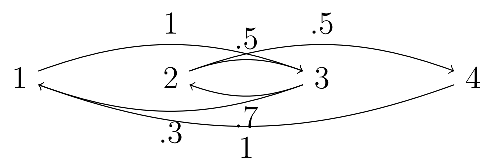
Homework 4

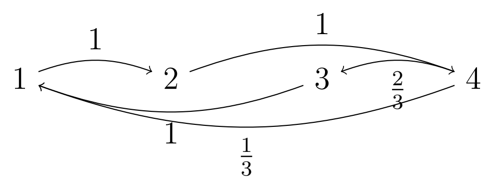
- **Exercise 1.14.**

(a)



From the graph, we can know $\{1, 2, 3, 4\}$ is an irreducible set, state 2 is periodic with period 2, by Lemma 68, we can know all states are periodic, thus, the chain **cannot** converge to equilibrium.

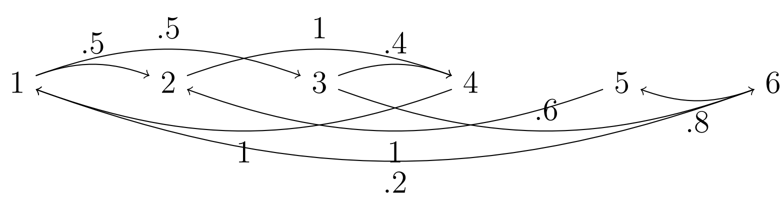
(b)



From the graph, we can know $\{1, 2, 3, 4\}$ is an irreducible set, state 2 will come back in 3 or 4 steps, gcd is 1, it is aperiodic, by Lemma 68, we can know all states are aperiodic as well, and by Theorem 66, we know the chain **can** converge to equilibrium.Use Matlab to calculate the left eigenvector and normalize, and $\pi P = \pi$ we can have

$$\pi(1) = .2727, \quad \pi(2) = .2727, \quad \pi(3) = .1818, \quad \pi(4) = .2727.$$

(c)



From the graph, we can know $\{1, 2, 3, 4, 5, 6\}$ is an irreducible set, state 2 is periodic, by Lemma 68, we can know all states are periodic with period 3, thus, the chain **cannot** converge to equilibrium.

- **Exercise 1.17.**
Probability transition matrix, $S = \{0, 1\}$, $0 =$without cable, $1 =$with cable:

$$P = \begin{array}{c} 0 \\ 1 \end{array} \begin{pmatrix} 0.74 & 0.26 \\ 0.08 & 0.92 \end{pmatrix}$$

We can use P to predict the percentage in 1995,

$$(0.436, 0.564)P = (0.436, 0.564) \begin{pmatrix} 0.74 & 0.26 \\ 0.08 & 0.92 \end{pmatrix} = (0.3678, 0.6322)$$

which is closed to the actual value 63.4%.
The percentage in 2000,

$$(0.436, 0.564)P^2 = (0.436, 0.564) \begin{pmatrix} 0.74 & 0.26 \\ 0.08 & 0.92 \end{pmatrix}^2 = (0.3227, 0.6773)$$

$$(0.366, 0.634)P = (0.436, 0.564) \begin{pmatrix} 0.74 & 0.26 \\ 0.08 & 0.92 \end{pmatrix} = (0.3216, 0.6784)$$

Both of these predictions are closed to actual percentage 68%. Since the chain is irreducible and aperiodic, we have limit distribution $(0.436, 0.564)$ P^n,$\pi = \pi P$, use the formula of $2-$state MC

$$\pi = (\frac{0.08}{0.08 + 0.26}, \frac{0.26}{0.08 + 0.26}) = (\frac{4}{17}, \frac{13}{17})$$

The long run proportion of people with cable is $\frac{13}{17}$.

1. Exercise 1.8(c).
We know irreducible sets are $\{2, 4\}$ and $\{1, 5\}$, we can have transition matrix and stationary distribution as below

$$P_1 = \begin{array}{c} 2 \\ 4 \end{array} \begin{pmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{pmatrix}$$

$$P_2 = \begin{array}{c} 1 \\ 5 \end{array} \begin{pmatrix} 0 & 1 \\ 0.3 & 0.7 \end{pmatrix}$$

use $2-$state MC formula

$$(\frac{0.6}{0.6 + 0.8}, \frac{0.8}{0.6 + 0.8}) = (\frac{3}{7}, \frac{4}{7})$$

$$\pi_1 = (0, \frac{3}{7}, 0, \frac{4}{7}, 0)$$

$$(\frac{0.3}{0.3 + 1}, \frac{1}{0.3 + 1}) = (\frac{3}{13}, \frac{10}{13})$$

$$\pi_2 = (\frac{3}{13}, 0, 0, 0, \frac{10}{13})$$

since $\alpha_1 + \alpha_2 = 1$, $\alpha_1, \alpha_2 > 0$

$$\pi = (\frac{3}{13}\alpha_2, \frac{3}{7}\alpha_1, 0, \frac{4}{7}\alpha_1, \frac{10}{13}\alpha_2).$$

2. Since there exists $n \geq 1$ such that every entry of the $n-$step transition matrix P^n is strictly positive (i.e. $p^n(x,y) > 0$ for all x, y), we can have

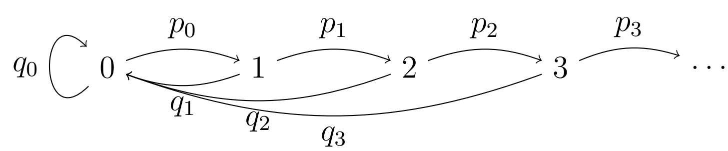
$$p^{n+1}(x,y) = \sum_{z \in S} p^n(x,z)p(z,y) > p^n(x,z)p(z,y) > 0$$

now, we can have $p^{n+1}(x,x) > 0$ and $p^n(x,x) > 0$,the period of x must divide n and $n+1$, only 1 meet this requirement, by Lemma 67, we can say x is aperiodic. Also, $p^n(x,y) > 0$ is equivalent to

$$\rho_{xy} > 0 \quad and \quad \rho_{yx} > 0$$

then DTMC is irreducible, all states communicate with each other, thus, by Lemma 68, all states are aperiodic.

3. Draw a transition graph as below



We can see all states communicate with each other, if state i is recurrent, then all states are recurrent, assume state 0 is recurrent

$$P_0(T_0 = \infty) = 0$$

$P_0(T_0 = \infty) = p_0p_1p_2p_3...$ holds, since

$$\begin{aligned} &p_0p_1p_2p_3... \\ &= \lim_{n\to\infty} p_0p_1p_2p_3...p_n \\ &= \lim_{n\to\infty} P_0(T_0 > n) \end{aligned}$$

by continuity of probability measures with intersections

$$\begin{aligned} &= P(\bigcup_{n=1}^{\infty} \{T_0 > n\}) \\ &= P_0(T_0 = \infty) \end{aligned}$$

then, we need to prove $p_0p_1p_2p_3... = 0$

$$\begin{aligned} &\lim_{n\to\infty} p_0p_1p_2p_3...p_n \\ &= \prod_{n=1}^{\infty} \frac{2^n+1}{2^n+2} \to 0 \end{aligned}$$

$$P_0(T_0 < \infty) = 1 - P_0(T_0 = \infty) = 1$$

thus, all states are recurrent.

4. (a) We can have

$$P(Y < \infty) = P(6^N < \infty) = P(N < \infty)$$

for proving $P(N < \infty) = 1$, which means after finite time N, we will toss Head, we can draw a transition graph

$$\begin{tikzcd} T \arrow[loop left, "\frac{1}{2}"] \arrow[r, bend left, "\frac{1}{2}"] & H \arrow[l, bend left, "\frac{1}{2}"] \arrow[loop right, "\frac{1}{2}"] \end{tikzcd}$$

we can know $\{T, H\}$ is irreducible, state H and T are recurrent. Then, assume we start from head

$$P_H(T_H < \infty) = 1$$
$$\{N < \infty\} = \{T_H < \infty\}$$

Thus, $P(Y < \infty) = 1$ holds.
Since

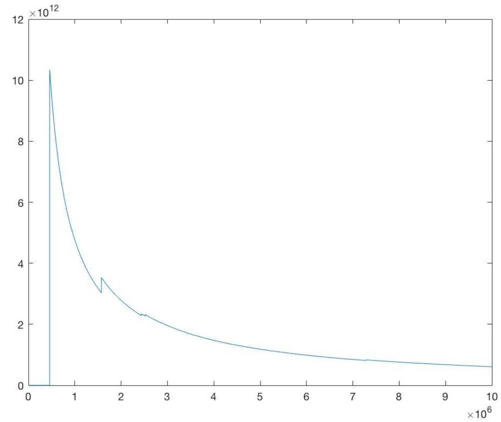
$$\begin{aligned} E[Y] &= \sum_{y=0}^{\infty} yP(Y = y) \\ &= \sum_{n=0}^{\infty} 6^n P(N = n) \\ &= \sum_{n=0}^{\infty} 6^n \frac{1}{2}^n \\ &= \sum_{n=0}^{\infty} 3^n \to \infty \end{aligned}$$

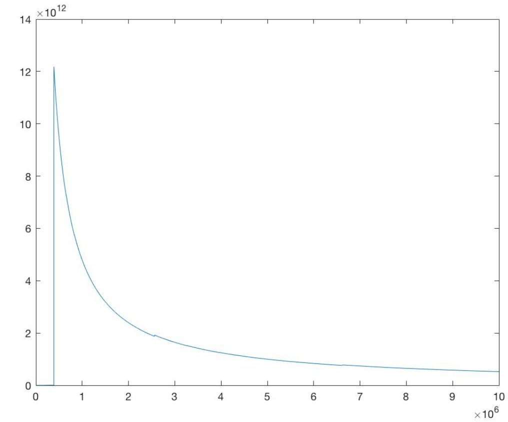
$E[Y] = \infty$ holds.

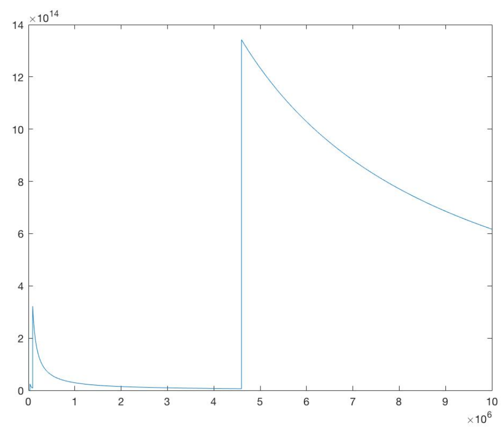
(b) Running those codes in Matlab, result as below
$N = 3$, $Y = 216$

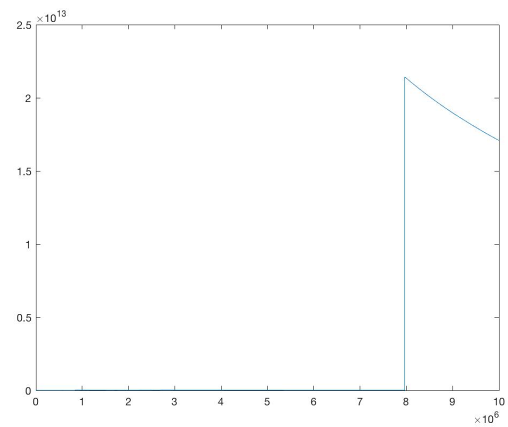
(c)(d) Running those codes in Matlab, plots showed **in the last page**.

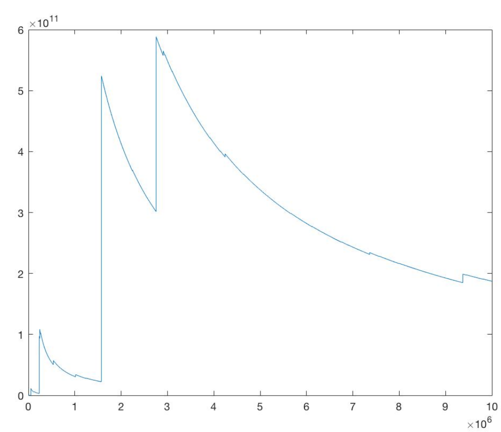
$A(m)$ will **not** converge for m going to infinity. From plots, we can see $A(m)$ fluctuates, after merging to a higher point, it will go down. Given $A(m) = \frac{1}{m}\sum_{i=1}^{m} Y(i)$, we know the merge happen resulting from the value of $Y(i)$ is large based on running above codes, the decrease come with m increase, thus, $A(m)$ will not converge.

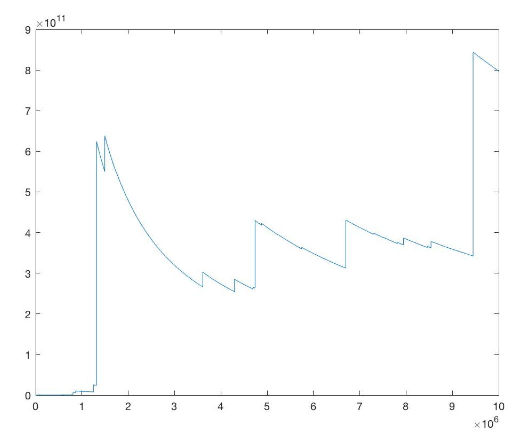

$\times 10^{12}$
0 2 4 6 8 10 12
0 1 2 3 4 5 6 7 8 9 10
$\times 10^{6}$


$\times 10^{12}$
0 2 4 6 8 10 12 14
0 1 2 3 4 5 6 7 8 9 10
$\times 10^{6}$


$\times 10^{14}$
0 2 4 6 8 10 12 14
0 1 2 3 4 5 6 7 8 9 10
$\times 10^{6}$


$\times 10^{13}$
0 0.5 1 1.5 2 2.5
0 1 2 3 4 5 6 7 8 9 10
$\times 10^{6}$


$\times 10^{11}$
0 1 2 3 4 5 6
0 1 2 3 4 5 6 7 8 9 10
$\times 10^{6}$


$\times 10^{11}$
0 1 2 3 4 5 6 7 8 9
0 1 2 3 4 5 6 7 8 9 10
$\times 10^{6}$