Thermodynamics of the early universe

A case-study of the electron-to-photon number density ratio during the early part of electron-positron annihilation era

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Abstract
The early universe has been an important phase in the history of its formation since it provides a direct link as to how the universe has expanded to its present stage. Looking back at 13.75 billion years ago, the universe existed in form of an energy singularity, having immeasurable density and temperature. This energy experienced a rapid expansion known as the Big-Bang, hence the formation of the universe, all matter, and forms of energy. My work towards the theoretical prediction of the ratio for the electron-to-the photon number density, especially as such ratio is temperature-dependent during the early part of the electron-positron annihilation era, showed a decrease in the electron-to-photon number density ratio, as the temperature decreased between the age of leptons and the age of nucleons respectively. Its shows further that there was a steady drop in the electron-to-photon number density ratio, per temperature decrease, during the beginning of the age of leptons but it achieved a steadier rate towards the end of the age of nucleons.

1.0 Introduction

1.1 Motivation
The early universe represents an environment where thermodynamics, quantum mechanics, and statistical physics interplay to explain various phenomena that took place during such phase. This project seeks to understand the nature of transformation that took place between photons and electrons during the age of leptons and age of nucleons. It also aims at comparing the number densities of photons and electrons during the aforesaid ages, by employing some statistical mechanics principles as well as some quantum mechanical procedures to arrive at a given conclusion about the nature of photons and electrons that existed during the ages of leptons and nucleons.

1.2 Theoretical Background
The earliest moments of the universe was characterized by high temperatures, high enough to reproduce species of relativistic particles-and-antiparticles pairs, with unique energy dispersion relation functions.

Assuming $\mathrm{kT} \gg \mathrm{mc}^2$, this implies the photon-photon interactions were liable to create particle-antiparticle pairs of given mass, with relativistic energy-momentum relationship given as $\varepsilon_k \approx \hbar \mathrm{ck}$; $\hbar \mathrm{k} \equiv \mathrm{p}$, magnitude of the momentum. This relation is applicable to photons, neutrinos, antineutrinos, electrons and positrons, since they are all pair-formation particles from photons. Also $\varepsilon_k \approx \hbar \mathrm{ck}$ is the right expression for relativistic energy because at very high density, degenerate relativistic fermi gases become so energetic that they no longer obey the $\varepsilon_k \approx \frac{\mathrm{p}^2}{2\mathrm{m}}$ expression.

For a particle moving with an arbitrary velocity, there is a special relativistic relation $\varepsilon_k = \mathrm{mc}^2(1 + (\frac{\mathrm{p}}{\mathrm{mc}})^2)^{\frac{1}{2}}$ [4][5][6] . For relativistic asymptotic limits, $\mathrm{p} \gg \mathrm{mc}$; $\varepsilon_k \approx \mathrm{pc}$ (ultra-relativistic limit)[9].

2.0 Age of Leptons: The World of Photons
2.1 Equation of State For Photon Number-density
Consider an elementary cell in a phases-pace with a volume $\Delta x \Delta y \Delta z \Delta p_x \Delta p_y \Delta p_z = \mathrm{h}^3$, where h = Planck's constant(6.63×10^{-27} ergs).

$\Delta x\Delta y\Delta z\Delta p_x\Delta p_y\Delta p_z$ = volume in ordinary space [3][9]
$\Delta p_x\Delta p_y\Delta p_z$ = volume in momentum space [3][9]

Assuming the universe obeyed, fully the laws of quantum mechanics, we can say that there is enough room for approximately one particle of any specie, within any elementary cell choice. This gives an average number of particles per cell as[1][6][8][9]:

$$n_{av} = \frac{g}{e^{\frac{(E-\mu)}{kT}} \pm 1} \qquad (1)$$

"+" is for fermions [g=1/2-spin]; "-" is for bosons [g=1, integer spin; 0,1,2,…]

Since the phenomenon involved electron-photon distribution, I am going to use both cases of "+" and "-" for electrons(fermions) and photons(bosons).

Assuming free ultra-relativistic particles, their energy is a function of momentum only, as described previously. Let the total momentum be p: $p^2 = p_x^2 + p_y^2 + p_z^2$ (3-dimensional universe)[3][9]

Let the number-density of particles in a unit volume, with momenta between p and p+dp be:

$$n(p)dp = \frac{g}{e^{\frac{(E-\mu)}{kT}} \pm 1}\left(\frac{4\pi p^2 dp}{h^3}\right) \qquad (2)$$

Here, the total number is:

$$n_d = \int_0^\infty n(p)dp\,; \qquad n_d = \frac{4\pi g}{h^3}\int_0^\infty \frac{p^2 dp}{e^{\frac{(E-\mu)}{kT}} \pm 1} \qquad (3)$$

Assuming the photons were distributed to a point of thermodynamic equilibrium, in the early universe, this implies the photons attained a phase of local thermodynamic equilibrium. Theoretically, we can say that the photons obeyed Planck's blackbody distribution for cavity-in-a-box experiment.

Furthermore, the local thermodynamic equilibrium assumes $\mu = 0$, for photons. Since they are bosons, their spin is 1. Again, we take g=2, for photons with the possibility of having two quantum states of polarization[2][4][6][9]. This implies:

$$n(p)dp = \frac{g}{e^{\frac{(E-\mu)}{kT}} \pm 1}\left(\frac{4\pi p^2 dp}{h^3}\right)$$

$$n_\gamma(p)dp = \frac{8\pi p^2 dp}{h^3(e^{\frac{E}{kT}} - 1)} \qquad (4)$$

Now we can estimate the total number density using the given expression below:

$n_\gamma = \int_0^\infty n_\gamma(p)dp \approx \frac{8\pi}{(hc)^3}\int_0^\infty \frac{E^2 dE}{\left(e^{E/kT}-1\right)}$; using $\varepsilon_k = E \approx pc$ (ultra-relativistic limit) . Let Z = E/kT, the integral transforms to:

$$n_\gamma \approx \frac{8\pi(kT)^3}{(hc)^3}\int_0^\infty \frac{Z^2 dZ}{\left(e^Z-1\right)} \qquad (5)$$

We are going to resolve the standard integral using the format below.
Recall:

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{Z^{s-1} dZ}{\left(e^Z-1\right)} \qquad (6)$$

where $\zeta(s)$ and $\Gamma(s)$ represent the Zeta and Gamma functions respectively.
Comparing this with the main integral, s-1=2, s=3. $\zeta(3)\Gamma(3) = \int_0^\infty \frac{Z^2 dZ}{\left(e^Z-1\right)}$

Here, the photon number-density during the age of leptons is given as:

$$n_\gamma \approx \frac{8\pi(kT)^3\zeta(3)\Gamma(3)}{(hc)^3} \qquad (7)$$

2.2 Creation of Electrons and Positrons

Sequel to the big-bang phenomenon, it took roughly about one second for the electron-positron pair to be created. This occurred when the temperature got closer to the threshold temperature T_e [9] for such pairs to be created. Here, $kT_e \approx m_e c^2 \approx 0.511$MeV[9]; $T_e \approx 5.93\times10^9$K [9].

When temperature fell below T_e the rate at which electron-positron pairs were created began to drop, below its annihilation rate. This changed the energy dispersion relationship for all pairs formed from the threshold photonic energy present in the universe at that point in time. The new energy-dispersion relation is given below[4][6][9]:

$E \approx mc^2 + \frac{p^2}{2m}$

2.3 Number density of Created Electrons during the Age of Nucleons

The number density of electrons with momentum, p, is given by the fermionic version of the compound number density expression above[4][5][6][8][9]:

$$n_d = \frac{4\pi g}{h^3}\int_0^\infty \frac{p^2 dp}{e^{\frac{(E-\mu)}{kT}} + 1}$$

Substituting the expression for E and for $\beta = \frac{1}{kT}$, and considering the fact that $\frac{(E-\mu)}{kT} \gg 1$, we have:

$$n_e \approx n_d \approx \frac{4\pi g}{h^3}\int_0^\infty \frac{p^2 dp}{e^{\beta mc^2} e^{\frac{\beta p^2}{2m}}\, e^{-\beta\mu}} \qquad (8)$$

For energy conservation, we assume the photon-electron-positron system were all at equilibrium before and after reaching the threshold temperature, T_e.

$\Rightarrow\ e^+ + e^- \rightleftharpoons \gamma + \gamma$ [9]

This also implies that the chemical potentials for the thermodynamic systems were negligible[9].

$\Rightarrow \mu_+ + \mu_- \rightleftharpoons 2\mu_\gamma = 0;\ \mu = \mu_- = 0$

Here, we need to simplify the electron number density equation above, taking g=1/2(fermions) as follows:

$$n_e \approx \frac{2\pi e^{-\beta mc^2}}{h^3}\int_0^\infty e^{-\frac{\beta p^2}{2m}}\, p^2 dp \qquad (9)$$

Substituting $y = \frac{\beta p^2}{2m}$ and simplifying the above integral, we have:

$n_e \approx \frac{\sqrt{2} m^{\frac{3}{2}} \pi e^{-\beta mc^2}}{\beta^{\frac{3}{2}} h^3}\int_0^\infty e^{-y}\sqrt{y}dy$. Using Gamma function; $\int_0^\infty e^{-y}\sqrt{y}dy = \Gamma(3/2) = \sqrt{\pi}/2$.

$n_e \approx \frac{\sqrt{2} m^{\frac{3}{2}} \pi e^{-\beta mc^2}(\sqrt{\pi})}{2\beta^{\frac{3}{2}} h^3} \approx \frac{m^{\frac{3}{2}}\pi^{\frac{3}{2}} e^{-\beta mc^2}}{\sqrt{2}\beta^{\frac{3}{2}} h^3}$;

$$n_e \approx \frac{m^{\frac{3}{2}}\, \pi^{\frac{3}{2}}\, e^{-\beta mc^2}}{\sqrt{2}\, \beta^{\frac{3}{2}}\, h^3} \qquad (10)$$

This is the expression for the number density of electrons present during the age of nucleons.

3.0 Calculating Electron-Photon number density ratio

To enable us to estimate the electron-photon number density ratio, we are going to consider the results for the photon number density as well as that of the electron number density, below.

Recall the expression for photon number density ratio:

$$n_\gamma \approx \frac{8\pi(kT)^3\zeta(3)\Gamma(3)}{(hc)^3}$$

Previously, we calculated the electron number density ratio as:

$$n_e \approx \frac{m^{\frac{3}{2}}\pi^{\frac{3}{2}}e^{-\beta mc^2}}{\sqrt{2}\,\beta^{\frac{3}{2}}h^3}$$

Taking the required ratio, we have:

$$n_T = \frac{n_e}{n_\gamma} \approx 0.0652(\beta mc)^{\frac{3}{2}}e^{-\beta mc^2}$$

$$n_T \approx 0.0652(\beta mc)^{\frac{3}{2}}e^{-\beta mc^2} \qquad (11)$$

Where:
$\zeta(3)$ = 1.2020569
$n_T =$ electron − photon number density ratio between the ages of leptons and nucleons respectively
$\beta = \frac{1}{kT}$, k= Boltzmann's constant (); T = absolute temperature in Kelvin
m = mass of the electrons during transformation
c = speed of light (3×10^8 ms^{-1})

4.0 Numerical Analysis For Electron-Photon Number density Ratio

During the electron-positron annihilation era, it is believed that the temperature started about 10^{10}K, around t=1 second and ended about 3×10^8K, which corresponded with a time, t=33minutes[9]. We are going to explore the temperature limits stated recently and see how these limits affected the electron-positron annihilation era, with respect to the electron-photon number density ratio.

Recall: $n_T \approx 0.0652(\beta mc)^{\frac{3}{2}}e^{-\beta mc^2}$; using k=$1.38064852\times10^{23}$ $m^2kgs^{-2}K^{-1}$; m=9.10938356×10^{-31}kg and temperature within the range, $T \in \{10^{10}\mathrm{K}, 3\times10^8\mathrm{K}\}$[9], the MATLAB codes and corresponding plot are applicable.

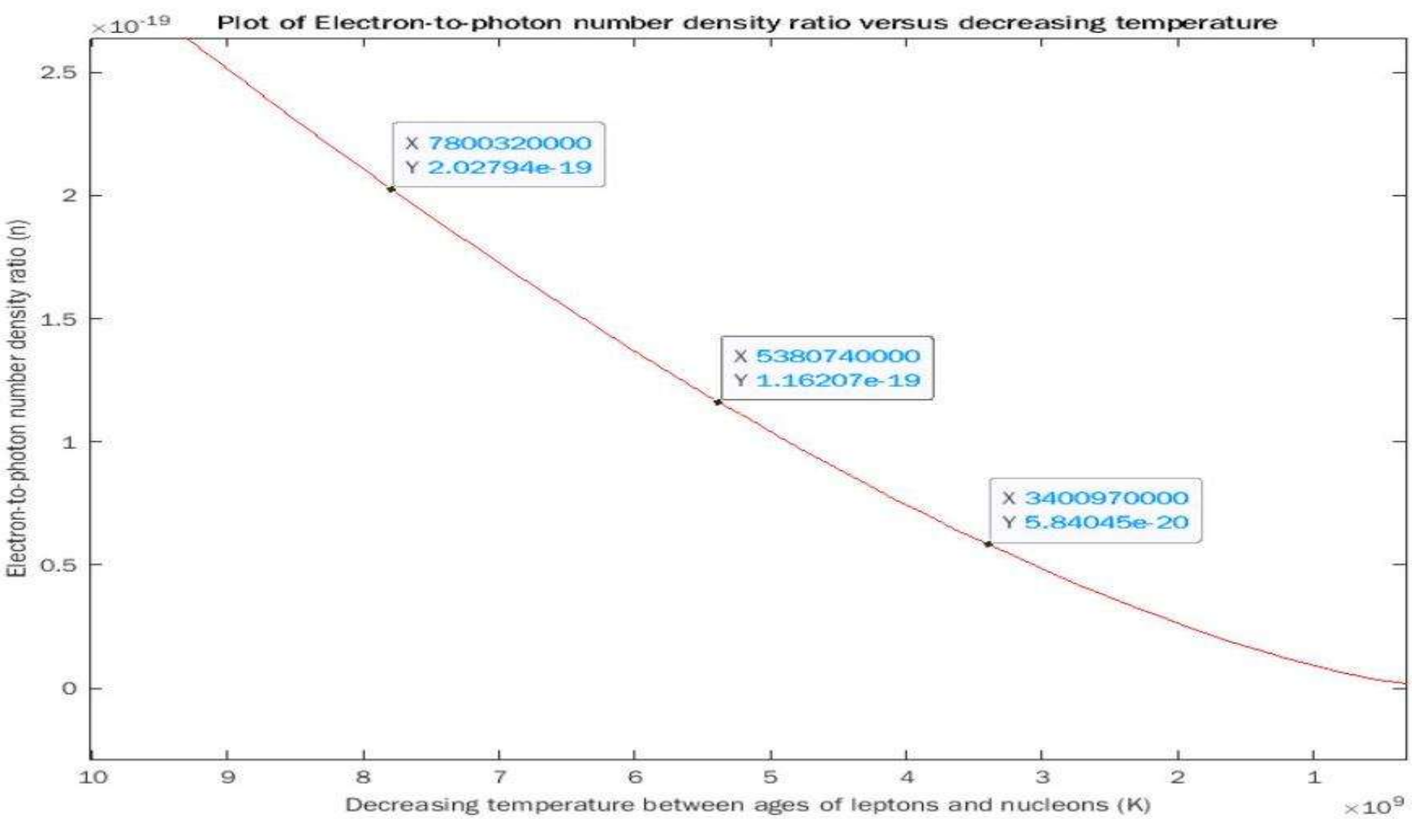

Figure1: Plot of Electron-to-photon number density ratio versus decreasing temperature

5.0 Results

Based on observed, the plot for the theoretical model depicts a decrease in the electron-to-photon number density ratio, as the temperature decreased between the age of leptons and the age of nucleons respectively. Its shows further that there was a fast drop in the electron-to-photon number density ratio, per temperature decrease, during the beginning of the age of leptons but achieved a more steady rate towards the end age of nucleons.

References

[1] R. S. Richard, R. Sanjay. "Neutron stars for undergraduates."

[2] D. William, W. G. Peter, K. Daniel, M. Gustavo, R. Surjeet. "Observable signatures of dark photons from supernovae."

[3] H. Lars. "On Effective Degrees of Freedom in the Early Universe."

[4] S. Bovino, D. Galli. "Thermodynamics and Chemistry of the Early Universe." arXiv:1807.05939v2 [astro-ph.GA]

[5] K. S. Abhik, D. Subhra. "Thermodynamics of irreversible particle creation phenomena and its cosmological consequence."

[6] M. A. Gorji, V. Hosseinzadeh, and K. Nozari. "Early universe thermostatistics in curved momentum spaces." arXiv:1602.05666v2 [gr-qc]

[7] G.L. Guardo, V. Greco, and M. Ruggieri. "Energy density fluctuations in Early Universe." arXiv:1401.7613 [hep-ph]

[8] A. Tawfik."Thermodynamics in the Viscous Early Universe." arXiv:1002.0296v1 [gr-qc]

[9] R. K. Pathria, P. D. Beale. "Statistical Mechanics." Third Edition.

MATLAB Codes:

```
clc;

clear;

format short

%.................................................................
%Variables
%.................................................................
k = 1.38064852*10^23; %m2 kgs^-2K^-1
m = 9.10938356*10^-31;%kg
c=3*10^8; % m/s

%...............................................................
f=@(b_o)(0.0652*(b_o.*m.*c).^(3/2).*(exp(-(b_o.*m.*c.^2))));
k = 1.38064852*10^23;
x=[3*10^8,10^10]; % x=T, temperature interval in Kelvin
b_o=1./(k*x);

%.................................................................
fplot(f,x,'Color','r')
% use fplot to plot an expression (an anonymous function)
set (gca, 'xdir','reverse')
%.................................................................
%xlim([0 Inf]) % Setting the limit of x to be from 0 to infinity
xlabel('Decreasing temperature between ages of leptons and nucleons (K)')
ylabel('Electron-to-photon number density ratio (n)')
title('Plot of Electron-to-photon number density ratio versus decreasing temperature')
```