Cálculo III
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Lista de exercícios – Derivada exterior

Questão 1. Calcule a derivada exterior das seguintes formas diferenciais em $\mathbb{R}^2$:

a) $\sqrt{x^2 + xy}$

b) $\dfrac{x}{y}\,dx + \operatorname{sen}(xy)\,dy$

c) $x^y\,dx$

Questão 2. Calcule a derivada exterior das seguintes formas diferenciais em $\mathbb{R}^3$:

a) $xy^2z^3\,dx - \dfrac{xy}{z}\,dz$

b) $x^4z^3\,dx{\wedge}dy + (x - 2y + 3z)^4\,dz{\wedge}dx$

Questão 3. Calcule a derivada exterior das seguintes formas diferenciais em $\mathbb{R}^4$:

a) $(2x^2z - 3y^3w^2)\,dx$

b) $x^2z^4\,dx{\wedge}dy + x^4y^3w\,dz{\wedge}dw$

Questão 4. Seja ω uma 0-forma em $\mathbb{R}^3$ dada por $\omega = f(x, y, z)$, onde f é de classe C^2 (isto é, as segundas derivadas parciais de f existem e são contínuas). Temos

$$d\omega = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz.$$

a) Mostre (fazendo a conta) que $d(d\omega) = 0$.

b) Mostre que $d\omega = \nabla f \cdot \langle dx,\, dy,\, dz\rangle$.

Questão 5. Seja ω uma 1-forma em $\mathbb{R}^3$ dada por

$$\omega = p(x, y, z)\,dx + q(x, y, z)\,dy + r(x, y, z)\,dz,$$

onde p, q e r são de classe C^2. Temos

$$d\omega = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx{\wedge}dy + \left(\frac{\partial r}{\partial y} - \frac{\partial q}{\partial z}\right) dy{\wedge}dz + \left(\frac{\partial p}{\partial z} - \frac{\partial r}{\partial x}\right) dz{\wedge}dx.$$

a) Mostre (fazendo a conta) que $d(d\omega) = 0$.

b) Mostre que $d\omega = \operatorname{rot}(\langle p,\, q,\, r\rangle) \cdot \langle dy{\wedge}dz,\, dz{\wedge}dx,\, dx{\wedge}dy\rangle$.

Questão 6. Seja ω uma 2-forma em $\mathbb{R}^3$ dada por

$$\omega = P(x, y, z)\,dy{\wedge}dz + Q(x, y, z)\,dz{\wedge}dx + R(x, y, z)\,dx{\wedge}dy,$$

onde P, Q e R são duas vezes deriváveis. Temos

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx{\wedge}dy{\wedge}dz.$$

a) Mostre (fazendo ou não a conta) que $d(d\omega) = 0$.

b) Mostre que $d\omega = \operatorname{div}(\langle P,\, Q,\, R\rangle) \cdot dx{\wedge}dy{\wedge}dz$.

Gabarito

1a) $\dfrac{2x+y}{2\sqrt{x^2+xy}}\,dx + \dfrac{x}{2\sqrt{x^2+xy}}\,dy$

1b) $\left(y\cos(xy) + \dfrac{x}{y^2}\right) dx{\wedge}dy$

1c) $-\ln(x)\,x^y\,dx{\wedge}dy$

2a) $-2xyz^3\,dx{\wedge}dy - \dfrac{x}{z}\,dy{\wedge}dz + \left(3xy^2z^2 + \dfrac{y}{z}\right) dz{\wedge}dx$

2b) $[3x^4z^2 - 8(x-2y+3z)^3]\,dx{\wedge}dy{\wedge}dz$

3a) $-9y^2w^2\,dy{\wedge}dx + 2x^2\,dz{\wedge}dx - 6y^3w\,dw{\wedge}dx$

3b) $4x^2z^3\,dz{\wedge}dx{\wedge}dy + 4x^3y^3w\,dx{\wedge}dz{\wedge}dw + 3x^4y^2w\,dy{\wedge}dz{\wedge}dw$

4a)

$$\begin{aligned} d(d\omega) &= \left(\frac{\partial^2 f}{\partial x\partial y}\,dx{\wedge}dy + \frac{\partial^2 f}{\partial x\partial z}\,dx{\wedge}dz\right) + \left(\frac{\partial^2 f}{\partial y\partial x}\,dy{\wedge}dx + \frac{\partial^2 f}{\partial y\partial z}\,dy{\wedge}dz\right) \\ &\quad + \left(\frac{\partial^2 f}{\partial z\partial x}\,dz{\wedge}dx + \frac{\partial^2 f}{\partial z\partial y}\,dz{\wedge}dy\right) \\ &= \left(\frac{\partial^2 f}{\partial x\partial y}\,dx{\wedge}dy - \frac{\partial^2 f}{\partial z\partial x}\,dz{\wedge}dx\right) + \left(-\frac{\partial^2 f}{\partial x\partial y}\,dx{\wedge}dy + \frac{\partial^2 f}{\partial y\partial z}\,dy{\wedge}dz\right) \\ &\quad + \left(\frac{\partial^2 f}{\partial z\partial x}\,dz{\wedge}dx - \frac{\partial^2 f}{\partial z\partial y}\,dz{\wedge}dy\right) = 0. \end{aligned}$$

4b) —

5a)

$$\begin{aligned} d(d\omega) &= \left(\frac{\partial^2 q}{\partial z\partial x} - \frac{\partial^2 p}{\partial z\partial y}\right) dz{\wedge}dx{\wedge}dy + \left(\frac{\partial^2 r}{\partial x\partial y} - \frac{\partial^2 q}{\partial x\partial z}\right) dx{\wedge}dy{\wedge}dz \\ &\quad + \left(\frac{\partial^2 p}{\partial y\partial z} - \frac{\partial^2 r}{\partial y\partial x}\right) dy{\wedge}dz{\wedge}dx \\ &= \left(\frac{\partial^2 q}{\partial z\partial x} - \frac{\partial^2 p}{\partial y\partial z}\right) dx{\wedge}dy{\wedge}dz + \left(\frac{\partial^2 r}{\partial x\partial y} - \frac{\partial^2 q}{\partial z\partial x}\right) dx{\wedge}dy{\wedge}dz \\ &\quad + \left(\frac{\partial^2 p}{\partial y\partial z} - \frac{\partial^2 r}{\partial x\partial y}\right) dx{\wedge}dy{\wedge}dz = 0. \end{aligned}$$

5b) —

6a) Como toda 4-forma em $\mathbb{R}^3$, $d(d\omega) = 0$.

6b) —