MS 120 In-class Problems

March 9, 2025

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Chapter R Section R1.2

1.2.001 Use the values in the following table.

x	-6	-1	0	3	4.2	9	12	14	15	22
y	0	0	1	5	9	12	38	22	22	70

1. Explain why the table defines y as a function of x.

   - ◯ For each value of y there are multiple values for x.
   - ◯ For each value of y there is only one x.
   - ◯ For each value of x there are multiple values for y.
   - ◯ For each value of x there is only one y.
   - ◯ For some values of y there are multiple values for x.

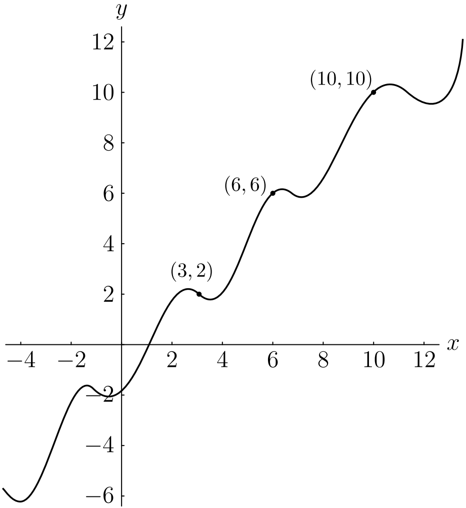
2. State the domain and range of this function.

   domain:
   range:

3. If the table expresses $y = f(x)$, find $f(0)$ and $f(12)$. (If the table does not express $y = f(x)$, enter DNE.)

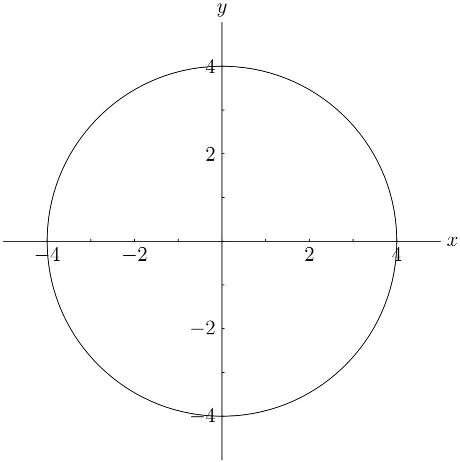
   $f(0) =$
   $f(12) =$

1.2.005a Determine whether the graph represents y as a function of x. Explain your answer.



- Yes, the vertical line test shows that the graph represents y as a function of x.
- Yes, the horizontal line test shows that the graph represents y as a function of x.
- No, the vertical line test shows that the graph does not represent y as a function of x.
- No, the horizontal line test shows that the graph does not represent y as a function of x.
- There is no way to determine this using the graph.

1.2.005b Determine whether the graph represents y as a function of x. Explain your answer.



- Yes, the vertical line test shows that the graph represents y as a function of x.
- Yes, the horizontal line test shows that the graph represents y as a function of x.
- No, the vertical line test shows that the graph does not represent y as a function of x.
- No, the horizontal line test shows that the graph does not represent y as a function of x.
- There is no way to determine this using the graph.

1.2.009 If $R(x) = 8x - 11$, find the following. (Give exact answers. Do not round.)

1. $R(0) =$

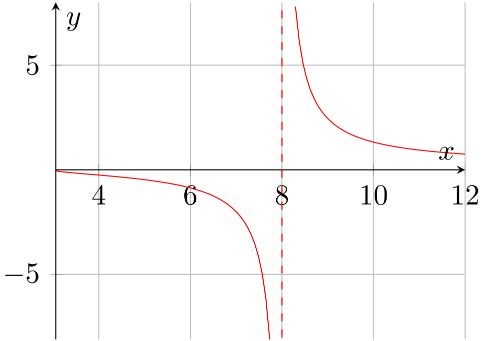
2. $R(2) =$

3. $R(-3) =$

4. $R(1.6) =$

1.2.029 A function and its graph are given. Find the domain. (Enter your answer using interval notation.)

$$f(x) = \frac{\sqrt{x-3}}{x-8}$$



Chapter R Section R1.3

1.3.001 Find the intercepts and graph.

$$5x + 8y = 40$$

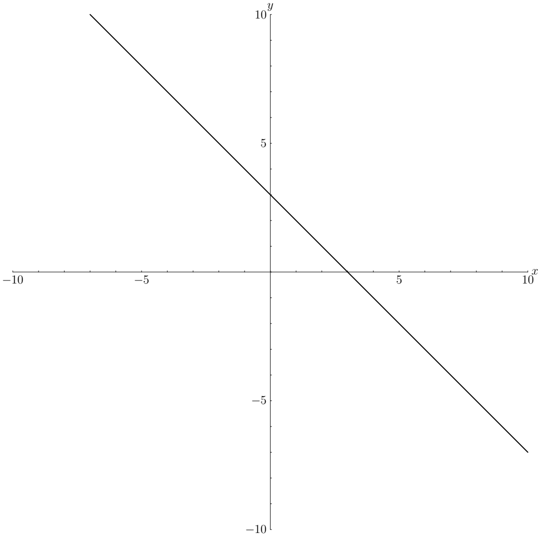
1.3.005 Find the slope m of the line passing through the given pair of points. (If an answer is undefined, enter UNDEFINED.)

$$(20, 21) \text{ and } (14, -3)$$

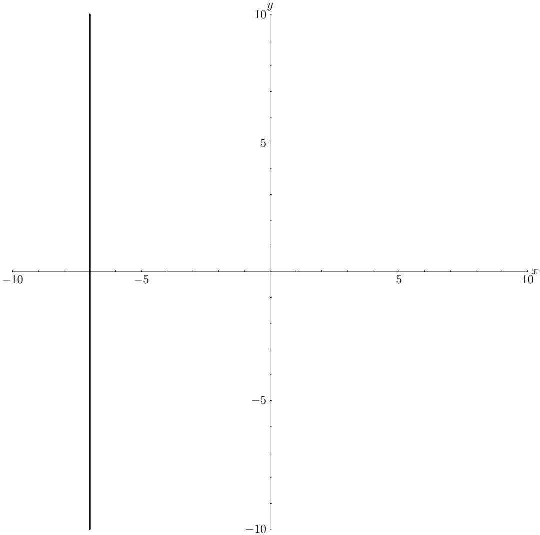
1.3.011 If a line is horizontal, then its slope is ______.

1.3.013 What is the rate of change of the function whose graph is a line passing through $(3, 4)$ and $(-1, 4)$?

1.3.015a For the given graph, determine whether the line has a slope that is positive, negative, 0, or undefined.



1.3.015b For the given graph, determine whether the line has a slope that is positive, negative, 0, or undefined.



1.3.017 Find the slope m and y-intercept b. (Give exact answers. Do not round. If an answer is undefined, enter UNDEFINED. If an answer does not exist, enter DNE.)

$$y = \frac{7}{3}x - \frac{1}{2}.$$

1.3.023 Find the slope m and y-intercept b. (Give exact answers. Do not round. If an answer is undefined, enter UNDEFINED. If an answer does not exist, enter DNE.)

$$2x + 7y = 14.$$

1.3.025 Write the slope-intercept form of the equation of the line that has the given slope and y-intercept.

$$\text{Slope } \frac{1}{3} \text{ and } y\text{-intercept } -3$$

1.3.033 Write the equation of the line that passes through the given point and has the given slope.

$$(-2, 2) \text{ with undefined slope}$$

1.3.035 Write the equation of the line described.

Through $(4, 5)$ and $(-1, -5)$

1.3.041 Determine whether the following pair of equations represents parallel lines, perpendicular lines, or neither of these.

$$3x + 8y = 24; \quad 8x - 3y = 24$$

1.3.045 Write the equation of the line passing through $(-2, -1)$ that is parallel to $3x + 5y = 11$.

Chapter R Section R1.6

In engineering and science, dimensional analysis is the analysis of the relationships between different physical quantities by identifying their base quantities (such as length, mass, time, and electric current) and units of measurement (such as metres and grams) and tracking these dimensions as calculations or comparisons are performed.
The term dimensional analysis is also used to refer to conversion of units from one dimensional unit to another, which can be used to evaluate scientific formulae.[1]

[1] https://en.wikipedia.org/wiki/Dimensional_analysis

1.6.005a A linear cost function is $C(x) = 3x + 750$. (Assume C is measured in dollars.)

1. What are the slope and the C-intercept?
2. What is the marginal cost C' ($\overline{MC}$)?
3. What does the marginal cost mean?

   a) Each additional unit produced costs this much (in dollars).

   b) If production is increased by this many units, the cost decreases by \$1.

   c) If production is increased by this many units, the cost increases by \$1.

   d) Each additional unit produced reduces the cost by this much (in dollars).

4. What are the fixed costs?
5. How are your answers to parts (1), (2), and (3) related?

   a) $\dfrac{C\text{-intercept}}{slope}$ = marginal cost

   b) slope = fixed costs, and C-intercept = marginal cost

   c) slope = marginal cost, and C-intercept = fixed costs

   d) $\dfrac{slope}{C\text{-intercept}}$ = marginal cost

6. What is the cost of producing one more item if 50 are currently being produced? What is the cost of producing one more item if 100 are currently being produced?

1.6.007 A linear revenue function is $R = 26x$. (Assume R is measured in dollars.)

1. What is the slope m?
2. What is the marginal revenue R'?
   What does the marginal revenue mean?

   a) Each additional unit sold decreases the revenue by this many dollars.
   b) If the number of units sold is increased by this amount, the revenue increases by $1.
   c) Each additional unit sold yields this many dollars in revenue.
   d) If the number of units sold is increased by this amount, the revenue decreases by $1.

3. What is the revenue received from selling one more item if 50 are currently being sold?
   What is the revenue received from selling one more item if 100 are being sold?

1.6.001 Suppose a calculator manufacturer has the total cost function $C(x) = 22x + 6600$ and the total revenue function $R(x) = 56x$.

1. What is the equation of the profit function $P(x)$ for the calculator?

2. What is the profit on 2800 units?

1.6.003 Suppose a ceiling fan manufacturer has the total cost function $C(x) = 34x + 560$ and the total revenue function $R(x) = 48x$.

1. What is the equation of the profit function P(x) for this commodity?
   $P(x) =$

2. What is the profit on 20 units?

   Interpret your result.
   - The total costs are less than the revenue.
   - The total costs are more than the revenue.
   - The total costs are exactly the same as the revenue.

3. How many fans must be sold to avoid losing money?

1.6.009 Let $C(x) = 3x + 750$ and $R(x) = 21x$.

1. Write the profit function $P(x)$.
2. What is the slope m of the profit function?
3. What is the marginal profit P'?
4. Interpret the marginal profit.

a) Each additional unit sold decreases the profit by this much.

b) Each additional unit sold increases the profit by this much.

c) This is the smallest number of units that can be sold in order to make a profit.

d) The profit is maximized when this many units are sold.

1.6.013 1-3 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

1. For this helmet, write the function for monthly total costs $C(x)$.
   $C(x) =$
2. Write the function for total revenue $R(x)$.
   $R(x) =$
3. Write the function for profit $P(x)$.
   $P(x) =$

1.6.013 4 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

4. Find $C(200)$.
   $C(200) =$

   Interpret $C(200)$.
   - For each \$1 increase in cost this many more helmets can be produced.
   - This is the cost (in dollars) of producing 200 helmets.
   - For every additional helmet produced the cost increases by this much.
   - When this many helmets are produced the cost is \$200.

1.6.013 4 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

4. Find $R(200)$.
   $R(200) =$

   Interpret $R(200)$.
   - When this many helmets are produced the revenue generated is \$200.
   - For each \$1 increase in revenue this many more helmets can be produced.
   - For every additional helmet produced the revenue generated increases by this much.
   - This is the revenue (in dollars) generated from the sale of 200 helmets.

1.6.013 4 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

4. Find $P(200)$.
   $P(200) =$

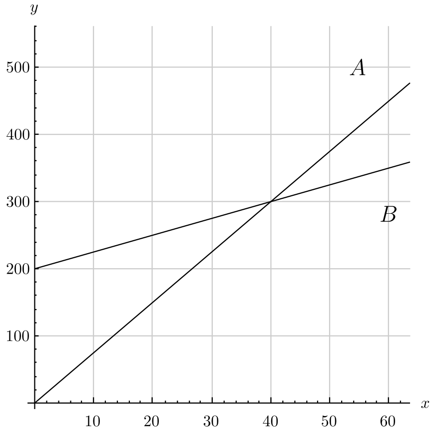
   Interpret $P(200)$.
   - This is the profit (in dollars) when 200 helmets are sold, but since it is negative it means that the company loses money when 200 helmets are sold.
   - For each additional helmet sold the profit (in dollars) increases by this much, but since it is positive it means that the company is producing too many helmets.
   - For each additional helmet sold the profit (in dollars) increases by this much, but since it is negative it means that the company needs to decrease the number of helmets sold in order to make a profit.
   - This is the profit (in dollars) when 200 helmets are sold, and since it is positive it means that the company makes money when 200 helmets are sold.

1.6.013 5,6 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

5. Same as the last part but with \$300 instead of \$200.
6. Find the marginal profit P'.
   $P' =$

   Write a sentence that explains its meaning.
   - When revenue is increased by this much the profit is increased by \$1.
   - For each \$1 increase in profit this many more helmets can be produced.
   - When costs are decreased by this much the profit is increased by \$1.
   - Each additional helmet sold increases the profit by this many dollars.

1.6.015 The figure shows graphs of the total cost function and the total revenue function for a commodity. (Assume cost and revenue are measured in dollars.)



1. Label each function correctly. Choose from *total revenue function*, *total cost function*

   a) function A

   b) function B

2. Determine the fixed costs.
3. Locate the break-even point. $(x, y) =$
   Determine the number of units sold to break even.
4. Estimate the marginal cost C' and marginal revenue R'.

Definition 1 (Market equilibrium)

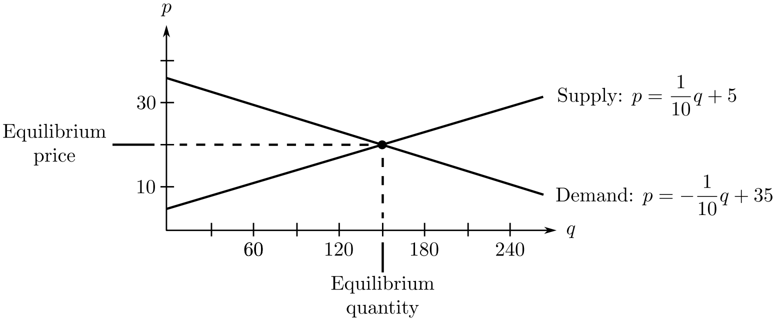
occurs when the quantity of a commodity demanded is equal to the quantity supplied.

Law 1 (Law of demand)

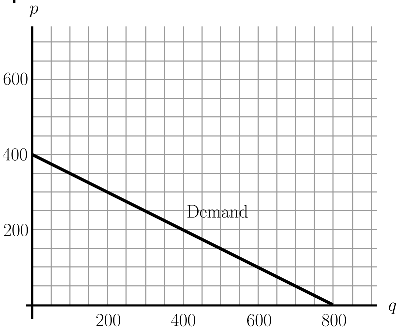
states that the quantity demanded will increase as price decreases and that the quantity demanded will decrease as price increases.

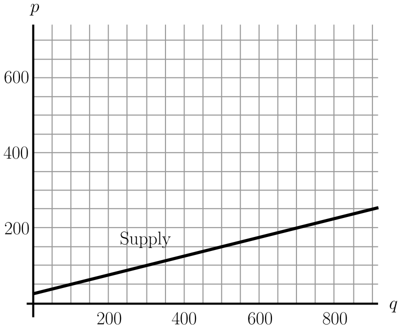
Law 2 (Law of supply)

states that the quantity supplied for sale will increase as the price of a product increases.


p
30
10
Equilibrium
price
60
120
180
240
q
Equilibrium
quantity
Supply: $p = \frac{1}{10}q + 5$
Demand: $p = -\frac{1}{10}q + 35$

1.6.031 The graphs of the demand function and supply function for a certain product, are given below. Use these graphs to answer the questions.





1. How many units q are demanded when the price p is \$50?
2. How many units q are supplied when the price p is \$50?
3. Will there be a market surplus (more supplied) or shortage (more demanded) when p = \$50?

1.6.033 If the demand for a pair of shoes is given by $2p + 5q = 200$ and the supply function for it is $p - 2q = 10$, compare the quantity demanded and the quantity supplied when the price is \$90.

quantity demanded _____ pairs of shoes

quantity supplied _____ pairs of shoes

Will there be a surplus or shortfall at this price?

Chapter 1 Section 5

Ex 1 Graphical method

$$\begin{cases} x & + & y & = & 2 \\ x & - & 2y & = & -1 \end{cases}$$

Ex 2 Graphical method

$$\begin{cases} x + y = 2 \\ x + y = -1 \end{cases}$$

Ex 3 Graphical method

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 4 Graphical method

$$\begin{cases} y &= x^2+1 \\ y &= x+1 \end{cases}$$

Ex 5 Graphical method

$$\begin{cases} y = x^2 - 1 \\ y = 0 \end{cases}$$

Ex 6 Substitution method

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

Ex 7 Substitution method

$$\left\{\begin{array}{rcrcr} x & + & y & = & 2 \\ x & + & y & = & -1 \end{array}\right.$$

Ex 8 Substitution method

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 9 Substitution method

$$\begin{cases} x - 3y + z = 0 \\ \quad\quad\; y - z = 3 \\ \quad\quad\quad\quad z = -2 \end{cases}$$

Ex 10 Elimination method

$$\begin{cases} x + y = 1 \\ x - y = 2 \end{cases}$$

Ex 11 Elimination method

$$\left\{ \begin{array}{rcrcr} x & + & y & = & 2 \\ x & - & 2y & = & -1 \end{array} \right.$$

Ex 12 Elimination method

$$\begin{cases} 2x & + & 3y & = & 9 \\ x & - & y & = & 2 \end{cases}$$

Ex 13 Elimination method

$$\begin{cases} 8x - 3y = -11 \\ 5x - 2y = -6 \end{cases}$$

Ex 14 Elimination method

$$\begin{cases} x & + & y & = & 2 \\ x & + & y & = & -1 \end{cases}$$

Ex 15 Elimination method

$$\begin{cases} \; x + \; y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 16 Elimination method

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 0 \\ & & y & - & z & = & 3 \\ & & & & z & = & -2 \end{array}\right.$$

Ex 17 Elimination method

$$\begin{cases} \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \end{cases}$$

Ex 1.5.039 A freight company has shipping orders for two products. The first product has a unit volume of 10 cu ft and weighs 50 lb. The second product's unit volume is 3 cu ft, and it weighs 40 lb. If the company's trucks have $2,290$ cu ft of space and can carry $20,700$ lb, how many units of each product can be transported in a single shipment with one truck using the entire volume and weight capacity?

Ex 1.5.044 A biologist has a 40% solution and a 10% solution of the same plant nutrient. How many cubic centimeters of each solution should be mixed to obtain 25 cc of a 22% solution?

Chapter 1 Section R1.6 Part Deux

1.6.044 Find the market equilibrium point for the following demand and supply functions.

Demand: $p = -2q + 318$

Supply: $p = 8q + 1$

$(q, p) =$

1.6.049.EP A group of retailers will buy 104 televisions from a wholesaler if the price is \$325 and 144 if the price is \$275. The wholesaler is willing to supply 84 if the price is \$255 and 164 if the price is \$345. Assume that the resulting supply and demand functions are linear. Let p represent price (in dollars) and q represent quantity.

State the two ordered pairs for the demand function in the form (q, p).

$(q, p) =$

$(q, p) =$

Write the demand function in terms of q.

$p =$

State the two ordered pairs for the supply function in the form (q, p).

$(q, p) =$

$(q, p) =$

Write the supply function in terms of q.

$p =$

Find the equilibrium point for the market in the form (q, p).

$(q, p) =$

Chapter 3 Section 3

How to avoid the graphical, substitution, and elimination methods?

How to avoid the graphical, substitution, and elimination methods? We need to learn how to rewrite a system of equations from equation form:

$$\left\{ \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \right.$$

How to avoid the graphical, substitution, and elimination methods? We need to learn how to rewrite a system of equations from equation form:

$$\begin{cases} \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \end{cases}$$

and into augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \\ 2 & 1 & 1 & 3 \end{array}\right]$$

Matrix

First what is a matrix?

Matrix

First what is a matrix?
It is a rectangular array of numbers (surrounded by some grouping symbols).
Note that it is standard to use capitol letters to name matrices: A, B, C, X, etc.

Matrix

First what is a matrix?
It is a rectangular array of numbers (surrounded by some grouping symbols).
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Example 1.1

a) $A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$

b) $B = \begin{bmatrix} 11 & 12 \\ 21 & 22 \\ 31 & 32 \end{bmatrix}$

c) $C = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}$

Matrix Size

Each matrix has a size. The size of a matrix is the number of rows and columns that it has.

Example 1.2

a) *This matrix*

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

has 2 *rows and* 3 *columns. So its size is* 2×3*.*

b) *This matrix*

$$B = \begin{bmatrix} 11 & 12 \\ 21 & 22 \\ 31 & 32 \end{bmatrix}$$

has 3 *rows and* 2 *columns. So its size is* 3×2*.*

c) *This matrix*

$$C = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}$$

has 2 *rows and* 2 *columns. So its size is* 2×2 *and is called an square matrix.*

Matrix elements I

It is often useful to be able to reference each element of a matrix. We do this by referring to its row and column position. To lazify this we use the lower case version of the matrix name with subscripts denoting the row first and then the column:

$a_{11} = 11$ Element in row 1, column 1

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

$a_{12} = 12$ Element in row 1, column 2

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

Matrix elements II

$a_{21} = 21$ Element in row 2, column 1

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

$a_{23} = 23$ Element in row 2, column 3

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

If the matrix is big enough to require two digit values for the rows or columns then the row and column is separated by a comma. So if you needed to reference an element in row 15 and column 7 you would write:

$$a_{15,7}$$

since a_{157} would be ambiguous.

The first step in writing the augmented matrix for

$$\begin{cases} \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \end{cases}$$

The first step in writing the augmented matrix for

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array}\right.$$

Is to make the coefficient matrix.

The first step in writing the augmented matrix for

$$\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$$

Is to make the coefficient matrix. To do this we look at the system of equations and only write down the coefficients on the left hand side:

$$\begin{cases} 1x + 1y + 1z = 4 \\ 1x + 3y + 3z = 10 \\ 2x + 1y + -1z = 3 \end{cases}$$

The first step in writing the augmented matrix for

$$\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$$

Is to make the coefficient matrix. To do this we look at the system of equations and only write down the coefficients on the left hand side:

$$\begin{cases} {\color{red}1}x + {\color{red}1}y + {\color{red}1}z = 4 \\ {\color{red}1}x + {\color{red}3}y + {\color{red}3}z = 10 \\ {\color{red}2}x + {\color{red}1}y + {\color{red}-1}z = 3 \end{cases}$$

Note that I moved the negative to the coefficient of the z variable in the last equation.

$$\begin{cases} {\color{red}1}x & + & {\color{red}1}y & + & {\color{red}1}z & = & 4 \\ {\color{red}1}x & + & {\color{red}3}y & + & {\color{red}3}z & = & 10 \\ {\color{red}2}x & + & {\color{red}1}y & + & {\color{red}-1}z & = & 3 \end{cases}$$

Writing these coefficients as a matrix looks like this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

Next we need to write the constants as a column vector. A vector is a kind of matrix that is either a single column or single row. Looking at the constants in the system of equations:

$$\begin{cases} \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \end{cases}$$

Next we need to write the constants as a column vector. A vector is a kind of matrix that is either a single column or single row. Looking at the constants in the system of equations:

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & \color{red}{4} \\ x & + & 3y & + & 3z & = & \color{red}{10} \\ 2x & + & y & - & z & = & \color{red}{3} \end{array}\right.$$

we get the column vector

$$\begin{bmatrix} 4 \\ 10 \\ 3 \end{bmatrix}$$

Finally to form the augmented matrix we take the coefficient matrix and the constant vector:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 10 \\ 3 \end{bmatrix}$$

Finally to form the augmented matrix we take the coefficient matrix and the constant vector:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 10 \\ 3 \end{bmatrix}$$

and combine them into the augmented matrix form separating the coefficients from the constants by a vertical line:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

Ex 01 Make the augmented matrix

$$\left\{\begin{array}{rcrcrcr} x & & & - & 4z & = & 1 \\ 2x & - & y & - & 6z & = & 4 \\ 2x & + & 3y & - & 2z & = & 8 \end{array}\right.$$

Ex 02 Make the augmented matrix.

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 0 \\ & & y & - & z & = & 3 \\ & & & & z & = & -2 \end{array}\right.$$

A system of linear equations and a reduced matrix for the system are given.

$$\begin{cases} x - 3y + z = 0 \\ y - z = 3 \\ z = -2 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

1. Use the reduced matrix to find the general solution of the system, if one exists. (If there is no solution, enter NO SOLUTION. If there are infinitely many solutions, express your answers in terms of z as in Example 3.)
2. If multiple solutions exist, find two specific solutions. (Enter your answers as a comma-separated list of ordered triples. If there is no solution, enter NO SOLUTION.)

Ex HarMathAp12 3.3.017 A system of linear equations and a reduced matrix for the system are given.

$$\left\{\begin{array}{rcrcrcl} x & + & 3y & + & 2z & = & 4 \\ 3x & - & y & & & = & 2 \\ x & + & 3y & + & 2z & = & 5 \end{array}\right. \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

1. Use the reduced matrix to find the general solution of the system, if one exists. (If there is no solution, enter NO SOLUTION. If there are infinitely many solutions, express your answers in terms of z as in Example 3.)
2. If multiple solutions exist, find two specific solutions. (Enter your answers as a comma-separated list of ordered triples. If there is no solution, enter NO SOLUTION.)

Ex HarMathAp12 3.3.020 A system of linear equations and a reduced matrix for the system are given.

$$\begin{cases} x - y + z = 5 \\ 3x + 2z = 13 \\ x - 4y + 2z = 7 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1. Use the reduced matrix to find the general solution of the system, if one exists. (If there is no solution, enter NO SOLUTION. If there are infinitely many solutions, express your answers in terms of z as in Example 3.)
2. If multiple solutions exist, find two specific solutions. (Enter your answers as a comma-separated list of ordered triples. If there is no solution, enter NO SOLUTION.)

Definition 1 (reduced row echelon form (reduced form))

If the matrix is in row echelon form, the first nonzero entry of each row is equal to 1 and the ones above it within the same column equal 0.

1. The first nonzero element in each row is 1.
2. Every column containing a first nonzero element for some row has zeros everywhere else.
3. The first nonzero element of each row is to the right of the first nonzero element of every row above it.
4. All rows containing zeros are grouped together below the rows containing nonzero entries.

So

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is in reduced row echelon form (rref)

If this is in rref,

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

what would an example (of many possible) of not being in rref look like?

If this is in rref,

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

what would an example (of many possible) of not being in rref look like?

This is not in rref:

$$\left[\begin{array}{ccc|c} 1 & 1 & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Chapter 3 Section 3 Numerical

Let's solve this system of equations numerically using the programming language Julia.

$$\begin{cases} 2x & - & 6y & - & 12z & = & -20 \\ 3x & - & 10y & - & 20z & = & -38 \\ 2x & & & - & 17z & = & -40 \end{cases}$$

We already know how to write the coefficient matrix and constant vector:

$$A = \begin{bmatrix} 2 & -6 & -12 \\ 3 & -10 & -20 \\ 2 & 0 & -17 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -20 \\ -38 \\ -40 \end{bmatrix}$$

The variable vector looks like:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

With A, $\vec{x}$, and $\vec{b}$ defined as:

$$A = \begin{bmatrix} 2 & -6 & -12 \\ 3 & -10 & -20 \\ 2 & 0 & -17 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -20 \\ -38 \\ -40 \end{bmatrix}$$

We can rewrite

$$\left\{ \begin{array}{rcrcrcr} 2x & - & 6y & - & 12z & = & -20 \\ 3x & - & 10y & - & 20z & = & -38 \\ 2x & & & - & 17z & = & -40 \end{array} \right.$$

as

$$A\vec{x} = \vec{b}$$

Matrix Multiplication

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Matrix Multiplication

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$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

To find their product

$$AB$$

Matrix Multiplication

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

To find their product

$$AB$$

We do

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Matrix Multiplication

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

To find their product

$$AB$$

We do

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

This is *messy*!

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

Have Julia multiply these for you.

Identity matrix

An identity matrix is a square matrix with 1s along the diagonal and zeros everywhere else.

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There is an identity matrix for every different size matrix:

$$2 \times 2, 3 \times 3, 4 \times 4, \ldots, n \times n, \ldots$$

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$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ldots$$

Inverse matrix

If you have a square matrix A sometimes you can find an inverse for that matrix A^{-1}.

Inverse matrix

If you have a square matrix A sometimes you can find an inverse for that matrix A^{-1}.
If A and A^{-1} are inverse matrices then

$$AA^{-1} = A^{-1}A = I$$

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2   -6  -12
3  -10  -20
2    0  -17
```

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2   -6  -12
3  -10  -20
2    0  -17

julia> Ainv = inv(A)
3×3 MatrixFloat64:
5.0        -3.0        2.22045e-16
0.323529  -0.294118   0.117647
0.588235  -0.352941  -0.0588235
```

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2   -6  -12
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5.0        -3.0        2.22045e-16
0.323529  -0.294118   0.117647
0.588235  -0.352941  -0.0588235

julia> Ainv*A
3×3 MatrixFloat64:
1.0           -3.55271e-15  -1.08802e-14
1.11022e-16    1.0            0.0
-1.52656e-16    0.0            1.0
```

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2   -6  -12
3  -10  -20
2    0  -17

julia> Ainv = inv(A)
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5.0        -3.0        2.22045e-16
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1.0           -3.55271e-15  -1.08802e-14
1.11022e-16    1.0            0.0
-1.52656e-16    0.0            1.0
```

The reason we aren't getting 0s in all the off diagonals is round off errors which is always a major concern when working with mathematics on computers.

Solving $ax = b$ using the above idea of a multiplicative inverse we get:

$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

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$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

So back to

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

To solve this system

$$\begin{cases} 2x - 6y - 12z = -20 \\ 3x - 10y - 20z = -38 \\ 2x - 17z = -40 \end{cases}$$

we first need to make the coefficient matrix

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2 -6 -12
3 -10 -20
2 0 -17
```

and the constant vector

```
julia> b = [-20, -38, -40]
3-element VectorInt64:
-20
-38
-40
```

Now we can solve for `x` using Julia

```
julia> x = Ainv*b
3-element VectorFloat64:
13.999999999999991
0.0
4.000000000000002
```

Since this is such an important and often used sequence of operations, Julia provides an operator to do this:

```
julia> x = A\b
3-element VectorFloat64:
14.000000000000009
1.0658141036401504e-15
4.000000000000001
```

To solve

$$\begin{cases} x + 2y + 3z = 1 \\ 2x - y = 2 \\ x + 2y + 3z = 2 \end{cases}$$

```
julia> A = [1 2 3; 2 -1 0; 1 2 3]
3×3 MatrixInt64:
1 2 3
2 -1 0
1 2 3

julia> b = [1;2;2]
3-element VectorInt64:
1
2
2
```

```
julia> x = A\b
ERROR: LinearAlgebra.SingularException(3)
Stacktrace:
[1] checknonsingular
@ \.julia\...\src\factorization.jl:68 [inlined]
...
```

To avoid this issue we use the determinant function

```
det
```

from the package `LinearAlgebra`. To add a package you can follow *doggo dot jl*'s video instructions at the clickable link: [10x27] How to use External Packages in Julia

Load the `LinearAlgebra` using:

```
using LinearAlgebra #Needed for the determinant function: det(A)
```

Looking at three examples:

```
A = [1 -3 1; 0 1 -1; 0 0 1]
det(A)
```

This gives the result

```
3×3 MatrixInt64:
1 -3 1
0 1 -1
0 0 1
1.0
```

Since the determinant $\det(A) \neq 0$ we can solve using the $\backslash$ operator.

But for the following two examples:

```
A = [1 1;1 1 ]
det(A)
```

This gives the result

```
2×2 MatrixInt64:
1 1
1 1

0.0
```

```
A = [1 1;2 2]
det(A)
```

This gives the result

```
2×2 MatrixInt64:
1 1
2 2

0.0
```

In both of these cases the determinant $\det(A) = 0$ we cannot solve using the $\backslash$ operator.
For now we will just say that there is no unique solution and will come back later to dissect what's happening.

It is now later...

Recall that

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

has one solution.

```
using LinearAlgebra
A = [1 1; 1 2]
b = [2, -1]
det(A)
> 1.0
x, y = A\b
> 2-element VectorFloat64:
 5.0
-3.0
```

And that

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

has infinitely many solutions since they are actually the same lines.

```
A = [1 1; 2 2]
b = [1, 2]
det(A)
> 0.0
```

And that

$$\begin{cases} x + y = 2 \\ x + y = -1 \end{cases}$$

has no solutions because the lines are parallel.

```
A = [1 1; 1 1]
b = [2, -1]

det(A)
> 0.0
```

There is a difference between the second and third systems, but we cannot tell that from the determinant. How can we tell the difference?

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We need to rewrite the system of equations into augmented matrix form and then run the command *rref* on the augmented matrix.

There is a difference between the second and third systems, but we cannot tell that from the determinant. How can we tell the difference?
We need to rewrite the system of equations into augmented matrix form and then run the command *rref* on the augmented matrix.
To use the *rref* command you will need to add and load (*using*) the package *RowEchelon.jl*.

How are we going to solve systems of linear equations?

1. Form the
   1.1 coefficient matrix A
   1.2 constant vector b
   1.3 augmented matrix A_{aug}
2. Take the determinant of A
3. If
   3.1 $\det(A) \neq 0$: Solve for x using $x = A\backslash b$
   3.2 $\det(A) = 0$: Solve using rref(A_{aug})
4. Is the system consistent or inconsistent? If inconsistent is it dependent or independent?
5. How many solutions does this system have?
6. What are they? If more than 1 then give two specific solutions.

Ex 1 Solve

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

Ex 2 Solve

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 3 Solve

$$\begin{cases} x + y = 2 \\ x + y = -1 \end{cases}$$

Ex 4 Solve

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 2 \\ 2x & + & y & + & 2z & = & 1 \\ 3x & + & 3y & + & 3z & = & 3 \end{array}\right.$$

Ex 5 Solve

$$\begin{cases} x + 2y + z = 2 \\ 2x + y + 2z = 1 \\ 3x + 3y + 3z = 0 \end{cases}$$

Give me a 3×3 augmented matrix that is

1. Independent

2. Dependent, Consistent

3. Dependent, Inconsistent

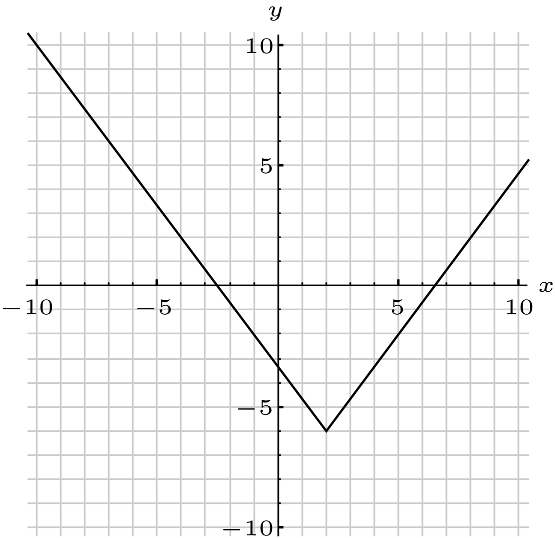
‘

Chapter 9 Section 1

Ex 9.1.002

a) $\lim_{x \to 2} f(x)$

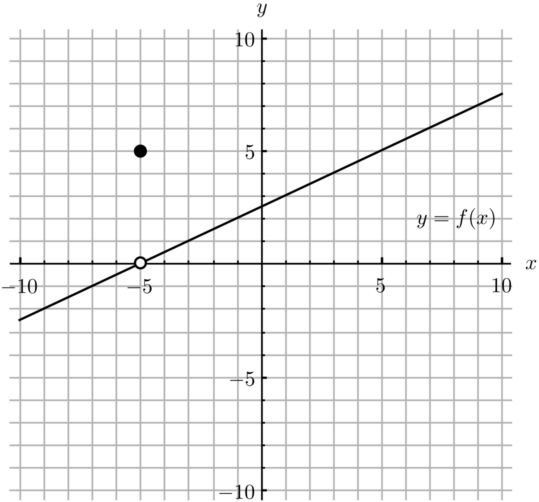
b) $f(2)$



Ex 9.1.005

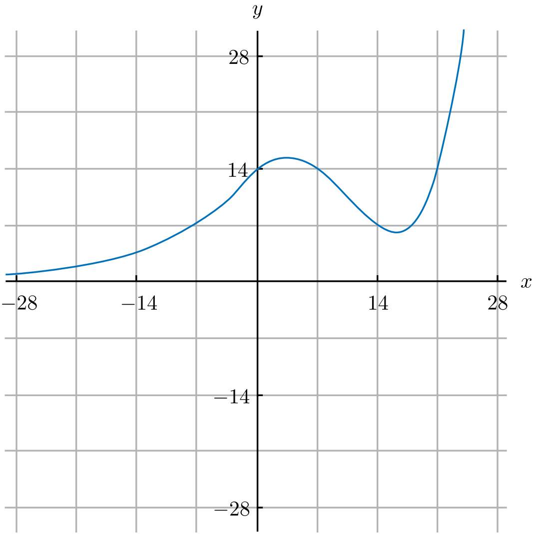
a) $\lim_{x \to -5} f(x)$

b) $f(-5)$



Ex WANEAC7 3.1.037

a) $\lim\limits_{x\to 0} f(x)$ b) $\lim\limits_{x\to 14} f(x)$ c) $\lim\limits_{x\to -\infty} f(x)$ d) $\lim\limits_{x\to \infty} f(x)$



Ex WANEAC7 3.1.039

a) $\lim_{x \to 12} f(x)$

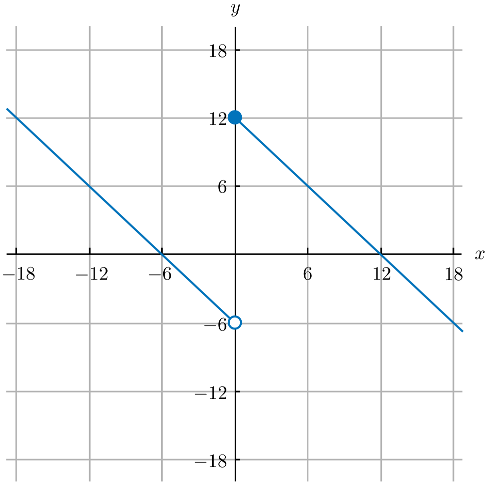
b) $\lim_{x \to 0^+} f(x)$

c) $\lim_{x \to 0^-} f(x)$

d) $\lim_{x \to 0} f(x)$

e) $f(0)$

f) $\lim_{x \to -\infty} f(x)$



Ex WANEAC7 3.1.045

a) $\lim_{x \to -2} f(x)$

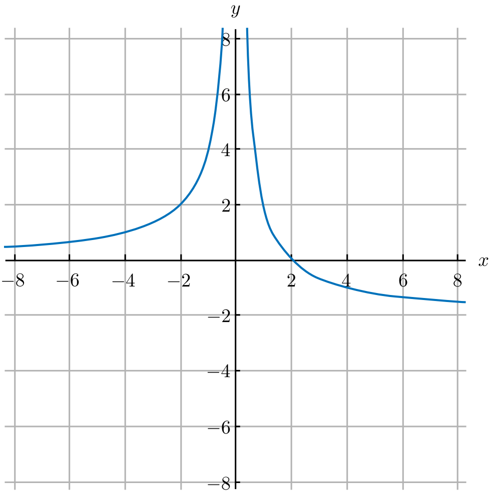
b) $\lim_{x \to 0^+} f(x)$

c) $\lim_{x \to 0^-} f(x)$

d) $\lim_{x \to 0} f(x)$

e) $f(0)$

f) $\lim_{x \to \infty} f(x)$



Ex WANEAC7 3.1.047

a) $\lim\limits_{x \to -7} f(x)$

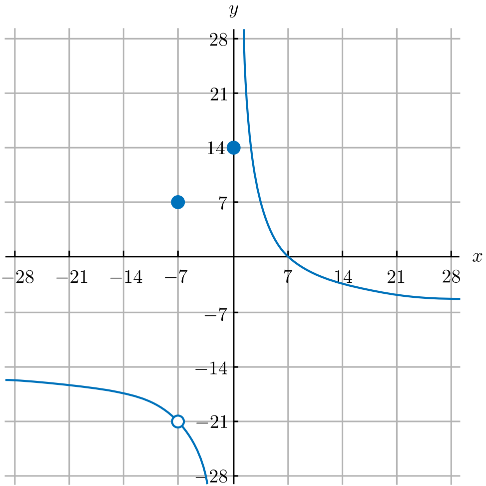
b) $\lim\limits_{x \to 0^+} f(x)$

c) $\lim\limits_{x \to 0^-} f(x)$

d) $\lim\limits_{x \to 0} f(x)$

e) $f(0)$

f) $f(-7)$

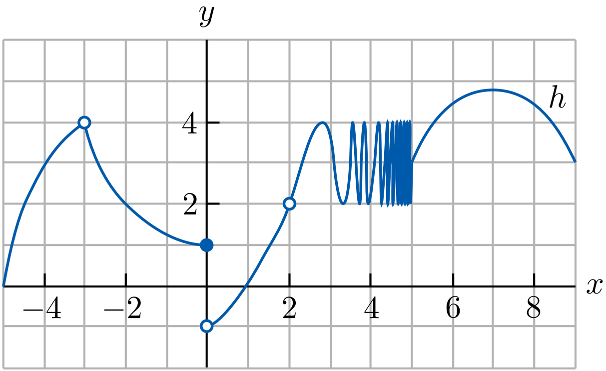


Ex SCALC9 1.5.006

a) $\lim_{x\to -3^-} h(x)$ b) $\lim_{x\to -3^+} h(x)$ c) $\lim_{x\to -3} h(x)$ d) $h(-3)$

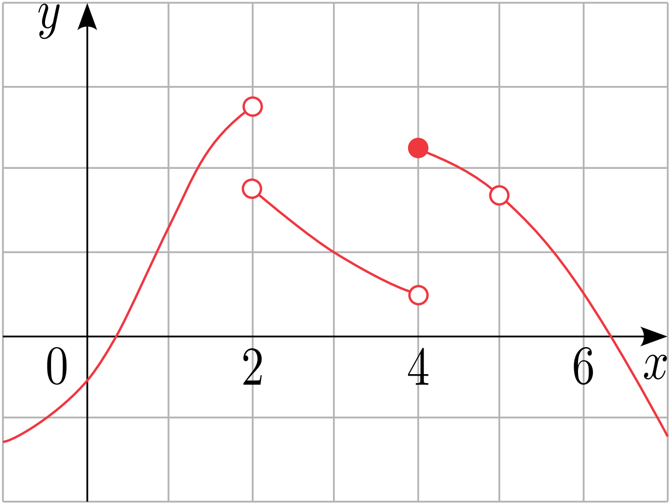
e) $\lim_{x\to 0^-} h(x)$ f) $\lim_{x\to 0^+} h(x)$ g) $\lim_{x\to 0} h(x)$ h) $h(0)$

i) $\lim_{x\to 2} h(x)$ j) $h(2)$ k) $\lim_{x\to 5^+} h(x)$ l) $\lim_{x\to 5^-} h(x)$



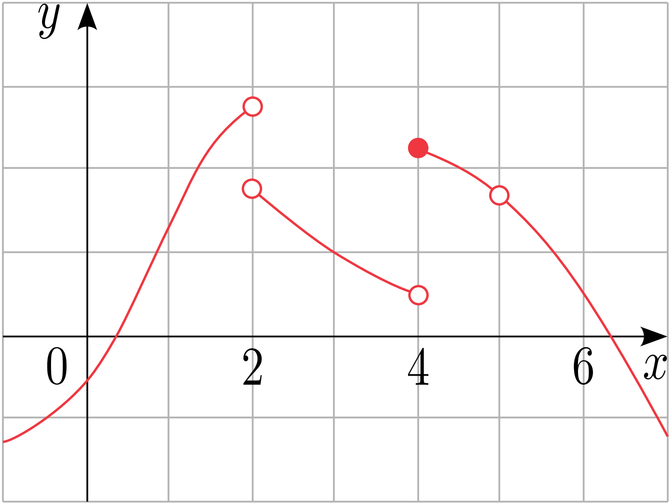
Ex SCALC9 1.5.007 (a) Find a such that the following is true.

$$\lim_{x\to a} g(x) \text{ does not exist but } g(a) \text{ is defined.}$$



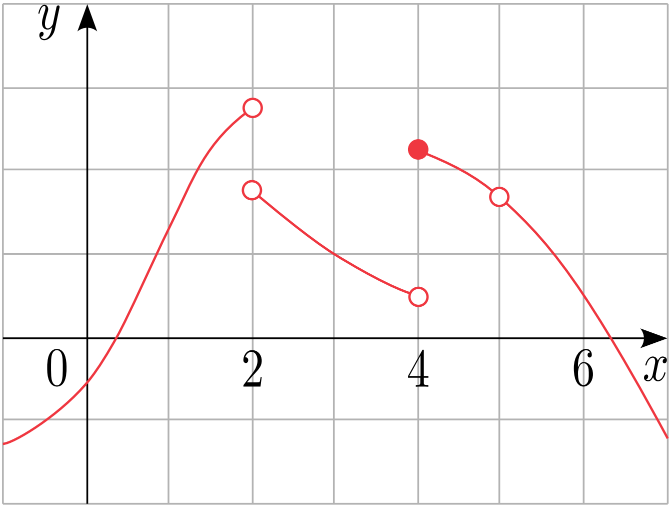
Ex SCALC9 1.5.007 (b) Find a such that the following is true.

$$\lim_{x \to a} g(x) \text{ exists but } g(a) \text{ is not defined.}$$



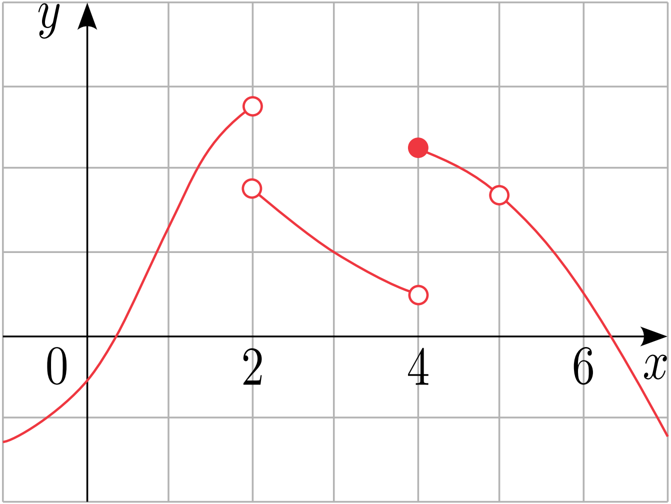
Ex SCALC9 1.5.007 (c) Find a such that the following is true.

$$\lim_{x\to a^-} g(x) \text{ and } \lim_{x\to a^+} g(x) \text{ both exist but } \lim_{x\to a} g(x) \text{ does not exist.}$$



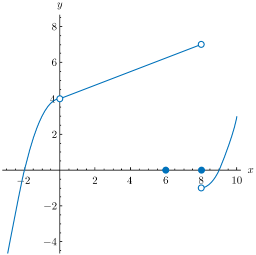
Ex SCALC9 1.5.007 (d) Find a such that the following is true.

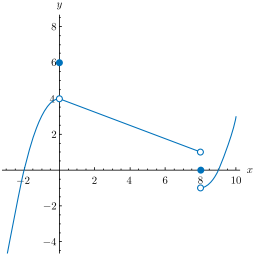
$$\lim_{x\to a^+} g(x) = g(a) \text{ but } \lim_{x\to a^-} g(x) \neq g(a).$$

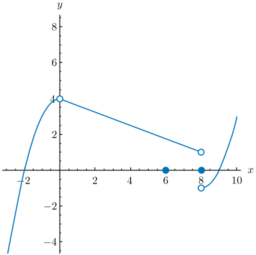


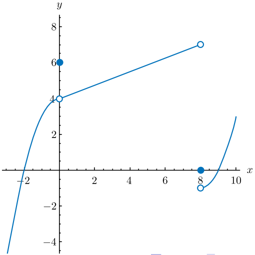
Ex SCALC9 1.5.016 Which graph satisfies the following:

$\lim_{x\to 0} f(x) = 4,\ \lim_{x\to 8^-} f(x) = 1,\ \lim_{x\to 8^+} f(x) = -1,\ f(0) = 6,\ f(8) = 0$

a) 

b) 

c) 

d) 

Properties of Limits If k is a constant, $\lim_{x \to c} f(x) = L$, and $\lim_{x \to c} g(x) = M$, then

1. $$\lim_{x \to c} k = k$$
2. $$\lim_{x \to c} x = c$$
3. $$\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$$
4. $$\lim_{x \to c} [f(x) \cdot g(x)] = LM$$
5. $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$$
6. $$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$$

   if $L > 0$ when n is even

- Let's find the limit of the following with these new rules:

$$\lim_{x \to 3} (2x - 5)$$

- Let's find the limit of

$$\lim_{x \to 5} (x^2 - 3x - 4)$$

- Let's find the limit of

$$f(x) = \begin{cases} 2x + 3 & , x < 0 \\ -x^2 - 2 & , x \geq 0 \end{cases}$$

as $x \to 0$:

$$\lim_{x \to 0} f(x)$$

- Let's find the limit of
$$\lim_{x \to 0} \frac{1}{x}$$
- Let's find the limit of
$$\lim_{x \to 0^+} \frac{1}{x}$$
- Let's find the limit of
$$\lim_{x \to 0^-} \frac{1}{x}$$
- Let's find the limit of
$$\lim_{x \to 0} \frac{1}{x^2}$$

Rational Functions: Limit of $\lim_{x \to c} \frac{f(x)}{g(x)}$ when $\lim_{x \to c} g(x) = 0$

$\lim_{x \to c} f(x) = 0$ when

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{0}{0}$$

this is called the *indeterminate form* at $x = c$. We can factor $x - c$ from both $f(x)$ and $g(x)$, simplify the fraction, and find the limit of the simplified form (if it exists).

$\lim_{x \to c} f(x) \neq 0$ then

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

does not exist. The fraction $f(x)/g(x)$ becomes unbounded as x approaches c. This results in a vertical asymptote which is the line $x = c$.

Rational Functions: Limit of $\lim_{x\to c}\frac{f(x)}{g(x)}$ when $\lim_{x\to c} g(x) = 0$

$\lim_{x\to c} f(x) = 0$ when

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{0}{0}$$

this is called the *indeterminate form* at $x = c$. We can factor $x - c$ from both $f(x)$ and $g(x)$, simplify the fraction, and find the limit of the simplified form (if it exists).

$\lim_{x\to c} f(x) \neq 0$ then

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

does not exist. The fraction $f(x)/g(x)$ becomes unbounded as x approaches c. This results in a vertical asymptote which is the line $x = c$.

Let's find the limit of

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

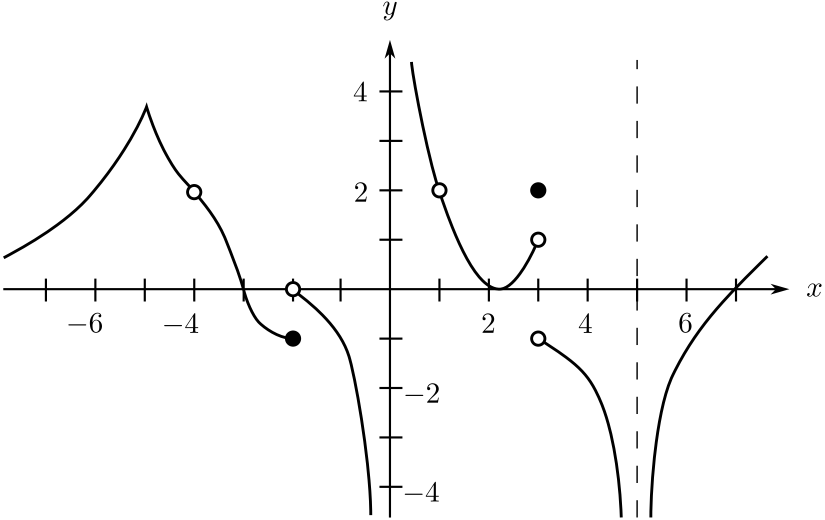
Chapter 9 Section 2

The function f is continuous at $x = c$ if all of the following conditions are satisfied.

1. $f(c)$ exists
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

If at least one of the above fails to be satisfied, then the function is discontinuous at $x = c$.

Ex 9.1.001 For each given x-value:



a) f is continuous

b) $\lim_{x \to c} f(x)$ does not exist

c) $f(c)$ does not exist

d) $\lim_{x \to c} f(x) = f(c)$ does not exist

Ex 9.2.007.EP Let

$$f(x) = \begin{cases} x - 5 & , x \leq 2 \\ 6x - 9 & , x > 2 \end{cases}$$

Is $f(x)$ continuous or discontinuous at $x = 2$.

Determine whether the given function is continuous. You can verify your conclusions by graphing the function with a graphing utility.

$$f(x) = 2x + 3$$

Determine whether the given function is continuous. You can verify your conclusions by graphing the function with a graphing utility.

$$f(x) = -5x^3 + 7x^2 - 9x + 3$$

- Determine whether the given function is continuous. You can verify your conclusions by graphing the function with a graphing utility.

$$f(x) = \frac{x^2 + 5x + 6}{x + 2}$$

- Like 9.2.013 Determine whether the given function is continuous. You can verify your conclusions by graphing the function with a graphing utility.

$$y = \frac{x}{x^2 + 2}$$

○ The function is continuous.
○ The function is not continuous.

If it is not, identify where it is discontinuous. You can verify your conclusion by graphing the function with a graphing utility. (If the function is continuous, enter CONTINUOUS.)

- Like 9.2.039.EP Suppose that the weekly sales volume (in thousands of units) for a product is given by

$$y = \frac{33}{(p+7)^{\frac{2}{5}}}$$

where p is the price in dollars per unit.

1. Is this function continuous for all values of p?
2. Is this function continuous at $p = 24$?
3. Is this function continuous for all $p \geq 0$?
4. What is the domain for this application? (Enter your answer using interval notation.)

Limits at Infinity and Horizontal Asymptotes

If $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$, where b is a constant, then the line $y = b$ is a horizontal asymptote for the graph of $y = f(x)$. Otherwise, $y = f(x)$ has no horizontal asymptotes.

- $$\lim_{x\to\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x}$$
- $$\lim_{x\to\infty} \frac{1}{x^2} \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^2}$$
- $$\lim_{x\to\infty} \frac{1}{x-1} \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x-1}$$
- $$\lim_{x\to\infty} \frac{x}{x-1} \quad \text{and} \quad \lim_{x\to-\infty} \frac{x}{x-1}$$
- $$\lim_{x\to\infty} \frac{2x^2-3x}{x^2+9} \quad \text{and} \quad \lim_{x\to-\infty} \frac{2x^2-3x}{x^2+9}$$

Chapter 7

Probability

Chapter 7 Section 1

Definition 2 (Set)

A set is a collection of objects. See section 0.1 in the book.

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Definition 3 (Elements)

The objects in a set are called elements.

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$$A = \{1, 2, 3, 4, 5\}$$

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Set bulider notation The set of positive even integers would be written as:

$$E = \{x | x \text{ is an integer and } x > 0 \text{ and } x \text{ is even}\}$$

and is read as the set of all x such that x is an integer and $x > 0$ and x is even.

For probability, one very important attribute of a set is how many elements it has. It's so important that it is given a name:

Definition 4 (Cardinality)

The cardinality of a set is how many elements it has. It is denoted as $n(A)$ *or* $|A|$.

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Definition 4 (Cardinality)

The cardinality of a set is how many elements it has. It is denoted as $n(A)$ *or* $|A|$.

The cardinality of a set can be either finite or infinite.
Finite:

$$n(\{2, 7, 11\}) = 3$$

$$n(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}) = 10$$

A special case is the set with no elements called the empty set and denoted either as: $\emptyset$ or $\{\}$.

$$n(\emptyset) = 0$$

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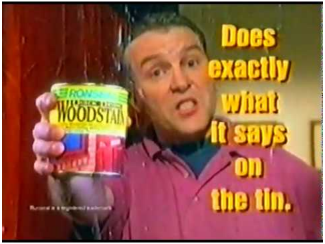
We just need to know that sets can be:

Denumerable Either finite or countably infinite.

Uncountably infinite It is exactly what it says on the tin.



Four key definitions:

1. Probability experiment

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- equally likely: e.g. fair coin or die

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1. Probability experiment
2. Outcome
3. Sample space
4. Event

Outcomes can be

- equally likely: e.g. fair coin or die
- not equally likely: e.g. weighted die

Probability of an Event

Let all outcomes be equally likely. If an event E has cardinality $n(E) = k$ and the sample space S has cardinality $n(S) = k$, then the probability of the event E is

$$\Pr(E) = \frac{n(E)}{n(S)} = \frac{k}{n}.$$

Sometimes we use $P(E)$ instead of $\Pr(E)$.

Ex 1 One ball is drawn at random from a bag containing 5 red balls and 9 white balls. (Enter your probabilities as fractions.)

1. What is the probability that the ball is red?

2. What is the probability that the ball is green?

3. What is the probability that the ball is red or white?

Ex 2 A six-sided die is rolled. (Enter your probabilities as fractions.)

1. What is the probability that a 6 will result?
2. What is the probability that an 8 will result?
3. What is the probability that an odd number will result?

Ex 3 An urn contains two red balls numbered 1, 2, five white balls numbered 3, 4, 5, 6, 7, and three black balls numbered 8, 9, 10. A ball is drawn from the urn. (Enter your probabilities as fractions.)

1. What is the probability that it is red?
2. What is the probability that it is odd-numbered?
3. What is the probability that it is red and odd-numbered?
4. What is the probability that it is red or odd-numbered?
5. What is the probability that it is not black?

Ex 4 From a deck of 52 ordinary playing cards, one card is drawn. (Enter your probabilities as fractions.)

1. Find the probability that it is a king.

2. Find the probability that it is a red card.

3. Find the probability that it is a heart.

Ex 5 If a pair of dice, one green and one red, is rolled, what is the probability that the following will occur? (Assume both dice are six-sided. Enter your probabilities as fractions.)

1. $\Pr(7 \leq S \leq 8)$, where S is the sum rolled

2. $\Pr(7 \leq S \leq 11)$, where S is the sum rolled

3. $\Pr(R_1 + R_2 = 4)$, where R_1 is the value on the first roll and R_2 is the value on the second roll

4. $\Pr(R_1 + R_2 > 9)$, where R_1 is the value on the first roll and R_2 is the value on the second roll

$$
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}
$$

Chapter 7 Section 5

Definition 5 (Fundamental Counting Principle)

If there are $n(A)$ *ways in which an event* A *can occur and if there are* $n(B)$ *ways in which a second event* B *can occur after the first event has occurred, then the two events can occur in* $n(A)n(B)$ *ways.*

Ex 1 Current Alabama license plates consist of a county code (there are 67 counties in Alabama - coded using the format 1 or 10 - but lets pretend we have a single character for each one, think hexadecimal - base 16 - but at least base 67) followed some number/letter combination depending on the year. See Vehicle registration plates of Alabama

1. In 1941 Alabama started using county codes in its license plates. The format was county code, followed by three digits. How many license plates can be issued.

2. Starting in January 2002 it consisted of the county code, a letter, three digits, a letter. How many license plates can be issued.

3. The current rule is it consists of the county code followed by 5 alphanumeric digits. How many license plates can be issued.

Ex 2 Telephone numbers consist of three parts: a three digit area code, a three digit exchange number, and a four digit line number. Only 555-0100 through 555-0199 are now specifically reserved for fictional use, but we will ignore this. How many exist?

Ex 3 How many sandwiches are possible if:

Bread	white, rye, sourdough, brioche, ciabatta, focaccia, wheat
Meat	ham, chicken, turkey, egg
Cheese	cheddar, pepper jack, Munster, Havarti, mozzarella
Condimemt	mayonnaise, mustard, wasabi

Ex 4 How many outfits can you wear if you have 10 shirts, 4 pants, 3 shoes, and 6 hats?

Definition 6 (Factorial)

For any positive integer n, we define n factorial as

$$n! = n(n-1)\cdots 3 \cdot 2 \cdot 1$$
$$0! = 1$$

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Ex 5

1. $5!$
2. $1!$
3. $0!$

Definition 7 (Permutations)

The number of possible ordered arrangements of r objects chosen from a set of n objects is called the number of permutations of n objects taken r at a time, and it equals

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

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Ex 6

1. $_3P_2$
2. $_4P_2$
3. $_{10}P_7$

Definition 8 (Combinations)

The number of ways in which r objects can be chosen from a set of n objects without regard to the order of selection is called the number of combinations of n objects taken r at a time, and it equals

$$_nC_r = \frac{_nP_r}{r!}$$

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Ex 7

1. $_3C_2$

2. $_4C_2$

3. $_{10}C_7$

Ex 8 In the 100-m dash 8 sprinters race each other. How many ways can they finish the race?

Ex 9 How many ways can a 10-question multiple-guess test be answered if each question has 4 possible answers?

Ex 10 Suppose 10 people want to go to the mountains but their only vehicle can only carry 6 at a time. How many different groups of 6 people can be chosen for the trip?

Ex 11 A game show has 7 finalist in its last segment. If the top 4 prizes are \$10000, \$5000, \$2000, \$1000, how many different ways are there to win these prizes?

Ex 12 A company wishes to use the services of 8 different temp agencies. If 15 temp agencies are available, in how many ways can it choose the 8 temp agencies?

Ex 13 The call letters for radio stations begin with K or W, followed by 3 additional letters. How many sets of call letters having 4 letters are possible?

Ex 14 The jeweler Amy Farrah Fowler wants to display 8 different rings, 5 different necklaces, and 3 different tiaras in a single row of her display case.

1. In how many ways can these pieces be displayed?
2. If she wants to group the tiaras to the left of the line, how many ways can she display the jewelry?

Pascal's triangle I

$$1$$

Pascal's triangle II

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

Pascal's triangle III

$$\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 2 & & 1 \end{array}$$

Pascal's triangle IV

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

Pascal's triangle V

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Pascal's triangle VI

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Pascal's triangle Combinations

n											
0						1					
1					1		1				
2				1		2		1			
3			1		3		3		1		
4		1		4		6		4		1	
5	1		5		10		10		5		1
0						${}_0C_0$					
1					${}_1C_0$		${}_1C_1$				
2				${}_2C_0$		${}_2C_1$		${}_2C_2$			
3			${}_3C_0$		${}_3C_1$		${}_3C_2$		${}_3C_3$		
4		${}_4C_0$		${}_4C_1$		${}_4C_2$		${}_4C_3$		${}_4C_4$	
5	${}_5C_0$		${}_5C_1$		${}_5C_2$		${}_5C_3$		${}_5C_4$		${}_5C_5$

Chapter 8

Distributions

Chapter 8 Section 1,3

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1. How many heads were flipped
2. How many tails were flipped

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2. How many tails were flipped
3. How many heads minus how many tails were flipped

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1. How many heads were flipped
2. How many tails were flipped
3. How many heads minus how many tails were flipped

We will look at how many heads were flipped.

Let is take a moment and look at an easier example and then we will return to the flip coin three times example.
Let us flip a fair coin once.

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Let X *count* the number of times we have seen a head for each flip.

Let is take a moment and look at an easier example and then we will return to the flip coin three times example.
Let us flip a fair coin once.
Sample space is $S = \{H, T\}$.
Let X *count* the number of times we have seen a head for each flip.

Outcome	Count
T	$X(\mathsf{T}) = 0$
H	$X(\mathsf{H}) = 1$

Back to three flips.

Back to three flips.

$$S =$$

Let's make a table of Outcomes, Count of heads, and the Random Variable of that outcome:

Outcome	# of H	RV

Why add this complexity?

Count	Outcomes	Probability
$X = 0$		$P(X = 0) =$
$X = 1$		$P(X = 1) =$
$X = 2$		$P(X = 2) =$
$X = 3$		$P(X = 3) =$

This is a probability distribution and one way of writing it is in table form:

x				
$P(X = x)$				

Definition 9 (Random variable)

is a function that relates the outcomes from a probability experiment to their probabilities.

Definition 10 (Probability distribution)

is a function that gives the probabilities of occurrence of different possible outcomes for an experiment and must satisfy:

1. $0 \leq P(X = x) \leq 1$
2. $\sum P(X = x) = 1$

Definition 11 (Discrete random variable)

is a random variable that has either a finite number of possible values or a countably infinite number.

Finite Number of Possible Values I

Recall when you roll a 2D6 the possible outcomes are:

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

Finite Number of Possible Values II

Roll a 2D6 and let r.v. $X = R_1 + R_2$.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

Infinite Number of Possible Values

From:
https://en.wikipedia.org/wiki/Poisson_distribution

Definition 12 (Poisson distribution)

is a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time if these events occur with a known constant mean rate and independently of the time since the last event.

Examples 1

- soldiers killed by horse-kicks each year in each corps in the Prussian cavalry. This example was used in a book by Ladislaus Bortkiewicz (1868–1931).
- yeast cells used when brewing Guinness beer. This example was used by William Sealy Gosset (1876–1937).
- phone calls arriving at a call center within a minute. This example was described by A.K. Erlang (1878–1929).
- goals in sports involving two competing teams.
- deaths per year in a given age group,
- jumps in a stock price in a given time interval,

Examples 2

- times a web server is accessed per minute (under an assumption of homogeneity,
- mutations in a given stretch of DNA after a certain amount of radiation,
- cells infected at a given multiplicity of infection,
- bacteria in a certain amount of liquid.
- photons arriving on a pixel circuit at a given illumination over a given time period,
- landing of V-1 flying bombs on London during World War II, investigated by R. D. Clarke in 1946.

Determine whether the table describes a discrete probability distribution. Explain.

x	$Pr(x)$
1	$3/10$
2	$1/4$
3	$-1/5$
4	$13/20$

Determine whether the table describes a discrete probability distribution. Explain.

x	$Pr(x)$
-1	$1/8$
1	$1/4$
2	$1/8$
4	$1/2$

Suppose that I roll a 1D6 behind a screen and ask you

$$\Pr(2) =$$

Suppose that I roll a 1D6 behind a screen and ask you

$$\Pr(2) =$$

Now suppose that I roll a 1D6 behind a screen, I tell you that I have rolled an even number, and ask you

$$\Pr(2|\{2, 4, 6\}) =$$

Suppose that I roll a 1D6 behind a screen and ask you

$$\Pr(2) =$$

Now suppose that I roll a 1D6 behind a screen, I tell you that I have rolled an even number, and ask you

$$\Pr(2|\{2, 4, 6\}) =$$

This is called a conditional probability.

Suppose that you chose a card from a deck of cards. What is

$$\Pr(\text{Red}) =$$

Suppose that you chose a card from a deck of cards. What is

$$\Pr(\text{Red}) =$$

Now without replacement, if you select a second card from the deck

$$\Pr(\text{Red}|\text{Red}) =$$

Suppose that you chose a card from a deck of cards. What is

$$\Pr(\mathsf{Red}) =$$

Now without replacement, if you select a second card from the deck

$$\Pr(\mathsf{Red}|\mathsf{Red}) =$$

These events are *not* independent since the outcome of the first one *does* affect the outcome of the second one.

Suppose that I flip a fair coin and ask you

$$\Pr(H) =$$

Suppose that I flip a fair coin and ask you

$$\Pr(H) =$$

Now suppose that I flip a fair coin again, I tell you that the first flip was a H, and ask you

$$\Pr(H|H) =$$

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$$\Pr(H) =$$

Now suppose that I flip a fair coin again, I tell you that the first flip was a H, and ask you

$$\Pr(H|H) =$$

These events *are* independent since the outcome of the first one *does not* affect the outcome of the second one.

This leads to the

Definition 13 (Independent Events)

The events A and B are independent if and only if

$$\Pr(A|B) = \Pr(A) \textit{ and } \Pr(B|A) = \Pr(B)$$

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We will investigate this more thoroughly later (this semester or when you take MS 302), but the key takeaway is

Theorem 1

The events A and B are independent if and only if

$$\Pr(A \textit{ and } B) = \Pr(A) \cdot \Pr(B)$$

As an example of how we will use this let

$$E_1 \equiv \text{flip a head on the first flip}$$

and

$$E_2 \equiv \text{flip a head on the second flip}$$

Since E_1 and E_2 are independent

$$\Pr(E_1 \text{ and } E_2) = \Pr(E_1) \cdot \Pr(E_2)$$

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$$\Pr(E_1 \text{ and } E_2) = \Pr(E_1) \cdot \Pr(E_2)$$

$$\Pr(E_1 \text{ and } E_2) = \frac{1}{2} \cdot \frac{1}{2}$$

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$$\Pr(E_1 \text{ and } E_2) = \frac{1}{4}$$

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$$\Pr(E_1 \text{ and } E_2) = \Pr(E_1) \cdot \Pr(E_2)$$

$$\Pr(E_1 \text{ and } E_2) = \frac{1}{2} \cdot \frac{1}{2}$$

$$\Pr(E_1 \text{ and } E_2) = \frac{1}{4}$$

Which makes sense because

$$\Pr(E_1 \text{ and } E_2) = \Pr(HH)$$

If A and B don't share any elements (they are disjoint $A \cap B = \emptyset$), then the Inclusion-Exclusion Principle says

$$\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$$

Again we will an explain later... but the key takeaway is

$$\begin{aligned}
\Pr(HT, TH) &= \Pr(HT \text{ or } TH) \\
&= \Pr(HT) + \Pr(TH) \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{2}{4} \\
&= \frac{1}{2}
\end{aligned}$$

Suppose we flip a fair coin three times and we are interested in the number of heads we flip (so our random variable X counts the number of heads). Recall that the sample space is:

$$S = \{TTT, TTH, THT, HTT, HHT, HTH, THH, HHH\}$$

Let us determine the probability of each outcome using the fact that our flips are independent. Let us start with TTT (no heads) and find $P(TTT)$. By independence and the fact that its a fair coin we get

$$P(TTT) = P(T)P(T)P(T) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

What about an event with one head: TTH?

$$P(TTH) = P(T)P(T)P(H) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

What about an event with one head: TTH?

$$P(TTH) = P(T)P(T)P(H) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

and we get the same probability for the other two permutations:

$$P(THT) = \frac{1}{8}$$

and

$$P(HTT) = \frac{1}{8}.$$

And two heads: HHT?

$$P(HHT) = P(H)P(H)P(T) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

And two heads: HHT?

$$P(HHT) = P(H)P(H)P(T) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

and again the other two permutations gives the same probabilities:

$$P(HTH) = \frac{1}{8}$$

and

$$P(THH) = \frac{1}{8}.$$

And finally three heads: HHH:

$$P(HHH) = P(H)P(H)P(H) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

Now let's look at $P(X = 0)$. Well $(X = 0) \equiv \{TTT\}$. So

$$P(X = 0) = P(TTT) = \frac{1}{8}$$

Now $(X = 1) \equiv \{TTH, THT, HTT\}$ and we get

$$\begin{aligned} P(X = 1) &= P(TTH, THT, HTT) \\ &= P(TTH) + P(THT) + P(HTT) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= 3 \cdot \frac{1}{8} \end{aligned}$$

If we are interested in two heads:
$(X = 2) \equiv \{HHT, HTH, THH\}$

$$\begin{aligned} P(X = 2) &= P(HHT, HTH, THH) \\ &= P(HHT) + P(HTH) + P(THH) \\ &= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \\ &= 3 \cdot \frac{1}{8} \end{aligned}$$

And finally three heads: $(X = 0) \equiv \{HHH\}$ gives

$$P(X = 3) = P(HHH) = \frac{1}{8}$$

What if its an unfair coin? Let's let

$$P(T) = .3$$

and

$$P(H) = .7$$

and run through the same steps as above. No heads:

$$P(TTT) = P(T)P(T)P(T) = 0.3 \cdot 0.3 \cdot 0.3 = (0.3)^3$$

$$P(T) = .3 \quad \text{and} \quad P(H) = .7$$

One head:

$$P(HTT) = P(H)P(T)P(T) = 0.7 \cdot 0.3 \cdot 0.3 = 0.7(0.3)^2$$

$$P(THT) = P(T)P(H)P(T) = 0.3 \cdot 0.7 \cdot 0.3 = 0.7(0.3)^2$$

$$P(TTH) = P(T)P(T)P(H) = 0.3 \cdot 0.3 \cdot 0.7 = 0.7(0.3)^2$$

$$P(T) = .3 \quad \text{and} \quad P(H) = .7$$

Two heads:

$$P(HHT) = P(H)P(H)P(T) = 0.7 \cdot 0.7 \cdot 0.3 = (0.7)^2 \cdot 0.3$$

$$P(HTH) = P(H)P(T)P(H) = 0.7 \cdot 0.3 \cdot 0.7 = (0.7)^2 \cdot 0.3$$

$$P(THH) = P(T)P(H)P(H) = 0.3 \cdot 0.7 \cdot 0.7 = (0.7)^2 \cdot 0.3$$

$$P(T) = .3 \quad \text{and} \quad P(H) = .7$$

Three heads

$$P(HHH) = P(H)P(H)P(H) = 0.7 \cdot 0.7 \cdot 0.7 = (0.7)^3$$

So now calculating the probability of getting 0, 1, 2, or 3 heads is:

$$P(X=0) = P(TTT) = (0.3)^3$$

$$\begin{aligned} P(X=1) &= P(TTH, THT, HTT) \\ &= P(TTH) + P(THT) + P(HTT) \\ &= 0.7(0.3)^2 + 0.7(0.3)^2 + 0.7(0.3)^2 \\ &= 3(0.7)(0.3)^2 \end{aligned}$$

$$\begin{aligned} P(X=2) &= P(HHT, HTH, THH) \\ &= P(HHT) + P(HTH) + P(THH) \\ &= (0.7)^2 \cdot 0.3 + (0.7)^2 \cdot 0.3 + (0.7)^2 \cdot 0.3 \\ &= 3(0.7)^2(0.3) \end{aligned}$$

$$P(X=3) = P(HHH) = (0.7)^3$$

So now we can make our probability distribution:

x	0	1	2	3
$P(X = x)$	$1(0.3)^3$	$3(0.7)(0.3)^2$	$3(0.7)^2(0.3)$	$1(0.7)^3$

Doing the same steps we can find that the probability distributions for flipping a coin twice and four times are:

$n = 2$

x	0	1	2
$P(X = x)$	$1(0.7)^0(0.3)^2$	$2(0.7)^1(0.3)^1$	$1(0.7)^2(0.3)^0$

$n = 3$

x	0	1	2	3
$P(X = x)$	$1(0.7)^0(0.3)^3$	$3(0.7)^1(0.3)^2$	$3(0.7)^2(0.3)^1$	$1(0.7)^3(0.3)^0$

$n = 4$

x	0	1	2	3	4
$P(X = x)$	$1(0.7)^0(0.3)^4$	$4(0.7)^1(0.3)^3$	$6(0.7)^2(0.3)^2$	$4(0.7)^3(0.3)^1$	$4(0.7)^4(0.3)^0$

I've add some information that we normally don't write to help recognize the patterns.

What patterns are we seeing?

Recalling that $P(H) = 0.7$ and $P(T) = 0.3$ we see that we have for $n = 2$:

Number of heads: x	0	1	2
Heads probability	$(0.7)^0$	$(0.7)^1$	$(0.7)^2$
Tails probability	$(0.3)^2$	$(0.3)^1$	$(0.3)^0$

x	0	1	2
$P(X = x)$	$1(0.7)^0(0.3)^2$	$2(0.7)^1(0.3)^1$	$1(0.7)^2(0.3)^0$

Recalling that $P(H) = 0.7$ and $P(T) = 0.3$ we see that we have for $n = 3$:

Number of heads: x	0	1	2	3
Heads probabilitiy	$(0.7)^0$	$(0.7)^1$	$(0.7)^2$	$(0.7)^3$
Tails probability	$(0.3)^3$	$(0.3)^2$	$(0.3)^1$	$(0.3)^0$

x	0	1	2	3
$P(X = x)$	$1(0.7)^0(0.3)^3$	$3(0.7)^1(0.3)^2$	$3(0.7)^2(0.3)^1$	$1(0.7)^3(0.3)^0$

Recalling that $P(H) = 0.7$ and $P(T) = 0.3$ we see that we have for $n = 4$:

Number of heads: x	0	1	2	3	4
Heads probabilitiy	$(0.7)^0$	$(0.7)^1$	$(0.7)^2$	$(0.7)^3$	$(0.7)^4$
Tails probability	$(0.3)^4$	$(0.3)^3$	$(0.3)^2$	$(0.3)^1$	$(0.3)^0$

x	0	1	2	3	4
$P(X = x)$	$1(0.7)^0(0.3)^4$	$4(0.7)^1(0.3)^3$	$6(0.7)^2(0.3)^2$	$4(0.7)^3(0.3)^1$	$4(0.7)^4(0.3)^0$

So in general if we flip a coin n times and x is the number of heads we have the following pattern:

Num heads: x	0	1	2	3	$\dots$	n
Heads prob	$(0.7)^0$	$(0.7)^1$	$(0.7)^2$	$(0.7)^3$	$\dots$	$(0.7)^n$
Tails prob	$(0.3)^{n-0}$	$(0.3)^{n-1}$	$(0.3)^{n-2}$	$(0.3)^{n-3}$	$\dots$	$(0.3)^{n-n} = (0.3)^0$

So in general if we flip a coin n times and x is the number of heads we have the following pattern:

Num heads: x	0	1	2	3	...	n
Heads prob	$(0.7)^0$	$(0.7)^1$	$(0.7)^2$	$(0.7)^3$	...	$(0.7)^n$
Tails prob	$(0.3)^{n-0}$	$(0.3)^{n-1}$	$(0.3)^{n-2}$	$(0.3)^{n-3}$	...	$(0.3)^{n-n} = (0.3)^0$

So far the formula for the probability of getting x heads when flipping a coin n times is:

$$P(X = x) = c_x(0.7)^x(0.3)^{n-x}$$

So in general if we flip a coin n times and x is the number of heads we have the following pattern:

Num heads: x	0	1	2	3	$\dots$	n
Heads prob	$(0.7)^0$	$(0.7)^1$	$(0.7)^2$	$(0.7)^3$	$\dots$	$(0.7)^n$
Tails prob	$(0.3)^{n-0}$	$(0.3)^{n-1}$	$(0.3)^{n-2}$	$(0.3)^{n-3}$	$\dots$	$(0.3)^{n-n} = (0.3)^0$

So far the formula for the probability of getting x heads when flipping a coin n times is:

$$P(X = x) = c_x(0.7)^x(0.3)^{n-x}$$

What is c_x?

If we can just find a pattern for the coefficients c_x we will have a formula for calculating the probability of getting x heads when flipping a coin n times. Lets go back to calculations:

x	0	1	2
$P(X=x)$	$1(0.7)^0(0.3)^2$	$2(0.7)^1(0.3)^1$	$1(0.7)^2(0.3)^0$

x	0	1	2	3
$P(X=x)$	$1(0.7)(0.3)^3$	$3(0.7)(0.3)^2$	$3(0.7)^2(0.3)$	$1(0.7)^3(0.3)^0$

x	0	1	2	3	4
$P(X=x)$	$1(0.7)^0(0.3)^4$	$4(0.7)^1(0.3)^3$	$6(0.7)^2(0.3)^2$	$4(0.7)^3(0.3)^1$	$1(0.7)^4(0.3)^0$

and we see that the coefficients are:

n									
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1

and we see that the coefficients are:

n									
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1

Well that looks familiar:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

And we have a formula for each of these values:

n									
2			1		2		1		
3		1		3		3		1	
4	1		4		6		4		1
2			${}_2C_0$		${}_2C_1$		${}_2C_2$		
3		${}_3C_0$		${}_3C_1$		${}_3C_2$		${}_3C_3$	
4	${}_4C_0$		${}_4C_1$		${}_4C_2$		${}_4C_3$		${}_4C_4$

And there are have it, the full formula for the probability of getting x heads when flipping a coin n times with $P(H) = 0.7$ and $P(T) = 0.3$ is

$$P(X = x) = {}_nC_x(0.7)^x(0.3)^{n-x}$$

So the fully general formula for the probability of getting x heads when flipping a coin n times with

$$P(H) = p$$

and

$$P(T) = q = (1 - p)$$

is

$$P(X = x) = {}_nC_x p^x q^{n-x}$$

Definition 14 (Binomial Probability Experiment)

A binomial probability experiment satisfies the following properties:

1. *There are n repeated trials of the experiment.*
2. *Each trial results in one of two outcomes, with one denoted success (S) and the other failure (F).*
3. *The trials are independent, with the probability of success, p, the same for every trial. The probability of failure is $q = 1 - p$ for every trial.*
4. *The probability of x successes in n trials is*

$$\Pr(x) = \binom{n}{x} p^x q^{n-x}$$

Note that they have changed notations for the combination:

$$\binom{n}{x} = {}_nC_x$$

Ex 1 Suppose a fair die is rolled 12 times.

1. What is the probability p that a 5 will occur any given time the die is rolled? (Enter your probability as a fraction.)

2. What is the probability q that a 5 will not occur any given time the die is rolled? (Enter your probability as a fraction.)

3. What is n for this experiment?

4. What is the probability that a 5 will occur 3 times in the 12 rolls? (Round your answer to three decimal places.)

Ex 2 Suppose the probability of success on each of 6 independent trials of an experiment is 0.2. (Round your answers to four decimal places.)

1. What is the probability of 4 successes?

2. What is the probability of at least 1 success?

Ex 3 7 balls are drawn with replacement from a bag containing 3 red balls and 7 black balls. (Round your answers to five decimal places.)

1. Find the probability that 2 of the balls will be red.

2. Find the probability that all 7 balls will be black.

3. Find the probability that at least 5 of the balls will be black.

Ex 4 A family has 2 children. If the probability that each child is a girl is 0.5, what is the probability that the following is true? (Round your answers to three decimal places.)

1. half of the children are girls

2. all of the children are girls

Ex 5 Suppose that 10% of the patients who have a certain disease die from it. If 10 patients have the disease, what is the probability of the following? (Round your answers to four decimal places.)

1. that exactly 2 patients will die from it

2. that no patients will die from it

3. that no more than 2 patients will die from it

Ex 6 A baseball player has a lifetime batting average of 0.267. If each at bat is independent and he comes to bat 6 times in a given game, what is the probability that the following occur? (Round your answers to five decimal places.)

1. he will get 4 hits

2. he will get more than 4 hits

Suppose we have the probability distribution

x	0	1	2
$P(X = x)$	0.5	0.2	0.3

If we could find a function such that:

$$\begin{aligned} f(0) &= 0.5 \\ f(1) &= 0.2 \\ f(2) &= 0.3 \end{aligned}$$

then this function $f(x)$ is called the probability mass function (PMF).
We have seen an example of this for the Binomial distribution:

$$f(x) = \binom{n}{x} p^x q^{1-x}$$

Another import concept is the *cumulative distribution function (CDF)*. For our PMF above the table associated with the CDF is

x	0	1	2
$P(X \leq x)$	0.5	0.7	1.0

In order to write this as a function we need to discuss the summation notation.

$$\sum_{i=s}^{n} i$$

How do we interpret

$$\sum_{i=1}^{4} i$$

It says to plug in the start value, increment up to the end value, and add those values:

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

Python

```
sum = 0
for i in range(1, 5):
____sum = sum + i
```

Julia

```
sum = 0
for i in 1:4
____sum = sum + i
end
```

So if the probability distribution is

x	0	1	2
$P(X = x)$	0.5	0.2	0.3

then the PMF is

$$f(0) = 0.5, \quad f(1) = 0.2, \quad f(2) = 0.3$$

It's CDF is

$$F(x) = \sum_{i=0}^{x} f(x)$$

so that the cumulative distribution table is

x	0	1	2
$P(X \leq x)$	0.5	0.7	1.0

Ex 7 You have a probability experiment that has a binomial distribution with $n = 2$ and $p = 0.5$.

1. Construct the binomial distribution that describes this experiment, with x indicating the number of successes.

x	0	1	2
$P(X = x)$			

2. Construct the cumulative distribution table for this experiment.

x	0	1	2
$P(X \leq x)$			

Ex 8 You have a probability experiment that has a binomial distribution with $n = 2$ and $p = 0.5$.

x	0	1	2
$P(X = x)$			

Draw the histogram for this distribution.

Ex 9 A die is rolled 3 times, and success is rolling a 2 or a 3.

1. Construct the binomial distribution that describes this experiment, with x indicating the number of successes. (Enter your probabilities as fractions.)

x	0	1	2	3
$P(X = x)$				

2. Construct the cumulative distribution table for this experiment.

x	0	1	2	3
$P(X \leq x)$				

Ex 10 A die is rolled 3 times, and success is rolling a 2 or a 3.

x	0	1	2	3
$P(X = x)$				

Draw the histogram for this distribution.

Ex 11 You have a probability experiment that has a binomial distribution with $n = 5$ and $p = 0.75$.

1. Construct the binomial distribution that describes this experiment, with x indicating the number of successes.

x	0	1	2	3	4	5
$P(X = x)$						

2. Construct the cumulative distribution table for this experiment.

x	0	1	2	3	4	5
$P(X \leq x)$						

Ex 12 You have a probability experiment that has a binomial distribution with $n = 5$ and $p = 0.75$.

x	0	1	2	3	4	5
$P(X = x)$						

Draw the histogram for this distribution.

Introduce binomial table.

Assume the random variable X has a binomial distribution with the given probability of obtaining a success. Find the following probability using the binomial table, given the number of trials and the probability of obtaining a success. Round your answer to four decimal places.

$$P(X \leq 4), n = 7, p = 0.6$$

Assume the random variable X has a binomial distribution with the given probability of obtaining a success. Find the following probability using the binomial table, given the number of trials and the probability of obtaining a success. Round your answer to four decimal places.

$$P(X < 5), n = 6, p = 0.7$$

Assume the random variable X has a binomial distribution with the given probability of obtaining a success. Find the following probability using the binomial table, given the number of trials and the probability of obtaining a success. Round your answer to four decimal places.

$$P(X = 4), n = 6, p = 0.3$$

Assume the random variable X has a binomial distribution with the given probability of obtaining a success. Find the following probability using the binomial table, given the number of trials and the probability of obtaining a success. Round your answer to four decimal places.

$$P(X > 4), n = 6, p = 0.4$$

A researcher wishes to conduct a study of the color preferences of new car buyers. Suppose that 40% of this population prefers the color green. If 12 buyers are randomly selected, what is the probability that exactly 2 buyers would prefer green? Round your answer to four decimal places.

A real estate agent has 18 properties that she shows. She feels that there is a 30% chance of selling any one property during a week. The chance of selling any one property is independent of selling another property. Compute the probability of selling at least 5 properties in one week. Round your answer to four decimal places.

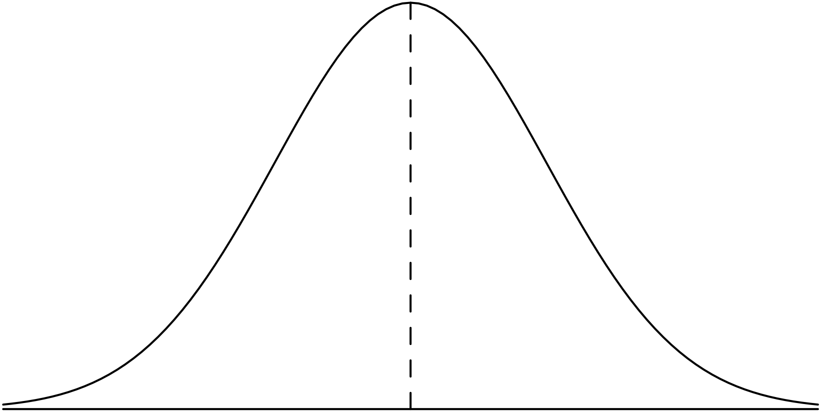
A quality control inspector has drawn a sample of 13 light bulbs from a recent production lot. If the number of defective bulbs is 1 or more, the lot fails inspection. Suppose 20% of the bulbs in the lot are defective. What is the probability that the lot will fail inspection? Round your answer to four decimal places.

A certain insecticide kills 70% of all insects in laboratory experiments. A sample of 9 insects is exposed to the insecticide in a particular experiment. What is the probability that exactly 4 insects will survive? Round your answer to four decimal places.

Chapter 8 Section 4 prevue

We want to switch from discrete distributions and start looking at continuous distributions.
Historically, the most important historically has been the normal distribution.

The normal (Gaussian) distribution looks like:



Recall that probability is area under the curve.

The normal distribution is controlled by two parameters:

Mean This is the center and is denoted by μ.

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- **Mean** This is the center and is denoted by μ.
- **Standard deviation** This is how spread out the data is and is denoted by σ.

A normal distribution with mean $\mu = 0$ and standard deviation of $\sigma = 1$ is denoted as

$$N(0, 1)$$

The normal distribution is controlled by two parameters:

Mean This is the center and is denoted by μ.

Standard deviation This is how spread out the data is and is denoted by σ.

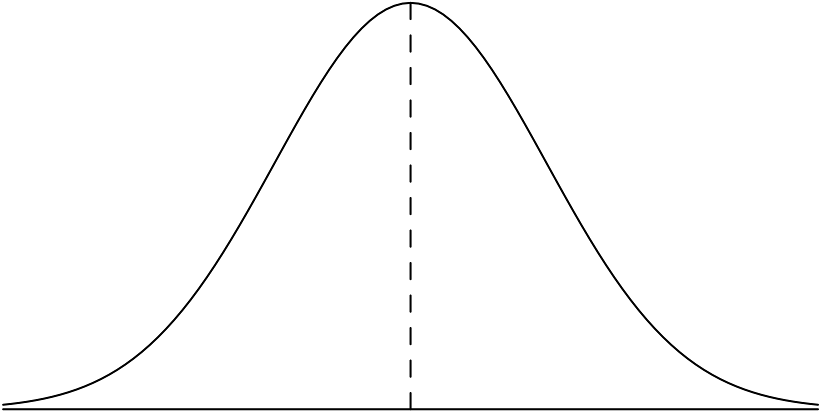
A normal distribution with mean $\mu = 0$ and standard deviation of $\sigma = 1$ is denoted as

$$N(0, 1)$$

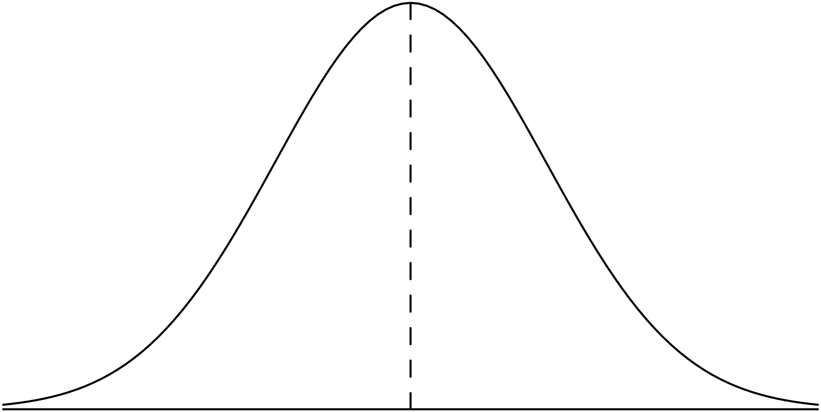
Or in general

$$N(\mu, \sigma)$$

The function that gives us the curve:



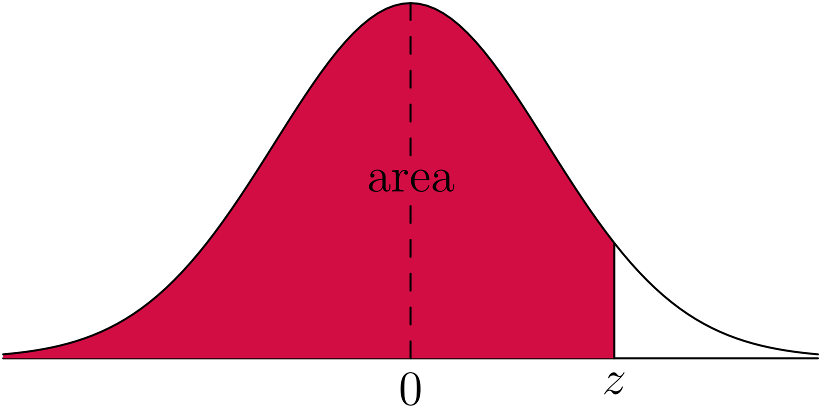
The function that gives us the curve:



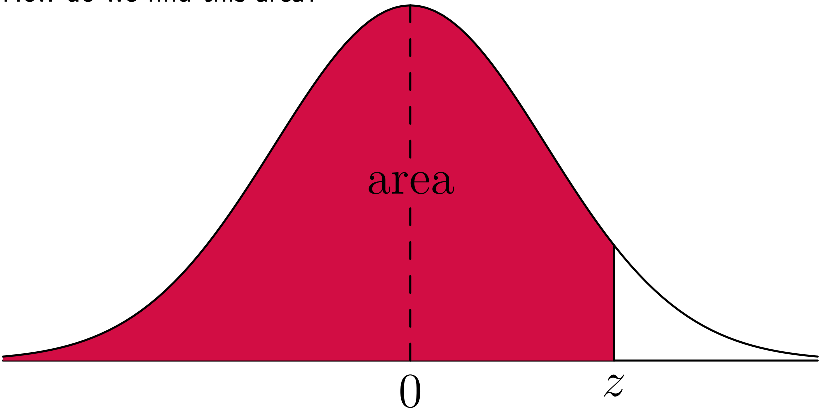
is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\dfrac{-(x-\mu)^2}{2\sigma^2}}$$

How do we find this area?



How do we find this area?



This leads us to...

Chapter 13, 12

Integration

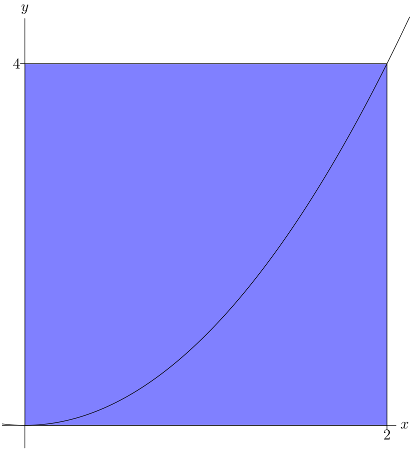
Chapter 13 Section 1

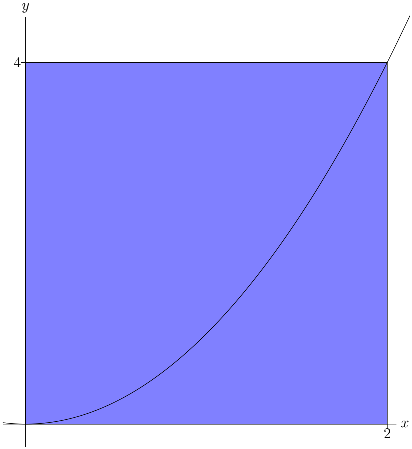
Find the area from $x = 0$ to $x = 2$under the curves:

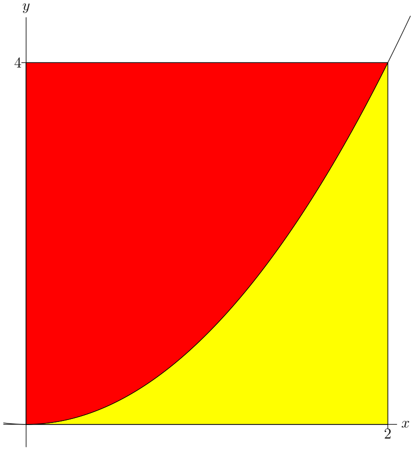
a) $f(x) = 3$

b) $f(x) = x$

c) $f(x) = x^2$


y
4
x
2

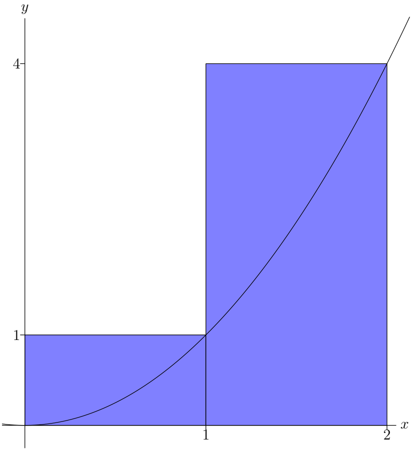


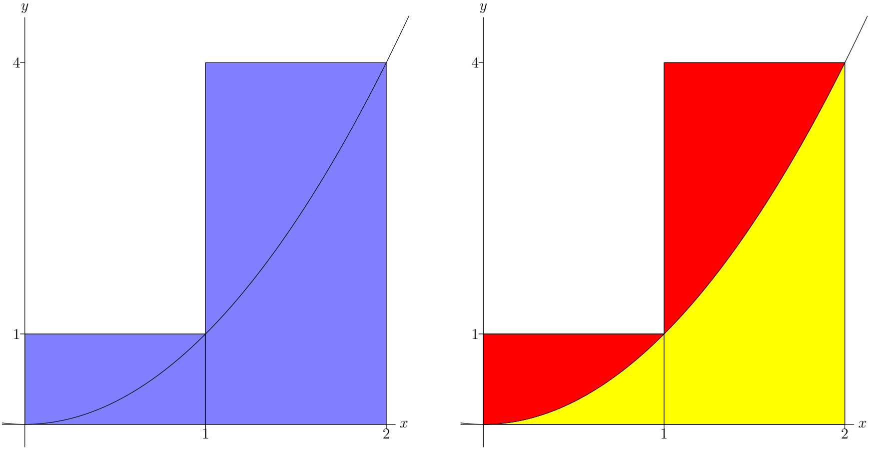


Rectangles right endpoints: 2

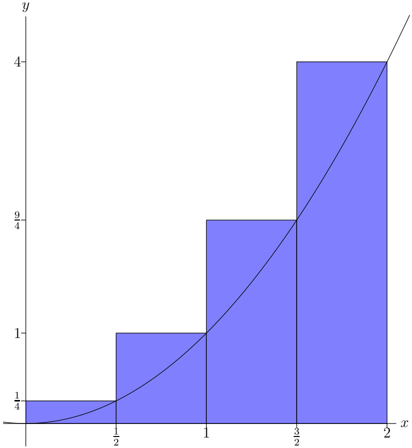
Rectangles width: 2

$A_1 = f(2) \cdot 2 = 4(2) = 8$

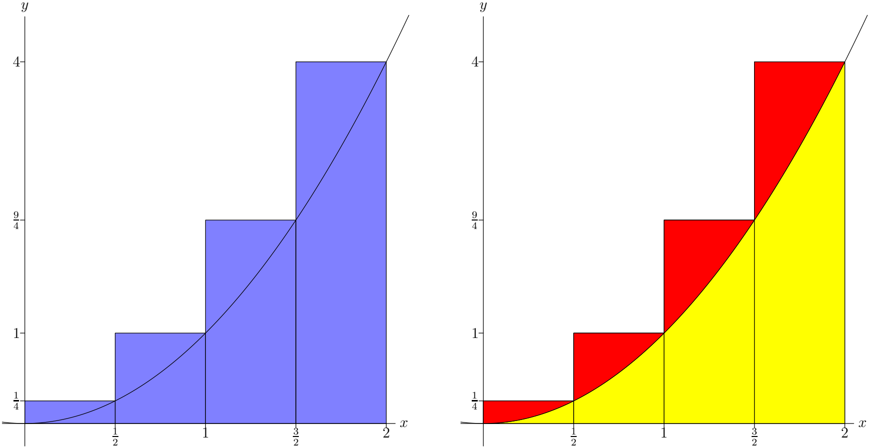

y
4
1
x
1
2



Rectangles right endpoints: $1, 2$
Rectangles width: 1
$A_2 = f(1) \cdot 1 + f(2) \cdot 1 = 1(1) + 4(1) = 5$



y
4
9/4
1
1/4
x
1/2
1
3/2
2

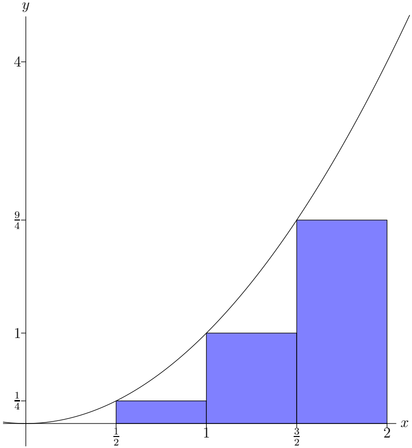


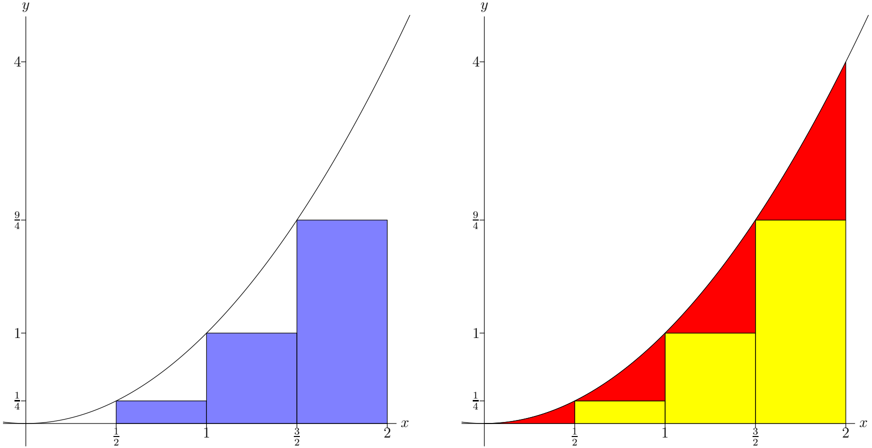
Rectangles right endpoints: $\dfrac{1}{2}, 1, \dfrac{3}{2}, 2$

Rectangles width: $\dfrac{1}{2}$

$$A_4 = f\left(\frac{1}{2}\right) \cdot \frac{1}{2} + f(1) \cdot \frac{1}{2} + f\left(\frac{3}{2}\right) \cdot \frac{1}{2} + f(2) \cdot \frac{1}{2} =$$

$$\frac{1}{4}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + \frac{9}{4}\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right) = \frac{15}{4}$$


y
4
$\frac{9}{4}$
1
$\frac{1}{4}$
$\frac{1}{2}$
1
$\frac{3}{2}$
2
x

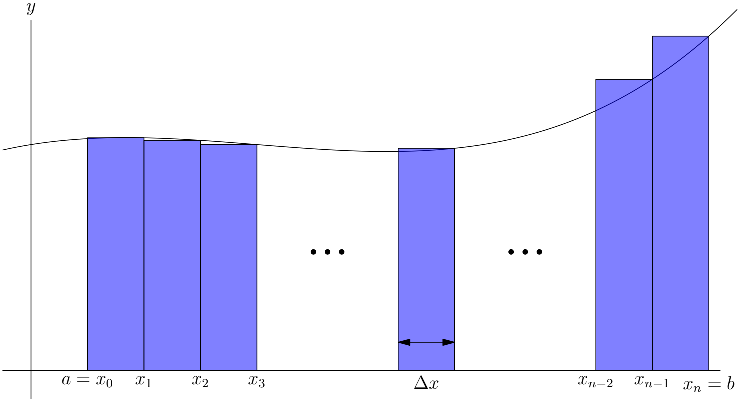


Rectangles left endpoints: $0, \dfrac{1}{2}, 1, \dfrac{3}{2}$

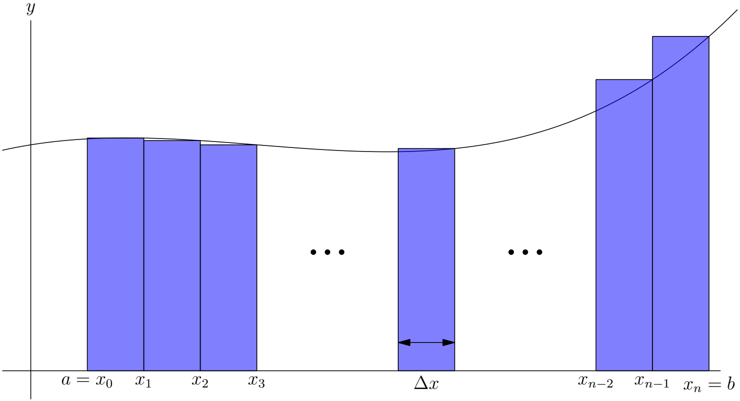
Rectangles width: $\dfrac{1}{2}$

$$A_4 = f(0)\cdot\frac{1}{2} + f\left(\frac{1}{2}\right)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f\left(\frac{3}{2}\right)\cdot\frac{1}{2} =$$

$$0\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right) + \frac{9}{4}\left(\frac{1}{2}\right) = \frac{7}{4}$$



$$\begin{aligned} A_n &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_{n-1})\Delta x + f(x_n)\Delta x \\ &= [f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{n-1}) + f(x_n)]\Delta x \\ &= \left[\sum_{i=1}^{n} f(x_i)\right]\Delta x \end{aligned}$$



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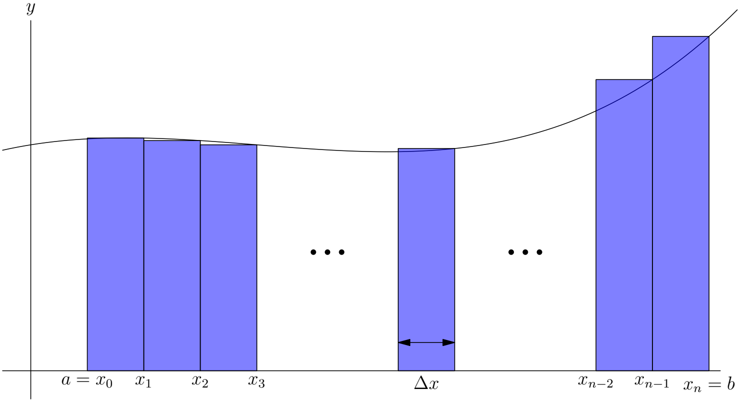
As we make more rectangles between a and b their widths must get smaller.



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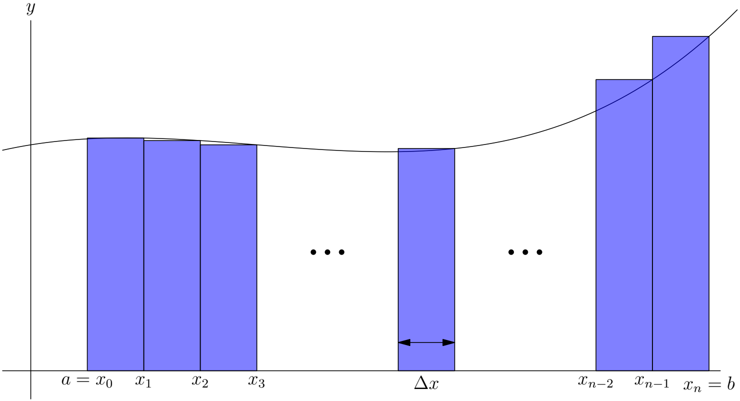

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As we make more rectangles between a and b their widths must get smaller. As their widths get smaller the errors (red areas in the previous slides) get smaller and therefore the better the better the approximation A_n is to the true area A.

These sums

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

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$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

as $n \to \infty$ where the area is between a and b..
This is denoted as

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$

and is the area under $f(x)$ from a to b.

Let's find the Riemann sum for

$$f(x) = x^2$$

from a to b using right endpoints.

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$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$A_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = *$

$f(x) = x^2$, $[a,b] \Rightarrow \Delta x = \frac{b-a}{n}$, $x_i = a + i\frac{b-a}{n}$

$* = \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)^2 \frac{b-a}{n}$

$= \sum_{i=1}^{n} \left[a^2 + 2ia\frac{b-a}{n} + i^2\left(\frac{b-a}{n}\right)^2\right]\frac{b-a}{n}$

$= \sum_{i=1}^{n} \left[a^2\frac{b-a}{n} + 2ia\left(\frac{b-a}{n}\right)^2 + i^2\left(\frac{b-a}{n}\right)^3\right]$

$= a^2\frac{b-a}{n}\sum_{i=1}^{n} 1 + 2a\left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^3\sum_{i=1}^{n} i^2$

$= a^2\frac{b-a}{\cancel{n}}\cancel{n} + \cancel{2}a\left(\frac{b-a}{n}\right)^2\frac{n(n+1)}{\cancel{2}} + \left(\frac{b-a}{n}\right)^3\frac{n(n+1)(2n+1)}{6}$

$= a^2(b-a) + a(b-a)^2\frac{n(n+1)}{n^2} + \frac{1}{6}(b-a)^3\frac{n(n+1)(2n+1)}{n^3}$

$= a^2(b-a) + a(b-a)^2\frac{n^2+n}{n^2} + \frac{1}{6}(b-a)^3\frac{2n^3+3n^2+n}{n^3}$

$= a^2(b-a) + a(b-a)^2\left(1+\frac{1}{n}\right) + \frac{1}{6}(b-a)^3\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$

$\left(a + i\frac{b-a}{n}\right)\left(a + i\frac{b-a}{n}\right)$
$= a^2 + 2ia\frac{b-a}{n} + i^2\left(\frac{b-a}{n}\right)^2$

$n(n+1)(2n+1) = n(2n^2+3n+1)$
$= 2n^3+3n^2+n$

$\lim_{n\to\infty} A_n$

$= \lim_{n\to\infty}\left[a^2(b-a) + a(b-a)^2\left(1+\frac{1}{n}\right) + \frac{1}{6}(b-a)^3\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\right]$

$= a^2(b-a) + a(b-a)^2\lim_{n\to\infty}\left(1+\frac{1}{n}\right) + \frac{1}{6}(b-a)^3\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)$

$(b-a)(b-a)(b-a)$
$= (b-a)(b^2-2ab+a^2)$
$= b^3-2ab^2+a^2b-ab^2+2a^2b-a^3$
$= b^3-3ab^2+3a^2b-a^3$

$= a^2b - a^3 + a(b^2-2ab+a^2) + \frac{1}{6}(b^3-3ab^2+3a^2b-a^3)\cdot 2$

$= a^2b - \cancel{a^3} + \cancel{ab^2} - 2a^2b + \cancel{a^3} + \frac{1}{3}b^3 - \cancel{ab^2} + a^2b - \frac{1}{3}a^3$

$= a^2b - 2a^2b + \frac{1}{3}b^3 + a^2b - \frac{1}{3}a^3$

$= \frac{1}{3}b^3 - \frac{1}{3}a^3 + 2a^2b - 2a^2b$

$= \frac{1}{3}b^3 - \frac{1}{3}a^3$

If we redid this for various exponents we could see the pattern

$$\begin{aligned}
\int_a^b x\,dx &= \tfrac{b^2}{2} - \tfrac{a^2}{2} \\
\int_a^b x^2\,dx &= \tfrac{b^3}{3} - \tfrac{a^3}{3} \\
\int_a^b x^3\,dx &= \tfrac{b^4}{4} - \tfrac{a^4}{4} \\
\int_a^b x^4\,dx &= \tfrac{b^5}{5} - \tfrac{a^5}{5}
\end{aligned}$$

And if we were slick enough we could show that for any real number n

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \quad \text{if } n \neq -1$$

Chapter 13 Section 2

Definition 15 (Fundamental Theorem of Calculus weaker version)

Let f be a continuous function on the closed interval $[a, b]$; then the definite integral of f exists on this interval, and

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is certain type of function that is defined for all x in $[a, b]$.

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where F is certain type of function that is defined for all x in $[a, b]$.

To make it easier to work problems we will introduce a new notation.

$$F(x)|_a^b = F(b) - F(a)$$

Some examples

1. $$x|_2^3 = 3 - 2 = 1$$

2. $$x^2|_2^3 = 3^2 - 2^2 = 9 - 4 = 5$$

For more complex expressions we let the closing square bracket take the place of the vertical line.

1. $$\begin{aligned} [x^2 - x]|_3^5 = \left[x^2 - x\right]_3^5 &= (5^2 - 5) - (3^2 - 3) \\ &= (25 - 5) - (9 - 3) \\ &= 20 - 6 = 14 \end{aligned}$$

2. $$\begin{aligned} [3x^2 + 2x]_4^5 &= (3(5)^2 + 2(5)) - (3(4)^2 + 2(4)) \\ &= (3(25) + 10) - (3(16) + 8) \\ &= (75 + 10) - (48 + 8) \\ &= 85 - 56 = 29 \end{aligned}$$

There are some rules we need to learn. Guided by the end of section 13.1 where we saw

$$\int_a^b x^n dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \quad \text{if } n \neq -1$$

So our first rule is

Rule 1 (Power rule)

For any real number n

$$\int_a^b x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_a^b, \quad \textit{if } n \neq -1$$

Examples I

1. $\displaystyle\int_1^3 x^2\,dx =$

2. $\displaystyle\int_1^3 x^5\,dx =$

Examples II

1. $\displaystyle\int_1^2 \frac{1}{x^3}\,dx =$

2. $\displaystyle\int_0^2 x^{1.5}\,dx =$

Examples III

1. $\displaystyle\int_1^2 \frac{1}{x}\,dx =$

2. $\displaystyle\int_0^4 x^{\frac{2}{3}}\,dx =$

Examples IV

1. $\displaystyle\int_0^4 \sqrt{x}\,dx =$

2. $\displaystyle\int_0^2 1\,dx =$