

NOTE ON THE IMPLEMENTATION AND EXAMPLES

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1. NOTE ON THE IMPLEMENTATION.

In this note, we recall briefly the setting of [CN26] and explain a slight change done in the computation to calculate $\text{ord}_p(u'_\ell(\omega))$ and $\log_{\beta_p}(u'_\ell(\omega))$. We keep the same notation.

Let F be a real quadratic field, p a prime number inert in F and $\ell \geq 5$ a prime different from p . Attached to $\omega \in F$ and ℓ is a p -unit $u_{\ell!}(\omega)$ that lives in the narrow ring class field H_ω over F . We recall the generating series involve to compute $u_{\ell!}(\omega)$ as explains in [CN26] Section 3.2.

Let

$$f_x(z) = \frac{e^{xz}}{e^x - 1} = \sum_{k \geq 0} b_k(z) x^{k-1}.$$

For any $z \in \mathbb{R}$, define its fractional part $\{z\}$ by

$$0 \leq \{z\} < 1 \text{ and } z - \{z\} \in \mathbb{Z}.$$

Let $c > 0, d$ be two coprime integers and $v = (v_1, v_2) \in \mathbb{Q}^2$, we now define the generating series in two variables:

$$(1) \quad g(c, d, v)(x, y) = \sum_{j=0}^{c-1} f_x\left(\left\{v_1 - \frac{d}{c}(j + v_2)\right\}\right) f_y\left(\frac{j + \{v_2\}}{c}\right) \in \frac{1}{xy} \mathbb{Q}[[x, y]].$$

Let ω be a real quadratic number and $\beta = c\omega + d$, we also define

$$h_\omega(t, c, d, v) = g(c, d, v)(t, \beta t) \in \frac{1}{t^2} \mathbb{Q}(\omega)[[t]].$$

Suppose that c is divisible by a prime $\ell \geq 5$, define the smoothed version of h_ω

$$(2) \quad h_\omega^{(\ell)}(t, c, d, v) := h_{\ell\omega}\left(\frac{t}{\ell}, \frac{c}{\ell}, d, (\ell v_1, v_2)\right) - \ell h_\omega(t, c, d, (v_1, v_2)).$$

Let $\ell \geq 5$ be a prime dividing c , define Let p be an odd prime number and define a measure $\mu_\ell(c, d)$ on $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ as follows. For any $(a, b) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ and $r \geq 1$, we define the measure on the compact open

$$(3) \quad U_{a,b,r} = (a, b) + p^r \mathbb{Z}_p^2$$

as

$$(4) \quad \mu_\ell(c, d)(U_{a,b,r}) := h_\omega^{(\ell)}\left(t, c, d, \left(\frac{a}{p^r}, \frac{b}{p^r}\right)\right) [t^0].$$

We can write explicitly the right hand side of Equation (4) as Dedekind sums:

$$(5) \quad \mu_\ell(c, d)(U_{a,b,r}) = \sum_{j=0}^{c/\ell-1} \tilde{b}_1\left(\ell v_1 - d \frac{(j + v_2)}{c/\ell}\right) \tilde{b}_1\left(\frac{j + v_2}{c/\ell}\right) - \ell \sum_{j=0}^{c-1} \tilde{b}_1\left(v_1 - d \frac{(j + v_2)}{c}\right) \tilde{b}_1\left(\frac{j + v_2}{c}\right),$$

where \tilde{b}_1 is the 1-periodic function equal to b_1 on $[0, 1[$ and $(v_1, v_2) = \left(\frac{a}{p^r}, \frac{b}{p^r}\right)$.

We give in [CN26] Section 3.1 formulas to compute $\log_p(u_\ell(\omega))$, $\text{ord}_p(u_\ell(\omega))$ and $\log_{\beta_p}(u_\ell(\omega))$ with M p -adic precision using the coefficients of $h_\omega^{(\ell)}$. Let $\gamma_\omega = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$ be a generator of $\text{stab}_{\Gamma_0(\ell)}(\omega)$. First, we have

$$(6) \quad \log_p(u'_\ell(\omega)) = 12(C_{\omega,c,d} + I') + O(p^{M+1}),$$

where

$$C_{\omega,c,d} := \sum_{\substack{a,b \mod p \\ a,b \neq 0,0}} \log_p(a+b\omega) \mu_\ell(c,d)(U_{a,b,1}),$$

and

$$(7) \quad I' = \sum_{\substack{a,b \mod p \\ a,b \neq 0,0}} \frac{1}{n} \sum_{n=1}^{M'} \sum_{j=0}^n \binom{n}{j} \frac{(-1)^{j-1}}{(a+b\omega)^j} j! p^j h_\omega^{(\ell)}(t, c, d, (\frac{a}{p}, \frac{b}{p})) [t^j] + O(p^{M+1}).$$

Let β_p be a primitive $p^2 - 1$ roots of unity in $K_p = K \otimes \mathbb{Q}_p$ and \log_{β_p} be the discrete logarithm with base β_p

$$\log_{\beta_p} : K_p^\times \rightarrow \mathbb{Z}/(p^2 - 1)\mathbb{Z},$$

such that

$$\frac{x}{\beta_p^{\log_{\beta_p}(x)} p^{\text{ord}_p(x)}} \in 1 + p\mathcal{O}_p \text{ for all } x \in K_p^\times.$$

Then, using [Das07] Equation (31) and (44) we have

$$(8) \quad \log_{\beta_p}(u'_\ell(\omega)) = 12 \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_{\beta_p}(x + \omega y) d\mu_\ell(c, d).$$

By splitting the integral, we obtain

$$(9) \quad \log_{\beta_p}(u'_\ell(\omega)) = 12 \sum_{\substack{a,b \mod p \\ a,b \neq 0,0}} \log_{\beta_p}(a+b\omega) h_\omega^{(\ell)} \left(t, c, d, \left(\frac{a}{p}, \frac{b}{p} \right) \right) [t^0].$$

Similarly, we have

$$(10) \quad \text{ord}_p(u'_\ell(\omega)) = 12 h_\omega^{(\ell)}(t, c, d, 0, 0) [t^0] + \left\lfloor \frac{\ell - 1}{4} \right\rfloor.$$

Equations (9) and (10) are true up to the normalization of \tilde{b}_1 at 0. The normalization in [Das07] is $\tilde{b}_1(0) = 0$. To match this normalization to compute $\text{ord}_p(u'_\ell(\omega))$ and $\log_{\beta_p}(u'_\ell(\omega))$, all we have to do is change $f_x(z)$ in the computations by

$$\tilde{f}_x(z) = \begin{cases} \frac{e^{xz}}{e^x - 1} & \text{if } z \neq 0, \\ \frac{e^{xz}}{e^x - 1} + \frac{1}{2} & \text{else.} \end{cases}$$

With this new normalization, Equation (6) still holds. Algorithm 1 given in [CN26] to compute the coefficients of h_ω works as well.

Furthermore, by taking $\ell \geq 5$ in the smoothing, we can drop the factor 12 in the computations (that is, computing $u_\ell(\omega)^{1/12}$ instead) as it is the case in [FL21].

2. NUMERICAL EXAMPLES

The computation of this section were done on the team Ouragan servers with the following characteristics RAM : 128 GB CPU: 32 13th Gen i9-13900.

In this section, we choose a fundamental discriminant D such that we can find $\omega \in F = \mathbb{Q}(\sqrt{D})$ with stabilizer $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$ with c small. Then we compute $u'_\ell(\omega) \in F_p$ for different p and precision p^M using Algorithm ???. As p grows larger, smaller M will be enough to recognize the coefficients of the minimal polynomial $P_{u'_\ell(\omega)}$. The restrictions on small c can be lifted by using the cocycles relation of the measure μ_ℓ . This is part of the strategy described in [Das07] (see Algorithm 4.1) and will be implemented later in our case.

The general strategy to compute $P_{u'_\ell(\omega)}$ is as follows. For each class in the narrow class group of F , we find a quadratic form $(a_i x^2 + b_i xy + c_i y^2)$ with $\ell | a_i$. We pick $\omega_i = \frac{-b_i + \sqrt{D}}{2a_i}$, the minimal polynomial over F is then given by

$$P_{u_\ell(\omega)} = P_{u'_\ell(\omega)} = \prod_i (x - u_\ell(\omega_i)).$$

Let $D = 24$ and $\ell = 5$, we take

$$\omega = \frac{\sqrt{D}}{10} + \frac{4}{5}, \quad \gamma_\omega = \begin{pmatrix} 13 & -4 \\ 10 & -3 \end{pmatrix}.$$

In particular, we have $(c, d) = (10, -3)$. We compute for the following primes the unit $u_\ell(-\omega)$ with precision $M = 100$. As the narrow class group of $F = \mathbb{Q}(\sqrt{24})$ is of order 2, we know that the minimal polynomial $P_{u'_\ell(\omega)}$ of $u'_\ell(\omega)$ with coefficients in K is of degree 2. In particular

$$P_{u_\ell(\omega)} = (x - u_\ell(\omega))(x - u'_\ell(\omega)) = (x - u_\ell(\omega))(x - u_\ell(\omega)^{-1}).$$

We use Sagemath to recognize the coefficient of x of $P_{u_\ell(\omega)}$. We have the following table.

p	$P_{u_\ell(\omega)}$	Computation time
7	$x^2 - \frac{11}{7}x + 1$	55s
11	$x^2 + \left(-\frac{3}{11}\sqrt{D} - \frac{7}{11}\right)x + 1$	2m 30s
13	$x^2 - \frac{1}{13}x + 1$	3m 37s
17	$x^2 + \left(-\frac{6}{17}\sqrt{D} + \frac{1}{17}\right)x + 1$	6m 39s
31	$x^2 + \frac{13}{31}x + 1$	25m 15s
37	$x^2 + \frac{26}{37}x + 1$	38m 12s
41	$x^2 + \left(\frac{12}{41}\sqrt{D} + \frac{23}{41}\right)x + 1$	45m 9s

TABLE 1. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 100$

As the prime p grows, we reduce the precision M that we need to recognize the coefficients as elements of $\mathbb{Q}(\omega)$.

p	$P_{u_\ell(\omega)}$	Computation time
59	$x^2 + \frac{82}{59}x + 1$	15m 26s
61	$x^2 - \frac{47}{61}x + 1$	14m 1s
79	$x^2 + \frac{142}{79}x + 1$	24m 8s
83	$x^2 + (\frac{9}{83}\sqrt{D} - \frac{79}{83})x + 1$	26m 49s
89	$x^2 + (-\frac{18}{89}\sqrt{D} + \frac{73}{89})x + 1$	31m 17s
103	$x^2 - \frac{194}{103}x + 1$	42m 8s
107	$x^2 + (-\frac{21}{107}\sqrt{D} - \frac{89}{107})x + 1$	45m 37s
109	$x^2 - \frac{143}{109}x + 1$	47m 55s

TABLE 2. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 50$

p	$P_{u_\ell(\omega)}$	Computation time
521	$x^2 + (\frac{48}{521}\sqrt{D} - \frac{503}{521})x + 1$	37m 5s
541	$x^2 - \frac{793}{541}x + 1$	39m 49s
563	$x^2 + \frac{226}{563}x + 1$	46m 30s
569	$x^2 + (-\frac{168}{569}\sqrt{D} + \frac{313}{569})x + 1$	47m 43s
587	$x^2 + (\frac{51}{587}\sqrt{D} + \frac{569}{587})x + 1$	50m 40s

TABLE 3. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 10$

p	$P_{\ell}(u_\ell(\omega))$	Computation time
7	$x^2 - \frac{11}{7}x + 1$	16h 40m 15s
11	$x^2 + (-\frac{3}{11}\sqrt{D} - \frac{7}{11})x + 1$	47h 59m 45s

TABLE 4. Minimal polynomial for $D = 24$, $\ell = 5$, $M = 1000$

We see in Table 3 that we are able to compute the values for large p . Our algorithm also allows us to work with very high p -adic precision when p is small. In Table 4, we give examples with 1000 p -adic digit accuracy, a comparable precision to similar computations done in the archimedean world see for example [BCG23].

The polynomials in these tables correspond to the same extension H of $F = \mathbb{Q}(\sqrt{24})$. Here

$$(11) \quad H = F(j) = \mathbb{Q}(\sqrt{24}, j) = \mathbb{Q}(\sqrt{24} + j),$$

where $j = e^{\frac{2i\pi}{3}}$. The field H is, as expected, the narrow class field of F .

Let $D = 156$ and $\ell = 5$, take

$$(12) \quad \omega_1 = \frac{\sqrt{D}}{20} + \frac{3}{10}, \quad \gamma_{\omega_1} = \begin{pmatrix} 37 & 12 \\ 40 & 13 \end{pmatrix}, \quad \omega_2 = \frac{\sqrt{D}}{10} - \frac{2}{5}, \quad \gamma_{\omega_2} = \begin{pmatrix} 17 & 28 \\ 20 & 33 \end{pmatrix}.$$

This time we have $(c_1, d_1) = (40, 13)$ and $(c_2, d_2) = (20, 33)$. As the computation time is linear in c , the computations for this examples should be six times longer. The narrow class group of $F = \mathbb{Q}(\sqrt{156})$ is of degree 4, $u_\ell(\omega_1)$ and $u_\ell(\omega_2)$ are conjugate, their minimal polynomial over F is given by

$$P_\omega = (x - u_\ell(\omega_1))(x - u_\ell(\omega_1)^{-1})(x - u_\ell(\omega_2))(x - u_\ell(\omega_2)^{-1}) \in K[x].$$

The polynomial $P_{u_\ell(\omega)}$ is palindromic, we write only the first coefficients.

p	$P_{u_\ell(\omega)}$	Computation time
11	$x^4 + \left(-\frac{24}{11^3}\sqrt{D} - \frac{2808}{11^3}\right)x^3 + \left(\frac{6}{11^2}\sqrt{D} + \frac{36137}{11^4}\right)x^2 + ..$	11m 23s
17	$x^4 + \left(-\frac{644}{17^3}\sqrt{D} + \frac{2898}{17^3}\right)x^3 + \left(-\frac{36}{17^2}\sqrt{D} + \frac{140387}{17^4}\right)x^2 + ..$	29m 7s

TABLE 5. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 100$

p	$P_{u_\ell(\omega)}$	Computation time
73	$x^4 - \frac{66332}{73^3}x^3 - \frac{43440954}{73^4}x^2 + ..$	37m 29s
79	$x^4 + \frac{1417444}{79^3}x^3 + \frac{147583686}{79^4}x^2 + ..$	44m 3s
83	$x^4 + \left(\frac{3128}{83^3}\sqrt{D} + \frac{150696}{83^3}\right)x^3 + \left(\frac{714}{83^2}\sqrt{D} + \frac{32851337}{83^4}\right)x^2 + ..$	48m 59s
97	$x^4 + \frac{324142}{97^3}x^3 + \frac{95350611}{97^4}x^2 + ..$	1h 8m 7s
101	$x^4 + \left(-\frac{263228}{101^3}\sqrt{D} + \frac{56406}{101^3}\right)x^3 + \left(-\frac{84}{101^2}\sqrt{D} + \frac{457391987}{101^4}\right)x^2 + ..$	1h 11m 50s
103	$x^4 - \frac{3631292}{103^3}x^3 + \frac{524542086}{103^4}x^2 + ..$	1h 13m 16s
109	$x^4 + \frac{1338766}{109^3}x^3 + \frac{48285291}{109^4}x^2 + ..$	1h 26m 52s

TABLE 6. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 50$

p	$P_{u_\ell(\omega)}$	Computation time
521	$x^4 + \frac{59731644}{521^3}x^3 - \frac{75691649914}{521^4}x^2 + ..$	3h 32m 37s
523	$x^4 + \frac{43862518}{523^3}x^3 - \frac{80826503229}{523^4}x^2 + ..$	3h 41m 51s
541	$x^4 + \frac{95348734}{541^3}x^3 + \frac{27184995291}{541^4}x^2 + ..$	4h 13m 15s
547	$x^4 + \frac{254172908}{547^3}x^3 + \frac{149879597286}{547^4}x^2 + ..$	4h 35m 43s
569	$x^4 + \left(\frac{32989928}{569^3}\sqrt{D} + \frac{66413934}{569^3}\right)x^3 + \left(\frac{23256}{569^2}\sqrt{D} + \frac{278087052707}{569^4}\right)x^2 + ..$	4h 38m 40s
571	$x^4 - \frac{593232794}{571^3}x^3 + \frac{467704609011}{571^4}x^2 + ..$	4h 54m 3s
577	$x^4 + \frac{46621582}{577^3}x^3 + \frac{211173573171}{577^4}x^2 + ..$	4h 58m 39s

TABLE 7. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 10$

All the polynomials in these tables generate the same unramified abelian extension H of F . The field H is generated over \mathbb{Q} by the polynomial

$$(13) \quad P(X) = 214358881 \times x^8 - 904462416 \times x^7 + 2001359602 \times x^6 - 3077062560 \times x^5 \\ + 3538768611 \times x^4 - 3077062560 \times x^3 + 2001359602 \times x^2 - 904462416 \times x + 214358881.$$

REFERENCES

- [BCG23] Nicolas Bergeron, Pierre Charollois, and Luis E. García. Elliptic units for complex cubic fields (on eisenstein’s jugendtraum). arXiv preprint, 2023. arXiv:2311.04110. [↑4](#).
- [CN26] Mateo Crabit Nicolau. Generating series techniques for computing darmon-dasgupta units. arXiv preprint, 2026. [↑1, 2](#).
- [Das07] Samit Dasgupta. Computations of elliptic units for real quadratic fields. *Can. J. Math.*, 59(3):553–574, 2007. [↑2, 3](#).
- [FL21] Max Fleischer and Yijia Liu. Computations of elliptic units. <https://github.com/liuyj8526/Computation-of-Elliptic-Units>, 2021. [↑2](#).

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