MS 302 In-class Problems

May 15, 2025

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Appendix

Definition 1 (Set)

A set is a collection of objects. We usually denote sets by capitol letters from the end of the alphabet: X, Y, Z.

Definition 2 (Element)

The objects in a set are called elements. Elements are usually denoted by lower case letters: x, y, z. If x is an element of the set X we denote this by $x \in X$.

There are two ways of showing what elements are in a set.

Bracket notation The first way we will represent the elements in a set is to list the elements in the set between curly braces, $\{\}$. Let us have the set consisting of the elements 1,2, and 3. This set is written in bracket notation as: $\{1, 2, 3\}$.

Set builder notation The other way that we represent the elements in a set is to use what is known as set builder notation. This looks like

$$\{x \mid P(x)\}$$

and is read as

$\{$	x	$\mid$	$P(x)$	$\}$
the set of all	x	such that	x satisfies property P	

The property P is a proposition.

Example 0.1

Suppose E is the set of all positive even integers. Let write this using both bracket and set builder notation. For bracket notation we write

$$E = \{2, 4, 6, 8, \dots\}$$

and in set builder notation we would write

$$E = \{x \mid x \textit{ is an integer } \wedge x > 0 \wedge x \textit{ is divisible by } 2\}$$

If E is the set of all positive even integers, then 2 is an element of E. So we can write $2 \in E$. Since 3 is odd we have $3 \notin E$.

Now suppose we have the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. We note that every element of A is also an element of B, since $1 \in A, 1 \in B$, $2 \in A, 2 \in B$ and $3 \in A, 3 \in B$. We say that A is a subset of B.

Definition 3 (Subset)

The set A is a subset of B, $A \subset B$, if for every $a \in A$, then $a \in B$.

So $\{1, 2, 3\} \subset \{1, 2, 3, 4\}$, but $\{1, 2, 3, 4\}$ is not a subset of $\{1, 2, 3\}$.
There is a special set that contains no elements.

Definition 4 (Empty set)

The empty set is the set that has no elements. It is denoted by $\emptyset$ or $\{\}$

Let $A = \{1, 2, 3\}$. Is $\emptyset \subset A$?

It turns out that the empty set is a subset of any set. Why? Well in order for A to be subset of B each element of A must also be an element of B. We can restate this as

$$\text{If } x \in A, \text{ then } x \in B.$$

So, if we ask, "Is the empty set a subset of the set X?", then we must have:

$$\text{If } x \in \emptyset, \text{ then } x \in X$$

Let us look at the statement $x \in \emptyset$. Is this true or false? Since the empty set has no elements, the statement that x is an element if $\emptyset$ must be false. We recall from the truth table for conditionals that if the hypothesis is false, then the conditional is true. So the empty set is a subset of every set.

Theorem 1

For any set A, $\emptyset \subset A$.

Proof.

Let A be any set. In order for $\emptyset \subset A$ then the following must be true, if $a \in \emptyset$, then $a \in A$. Since it is false that $a \in \emptyset$ by the definition of the empty set, the conditional must be true. This is because any time the antecedent of a conditional is false the conditional must be true. Hence $\emptyset \subset A$. $\square$

Suppose we want to combine two sets together. So for example we want to combine the sets

$$\{2, 4, 6, 8, 10\}$$

and

$$\{1, 3, 5, 7, 9\}.$$

We want this combined set to be

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

The way we do this is to take the union of the sets.

Definition 5 (Union)

The union of the sets A and B is

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

So x is an element of $A \cup B$ if x is in A or x is in B.

Let us see some examples.

Example 0.2

1. *Let* $A = \{1, 2, 3\}$ *and* $B = \{4, 5, 6\}$
   Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
2. *Let* $A = \{1, 2, 4\}$ *and* $B = \{3, 5, 6\}$
   Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
3. *Let* $A = \{1, 2, 3\}$ *and* $B = \{3, 4, 5\}$
   Then $A \cup B = \{1, 2, 3, 4, 5\}$.
4. *Let* $A = \{2, 4, 6, \ldots\}$ *and* $B = \{1, 3, 5, \ldots\}$
   Then $A \cup B = \{1, 2, 3, 4, \ldots\}$.

Now what if we wanted a set that only contained elements that are in both sets A and B. That is called the intersection.

Definition 6 (Intersection)

The intersection of the sets A and B is

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

So x is an element of $A \cap B$ if x is in A and x is in B.

Let us see some examples.

Example 0.3

1. *Let* $A = \{1, 2, 3\}$ *and* $B = \{2, 3, 4\}$
   Then $A \cap B = \{2, 3\}$.
2. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{3, 4, 5, 6\}$
   Then $A \cap B = \{2, 3, 4\}$.
3. *Let* $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ *and*
   $B = \{2, 3, 5, 7, 11, 13, 17, 19\}$
   Then $A \cap B = \{3, 5, 7, 11, 13, 17, 19\}$.
4. *Let* $A = \{2, 4, 6, 8, 10\}$ *and* $B = \{1, 3, 5, 7, 9\}$
   Then $A \cap B = \emptyset$.

Let us set up some notation for the different sets of numbers that we commonly use.

Name(Symbol)	Set
Natural numbers($\mathbb{N}$)	$\{1, 2, 3, \dots\}$
Integers($\mathbb{Z}$)	$\{\dots, -2, -1, 0, 1, 2, \dots\}$
Rational numbers($\mathbb{Q}$)	$\{\frac{m}{n} \mid m, n \in \mathbb{Z} \wedge n \neq 0\}$
Irrational numbers($\mathbb{J}$)	$\{x \mid x$ has nonrepeat. nonterminat. decimal$\}$
Real numbers($\mathbb{R}$)	$\mathbb{Q} \cup \mathbb{J}$

Since some sets are infinite, let us think about what it means to be infinite. For the moment we will restrict ourselves to the natural numbers, $\mathbb{N}$. The first thing we notice is that there is no largest natural number. Why? Well suppose there is a largest natural number. Lets call it L. But $L+1$ is bigger than L. So L cannot have been the biggest natural number. Hence there is no biggest natural number.

Now there is a smallest natural number, 1. What about the positive real numbers? Is there a smallest positive real number? Suppose there is a smallest real number. Let us call it a. So we have $0 < a$ and there does not exist a positive real number b such that $0 < b < a$, otherwise b would be the smallest positive real number. Now we recall that given any positive number if we take half of it it becomes smaller. So $0 < \frac{a}{2} < a$. But this contradicts our assumption that a was the smallest positive real number. Therefore there must not be a smallest positive real number.

Let us look at another interesting thing that happens with infinity. Suppose we have an urn and an infinite number of balls numbered $1, 2, 3, \ldots$. Let us also suppose that we can move as fast as we want to. Now here is the situation. At 11:00 we put the first ten balls, $1, 2, \ldots, 10$, in the urn and draw out ball numbered 1. Next at 11:30 (half way to 12:00) we put the next ten balls, $11, 12, \ldots, 20$, in the urn and draw out ball number 11. Continuing the process of putting in the next ten balls and pulling out the last ball we put in half way to 12:00 we get:

Time	Put in balls	Removed ball
11:00	$\{1, \ldots, 10\}$	10
11:30	$\{11, \ldots, 20\}$	20
11:45	$\{21, \ldots, 30\}$	30
11:52:30	$\{31, \ldots, 40\}$	40
$\vdots$	$\vdots$	$\vdots$

How many balls are in the urn at 12:00?

Now suppose we do the same process as above except we first remove ball number 1, then ball number 2, then ball number 3, etc. So

Time	Put in balls	Removed ball
11:00	$\{1, \ldots, 10\}$	1
11:30	$\{11, \ldots, 20\}$	2
11:45	$\{21, \ldots, 30\}$	3
11:52:30	$\{31, \ldots, 40\}$	4
$\vdots$	$\vdots$	$\vdots$

Now how many balls will be in the urn at 12:00?

Theorem 2 (Distributive law)

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Theorem 3 (De Morgan's law)

- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$