NOTE ON THE IMPLEMENTATION AND EXAMPLES

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Contents

1. Note on the implementation.

In this note, we recall briefly the setting of [CN26] and explain a slight change done in the computation to calculate $\mathrm{ord}_p(u'_\ell(\omega))$ and $\log_{\beta_p}(u'_\ell(\omega))$. We keep the same notation.

Let F be a real quadratic field, p a prime number inert in F and $\ell \geq 5$ a prime different from p. Attached to $\omega \in F$ and ℓ is a p-unit $u_{\ell'}(\omega)$ that lives in the narrow ring class field H_ω over F. We recall the generating series involve to compute $u_{\ell'}(\omega)$ as explains in [CN26] Section 3.2.

Let

$$f_x(z) = \frac{e^{xz}}{e^x - 1} = \sum_{k \geq 0} b_k(z) x^{k-1}.$$

For any $z \in \mathbb{R}$, define its fractional part $\{z\}$ by

$$0 \leq \{z\} < 1 \text{ and } z - \{z\} \in \mathbb{Z}.$$

Let $c > 0, d$ be two coprime integers and $v = (v_1, v_2) \in \mathbb{Q}^2$, we now define the generating series in two variables:

$$g(c,d,v)(x,y) = \sum_{j=0}^{c-1} f_x\left(\left\{v_1 - \frac{d}{c}(j+v_2)\right\}\right) f_y\left(\frac{j+\{v_2\}}{c}\right) \in \frac{1}{xy}\mathbb{Q}[[x,y]]. \tag{1}$$

Let ω be a real quadratic number and $\beta = c\omega + d$, we also define

$$h_\omega(t,c,d,v) = g(c,d,v)(t,\beta t) \in \frac{1}{t^2}\mathbb{Q}(\omega)[[t]].$$

Suppose that c is divisible by a prime $\ell \geq 5$, define the smoothed version of h_ω

$$h_\omega^{(\ell)}(t,c,d,v) := h_{\ell\omega}\left(\frac{t}{\ell}, \frac{c}{\ell}, d, (\ell v_1, v_2)\right) - \ell h_\omega(t,c,d,(v_1,v_2)). \tag{2}$$

Let $\ell \geq 5$ be a prime dividing c, define Let p be an odd prime number and define a measure $\mu_\ell(c,d)$ on $\mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ as follows. For any $(a,b) \in \mathbb{Z}_p^2 - p\mathbb{Z}_p^2$ and $r \geq 1$, we define the measure on the compact open

$$U_{a,b,r} = (a,b) + p^r\mathbb{Z}_p^2 \tag{3}$$

as

$$\mu_\ell(c,d)(U_{a,b,r}) := h_\omega^{(\ell)}\left(t,c,d,\left(\frac{a}{p^r}, \frac{b}{p^r}\right)\right)[t^0]. \tag{4}$$

We can write explicitly the right hand side of Equation (4) as Dedekind sums:

$$\mu_\ell(c,d)(U_{a,b,r}) = \sum_{j=0}^{c/\ell-1} \tilde{b_1}\left(\ell v_1 - d\frac{(j+v_2)}{c/\ell}\right)\tilde{b_1}\left(\frac{j+v_2}{c/\ell}\right) - \ell\sum_{j=0}^{c-1}\tilde{b_1}\left(v_1 - d\frac{(j+v_2)}{c}\right)\tilde{b_1}\left(\frac{j+v_2}{c}\right), \tag{5}$$

where $\tilde{b_1}$ is the 1-periodic function equal to b_1 on $[0,1[$ and $(v_1,v_2) = \left(\frac{a}{p^r}, \frac{b}{p^r}\right)$.

We give in [CN26] Section 3.1 formulas to compute $\log_p(u_\ell(\omega))$, $\mathrm{ord}_p(u_\ell(\omega))$ and $\log_{\beta_p}(u_\ell(\omega))$ with M p-adic precision using the coefficients of $h_\omega^{(\ell)}$. Let $\gamma_\omega = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$ be a generator of $\mathrm{stab}_{\Gamma_0(\ell)}(\omega)$. First, we have

$$\log_p(u'_\ell(\omega)) = 12(C_{\omega,c,d} + I') + O(p^{M+1}), \tag{6}$$

where

$$C_{\omega,c,d} := \sum_{\substack{a,b \mod p \\ a,b \neq 0,0}} \log_p(a+b\omega)\mu_\ell(c,d)(U_{a,b,1}),$$

and

$$I' = \sum_{\substack{a,b \mod p \\ a,b \neq 0,0}} \sum_{n=1}^{M'} \frac{1}{n} \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^{j-1}}{(a+b\omega)^j} j! p^j h_\omega^{(\ell)}(t,c,d,(\frac{a}{p},\frac{b}{p}))[t^j] + O(p^{M+1}). \tag{7}$$

Let β_p be a primitive p^2-1 roots of unity in $K_p = K \otimes \mathbb{Q}_p$ and $\log_{\beta_p}$ be the discrete logarithm with base β_p

$$\log_{\beta_p} : K_p^\times \to \mathbb{Z}/(p^2-1)\mathbb{Z},$$

such that

$$\frac{x}{\beta_p^{\log_{\beta_p}(x)} p^{\mathrm{ord}_p(x)}} \in 1 + p\mathcal{O}_p \text{ for all } x \in K_p^\times.$$

Then, using [Das07] Equation (31) and (44) we have

$$\log_{\beta_p}(u'_\ell(\omega)) = 12 \int_{\mathbb{Z}_p^2 - p\mathbb{Z}_p^2} \log_{\beta_p}(x+\omega y) d\mu_\ell(c,d). \tag{8}$$

By splitting the integral, we obtain

$$\log_{\beta_p}(u'_\ell(\omega)) = 12 \sum_{\substack{a,b \mod p \\ a,b \neq 0,0}} \log_{\beta_p}(a+b\omega) h_\omega^{(\ell)}\left(t,c,d,\left(\frac{a}{p},\frac{b}{p}\right)\right)[t^0]. \tag{9}$$

Similarly, we have

$$\mathrm{ord}_p(u'_\ell(\omega)) = 12 h_\omega^{(\ell)}(t,c,d,0,0)[t^0] + \left\lfloor \frac{\ell-1}{4} \right\rfloor. \tag{10}$$

Equations (9) and (10) are true up to the normalization of $\tilde{b_1}$ at 0. The normalization in [Das07] is $\tilde{b_1}(0) = 0$. To match this normalization to compute $\mathrm{ord}_p(u'_\ell(\omega))$ and $\log_{\beta_p}(u'_\ell(\omega))$, all we have to do is change $f_x(z)$ in the computations by

$$\tilde{f_x}(z) = \begin{cases} \frac{e^{xz}}{e^x - 1} & \text{if } z \neq 0, \\ \frac{e^{xz}}{e^x-1} + \frac{1}{2} & \text{else.} \end{cases}$$

Algorithm 1 given in [CN26] to compute the coefficients of h_ω works as well.
Furthermore, by taking $\ell \geq 5$ in the smoothing, we can drop the factor 12 in the computations (that is, computing $u_\ell(\omega)^{1/12}$ instead) as it is the case in [FL21].

2. Numerical examples

The computation of this section were done on the team Ouragan servers with the following characteristics RAM : 128 GB CPU: 32 13th Gen i9-13900.
In this section, we choose a fundamental discriminant D such that we can find $\omega \in F = \mathbb{Q}(\sqrt{D})$ with stabilizer $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\ell)$ with c small. Then we compute $u'_\ell(\omega) \in F_p$ for different p and precision p^M using Algorithm **??**. As p grows larger, smaller M will be enough to recognize the coefficients of the minimal polynomial $P_{u'_\ell(\omega)}$. The restrictions on small c can be lifted by using the cocycles relation of the measure μ_ℓ. This is part of the strategy described in [Das07] (see Algorithm 4.1) and will be implemented later in our case.

The general strategy to compute $P_{u'_\ell(\omega)}$ is as follows. For each class in the narrow class group of F, we find a quadratic form $(a_i x^2 + b_i xy + c_i y^2)$ with $\ell | a_i$. We pick $\omega_i = \frac{-b_i + \sqrt{D}}{2a_i}$, the minimal polynomial over F is then given by

$$P_{u_\ell(\omega)} = P_{u'_\ell(\omega)} = \prod_i (x - u_\ell(\omega_i))).$$

Let $D = 24$ and $\ell = 5$, we take

$$\omega = \frac{\sqrt{D}}{10} + \frac{4}{5}, \ \ \gamma_\omega = \begin{pmatrix} 13 & -4 \\ 10 & -3 \end{pmatrix}.$$

In particular, we have $(c, d) = (10, -3)$. We compute for the following primes the unit $u_\ell(-\omega)$ with precision $M = 100$. As the narrow class group of $F = \mathbb{Q}(\sqrt{24})$ is of order 2, we know that the minimal polynomial $P_{u'_\ell(\omega)}$ of $u'_\ell(\omega)$ with coefficients in K is of degree 2. In particular

$$P_{u_\ell(\omega))} = (x - u_\ell(\omega))(x - u'_\ell(\omega)) = (x - u_\ell(\omega))(x - u_\ell(\omega)^{-1}).$$

We use Sagemath to recognize the coefficient of x of $P_{u_\ell(\omega)}$. We have the following table.

p	$P_{u_\ell(\omega)}$	Computation time
7	$x^2 - \frac{11}{7}x + 1$	55s
11	$x^2 + \left(-\frac{3}{11}\sqrt{D} - \frac{7}{11}\right)x + 1$	2m 30s
13	$x^2 - \frac{1}{13}x + 1$	3m 37s
17	$x^2 + \left(-\frac{6}{17}\sqrt{D} + \frac{1}{17}\right)x + 1$	6m 39s
31	$x^2 + \frac{13}{31}x + 1$	25m 15s
37	$x^2 + \frac{26}{37}x + 1$	38m 12s
41	$x^2 + \left(\frac{12}{41}\sqrt{D} + \frac{23}{41}\right)x + 1$	45m 9s

Table 1. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 100$

As the prime p grows, we reduce the precision M that we need to recognize the coefficients as elements of $\mathbb{Q}(\omega)$.

p	$P_{u_\ell(\omega)}$	Computation time
59	$x^2+\frac{82}{59}x+1$	15m 26s
61	$x^2-\frac{47}{61}x+1$	14m 1s
79	$x^2+\frac{142}{79}x+1$	24m 8s
83	$x^2+\left(\frac{9}{83}\sqrt{D}-\frac{79}{83}\right)x+1$	26m 49s
89	$x^2+\left(-\frac{18}{89}\sqrt{D}+\frac{73}{89}\right)x+1$	31m 17s
103	$x^2-\frac{194}{103}x+1$	42m 8s
107	$x^2+\left(-\frac{21}{107}\sqrt{D}-\frac{89}{107}\right)x+1$	45m 37s
109	$x^2-\frac{143}{109}x+1$	47m 55s

Table 2. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 50$

p	$P_{u_\ell(\omega)}$	Computation time
521	$x^2+\left(\frac{48}{521}\sqrt{D}-\frac{503}{521}\right)x+1$	37m 5s
541	$x^2-\frac{793}{541}x+1$	39m 49s
563	$x^2+\frac{226}{563}x+1$	46m 30s
569	$x^2+\left(-\frac{168}{569}\sqrt{D}+\frac{313}{569}\right)x+1$	47m 43s
587	$x^2+\left(\frac{51}{587}\sqrt{D}+\frac{569}{587}\right)x+1$	50m 40s

Table 3. Minimal polynomial for $D = 24$, $\ell = 5$ and $M = 10$

p	$P_\ell(u_\ell(\omega))$	Computation time
7	$x^2-\frac{11}{7}x+1$	16h 40m 15s
11	$x^2+\left(-\frac{3}{11}\sqrt{D}-\frac{7}{11}\right)x+1$	47h 59m 45s

Table 4. Minimal polynomial for $D = 24$, $\ell = 5$, $M = 1000$

We see in Table 3 that we are able to compute the values for large p. Our algorithm also allows us to work with very high p-adic precision when p is small. In Table 4, we give examples with 1000 p-adic digit accuracy, a comparable precision to similar computations done in the archimedean world see for example [BCG23].

The polynomials in these tables correspond to the same extension H of $F = \mathbb{Q}(\sqrt{24})$. Here

$$H = F(j) = \mathbb{Q}(\sqrt{24}, j) = \mathbb{Q}(\sqrt{24} + j), \tag{11}$$

where $j = e^{\frac{2i\pi}{3}}$. The field H is, as expected, the narrow class field of F.

Let $D = 156$ and $\ell = 5$, take

$$\omega_1 = \frac{\sqrt{D}}{20} + \frac{3}{10},\ \ \gamma_{\omega_1} = \begin{pmatrix} 37 & 12 \\ 40 & 13 \end{pmatrix},\ \ \omega_2 = \frac{\sqrt{D}}{10} - \frac{2}{5},\ \ \gamma_{\omega_2} = \begin{pmatrix} 17 & 28 \\ 20 & 33 \end{pmatrix}. \tag{12}$$

This time we have $(c_1, d_1) = (40, 13)$ and $(c_2, d_2) = (20, 33)$. As the computation time is linear in c, the computations for this examples should be six times longer. The narrow class group of $F = \mathbb{Q}(\sqrt{156})$ is of degree 4, $u_\ell(\omega_1)$ and $u_\ell(\omega_2)$ are conjugate, their minimal polynomial over F is given by

$$P_\omega = (x - u_\ell(\omega_1))(x - u_\ell(\omega_1)^{-1})(x - u_\ell(\omega_2))(x - u_\ell(\omega_2)^{-1}) \in K[x].$$

The polynomial $P_{u_\ell(\omega)}$ is palindromic, we write only the first coefficients.

p	$P_{u_\ell(\omega)}$	Computation time
11	$x^4+\left(-\frac{24}{11^3}\sqrt{D}-\frac{2808}{11^3}\right)x^3+\left(\frac{6}{11^2}\sqrt{D}+\frac{36137}{11^4}\right)x^2+..$	11m 23s
17	$x^4+\left(-\frac{644}{17^3}\sqrt{D}+\frac{2898}{17^3}\right)x^3+\left(-\frac{36}{17^2}\sqrt{D}+\frac{140387}{17^4}\right)x^2+..$	29m 7s

TABLE 5. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 100$

p	$P_{u_\ell(\omega)}$	Computation time
73	$x^4-\frac{66332}{73^3}x^3-\frac{43440954}{73^4}x^2+..$	37m 29s
79	$x^4+\frac{1417444}{79^3}x^3+\frac{147583686}{79^4}x^2+..$	44m 3s
83	$x^4+\left(\frac{3128}{83^3}\sqrt{D}+\frac{150696}{83^3}\right)x^3+\left(\frac{714}{83^2}\sqrt{D}+\frac{32851337}{83^4}\right)x^2+..$	48m 59s
97	$x^4+\frac{324142}{97^3}x^3+\frac{95350611}{97^4}x^2+..$	1h 8m 7s
101	$x^4+\left(-\frac{263228}{101^3}\sqrt{D}+\frac{56406}{101^3}\right)x^3+\left(-\frac{84}{101^2}\sqrt{D}+\frac{457391987}{101^4}\right)x^2+..$	1h 11m 50s
103	$x^4-\frac{3631292}{103^3}x^3+\frac{524542086}{103^4}x^2+..$	1h 13m 16s
109	$x^4+\frac{1338766}{109^3}x^3+\frac{48285291}{109^4}x^2+..$	1h 26m 52s

TABLE 6. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 50$

p	$P_{u_\ell(\omega)}$	Computation time
521	$x^4+\frac{59731644}{521^3}x^3-\frac{75691649914}{521^4}x^2+..$	3h 32m 37s
523	$x^4+\frac{43862518}{523^3}x^3-\frac{80826503229}{523^4}x^2+..$	3h 41m 51s
541	$x^4+\frac{95348734}{541^3}x^3+\frac{27184995291}{541^4}x^2+..$	4h 13m 15s
547	$x^4+\frac{254172908}{547^3}x^3+\frac{149879597286}{547^4}x^2+..$	4h 35m 43s
569	$x^4+\left(\frac{32989928}{569^3}\sqrt{D}+\frac{66413934}{569^3}\right)x^3+\left(\frac{23256}{569^2}\sqrt{D}+\frac{278087052707}{569^4}\right)x^2+..$	4h 38m 40s
571	$x^4-\frac{593232794}{571^3}x^3+\frac{467704609011}{571^4}x^2+..$	4h 54m 3s
577	$x^4+\frac{46621582}{577^3}x^3+\frac{211173573171}{577^4}x^2+..$	4h 58m 39s

TABLE 7. Minimal polynomial for $D = 156$, $\ell = 5$ and $M = 10$

All the polynomials in these tables generate the same unramified abelian extension H of F. The field H is generated over $\mathbb{Q}$ by the polymomial

$$\begin{aligned}P(X) = {} & 214358881\times x^8-904462416\times x^7+2001359602\times x^6-3077062560\times x^5\\ & +3538768611\times x^4-3077062560\times x^3+2001359602\times x^2-904462416\times x+214358881.\end{aligned} \tag{13}$$

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