Part III-B: Probability Theory and Mathematical Statistics

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Example. $(X,Y) \sim N_2\left(0,1\right), \phi\left(x,y\right) = \frac{1}{2\pi}\mathrm{e}^{-x^2+y^2/2}$

$Z = \sqrt{X^2+Y^2}$，求 $E\left(Z\right)$

解：由定理：

Rule.

$$E\left(g\left(X,Y\right)\right) = \begin{cases} \displaystyle\sum_{i=1}^{+\infty}\sum_{j=1}^{+\infty} g\left(x_i,y_j\right)P\left\{X=x_i,Y=y_j\right\}, & \text{离散} \\ \displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g\left(x,y\right)f\left(x,y\right)\mathrm{d}x\mathrm{d}y, & \text{连续} \end{cases}.$$

可得数学期望：

$$E\left(Z\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}Z\cdot f\left(x,y\right)\mathrm{d}x\mathrm{d}y.$$

0.1 数学期望的性质

$$\begin{cases}\text{线性可加性}\\\text{独立性}\end{cases}.$$

- ○ $E\left(aX+bY+c\right)=E\left(aX+bY\right)+c$：常数的数学期望为其本身

Notation. 什么是数学期望：一个随机变量的中心

方差：去中心化的随机变量

常数的中心为其本身

- ○ $E\left(aX+bY\right)=E\left(aX\right)+E\left(bY\right)=aE\left(X\right)+bE\left(Y\right)$：线性性

证明. 已知：

$$\int_{-\infty}^{+\infty}f\left(x,y\right)\mathrm{d}y=F_X\left(x\right).$$

$$\int_{-\infty}^{+\infty}xf\left(x,y\right)\mathrm{d}x=E\left(X\right).$$

$$\begin{aligned}E\left(aX+bY\right)&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(ax+by\right)f\left(x,y\right)\mathrm{d}x\mathrm{d}y\\&=a\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}xf\left(x,y\right)\mathrm{d}x\mathrm{d}y+b\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}yf\left(x,y\right)\mathrm{d}x\mathrm{d}y\\&=aE\left(X\right)+bE\left(Y\right).\end{aligned}$$

□

Example. $E\left(X\right)\pm E\left(Y\right)=E\left(X\pm Y\right)$

- ○ 对于独立的随机变量：$E\left(XY\right)=E\left(X\right)\cdot E\left(Y\right)$

证明. 二重积分转换为二次积分:

$$\begin{aligned}
E(XY) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xyf_X(x)\,f_Y(y)\,\mathrm{d}x\mathrm{d}y \\
&= \left(\int_{-\infty}^{+\infty} xf_X(x)\,\mathrm{d}x\right)\left(\int_{-\infty}^{+\infty} yf_Y(y)\,\mathrm{d}y\right) \\
&= E(X)\cdot E(Y).
\end{aligned}$$

□

Notation.

$$\begin{aligned}
\operatorname{cov}(X,Y) = E(XY) - E(X)\,E(Y) = 0 \\
\implies \rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sqrt{DX}\sqrt{DY}} = 0 \\
\implies X,Y\text{无相关(独立)}.
\end{aligned}$$

Notation. 线性可加性:

$$E\left(\sum_{i=1}^{n} a_iX_i + c\right) = \sum_{i=1}^{n} a_iE(X_i) + c.$$

将 n 重积分转换为一重积分

0.2 方差的性质

$$D(X) = E(X - EX)^2.$$

Notation. 方差是描述数据偏离中心的程度值

- 常数的方差等于 0: $D(c) = 0$

Notation. 正态分布的方差 σ 不大: 3σ 准则保证数据方差在可控范围内

- $D(aX+b) = D(aX) = \boldsymbol{a^2}D(X)$: 离散程度与整体移动无关

证明.

$$\begin{aligned}D(aX+b) &= E(aX+b-E(aX+b))^2\\ &= E(aX+b-aE(X)-b)^2\\ &= E(aX-aE(X))^2 = D(aX)\\ &= a(X-E(X))\cdot aE(X-E(X))\\ &= a^2E(X-E(X))^2\\ &= \boldsymbol{a^2}D(X).\end{aligned}$$

□

∘ $D(X\pm Y) = D(X)+D(Y)\pm E((X-EX)\cdot(Y-EY))$

证明.

$$\begin{aligned}D(X-Y) &= E(X-Y-E(X-Y))^2\\ &= E(X-Y-(EX-EY))^2\\ &= E((X-EX)-(Y-EY))^2\\ &= E\left((X-EX)^2-2(X-EX)(Y-EY)+(Y-EY)^2\right)\\ &= E(X-EX)^2-2E(X-EX)(Y-EY)+E(Y-EY)^2\\ &= D(X)+D(Y)-2\mathrm{cov}(X,Y).\end{aligned}$$

$$\begin{aligned}\mathrm{cov}(X,Y) &= E(X-EX)(Y-EY)\\ &= E(XY-X\cdot EY-Y\cdot EX+EX\cdot EY)\\ &= E(XY)-E(X\cdot EY)-E(Y\cdot EX)+E(EX\cdot EY)\\ &= E(XY)-EY\cdot E(X)-EX\cdot E(Y)+EX\cdot EY\\ &= E(XY)-E(X)\cdot E(Y).\end{aligned}$$

当 X,Y 独立时：$\mathrm{cov}(X,Y)=0$，即 $D(X-Y)=D(X)+D(Y)$，加法同理 □

Notation. 当 $X=Y$ 时:

$$\begin{aligned}\operatorname{cov}(X,Y)&=\operatorname{cov}(X,X)\\&=E(X-EX)(X-EX)\\&=E(X-EX)^2\\&=D(X).\end{aligned}$$

即协方差退化为方差

Notation. 均方偏离函数:$f(x)=E(X-x)^2\geq D(X)$,当且仅当 $x=E(X)$ 时 $f(X)=D(X)$

◦ 切比雪夫不等式(概率论最基础的不等式)

$$P\{|X-EX|\geq\varepsilon\}\leq\frac{D(X)}{\varepsilon^2}.$$

或:

$$P\{|X-EX|>\varepsilon\}\geq 1-\frac{D(X)}{\varepsilon}.$$

证明时使用:

$$P\left\{(X-EX)^2\leq\varepsilon^2\right\}\leq\frac{D(X)}{\varepsilon^2}.$$

证明.

$$\begin{aligned}P\{|X-EX|\geq\varepsilon\}&=\int_{|x-EX|\geq\varepsilon}f(x)\,\mathrm{d}x\\&\leq\int_{|x-EX|\geq\varepsilon}\frac{|x-EX|^2}{\varepsilon^2}f(x)\,\mathrm{d}x\\&\leq\int_{-\infty}^{+\infty}\frac{|x-EX|^2}{\varepsilon^2}f(x)\,\mathrm{d}x\\&=\frac{1}{\varepsilon^2}\int_{-\infty}^{+\infty}(x-EX)^2f(x)\,\mathrm{d}x\\&=\frac{1}{\varepsilon^2}E(X-EX)^2\\&=\frac{D(X)}{\varepsilon^2}.\end{aligned}$$

□

Notation. 切比雪夫不等式 $\Longrightarrow$ 马尔可夫不等式 $\Longrightarrow$ 协方差不等式 $\Longrightarrow$ 阶乘不等式 $\Longrightarrow$ ...

$D(X)=0$ 的充要条件为 $P=1$

Lecture 11

10.24

Review:

Notation. 数学期望的性质:

1. $E(c)=c$
2. $E(cX)=cE(X)$
3. $E(X+Y)=E(X)+E(Y)$

3.1 $E(E(Y)X)=E(Y)E(X)$

4. X,Y 相互独立，$E(XY)=E(X)E(Y)$

协方差：$\operatorname{cov}(X,Y)=E(X-EX)(Y-EY)=E(XY)-E(X)E(Y)$

若 X,Y 独立则 $\operatorname{cov}(X,Y)=0$

Notation. 方差的性质:

1. $D(c)=0$
2. $D(cX)=c^2D(X)$

2.1. $D(X)=E(X-EX)^2=E(X^2)-E(X)^2$

3. X,Y 相互独立，$D(X+Y)=D(X)+D(Y)$

$\operatorname{cov}(X,Y)=E(X-EX)(Y-EY)$

当 $X=Y$ ，$\operatorname{cov}(X,Y)=\operatorname{cov}(X,X)=E(X-EX)^2=D(X)$

或：$\operatorname{cov}(X,Y)=E(XY)-E(X)E(Y)=E(X^2)-E(X)^2$

Example. $D(aX+bY+c)=D(aX+bY)$

$$\begin{aligned}
D(aX+bY)&=E((aX+bY)-E(aX+bY))^2\\
&=E(a(X-EX)+b(Y-EY))^2\\
&=E\left(a^2(X-EX)^2+2ab(X-EX)(Y-EY)+b^2(Y-EY)^2\right)\\
&=a^2D(X)+b^2D(Y)+2ab\operatorname{cov}(X,Y).
\end{aligned}$$

◦ 切比雪夫不等式：已知一个随机变量的方差可以估算出数学期望

Question. 一个随机变量 X 分布未知, 已知 $\mu = 18, \sigma = 2.5$, 求 $P\left\{X \in (8, 28)\right\}$

解：由切比雪夫不等式：

$$\begin{aligned} P\left\{X \in (8, 28)\right\} &= P\left\{X - 18 \in (-10, 10)\right\} \\ &= P\left\{|X - 18| < 10\right\} \\ &= P\left\{|X - \mu| < \varepsilon\right\} \\ &\geq 1 - \frac{\sigma^2}{\varepsilon^2} \\ &= 1 - \frac{2.5^2}{10^2} = 0.9375. \end{aligned}$$

◦ 马尔可夫不等式

Example. $X_1, X_2, \ldots, X_n : i.i.d, X \sim N\left(\mu, \sigma^2\right)$，证明：

1. $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$
2. 设 $Y_i = \frac{X_i - \mu}{\sigma}, i = 1, 2, \ldots, n$ 则 $E\left(\sum_{i=1}^{n} Y_i^2\right) = n$

证明. 1. 由线性性：

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(E\overline{X}, D\overline{X}\right).$$

由于 X 之间相互独立，有 $D\left(X_1 + X_2\right) = D\left(X_1\right) + D\left(X_2\right)$

$$E\overline{X} = \frac{1}{n}\sum_{i=1}^{n} EX_i = \mu, \quad D\overline{X} = \frac{1}{n^2}\sum_{i=1}^{n} DX_i = \frac{\sigma^2}{n}.$$

2. 由题：$EY_i = 0, DY_i = 1$

$$E\left(\sum_{i=1}^{n} Y_i^2\right) = \sum_{i=1}^{n} EY_i^2.$$

Notation. Y_i^2 符合自由度为 1 的卡方分布：$Y_i^2 \sim X^2\left(1\right)$

即：$\sum_{i=1}^{n} E\left(Y_i^2\right) = nE\left(Y_i^2\right)$

由方差的定义：$D\left(Y_i\right) = E\left(Y_i^2\right) - E\left(Y_i\right)^2$ ：

$$EY_i^2 = D\left(Y_i\right) + E\left(Y_i\right)^2 = 1 + 0^2 = 1$$
$$\sum_{i=1}^{n} E\left(Y_i^2\right) = nE\left(Y_i^2\right) = n.$$

□

0.3 协方差的性质

- $\operatorname{cov}\left(X, Y\right) = \operatorname{cov}\left(Y, X\right)$ （对称性）
- $\operatorname{cov}\left(aX, bY\right) = ab\operatorname{cov}\left(X, Y\right)$

证明. 已知：$\operatorname{cov}\left(X, Y\right) = E\left(XY\right) - E\left(X\right)E\left(Y\right)$

$$\begin{aligned}\operatorname{cov}\left(aX, bY\right) &= E\left(aXbY\right) - E\left(aX\right)E\left(bY\right)\\ &= abE\left(XY\right) - abE\left(X\right)E\left(Y\right)\\ &= ab\operatorname{cov}E\left(X, Y\right).\end{aligned}$$

□

- $\operatorname{cov}\left(c, X\right) = 0$

Notation. 协方差用于衡量随机变量之间的线性关系，常数和其他随机变量不存在线性关系

证明.

$$\begin{aligned}\operatorname{cov}\left(cX\right) &= E\left(cX\right) - E\left(c\right)E\left(X\right)\\ &= cE\left(X\right) - cE\left(X\right)\\ &= 0.\end{aligned}$$

□

Notation. $\operatorname{cov}\left(c, c\right) = D\left(c\right) = 0$

$\circ$ $\mathrm{cov}\left(aX+bY,cZ\right)=ac\mathrm{cov}\left(X+Y\right)+bc\mathrm{cov}\left(Y+Z\right)$ （分配律）

证明.

$$\begin{aligned}\mathrm{cov}\left(aX+bY,cZ\right)&=E\left(\left(aX+bY\right)cZ\right)-E\left(aX+bY\right)E\left(cZ\right)\\&=E\left(acXZ+bcYZ\right)-cEZ\left(aEX+bEY\right)\\&=acE\left(XZ\right)+bcE\left(YZ\right)-acEXEZ-bcEYEZ\\&=ac\mathrm{cov}\left(X,Z\right)+bc\mathrm{cov}\left(Y,Z\right).\end{aligned}$$

□

Notation. $\mathrm{cov}\left(\sum_{i=1}^{n}a_iX_i,b_iZ\right)=\sum_{i=1}^{n}a_ib_i\mathrm{cov}\left(X_i,Z\right)$

Notation. $D\left(\sum_{i=1}^{n}a_iX_i\right)=\sum_{i=1}^{n}a_i^2DX_i+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}a_ia_j\mathrm{cov}\left(X_i,X_j\right)$

0.4 相关系数

0.4.1 标准化

$$X^*=\frac{X-EX}{\sqrt{DX}}.$$

标准化后的变量 $EX^*=0,DX^*=1$

Definition. X^*,Y^* 的协方差 $\mathrm{cov}\left(X^*,Y^*\right)$ 为 X,Y 的相关系数 $\rho\left(X,Y\right)$

$$\begin{aligned}\mathrm{cov}\left(X^*,Y^*\right)&=\mathrm{cov}\left(\frac{X-EX}{\sqrt{D\left(X\right)}},\frac{Y-EY}{\sqrt{DX}}\right)\\&=\frac{1}{\sqrt{DX}\sqrt{DY}}\mathrm{cov}\left(X-EX,Y-EY\right).\end{aligned}$$

易得 $\mathrm{cov}\left(X-EX,Y-EY\right)=\mathrm{cov}\left(X,Y\right)$

$$\begin{aligned}\mathrm{cov}\left(X^*,Y^*\right)&=\frac{\mathrm{cov}\left(X,Y\right)}{\sqrt{DX}\sqrt{DY}}\\&=\rho\left(X,Y\right).\end{aligned}$$

0.4.2 性质

- $|\rho(X,Y)| \leq 1$
- $P\{X^* = \pm Y^*\} = 1$ 是 $\rho(X,Y) = \pm_1$ 的充要条件