Math 125
Lecture notes — Binomial Theorem and Graphs

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Chapter 1

Binomial Theorem

Recall, $\binom{n}{r}$ = the number of r-element subsets of an n-element set.

$$\binom{n}{0} = 1 = \binom{n}{n}$$

$$\binom{n}{1} = n = \binom{n}{n-1}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

Theorem 1.0.1 The Binomial Theorem

Given any real numbers a and b and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

For example, $(a+b)^3$
$= \binom{}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3$ $(a+b)^3 = (a+b)(a+b)(a+b)$ multiplied out is the sum of all possible ordered products of a's and b's. It's $aaa+baa+aba+aab+bba+bab+abb+bbb$. The variable in the ith position is taken from the ith parenthesis.

Example 1.0.1 (Inductive Proof of the Binomial Theorem)

Clearly, $(a+b)^0 = 1 = \sum_{k=0}^{0} \binom{0}{k} a^{0-k} b^0$.
Suppose, $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$ for some integer $m \geqslant 0$.
Then $(a+b)^{m+1} = (a+b)(a+b)^m$
$= (a+b)\sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k$
$= \sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{j=0}^{m} \binom{m}{j} a^{m-j} b^{j+1}$
For the second sum, let $k = j+1$ so that the sum becomes,
$= \sum_{k=0}^{m} \binom{m}{k} a^{m-k+1} b^k + \sum_{k=1}^{m+1} \binom{m}{k-1} a^{m-k+1} b^{k+1}$

Chapter 2

Graph Theory

A graph G consists of two finite sets: a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges where each edge is associated with either one or two vertices called its endpoints. If edge e is associated with vertex v, then e and v are said to be incident.

Graphs have pictorial representations in which the vertices are represented by dots and the edges by line segments.

Example 2.0.1

$V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{e_1, e_2, e_3, e_4, e_5\}$.$e_3$ and e_4 have the same set of endpoints. They are said to parallel. Edge e_5 has only one endpoint. e_5 is called a loop. Vertex v_4 is said to be isolated.

The degree of a vertex in a graph is the number of edges incident to it. If a loop is incident to a vertex, then the loop contributes 2 to the degree.

In the graph of the previous slide, $\deg(v_1) = 2$, $\deg(v_2) = 3$, $\deg(v_3) = 5$ and $\deg(v_4) = 0$. Note that $2 + 3 + 5 + 0 = 10$ is twice the number of edges.

Theorem 2.0.1 Handshake Theorem

In a graph G, the sum of the degrees of the vertices of G equals twice the number of edges of G.

Proof of Theorem 2.0.1: Each edge contributes 2 to the sum of the degrees. 😄

Corollary 2.0.1

Every graph has an even number of vertices of odd degree.

Example 2.0.2 (Königsberg Bridge Problem, 1736)

The problem is to walk along the edges of the graph, traversing every edge exactly once and coming back to the starting point.

Let G be a graph and let v and w be vertices of G.

A walk in G from v to w is a finite alternating sequence of vertices and edges of G. Thus a walk has the form $v_0e_1v_1e_2\cdots v_{n-1}e_nv_n$ where the v's represent the vertices, and the e's represent the edges.

A trail from v to w is a walk from v to w that does not contain a repeated edge.

A path from v to w is a trail that does not contain a repeated vertex.

A closed walk is a walk that starts and ends at the same vertex.

A circuit is a closed walk that contains at least one edge and does not contain a repeated edge.

A graph H is a subgraph of a graph G if, and only if, every vertex of H is also a vertex in G, and evry edge in H is also an edge of G, and every edge in H has the same endpoints as it has in G.

Two vertices v and w of a graph G are connected if, and only if, there is a walk from v to w.

A graph H is a connected component of a graph G if, and only if,

1. H is a subgraph of G.
2. H is connected.
3. no connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H.

Let G be a graph. An Euler Circuit for G is a circuit that contains every vertex and every edge of G.

Theorem 2.0.2
If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

Proof of Theorem 2.0.2: Consider and Euler circuit in a graph G and a vertex v of G. The degree of v is positive because v is in the Euler circuit and each vertex of a circuit is associated with an edge of the circuit. For each appearance of v in the circuit other than as the first or last vertex, say $v = v_i$, there are two edges incident to v in the circuit, e_i and e_{i+1}. If v is the first and last vertex in the circuit then the first and last edges together contribute 2 to the degree. 😄