MS 120 In-class Problems

February 7, 2025

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Chapter R Section R1.2

1.2.001 Use the values in the following table.

x	-6	-1	0	3	4.2	9	12	14	15	22
y	0	0	1	5	9	12	38	22	22	70

1. Explain why the table defines y as a function of x.

   - ◯ For each value of y there are multiple values for x.
   - ◯ For each value of y there is only one x.
   - ◯ For each value of x there are multiple values for y.
   - ◯ For each value of x there is only one y.
   - ◯ For some values of y there are multiple values for x.

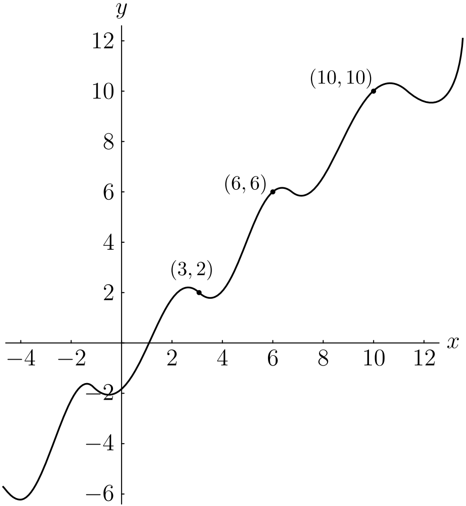
2. State the domain and range of this function.

   domain:
   range:

3. If the table expresses $y = f(x)$, find $f(0)$ and $f(12)$. (If the table does not express $y = f(x)$, enter DNE.)

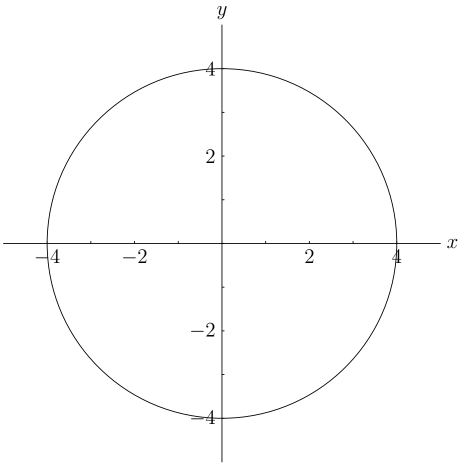
   $f(0) =$
   $f(12) =$

1.2.005a Determine whether the graph represents y as a function of x. Explain your answer.



- Yes, the vertical line test shows that the graph represents y as a function of x.
- Yes, the horizontal line test shows that the graph represents y as a function of x.
- No, the vertical line test shows that the graph does not represent y as a function of x.
- No, the horizontal line test shows that the graph does not represent y as a function of x.
- There is no way to determine this using the graph.

1.2.005b Determine whether the graph represents y as a function of x. Explain your answer.



- Yes, the vertical line test shows that the graph represents y as a function of x.
- Yes, the horizontal line test shows that the graph represents y as a function of x.
- No, the vertical line test shows that the graph does not represent y as a function of x.
- No, the horizontal line test shows that the graph does not represent y as a function of x.
- There is no way to determine this using the graph.

1.2.009 If $R(x) = 8x - 11$, find the following. (Give exact answers. Do not round.)

1. $R(0) =$

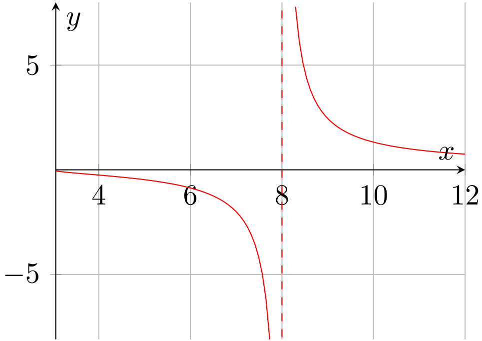
2. $R(2) =$

3. $R(-3) =$

4. $R(1.6) =$

1.2.029 A function and its graph are given. Find the domain. (Enter your answer using interval notation.)

$$f(x) = \frac{\sqrt{x-3}}{x-8}$$



Chapter R Section R1.3

1.3.001 Find the intercepts and graph.

$$5x + 8y = 40$$

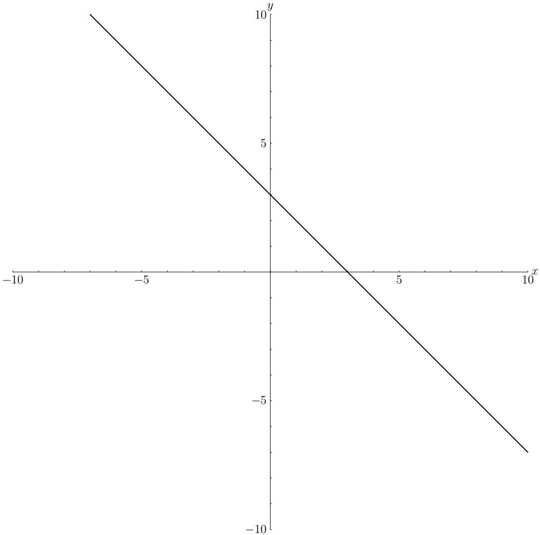
1.3.005 Find the slope m of the line passing through the given pair of points. (If an answer is undefined, enter UNDEFINED.)

$$(20, 21) \text{ and } (14, -3)$$

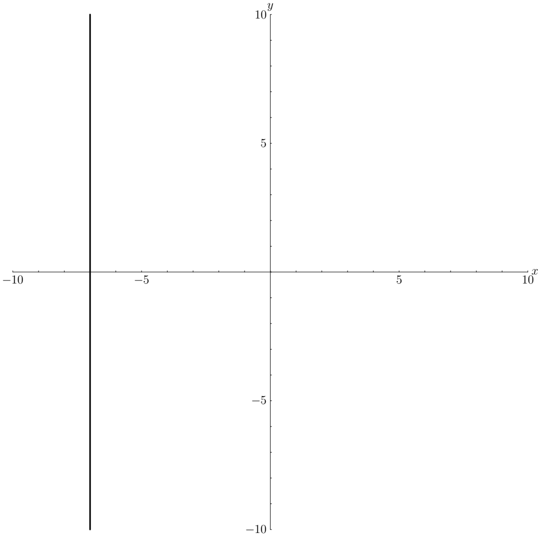
1.3.011 If a line is horizontal, then its slope is ______.

1.3.013 What is the rate of change of the function whose graph is a line passing through $(3, 4)$ and $(-1, 4)$?

1.3.015a For the given graph, determine whether the line has a slope that is positive, negative, 0, or undefined.



1.3.015b For the given graph, determine whether the line has a slope that is positive, negative, 0, or undefined.



1.3.017 Find the slope m and y-intercept b. (Give exact answers. Do not round. If an answer is undefined, enter UNDEFINED. If an answer does not exist, enter DNE.)

$$y = \frac{7}{3}x - \frac{1}{2}.$$

1.3.023 Find the slope m and y-intercept b. (Give exact answers. Do not round. If an answer is undefined, enter UNDEFINED. If an answer does not exist, enter DNE.)

$$2x + 7y = 14.$$

1.3.025 Write the slope-intercept form of the equation of the line that has the given slope and y-intercept.

$$\text{Slope } \tfrac{1}{3} \text{ and } y\text{-intercept } -3$$

1.3.033 Write the equation of the line that passes through the given point and has the given slope.

$(-2, 2)$ with undefined slope

1.3.035 Write the equation of the line described.

Through $(4, 5)$ and $(-1, -5)$

1.3.041 Determine whether the following pair of equations represents parallel lines, perpendicular lines, or neither of these.

$$3x + 8y = 24; \quad 8x - 3y = 24$$

1.3.045 Write the equation of the line passing through $(-2, -1)$ that is parallel to $3x + 5y = 11$.

Chapter R Section R1.6

In engineering and science, dimensional analysis is the analysis of the relationships between different physical quantities by identifying their base quantities (such as length, mass, time, and electric current) and units of measurement (such as metres and grams) and tracking these dimensions as calculations or comparisons are performed.

The term dimensional analysis is also used to refer to conversion of units from one dimensional unit to another, which can be used to evaluate scientific formulae.[1]

[1] https://en.wikipedia.org/wiki/Dimensional_analysis

1.6.005a A linear cost function is $C(x) = 3x + 750$. (Assume C is measured in dollars.)

1. What are the slope and the C-intercept?
2. What is the marginal cost C' ($\overline{MC}$)?
3. What does the marginal cost mean?

   a) Each additional unit produced costs this much (in dollars).

   b) If production is increased by this many units, the cost decreases by \$1.

   c) If production is increased by this many units, the cost increases by \$1.

   d) Each additional unit produced reduces the cost by this much (in dollars).

4. What are the fixed costs?
5. How are your answers to parts (1), (2), and (3) related?

   a) $\dfrac{C\text{-intercept}}{slope}$ = marginal cost

   b) slope = fixed costs, and C-intercept = marginal cost

   c) slope = marginal cost, and C-intercept = fixed costs

   d) $\dfrac{slope}{C\text{-intercept}}$ = marginal cost

6. What is the cost of producing one more item if 50 are currently being produced? What is the cost of producing one more item if 100 are currently being produced?

1.6.007 A linear revenue function is $R = 26x$. (Assume R is measured in dollars.)

1. What is the slope m?
2. What is the marginal revenue R'?
   What does the marginal revenue mean?

   a) Each additional unit sold decreases the revenue by this many dollars.

   b) If the number of units sold is increased by this amount, the revenue increases by $1.

   c) Each additional unit sold yields this many dollars in revenue.

   d) If the number of units sold is increased by this amount, the revenue decreases by $1.

3. What is the revenue received from selling one more item if 50 are currently being sold?
   What is the revenue received from selling one more item if 100 are being sold?

1.6.001 Suppose a calculator manufacturer has the total cost function $C(x) = 22x + 6600$ and the total revenue function $R(x) = 56x$.

1. What is the equation of the profit function $P(x)$ for the calculator?

2. What is the profit on 2800 units?

1.6.003 Suppose a ceiling fan manufacturer has the total cost function $C(x) = 34x + 560$ and the total revenue function $R(x) = 48x$.

1. What is the equation of the profit function P(x) for this commodity?
   $P(x) =$
2. What is the profit on 20 units?

   Interpret your result.
   - The total costs are less than the revenue.
   - The total costs are more than the revenue.
   - The total costs are exactly the same as the revenue.
3. How many fans must be sold to avoid losing money?

1.6.009 Let $C(x) = 3x + 750$ and $R(x) = 21x$.

1. Write the profit function $P(x)$.
2. What is the slope m of the profit function?
3. What is the marginal profit P'?
4. Interpret the marginal profit.

a) Each additional unit sold decreases the profit by this much.

b) Each additional unit sold increases the profit by this much.

c) This is the smallest number of units that can be sold in order to make a profit.

d) The profit is maximized when this many units are sold.

1.6.013 1-3 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

1. For this helmet, write the function for monthly total costs $C(x)$.
   $C(x) =$
2. Write the function for total revenue $R(x)$.
   $R(x) =$
3. Write the function for profit $P(x)$.
   $P(x) =$

1.6.013 4 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

4. Find $C(200)$.
   $C(200) =$

   Interpret $C(200)$.
   - For each \$1 increase in cost this many more helmets can be produced.
   - This is the cost (in dollars) of producing 200 helmets.
   - For every additional helmet produced the cost increases by this much.
   - When this many helmets are produced the cost is \$200.

1.6.013 4 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

4. Find $R(200)$.
   $R(200) =$

   Interpret $R(200)$.
   - When this many helmets are produced the revenue generated is \$200.
   - For each \$1 increase in revenue this many more helmets can be produced.
   - For every additional helmet produced the revenue generated increases by this much.
   - This is the revenue (in dollars) generated from the sale of 200 helmets.

1.6.013 4 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

4. Find $P(200)$.
   $P(200) =$

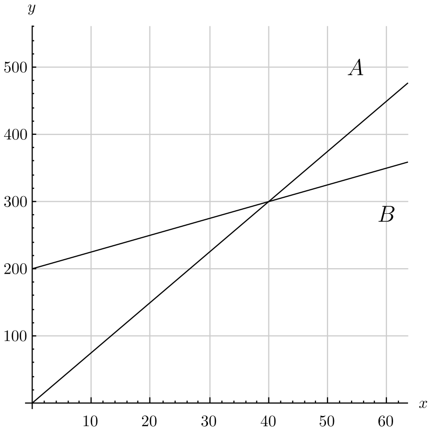
   Interpret $P(200)$.
   - This is the profit (in dollars) when 200 helmets are sold, but since it is negative it means that the company loses money when 200 helmets are sold.
   - For each additional helmet sold the profit (in dollars) increases by this much, but since it is positive it means that the company is producing too many helmets.
   - For each additional helmet sold the profit (in dollars) increases by this much, but since it is negative it means that the company needs to decrease the number of helmets sold in order to make a profit.
   - This is the profit (in dollars) when 200 helmets are sold, and since it is positive it means that the company makes money when 200 helmets are sold.

1.6.013 5,6 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C, R, and P be measured in dollars.)

5. Same as the last part but with \$300 instead of \$200.
6. Find the marginal profit P'.
   $P' =$

   Write a sentence that explains its meaning.
   - When revenue is increased by this much the profit is increased by \$1.
   - For each \$1 increase in profit this many more helmets can be produced.
   - When costs are decreased by this much the profit is increased by \$1.
   - Each additional helmet sold increases the profit by this many dollars.

1.6.015 The figure shows graphs of the total cost function and the total revenue function for a commodity. (Assume cost and revenue are measured in dollars.)



1. Label each function correctly. Choose from *total revenue function*, *total cost function*

   a) function A b) function B

2. Determine the fixed costs.

3. Locate the break-even point. $(x, y) =$
   Determine the number of units sold to break even.

4. Estimate the marginal cost C' and marginal revenue R'.

Definition 1 (Market equilibrium)

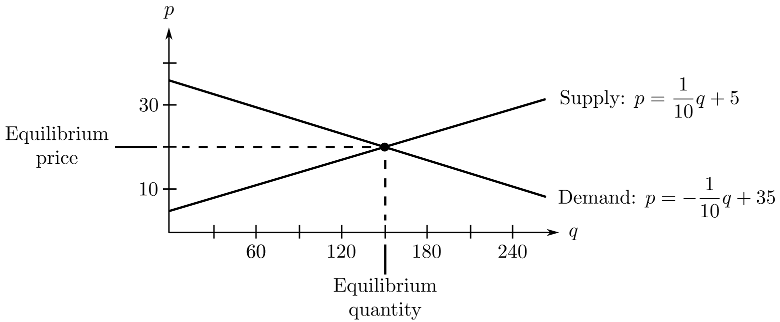
occurs when the quantity of a commodity demanded is equal to the quantity supplied.

Law 1 (Law of demand)

states that the quantity demanded will increase as price decreases and that the quantity demanded will decrease as price increases.

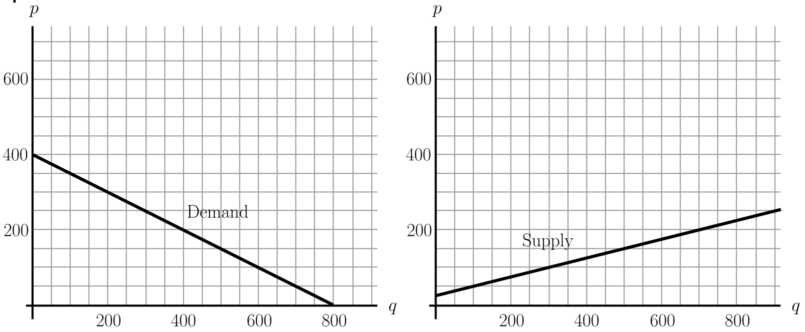
Law 2 (Law of supply)

states that the quantity supplied for sale will increase as the price of a product increases.



p
30
10
Equilibrium price
Supply: $p = \frac{1}{10}q + 5$
Demand: $p = -\frac{1}{10}q + 35$
q
60
120
180
240
Equilibrium quantity

1.6.031 The graphs of the demand function and supply function for a certain product, are given below. Use these graphs to answer the questions.



1. How many units q are demanded when the price p is \$50?
2. How many units q are supplied when the price p is \$50?
3. Will there be a market surplus (more supplied) or shortage (more demanded) when p = \$50?

1.6.033 If the demand for a pair of shoes is given by $2p + 5q = 200$ and the supply function for it is $p - 2q = 10$, compare the quantity demanded and the quantity supplied when the price is \$90.

quantity demanded _____ pairs of shoes

quantity supplied _____ pairs of shoes

Will there be a surplus or shortfall at this price?

Chapter 1 Section 5

Ex 1 Graphical method

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

Ex 2 Graphical method

$$\left\{ \begin{array}{rcrcr} x & + & y & = & 2 \\ x & + & y & = & -1 \end{array} \right.$$

Ex 3 Graphical method

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 4 Graphical method

$$\begin{cases} y = x^2 + 1 \\ y = x + 1 \end{cases}$$

Ex 5 Graphical method

$$\begin{cases} y = x^2 - 1 \\ y = 0 \end{cases}$$

Ex 6 Substitution method

$$\left\{\begin{array}{rcrcr} x & + & y & = & 2 \\ x & - & 2y & = & -1 \end{array}\right.$$

Ex 7 Substitution method

$$\begin{cases} x + y = 2 \\ x + y = -1 \end{cases}$$

Ex 8 Substitution method

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 9 Substitution method

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 0 \\ & & y & - & z & = & 3 \\ & & & & z & = & -2 \end{array}\right.$$

Ex 10 Elimination method

$$\begin{cases} x & + & y & = & 1 \\ x & - & y & = & 2 \end{cases}$$

Ex 11 Elimination method

$$\begin{cases} x & + & y & = & 2 \\ x & - & 2y & = & -1 \end{cases}$$

Ex 12 Elimination method

$$\left\{\begin{array}{rcrcr} 2x & + & 3y & = & 9 \\ x & - & y & = & 2 \end{array}\right.$$

Ex 13 Elimination method

$$\begin{cases} 8x - 3y = -11 \\ 5x - 2y = -6 \end{cases}$$

Ex 14 Elimination method

$$\begin{cases} x + y = 2 \\ x + y = -1 \end{cases}$$

Ex 15 Elimination method

$$\left\{\begin{array}{rcrcl} x & + & y & = & 1 \\ 2x & + & 2y & = & 2 \end{array}\right.$$

Ex 16 Elimination method

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 0 \\ & & y & - & z & = & 3 \\ & & & & z & = & -2 \end{array}\right.$$

Ex 17 Elimination method

$$\begin{cases} \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \end{cases}$$

Ex 1.5.039 A freight company has shipping orders for two products. The first product has a unit volume of 10 cu ft and weighs 50 lb. The second product's unit volume is 3 cu ft, and it weighs 40 lb. If the company's trucks have $2,290$ cu ft of space and can carry $20,700$ lb, how many units of each product can be transported in a single shipment with one truck using the entire volume and weight capacity?

Ex 1.5.044 A biologist has a 40% solution and a 10% solution of the same plant nutrient. How many cubic centimeters of each solution should be mixed to obtain 25 cc of a 22% solution?

Chapter 1 Section R1.6 Part Deux

1.6.044 Find the market equilibrium point for the following demand and supply functions.

Demand: $p = -2q + 318$

Supply: $p = 8q + 1$

$(q, p) =$

1.6.049.EP A group of retailers will buy 104 televisions from a wholesaler if the price is \$325 and 144 if the price is \$275. The wholesaler is willing to supply 84 if the price is \$255 and 164 if the price is \$345. Assume that the resulting supply and demand functions are linear. Let p represent price (in dollars) and q represent quantity.

State the two ordered pairs for the demand function in the form (q, p).

$(q, p) =$

$(q, p) =$

Write the demand function in terms of q.

$p =$

State the two ordered pairs for the supply function in the form (q, p).

$(q, p) =$

$(q, p) =$

Write the supply function in terms of q.

$p =$

Find the equilibrium point for the market in the form (q, p).

$(q, p) =$

Chapter 3 Section 3

How to avoid the graphical, substitution, and elimination methods?

How to avoid the graphical, substitution, and elimination methods? We need to learn how to rewrite a system of equations from equation form:

$$\begin{cases} \begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array} \end{cases}$$

How to avoid the graphical, substitution, and elimination methods? We need to learn how to rewrite a system of equations from equation form:

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array}\right.$$

and into augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \\ 2 & 1 & 1 & 3 \end{array}\right]$$

Matrix

First what is a matrix?

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First what is a matrix?
It is a rectangular array of numbers (surrounded by some grouping symbols).
Note that it is standard to use capitol letters to name matrices: A, B, C, X, etc.

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Example 1.1

a) $A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$ *b)* $B = \begin{bmatrix} 11 & 12 \\ 21 & 22 \\ 31 & 32 \end{bmatrix}$

c) $C = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}$

Matrix Size

Each matrix has a size. The size of a matrix is the number of rows and columns that it has.

Example 1.2

a) *This matrix*

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

has 2 *rows and* 3 *columns. So its size is* 2×3*.*

b) *This matrix*

$$B = \begin{bmatrix} 11 & 12 \\ 21 & 22 \\ 31 & 32 \end{bmatrix}$$

has 3 *rows and* 2 *columns. So its size is* 3×2*.*

c) *This matrix*

$$C = \begin{bmatrix} 11 & 12 \\ 21 & 22 \end{bmatrix}$$

has 2 *rows and* 2 *columns. So its size is* 2×2 *and is called an square matrix.*

Matrix elements I

It is often useful to be able to reference each element of a matrix. We do this by referring to its row and column position. To lazify this we use the lower case version of the matrix name with subscripts denoting the row first and then the column:

$$a_{11} = 11 \quad \text{Element in row 1, column 1}$$

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

$$a_{12} = 12 \quad \text{Element in row 1, column 2}$$

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

Matrix elements II

$$a_{21} = 21 \quad \text{Element in row 2, column 1}$$

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

$$a_{23} = 23 \quad \text{Element in row 2, column 3}$$

$$A = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \end{bmatrix}$$

If the matrix is big enough to require two digit values for the rows or columns then the row and column is separated by a comma. So if you needed to reference an element in row 15 and column 7 you would write:

$$a_{15,7}$$

since a_{157} would be ambiguous.

The first step in writing the augmented matrix for

$$\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$$

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Is to make the coefficient matrix.

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$$\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$$

Is to make the coefficient matrix. To do this we look at the system of equations and only write down the coefficients on the left hand side:

$$\begin{cases} \color{red}{1}x + \color{red}{1}y + \color{red}{1}z = 4 \\ \color{red}{1}x + \color{red}{3}y + \color{red}{3}z = 10 \\ \color{red}{2}x + \color{red}{1}y + \color{red}{-1}z = 3 \end{cases}$$

The first step in writing the augmented matrix for

$$\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$$

Is to make the coefficient matrix. To do this we look at the system of equations and only write down the coefficients on the left hand side:

$$\begin{cases} {\color{red}1}x + {\color{red}1}y + {\color{red}1}z = 4 \\ {\color{red}1}x + {\color{red}3}y + {\color{red}3}z = 10 \\ {\color{red}2}x + {\color{red}1}y + {\color{red}-1}z = 3 \end{cases}$$

Note that I moved the negative to the coefficient of the z variable in the last equation.

$$\begin{cases} 1x & + & 1y & + & 1z & = & 4 \\ 1x & + & 3y & + & 3z & = & 10 \\ 2x & + & 1y & + & -1z & = & 3 \end{cases}$$

Writing these coefficients as a matrix looks like this:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & -1 \end{bmatrix}$$

Next we need to write the constants as a column vector. A vector is a kind of matrix that is either a single column or single row. Looking at the constants in the system of equations:

$$\begin{cases} x + y + z = 4 \\ x + 3y + 3z = 10 \\ 2x + y - z = 3 \end{cases}$$

Next we need to write the constants as a column vector. A vector is a kind of matrix that is either a single column or single row. Looking at the constants in the system of equations:

$$\left\{\begin{array}{rcrcrcr} x & + & y & + & z & = & 4 \\ x & + & 3y & + & 3z & = & 10 \\ 2x & + & y & - & z & = & 3 \end{array}\right.$$

we get the column vector

$$\begin{bmatrix} 4 \\ 10 \\ 3 \end{bmatrix}$$

Finally to form the augmented matrix we take the coefficient matrix and the constant vector:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 10 \\ 3 \end{bmatrix}$$

Finally to form the augmented matrix we take the coefficient matrix and the constant vector:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 \\ 10 \\ 3 \end{bmatrix}$$

and combine them into the augmented matrix form separating the coefficients from the constants by a vertical line:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 1 & 3 & 3 & 10 \\ 2 & 1 & -1 & 3 \end{array}\right]$$

Ex 01 Make the augmented matrix

$$\left\{\begin{array}{rcrcrcr} x & & & - & 4z & = & 1 \\ 2x & - & y & - & 6z & = & 4 \\ 2x & + & 3y & - & 2z & = & 8 \end{array}\right.$$

Ex 02 Make the augmented matrix.

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 0 \\ & & y & - & z & = & 3 \\ & & & & z & = & -2 \end{array}\right.$$

A system of linear equations and a reduced matrix for the system are given.

$$\left\{\begin{array}{rcrcrcr} x & - & 3y & + & z & = & 0 \\ & & y & - & z & = & 3 \\ & & & & z & = & -2 \end{array}\right. \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

1. Use the reduced matrix to find the general solution of the system, if one exists. (If there is no solution, enter NO SOLUTION. If there are infinitely many solutions, express your answers in terms of z as in Example 3.)
2. If multiple solutions exist, find two specific solutions. (Enter your answers as a comma-separated list of ordered triples. If there is no solution, enter NO SOLUTION.)

Ex HarMathAp12 3.3.017 A system of linear equations and a reduced matrix for the system are given.

$$\left\{\begin{array}{rcrcrcl} x & + & 3y & + & 2z & = & 4 \\ 3x & - & y & & & = & 2 \\ x & + & 3y & + & 2z & = & 5 \end{array}\right. \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & \frac{3}{5} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

1. Use the reduced matrix to find the general solution of the system, if one exists. (If there is no solution, enter NO SOLUTION. If there are infinitely many solutions, express your answers in terms of z as in Example 3.)
2. If multiple solutions exist, find two specific solutions. (Enter your answers as a comma-separated list of ordered triples. If there is no solution, enter NO SOLUTION.)

Ex HarMathAp12 3.3.020 A system of linear equations and a reduced matrix for the system are given.

$$\begin{cases} x - y + z = 5 \\ 3x + 2z = 13 \\ x - 4y + 2z = 7 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

1. Use the reduced matrix to find the general solution of the system, if one exists. (If there is no solution, enter NO SOLUTION. If there are infinitely many solutions, express your answers in terms of z as in Example 3.)
2. If multiple solutions exist, find two specific solutions. (Enter your answers as a comma-separated list of ordered triples. If there is no solution, enter NO SOLUTION.)

Definition 1 (reduced row echelon form (reduced form))

If the matrix is in row echelon form, the first nonzero entry of each row is equal to 1 and the ones above it within the same column equal 0.

1. The first nonzero element in each row is 1.
2. Every column containing a first nonzero element for some row has zeros everywhere else.
3. The first nonzero element of each row is to the right of the first nonzero element of every row above it.
4. All rows containing zeros are grouped together below the rows containing nonzero entries.

So

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is in reduced row echelon form (rref)

If this is in rref,

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

what would an example (of many possible) of not being in rref look like?

If this is in rref,

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{3} & \frac{13}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

what would an example (of many possible) of not being in rref look like?
This is not in rref:

$$\left[\begin{array}{ccc|c} 1 & \color{red}{1} & \frac{1}{3} & \frac{11}{3} \\ 0 & 1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Chapter 3 Section 3 Numerical

Let's solve this system of equations numerically using the programming language Julia.

$$\left\{\begin{array}{rcrcrcr} 2x & - & 6y & - & 12z & = & -20 \\ 3x & - & 10y & - & 20z & = & -38 \\ 2x & & & - & 17z & = & -40 \end{array}\right.$$

We already know how to write the coefficient matrix and constant vector:

$$A = \begin{bmatrix} 2 & -6 & -12 \\ 3 & -10 & -20 \\ 2 & 0 & -17 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -20 \\ -38 \\ -40 \end{bmatrix}$$

The variable vector looks like:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

With A, $\vec{x}$, and $\vec{b}$ defined as:

$$A = \begin{bmatrix} 2 & -6 & -12 \\ 3 & -10 & -20 \\ 2 & 0 & -17 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -20 \\ -38 \\ -40 \end{bmatrix}$$

We can rewrite

$$\left\{ \begin{array}{rcrcrcr} 2x & - & 6y & - & 12z & = & -20 \\ 3x & - & 10y & - & 20z & = & -38 \\ 2x & & & - & 17z & = & -40 \end{array} \right.$$

as

$$A\vec{x} = \vec{b}$$

Matrix Multiplication

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

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To find their product

$$AB$$

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$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

To find their product

$$AB$$

We do

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Matrix Multiplication

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

To find their product

$$AB$$

We do

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

This is *messy*!

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

Have Julia multiply these for you.

Identity matrix

An identity matrix is a square matrix with 1s along the diagonal and zeros everywhere else.

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$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ldots$$

Inverse matrix

If you have a square matrix A sometimes you can find an inverse for that matrix A^{-1}.

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If you have a square matrix A sometimes you can find an inverse for that matrix A^{-1}.
If A and A^{-1} are inverse matrices then

$$AA^{-1} = A^{-1}A = I$$

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2   -6  -12
3  -10  -20
2    0  -17
```

```
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3×3 MatrixInt64:
2   -6  -12
3  -10  -20
2    0  -17

julia> Ainv = inv(A)
3×3 MatrixFloat64:
5.0        -3.0         2.22045e-16
0.323529   -0.294118    0.117647
0.588235   -0.352941   -0.0588235
```

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
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julia> Ainv*A
3×3 MatrixFloat64:
1.0           -3.55271e-15  -1.08802e-14
1.11022e-16    1.0            0.0
-1.52656e-16    0.0            1.0
```

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
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```

The reason we aren't getting 0s in all the off diagonals is round off errors which is always a major concern when working with mathematics on computers.

Solving $ax = b$ using the above idea of a multiplicative inverse we get:

$$\begin{aligned} ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

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So back to

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ Ix &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

To solve this system

$$\begin{cases} 2x - 6y - 12z = -20 \\ 3x - 10y - 20z = -38 \\ 2x - 17z = -40 \end{cases}$$

we first need to make the coefficient matrix

```
julia> A = [2 -6 -12; 3 -10 -20; 2 0 -17]
3×3 MatrixInt64:
2 -6 -12
3 -10 -20
2 0 -17
```

and the constant vector

```
julia> b = [-20, -38, -40]
3-element VectorInt64:
-20
-38
-40
```

Now we can solve for x using Julia

```
julia> x = Ainv*b
3-element VectorFloat64:
13.999999999999991
0.0
4.000000000000002
```

Since this is such an important and often used sequence of operations, Julia provides an operator to do this:

```
julia> x = A\b
3-element VectorFloat64:
14.000000000000009
1.0658141036401504e-15
4.000000000000001
```

To solve

$$\begin{cases} x + 2y + 3z = 1 \\ 2x - y = 2 \\ x + 2y + 3z = 2 \end{cases}$$

```
julia> A = [1 2 3; 2 -1 0; 1 2 3]
3×3 MatrixInt64:
1 2 3
2 -1 0
1 2 3

julia> b = [1;2;2]
3-element VectorInt64:
1
2
2
```

```
julia> x = A\b
ERROR: LinearAlgebra.SingularException(3)
Stacktrace:
[1] checknonsingular
@ \.julia\...\src\factorization.jl:68 [inlined]
...
```

To avoid this issue we use the determinant function

```
det
```

from the package `LinearAlgebra`. To add a package you can follow *doggo dot jl*'s video instructions at the clickable link: [10x27] How to use External Packages in Julia

Load the `LinearAlgebra` using:

```
using LinearAlgebra #Needed for the determinant function: det(A)
```

Looking at three examples:

```
A = [1 -3 1; 0 1 -1; 0 0 1]
det(A)
```

This gives the result

```
3×3 MatrixInt64:
1 -3 1
0 1 -1
0 0 1
1.0
```

Since the determinant $\det(A) \neq 0$ we can solve using the $\backslash$ operator.

But for the following two examples:

```
A = [1 1;1 1 ]
det(A)
```

This gives the result

```
2×2 MatrixInt64:
1 1
1 1

0.0
```

```
A = [1 1;2 2]
det(A)
```

This gives the result

```
2×2 MatrixInt64:
1 1
2 2

0.0
```

In both of these cases the determinant $\det(A) = 0$ we cannot solve using the $\backslash$ operator.
For now we will just say that there is no unique solution and will come back later to dissect what's happening.

It is now later...

Recall that

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

has one solution.

```
using LinearAlgebra
A = [1 1; 1 2]
b = [2, -1]
det(A)
> 1.0
x, y = A\b
> 2-element VectorFloat64:
 5.0
-3.0
```

And that

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

has infinitely many solutions since they are actually the same lines.

```
A = [1 1; 2 2]
b = [1, 2]
det(A)
> 0.0
```

And that

$$\begin{cases} x + y = 2 \\ x + y = -1 \end{cases}$$

has no solutions because the lines are parallel.

```
A = [1 1; 1 1]
b = [2, -1]

det(A)
> 0.0
```

There is a difference between the second and third systems, but we cannot tell that from the determinant. How can we tell the difference?

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We need to rewrite the system of equations into augmented matrix form and then run the command *rref* on the augmented matrix.

There is a difference between the second and third systems, but we cannot tell that from the determinant. How can we tell the difference?
We need to rewrite the system of equations into augmented matrix form and then run the command *rref* on the augmented matrix.
To use the *rref* command you will need to add and load (*using*) the package *RowEchelon.jl*.

How are we going to solve systems of linear equations?

1. Form the
    1.1 coefficient matrix A
    1.2 constant vector b
    1.3 augmented matrix A_{aug}
2. Take the determinant of A
3. If
    3.1 $\det(A) \neq 0$: Solve for x using $x = A\backslash b$
    3.2 $\det(A) = 0$: Solve using rref(A_{aug})
4. Is the system consistent or inconsistent? If inconsistent is it dependent or independent?
5. How many solutions does this system have?
6. What are they? If more than 1 then give two specific solutions.

Ex 1 Solve

$$\left\{\begin{array}{rcrcr} x & + & y & = & 2 \\ x & - & 2y & = & -1 \end{array}\right.$$

Ex 2 Solve

$$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Ex 3 Solve

$$\begin{cases} x & + & y & = & 2 \\ x & + & y & = & -1 \end{cases}$$

Ex 4 Solve

$$\begin{cases} x + 2y + z = 2 \\ 2x + y + 2z = 1 \\ 3x + 3y + 3z = 3 \end{cases}$$

Ex 5 Solve

$$\begin{cases} x + 2y + z = 2 \\ 2x + y + 2z = 1 \\ 3x + 3y + 3z = 0 \end{cases}$$

Give me a 3×3 augmented matrix that is

1. Independent

2. Dependent, Consistent

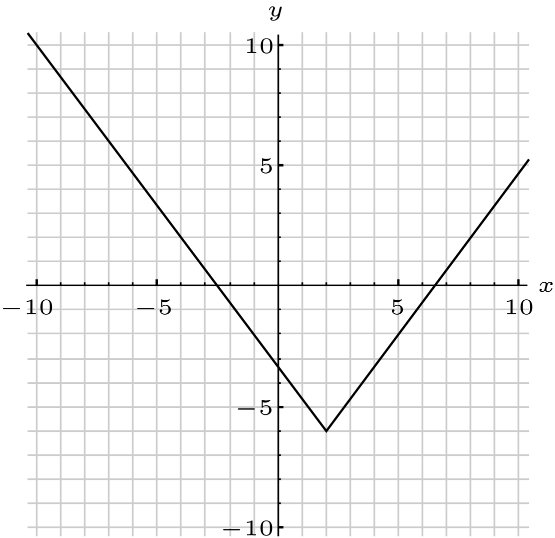
3. Dependent, Inconsistent

Chapter 9 Section 1

Ex 9.1.002

a) $\lim_{x \to 2} f(x)$

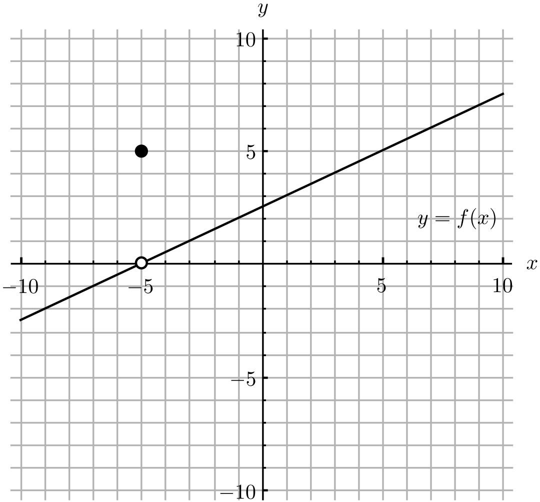
b) $f(2)$



Ex 9.1.005

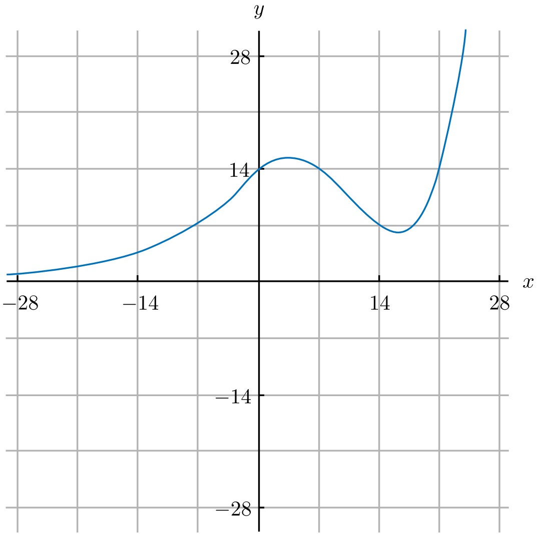
a) $\lim_{x \to -5} f(x)$

b) $f(-5)$



Ex WANEAC7 3.1.037

a) $\lim\limits_{x\to 0} f(x)$ b) $\lim\limits_{x\to 14} f(x)$ c) $\lim\limits_{x\to -\infty} f(x)$ d) $\lim\limits_{x\to \infty} f(x)$



Ex WANEAC7 3.1.039

a) $\lim\limits_{x\to 12} f(x)$

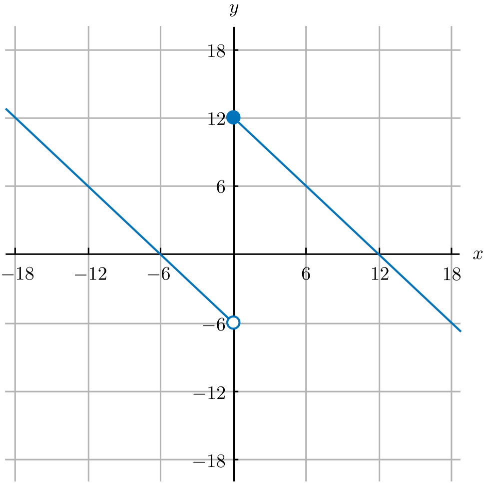
b) $\lim\limits_{x\to 0^+} f(x)$

c) $\lim\limits_{x\to 0^-} f(x)$

d) $\lim\limits_{x\to 0} f(x)$

e) $f(0)$

f) $\lim\limits_{x\to -\infty} f(x)$



Ex WANEAC7 3.1.045

a) $\lim_{x \to -2} f(x)$

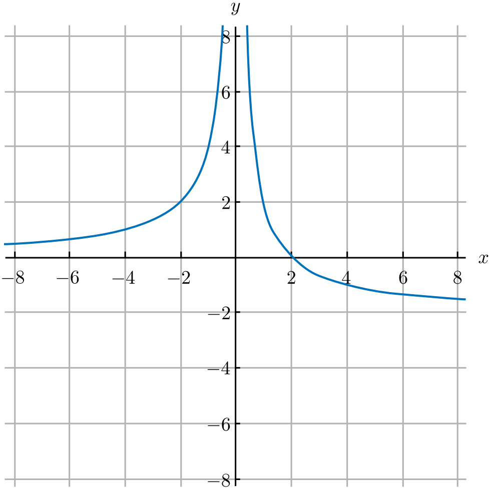
b) $\lim_{x \to 0^+} f(x)$

c) $\lim_{x \to 0^-} f(x)$

d) $\lim_{x \to 0} f(x)$

e) $f(0)$

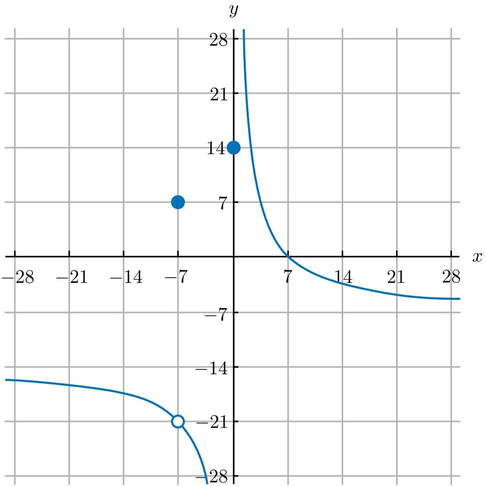
f) $\lim_{x \to \infty} f(x)$



Ex WANEAC7 3.1.047

a) $\lim_{x \to -7} f(x)$ b) $\lim_{x \to 0^+} f(x)$ c) $\lim_{x \to 0^-} f(x)$

d) $\lim_{x \to 0} f(x)$ e) $f(0)$ f) $f(-7)$

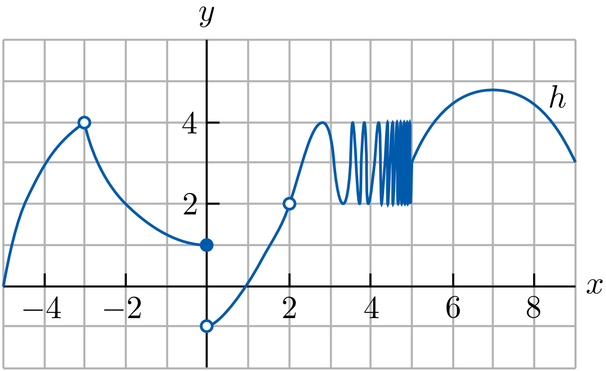


Ex SCALC9 1.5.006

a) $\lim_{x \to -3^-} h(x)$ b) $\lim_{x \to -3^+} h(x)$ c) $\lim_{x \to -3} h(x)$ d) $h(-3)$

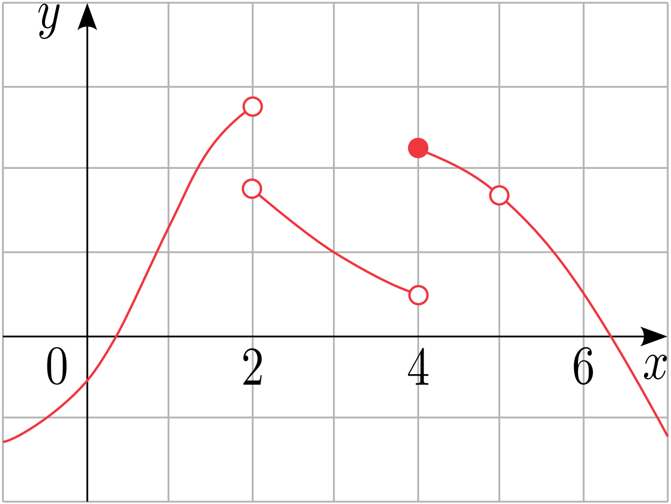
e) $\lim_{x \to 0^-} h(x)$ f) $\lim_{x \to 0^+} h(x)$ g) $\lim_{x \to 0} h(x)$ h) $h(0)$

i) $\lim_{x \to 2} h(x)$ j) $h(2)$ k) $\lim_{x \to 5^+} h(x)$ l) $\lim_{x \to 5^-} h(x)$



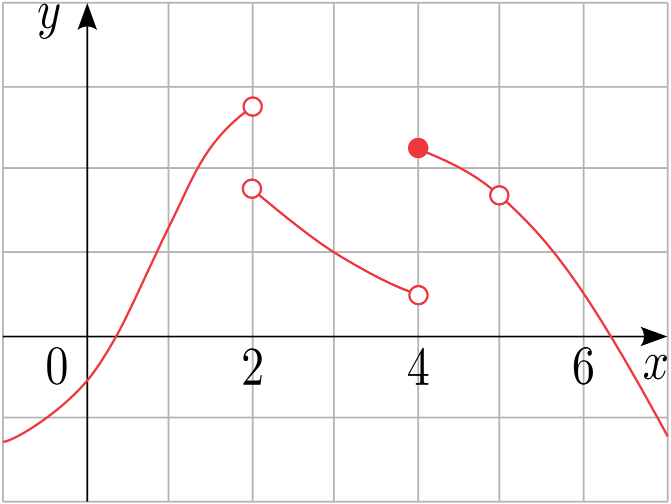
Ex SCALC9 1.5.007 (a) Find a such that the following is true.

$$\lim_{x \to a} g(x) \text{ does not exist but } g(a) \text{ is defined.}$$



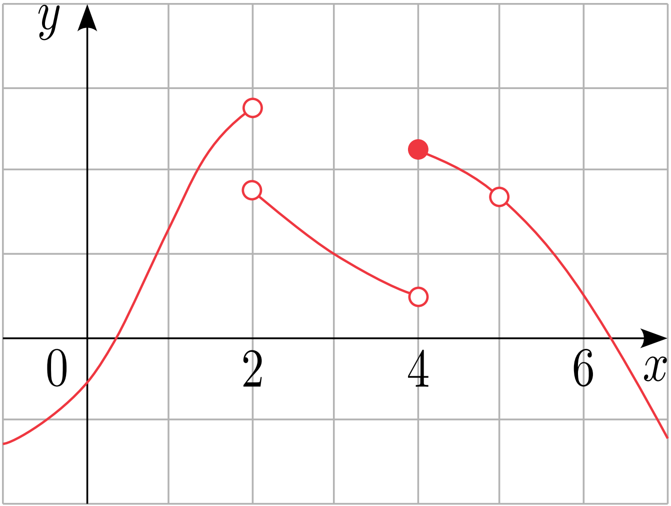
Ex SCALC9 1.5.007 (b) Find a such that the following is true.

$$\lim_{x \to a} g(x) \text{ exists but } g(a) \text{ is not defined.}$$



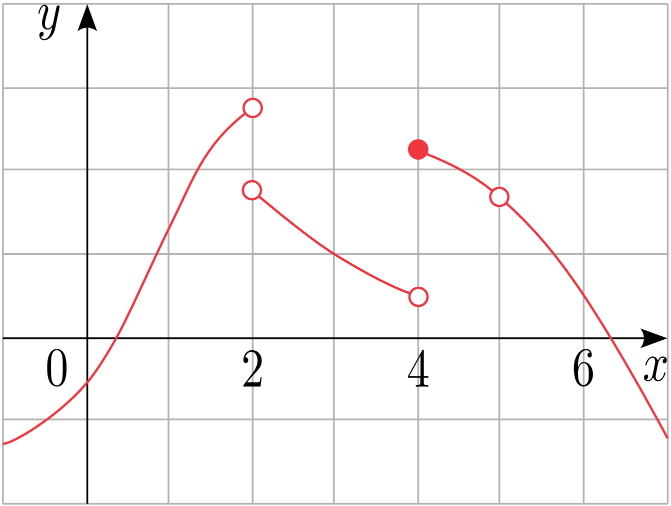
Ex SCALC9 1.5.007 (c) Find a such that the following is true.

$$\lim_{x\to a^-} g(x) \text{ and } \lim_{x\to a^+} g(x) \text{ both exist but } \lim_{x\to a} g(x) \text{ does not exist.}$$



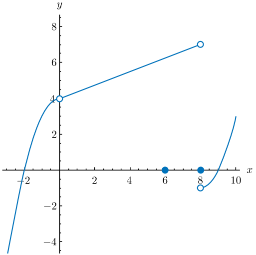
Ex SCALC9 1.5.007 (d) Find a such that the following is true.

$$\lim_{x \to a^+} g(x) = g(a) \text{ but } \lim_{x \to a^-} g(x) \neq g(a).$$

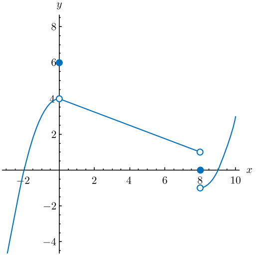


Ex SCALC9 1.5.016 Which graph satisfies the following:

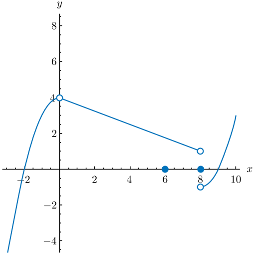
$$\lim_{x\to 0} f(x) = 4,\ \lim_{x\to 8^-} f(x) = 1,\ \lim_{x\to 8^+} f(x) = -1,\ f(0) = 6,\ f(8) = 0$$



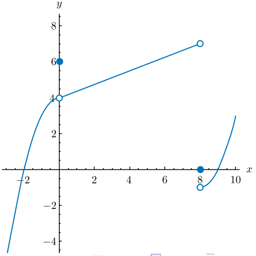
a)



b)



c)



d)

Properties of Limits

If k is a constant, $\lim_{x \to c} f(x) = L$, and $\lim_{x \to c} g(x) = M$, then the following are true.

I. $\lim_{x \to c} k = k$

II. $\lim_{x \to c} x = c$

III. $\lim_{x \to c} [f(x) \pm g(x)] = L \pm M$

IV. $\lim_{x \to c} [f(x) \cdot g(x)] = LM$

V. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M} \quad \text{if} \quad M \neq 0$

VI. $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)} = \sqrt[n]{L}$, provided $L > 0$ when n is even.

- Let's find the limit of the following with these new rules:

$$\lim_{x \to 3} (2x - 5)$$

- Let's find the limit of

$$\lim_{x \to 5} (x^2 - 3x - 4)$$

- Let's find the limit of

$$f(x) = \begin{cases} 2x + 3 & , x < 0 \\ -x^2 - 2 & , x \geq 0 \end{cases}$$

as $x \to 0$:

$$\lim_{x \to 0} f(x)$$

- Let's find the limit of
$$\lim_{x\to 0} \frac{1}{x}$$
- Let's find the limit of
$$\lim_{x\to 0^+} \frac{1}{x}$$
- Let's find the limit of
$$\lim_{x\to 0^-} \frac{1}{x}$$
- Let's find the limit of
$$\lim_{x\to 0} \frac{1}{x^2}$$

Rational Functions: Evaluating Limits of the Form $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ where $\lim\limits_{x\to c} g(x) = 0$

Type I. If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ has the **0/0 indeterminate form** at $x = c$. We can factor $x - c$ from $f(x)$ and $g(x)$, reduce the fraction, and then find the limit of the resulting expression, if it exists.

Type II. If $\lim\limits_{x\to c} f(x) \neq 0$ and $\lim\limits_{x\to c} g(x) = 0$, then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}$ does not exist. In this case, the values of $f(x)/g(x)$ become unbounded as x approaches c; the line $x = c$ is a vertical asymptote.

Let's find the limit of

$$\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$$