MS 302 In-class Problems

June 10, 2025

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Chapter 1 Section 1

Suppose that you produce structural metal bars by heat treating them using induction heating and that they must withstand 10,000 lbs.

You set as your goal a breaking point of 15,000 lbs.

Will the breaking point be exactly 15,000 lbs?

Almost every procedure we deal with has some form of variation built into it.

1. The failure point of structral metal bars
2. How long it takes to brush your teeth in the morning
3. The location that a basketball hits as you practice your free throw shot.

In order to understand and maybe even control the *uncertainty* and *variation* we use

Definition 1 (statistical inference)

*The drawing of inferences about a population based on data taken from a sample of that population**.

Two *factors* that contribute to the strength of a bar in our process are temperature and time in the heat treatment.

Definition 2 (factors)

are properties or characteristics of a population.

*OED

To determine how much weight our bars can support we could put them into a machine that destructively tests them. But...?

Definition 3 (Population)

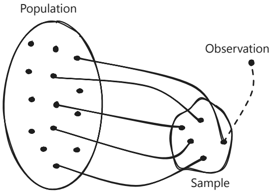
collection of all individuals in a particular scientific system or process.

Definition 4 (Sample)

collection of observations from the population.

Definition 5 (Observation)

individual or item of a particular type involved in the scientific system or process.



In addition to *Inferential statistics* there is *Descriptive statistics*.

Definition 6 (Descriptive statistics)

are used when seeking only to gain some summary of a set of data represented by a sample (single-number statistics that provide a sense of center, variability, or general nature of the distribution of the sample data).

An understanding of probability provides a basis to understand statistical inference.

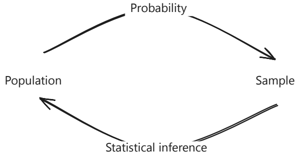
Ex The average height (in inches) of a male JSU student is $\mu_M = 68$ and the standard deviation is $\sigma_M = 3$ and for women $\mu_F = 63$ and $\sigma_F = 2.7$. Now suppose that someone believes that the average male JSU student height is $\mu_M = 63$. How can we show that this is not true without measuring almost every male JSU student? We can show that it is *probably* not true using statistical inference. Take a sample of heights

$$69.8, 68.0, 70.2, 70.6, 65.4, 72.0, 73.3, 63.1, 73.5, 69.1, 66.8, 64.8,$$

$$64.0, 66.4, 65.1, 70.8, 75.4, 71.5, 67.1, 66.0, 64.1, 64.9, 69.4, 68.2, 69.0$$

This $n = 25$ sample data set has a sample mean $\bar{x} = 68.4$ and sample standard deviation $s = 3.3$, but let's use $\sigma_M = 3$.

Population with know features + Probability: allow us to draw conclusions about characteristics of hypothetical data taken from the population.



Sample + Inferential Statistics: allow us to draw conclusions about the population.

Chapter 1 Section 2

Note to myself: Open mpg.csv

Definition 7 (Simple Random Sample)

Given a specified sample size, then every sample is as likely to be chosen as any other sample.

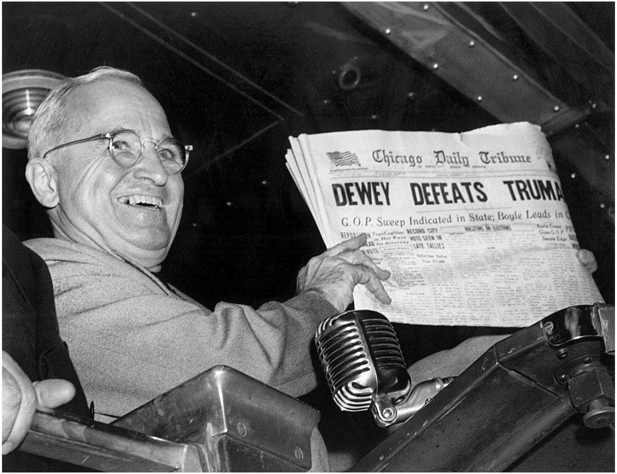
Ex Suppose that a city has two malls and you want to determine on average how many people visit each store per day. You could chose to use a sample of size of 10 and then

1. chose a mall at random
2. pick 10 stores at random.

While this is a *random sample* it is not a simple random sample. Why?
Because while every store has an equally likely chance of being chosen, there is no way to get a store from mall 1 and mall 2 in a sample at the same time. Therefore there are samples that could never be chosen.

Definition 8 (Biased Sample)

is a sample that does not accurately represent the population (it over/under-represents some segment of the population).



Chapter 1 Section 3
Measures of Location

Definition 9 (Sample mean)

If the observations in a sample are $x_1, x_2, \ldots, x_n$, then the sample mean is

$$\bar{x} = \frac{\sum x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Problem is that it is influenced by outliers. Note that as a single value gets larger so does the mean.

Ex Find the sample mean of

$$50000, 30000, 45000, 33000, 47000, 51000, 6744000$$

Definition 10 (Sample median)

If the observations in a sample are $x_1, x_2, \ldots, x_n$ *are arranged in* **increasing** *order, then the sample median is*

$$\tilde{x} = \begin{cases} x_{\frac{n+1}{2}}, & \text{if } n \text{ odd} \\ \frac{1}{2}\left(x_{\frac{n}{2}} + x_{\frac{n}{2}+1}\right), & \text{if } n \text{ even} \end{cases}$$

Ex Find the sample median of

$$50000, 30000, 45000, 33000, 47000, 51000, 6744000$$

Chapter 1 Section 4
Measures of Variability

Both data sets

$$1, 2, 3, 7, 8, 9$$

and

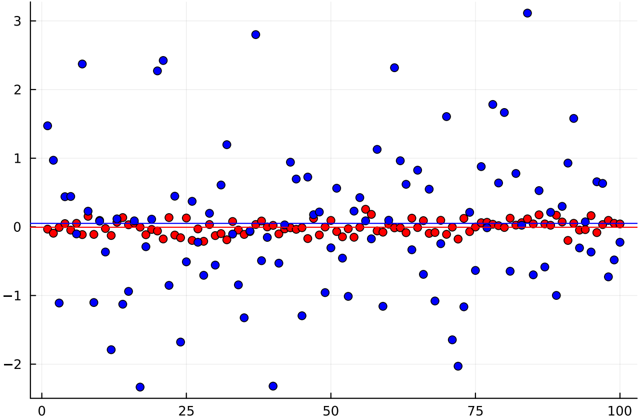
$$1, 1, 1, 9, 9, 9$$

have a mean of 5.
But the data sets are clearly different.

These two data sets of 100 points each have slightly different centers (means)

$$\mu_{red} = -0.006, \quad \mu_{blue} = 0.051$$

but there spreads are very different.



How should we measure this variation in the data?

Definition 11

If the observations in a sample are $x_1, x_2, \ldots, x_n$*, then*

sample range $x_{\mathit{max}} - x_{\mathit{min}}$

sample variance

$$s^2 = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{n-1}$$

sample standard deviation

$$s = \sqrt{s^2}$$

Ex Find the sample range, variance, and standard deviation for

$$1.7, 2.2, 3.9, 3.11, 14.7$$

Chapter 1 Section 5
Discrete and Continuous Distributions

Definition 12 (Discrete data)

can only take on certain values.

Example 5.1

1. *Number of students in my class, e.g.* $0, 1, 2, \ldots, 35$
2. *FM radio station frequencies can be from 88.1 MHz to 108.1 (or 107.9) MHz with a step size of 0.2 MHz, e.g. 106.9, 100.7, 103.5, 95.3, 95.1*

Definition 13 (Continuous data)

can take on any value in a range.

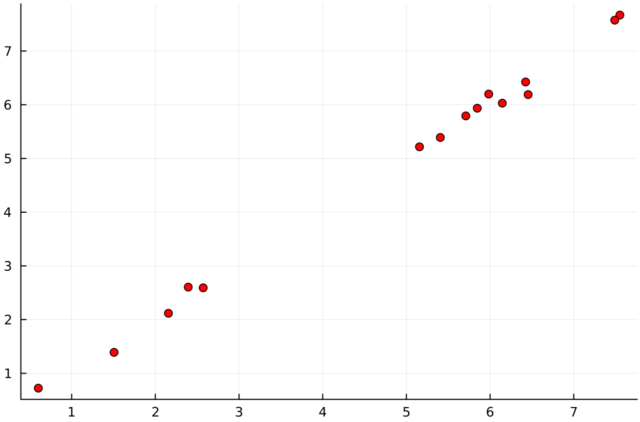
Example 5.2

Heights of adult humans can be anything from 20 *inches to* 110 *inches.*

Chapter 1 Section 6

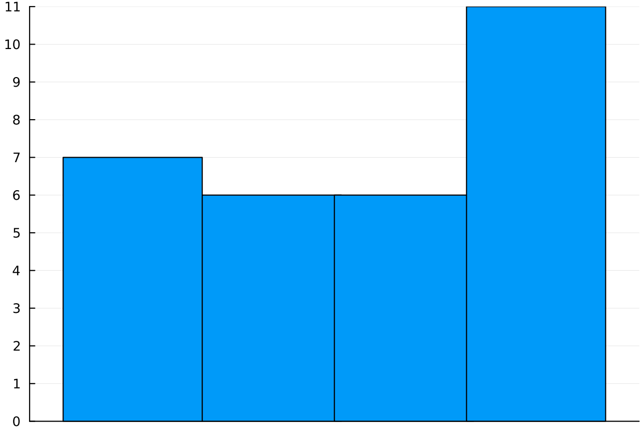
Scatter plot

x	y
0.601	0.723
1.507	1.390
2.156	2.118
2.393	2.605
2.572	2.592
5.156	5.216
5.405	5.391
5.710	5.793
5.847	5.935
5.984	6.199
6.146	6.030
6.425	6.424
6.455	6.190
7.490	7.575
7.551	7.670



Histogram

Sam data as before:
$20, 20, 20, 21, 22, 23, 27, 33, 33, 35, 38, 39, 39, 45, 46, 47, 47, 48, 48,$
$50, 52, 52, 53, 53, 54, 54, 56, 56, 56, 58$



Steam and leaf

Ordered data:
$20, 20, 20, 21, 22, 23, 27, 33, 33, 35, 38, 39, 39, 45, 46, 47, 47, 48, 48, 50, 52, 52, 53, 53, 54, 54, 56, 56, 56, 58$
The stem are the digits excluding the least significant digit and they lie on the left of the plot.
The leaves are what remains after removing the stem and are written to the right and are vertically aligned.

2	0001237
3	335899
4	567788
5	02233446668

Key $3|5 = 35$

Chapter 2 Section 1

Definition 14 (Experiment)

is any process that generates a set of data.

Definition 15 (Sample space)

is the set of all possible outcomes of a statistical experiment.

It is denoted by S.

Ex Flip a coin. What is the sample space as a set and a tree.

Ex Flip a coin twice. What is the sample space as a set and a tree.

Ex Flip a coin, if H roll a 4-sided die, else flip the coin again. What is the sample space as a set and a tree.

Ex Flip a coin until you get a H. What is the sample space as a set and a tree.

Ex Consider the set S to be all of the points that make up a circle with radius 5 that is centered at the origin.
List the sample space.

Chapter 2 Section 2

Definition 16 (Event)
is a subset of a sample space.

Definition 17 (Null (Empty) Set)
is the set with no elements.

Denoted by $\emptyset$.

Ex Flip a coin. List all events.

Ex Flip a coin twice.

1. $E_1 \equiv$ flip exactly one H.
2. $E_2 \equiv$ flip at least one H.
3. $E_3 \equiv$ the second flip is the same as the first.
4. $E_4 \equiv$ the coin lands on it's edge.

Ex Let $S = \{x : x \geq 0\}$ be the distance Forrest Gump ran before he tripped.
What is the event:

1. A that he trips after the 50^{th} mile and before or at the 90^{th} mile?
2. B that he never trips?

Definition 18 (Complement)

of the event A with respect to S is the set of all elements in S not in A.

Denoted as A'

Ex 2.15 Consider the sample space

$$S = \{\text{copper, sodium, nitrogen, potassium, uranium, oxygen, zinc}\}$$

and the events

$$A = \{\text{copper, sodium, zinc}\},$$
$$B = \{\text{sodium, nitrogen, potassium}\},$$
$$C = \{\text{oxygen}\}.$$

List the elements of the sets corresponding to the following events:

(a) A'

Definition 19 (Intersection of the events A and B)

is the set

$$A \cap B = \{x | x \in A \text{ and } x \in B\}$$

Ex 2.15 Consider the sample space

$S = \{\text{copper, sodium, nitrogen, potassium, uranium, oxygen, zinc}\}$

and the events

$$A = \{\text{copper, sodium, zinc}\},$$
$$B = \{\text{sodium, nitrogen, potassium}\},$$
$$C = \{\text{oxygen}\}.$$

List the elements of the sets corresponding to the following events:

1. $A \cap B$

Definition 20 (Mutually disjoint)

Events A and B are mutually disjoint if $A \cap B = \emptyset$.

Ex 2.15 Consider the sample space

$$S = \{\text{copper, sodium, nitrogen, potassium, uranium, oxygen, zinc}\}$$

and the events

$$A = \{\text{copper, sodium, zinc}\},$$
$$B = \{\text{sodium, nitrogen, potassium}\},$$
$$C = \{\text{oxygen}\}.$$

Which pairs of A, B, and C are disjoint?

Definition 21 (Union of the events A and B)

is the set

$$A \cup B = \{x | x \in A \text{ or } x \in B\}$$

Ex 2.15 Consider the sample space

Consider the sample space

$$S = \{\text{copper, sodium, nitrogen, potassium, uranium, oxygen, zinc}\}$$

and the events

$$A = \{\text{copper, sodium, zinc}\},$$
$$B = \{\text{sodium, nitrogen, potassium}\},$$
$$C = \{\text{oxygen}\}.$$

List the elements of the sets corresponding to the following events:

(b) $A \cup C$

(c) $(A \cap B') \cup C'$

Ex 2.15

(d) $B' \cap C'$

(e) $A \cap B \cap C$

(f) $(A' \cup B') \cap (A' \cap C)$

Chapter 2 Section 3
Counting

If we flip a fair coin, what is the probability of getting a head?

$$P(H) = \frac{1}{2}$$

We will see that the way we calculate probabilities is that

$$P(E) = \frac{\text{how many ways to get the event } E}{\text{total number of outcomes}}$$

So what we need to figure out how to do is count.

If we flip a coin and then roll a 20-sided die, how many outcomes are possible?

Definition 22 (Multiplication rule)

If there are n_1 *ways to perform an operation and each of these are followed by* n_2 *ways to complete a second operation, then there is a total of*

$$n_1 \cdot n_2$$

ways to perform both operations.

Ex Flip a coin and then roll a 20-sided die.

Ex 2.24 Students at a private liberal arts college are classified as being freshmen, sophomores, juniors, or seniors, and also according to whether they are male or female. Find the total number of possible classifications for the students of that college.

Ex 2.25 A certain brand of shoes comes in 5 different styles, with each style available in 4 distinct colors. If the store wishes to display pairs of these shoes showing all of its various styles and colors, how many different pairs will the store have on display?

Definition 23 (Generalized multiplication rule)

If theree are k different tasks to be accomplished in sequence with n_1 ways to do the first, n_2 ways to do the second, ..., n_k ways to do the k^{th}, then the total number of ways to accomplish these tasks is

$$n_1 \cdot n_2 \cdots n_k.$$

Ex How many ways can you pick a lower case letter, pick a digit, and then flip a coin to get H or T?

Ex 2.27 A developer of a new subdivision offers a prospective home buyer a choice of 4 designs, 3 different heating systems, a garage or carport, and a patio or screened porch. How many different plans are available to this buyer?

Ex Current Alabama license plates consist of a county code (there are 67 counties in Alabama - coded using the format 1 or 10 - but lets pretend we have a single character for each one, think hexadecimal - base 16 - but at least base 67) followed some number/letter combination depending on the year. See Vehicle registration plates of Alabama

1. In 1941 Alabama started using county codes in its license plates. The format was county code, followed by three digits. How many license plates can be issued.

2. Starting in January 2002 it consisted of the county code, a letter, three digits, a letter. How many license plates can be issued.

3. The current rule is it consists of the county code followed by 5 alphanumeric digits. How many license plates can be issued.

Suppose you have 3 books that you want to put on a shelf

Math, Computer Science, Physics

In how many ways can we do this?

Definition 24 (Permutation)

is an arrangement of a set of objects, either all or part of them.

Definition 25 (Factorial)

$$n! = n(n-1)\cdots 1; \quad n \in \mathbb{N}$$

and

$$0! = 1$$

Theorem 1

The number of permutations of n *objects is* $n!$.

Ex 2.32 (a) In how many ways can 6 people be lined up to get on a bus?

Ex 2.34

(a) How many distinct permutations can be made from the letters of the word COLUMNS?

(b) How many of these permutations start with the letter M?

Ex 2.35 A contractor wishes to build 9 houses, each different in design. In how many ways can he place these houses on a street if 6 lots are on one side of the street and 3 lots are on the opposite side?

Ex 2.37 In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

If there are 5 boys and 5 girls.

Sometimes you don't need all n objects.

Definition 26 (Permutation of n objects taken r at a time)

$$_{n}P_{r} = \frac{n!}{(n-r)!}$$

Ex 2.40 In how many ways can 5 starting positions on a basketball team be filled with 8 men who can play any of the positions?

When order does not matter

Definition 27 (Combination of n objects taken r at a time)

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

Ex 2.26 A California study concluded that following 7 simple health rules can extend a man's life by 11 years on the average and a woman's life by 7 years. These 7 rules are as follows: no smoking, get regular exercise, use alcohol only in moderation, get 7 to 8 hours of sleep, maintain proper weight, eat breakfast, and do not eat between meals. In how many ways can a person adopt 5 of these rules to follow

(a) if the person presently violates all 7 rules?

(b) if the person never drinks and always eats breakfast?

Ex 2.22 In a medical study, patients are classified in 8 ways according to whether they have blood type AB^+, AB^-, A^+, A^-, B^+, B^-, O^+, or O^-, and also according to whether their blood pressure is low, normal, or high. Find the number of ways in which a patient can be classified.

Ex How many odd four-digit numbers can be formed from the digits 8, 1, 2, 5, 6, and 9 where each digit is used once?

Ex A traditional poker hand has how many options?

Chapter 2 Section 4
Probability

Definition 28 (Probability)

is a fucntion $P : \mathcal{E} \to [0, 1]$ such that for any event A in $\mathcal{E}$ such that the following are satisfied:

1. $0 \leq P(A) \leq 1$
2. $P(\emptyset) = 0$
3. $P(S) = 1$

and if $A_1, A_2, \ldots$ are mutually exclusive then

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots$$

Ex 2.49 Find the errors in each of the following statements:

1. The probabilities that an automobile salesperson will sell 0, 1, 2, or 3 cars on any given day in February are, respectively,

$$0.19, 0.38, 0.29, \text{ and } 0.15$$

2. The probability that it will rain tomorrow is 0.40, and the probability that it will not rain tomorrow is 0.52.
3. The probabilities that a printer will make 0, 1, 2, 3, or 4 or more mistakes in setting a document are, respectively,

$$0.19, 0.34, -0.25, 0.43, \text{ and } 0.29$$

4. On a single draw from a deck of playing cards, the probability of selecting a heart is $\frac{1}{4}$, the probability of selecting a black card is $\frac{1}{2}$, and the probability of selecting both a heart and a black card is $\frac{1}{8}$.

In this chapter all sample spaces S are finite.
The probability of an event A is the sum of the probability of every element in A.

Ex Flip a fair coin once.

1. Find the sample space.

2. State the set of all events.

3. What are
   3.1 $P(\emptyset) =$

   3.2 $P(\{H, T\}) = P(H, T) =$
4. Derive $P(H)$ and $P(T)$.

Example (pg 53) 2.25 A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If E is the event that a number less than 4 occurs on a single toss of the die, find $P(E)$.

Theorem 2

Let an experiment have N different equally likely outcomes. If the event A consists of exactly n of these outcomes, then

$$P(A) = \frac{n}{N}$$

Ex Flip a coin twice and let $A = \{HT, TH, TT\}$.
Find $P(A)$.

Ex 2.51 A box contains 500 envelopes, of which 75 contain \$100 in cash, 150 contain \$25, and 275 contain \$10. An envelope may be purchased for \$25. What is the sample space for the different amounts of money? Assign probabilities to the sample points and then find the probability that the first envelope purchased contains less than \$100.

Ex 2.52 Statement Suppose that in a senior college class of 500 students it is found that

- 210 smoke,
- 258 drink alcoholic beverages,
- 216 eat between meals,
- 122 smoke and drink alcoholic beverages,
- 83 eat between meals and drink alcoholic beverages,
- 97 smoke and eat between meals, and
- 52 engage in all three of these bad health practices.

Ex 2.52 Questions If a member of this senior class is selected at random, find the probability that the student

(a) smokes but does not drink alcoholic beverages;

(b) eats between meals and drinks alcoholic beverages but does not smoke;

Example (pg 55) 2.28 In a poker hand consisting of 5 cards, find the probability of holding 2 aces and 3 jacks.

Chapter 2 Section 5
Additive Rules

Suppose you have a set $S = \{1, 2, 3, 4, 5, 6\}$.
Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ be two subsets of S.
How many elements are in $A \cup B$?
It's not $|A| + |B| = 6$. Why not?

Suppose you roll a fair 6-sided die.
Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ be two events.
What is $P(A \cup B)$?

Theorem 3

If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Ex 2.56 An automobile manufacturer is concerned about a possible recall of its best-selling four-door sedan. If there were a recall, there is a probability of 0.25 of a defect in the brake system, 0.18 of a defect in the transmission, 0.17 of a defect in the fuel system, and 0.40 of a defect in some other area.

(a) What is the probability that the defect is the brakes or the fueling system if the probability of defects in both systems simultaneously is 0.15?

If A and B are mutually exclusive, what is $P(A \cap B)$?

Corollary 1

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B)$$

Corollary 2

If $A_1, A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Definition 29 (Partition)

of a set S is a collection of subsets

$$\{A_1, \dots, A_n\}$$

such that

$$A_1 \cup \cdots \cup A_n = S$$

and $A_1, \cdots, A_n$ are mutually exclusive.

Theorem 4

If A, B, C are any three events, then

$$\begin{aligned} P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ + P(A \cap B \cap C) \end{aligned}$$

What can you tell me about the two events A and A' of S?
They partition S.

Theorem 5

Let A and A' be events of S, then

$$P(A) + P(A') = 1$$

Ex 2.52 Suppose that in a senior college class of 500 students it is found that 210 smoke, 258 drink alcoholic beverages, 216 eat between meals, 122 smoke and drink alcoholic beverages, 83 eat between meals and drink alcoholic beverages, 97 smoke and eat between meals, and 52 engage in all three of these bad health practices.
If a member of this senior class is selected at random, find the probability that the student

(c) neither smokes nor eats between meals.

Ex 2.56 An automobile manufacturer is concerned about a possible recall of its best-selling four-door sedan. If there were a recall, there is a probability of 0.25 of a defect in the brake system, 0.18 of a defect in the transmission, 0.17 of a defect in the fuel system, and 0.40 of a defect in some other area.

(b) What is the probability that there are no defects in either the brakes or the fueling system?

Ex 2.58 A pair of fair dice is tossed. Find the probability of getting

(a) a total of 8;

(b) at most a total of 5.

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)
\end{array}$$

Ex 2.59 In a poker hand consisting of 5 cards, find the probability of holding

(a) 3 aces;

(b) 4 hearts and 1 club.

Ex 2.62 Dom's Pizza Company uses taste testing and statistical analysis of the data prior to marketing any new product. Consider a study involving three types of crusts (thin, thin with garlic and oregano, and thin with bits of cheese). Dom's is also studying three sauces (standard, a new sauce with more garlic, and a new sauce with fresh basil).

(a) How many combinations of crust and sauce are involved?

(b) What is the probability that a judge will get a plain thin crust with a standard sauce for his first taste test?

Ex 2.64 Interest centers around the life of an electronic component. Suppose it is known that the probability that the component survives for more than 6000 hours is 0.42. Suppose also that the probability that the component survives no longer than 4000 hours is 0.04.

(a) What is the probability that the life of the component is less than or equal to 6000 hours?

(b) What is the probability that the life is greater than 4000 hours?

Ex 2.65 Consider the situation of Exercise 2.64. Let A be the event that the component fails a particular test and B be the event that the component displays strain but does not actually fail. Event A occurs with probability 0.20, and event B occurs with probability 0.35.

(a) What is the probability that the component does not fail the test?

(b) What is the probability that the component works perfectly well (i.e., neither displays strain nor fails the test)?

(c) What is the probability that the component either fails or shows strain in the test?

Ex 2.72 Prove that

$$P(A' \cap B') = 1 + P(A \cap B) - P(A) - P(B)$$

Chapter 2 Section 6
Conditional Probability, Independence, Product Rule

Suppose that I have a screen setup so that you cannot see me rolling a fair 6-sided die. I roll the die.
What is $P(2) =$?

Now suppose that I rolled the 6-sided die and tell you that I rolled an even number. Now what is $P(2) =$?

Definition 30 (Conditional probability of B given A)

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Ex 2.77 In the senior year of a high school graduating class of 100 students, 42studied mathematics, 68 studied psychology, 54 studied history, 22 studied both mathematics and history, 25 studied both mathematics and psychology, 7 studied history but neither mathematics nor psychology, 10 studied all three subjects, and 8 did not take any of the three. Randomly select a student from the class and find the probabilities of the following events.

(a) A person enrolled in psychology takes all three subjects.

(b) A person not taking psychology is taking both history and mathematics.

Ex 2.79 Statement In USA Today (Sept. 5, 1996), the results of a survey involving the use of sleepwear while traveling were listed as follows:

	Male	Female	Total
Underwear	0.220	0.024	0.244
Nightgown	0.002	0.180	0.182
Nothing	0.160	0.018	0.178
Pajamas	0.102	0.073	0.175
T-shirt	0.046	0.088	0.134
Other	0.084	0.003	0.087

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It is often convenient to also have column totals so...

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Nightgown	0.002	0.180	0.182
Nothing	0.160	0.018	0.178
Pajamas	0.102	0.073	0.175
T-shirt	0.046	0.088	0.134
Other	0.084	0.003	0.087
Total	0.614	0.386	1.000

Ex 2.79 Questions

(a) What is the probability that a traveler is a female who sleeps in the nude?

(b) What is the probability that a traveler is male?

(c) Assuming the traveler is male, what is the probability that he sleeps in pajamas?

(d) What is the probability that a traveler is male if the traveler sleeps in pajamas or a T-shirt?

Ex 2.80 The probability that an automobile being filled with gasoline also needs an oil change is 0.25; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and the filter need changing is 0.14.

(a) If the oil has to be changed, what is the probability that a new oil filter is needed?

(b) If a new oil filter is needed, what is the probability that the oil has to be changed?

Theorem 6

If events A and B can both occur in any exeriment, then

$$P(A \cap B) = P(B|A)P(A), \quad \text{if } P(A) > 0.$$

Ex 2.81 The probability that a married man watches a certain television show is 0.4, and the probability that a married woman watches the show is 0.5. The probability that a man watches the show, given that his wife does, is 0.7. Find the probability that

(a) a married couple watches the show;

(b) a wife watches the show, given that her husband does;

(c) at least one member of a married couple will watch the show.

Roll a fair 6-sided die, let $A = \{2\}$, and $B = \{2, 4, 6\}$.

$$P(A) = \qquad P(B|A) =$$

Flip a fair coin twice. Let

$$H_1 = \text{head on 1st flip}, \qquad H_2 = \text{head on 2nd flip}$$

$$P(H_1) = \qquad P(H_2) = \qquad P(H_2|H_1) =$$

Definition 31 (Independent)

Events A and B are independent iff

$$P(B|A) = P(B) \quad \textit{or} \quad P(A|B) = P(A)$$

provided the conditional probability exists.

Otherwise they are dependent.

Theorem 7

Events A and B are independent iff

$$P(A \cap B) = P(A)P(B)$$

Ex 2.36 Suppose that we have a fuse box containing 20 fuses, of which 5 are defective. If 2 fuses are selected at random and removed from the box in succession without replacing the first, what is the probability that both fuses are defective?

Ex 2.38 A small town has one fire engine and one ambulance available for emergencies. The probability that the fire engine is available when needed is 0.98, and the probability that the ambulance is available when called is 0.92. In the event of an injury resulting from a burning building, find the probability that both the ambulance and the fire engine will be available, assuming they operate independently.

Ex Suppose you are flipping a fair coin.

1. If you have flipped 4 heads what is the probability that the 5th flip is a head?

2. What is the probability of flipping 5 heads in a row?

Chapter 2 Section 7
Bayes' Rule

Suppose you get checked for prostrate (or breast) cancer.
You are told that the test's sensitivity is 90%.
This means that if you have cancer (C), then the test will return a positive result ($+$) 90% of the time

$$P(+|C) = 0.90$$

If the test comes back positive, how concerned should you be?

Ex pg 72 Suppose a town can be divided into the categories:

E: employed, $U = E'$: unemployed; M: male, F: female

as seen in the table:

	E	E'	
M	460	40	500
F	140	260	400
	600	300	900

We also know that 36 employed and 12 unemployed are members of the Rotary Club.

Let $A \equiv$ is a Rotary Club member, find $P(A)$.

Theorem 8

If events $B_1, B_2, \ldots, B_k$ *are a partition of* S *such that* $P(B_i) \neq 0 \; \forall i$, *then for any event* A *of* S

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{n} P(B_i) P(A|B_i)$$

Ex 2.41 pg 74 In a certain assembly plant, three machines, B_1, B_2, and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Back to our cancer discussion.
We know the sensitivity,

$$P(+|C) = 0.90,$$

but that's not what we want to know.
We want to know is

$$P(C|+).$$

Theorem 9 (Bayes' Rule)

If $B_1, B_2, \ldots, B_k$ are a partition of S such that $P(B_i) \neq 0 \ \forall i$, then for any event A of S such that $P(A) \neq 0$

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

Ex 2.101 A paint-store chain produces and sells latex and semigloss paint. Based on long-range sales, the probability that a customer will purchase latex paint is 0.75. Of those that purchase latex paint, 60% also purchase rollers. But only 30% of semigloss paint buyers purchase rollers. A randomly selected buyer purchases a roller and a can of paint. What is the probability that the paint is latex?

Ex 2.99 Suppose that the four inspectors at a film factory are supposed to stamp the expiration date on each package of film at the end of the assembly line. John, who stamps 20% of the packages, fails to stamp the expiration date once in every 200 packages; Tom, who stamps 60% of the packages, fails to stamp the expiration date once in every 100 packages; Jeff, who stamps 15% of the packages, fails to stamp the expiration date once in every 90 packages; and Pat, who stamps 5% of the packages, fails to stamp the expiration date once in every 200 packages. If a customer complains that her package of film does not show the expiration date, what is the probability that it was inspected by John?

Ex 2.100 A regional telephone company operates three identical relay stations at different locations. During a one-year period, the number of malfunctions reported by each station and the causes are shown below.

Station	A	B	C
Problems with electricity supplied	2	1	1
Computer malfunction	4	3	2
Malfunctioning electrical equipment	5	4	2
Caused by other human errors	7	7	5

Suppose that a malfunction was reported and it was found to be caused by other human errors. What is the probability that it came from station C?

Finally, lets work the prostrate cancer problem.
Let C represent the event that you have cancer, C' that you do not have cancer, $+$ that the prostate cancer test came back positive, and $-$ that the prostate cancer test came back negative.

	$+$	$-$	
C Yes	1688	187	1875
C' No	32381	65744	98125
	34069	65931	100000

So

$$P(C) = \qquad , \quad P(C') =$$

$$P(+|C) = \qquad , \quad P(+|C') =$$

Now we know the sensitivity, $P(+|C)$, and the specificity, $P(+|C')$.

Now we know the sensitivity, $P(+|C)$, and the specificity, $P(+|C')$. But you don't really care about either of these. What you want to know is:

Now we know the sensitivity, $P(+|C)$, and the specificity, $P(+|C')$. But you don't really care about either of these. What you want to know is:
$P(C|+) =$

Now when a male is 60 year old or older, then $P(C) = .4$.
This results in

$$P(C|+) = .645.$$

Chapter 3 Section 1

Flip a fair coin 3 times. What is the sample space?

$$S = \{HHH, HHT, \ldots, TTT\}$$

What is the average of these coin flips?

Definition 32 (Random variable)

is a function that associates a real number with each element in S.

We could let the function be

1. the number of heads
2. the number of tails
3. the number of heads minus the number of tails

Let's let our random variable count the number of heads. We will call this function X.
So for example,

$$X(HHH) = 3, X(HTT) = 1, X(TTT) = 0$$

What you usually see is:

$$X = 0, X = 1, X = 2, X = 3$$

Which is asking for the inverse of the function X, so $X = 0$ is interpreted as

$$X^{-1}(0) =$$

What you usually see is:

$$X = 0, X = 1, X = 2, X = 3$$

Which is asking for the inverse of the function X, so $X = 0$ is interpreted as

$$X^{-1}(0) = \{TTT\}$$

Ex 3.2 An overseas shipment of 5 foreign automobiles contains 2 that have slight paint blemishes. If an agency receives 3 of these automobiles at random, list the elements of the sample space S, using the letters B and N for blemished and nonblemished, respectively; then to each sample point assign a value x of the random variable X representing the number of automobiles with paint blemishes purchased by the agency.

Definition 33 (Bernoulli random variable)

is a random variable with 0 *and* 1 *as the only possible values.*

Ex 3.2, pg 82 Consider the simple condition in which components are arriving from the production line and they are stipulated to be defective or not defective. Can we find a Bernoulli random variable for this?

Ex Suppose we flip a coin until we get a H. What is the sample space?

Definition 34 (Discrete sample space)

is a sample space that contains a finite number of elements or is an infinite sequence with the same number of elements as $\mathbb{N}$*, that is they can be listed.*

Definition 35 (Continuous sample space)

is a sample space that contains as many points as a line segment.

Ex Let T be a random variable defined by the amount of time between two mouse clicks. So T can be any value $t \geq 0$.

Ex Let W be defined as the length of time it takes a student to complete an hour long test. Then W can be any value $0 \leq w \leq 1$.

Chapter 3 Section 2

We will concentrate on discrete probability distributions for the time being.

Definition 36 (Probability mass function)

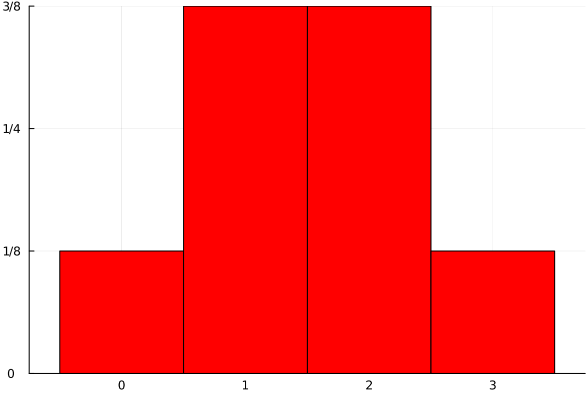
is a set of $(x, f(x))$ *of a random variable* X *if* $\forall x$*:*

1. $f(x) \geq 0$
2. $\sum_x f(x) = 1$
3. $P(X = x) = f(x)$

A pmf is also called a *probability function* and a *probability distribution*.

Looking again at our flip a fair coin three times with $X \equiv \#H$

$$f(x) = \begin{cases} \frac{1}{8}, & x = 0 \\ \frac{3}{8}, & x = 1 \\ \frac{3}{8}, & x = 2 \\ \frac{1}{8}, & x = 3 \end{cases}$$



Ex 3.8 pg 84 A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Sometimes it is convenient to have the probability computed for all values $\leq x$. This is often how tables of probabilities are given.

Definition 37 (Cumulative distribution function)

of a random variable X with pmf $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \text{ for } -\infty < x < \infty$$

Flip a fair coin three times with $X \equiv \#H$ pmf:

$$f(x) = \begin{cases} \frac{1}{8}, & x = 0 \\ \frac{3}{8}, & x = 1 \\ \frac{3}{8}, & x = 2 \\ \frac{1}{8}, & x = 3 \end{cases}$$

It's cdf

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Chapter 3 Section 3

Recall that an integral is just an infinite sum.
For a discrete rv $P(X \leq x) \neq P(X < x)$.
But for a constant rv $P(X \leq x) = P(X < x)$, since excluding a finite number of points from an integral does not change the integral.

Definition 38 (Probability density function)

of a continuous random variable X, $f(x)$, *satisfies*

1. $f(x) \geq 0, \forall x \in \mathbb{R}$
2. $$\int_{-\infty}^{\infty} f(x)dx = 1$$
3. $$P(a < X < b) = \int_{a}^{b} f(x)dx$$

Definition 39 (Cumulative distribution function)

of a continuous random variable X *with pdf* $f(x)$ *is*

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt, x \in (-\infty, \infty)$$

Note $f(x) = \dfrac{dF}{dx}$ and $P(a < X < b) = F(b) - F(a)$.

Ex 3.11 pg 89; part 1

Suppose that the error in the reaction temperature, in degrees celsius, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

1) Verify that $f(x)$ is a density function.

Ex 3.11 pg 89; part 2

Suppose that the error in the reaction temperature, in degrees celsius, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

2) Find $P(0 < X \leq 1)$.

Ex For

$$f(x) = \frac{1}{24}(2x + 3), x \in [1, 4]$$

1) Check properties 1) and 2) of pdf definition;

Ex For

$$f(x) = \frac{1}{24}(2x+3), x \in [1,4]$$

2) find the cdf.

Chapter 4 Section 1

Suppose we have a fair 1D3 die.
Let the random variable $X \equiv$ the number rolled.
What is the average?

$$\frac{1+2+3}{3} = \frac{6}{3} = 2$$

The probability distribution is

x	1	2	3
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Suppose that the die is unfair and the probability distribution is

x	1	2	3
$P(X = x)$	0.25	0.25	0.5

What is the average of this die?

$$0.25(1) + 0.25(2) + 0.5(3) = 2.25$$

This value, 2.25, is in a sense the center of the distribution

Definition 40 (Expected value)

of a random variable X with probability distribution $f(x)$ is:

discrete

$$\mu = E(X) = \sum_{x} x f(x)$$

continuous

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

a.k.a. the mean.

Ex 4.4 A coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

Ex Find $E(X)$ for

$$f(x) = \frac{1}{24}(2x+3), \quad x \in [1,4]$$

Sometimes it is useful/necessary to find the expected value where the random value is a function of X.

Theorem 10

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

discrete

$$E(g(X)) = \sum_x g(x)f(x)$$

continuous

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Ex 4.17 Let X be a random variable with the following probability distribution:

x	-3	6	9
$f(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find $\mu_{g(X)}$, where $g(X) = (2X+1)^2$.

Ex 4.20 A continuous random variable X has the density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{o.w.} \end{cases}$$

Find the expected value of

$$g(X) = e^{\frac{2X}{3}}$$

Chapter 4 Section 2

The expected value is a measure of centrality.
Knowing this helps us to understand our data.
The other major measure is a measure of variability.
The will tell us how spread out the data is from the center.
Let's look at the data

$$1, 1, 2, 5, 8, 9, 9$$

It has a expected value of

$$\mu = \frac{1+1+2+5+8+9+9}{7} = 5$$

To find the variance:

x	$x-\mu$	$(x-\mu)^2$
1	-4	16
1	-4	16
2	-3	9
5	0	0
8	3	9
9	4	16
9	4	16

To find the variance:

x	$x - \mu$	$(x - \mu)^2$
1	-4	16
1	-4	16
2	-3	9
5	0	0
8	3	9
9	4	16
9	4	16
	0	82

$\sigma^2 = \frac{82}{7} = 11.7$

Definition 41 (Variance)

of the random variable X with probability function $f(x)$ and mean μ is

discrete

$$\sigma^2 = E[(X-\mu)^2] = \sum_x (x-\mu)^2 f(x)$$

continuous

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

Definition 42 (Standard deviation)

$\sigma = \sqrt{\sigma^2}$

Ex 4.36 Suppose that the probabilities are 0.4, 0.3, 0.2, and 0.1, respectively, that 0, 1, 2, or 3 power failures will strike a certain subdivision in any given year. Find the mean and variance of the random variable X representing the number of power failures striking this subdivision.

Theorem 11

The variance of a random variable X is

$$\sigma^2 = E[X^2] - \mu^2$$

Ex 4.36 again

x	0	1	2	3
$f(x)$	0.4	0.3	0.2	0.1

Ex 4.41 Find the standard deviation of the random variable $g(X) = (2X + 1)^2$ in Exercise 4.17 on page 118.

Chapter 4 Section 3

Theorem 12

If a and b are constants, then

$$E[aX + b] = aE[X] + b$$

Theorem 13

If a and b are constants, then

$$\sigma^2_{aX+b} = a^2\sigma^2_X$$

Chapter 5 Section 2

Let's look at $(p+q)^n$ for $n = 0, 1, 2, \ldots$.

$$\begin{aligned}
&(p+q)^0 = 1 \\
&(p+q)^1 = 1p^1 + 1q^1 \\
&(p+q)^2 = 1p^2 + 2pq + 1q^2 \\
&(p+q)^3 = 1p^3 + 3p^2q + 3pq^2 + 1q^3 \\
&\vdots
\end{aligned}$$

That looks familiar.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

That's Pascal's triangle.
Note that each line is a combination:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
 & & & & \vdots & & & & \\
\binom{n}{0} & \binom{n}{1} & & & \cdots & & & \binom{n}{n-1} & \binom{n}{n}
\end{array}
$$

Recall that a Bernoulli trial is a random process whose random variable X can be written

$$X = \begin{cases} 0, & \text{failure} \\ 1, & \text{success} \end{cases}$$

Ex Flip a coin once. How should we define the random variable X?

Ex Let the random variable $Z \equiv$ patient died after surgery. How should we define the random variable Z?

If we flip a coin multiple times then we get a Bernoulli Process:

1. experiment consists of repeated trials,
2. each outcome is either a success or failure
3. probability of success, p, remains constant for all trials
4. repeated trials are independent

Let's flip a fair coin twice and let the random variable $X \equiv$ #Hs. Does this satisfy the properties of a Bernoulli Process? Yes, so $p = \frac{1}{2}$ is the probability of success and letting q be the probability of failure $q = \frac{1}{2}$.
How to find the pmf

x	0	1	2
$f(x)$			

S	HH	HT	TH	TT
x	0	1	1	2
prob				

What if we flipping a fair coin three times with $X \equiv \#\text{Hs}$

S	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
x	3	2	2	2	1	1	1	0
prob	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$

So the pmf is

x	0	1	2	3
$f(x)$	$1\left(\frac{1}{8}\right)$	$3\left(\frac{1}{8}\right)$	$3\left(\frac{1}{8}\right)$	$1\left(\frac{1}{8}\right)$

Ex pg 144 Consider the set of Bernoulli trials where three items are selected at random from a manufacturing process, inspected, and classified as defective or nondefective. A defective item is designated a success, $X \equiv \#$Ds assuming integral values from 0 through 3. The items are selected independently and we assume that the process produces 25% defectives.

Definition 43 (Binomial Distribution)

Let X be a random variable of the number of successes in a Bernoulli Process with the probability of success p and failure $q = 1 - p$ and n independent trials, then the probability mass function is

$$b(x; n, p) = f(x) = \binom{n}{x} p^x q^{n-x}, x \in \{0, 1, 2, \ldots, n\}$$

Ex 5.3 An employee is selected from a staff of 10 to supervise a certain project by selecting a tag at random from a box containing 10 tags numbered from 1 to 10. Find the formula for the probability distribution of X representing the number on the tag that is drawn. What is the probability that the number drawn is less than 4?

Ex 5.5 According to *Chemical Engineering Progress* (November 1990), approximately 30% of all pipework failures in chemical plants are caused by operator error.

1. What is the probability that out of the next 20 pipework failures at least 10 are due to operator error?
2. What is the probability that no more than 4 out of 20 such failures are due to operator error?

Ex 5.6 According to a survey by the Administrative Management Society, one-half of U.S. companies give employees 4 weeks of vacation after they have been with the company for 15 years. Find the probability that among 6 companies surveyed at random, the number that give employees 4 weeks of vacation after 15 years of employment is

1. anywhere from 2 to 5;
2. fewer than 3.

Ex 5.10 A nationwide survey of college seniors by the University of Michigan revealed that almost 70% disapprove of daily pot smoking, according to a report in Parade. If 12 seniors are selected at random and asked their opinion, find the probability that the number who disapprove of smoking pot daily is

1. anywhere from 7 to 9;
2. at most 5;
3. not less than 8.

Theorem 14

for the binomial distribution $b(x;n,p)$
Expected value:

$$\mu = np$$

Variance:

$$\sigma^2 = npq$$

Chapter 5 Section 3

For a Bernoulli process there are four properties that must be satisfied:

1. experiment consists of repeated trials,
2. each outcome is either a success or failure
3. probability of success, p, remains constant for all trials
4. repeated trials are independent

The fourth of them is that the trials are independent. So if we are picking cards from a deck, then we must do so with replacement.

If we relax this property the we get a hypergeometric experiment that satisfies

1. a random sample of size n is selected without replacement from N items.
2. Of the N items, k are considered as a success and the rest, $N - k$, as failures.

The random variable $X \equiv \#$ of successes is a hypergeometric experiment is called a hypergeometric random variable and its distribution is

Definition 44 (Hypergeometric distribution)

of a hypergeometric random variable X (the number of successes) of sample size n selected from N (k successes, $N - k$ failures) is

$$h(x; N, n, k) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}; \quad \max\{0, n-(N-k)\} \leq x \leq min\{n, k\}$$

Ex 5.33 If 7 cards are dealt from an ordinary deck of 52 playing cards, what is the probability that

(a) exactly 2 of them will be face cards?

(b) at least 1 of them will be a queen?

Ex 5.34 What is the probability that a waitress will refuse to serve alcoholic beverages to only 2 minors if she randomly checks the IDs of 5 among 9 students, 4 of whom are minors?

Ex 5.35 A company is interested in evaluating its current inspection procedure for shipments of 50 identical items. The procedure is to take a sample of 5 and pass the shipment if no more than 2 are found to be defective. What proportion of shipments with 20% defectives will be accepted?

Theorem 15

The hypergeometric distribution $h(x; N, n, k)$ *has*

$$\mu = \frac{nk}{N} \quad \textit{and} \quad \sigma^2 = \frac{N-n}{N-1}\left(n\frac{k}{N}\right)\left(1-\frac{k}{N}\right)$$

Let us look at the probability of a success at each step

if $N = 1000$ and $k = 5$:

if $N = 1000$ and $k = 500$:

As long as we don't pick too many times these probabilities will remain approximately the same.
In fact, as long as

$$\frac{\text{sample size}}{\text{population size}} = \frac{n}{N} \leq 0.05$$

we can approximate a hypergeometric distribution with a binomial distribution where $p = \frac{k}{N}$.

Ex 5.39 An annexation suit against a county subdivision of 1200 residences is being considered by a neighboring city. If the occupants of half the residences object to being annexed, what is the probability that in a random sample of 10 at least 3 favor the annexation suit?

If N items can be divided into k different cells that form a partition:

$$A_1, A_2, \ldots, A_k$$

where there are a_i elements in A_i. Then we can find the probability that a sample of size n selects x_i elements from A_i for $i \in \{1, \ldots, k\}$.

Definition 45 (Multivariate hypergeometric distribution)

If N items can be partitioned into k cells $A_1, A_2, \ldots, A_k$ with cell A_i having a_i elements, then the probability distribution of random variables $X_1, X_2, \ldots, X_k$ which represents the number of elements selected from $A_1, A_2, \ldots, A_k$ in a random sample of size n is

$$f(x_1, \ldots, x_k, a_1, \ldots, a_k, N, n) = \frac{\binom{a_1}{x_1} \cdots \binom{a_k}{x_k}}{\binom{N}{n}}$$

Ex 5.13 pg 156 A group of 10 individuals is used for a biological case study. The group contains 3 people with blood type O, 4 with blood type A, and 3 with blood type B. What is the probability that a random sample of 5 will contain 1 person with blood type O, 2 people with blood type A, and 2 people with blood type B?

Chapter 5 Section 4

Suppose we flip a coin until we get three Hs. Let's find the sample space. What is the smallest number of times we flip and get 3 Hs?

$$3$$

So the next smallest is 4. What are the outcomes? What about 5?

$$
\begin{array}{cccc}
3 & 4 & 5 & 6 \\
\hline
HHH & THHH & TTHHH & TTTHHH \\
 & HTHH & THTHH & TTHTHH \\
 & HHTH & THHTH & TTHHTH \\
 & & HTTHH & THTTHH \\
 & & HTHTH & THTHTH \\
 & & HHTTH & THHTTH \\
 & & & HTTTHH \\
 & & & HTTHTH \\
 & & & HTHTTH \\
 & & & HHTTTH \\
\hline
\binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \binom{5}{2}
\end{array}
$$

Definition 46 (Negative Binomial distribution)

If repeated independent trials are preformed with probability of success p and failure $q = 1 - p$, then the pmf of the random variable $X \equiv \#$ if trials on which the kth success occurs is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, x \in \{k, k+1, \dots\}$$

Ex 5.50 Find the probability that a person flipping a coin gets

(a) the third head on the seventh flip;

(b) the first head on the fourth flip.

Ex 5.52 A scientist inoculates mice, one at a time, with a disease germ until he finds 2 that have contracted the disease. If the probability of contracting the disease is $\frac{1}{6}$, what is the probability that 8 mice are required?

Looking for the occurrence of a 1st success is important enough to get special treatment. If we let $k = 1$ in the negative binomial distribution we get

$$b^*(x; 1, p) = \binom{x-1}{1-1} p^1 q^{x-1} = pq^{x-1}$$

Definition 47 (Geometric distribution)

If repeated independent trials can result a success with probability p *and failure* $q = 1 - p$*, then the pmf of the random variable* $X \equiv$ $\#$ *if trials on which the 1st success occurs is*

$$g(x; p) = pq^{x-1}, x \in \{1, 2, \dots\}$$

Ex 5.50 Find the probability that a person flipping a coin gets

(b) the first head on the fourth flip.

Ex 5.51 Three people toss a fair coin and the odd one pays for coffee. If the coins all turn up the same, they are tossed again. Find the probability that fewer than 4 tosses are needed.

Theorem 16

For a geometric distribution

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}$$

Chapter 4 review
Borrowed from:
Applied Statistics and Probability for Engineers,
Montgomery & Runger, 5th Ed

3-59 Problem An article in the *Journal of Database Management* ["Experimental Study of a Self-Tuning Algorithm for DBMS Buffer Pools" (2005, Vol. 16, pp. 1–20)] provided the workload used in the TPC-C OLTP (Transaction Processing Performance Council's Version C On-Line Transaction Processing) benchmark, which simulates a typical order entry application. The frequency of each type of transaction (in the second column) can be used as the percentage of each type of transaction. The average number of *selects* operations required for each type transaction is shown.

1. Determine the mean and standard deviation of the number of *selects* operations for a transaction from the distribution of types shown in the table.
2. Determine the mean and standard deviation of the total number of operations (*selects*, *updates*, ... , and *joins*) for a transaction from the distribution of types shown in the table.

3-59 Table

Transaction	Frequency	Selects	Updates	Inserts	Deletes	Non-Unique Selects	Joins
New Order	43	23	11	12	0	0	0
Payment	44	4.2	3	1	0	0.6	0
Order Status	4	11.4	0	0	0	0.6	0
Delivery	5	130	120	0	10	0	0
Stock Level	4	0	0	0	0	0	1

Table: Average Frequencies and Operations in TPC-C

Ex 3-8 Suppose that a day's production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. What is the cumulative distribution function of X given the probability mass function below?

x	0	1	2
$f(x)$	0.886	0.111	0.003

4-13 Suppose the cumulative distribution function of the continuous random variable X is

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.2x, & 0 \leq x < 5 \\ 1, & x \geq 5 \end{cases}$$

Determine the following:

1. $P(X < 2.8)$
2. $P(X > 1.5)$
3. $P(X < -2)$
4. $P(X > 6)$

Ex 4-5 The time until a chemical reaction is complete (in milliseconds) is approximated by the probability mass function

$$f(x) = \begin{cases} 0, & x < 0 \\ 0.01e^{-0.01x}, & x \geq 0 \end{cases}$$

Find the cumulative distribution function.

Ex 4-1 Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is $[0, 20\text{mA}]$, and assume that the probability density function of X is $f(x) = 0.05$ for $0 \leq x \leq 20$. What is the probability that a current measurement is less than 10 milliamperes?

Ex 4-3 Now find it's cdf

Ex 4-6 For the copper current measurement in Example 4-1, find the mean and variance of X:

Chapter 5 Section 5

$$\binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!x!} \frac{\lambda^x}{n^x} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$
$$= \frac{n!}{(n-x)!n^x} \frac{\lambda^x}{x!} \frac{1}{\left(1-\frac{\lambda}{n}\right)^x} \left(1+\frac{-\lambda}{n}\right)^n$$

Since $p = \frac{\lambda}{n} \approx 0$, then $1 - \frac{\lambda}{n} \approx 1$.

$$\lim_{n\to\infty} \left(1+\frac{-\lambda}{n}\right)^n = e^{-\lambda}$$

Finally

$$\frac{n!}{(n-x)!n^x} = \frac{n(n-1)\cdots(n-x+1)}{n^x}$$

$$= \frac{n(n-1)\cdots(n-(x-1))}{n^x} \qquad = \frac{n^x + \mathsf{LOTs}}{n^x}$$

$$\lim_{n\to\infty} \frac{n^x + \mathsf{LOTs}}{n^x} = 1$$

In general, consider an interval T of real numbers partitioned into subintervals of small length t and assume that as Δt tends to zero,

1. the probability of more than one event in a subinterval tends to zero,
2. the probability of one event in a subinterval tends to $\lambda\frac{\Delta t}{T}$,
3. the event in each subinterval is independent of other subintervals. (No memory)

A random experiment with these properties is called a Poisson process. The time interval can be replaced with a length, area etc. above.

Definition 48 (Poisson distribution)

The pmf of a Poisson random variable $X \equiv \#$ of outcomes in an interval with parameter $\lambda > 0$ is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \in \{0, 1, 2, \dots\}$$

Ex 3-32 You have a thin copper wire and suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter.

1. Determine the probability of exactly two flaws in 1 millimeter of wire.
2. Determine the probability of 10 flaws in 5 millimeters of wire.

Ex 3-132 The number of telephone calls that arrive at a phone exchange is often modeled as a Poisson random variable. Assume that on the average there are 10 calls per hour.

1. What is the probability that there are exactly five calls in one hour?
2. What is the probability that there are three or fewer calls in one hour?
3. What is the probability that there are exactly 15 calls in two hours?
4. What is the probability that there are exactly five calls in 30 minutes?

Some examples[†]:

1. The number of misprints on a page (or a group of pages) of a book
2. The number of people in a community who survive to age 100
3. The number of wrong telephone numbers that are dialed in a day
4. The number of packages of dog biscuits sold in a particular store each day
5. The number of customers entering a post office on a given day
6. The number of vacancies occurring during a year in the federal judicial system
7. The number of α-particles discharged in a fixed period of time from some radioactive material

[†]A First Course in Probability, Ross, 8th Edition

Theorem 17 (Mean and Variance Poisson distribution)

If X is a Poisson random variable with parameter λ, then

$$\mu = E[X] = \lambda, \quad \sigma^2 = \lambda$$

Ex 3-135 In 1898 L. J. Bortkiewicz published a book entitled The Law of Small Numbers. He used data collected over 20 years to show that the number of soldiers killed by horse kicks each year in each corps in the Prussian cavalry followed a Poisson distribution with a mean of 0.61.

1. What is the probability of more than one death in a corps in a year?
2. What is the probability of no deaths in a corps over five years?

Ex 3-140 The number of failures of a testing instrument from contamination particles on the product is a Poisson random variable with a mean of 0.02 failure per hour.

1. What is the probability that the instrument does not fail in an eight-hour shift?
2. What is the probability of at least one failure in a 24-hour day?

Chapter 5 review

5.77 During a manufacturing process, 15 units are randomly selected each day from the production line to check the percent defective. From historical information it is known that the probability of a defective unit is 0.05. Any time 2 or more defectives are found in the sample of 15, the process is stopped. This procedure is used to provide a signal in case the probability of a defective has increased.

(a) What is the probability that on any given day the production process will be stopped? (Assume 5% defective.)

(b) Suppose that the probability of a defective has increased to 0.07. What is the probability that on any given day the production process will not be stopped?

5.80 Service calls come to a maintenance center with an average of 2.7 calls are received per minute. Find the probability that

(a) no more than 4 calls come in any minute;

(b) fewer than 2 calls come in any minute;

(c) more than 10 calls come in a 5-minute period.

5.78 An automatic welding machine is being considered for use in a production process. It will be considered for purchase if it is successful on 99% of its welds. Otherwise, it will not be considered efficient. A test is to be conducted with a prototype that is to perform 100 welds. The machine will be accepted for manufacture if it misses no more than 3 welds.

(a) What is the probability that a good machine will be rejected?

(b) What is the probability that an inefficient machine with 95% welding success will be accepted?

5.79 A car rental agency at a local airport has available 5 Fords, 7 Chevrolets, 4 Dodges, 3 Hondas, and 4 Toyotas. If the agency randomly selects 9 of these cars to chauffeur delegates from the airport to the downtown convention center, find the probability that 2 Fords, 3 Chevrolets, 1 Dodge, 1 Honda, and 2 Toyotas are used.

5.81 An electronics firm claims that the proportion of defective units from a certain process is 5%. A buyer has a standard procedure of inspecting 15 units selected randomly from a large lot. On a particular occasion, the buyer found 5 items defective.

(a) What is the probability of this occurrence, given that the claim of 5% defective is correct?

(b) What would be your reaction if you were the buyer?

5.82 An electronic switching device occasionally malfunctions, but the device is considered satisfactory if it makes, on average, no more than 0.20 error per hour. A particular 5-hour period is chosen for testing the device. If no more than 1 error occurs during the time period, the device will be considered satisfactory.

(a) What is the probability that a satisfactory device will be considered unsatisfactory on the basis of the test?

(b) What is the probability that a device will be accepted as satisfactory when, in fact, the mean number of errors is 0.25?

5.95 A production process outputs items in lots of 50. Sampling plans exist in which lots are pulled aside periodically and exposed to a certain type of inspection. It is usually assumed that the proportion defective is very small. It is important to the company that lots containing defectives be a rare event. The current inspection plan is to periodically sample randomly 10 out of the 50 items in a lot and, if none are defective, to perform no intervention.

(a) Suppose in a lot chosen at random, 2 out of 50 are defective. What is the probability that at least 1 in the sample of 10 from the lot is defective?

(c) What is the mean number of defects found out of 10 items sampled?

5.91 An oil drilling company ventures into various locations, and its success or failure is independent from one location to another. Suppose the probability of a success at any specific location is 0.25. The drilling company feels that it will "hit it big" if the second success occurs on or before the sixth attempt. What is the probability that the driller will hit it big?

5.83 A company generally purchases large lots of a certain kind of electronic device. A method is used that rejects a lot if 2 or more defective units are found in a random sample of 100 units.

(a) What is the probability of rejecting a lot that is 1% defective?

(b) What is the probability of accepting a lot that is 5% defective?

5.84 A local drugstore owner knows that, on average, 100 people enter his store each hour.

(a) Find the probability that in a given 3-minute period nobody enters the store.

(b) Find the probability that in a given 3-minute period more than 5 people enter the store.

5.89 The acceptance scheme for purchasing lots containing a large number of batteries is to test no more than 75 randomly selected batteries and to reject a lot if a single battery fails. Suppose the probability of a failure is 0.001.

(a) What is the probability that a lot is accepted?

(b) What is the probability that a lot is rejected on the 20th test?

5.92 A couple decides to continue to have children until they have two males. Assuming that $P(\text{male}) = 0.5$, what is the probability that their second male is their fourth child?

5.100 There are two vacancies in a certain university statistics department. Five individuals apply. Two have expertise in linear models, and one has expertise in applied probability. The search committee is instructed to choose the two applicants randomly.

(a) What is the probability that the two chosen are those with expertise in linear models?

(b) What is the probability that of the two chosen, one has expertise in linear models and one has expertise in applied probability?

Chapter 6 Section 1

Discrete distributions have sample spaces such as:

Binomial $S = \{HHH, HHT, \dots\}$

Negative Binomial $S = \{H, TH, TTH, \dots\}$

Continuous distributions have sample spaces such as:

Adult heights $S = \{x | 1 \leq x \leq 10\}$

GPAs $G = \{g | 0 \leq g \leq 4.0\}$

Time interval $T = \{t | t \geq 0\}$

Suppose you want all outcomes to be equally likely and the possible values to be $0 \leq x \leq 2$. What would $f(x) =$? Recall that

$$P(a \leq x \leq b) = \int_1^b f(x)dx$$

Definition 49 (Uniform distribution)

The pdf of the continuous uniform r.v. X on the interval $[A, B]$ is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \leq x \leq B \\ 0, & o.w. \end{cases}$$

Ex 30 When pearl oysters are opened, pearls of various sizes are found. Suppose that each oyster contains a pearl with a diameter in mm that has a $U(0, 10)$ distribution.

1. What is the probability of getting a pearl with a diameter of 7?
2. What is the probability of getting a pearl with a diameter in $7 \leq d < 10$?
3. What is the probability of getting a pearl with a diameter in $7 \leq d \leq 10$?
4. What is the probability of getting a pearl with a diameter in $9 \leq d < 12$?

Ex Suppose that a dial is spun and the angle θ that it makes with a fixed mark is measured once it has come to a halt, shown below. Let the angle θ be measured so that it lies between 0° and 180°. The value of θ obtained from a spin is then a continuous random variable taking any value within the interval $[0, 180]$. The angle θ has a uniform distributions, $\Theta \sim U(0, 180)$.

1. What is the probability of getting an angle between $0 \leq \theta \leq 180$?
2. What is the probability of getting an angle between $180 \leq \theta \leq 360$?
3. What is the probability of getting an angle between $90 \leq d < 120$?

Theorem 18

A uniform distribution has

$$\mu = \frac{A+B}{2}, \quad \sigma^2 = \frac{(B-A)^2}{12}$$

Proof.

$\square$

Ex 30again When pearl oysters are opened, pearls of various sizes are found. Suppose that each oyster contains a pearl with a diameter in mm that has a $U(0, 10)$ distribution. What is the expected value of a pearl and its standard deviation?

Ex Recall that the dial spinning game has a uniform distribution based on the angle θ and is $\Theta \sim U(0, 180)$. What is the expected value and variance of this game?

Ex 4.1.2 A new battery supposedly with a charge of 1.5 volts actually has a voltage with a uniform distribution between 1.43 and 1.60 volts.

a) What is the expectation of the voltage?
b) What is the standard deviation of the voltage?
c) What is the cumulative distribution function of the voltage?
d) What is the probability that a battery has a voltage less than 1.48 volts?

Chapter 6 Section 2

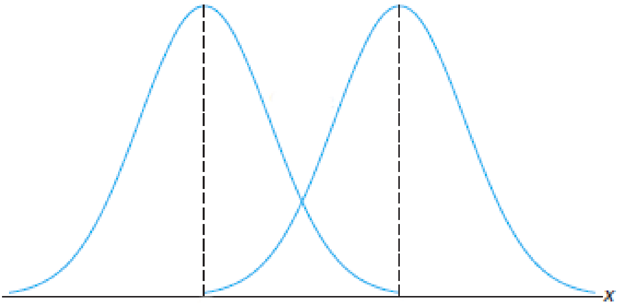
The most famous and used distribution is the normal distribution (using the frequentist model of statistics).

Definition 50 (Normal distribution)

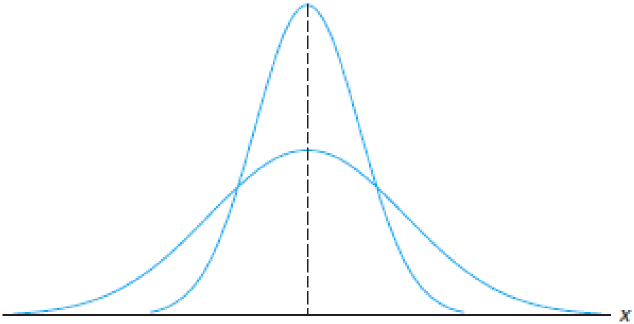
The pdf of the normal r.v. X with mean μ and varianceσ^2 is

$$n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

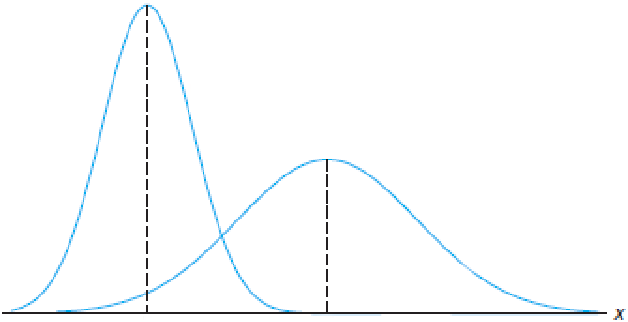
Ex Compare the means and standard deviations of these two normal distributions.



Ex Compare the means and standard deviations of these two normal distributions.



Ex Compare the means and standard deviations of these two normal distributions.



Property 1

of $n(x; \mu, \sigma)$

1. *the mode is located at* μ
2. *the curve is symmetric about* μ
3. *the curve has inflection points at* $\mu \pm \sigma$
4. $$\lim_{x \to \pm\infty} n(x) = 0$$
5. $$\int_{-\infty}^{\infty} n(x) dx = 1$$

Theorem 19

The mean and variance of $n(x;\mu,\sigma)$ is μ and σ.

Proof.

The expected value is calculated by

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

But this is messy, so instead we do

$$E[X-\mu] = \int_{-\infty}^{\infty} \frac{x-\mu}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

and we can do a change of variable

$$z = \frac{x-\mu}{\sigma}$$

and get

$$E[X-\mu] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz$$

Which can be used to show that $E[X] = \mu$. □

Chapter 6 Section 3

Recalling from section 3.3

$$P(a < X < b) = \int_a^b f(x)dx$$

Suppose $\mu = 68$ and $\sigma = 3$.
What is the probability that a man is between $a = 65$ and $b = 71$ inches where $X \equiv$ height of a man?

$$P(65 < X < 71) = \int_{65}^{71} n(x; 68, 3)dx = \int_{65}^{71} \frac{1}{\sqrt{2\pi} \cdot 3} e^{-\frac{1}{2}\left(\frac{x-68}{3}\right)^2} dx$$

Let's simplify and look at how to integrate

$$\int e^{-x^2}\,dx$$

There is does not exist a solution using elementary functions for this integral.
So what do we do? Use a table. How many tables will we need?
Recall from the proof that $E[X] = \mu$ we used the change of variables (substitution)

$$z = \frac{x - \mu}{\sigma}$$

We can do this for any normal random variable X using

$$Z = \frac{X - \mu}{\sigma}$$

Let think about what this does to our original data using $\mu = 68$ and $\sigma = 3$.

Which leads to the definition

Definition 51 (Standard normal distribution)

is the distribution of a normal random variable with $\mu = 0$ and $\sigma = 1$, $n(x, 0, 1)$.

Find the area under the standard normal curve to the left of $z = -1.67$. Round your answer to four decimal places, if necessary.

Find the area under the standard normal curve to the left of $z = -1.67$. Round your answer to four decimal places, if necessary.

$$z = -1.6 + 0.07^{\dagger}$$

†Technically this should say $-(1.6 + 0.07)$ or $-1.6 - 0.07$. Think of the $+$ as the word "and".

Find the area under the standard normal curve to the left of $z = -1.67$. Round your answer to four decimal places, if necessary.

$$z = -1.6 + 0.07 = \text{row value} + \text{column value}$$

Find the area under the standard normal curve to the left of $z = -1.67$. Round your answer to four decimal places, if necessary.

$$z = -1.6 + 0.07 = \text{row value} + \text{column value}$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09

Find the area under the standard normal curve to the left of $z = -1.67$. Round your answer to four decimal places, if necessary.

$$z = -1.6 + 0.07 = \text{row value} + \text{column value}$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09

Find the area under the standard normal curve to the right of $z = 1.38$. Round your answer to four decimal places, if necessary.

Find the area under the standard normal curve to the right of $z = 2.38$. Round your answer to four decimal places, if necessary.

Find the area under the standard normal curve to the right of $z = -2.37$. Round your answer to four decimal places, if necessary.

Find the area under the standard normal curve between $z = 1.38$ and $z = 2.92$. Round your answer to four decimal places, if necessary.

Find the area under the standard normal curve between $z = -1.44$ and $z = 1.44$. Round your answer to four decimal places, if necessary.

Find the area under the standard normal curve to the left of $z = -1.44$ and to the right of $z = 1.44$. Round your answer to four decimal places, if necessary.

Find the specified probability. Round your answer to four decimal places, if necessary.

$$P(0 < z < 2.03)$$

What value of z divides the standard normal distribution so that half the area is on one side and half is on the other? Round your answer to two decimal places.

Find the value of z such that 0.1401 of the area lies to the left of z. Round your answer to two decimal places.

Find the value of z such that 0.14 of the area lies to the left of z. Round your answer to two decimal places.

Find the value of z such that 0.03 of the area lies to the right of z. Round your answer to two decimal places.

Find the value of z such that 0.05 of the area lies to the right of z. Round your answer to two decimal places.

Find the value of z such that 0.8664 of the area lies between $-z$ and z. Round your answer to two decimal places.

Chapter 6 Section 4

Ex 6.10 According to Chebyshev's theorem, the probability that any random variable assumes a value within 3 standard deviations of the mean is at least $\frac{8}{9}$. If it is known that the probability distribution of a random variable X is normal with mean μ and variance σ^2, what is the exact value of $P(\mu - 3\sigma < X < \mu + 3\sigma)$?

Theorem 20 (Chebyshev's Theorem)

The probability that any random variable X will assume a value within k standard deviations of the mean is at least $1 - \frac{1}{k^2}$. That is,

$$P(\mu - k\sigma < X\mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

Ex 6.14 (a) The finished inside diameter of a piston ring is normally distributed with a mean of 10 centimeters and a standard deviation of 0.03 centimeter.

(a) What proportion of rings will have inside diameters exceeding 10.075 centimeters?

Ex 6.14 (b) The finished inside diameter of a piston ring is normally distributed with a mean of 10 centimeters and a standard deviation of 0.03 centimeter.

(b) What is the probability that a piston ring will have an inside diameter between 9.97 and 10.03 centimeters?

Ex 6.14 (c) The finished inside diameter of a piston ring is normally distributed with a mean of 10 centimeters and a standard deviation of 0.03 centimeter.

(c) Below what value of inside diameter will 15% of the piston rings fall?

Ex 6.15 (a) A lawyer commutes daily from his suburban home to his midtown office. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume the distribution of trip times to be normally distributed.

(a) What is the probability that a trip will take at least $1/2$ hour?

Ex 6.15 (b) A lawyer commutes daily from his suburban home to his midtown office. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume the distribution of trip times to be normally distributed.

(b) If the office opens at 9:00 A.M. and the lawyer leaves his house at 8:45 A.M. daily, what percentage of the time is he late for work?

Ex 6.15 (c) A lawyer commutes daily from his suburban home to his midtown office. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume the distribution of trip times to be normally distributed.

(c) If he leaves the house at 8:35 A.M. and coffee is served at the office from 8:50 A.M. until 9:00 A.M., what is the probability that he misses coffee?

Ex 6.15 (d) A lawyer commutes daily from his suburban home to his midtown office. The average time for a one-way trip is 24 minutes, with a standard deviation of 3.8 minutes. Assume the distribution of trip times to be normally distributed.

(d) Find the length of time above which we find the slowest 15% of the trips.

Ex 6.16 (a) In the November 1990 issue of Chemical Engineering Progress, a study discussed the percent purity of oxygen from a certain supplier. Assume that the mean was 99.61 with a standard deviation of 0.08. Assume that the distribution of percent purity was approximately normal.

(a) What percentage of the purity values would you expect to be between 99.5 and 99.7?

Ex 6.16 (b) In the November 1990 issue of Chemical Engineering Progress, a study discussed the percent purity of oxygen from a certain supplier. Assume that the mean was 99.61 with a standard deviation of 0.08. Assume that the distribution of percent purity was approximately normal.

(b) What purity value would you expect to exceed exactly 5% of the population?

Chapter 8 Section 1

Definition 52 (Population)

The totality of observations with which we are concerned.

Each observation in the population is a value of a random variable X having a probability distribution $f(x)$.
We usually cannot observe the totality.
Why?

Definition 53 (Sample)

A sample is a subset of a population.

Samples are used it to make inferences about populations.
Samples must be representative of the population.
Beware of choosing convenient samples. They often lead to erroneous inferences because they are:

Definition 54 (Bias)

Any sampling procedure that produces inferences that consistently overestimate or consistently underestimate some characteristic of the population is said to be biased.

To eliminate bias in the sampling procedure, choose samples with observations that are independent and random.

Definition 55 (Joint probability distribution)

The function $f(x, y)$ is a joint probability distribution or probability mass function of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ *for all* (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = x) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum\sum_{A} f(x, y)$

Ex Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

1. the joint probability function $f(x, y)$,
2. $P[(X, Y) \in A]$, where A is the region $\{(x, y)|x + y \leq 1\}$.

Ex Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

1. the joint probability function $f(x, y)$,
2. $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

1.
$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}}, x = 0, 1, 2; y = 0, 1, 2; \text{ and } 0 \leq x+y \leq 2$$

2. Since $A = \{(x, y) | x + y \leq 1\} = \{(0, 0), (0, 1), (1, 0)\}$

$$\begin{aligned} P[(X, Y) \in A] &= P(X + Y \leq 1) \\ &= f(0, 0) + f(0, 1) + f(1, 0) \\ &= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14} \end{aligned}$$

Definition 56 (Joint probability distribution)

The function $f(x, y)$ is a joint density function of the continuous random variables X and Y if

1. $f(x, y) \geq 0$ *for all* (x, y),
2. $$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1,$$
3. $$P[(X, Y) \in A] = \int\int_{A} f(x, y) dx dy,$$

   for any regions A in the xy plane.

Definition 57 (Random sample)

Let $X_1, X_2, \ldots, X_n$ be n independent random variables, each having the same probability distribution $f(x)$. Define $X_1, X_2, \ldots, X_n$ to be a random sample of size n from the population $f(x)$ and write its joint probability distribution as

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

Chapter 8 Section 2

Suppose we want to know what proportion of people in the US has crimson as their favorite color.
If we could ask everyone in the US "at the same time" whether crimson is their favorite color or not then we could find the population parameter of the proportion that prefer crimson (totally not made up numbers)

$$p = \frac{200}{340.1} = 0.588$$

But this is essentially impossible to do.
So instead we select a large ransom sample let's say $n = 100000$ and ask them if crimson is their favorite color (58200 say yes).
The proportion of this sample that prefers crimson is written as

$$\widehat{p}_1 = \frac{58200}{100000} = 0.582$$

We can use this value $\widehat{p}$ to make an inference about the true proportion p.

If we were to take another random sample we might get

$$\widehat{p}_2 = \frac{59300}{100000} = 0.593$$

And if we keep picking random samples from the population we expect $\widehat{p}$ to be different each time.
We see that $\widehat{p}$ is a function of the observed values of the random sample.
So $\widehat{p}$ is a value of a random variable that we can call P.

This idea is so important that it gets a name:

Definition 58 (Statistic)

Any function of the random variables constituting a random sample is called a statistic.

Some common statistics are:

1. Sample Mean
2. Sample Median
3. Sample Mode
4. Sample Variance
5. Sample Standard Deviation
6. Sample Range

Rephrasing from sections 1.3 and 1.4 Location Measures of a Sample

1. Sample Mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

2. Sample Median

$$\widetilde{X} = \begin{cases} X_{\frac{n+1}{2}}, n \text{ odd} \\ \frac{1}{2}\left(X_{\frac{n}{2}} + X_{\frac{n}{2}+1}\right), n \text{ even} \end{cases}$$

3. Sample Mode is the value of the sample that occurs most often.

When X_1 assumes the value x_1, X_2 assumes the value x_2, etc. then $\overline{X}$ assumes the value

$$\bar{x} = \frac{\sum x_i}{n}$$

and the term *sample mean* is used for both the statistic $\overline{X}$ and $\overline{x}$.

Variability Measures of a Sample

1. Sample Variance

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

2. Sample Standard Deviation

$$S = \sqrt{S^2}$$

3. Sample Range Let X_{max} denote the largest observation in the sample and X_{min} denote the smallest observation in the sample, then $R = X_{\mathsf{max}} - X_{\mathsf{min}}$.

As with the sample mean, the computed value of S^2 is denoted s^2. Note that S^2 is almost mean of the squares of the deviations of the observations from their mean except that we divide by $n-1$ instead of n.

Ex Find the location measures and the variability measures for the following data:

$$3, 4, 6, 6, 6, 7$$

One more variant of the variance formula:

Theorem 21

If S^2 is the variance of a random sample of size n, we may write

$$S^2 = \frac{1}{n(n-1)} \left[n \sum_{i=1}^{n} X_1^2 - \left(\sum_{i=1}^{n} X_i \right)^2 \right]$$

Chapter 6 Section 6

While the normal distribution is pretty awesome we still need other distributions for many applications. Both queuing theory and reliability problems use the exponential and gamma distributions. Exponential distributions are important for interarrival times.
We start with

Definition 59 (Gamma function)

The gamma function is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \qquad \textit{for } \alpha > 0.$$

which has properties where n is a positive integer:

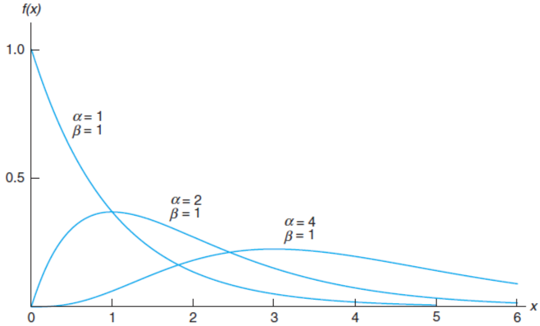
1. $\Gamma(n) = (n-1)(n-2)\cdots(1)\Gamma(1)$
2. $\Gamma(1) = 1$
3. $\Gamma(n) = (n-1)!$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Definition 60 (Gamma distribution)

The continuous random variable X has a gamma distribution, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} & , x > 0 \\ 0 & , o.w. \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.



Let us count the number of flaws in x millimeters of wire given that the mean number of flaws per millimeter is λ. This is a Poisson distribution, N, with mean λx and has pmf

$$f(n) = \frac{e^{-\lambda x}(\lambda x)^n}{n!}$$

Suppose that instead of counting the number of flaws in some length of wire we want to measure the distance between flaws.
Let X be the r.v. that is the length from any starting point on the wire until a flaw is detected.
(Assume that the wire is longer than the value of x.) Let's find the probability that the first flaw is longer than x:

$$P(X > x) = P(N = 0) = \frac{e^{-\lambda x}(\lambda x)^0}{0!} = e^{-\lambda x}$$

We can find the cdf

$$F(x) = P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

Finally we can find the pdf of X by differentiating the cdf:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

Definition 61 (Exponential distribution)

The continuous random variable X has an exponential distribution, with parameter λ, if its density function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$\mu = E[X] = \frac{1}{\lambda}, \quad \sigma^2 = \frac{1}{\lambda^2}$$

This is a special case of the gamma distribution with $\alpha = 1$ and $\beta = \lambda^{-1}$.

Ex In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour.
What is the probability that there are no log-ons in an interval of 6 minutes?

Ex In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour.
What is the probability that the time until the next log-on is between 2 and 3 minutes?

One of the cool properties is the lack of memory property that the exponential distribution has.

Ex Let X denote the time between detections of a particle with a Geiger counter and assume that X has an exponential distribution with $E[X] = 1.4$ minutes.
What is the probability that we detect a particle within 30 seconds of starting the counter?

Ex Let X denote the time between detections of a particle with a Geiger counter and assume that X has an exponential distribution with $E[X] = 1.4$ minutes.
Suppose we turn on the Geiger counter and wait 3 minutes without detecting a particle. What is the probability that a particle is detected in the next 30 seconds?

Property 2 (Lack of memory property)

For an exponential random variable X,

$$P(X < x + t | X > x) = P(X < t)$$

or

$$P(X \geq x + t | X \geq x) = P(X \geq t)$$

Ex The life of automobile voltage regulators has an exponential distribution with a mean life of six years. You purchase an automobile that is six years old, with a working voltage regulator, and plan to own it for six years.

1. What is the probability that the voltage regulator fails during your ownership?
2. If your regulator fails after you own the automobile three years and it is replaced, what is the mean time until the next failure?

Ex The time between the arrival of electronic messages at your computer is exponentially distributed with a mean of two hours.

1. What is the probability that you do not receive a message during a two-hour period?
2. If you have not had a message in the last four hours, what is the probability that you do not receive a message in the next two hours?
3. What is the expected time between your fifth and sixth messages?

Chapter 8 Section 3

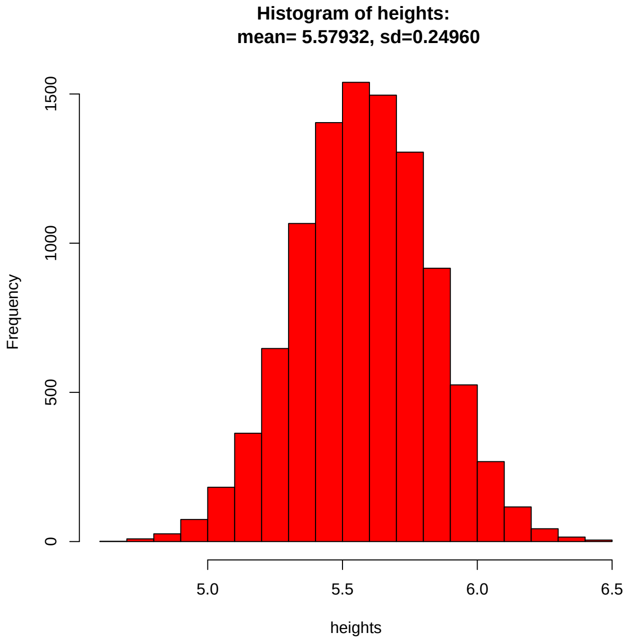
Let the random variable $X \sim N(5.57932, 0.24960)$. Suppose that there are 10000 data points in the population:

5.233732	5.589581	5.389242	5.633077	5.936384	5.766120	5.755057	5.522661
5.659584	5.224050	5.478727	5.828847	5.819704	5.809522	5.542258	5.274233
5.304091	5.691046	5.528762	5.998908	5.547169	5.530029	5.593728	5.409459
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5.570619	5.172144	5.624292	5.576875	5.481421	5.667891	5.799692	5.631164

The summary statistics for this population is:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Std Dev
4.692	5.410	5.580	5.579	5.751	6.467	0.24960

The histogram for this data set is:



Let us take a sample of size $n = 10$ from the population:

x_1	x_2	x_3	x_4	x_5	$\cdots$	x_9	x_{10}
5.318737	5.920416	5.554704	5.415792	5.277280	$\cdots$	5.235289	5.207904

We can take the average of these values:

x_1	x_2	x_3	x_4	$\ldots$	x_{10}	$\overline{x}$
5.318737	5.920416	5.554704	5.415792	$\ldots$	5.207904	5.441996

Now let us repeat this process 1,000,000 times:

x_1	x_2	x_3	x_4	$\ldots$	x_{10}	$\overline{x}$
5.318737	5.920416	5.554704	5.415792	$\ldots$	5.207904	5.441996
6.035656	5.547326	5.307189	6.095374	$\ldots$	6.004688	5.663764
5.632127	5.372547	5.694491	5.523235	$\ldots$	5.557954	5.549524
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
5.714549	5.700857	5.288253	5.363723	$\ldots$	5.776595	5.587488

Since these means, $\overline{x}$, are random they can be considered as the values of a random variable.

Definition 62 (Sampling distribution)

The probability distribution of a statistic is called a sampling distribution.

Let us call this random variable $\overline{X}$ and in this case it has the values:

$$5.441996, 5.663764, 5.549524, \ldots, 5.587488$$

It is called the *sampling distribution of the mean*.

Chapter 8 Section 4

Suppose that the population of interest has a normal distribution with mean μ and variance σ^2.
Let us suppose that we randomly sampled n (identical and independent) random variables from the population

$$X_1, X_2, \ldots, X_n$$

and $X_i \sim N(\mu, \sigma^2)$

Then

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

has

$$E[\overline{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}(\mu + \mu + \cdots + \mu)$$

so

$$E[\overline{X}] = \frac{1}{n}(n\mu) = \mu$$

and

$$V[\overline{X}] = V\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} V[X_i] = \frac{1}{n^2}(\sigma^2 + \sigma^2 + \cdots + \sigma^2)$$

so

$$V[\overline{X}] = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

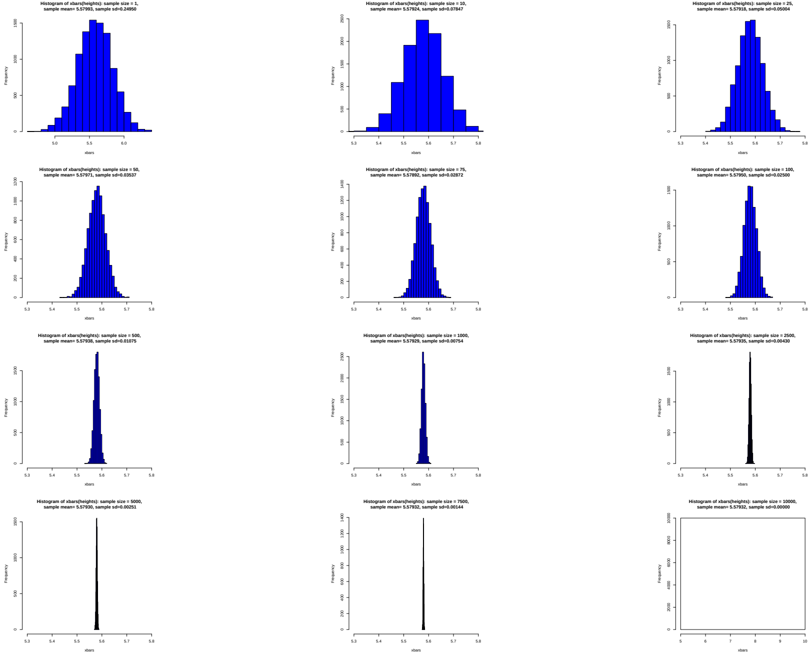
If we take a sample of sizes

$$n = 1, 10, 25, 50, 75, 100, 500, 1000, 2500, 5000, 10000$$

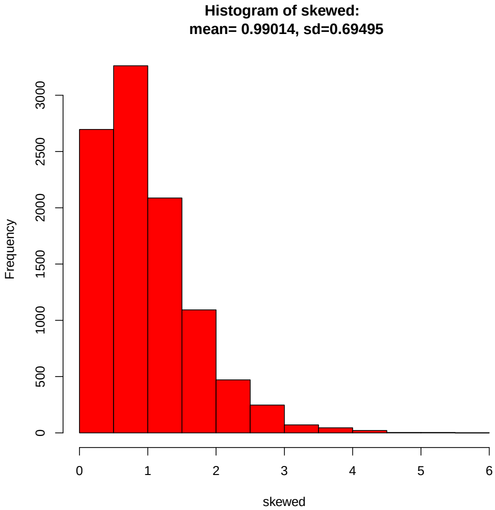
and then find their means and standard deviations we get

Sample size	$\overline{X}$	S
1	5.579931	0.249499947
10	5.579237	0.078466556
25	5.579184	0.050039821
50	5.579710	0.035372314
75	5.578922	0.028724527
100	5.579497	0.024999452
500	5.579379	0.010747983
1000	5.579292	0.007540395
2500	5.579346	0.004303151
5000	5.579302	0.002511437
7500	5.579323	0.001438324
10000	5.579316	0.000000000

Note that the sample means $\overline{X}$ hover around the population mean of $\mu = 5.57932$ for every sample size.
But as the sample size increases the sample standard deviations S approach 0.
When the sample size becomes the population size of $n = 10000$ the sample mean is the same as the population mean and the sample standard deviation equals 0.
We interpret these facts as saying that when we take a random sample its mean should be fairly close to the population mean and as the sample size increases the approximation gets better.
Also note that the distribution for each sample size is also normally distributed.


Histogram of xbars(heights): sample size = 1, sample mean= 5.57993, sample sd=0.24950
Histogram of xbars(heights): sample size = 10, sample mean= 5.57924, sample sd=0.07847
Histogram of xbars(heights): sample size = 25, sample mean= 5.57918, sample sd=0.05004
Histogram of xbars(heights): sample size = 50, sample mean= 5.57971, sample sd=0.03537
Histogram of xbars(heights): sample size = 75, sample mean= 5.57892, sample sd=0.02872
Histogram of xbars(heights): sample size = 100, sample mean= 5.57950, sample sd=0.02500
Histogram of xbars(heights): sample size = 500, sample mean= 5.57938, sample sd=0.01075
Histogram of xbars(heights): sample size = 1000, sample mean= 5.57929, sample sd=0.00754
Histogram of xbars(heights): sample size = 2500, sample mean= 5.57935, sample sd=0.00430
Histogram of xbars(heights): sample size = 5000, sample mean= 5.57930, sample sd=0.00251
Histogram of xbars(heights): sample size = 7500, sample mean= 5.57932, sample sd=0.00144
Histogram of xbars(heights): sample size = 10000, sample mean= 5.57932, sample sd=0.00000
Frequency
xbars

Let us repeat this experiment, but this time we will begin with a distribution that is not normal but skewed right with mean $\mu = 0.99014$ and standard deviation $\sigma = 0.69495$:



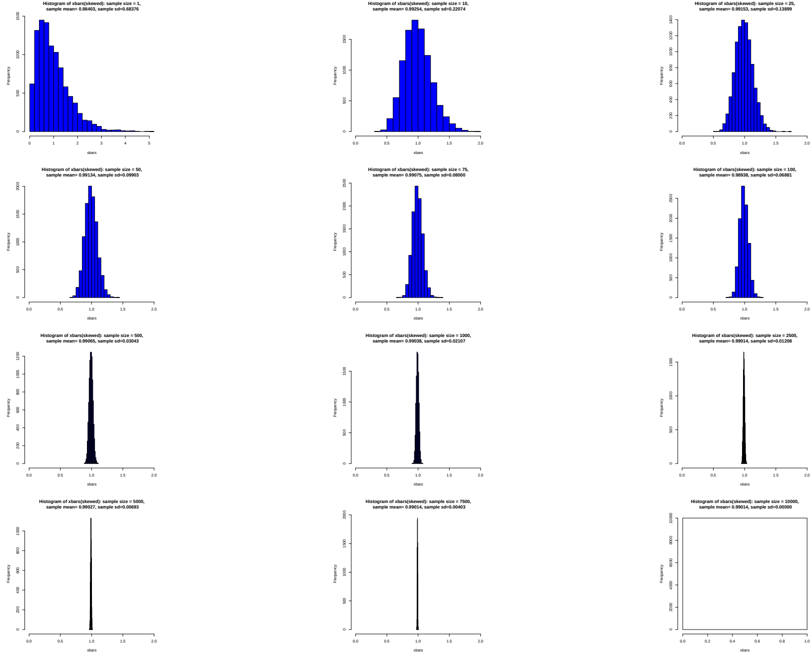
and with the following summary stats:

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Std Dev
0.0142	0.4763	0.8303	0.9901	1.3382	5.6951	0.69495

Finding samples of sizes

$$n = 1, 10, 25, 50, 75, 100, 500, 1000, 2500, 5000, 10000$$

and graphing their histograms we get:


Histogram of xbars(skewed): sample size = 1, sample mean= 0.98403, sample sd=0.68376
Histogram of xbars(skewed): sample size = 10, sample mean= 0.99254, sample sd=0.22074
Histogram of xbars(skewed): sample size = 25, sample mean= 0.99153, sample sd=0.13899
Histogram of xbars(skewed): sample size = 50, sample mean= 0.99134, sample sd=0.09903
Histogram of xbars(skewed): sample size = 75, sample mean= 0.99075, sample sd=0.08000
Histogram of xbars(skewed): sample size = 100, sample mean= 0.98938, sample sd=0.06881
Histogram of xbars(skewed): sample size = 500, sample mean= 0.99065, sample sd=0.03043
Histogram of xbars(skewed): sample size = 1000, sample mean= 0.99038, sample sd=0.02107
Histogram of xbars(skewed): sample size = 2500, sample mean= 0.99014, sample sd=0.01208
Histogram of xbars(skewed): sample size = 5000, sample mean= 0.99027, sample sd=0.00693
Histogram of xbars(skewed): sample size = 7500, sample mean= 0.99014, sample sd=0.00403
Histogram of xbars(skewed): sample size = 10000, sample mean= 0.99014, sample sd=0.00000
Frequency
xbars

The first thing to notice is that even though our original distribution was not normal but skewed right the sampling distributions start to look more normal as the sample size increases. Next looking at the sample means $\overline{X} = 0.9901385$ and sample standard deviations $S = 0.69495$:

Sample size	$\overline{X}$	S
1	0.9840327	0.683760893
10	0.9925392	0.220744517
25	0.9915268	0.138985857
50	0.9913373	0.099026387
75	0.9907530	0.080001154
100	0.9893763	0.068806402
500	0.9906488	0.030432479
1000	0.9903801	0.021065101
2500	0.9901397	0.012081798
5000	0.9902687	0.006932911
7500	0.9901378	0.004033101
10000	0.9901385	0.000000000

we notice that the sample means hover around the population mean of $\mu = 0.9901385$.

And because the sample standard deviations are going to 0 as the sample size increases we know that the sample means become better approximations to the population mean.
When the sample size becomes the population size the sample mean is exactly the population mean $\mu = \overline{X}$ and the sample standard deviation is $S = 0$.

So this property holds even when X_i come from some non-normal population with mean μ and variance σ^2.

Theorem 22 (Central Limit Theorem)

If $\overline{X}$ is the mean of a random sample of size n taken from a population with mean μ and finite variance σ^2, which the limiting form of a distribution of

$$Z = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Rule of thumb: $n \geq 30$ or if $n < 30$ then the population needs to be approximately a normal distribution.

Suppose the horses in a large stable have a mean weight of 873 lbs, and a variance of 17,161 lbs.

What is the probability that the mean weight of the sample of horses would be less than 856 lbs if 50 horses are sampled at random from the stable? Round your answer to four decimal places.

Suppose a batch of metal shafts produced in a manufacturing company have a standard deviation of 1.5 and a mean diameter of 208 inches.

If 60 shafts are sampled at random from the batch, what is the probability that the mean diameter of the sample shafts would be greater than 208.1 inches? Round your answer to four decimal places.

Appendix

Definition 63 (Set)

A set is a collection of objects. We usually denote sets by capitol letters from the end of the alphabet: X*,* Y*,* Z*.*

Definition 64 (Element)

The objects in a set are called elements. Elements are usually denoted by lower case letters: x*,* y*,* z*. If* x *is an element of the set* X *we denote this by* $x \in X$*.*

There are two ways of showing what elements are in a set.

Bracket notation The first way we will represent the elements in a set is to list the elements in the set between curly braces, $\{\}$. Let us have the set consisting of the elements 1,2, and 3. This set is written in bracket notation as: $\{1, 2, 3\}$.

Set builder notation The other way that we represent the elements in a set is to use what is known as set builder notation. This looks like

$$\{x \mid P(x)\}$$

and is read as

$\{$	x	$\mid$	$P(x)$	$\}$
the set of all	x	such that	x satisfies property P	

The property P is a proposition.

Example 0.1

Suppose E is the set of all positive even integers. Let write this using both bracket and set builder notation. For bracket notation we write

$$E = \{2, 4, 6, 8, \dots\}$$

and in set builder notation we would write

$$E = \{x \mid x \textit{ is an integer } \wedge x > 0 \wedge x \textit{ is divisible by } 2\}$$

If E is the set of all positive even integers, then 2 is an element of E. So we can write $2 \in E$. Since 3 is odd we have $3 \notin E$.

Now suppose we have the sets $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. We note that every element of A is also an element of B, since $1 \in A, 1 \in B$, $2 \in A, 2 \in B$ and $3 \in A, 3 \in B$. We say that A is a subset of B.

Definition 65 (Subset)

The set A is a subset of B, $A \subset B$, if for every $a \in A$, then $a \in B$.

So $\{1, 2, 3\} \subset \{1, 2, 3, 4\}$, but $\{1, 2, 3, 4\}$ is not a subset of $\{1, 2, 3\}$.
There is a special set that contains no elements.

Definition 66 (Empty set)

The empty set is the set that has no elements. It is denoted by $\emptyset$ or $\{\}$

Let $A = \{1, 2, 3\}$. Is $\emptyset \subset A$?

It turns out that the empty set is a subset of any set. Why? Well in order for A to be subset of B each element of A must also be an element of B. We can restate this as

If $x \in A$, then $x \in B$.

So, if we ask, "Is the empty set a subset of the set X?", then we must have:

If $x \in \emptyset$, then $x \in X$

Let us look at the statement $x \in \emptyset$. Is this true or false? Since the empty set has no elements, the statement that x is an element if $\emptyset$ must be false. We recall from the truth table for conditionals that if the hypothesis is false, then the conditional is true. So the empty set is a subset of every set.

Theorem 23

For any set A, $\emptyset \subset A$.

Proof.

Let A be any set. In order for $\emptyset \subset A$ then the following must be true, if $a \in \emptyset$, then $a \in A$. Since it is false that $a \in \emptyset$ by the definition of the empty set, the conditional must be true. This is because any time the antecedent of a conditional is false the conditional must be true. Hence $\emptyset \subset A$. □

Suppose we want to combine two sets together. So for example we want to combine the sets

$$\{2, 4, 6, 8, 10\}$$

and

$$\{1, 3, 5, 7, 9\}.$$

We want this combined set to be

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

The way we do this is to take the union of the sets.

Definition 67 (Union)

The union of the sets A and B is

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

So x is an element of $A \cup B$ if x is in A or x is in B.

Let us see some examples.

Example 0.2

1. *Let* $A = \{1, 2, 3\}$ *and* $B = \{4, 5, 6\}$
   Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
2. *Let* $A = \{1, 2, 4\}$ *and* $B = \{3, 5, 6\}$
   Then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.
3. *Let* $A = \{1, 2, 3\}$ *and* $B = \{3, 4, 5\}$
   Then $A \cup B = \{1, 2, 3, 4, 5\}$.
4. *Let* $A = \{2, 4, 6, \dots\}$ *and* $B = \{1, 3, 5, \dots\}$
   Then $A \cup B = \{1, 2, 3, 4, \dots\}$.

Now what if we wanted a set that only contained elements that are in both sets A and B. That is called the intersection.

Definition 68 (Intersection)

The intersection of the sets A *and* B *is*

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$

So x *is an element of* $A \cap B$ *if* x *is in* A *and* x *is in* B*.*

Let us see some examples.

Example 0.3

1. *Let* $A = \{1, 2, 3\}$ *and* $B = \{2, 3, 4\}$
   Then $A \cap B = \{2, 3\}$*.*
2. *Let* $A = \{1, 2, 3, 4\}$ *and* $B = \{3, 4, 5, 6\}$
   Then $A \cap B = \{2, 3, 4\}$*.*
3. *Let* $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$ *and*
   $B = \{2, 3, 5, 7, 11, 13, 17, 19\}$
   Then $A \cap B = \{3, 5, 7, 11, 13, 17, 19\}$*.*
4. *Let* $A = \{2, 4, 6, 8, 10\}$ *and* $B = \{1, 3, 5, 7, 9\}$
   Then $A \cap B = \emptyset$*.*

Let us set up some notation for the different sets of numbers that we commonly use.

Name(Symbol)	Set
Natural numbers($\mathbb{N}$)	$\{1, 2, 3, \ldots\}$
Integers($\mathbb{Z}$)	$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Rational numbers($\mathbb{Q}$)	$\{\frac{m}{n} \mid m, n \in \mathbb{Z} \wedge n \neq 0\}$
Irrational numbers($\mathbb{J}$)	$\{x \mid x$ has nonrepeat. nonterminat. decimal$\}$
Real numbers($\mathbb{R}$)	$\mathbb{Q} \cup \mathbb{J}$

Since some sets are infinite, let us think about what it means to be infinite. For the moment we will restrict ourselves to the natural numbers, $\mathbb{N}$. The first thing we notice is that there is no largest natural number. Why? Well suppose there is a largest natural number. Lets call it L. But $L + 1$ is bigger than L. So L cannot have been the biggest natural number. Hence there is no biggest natural number.

Now there is a smallest natural number, 1. What about the positive real numbers? Is there a smallest positive real number? Suppose there is a smallest real number. Let us call it a. So we have $0 < a$ and there does not exist a positive real number b such that $0 < b < a$, otherwise b would be the smallest positive real number. Now we recall that given any positive number if we take half of it it becomes smaller. So $0 < \frac{a}{2} < a$. But this contradicts our assumption that a was the smallest positive real number. Therefore there must not be a smallest positive real number.

Let us look at another interesting thing that happens with infinity. Suppose we have an urn and an infinite number of balls numbered $1, 2, 3, \ldots$. Let us also suppose that we can move as fast as we want to. Now here is the situation. At 11:00 we put the first ten balls, $1, 2, \ldots, 10$, in the urn and draw out ball numbered 1. Next at 11:30 (half way to 12:00) we put the next ten balls, $11, 12, \ldots, 20$, in the urn and draw out ball number 11. Continuing the process of putting in the next ten balls and pulling out the last ball we put in half way to 12:00 we get:

Time	Put in balls	Removed ball
11:00	$\{1, \ldots, 10\}$	10
11:30	$\{11, \ldots, 20\}$	20
11:45	$\{21, \ldots, 30\}$	30
11:52:30	$\{31, \ldots, 40\}$	40
$\vdots$	$\vdots$	$\vdots$

How many balls are in the urn at 12:00?

Now suppose we do the same process as above except we first remove ball number 1, then ball number 2, then ball number 3, etc. So

Time	Put in balls	Removed ball
11:00	$\{1, \ldots, 10\}$	1
11:30	$\{11, \ldots, 20\}$	2
11:45	$\{21, \ldots, 30\}$	3
11:52:30	$\{31, \ldots, 40\}$	4
$\vdots$	$\vdots$	$\vdots$

Now how many balls will be in the urn at 12:00?

Theorem 24 (Distributive law)

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Theorem 25 (De Morgan's law)

- $(A \cap B)' = A' \cup B'$
- $(A \cup B)' = A' \cap B'$