George Mason University

Department of Mathematical Sciences

Math 125: Discrete Math

Dr. Morris

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Practice Exam # 2

Matteo Costagliola

(Student ID: G01488318)

Problem 1

Given

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$. Find:

(a) $A \times B$

(b) $\mathcal{P}(A)$, the power set of A

Solution

(a) $A \times B = \{(1,1), (1,3), (1,5), (1,7), (2,1), (2,3), (2,5), (2,7), (3,1), (3,3), (3,5), (3,7)\}$

(b) $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Problem 2

Given

Prove or give a counterexample to the statement: For all sets A, B, C, $(A - B) \cup C = A - (B \cap C)$.

Solution

We propose a counterexample: Let $A = \{1\}, B = \{2\},$ and $C = \emptyset$

$(A - B) \cup C =$

$$A - B = 1$$

$$(A - B) \cup C = \{\emptyset, 1\}$$

$A - (B \cap C) =$

$$B \cap C = \emptyset$$

$$A - (B \cap C) = 1$$

Thus we have $\{\emptyset, \{1\}\} \neq 1$.

Problem 3

Given

Let $f = \{(a,a),(b,c),(c,b)\}$ and $g = \{(a,c),(b,c),(c,a)\}$ be two functions from $\{a,b,c\}$ to $\{a,b,c\}$.

(a) Find f^{-1}. Does g have an inverse?

(b) Find $g \circ f$.

Solution

(a.1) $f^{-1} = \{(a,a),(c,b),(b,c)\}$

(a.2) No g does not have an inverse as g is not a bijective function.

(b) $g \circ f = \{(a,c),(b,a),(c,c)\}$

Problem 4

Given

Define $f : \{x \in \mathbb{R} \mid x > 0\} \mapsto \{y \in \mathbb{R} \mid y > 1\}$ by $f(x) = 1 + \frac{1}{x}$.

(a) Show that f is one-to-one and onto.

(b) Find the inverse function $f^{-1}(x)$.

Solution

(a.1) **Proof: f is one-to-one.**

Let x_1, $x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$, then by the definition of f

$$1 + \frac{1}{x_1} = 1 + \frac{1}{x_2}$$

Then, by simple algebra

$$\frac{1}{x_1} = \frac{1}{x_2},\ x_1 = x_2$$

Thus we have that f is one-to-one.

(a.2) **Proof: f is onto.**

Let y be an element of the range of f such that there exists an x where $f(x) = y$.

Then by the rule of f

$$y = 1 + \frac{1}{x},\ y - 1 = \frac{1}{x},\ x = \frac{1}{y-1}$$

Plugging x into $f(x) = y$ gives

$$f(\frac{1}{y-1}) = 1 + \frac{1}{\frac{1}{y-1}} = 1 + y - 1 = y = y$$

Thus f is onto.

(b) $f^{-1} =$

$$y = 1 + \frac{1}{x},\ x = 1 + \frac{1}{y},\ x - 1 = \frac{1}{y},\ y = \frac{1}{x-1}$$

$$f^{-1} = \frac{1}{x-1}$$

Problem 5

Given

Define a transitive relation R on $\mathbb{Z}$ by xRy if and only if 3 divides $x - y$. Prove that R is transitive.

Solution

Proof: R is transitive.

Let $x, y, x \in R$ such that xRy and yRz. Then by the definition of R we have

$$x - y = 3m \text{ for some integer } m$$

$$y - z = 3l \text{ for some integer } l$$

For R to be transitive we must have xRz, so we solve the second equation for y and substitute this into the first equation.

$$y = 3l + z$$

$$x - (3l + z) = 3m, \; x - 3l - z = 3m$$

$$x - z = 3m + 3l, \; x - z = 3(m + l), \; \text{where } (m + l) \in \mathbb{Z}$$

Thus R is transitive.

Problem 6

Given

Let $R = \{(1, 2), (2, 1), (2, 2), (1, 1), (3, 3)\}$ be a relation on $\{1, 2, 3, 4\}$. For each of the properties: reflexivity, symmetry, transitivity, antisymmetry, state whether or not R has the property.

Solution

R is not reflexive as $(4, 4) \notin R$.

R is symmetric as $(a, b) \in R$ if (b, a) is in R.

R is transitive as $(a, b) \in R \wedge (b, c) \in R \implies (a, c) \in R$.

R is not antisymmetric as $(1, 2) \in R$ and $(2, 1) \in R$ but $1 \neq 2$.

Problem 7

Given

Let $A = \{a, b, c, d\}$ and let $R = \{(a, a), (a, c), (b, b), (c, c), (c, a), (d, d)\}$

(a) Prove that R is an equivalence relation on A.

(b) What are the equivalence classes for R?

Solution

(a) R is reflexive as $\forall x \in A, (x, x) \in R$. R is symmetric as $\forall (x, y) \in R, (y, x) \in R$. R is transitive as $\forall x, y, x \in R, (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$.

(b) The equivalence classes for R are as follows

$$\{a, c\}, \ \{b\}, \ \{d\}$$

Problem 8

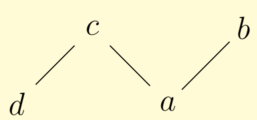
Given

Consider the following partial order on
$\{a, b, c, d\} : \ (a, a), (b, b), (c, c), (a, b), (a, c), (d, c), (d, d)$.

(a) Draw the Hasse diagram of the partial order.

(b) List the minimal and the least elements.

Solution

(a) 

(b) The minimal elements are: a and c, as no elements are below these. There is no least element in this partial order.

Problem 9

Given

(a) How many odd integers are there from 3000 through 9999?

(b) How many odd integers from 3000 through 9999 have distinct digits?

Solution

(a) $N(9999 - 3000) = 7000$, $\frac{7000}{2} = 3500$, thus there are 3500 odd integers from 3000 to 9999.

(b) We will divide this problem into two cases. First, consider if the last digit is 1. This leaves us with 7 choices for the first digit, 8 choices for the second, and 7 choices for the third. For a total of 392 integers. For the second case, consider numbers ending with 3,5,7, and 9. There are 4 choices for the last digit, 6 choices for the first, 8 choices for the second, and 7 choices for the third. For a total of 1334 integers. Giving a total between the two cases of 1736 odd integers with distinct digits between 3000 and 9999.

Problem 10

Given

Suppose that A, B, and C are sets and that $N(A) = 28$, $N(B) = 26$, $N(C) = 14$, $N(A \cap B) = 8$, $N(A \cap C) = 4N(B \cap C) = 3$, and $N(A \cap B \cap C) = 2$. What is $N(A \cup B \cup C)$?

Solution

$N(A \cup B \cup C) = 28 + 26 + 14 - 8 - 4 - 3 + 2 = 55$.

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*End of **Practice Exam # 2***