2º CASO : $\boxed{\Delta = 0}$ $\Rightarrow$ m $\Rightarrow$ RAIZ REAL DUPLA.

TEOREMA II :

SE A EQ. AUXILIAR TEM UMA RAIZ DUPLA m, A SOLUÇÃO GERAL É:

$$\boxed{y = c_1 . e^{m.x} + c_2 . x . e^{m.x} = (c_1 + c_2 . x) e^{m.x}} \quad (13)$$

EX2 - RESOLVA $y'' - 6y' + 9y = 0$

$$y = e^{m.x}, \quad y' = m . e^{mx}, \quad y'' = m^2 . e^{m.x} . \Rightarrow \begin{cases} m^2 . e^{mx} - 6 m e^{mx} + 9 e^{m.x} ; \\ (m^2 - 6m + 9) . e^{m.x} = 0 ; \\ \therefore \boxed{m^2 - 6m + 9 = 0} \quad (14) \end{cases}$$

RESOLVENDO (14)

$\Delta = b^2 - 4.1.c$;

$\Delta = 36 - 4 . 9$;

$\Delta = 36 - 36$;

$\boxed{\Delta = 0 .}$ (15)

$m = \dfrac{-b \pm \sqrt{\Delta}}{2.1} = \dfrac{-(-6) \pm 0}{2}$;

$\boxed{m = \dfrac{6}{2} = 3}$ (16)

SOLUÇÃO GERAL: $\boxed{y = (c_1 + c_2 x) e^{3.x}}$ (17)

OBS: PODE-SE RESOLVER (14) FATORANDO:

$m^2 - 6m + 9 = (m-3).(m-3) = (m-3)^2 = 0$;

$\therefore \boxed{m = 3}$ (18)

OBS:

PODE-SE CONSIDERAR EQS. NA FORMA:

$$a.y'' + by' + cy = 0 \; ; \; (a \neq 1) \Rightarrow \text{FAZEMOS:}$$

$$\boxed{y'' + \frac{b}{a} y' + \frac{c}{a} y = 0} \quad (19)$$

3º CASO: $\boxed{\Delta < 0} \Rightarrow$ RAÍZES COMPLEXAS

- CONSIDERAÇÕES: $x^2 - 4x + 13 = 0 \Rightarrow \begin{cases} a = 1 \\ b = -4 \\ c = 13 \end{cases}$

$\Delta = b^2 - 4ac$,

$\Delta = 16 - 4.1.13$,

$\Delta = 16 - 52$;

$\boxed{\Delta = -36}$ (20)

$$x = \frac{-b \pm \sqrt{\Delta}}{2.a} \; ; \quad (21)$$

$$x = \frac{-(-4) \pm \sqrt{-36}}{2.1} = \frac{4 \pm \sqrt{i^2 . 36}}{2} \; ;$$

$$x = \frac{4 \pm \sqrt{36} . \sqrt{i^2}}{2} = \frac{4 \pm 6i}{2} \; ;$$

$x_1 = 2 + 3i$;

$x_2 = 2 - 3i$; (22)

ONDE: $\boxed{i = \sqrt{-1}}$

OU $\boxed{i^2 = -1}$

FORMA GERAL DE UM Nº COMPLEXO:

$$\boxed{\begin{aligned} z_1 &= a + b.i \\ z_2 &= a - b.i = z_1^* \end{aligned}}$$

ONDE $a, b \in \mathbb{R}$

(z_2: COMPLEXO CONJUGADO DE z_1)

(23)

LEMBRANDO: SE $m_1 \neq m_2$

SOL. GERAL: $$\boxed{y = C_1 . e^{m_1 . x} + C_2 . e^{m_2 . x}} \qquad (24)$$

ANALOGIA: $z_1 \neq z_2 \Rightarrow$ SOL. GERAL:

$$y = C_1 . e^{z_1 . x} + C_2 . e^{z_2 . x} \; ; \qquad (25)$$

$$y = C_1 . e^{(a+ib)x} + C_2 \, e^{(a-ib)x} \; ; \qquad (26)$$

$$y = C_1 . e^{ax} . e^{ibx} + C_2 . e^{ax} . e^{-ibx} \; ; \qquad (27)$$

$$\boxed{y = e^{a.x} \left\{ C_1 . e^{i.bx} + C_2 . e^{-i.bx} \right\}} \; ; \qquad (28)$$

- EM (28) FAZEMOS ESCOLHAS ADEQUADAS PARA C_1 E C_2 E USAMOS AS IDENTIDADES:

$$\boxed{\cos(bx) = \frac{e^{ibx} + e^{-ibx}}{2}} \; ; \qquad (29)$$

$$\boxed{\operatorname{sen}(bx) = \frac{e^{ibx} - e^{-ibx}}{2.i}} \; ; \qquad (30)$$

REPETINDO 28:

$$y = e^{ax}\left\{C_1 . e^{ibx} + C_2 . e^{-ibx}\right\} \quad (28)$$

A $\boxed{C_1 = C_2 = 1/2}$ => EM 28:

$$y = e^{ax}\left\{\frac{e^{ibx} + e^{-ibx}}{2}\right\} = e^{ax} . \cos(bx) ;$$

USA EQ. 29

$$\therefore \boxed{y = e^{ax} . \cos(bx)} \quad (31)$$ É UMA SOLUÇÃO.

B $\boxed{C_1 = -C_2 = \frac{1}{2 . i}}$ => EM 28:

$$y = e^{ax}\left\{\frac{e^{ibx}}{2i} - \frac{e^{-ibx}}{2i}\right\} = e^{ax}\left\{\frac{e^{ibx} - e^{-ibx}}{2i}\right\} ;$$

USA 30

$$\boxed{y = e^{ax} . \operatorname{SEN}(b.x)} \quad (32)$$ É OUTRA SOLUÇÃO.

ENTÃO A SOL. GERAL É:

$$y = C_1 . e^{ax} . \cos(bx) + C_2 e^{ax} . \operatorname{SEN}(b.x) ; \quad (33)$$

$$\boxed{y = e^{ax}\left[C_1 . \cos(bx) + C_2 \operatorname{SEN}(bx)\right]} \quad (34)$$

$\boxed{\text{TEOREMA III}}$:

SE A EQ. AUXILIAR TEM RAÍZES COMPLEXAS $\boxed{a \pm b i}$, A SOL. GERAL É:

$$\boxed{y = e^{a.x}\left[C_1 \cos(b x) + C_2 \operatorname{sen}(b.x)\right]} \quad (34)$$

OBS: $z = a \pm i b \Rightarrow \begin{cases} \operatorname{Re}[z] = a, \\ \operatorname{Im}[z] = b, \end{cases}$ (35)

EX3 - RESOLVA $\boxed{y'' - 10 y' + 41 y = 0}$

$\boxed{m^2 + b m + c = 0}$

$\boxed{m^2 - 10 m + 41 = 0}$ (36) $\Rightarrow \Delta = (10)^2 - 4.(41) = 100 - 164;$

$\boxed{\Delta = -64}$

$$m = \frac{-(-10) \pm \sqrt{-64}}{2} = \frac{10 \pm \sqrt{64}.\sqrt{-1}}{2};$$

$$m = \frac{10}{2} \pm \frac{8}{2} i \Rightarrow \begin{cases} m_1 = z_1 = 5 + 4i \\ m_2 = z_2 = 5 - 4i \end{cases}; \quad \begin{matrix} a = 5 \\ b = 4 \end{matrix}$$

DE (34):

$$\boxed{y = e^{5x}\left[C_1 \cos(4.x) + C_2 \operatorname{sen}(4.x)\right]} \quad (37)$$

PARA OBTER FÓRMULAS DE COSSENO E SENO, EQS. 29 E 30:

RELEMBRANDO A EQ. 28 $\Rightarrow$ $\boxed{y = e^{ax}\left\{C_1 e^{ibx} + C_2 e^{-ibx}\right\}}$

FÓRMULA DE EULER:

$$e^{i\theta} = \cos(\theta) + i \cdot \text{sen}(\theta) \; ; \quad (A)$$

ENTÃO:

$$\boxed{e^{ibx} = \cos(bx) + i\,\text{sen}(bx)} \quad (B)$$

$$e^{-ibx} = e^{i(-b \cdot x)} = \cos(-bx) + i\,\text{sen}(-bx)$$

$$\boxed{e^{-ibx} = \cos(bx) - i\,\text{sen}(bx)} \quad (C)$$

USA QUE: $\boxed{\cos(-\theta) = \cos(\theta)}$ PAR

$\boxed{\text{sen}(-\theta) = -\text{sen}(\theta)}$

IMPAR

- SOMANDO B E C :

$$e^{ibx} = \cos(bx) + i\,\text{sen}(bx)$$

$$e^{-ibx} = \cos(bx) - i\,\text{sen}(bx) \quad +$$

$$e^{ibx} + e^{-ibx} = 2\cos(bx) \;;$$

$$\therefore \boxed{\cos(bx) = \frac{e^{ibx} + e^{-ibx}}{2}}$$

- SUBTRAINDO C DE B : B - C :

$$e^{ibx} = \cos(bx) + i\,\text{sen}(b.x)$$

$$-e^{-ibx} = -\cos(bx) + i\,\text{sen}(b.x)$$

$$e^{ibx} - e^{-ibx} = 2i\,\text{sen}(bx) \;;$$

$$\therefore \boxed{\text{sen}(b.x) = \frac{e^{ibx} - e^{-ibx}}{2.i}}$$