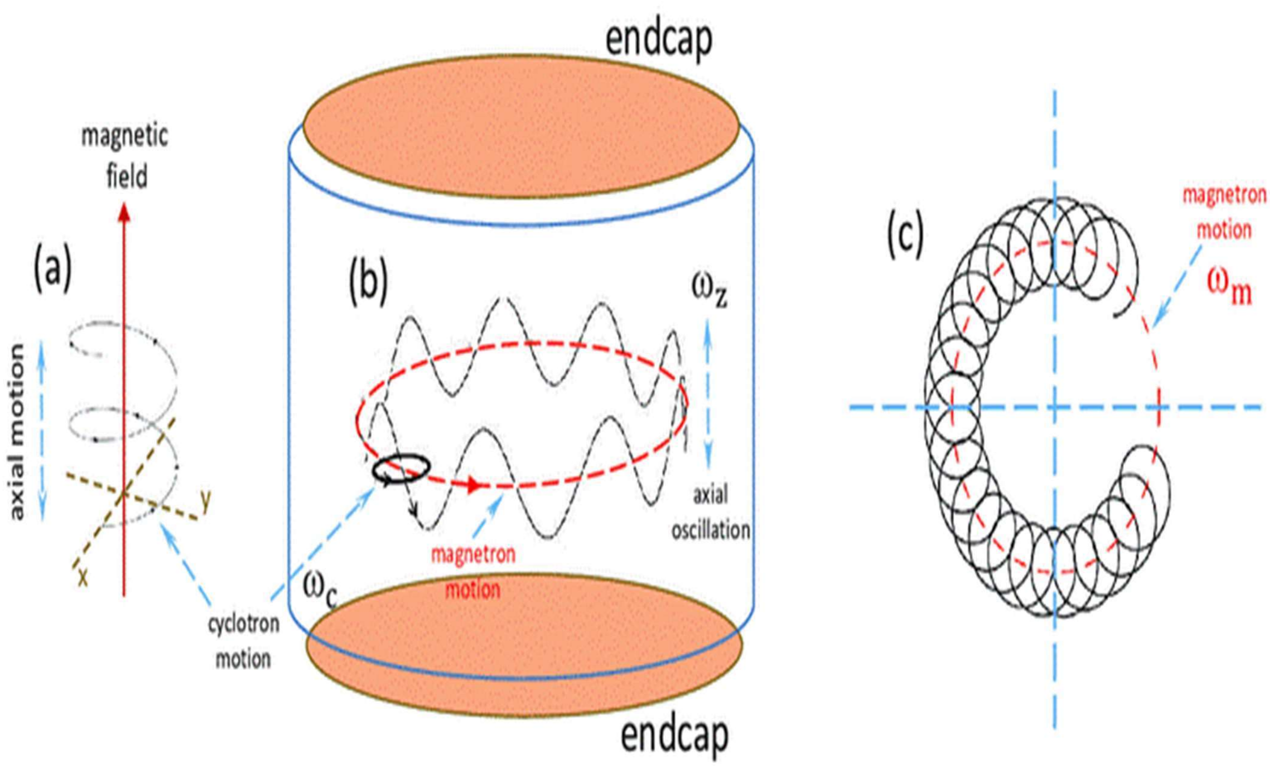

Figure 5: Motions of a confined ion in a Penning trap [21].

3.0 METHODS

3.1 Thought-Experiment with an ideal hyperbolic Penning trap

Given the Lorenz force on a test positive magnetic charge, g,

$$\overrightarrow{F_L} = g\left[\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right] \text{ [4]} \qquad (1)$$

$\vec{B} = B_0\hat{z}$ [16] represents the external magnetic field within the Penning trap configuration

$\vec{E} = -\nabla U$ [16], U is the quadrupolar potential of the Penning Trap device. We assume that the possible $\vec{E}$ and $\vec{B}$ fields of the magnetic monopole are negligible, for this casestudy. We will also use the cylindrical coordinate system as applicable to the symmetry of the Penning Trap device.

Recall: The gradient in cylindrical symmetry is given as:

$$\nabla = \left(\frac{\partial}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial}{\partial\varphi}\hat{\varphi} + \frac{\partial}{\partial z}\hat{z}\right)\text{[22]} \qquad (2)$$

The three-dimensional quadrupolar potential for an ideal hyperbolic Penning trap is given as:

$$U = U_0\frac{(2z^2 - \rho^2)}{(2z_0^2 + r_0^2)} \text{ [16]} \qquad (3)$$

The equipotential surfaces of U are normal hyperboloids, and the consist of the potential difference, U_0, between the two surfaces. Implementing the expression for the electric field, using equations 2 and 3, results to the equation below:

$$\vec{E} = U_0\frac{(2\rho\hat{\rho} - 4z\hat{z})}{(2z_0^2 + r_0^2)}$$

This is the expression for the basic electric field of the Penning trap, using cylindrical coordinates and the symmetry of the quadrupolar potential.

Here, for simplicity, we are going to establish some constants to make further computations achievable.

Let
$$\beta = \frac{U_0}{c^2(2z_0^2 + r_0^2)} \qquad (5)$$

Also, we establish that,

$$\frac{\vec{E}}{c^2} = \beta(2\rho\hat{\rho} - 4z\hat{z}) \qquad (6)$$

For a position vector in cylindrical coordinates system, $\vec{r} = \rho\hat{\rho} + z\hat{z}$ (7)

Here, we can derive the components of its velocity in the cylindrical symmetry, given as

$$\vec{v} = \dot{\vec{r}} = \dot{\rho}\hat{\rho} + \rho\dot{\varphi}\hat{\varphi} + \dot{z}\hat{z} \qquad (8)$$

Furthermore, we can proceed to derive the components of the acceleration by taking a further time-derivative of equation 7, as described below:

$$\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z} \qquad (9)$$

The afore-derived propositions are based on the time derivatives of cylindrical component basis vectors given as:

$$\dot{\hat{\rho}} = \dot{\varphi}\hat{\varphi} \qquad (10)$$

$$\hat{\varphi} = -\dot{\varphi}\hat{\rho} \qquad (11)$$

$$\dot{\hat{z}} = 0 \qquad (12)$$

Recall, $\hat{\varphi}$ is not needed to define the aforementioned position vector $\vec{r}$, because $\hat{\varphi}$ and $\hat{\rho}$ change to describe the given position of a given point in cylindrically symmetric system. It actually resurfaced in the expression for velocity and acceleration, respectively.

Recall, from Newtonian mechanics, the force component acting on an object is simply the product 0f the force as well as the acceleration of the object.

This implies that, typically, the Lorenz force on the hypothetical magnetic monopole is given as:

$\overrightarrow{F_L} = m\vec{a}$,   here m = mass of the magnetic monopole

Also, $\overrightarrow{F_L} = \mathbf{g}\left[\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right]$

$$m\vec{a} = \mathbf{g}\left[\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right]$$

$$\vec{a} = \frac{\mathbf{g}}{m}\left[\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right] \qquad (13)$$

Recalling equation 6, and using a matrix configuration to resolve the cross-products:

$$\frac{\vec{v}\times\vec{E}}{c^2} = \begin{vmatrix} \hat{\rho} & \hat{\varphi} & \hat{z} \\ \dot{\rho} & \rho\dot{\varphi} & \dot{z} \\ 2\beta\rho & 0 & -4z \end{vmatrix} \qquad (14)$$

We find the accurate terms for the cross products by finding the determinant of the 3×3 matrix.

$$\frac{\vec{v}\times\vec{E}}{c^2} = -4\rho z\dot{\varphi}\hat{\rho} + (4z\dot{\rho} + 2\beta\rho\dot{z})\hat{\varphi} - 2\beta\rho^2\dot{\varphi}\hat{z} \qquad (15)$$

Since $\vec{B} = B_0\hat{z}$,

$$\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2} = 4\rho z\dot{\varphi}\hat{\rho} - (4z\dot{\rho} + 2\beta\rho\dot{z})\hat{\varphi} + (B_0 + 2\beta\rho^2\dot{\varphi})\hat{z} \qquad (16)$$

Recall:
$\vec{a} = \frac{\mathbf{g}}{m}\left[\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right]$
Putting equation 16 into the expression for acceleration, we will arrive at the following:

$$\vec{a} = \frac{\mathbf{g}}{m}[4\rho z\dot{\varphi}\hat{\rho} - (4z\dot{\rho} + 2\beta\rho\dot{z})\hat{\varphi} + (B_0 + 2\beta\rho^2\dot{\varphi})\hat{z}] \qquad (17)$$

Let $\gamma = \frac{\mathbf{g}}{m}$, This is the charge-to-mass ratio for the hypothetical test positive magnetic charge

$$\vec{a} = \gamma[4\rho z\dot{\varphi}\hat{\rho} - (4z\dot{\rho} + 2\beta\rho\dot{z})\hat{\varphi} + (B_0 + 2\beta\rho^2\dot{\varphi})\hat{z}] \qquad (18)$$

Since, $\vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z}$ (from equation 9),

We are going to simplify the acceleration based on the directions involved.

3.1.1 Acceleration in the $\hat{\rho}$ direction

$$\ddot{\rho} - \rho\dot{\varphi}^2 = 4\gamma\rho z\dot{\varphi} \qquad (19)$$

Radial acceleration of the magnetic monopole is slightly harmonic. It also depends on the rate of change in its $\hat{\varphi}$ component, which also is related to the monopole's cyclotron frequency within the confines of the Penning trap. The charge-to-mass ratio also influences acceleration in this direction, depending on its magnitude. It is clearly obvious that the acceleration in this direction is prone to aliquot amount of instability, since the magnetic charge is likely to move freely in this direction without influence from the E and B fields of the ideal hyperbolic Penning Trap device.

3.1.2 Acceleration in the $\hat{\varphi}$ direction

$$\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} = -\gamma(4z\dot{\rho} + 2\beta\rho\dot{z}) \qquad (20)$$

Simplifying equation 20, we have:

$$\ddot{\varphi} + 2\frac{\dot{\rho}\dot{\varphi}}{\rho} = -\gamma\left(4z\frac{\dot{\rho}}{\rho} + 2\beta\dot{z}\right) \qquad (21)$$

Here, acceleration along this direction holds a promising course for stability, but there are some velocity-dependent resistive forces along the $\hat{\rho}$ and $\hat{z}$ directions. Theses forces tend to dampen the motion of the hypothetical magnetic monopole, as it propagates within the confinement of the hyperbolic Penning trap.

This implies that, for a magnetic monopole with constant magnetic charge-to-mass ratio, γ, the resistive component of force involving $\dot{z}$ decreases as the potential difference, U_0, decreases.

Also, its acceleration along this direction is not influenced by the internal magnetic field of the Penning trap.

3.1.3 Acceleration in the $\hat{z}$ direction

$$\ddot{z} = \gamma(B_0 + 2\beta\rho^2\dot{\varphi}) \qquad (22)$$

Along this direction, it shows theoretically that the acceleration of the hypothetical magnetic monopole simply undergoes minimum amount of drag-force, a promising evidence for stability in its motion.

We can see that, along this direction, the aforesaid magnetic charge tends to interact constantly with the electromagnetic field of the Penning trap, which is likely to guarantee a higher chance of confinement about this direction.

3.2 Boundary Conditions Considered

To enable us to understand the true dynamics of a hypothetical magnetic charge in the presence of an electric and magnetic field, applicable to an ideal hyperbolic Penning trap, we shall carefully explore the three equations of the monopole's acceleration, on the basis of some applicable boundary conditions.

3.2.1 For the $\hat{\rho}$ direction

We are going to assume a condition where the magnetic monopole is not allowed to accelerate so as not to lose its energy through radiations. This implies that its acceleration becomes negligible, along this direction.

$$\ddot{\rho} - \rho\dot{\varphi}^2 = 4\gamma\rho z\dot{\varphi}$$

Here, $\ddot{\rho} = 0$ and the magnitude for equation 19 reduces to:

$$\dot{\varphi} = 4\gamma z \qquad (23)$$

This expression could be seen as the monopole analog of the electron's cyclotron frequency

3.2.2 For the $\hat{\varphi}$ direction

Since we have a monopole analog for the cyclotron frequency applicable to confined particles, it then implies that any change in position with respect to time, along the $\hat{\rho}$ direction, will be negligible(constant circle condition).

This implies that, $\dot{\rho} = 0$, likewise acceleration, $\ddot{\varphi} = 0$. The equation 21 has no promising trait for parametric substitutions. It does not imply that the overall equation is not useful. We shall explore this equation again, in the course of our perturbative analysis(below).

$$\ddot{\varphi} + 2\frac{\dot{\rho}\dot{\varphi}}{\rho} = -\gamma\left(4z\frac{\dot{\rho}}{\rho} + 2\beta\dot{z}\right) \qquad (21)$$

3.2.3 For the $\hat{z}$ direction

Recall equation 23, which contains an expression for $\dot{\varphi}$. Here, we are going to substitute the expression for $\dot{\varphi}$ into equation 22 above

$$\ddot{z} = \gamma(B_0 + 2\beta\rho^2\dot{\varphi}) \qquad (22)$$

This simplifies to the expression:

$$\ddot{z} + (8\gamma^2\beta\rho^2)z = \gamma B_0 \qquad (24)$$

It is obvious that the acceleration along the z direction of the Penning trap, corresponds to a solvable time-bound second order non-homogenous ordinary differential equation(ODE).

The boundary condition for stability along the $\hat{z}$ direction shall not be substituted yet but shall be used later to proffer a general solution to the aforesaid ordinary differential equation.

Solving the Time-bound Second Order Non-Homogenous Ordinary Differential Equation

Generally, it has the form:

$$P\ddot{z} + Q\dot{z} + Rz = G, \qquad G \neq 0$$

Here, $G(z) = Z_H + Z_P$

Based on symmetry, we assume that the complementary function is a combination of the magnetic monopole's amplitudes of possible oscillation, in response to the electromagnetic field of the Penning trap.

Again, transforming the ordinary differential equation, we have:

$$r^2 + 8\gamma^2\beta\rho^2 = 0; \quad r^2 = -8\gamma^2\beta\rho^2 = i^2 8\gamma^2\beta\rho^2; \quad r = \pm i2\gamma\rho\sqrt{2\beta} \qquad (25)$$

Let Z_H = Complimentary function

$$Z_H = Ae^{-i(2\gamma\rho\sqrt{2\beta})t} + Be^{i(2\gamma\rho\sqrt{2\beta})t}; \qquad (26)$$

A and B are amplitudes of oscillation for the hypothetical magnetic charge.

Let the particular solution be $Z_P = C$, where C = constant, since the product, γB_0 is also a constant.

Generally, $Z(t) = Z_H(t) + Z_P$

To find C, we are going to use $\ddot{Z}_P$ and Z_p as $\ddot{Z}$ and Z in the equation:

$$\ddot{z} = \gamma(B_0 + 2\beta\rho^2\dot{\varphi}) \quad \text{OR} \quad \ddot{z} + (8\gamma^2\beta\rho^2)z = \gamma B_0$$

This implies:

$$8\gamma^2\beta\rho^2 C = \gamma B_0; \quad C = \frac{B_0}{8\gamma\beta\rho^2}\ ; \quad Z_P = \frac{B_0}{8\gamma\beta\ {}^2} \tag{27}$$

The general solution along the $\hat{z}$ direction is given as: $Z(t) = Z_H(t) + Z_P$

$$Z(t) = Ae^{i(2\gamma\rho\sqrt{2\beta})t} + Be^{-i(2\gamma\rho\sqrt{2\beta})t} + \frac{B_0}{8\gamma\beta\rho^2} \tag{28}$$

Recall, we discussed two constraints applicable to the $\hat{z}$ direction, which we are going to implement, to enable us to arrive at the values for the amplitudes of oscillation; A and B.

Constraint 1

$Z(0) = 0$ (Symmetry related constraint). It assumes the magnetic charge will have no displacement along the $\hat{z}$ direction, on entry into the Penning Trap. But its constituent amplitudes of oscillation tend to undergo some reorientation, by virtue of the direction of the electromagnetic field within the trap.

$$A + B = -\frac{B_0}{8\gamma\beta\rho^2}$$

Constraint 2

Expressing the term for the acceleration along the $\hat{z}$ direction, using the general solution, we have:

$$\ddot{Z}(t) = -A(2\gamma\rho\sqrt{2\beta})^2 e^{i(2\gamma\rho\sqrt{2\beta})t} - B(2\gamma\rho\sqrt{2\beta})^2 e^{-i(2\gamma\rho\sqrt{2\beta})t}$$

Here, we imply that $\ddot{Z}(0) = 0$, for equilibrium constraint along the $\hat{z}$ direction.
This implies that,

$$\ddot{Z}(0) = -A(2\gamma\rho\sqrt{2\beta})^2 - B(2\gamma\rho\sqrt{2\beta})^2 = 0$$

$$-A - B = 0$$

Using the positive magnitudes for the magnetic monopole's amplitudes of oscillation, to avoid complex oscillations along the imaginary plane, we have:

$$|A| = |B| \tag{29}$$

Recall:

$$A + B = -\frac{B_0}{8\gamma\beta\rho^2} \tag{30}$$

We will substitute for B in the expression relating A and B to the field variables and constants, stated previously.

This gives the following results:

$$|B| = \frac{B_0}{16\gamma\beta\rho^2}\ ; \quad |A| = \frac{B_0}{16\gamma\beta\rho^2}$$

Finally, we are going to substitute the absolute-value expressions for A and B into the general solution for the magnetic monopole movement along the $\hat{z}$ direction.

$$Z(t) = \frac{B_0}{16\gamma\beta\rho^2} e^{i(2\gamma\rho\sqrt{2\beta})t} + \frac{B_0}{16\gamma\beta\rho^2} e^{-i(2\gamma\rho\sqrt{2\beta})t} + \frac{B_0}{8\gamma\beta\rho^2} \tag{31}$$

In simplified terms, we have:

$$Z(t) = \frac{B_0}{8\gamma\beta\rho^2}\left[\frac{1}{2}\left(e^{-i(2\gamma\rho\sqrt{2\beta})t} + e^{i(2\gamma\rho\sqrt{2\beta})t}\right) + 1\right]$$

Using a skeletal form of the equation for the trajectory of the magnetic monopole, we have:

$$Z(t) = \theta\left[\frac{1}{2}\left(e^{-i\mu t} + e^{i\mu t}\right) + 1\right] \qquad (32)$$

Recall Euler forms of writing real and imaginary variables:
$e^{-i\mu t} = \cos\mu t - i\sin\mu t$; $e^{i\mu t} = \cos\mu t + i\sin\mu t$
Simplifying the transformed equation for the monopole's trajectory along the $\hat{z}$ direction, we have:

$$Z(t) = \theta[1 + \cos\mu t]$$

Here, the previous equation can be simplified using half-angle property for cosine functions. This result to:

$$Z(t) = 2\theta\cos^2\left(\frac{\mu t}{2}\right) \qquad (33)$$

Recall that:

$$\theta = \frac{B_0}{8\gamma\beta\rho^2} \text{ and } \mu = 2\gamma\rho\sqrt{2\beta} \qquad (34)$$

This implies that;

$$Z(t) = \frac{B_0}{4\gamma\beta\rho^2}\cos^2\left(\frac{\mu t}{2}\right) \qquad (35)$$

The expression for the monopole's trajectory along the $\hat{z}$ direction depicts a sinusoidal behaviour about the direction of the magnetic field for the idea hyperbolic Penning trap.

Here, $\mu = 2\gamma\rho\sqrt{2\beta}$ represents the monopole's frequency for axial oscillation about the direction of the aforesaid magnetic field.

Again, a considerable change in the potential difference of the E-field, at constant B-field, simply increases the frequency of axial oscillation of the magnetic monopole and thereby decreases its amplitude of axial oscillation along the $\hat{z}$ direction. This is evidence in the sketch below.

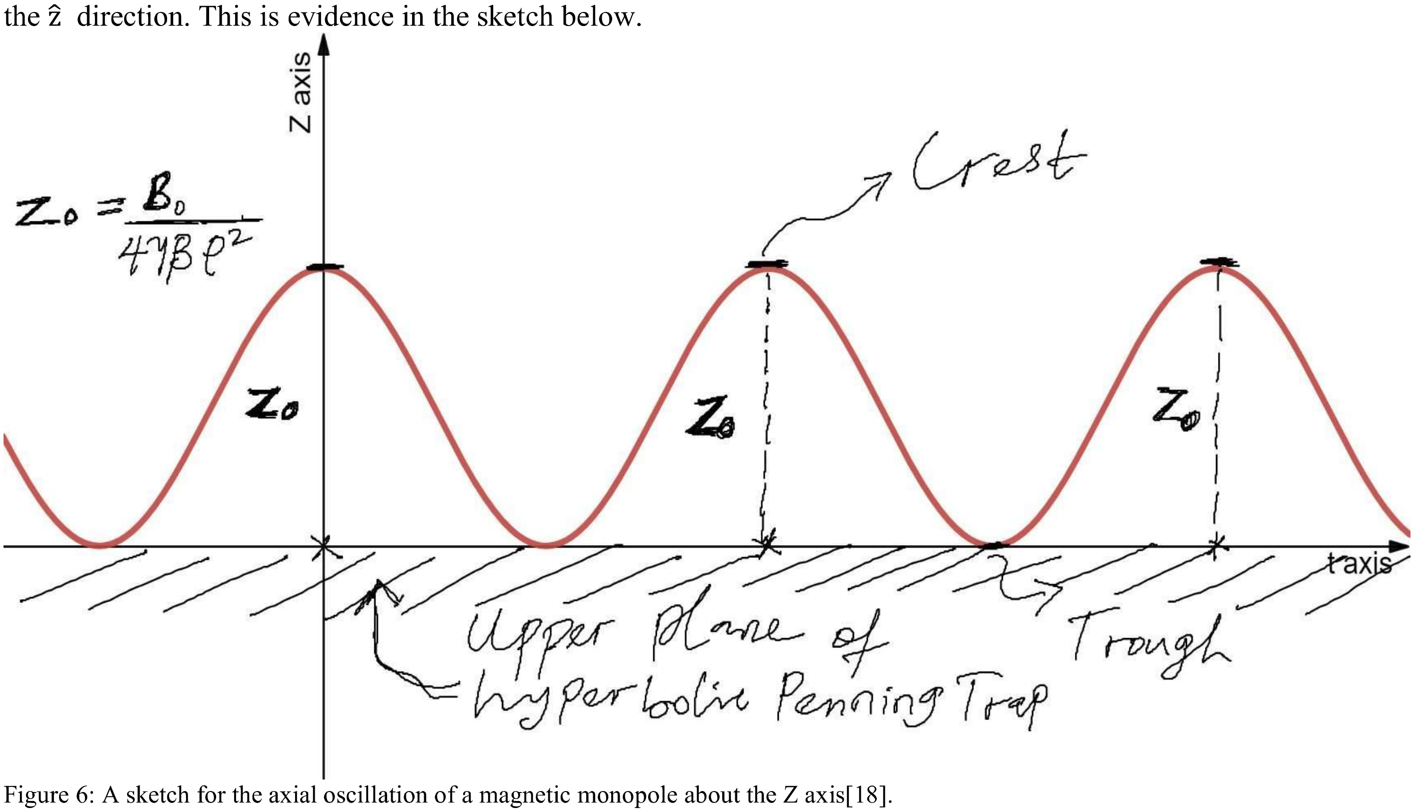

Figure 6: A sketch for the axial oscillation of a magnetic monopole about the Z axis[18].

At t=0, amplitude of monopole oscillation, $Z(0) = \frac{B_0}{4\gamma\beta\rho^2}\cos^2(0) = \frac{B_0}{4\gamma\beta\rho^2} = Z_0 \qquad (36)$

Again, the trajectory for the magnetic monopole along the $\hat{z}$ direction can be re-expressed as:

$$Z(t) = Z_0 \cos^2\left(\frac{\mu t}{2}\right) \tag{37}$$

To enable us to gauge the amount of magnetic field needed, we can use the geometric configurations of the hyperbolic Penning trap as well as a sample potential difference. The given expression below depicts the required relationship.

$$B_0 = 4\gamma\beta\rho^2 Z_0$$

It also allows for the degree of freedom in variation of the potential difference U_0 and the magnetic field strength B_0, to an equilibrium point, to enable us to have a balanced force-environment for the hypothetical monopole. This is the ideal case for the Lorentz force law on a magnetic charge, compared to its electric counterpart.

Another possible case is to reduce the potential difference, which leaves only the magnetic field of the Penning trap active. This procedure will result to a purely magnetic force on the hypothetical magnetic charge, as stated below:

$$\ddot{Z} = \gamma B_0$$

Using the expression for the charge-to-mass ratio for the magnetic charge, further simplification leads to:

$$m\ddot{Z} = gB_0 \tag{38}$$

$$F_m = gB_0$$

Lastly, for an extremely low B-field, the acceleration along the $\hat{z}$ direction is largely influenced by the E-filed as stated below:

$$m\ddot{Z} \approx g\left[\frac{2U_0\rho^2\dot{\varphi}}{c^2(2z_0^2 + r_0^2)}\right] \tag{39}$$

$$F_E \approx g\left[\frac{2U_0\rho^2\dot{\varphi}}{c^2(2z_0^2 + r_0^2)}\right] \tag{40}$$

3.3 Computations for Perturbations Along the Three Distinct Directions

To enable us to gauge the stability of such hypothetical test-positive magnetic charge, we are going to explore the effects of small perturbations on the three groups of equations for its acceleration along the three distinct directions.

Basically, we are going to assume small positive perturbative constant magnitudes, for simplicity. Here, let the parameters vary as follows:

$$\rho = b + \delta\rho;\ \ddot{\rho} = \delta\ddot{\rho} \tag{41}$$

$$\varphi = a + \delta\varphi;\ \ddot{\varphi} = \delta\ddot{\varphi} \tag{42}$$

$$Z = c + \delta Z;\ \ddot{Z} = \delta\ddot{Z} \tag{43}$$

3.3.1 Small Perturbations Along the $\hat{\rho}$ direction

$$\ddot{\rho} - \rho\dot{\varphi}^2 = 4\gamma\rho z\dot{\varphi}$$

$$\delta\ddot{\rho} - (b + \delta\rho)\delta\dot{\varphi}^2 = 4\gamma(b + \delta\rho)(c + \delta Z)\delta\dot{\varphi}$$

Ignoring extremely small perturbations, we have the given equation below:

$$\delta\ddot{\rho} \approx b\delta\dot{\varphi}^2 + 4\gamma bc\delta\dot{\varphi} \tag{44}$$

This result shows no evidence of stability along the $\hat{\rho}$ direction, for the test-positive magnetic charge.

3.3.2 Small Perturbations Along the $\hat{\varphi}$ direction

$$\ddot{\varphi} + 2\frac{\dot{\rho}\dot{\varphi}}{\rho} = -\gamma\left(4z\frac{\dot{\rho}}{\rho} + 2\beta\dot{z}\right) \quad (45)$$

$$\delta\ddot{\varphi} + 2\frac{\delta\dot{\rho}\delta\dot{\varphi}}{(b+\delta\rho)} = -\gamma\left[\frac{4(c+\delta Z)\delta\dot{\rho}}{(b+\delta\rho)} + 2\beta\delta\dot{z}\right] \quad (46)$$

Again, ignoring extremely small perturbations and simplifying the aforesaid expression, we have:

$$\delta\ddot{\varphi} \approx -\gamma\left[\frac{4c\delta\dot{\rho}}{b} + 2\beta\delta\dot{z}\right] \quad (47)$$

The simplified equation for perturbation along the $\hat{\varphi}$ direction shows a promising stability for a magnetic monopole.

3.3.3 Small Perturbations Along the $\hat{z}$ direction

Using the derived general solution and the perturbative constants along the $\hat{\mathbf{z}}$ direction, we have:

$$Z(t) = \frac{B_0}{4\gamma\beta\rho^2}\cos^2\left(\frac{\mu t}{2}\right)$$

Here, the expression for acceleration along the $\hat{z}$ direction is given as:

$$\ddot{Z}(t) = -\frac{\mu^2 B_0}{8\gamma\beta\rho^2}\cos\left(\frac{\mu t}{2}\right) \quad (48)$$

By direct substitution and simplification:

$$\delta\ddot{Z} = -\frac{\mu^2 B_0}{8\gamma\beta b^2}\left(1 + \frac{\delta\rho}{b}\right)^{-2}\cos\left(\frac{\mu t}{2}\right) \quad (49)$$

We can proceed to simplify the previous equation as stated below:

$$\delta\ddot{Z} = -\frac{\mu^2 B_0}{8\gamma\beta b^2}\left(1 - \frac{2\delta\rho}{b} + \ldots\right)\cos\left(\frac{\mu t}{2}\right)$$

$$\delta\ddot{Z} \approx -\frac{\mu^2 B_0}{8\gamma\beta b^2}\cos\left(\frac{\mu t}{2}\right) + \frac{2\delta\rho}{b^3}\cos\left(\frac{\mu t}{2}\right); \quad \frac{1}{b^3} \ll 1 \text{ as } b \to \infty$$

$$\delta\ddot{Z} \approx -\frac{\mu^2 B_0}{8\gamma\beta b^2}\cos\left(\frac{\mu t}{2}\right) \quad (50)$$

Again, the perturbed equation for acceleration along the $\hat{\mathbf{z}}$ direction shows promising evidence for the stability of the hypothetical magnetic monopole.

4.0 Results and Discussion

So far, the theoretical computations simply show that an ideal hyperbolic Penning trap could potentially confine a hypothetical magnetic monopole within its electric and magnetic field. In its confined state, the aforesaid particle oscillates axially about the upper plane of the $\hat{z}$ direction, with an axial frequency, μ. Also, as it oscillates axially about the $\hat{z}$ direction, it exhibits a circular motion about the the center of the axis of the Penning trap's magnetic field, which corresponds to the magnetron motion of an ion in a Penning trap, with an angular velocity given as $\dot{\varphi}$. The resultant motion, which is caused by the influence of the Penning trap's electric and magnetic forces, is a direct combination of the oscillatory motion as well as its circular motion.

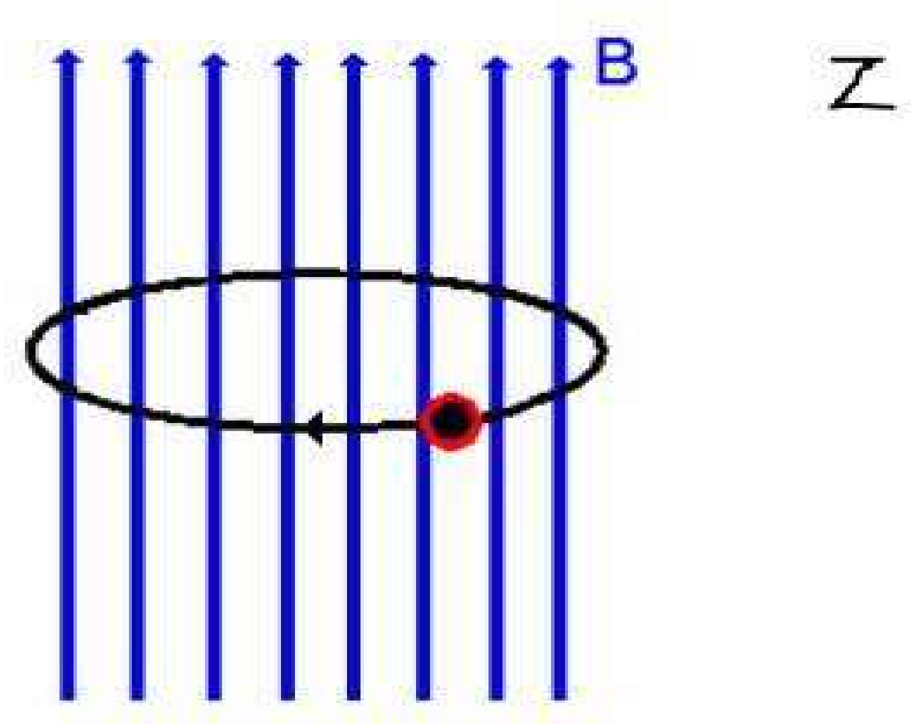

Figure7: Circular motion of a magnetic monopole about the Z axis, with angular frequency of $\dot{\varphi}$ [22]

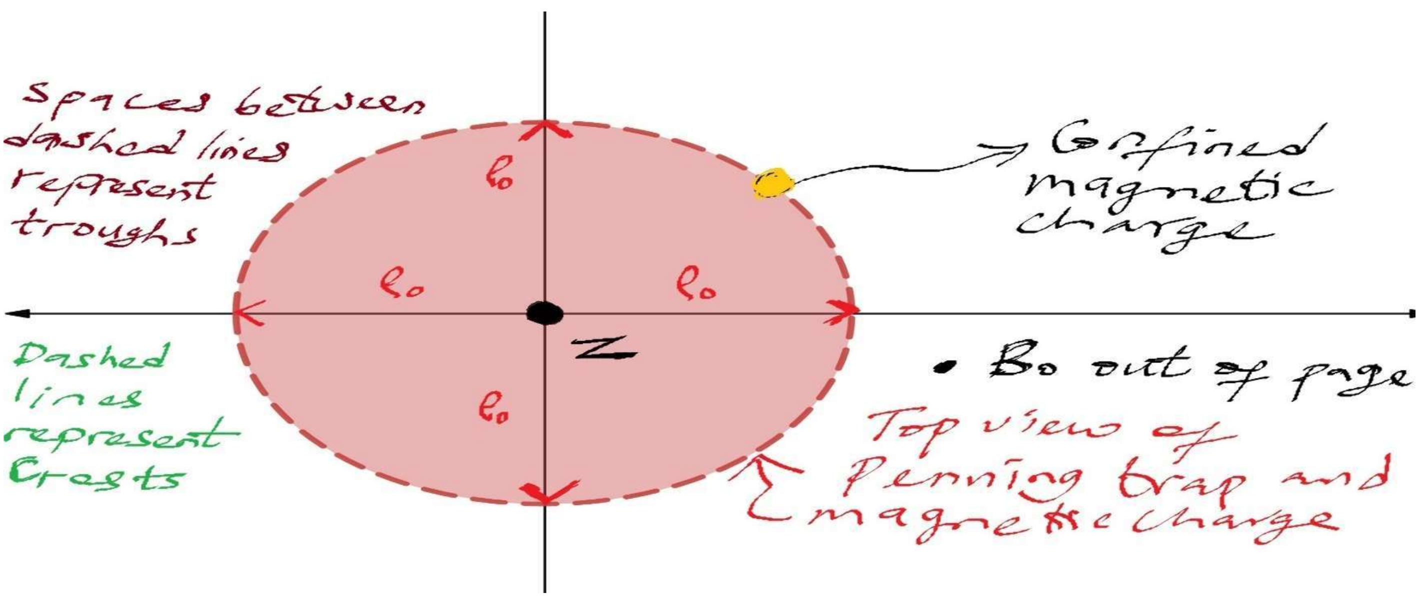

Figure 8: A combination of the magnetic monopole's axial oscillation and circular motion in its confined state within the Penning trap[18]

Furthermore, a detector programmed to check signals for the magnetic monopole within the hyperbolic Penning trap, is likely to record magnetic monopole signatures along the directions of stability, the $\hat{\varphi}$ and $\hat{z}$ directions respectively. Building a hyperbolic Penning trap for such magnetic monopole confinement will require a greater form of sophistication, since we are looking at confining an entity that has not yet been discovered but has shown promising theoretical evidence of its likely confinement within the hyperbolic Penning trap. To enable us to arrive at an accurate prediction for further properties of the aforesaid entity, further simulations shall be carried out using SIMION particle confinement application, alongside the required codes for operation the Penning trap version of this simulation[17]. Another tricky aspect of this research is the quest for an actual value for the charge-to-mass ratio of the magnetic monopole. Difference models for monopole research, have different values for the charge-to-mass ratio of a magnetic monopole. The next stage is to work towards a unified value for the charge-to-mass ratio of the magnetic monopole, so as to enable us arrive at a favorable condition on the search for other parameters required for the numerical analysis of the afore-mentioned results. Once the difference in charge-to-mass ratio has been rectified, and experimental tests are carried out, possibly through hyflex techniques, further checks shall be carried out to determine random errors in observations and results.

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Figures
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