

MATH 125  
Random Examples

Your Name

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# Chapter 1

## 1.1 Random Examples

### Definition 1.1.1: Limit of Sequence in $\mathbb{R}$

Let  $\{s_n\}$  be a sequence in  $\mathbb{R}$ . We say

$$\lim_{n \rightarrow \infty} s_n = s$$

where  $s \in \mathbb{R}$  if  $\forall$  real numbers  $\epsilon > 0 \exists$  natural number  $N$  such that for  $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

### Question 1

Is the set  $x\text{-axis} \setminus \{\text{Origin}\}$  a closed set

**Solution:** We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

### Note:-

We will do topology in Normed Linear Space (Mainly  $\mathbb{R}^n$  and occasionally  $\mathbb{C}^n$ ) using the language of Metric Space

### Claim 1.1.1 Topology

Topology is cool

### Example 1.1.1 (Open Set and Close Set)

- Open Set:
- $\phi$
  - $\bigcup_{x \in X} B_r(x)$  (Any  $r > 0$  will do)
  - $B_r(x)$  is open
- Closed Set:
- $X, \phi$
  - $\overline{B_r(x)}$
- $x\text{-axis} \cup y\text{-axis}$

### Theorem 1.1.1

If  $x \in$  open set  $V$  then  $\exists \delta > 0$  such that  $B_\delta(x) \subset V$

**Proof:** By openness of  $V$ ,  $x \in B_r(u) \subset V$



Given  $x \in B_r(u) \subset V$ , we want  $\delta > 0$  such that  $x \in B_\delta(x) \subset B_r(u) \subset V$ . Let  $d = d(u, x)$ . Choose  $\delta$  such that  $d + \delta < r$  (e.g.  $\delta < \frac{r-d}{2}$ )

If  $y \in B_\delta(x)$  we will be done by showing that  $d(u, y) < r$  but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

☺

### Corollary 1.1.1

By the result of the proof, we can then show...

### Lemma 1.1.1

Suppose  $\vec{v}_1, \dots, \vec{v}_n \in \mathbb{R}^n$  is subspace of  $\mathbb{R}^n$ .

### Proposition 1.1.1

$1 + 1 = 2$ .

## 1.2 Random

### Definition 1.2.1: Normed Linear Space and Norm $\|\cdot\|$

Let  $V$  be a vector space over  $\mathbb{R}$  (or  $\mathbb{C}$ ). A norm on  $V$  is function  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  satisfying

- ①  $\|x\| = 0 \iff x = 0 \ \forall x \in V$
- ②  $\|\lambda x\| = |\lambda| \|x\| \ \forall \lambda \in \mathbb{R}(\text{or } \mathbb{C}), x \in V$
- ③  $\|x + y\| \leq \|x\| + \|y\| \ \forall x, y \in V$  (Triangle Inequality/Subadditivity)

And  $V$  is called a normed linear space.

• Same definition works with  $V$  a vector space over  $\mathbb{C}$  (again  $\|\cdot\| \rightarrow \mathbb{R}_{\geq 0}$ ) where ② becomes  $\|\lambda x\| = |\lambda| \|x\| \ \forall \lambda \in \mathbb{C}, x \in V$ , where for  $\lambda = a + ib, |\lambda| = \sqrt{a^2 + b^2}$

### Example 1.2.1 ( $p$ -Norm)

$V = \mathbb{R}^m, p \in \mathbb{R}_{\geq 0}$ . Define for  $x = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left( |x_1|^p + |x_2|^p + \dots + |x_m|^p \right)^{\frac{1}{p}}$$

(In school  $p = 2$ )

**Special Case  $p = 1$ :**  $\|x\|_1 = |x_1| + |x_2| + \dots + |x_m|$  is clearly a norm by usual triangle inequality.

**Special Case  $p \rightarrow \infty$  ( $\mathbb{R}^m$  with  $\|\cdot\|_\infty$ ):**  $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_m|\}$

For  $m = 1$  these  $p$ -norms are nothing but  $|x|$ . Now exercise

## Question 2

Prove that triangle inequality is true if  $p \geq 1$  for  $p$ -norms. (What goes wrong for  $p < 1$  ?)

**Solution:** For Property ③ for norm-2

When field is  $\mathbb{R}$  :

We have to show

$$\begin{aligned}\sum_i (x_i + y_i)^2 &\leq \left( \sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2 \\ \Rightarrow \sum_i (x_i^2 + 2x_i y_i + y_i^2) &\leq \sum_i x_i^2 + 2\sqrt{\left[ \sum_i x_i^2 \right] \left[ \sum_i y_i^2 \right]} + \sum_i y_i^2 \\ \Rightarrow \left[ \sum_i x_i y_i \right]^2 &\leq \left[ \sum_i x_i^2 \right] \left[ \sum_i y_i^2 \right]\end{aligned}$$

So in other words prove  $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$  where

$$\langle x, y \rangle = \sum_i x_i y_i$$

### Note:-

1.  $\|x\|^2 = \langle x, x \rangle$
2.  $\langle x, y \rangle = \langle y, x \rangle$
3.  $\langle \cdot, \cdot \rangle$  is  $\mathbb{R}$ -linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

Here in  $\langle x, y \rangle$   $x$  is in first slot and  $y$  is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

expand everything of  $\langle x - \lambda y, x - \lambda y \rangle$  which is going to give a quadratic equation in variable  $\lambda$

$$\begin{aligned}\langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle\end{aligned}$$

Now unless  $x = \lambda y$  we have  $\langle x - \lambda y, x - \lambda y \rangle > 0$  Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is  $\mathbb{C}$  :

Modify the definition by

$$\langle x, y \rangle = \sum_i \bar{x}_i y_i$$

Then we still have  $\langle x, x \rangle \geq 0$

## 1.3 Algorithms

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**Algorithm 1:** what

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**Input:** This is some input

**Output:** This is some output

*/\* This is a comment \*/*

```
1 some code here;
2  $x \leftarrow 0$ ;
3  $y \leftarrow 0$ ;
4 if  $x > 5$  then
5   |  $x$  is greater than 5 ;                                // This is also a comment
6 else
7   |  $x$  is less than or equal to 5;
8 end
9 foreach  $y$  in 0..5 do
10  |  $y \leftarrow y + 1$ ;
11 end
12 for  $y$  in 0..5 do
13  |  $y \leftarrow y - 1$ ;
14 end
15 while  $x > 5$  do
16  |  $x \leftarrow x - 1$ ;
17 end
18 return Return something here;
```

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