

# NOTE ON THE IMPLEMENTATION AND EXAMPLES

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### 1. NOTE ON THE IMPLEMENTATION.

In this note, we recall briefly the setting of [CN26] and explain a slight change done in the computation to calculate  $\text{ord}_p(u'_\ell(\omega))$  and  $\log_{\beta_p}(u'_\ell(\omega))$ . We keep the same notation.

Let  $F$  be a real quadratic field,  $p$  a prime number inert in  $F$  and  $\ell \geq 5$  a prime different from  $p$ . Attached to  $\omega \in F$  and  $\ell$  is a  $p$ -unit  $u_\ell t(\omega)$  that lives in the narrow ring class field  $H_\omega$  over  $F$ . We recall the generating series involve to compute  $u_\ell t(\omega)$  as explains in [CN26] Section 3.2.

Let

$$f_x(z) = \frac{e^{xz}}{e^x - 1} = \sum_{k \geq 0} b_k(z) x^{k-1}.$$

For any  $z \in \mathbb{R}$ , define its fractional part  $\{z\}$  by

$$0 \leq \{z\} < 1 \text{ and } z - \{z\} \in \mathbb{Z}.$$

Let  $c > 0, d$  be two coprime integers and  $v = (v_1, v_2) \in \mathbb{Q}^2$ , we now define the generating series in two variables:

$$(1) \quad g(c, d, v)(x, y) = \sum_{j=0}^{c-1} f_x \left( \left\{ v_1 - \frac{d}{c}(j + v_2) \right\} \right) f_y \left( \frac{j + \{v_2\}}{c} \right) \in \frac{1}{xy} \mathbb{Q}[[x, y]].$$

Let  $\omega$  be a real quadratic number and  $\beta = c\omega + d$ , we also define

$$h_\omega(t, c, d, v) = g(c, d, v)(t, \beta t) \in \frac{1}{t^2} \mathbb{Q}(\omega)[[t]].$$

Suppose that  $c$  is divisible by a prime  $\ell \geq 5$ , define the smoothed version of  $h_\omega$

$$(2) \quad h_\omega^{(\ell)}(t, c, d, v) := h_{\ell\omega} \left( \frac{t}{\ell}, \frac{c}{\ell}, d, (\ell v_1, v_2) \right) - \ell h_\omega(t, c, d, (v_1, v_2)).$$

Let  $\ell \geq 5$  be a prime dividing  $c$ , define Let  $p$  be an odd prime number and define a measure  $\mu_\ell(c, d)$  on  $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$  as follows. For any  $(a, b) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$  and  $r \geq 1$ , we define the measure on the compact open

$$(3) \quad U_{a,b,r} = (a, b) + p^r \mathbb{Z}_p^2$$

as

$$(4) \quad \mu_\ell(c, d)(U_{a,b,r}) := h_\omega^{(\ell)} \left( t, c, d, \left( \frac{a}{p^r}, \frac{b}{p^r} \right) \right) [t^0].$$

We can write explicitly the right hand side of Equation (4) as Dedekind sums:

$$(5) \quad \mu_\ell(c, d)(U_{a,b,r}) = \sum_{j=0}^{c/\ell-1} \tilde{b}_1 \left( \ell v_1 - d \frac{(j + v_2)}{c/\ell} \right) \tilde{b}_1 \left( \frac{j + v_2}{c/\ell} \right) - \ell \sum_{j=0}^{c-1} \tilde{b}_1 \left( v_1 - d \frac{(j + v_2)}{c} \right) \tilde{b}_1 \left( \frac{j + v_2}{c} \right),$$

where  $\tilde{b}_1$  is the 1-periodic function equal to  $b_1$  on  $[0, 1[$  and  $(v_1, v_2) = \left( \frac{a}{p^r}, \frac{b}{p^r} \right)$ .

We give in [CN26] Section 3.1 formulas to compute  $\log_p(u_\ell(\omega))$ ,  $\text{ord}_p(u_\ell(\omega))$  and  $\log_{\beta_p}(u_\ell(\omega))$  with  $M$   $p$ -adic precision using the coefficients of  $h_\omega^{(\ell)}$ . Let  $\gamma_\omega = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$  be a generator of  $\text{stab}_{\Gamma_0(\ell)}(\omega)$ . First, we have

$$(6) \quad \log_p(u'_\ell(\omega)) = 12(C_{\omega,c,d} + I') + O(p^{M+1}),$$

where

$$C_{\omega,c,d} := \sum_{\substack{a,b \pmod p \\ a,b \neq 0,0}} \log_p(a + b\omega) \mu_\ell(c, d)(U_{a,b,1}),$$

and

$$(7) \quad I' = \sum_{\substack{a,b \pmod p \\ a,b \neq 0,0}} \sum_{n=1}^{M'} \frac{1}{n} \sum_{j=0}^n \binom{n}{j} \frac{(-1)^{j-1}}{(a + b\omega)^j} j! p^j h_\omega^{(\ell)}(t, c, d, (\frac{a}{p}, \frac{b}{p})) [t^j] + O(p^{M+1}).$$

Let  $\beta_p$  be a primitive  $p^2 - 1$  roots of unity in  $K_p = K \otimes \mathbb{Q}_p$  and  $\log_{\beta_p}$  be the discrete logarithm with base  $\beta_p$

$$\log_{\beta_p} : K_p^\times \rightarrow \mathbb{Z}/(p^2 - 1)\mathbb{Z},$$

such that

$$\frac{x}{\beta_p^{\log_{\beta_p}(x)} p^{\text{ord}_p(x)}} \in 1 + p\mathcal{O}_p \text{ for all } x \in K_p^\times.$$

Then, using [Das07] Equation (31) and (44) we have

$$(8) \quad \log_{\beta_p}(u'_\ell(\omega)) = 12 \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_{\beta_p}(x + \omega y) d\mu_\ell(c, d).$$

By splitting the integral, we obtain

$$(9) \quad \log_{\beta_p}(u'_\ell(\omega)) = 12 \sum_{\substack{a,b \pmod p \\ a,b \neq 0,0}} \log_{\beta_p}(a + b\omega) h_\omega^{(\ell)}\left(t, c, d, \left(\frac{a}{p}, \frac{b}{p}\right)\right) [t^0].$$

Similarly, we have

$$(10) \quad \text{ord}_p(u'_\ell(\omega)) = 12 h_\omega^{(\ell)}(t, c, d, 0, 0) [t^0] + \left\lfloor \frac{\ell - 1}{4} \right\rfloor.$$

Equations (9) and (10) are true up to the normalization of  $\tilde{b}_1$  at 0. The normalization in [Das07] is  $\tilde{b}_1(0) = 0$ . To match this normalization to compute  $\text{ord}_p(u'_\ell(\omega))$  and  $\log_{\beta_p}(u'_\ell(\omega))$ , all we have to do is change  $f_x(z)$  in the computations by

$$\tilde{f}_x(z) = \begin{cases} \frac{e^{xz}}{e^x - 1} & \text{if } z \neq 0, \\ \frac{e^{xz}}{e^x - 1} + \frac{1}{2} & \text{else.} \end{cases}$$

With this new normalization, Equation (6) still holds. Algorithm 1 given in [CN26] to compute the coefficients of  $h_\omega$  works as well.

Furthermore, by taking  $\ell \geq 5$  in the smoothing, we can drop the factor 12 in the computations (that is, computing  $u_\ell(\omega)^{1/12}$  instead) as it is the case in [FL21].

## 2. NUMERICAL EXAMPLES

The computation of this section were done on the team Ouragan servers with the following characteristics RAM : 128 GB CPU: 32 13th Gen i9-13900.

In this section, we choose a fundamental discriminant  $D$  such that we can find  $\omega \in F = \mathbb{Q}(\sqrt{D})$  with stabilizer  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$  with  $c$  small. Then we compute  $u'_\ell(\omega) \in F_p$  for different  $p$  and precision  $p^M$  using Algorithm ??. As  $p$  grows larger, smaller  $M$  will be enough to recognize the coefficients of the minimal polynomial  $P_{u'_\ell(\omega)}$ . The restrictions on small  $c$  can be lifted by using the cocycles relation of the measure  $\mu_\ell$ . This is part of the strategy described in [Das07] (see Algorithm 4.1) and will be implemented later in our case.

The general strategy to compute  $P_{u'_\ell(\omega)}$  is as follows. For each class in the narrow class group of  $F$ , we find a quadratic form  $(a_i x^2 + b_i x y + c_i y^2)$  with  $\ell | a_i$ . We pick  $\omega_i = \frac{-b_i + \sqrt{D}}{2a_i}$ , the minimal polynomial over  $F$  is then given by

$$P_{u_\ell(\omega)} = P_{u'_\ell(\omega)} = \prod_i (x - u_\ell(\omega_i)).$$

Let  $D = 24$  and  $\ell = 5$ , we take

$$\omega = \frac{\sqrt{D}}{10} + \frac{4}{5}, \quad \gamma_\omega = \begin{pmatrix} 13 & -4 \\ 10 & -3 \end{pmatrix}.$$

In particular, we have  $(c, d) = (10, -3)$ . We compute for the following primes the unit  $u_\ell(-\omega)$  with precision  $M = 100$ . As the narrow class group of  $F = \mathbb{Q}(\sqrt{24})$  is of order 2, we know that the minimal polynomial  $P_{u'_\ell(\omega)}$  of  $u'_\ell(\omega)$  with coefficients in  $K$  is of degree 2. In particular

$$P_{u_\ell(\omega)} = (x - u_\ell(\omega))(x - u'_\ell(\omega)) = (x - u_\ell(\omega))(x - u_\ell(\omega)^{-1}).$$

We use Sagemath to recognize the coefficient of  $x$  of  $P_{u_\ell(\omega)}$ . We have the following table.

$p$	$P_{u_\ell(\omega)}$	Computation time
7	$x^2 - \frac{11}{7}x + 1$	55s
11	$x^2 + \left(-\frac{3}{11}\sqrt{D} - \frac{7}{11}\right)x + 1$	2m 30s
13	$x^2 - \frac{1}{13}x + 1$	3m 37s
17	$x^2 + \left(-\frac{6}{17}\sqrt{D} + \frac{1}{17}\right)x + 1$	6m 39s
31	$x^2 + \frac{13}{31}x + 1$	25m 15s
37	$x^2 + \frac{26}{37}x + 1$	38m 12s
41	$x^2 + \left(\frac{12}{41}\sqrt{D} + \frac{23}{41}\right)x + 1$	45m 9s

TABLE 1. Minimal polynomial for  $D = 24$ ,  $\ell = 5$  and  $M = 100$

As the prime  $p$  grows, we reduce the precision  $M$  that we need to recognize the coefficients as elements of  $\mathbb{Q}(\omega)$ .

$p$	$P_{u_\ell(\omega)}$	Computation time
59	$x^2 + \frac{82}{59}x + 1$	15m 26s
61	$x^2 - \frac{47}{61}x + 1$	14m 1s
79	$x^2 + \frac{142}{79}x + 1$	24m 8s
83	$x^2 + \left(\frac{9}{83}\sqrt{D} - \frac{79}{83}\right)x + 1$	26m 49s
89	$x^2 + \left(-\frac{18}{89}\sqrt{D} + \frac{73}{89}\right)x + 1$	31m 17s
103	$x^2 - \frac{194}{103}x + 1$	42m 8s
107	$x^2 + \left(-\frac{21}{107}\sqrt{D} - \frac{89}{107}\right)x + 1$	45m 37s
109	$x^2 - \frac{143}{109}x + 1$	47m 55s

TABLE 2. Minimal polynomial for  $D = 24$ ,  $\ell = 5$  and  $M = 50$ 

$p$	$P_{u_\ell(\omega)}$	Computation time
521	$x^2 + \left(\frac{48}{521}\sqrt{D} - \frac{503}{521}\right)x + 1$	37m 5s
541	$x^2 - \frac{793}{541}x + 1$	39m 49s
563	$x^2 + \frac{226}{563}x + 1$	46m 30s
569	$x^2 + \left(-\frac{168}{569}\sqrt{D} + \frac{313}{569}\right)x + 1$	47m 43s
587	$x^2 + \left(\frac{51}{587}\sqrt{D} + \frac{569}{587}\right)x + 1$	50m 40s

TABLE 3. Minimal polynomial for  $D = 24$ ,  $\ell = 5$  and  $M = 10$ 

$p$	$P_\ell(u_\ell(\omega))$	Computation time
7	$x^2 - \frac{11}{7}x + 1$	16h 40m 15s
11	$x^2 + \left(-\frac{3}{11}\sqrt{D} - \frac{7}{11}\right)x + 1$	47h 59m 45s

TABLE 4. Minimal polynomial for  $D = 24$ ,  $\ell = 5$ ,  $M = 1000$ 

We see in Table 3 that we are able to compute the values for large  $p$ . Our algorithm also allows us to work with very high  $p$ -adic precision when  $p$  is small. In Table 4, we give examples with 1000  $p$ -adic digit accuracy, a comparable precision to similar computations done in the archimedean world see for example [BCG23].

The polynomials in these tables correspond to the same extension  $H$  of  $F = \mathbb{Q}(\sqrt{24})$ . Here

$$(11) \quad H = F(j) = \mathbb{Q}(\sqrt{24}, j) = \mathbb{Q}(\sqrt{24} + j),$$

where  $j = e^{\frac{2i\pi}{3}}$ . The field  $H$  is, as expected, the narrow class field of  $F$ .

Let  $D = 156$  and  $\ell = 5$ , take

$$(12) \quad \omega_1 = \frac{\sqrt{D}}{20} + \frac{3}{10}, \quad \gamma_{\omega_1} = \begin{pmatrix} 37 & 12 \\ 40 & 13 \end{pmatrix}, \quad \omega_2 = \frac{\sqrt{D}}{10} - \frac{2}{5}, \quad \gamma_{\omega_2} = \begin{pmatrix} 17 & 28 \\ 20 & 33 \end{pmatrix}.$$

This time we have  $(c_1, d_1) = (40, 13)$  and  $(c_2, d_2) = (20, 33)$ . As the computation time is linear in  $c$ , the computations for this examples should be six times longer. The narrow class group of  $F = \mathbb{Q}(\sqrt{156})$  is of degree 4,  $u_\ell(\omega_1)$  and  $u_\ell(\omega_2)$  are conjugate, their minimal polynomial over  $F$  is given by

$$P_\omega = (x - u_\ell(\omega_1))(x - u_\ell(\omega_1)^{-1})(x - u_\ell(\omega_2))(x - u_\ell(\omega_2)^{-1}) \in K[x].$$

The polynomial  $P_{u_\ell(\omega)}$  is palindromic, we write only the first coefficients.

$p$	$P_{u_\ell(\omega)}$	Computation time
11	$x^4 + \left(-\frac{24}{11^3}\sqrt{D} - \frac{2808}{11^3}\right)x^3 + \left(\frac{6}{11^2}\sqrt{D} + \frac{36137}{11^4}\right)x^2 + ..$	11m 23s
17	$x^4 + \left(-\frac{644}{17^3}\sqrt{D} + \frac{2898}{17^3}\right)x^3 + \left(-\frac{36}{17^2}\sqrt{D} + \frac{140387}{17^4}\right)x^2 + ..$	29m 7s

TABLE 5. Minimal polynomial for  $D = 156$ ,  $\ell = 5$  and  $M = 100$ 

$p$	$P_{u_\ell(\omega)}$	Computation time
73	$x^4 - \frac{66332}{73^3}x^3 - \frac{43440954}{73^4}x^2 + ..$	37m 29s
79	$x^4 + \frac{1417444}{79^3}x^3 + \frac{147583686}{79^4}x^2 + ..$	44m 3s
83	$x^4 + \left(\frac{3128}{83^3}\sqrt{D} + \frac{150696}{83^3}\right)x^3 + \left(\frac{714}{83^2}\sqrt{D} + \frac{32851337}{83^4}\right)x^2 + ..$	48m 59s
97	$x^4 + \frac{324142}{97^3}x^3 + \frac{95350611}{97^4}x^2 + ..$	1h 8m 7s
101	$x^4 + \left(-\frac{263228}{101^3}\sqrt{D} + \frac{56406}{101^3}\right)x^3 + \left(-\frac{84}{101^2}\sqrt{D} + \frac{457391987}{101^4}\right)x^2 + ..$	1h 11m 50s
103	$x^4 - \frac{3631292}{103^3}x^3 + \frac{524542086}{103^4}x^2 + ..$	1h 13m 16s
109	$x^4 + \frac{1338766}{109^3}x^3 + \frac{48285291}{109^4}x^2 + ..$	1h 26m 52s

TABLE 6. Minimal polynomial for  $D = 156$ ,  $\ell = 5$  and  $M = 50$ 

$p$	$P_{u_\ell(\omega)}$	Computation time
521	$x^4 + \frac{59731644}{521^3}x^3 - \frac{75691649914}{521^4}x^2 + ..$	3h 32m 37s
523	$x^4 + \frac{43862518}{523^3}x^3 - \frac{80826503229}{523^4}x^2 + ..$	3h 41m 51s
541	$x^4 + \frac{95348734}{541^3}x^3 + \frac{27184995291}{541^4}x^2 + ..$	4h 13m 15s
547	$x^4 + \frac{254172908}{547^3}x^3 + \frac{149879597286}{547^4}x^2 + ..$	4h 35m 43s
569	$x^4 + \left(\frac{32989928}{569^3}\sqrt{D} + \frac{66413934}{569^3}\right)x^3 + \left(\frac{23256}{569^2}\sqrt{D} + \frac{278087052707}{569^4}\right)x^2 + ..$	4h 38m 40s
571	$x^4 - \frac{593232794}{571^3}x^3 + \frac{467704609011}{571^4}x^2 + ..$	4h 54m 3s
577	$x^4 + \frac{46621582}{577^3}x^3 + \frac{211173573171}{577^4}x^2 + ..$	4h 58m 39s

TABLE 7. Minimal polynomial for  $D = 156$ ,  $\ell = 5$  and  $M = 10$ 

All the polynomials in these tables generate the same unramified abelian extension  $H$  of  $F$ . The field  $H$  is generated over  $\mathbb{Q}$  by the polynomial

$$(13) \quad \begin{aligned} P(X) = & 214358881 \times x^8 - 904462416 \times x^7 + 2001359602 \times x^6 - 3077062560 \times x^5 \\ & + 3538768611 \times x^4 - 3077062560 \times x^3 + 2001359602 \times x^2 - 904462416 \times x + 214358881. \end{aligned}$$

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