Math 125
Course Notes

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Contents

Chapter 1

Set Theory

1.1 Definitions and The Element Method of Proof

1.1.1 Definitions

Definition 1.1.1

Let S denote a set and a an element of set S. $a \in S$ means that a is an element of S. $a \notin S$ means that a is not an element of S. If S is a set and $P(x)$ is a property that elements of S may or may not satisfy, then a set A may be defined by writing

$$A = \{x \in S \mid P(x)\}$$

Read "A is the set of all x in S such that $P(x)$."

Definition 1.1.2: Subsets

$A \subseteq B \Leftrightarrow \forall x,$ if $x \in A$ then $x \in B$

Note:-

Since the definition is a universal statement, the negation is existential.

$$A \nsubseteq B \Leftrightarrow \exists x \text{ such that } x \in A \text{ and } x \notin B$$

Definition 1.1.3: Proper Subsets

$A \subset B \Leftrightarrow$

(1) $A \subseteq B$

(2) $\exists x$ such that $x \in B$ and $x \notin A$

Example 1.1.1

Let $A = \{1\}, B = \{1, \{1\}\}$

Note:-

A set with one element is a singleton

(a) Is $A \subseteq B$?

(b) Is $A \subset B$?

Solution:

(a) Yes, $\forall x$, if $x \in A$ then $x \in B$

(b) Yes, $\exists x$ such that $x \in B$ and $x \notin A$

1.1.2 Element Method of Proof

Definition 1.1.4: Element Method of Proof

Let sets X and Y be given. To prove that $X \subseteq Y$,

(1) Suppose that x is a particular but arbitrarily chosen element of X

(2) Show that x is and element of Y

Example 1.1.2

Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 6r + 12 \text{ for some } r \in \mathbb{Z}\}$$

$$B = \{n \in \mathbb{Z} \mid n = 3s \text{ for some } s \in \mathbb{Z}\}$$

(a) Prove that $A \subseteq B$

(b) Disprove that $B \subseteq A$

Proof: Show that $A \subseteq B$

Let $x \in A$ (We must show that $x \in B$, by definition of B we must show that $x = 3\cdot$ some integer) By definition of A, there is an integer r, such that $x = 6r + 12$. Let $s = 2r + 4$, then $s \in \mathbb{Z}$ as products and sums of integers are integers, and so $3s \in B$ by definition of B. Also, $3s = 3(2r + 4) = 6r + 12 = x$, thus by definition of B, x is an element of B.

😊

Proof: Show that $B \nsubseteq A$

Let $x \in B$, $x = 3s$ by definition of B. Let $x = 3 = 3 \cdot 1$, and set this equal to the definition of A. $6r + 12 = 3$

$$\begin{aligned} 6r + 12 &= 3 \text{ by assumption} \\ 2r + 4 &= 1 \text{ by diving both sides by } 3 \\ 2r &= -3 \text{ by subtracting } 4 \text{ from both sides} \\ r &= \frac{-3}{2} \text{ by dividing both sides by } 2 \end{aligned}$$

😊

1.1.3 Set Equality

Definition 1.1.5: Set Equality

Given sets A and B, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$.

Example 1.1.3

Define sets A and B as follows:

$$A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some integer a}\}$$

$$B = \{n \in \mathbb{Z} \mid 2b - 2 \text{ for some integer b}\}$$

Show that $A = B$

(a) ***Proof:*** $A \subseteq B$ Let $x \in A$, by the definition of A there exists an integer a such that $x = 2a$. Let $b = a + 1$, then $b \in \mathbb{Z}$ as it is a sum of integers. Also $2b - 2 = 2(a+1) - 2 = 2a + 2 - 2 = 2a = x$. Thus by definition of B, $x \in B$. Hence $A \subseteq B$. ☺

(b) ***Proof:*** Let $x \in B$, by the definition of B there exists an integer b such that $x = 2b - 2$. Let $a = b - 1$, then $a \in \mathbb{Z}$ as it is a difference of integers. Also $2a = 2(b-1) = 2b - 2 = x$ Thus by the definition of A, $x \in A$. Hence $B \subseteq A$. ☺

Thus we have shown that $A = B$.

1.1.4 Operations On Sets

Definition 1.1.6: Set Operations

Let A and B be subsets of a universal set U.

- The union of A and B, denoted $A \cup B$, is the set of all elements in either A or B.
- The intersection of A and B, denoted $A \cap B$, is the set of all elements common to both A and B.
- The difference of B minus A, denoted $B - A$, is the set of all elements in B not in A.
- The complement of A, denoted A^c, is the set off all elements of U that are not in A. Symbolically,

$$\begin{aligned} A \cup B &= \{x \in U \mid x \in A \text{ or } x \in B\} \\ A \cap B &= \{x \in U \mid x \in A \text{ and } x \in B\} \\ B - A &= \{x \in U \mid x \in B \text{ and } x \notin A\} \\ A^c &= \{x \in U \mid x \notin A\} \end{aligned}$$

Definition 1.1.7: Repeated Operations On Sets

Given sets $A_0, A_1, A_2, \dots$ that are subsets of a universal set U and given a nonnegative integer n,

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 1, 2, \dots, n\}$$

$$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one positive integer } n\}$$

$$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for every } i = 1, 2, \dots, n\}$$

$$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for every positive integer } i\}$$

1.1.5 The Empty Set

Definition 1.1.8: The Empty Set

The empty set, denoted $\emptyset$, is a set containing no elements.

$$\emptyset = \{\}$$

1.1.6 Partitions of Sets

Definition 1.1.9: Disjoint Sets

Two sets are called <u>disjoint</u> if, and only if, they have no elements in common.

$$A \text{ and } B \text{ are disjoint } \Leftrightarrow A \cap B = \emptyset$$

Example 1.1.4

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?
Solution:

$A \cap B = \{\} = \emptyset$, yes A and B are disjoint.

Definition 1.1.10: Mutually Disjoint Sets

Sets $A_1, A_2, A_3, \dots$ are <u>mutually disjoint</u> (or pairwise disjoint or nonoverlapping) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all integers i and $j = 1, 2, 3, \dots$

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j$$

Example 1.1.5

Let $A_1 = \{3, 5\}, A_2 = \{1, 4, 6\},$ and $A_3 = \{2\}$. Are $A_1, A_2,$ and A_3 mutually disjoint?
Solution:

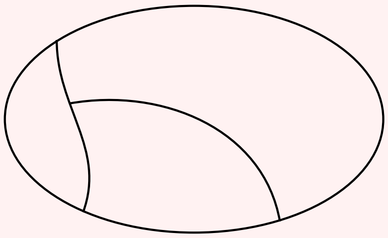
Yes they are disjoint as $A_1 \cap A_2 \cap A_3 = \emptyset$.

Definition 1.1.11: Partitions

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, ...\}$ is a partition of a set A if, and only if,

(1) A is the union of all the A_i.

(2) The sets $A_1, A_2, A_3, ...$ are mutually disjoint.



1.1.7 Power Sets

Definition 1.1.12: Power Set

Given a set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Note:-

The power set of A with length n, has length 2^n.

Example 1.1.6

Let $A = \{x, y\}$, find $\mathcal{P}(A)$.

Solution:

$\mathcal{P}(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$

Question 1: Proof Practice

Let sets R, S and T be defined as follows:

$$R = \{x \in \mathbb{Z} \mid x \text{ is divisible by } 2\}$$
$$S = \{y \in \mathbb{Z} \mid y \text{ is divisible by } 3\}$$
$$T = \{z \in \mathbb{Z} \mid z \text{ is divisible by } 6\}$$

Prove or Disprove each of the following:

(a) $R \subseteq T$

(b) $T \subseteq R$

(c) $T \subseteq S$

Solution:

(a) ***Proof:*** Suppose $R \subseteq T$, then by definition of a subset, every element in R must be common to T. However, $2 \in R$ but $2 \notin T$ as $\frac{2}{6} \notin \mathbb{Z}$. Thus $R \nsubseteq T$. 😊

(b) ***Proof:*** Let $z \in T$ then by definition of T, $z = 6\ell$ for some integer ℓ. By basic algebra we can see that $z = 2(3\ell)$, where $3\ell \in \mathbb{Z}$ as it is a product of integers. Thus $z = 2m$ for some integer m. Hence, by definition of set R, $T \subseteq R$. 😊

(c) ***Proof:*** Let $z \in T$ then by definition of T, $z = 6\ell$ for some integer ℓ. By basic algebra we can see that $z = 3(2\ell)$, where $2\ell \in \mathbb{Z}$ as it is a product of integers. Thus $z = 3m$ for some integer m. Hence, by definition of set S, $T \subseteq S$. 😊

1.2 Properties of Sets

1.2.1 Some Subset Relations

Theorem 1.2.1 Some Subset Relations

(1) Inclusion of Intersection: For all sets A and B,

  (a) $A \cap B \subseteq A$ and (b) $A \cap B \subseteq B$

(2) Inclusion of Union: For all sets A and B,

  (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$

(3) Transitive Property of Subsets: For all sets A and B and C,

  (a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Definition 1.2.1: Procedural Versions of Set Definitions

(1) $x \in X \cup Y \Leftrightarrow x \in X$ or $x \in Y$

(2) $x \in X \cap Y \Leftrightarrow x \in X$ and $x \in Y$

(3) $x \in X - Y \Leftrightarrow x \in X$ and $x \notin Y$

(4) $x \in X^c \Leftrightarrow x \notin X$

(5) $(x, y) \in X \times Y \Leftrightarrow x \in X$ and $y \in Y$

1.2.2 Proving a Subset Relation

Proof of Theorem 6.2.1-1: Suppose that A and B are any sets. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. In particular $x \in A$. Thus $\forall x \in A \cap B, x \in A$. Hence $A \cap B \subseteq A$. ☺

Proof of Theorem 6.2.1-2: Suppose that A and B are any sets. Let $x \in A$, then by definition of union, $x \in A \cup B$. Therefore $A \subseteq A \cup B$. ☺

1.2.3 Set Identities

Theorem 1.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

(1) Cummutative Laws: For all sets A and B,

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

(2) Associative Laws: For all sets A,B and C.

- $(A \cup B) \cup C = A \cup (B \cup C)$
- $(A \cap B) \cap C = A \cap (B \cap C)$

(3) Distributive Laws: For all sets A,B and C

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(4) Identity Laws: For every set A,

- $A \cup \emptyset = A$
- $A \cap U = A$

(5) Complement Laws: For every set A,

- $A \cup A^c = U$
- $A \cap A^c = \emptyset$

(6) Double Complement Law: For every set A,

- $(A^c)^c = A$

(7) DeMorgan's Laws: For all sets A and B,

- $(A \cup B)^c = A^c \cap B^c$
- $(A \cap B)^C = A^c \cup B^c$

(8) Absorbtion Laws: For all sets A and B,

- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

(9) Set Difference Law: For all sets A and B,

- $A - B = A \cap B^c$