MS 114 In-class Problems

November 11, 2025

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The table below is taken from the website of the U.S. Department of Labor. It shows the minimum wage for each decade from 1950 to 2010. The figures are adjusted for inflation and expressed in constant 2012 dollars.

$y =$ Year	$m =$ Minimum wage
1950	\$7.01
1960	\$7.59
1970	\$9.28
1980	\$8.46
1990	\$6.66
2000	\$6.90
2010	\$7.67

The table below shows the highest grossing movies of the given year. The amount is the domestic box office gross, in millions of dollars. Let $G(y)$ be the gross of the highest grossing movies of that given year.

Year	Movie	Amount (millions)
2006	Pirates of the Caribbean: Dead Man's Chest	423.32
2007	Spider-Man 3	336.53
2008	The Dark Knight	533.35
2009	Avatar	760.51
2010	Toy Story 3	415.00
2011	Harry Potter and the Deathly Hallows: Part 2	381.01
2012	The Avengers	623.28
2013	The Hunger Games: Catching Fire	424.67
2014	American Sniper	350.13

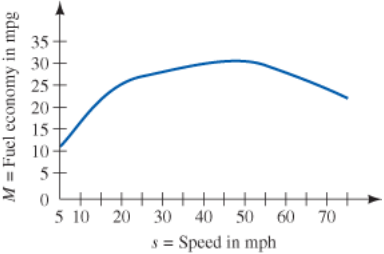
A chart from Dick's Sporting Goods gives the recommended bat length B in inches for a man weighing between 161 and 170 pounds as a function of his height h in inches. The table is partially reproduced below.

h = Height	B = Bat length
45–48	30
49–52	31
53–56	31
57–60	32
61–64	32
65–68	33
69–72	33
73+	33

Chapter 1 Section 3

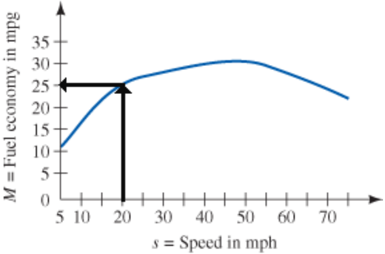
Many factors affect fuel economy, but a website maintained by the U.S. government warns that "gas mileage usually decreases rapidly at speeds above miles per hour...."

The site includes the graph below (Fig 1.7 in our text book). It shows that the fuel economy M as a function of the speed s, so that $M = M(s)$.



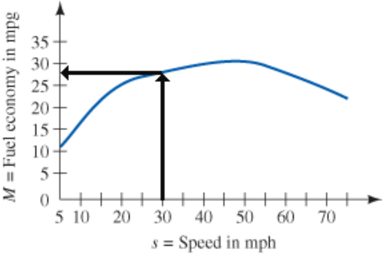
It is customary to describe this as a graph of M versus s or as a graph of M against s. These phrases indicate that the horizonal axis corresponds to s and the vertical axis corresponds to M.

What is the fuel economy at 20 mph?
Find the value 20 on the horizontal axis then go vertical until you hit the graph. Next go horizontal from that point until you hit the vertical axis.



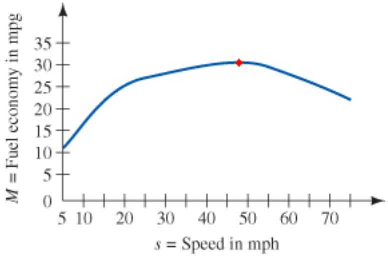
We can see that this hits the vertical axis at 25.
Written in functional notation we have $M(20) = 25$.

What if we repeat this at $s = 30$?



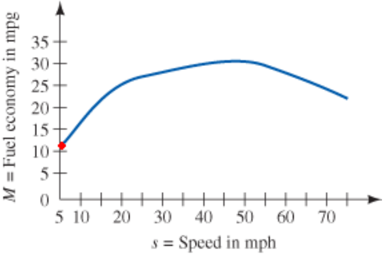
Now it is harder to see where you are hitting the vertical axis, but it looks to be close to 28, so $M(30) = 28$.
The graphical representation of a function may only allow us to make approximations of the function values. But they also allow us to see important features of the graph that cannot be seen from an equation.

One of these is the maximum value of the function.



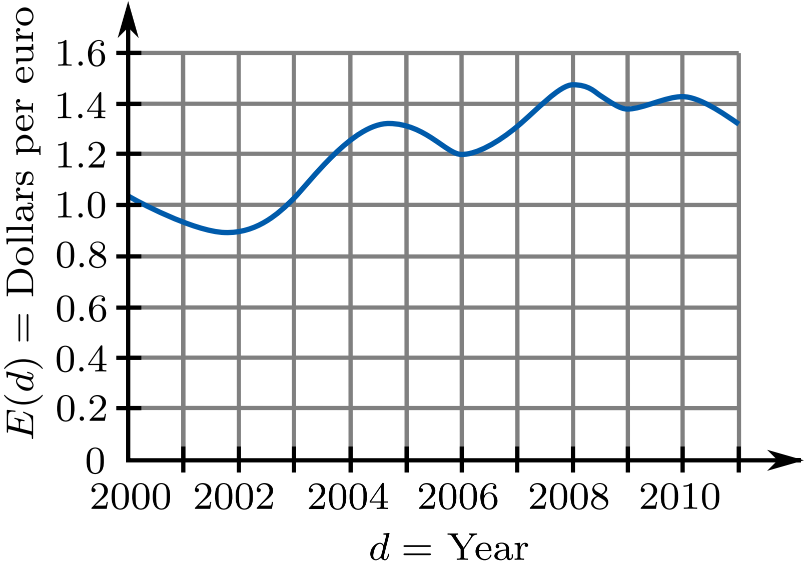
From this we can see that the maximum fuel economy is around 30 mpg and this occurs at around 50 mph.
When a function increases to a point and then decreases from that point that will always give us a maximum.
When a function decreases to a point and then increases that point will be a minimum.

Another place where a max or min can occur is at the endpoints of the function.

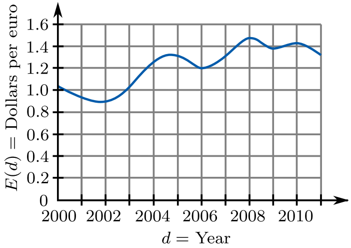


We see that the function has a minimum value of around 11 mpg at a speed of 5 mph.

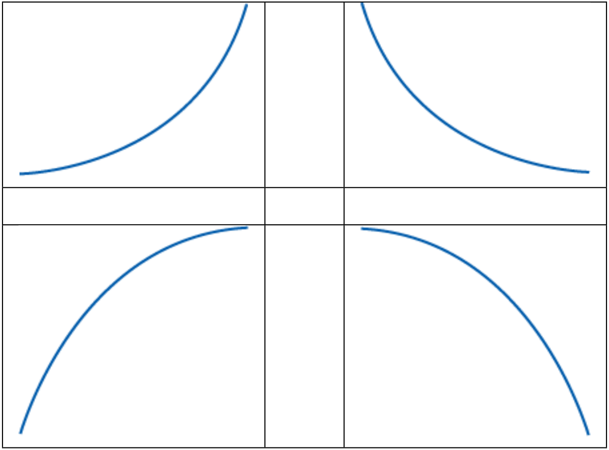
The value $E = E(d)$, in U.S. dollars, of the euro as a function of the date d is shown in the graph below.



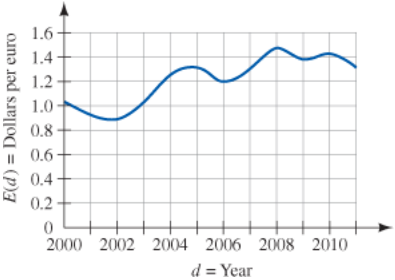
The value $E = E(d)$, in U.S. dollars, of the euro as a function of the date d is shown in the graph below.



1. Explain the meaning of $E(2003)$ and estimate its value.
2. From 2000 through 2011, what was the largest value the euro attained? When did that happen?
3. What was the average yearly increase in the value of the euro from 2006 to 2009?
4. During which one-year period was the graph increasing most rapidly?
5. As an American investor, would you have made money if you bought euros in 2002 and sold them in 2008?



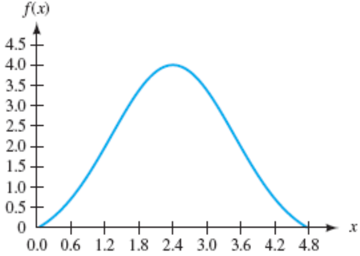
The value $E = E(d)$, in U.S. dollars, of the euro as a function of the date d is shown in the graph below.



1. From 2000 to late 2003, is the graph concave up or concave down?

2. Explain in practical terms what the concavity means about the value of the euro during this period.

1.3.SB.005 The following is the graph of a function $f = f(x)$.

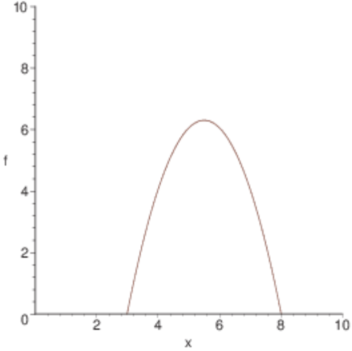


1. What is the value of $f(0.6)$?

2. What is the smallest value of x for which $f(x) = 1.5$?

3. What is the largest value of x for which $f(x) = 2.5$?

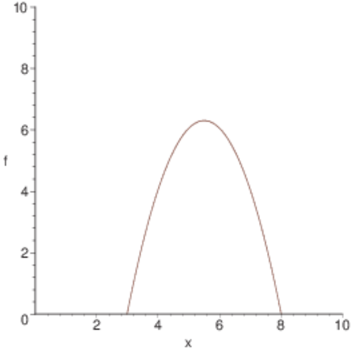
1.3.SB.007* The following is the graph of a function $f = f(x)$. Where does the graph reach a maximum, and what is that maximum value?



1. $x =$

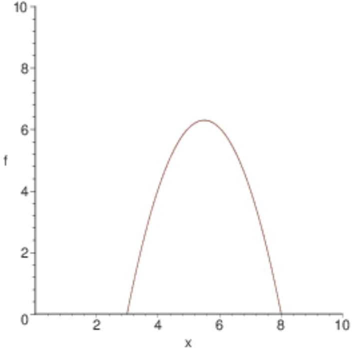
2. $f(x) =$

1.3.SB.008 The following is the graph of a function $f = f(x)$. Where is the graph increasing?



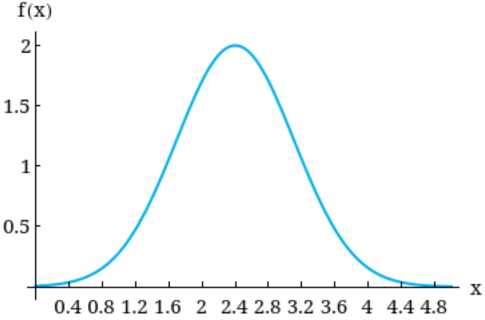
- $3.0 < x < 5.5$
- $x > 3.0$
- $x < 5.5$
- $x < 3.0 \cup x > 5.5$
- $-\infty < x < \infty$

1.3.SB.009 The following is the graph of a function $f = f(x)$. Where is the graph decreasing?



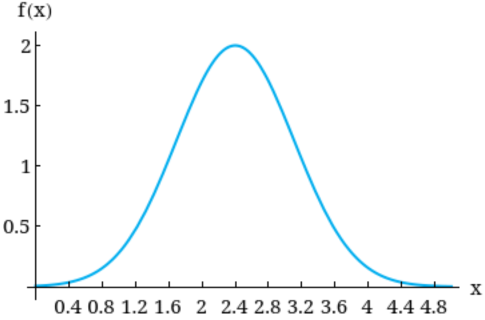
- $5.5 < x < 8.0$
- $x < 8.0$
- $x > 5.5$
- $x < 5.5 \cup x > 8.0$
- $-\infty < x < \infty$

1.3.SB.011* The following is the graph of a function $f = f(x)$. What is the concavity of the graph between $x = 0.4$ and $x = 1.6$?



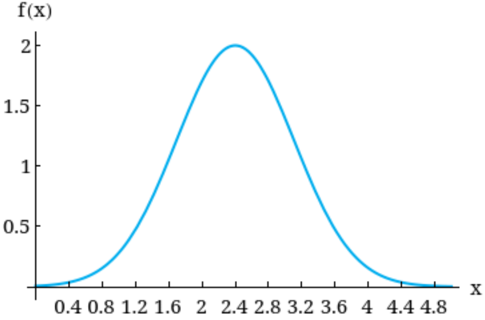
- concave up
- concave down

1.3.SB.012* The following is the graph of a function $f = f(x)$. What is the concavity of the graph between $x = 3.2$ and $x = 4.8$?



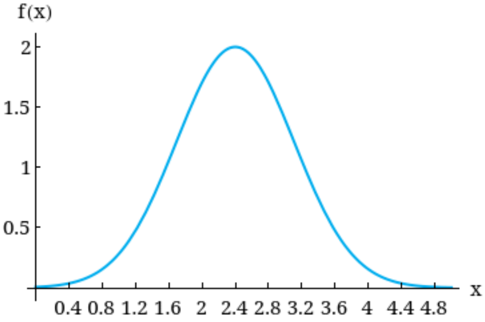
- concave up
- concave down

1.3.SB.013* The following is the graph of a function $f = f(x)$. What is the concavity of the graph between $x = 2.0$ and $x = 2.8$?



- concave up
- concave down

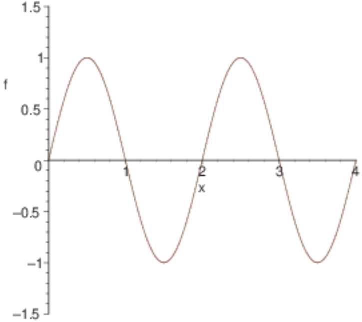
1.3.SB.014* The following is the graph of a function $f = f(x)$. Where on the graph are there points of inflection?



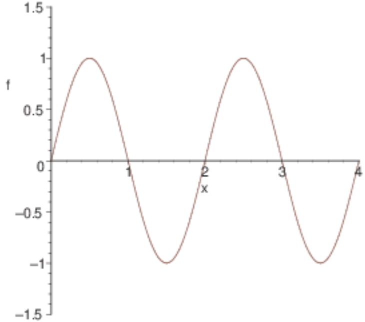
1. $x =$ (smaller x-value)
2. $x =$ (larger x-value)

1.3.SB.017 The following is the graph of a function $f = f(x)$. At what values of x does the graph reach

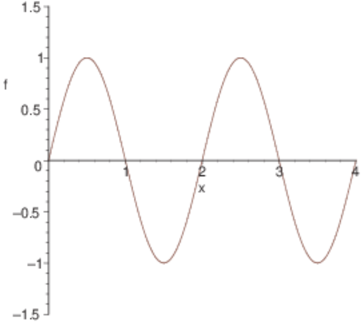
1. a maximum value?
2. a minimum value?



1.3.SB.023* The following is the graph of a function $f = f(x)$. Where is the graph increasing and concave up?



1.3.SB.019* The following is the graph of a function $f = f(x)$. Find the x-value of the inflection points of $f(x)$.



Chapter 1 Section 4

Suppose there are initially 2000 bacteria in a petri dish. The bacteria reproduce by cell division, and each hour the number of bacteria doubles. This is a verbal description of a function $N = N(t)$, where N is the number of bacteria present at time t.
It is common in situations like this to begin at time $t = 0$.
Thus, $N(0)$ is the number of bacteria we started with, 2000.

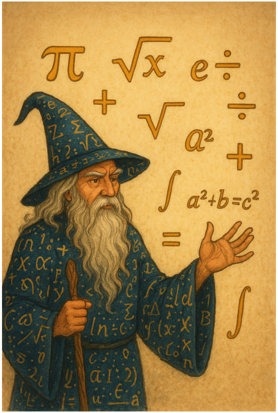
$N(0) =$

$N(1) =$

$N(2) =$

$N(3) =$

Finding $N(6)$ or any larger integer wouldn't be difficult, though tedious, how would we find the value for $t = 4.5$, $N(4.5) =$? Using "mathemagics"



we get the function

$$N(t) = 2000 \cdot 2^t$$

Now given

$$N(t) = 2000 \cdot 2^t$$

$N(0) =$

$N(1) =$

$N(2) =$

$N(3) =$

and now we can find:

$N(4.5) =$

A rental car agency charges \$15 per day plus \$0.27 per mile to rent a car. Determine a function that can be used to calculate the cost of daily car rentals.

Step 1 Identify the function and the things on which it depends, and write the relationships you know in a formula using words.

Step 2 Select and record letter names for the function and for each of the variables involved, and state their units.

Step 3 Replace the words in Step 1 by the letters identified in step 2 and appropriate information from the verbal description.

A rental car agency charges \$15 per day plus \$0.27 per mile to rent a car. Determine a function that can be used to calculate the cost of daily car rentals.

Step 1

Step 2

Step 3

Ex 1 Extreme Protection, Inc. manufactures helmets for skiing and snowboarding. The fixed costs for one model of helmet are \$4700 per month. Materials and labor for each helmet of this model are \$50, and the company sells this helmet to dealers for \$70 each. (Let x represent the number of helmets sold. Let C and R be measured in dollars.)

1. For this helmet, write the function for monthly total costs $C(x)$.

2. Write the function for total revenue $R(x)$.

Ex 2 A manufacturer of DVD players has weekly fixed costs of \$1,520 and variable costs of \$13.50 per unit for one particular model. The company sells this model to dealers for \$18.50 each.

1. For this model DVD player, write the function for weekly total costs $C(x)$.

2. Write the function for total revenue $R(x)$.

Ex 3 The federal tax code allows some items used for business purposes to be *depreciated*. That is, their taxable value decreases over time. A new van used in your delivery business has a taxable value of \$22,000. The tax code allows you to depreciate this van by \$2300 per year. Find a formula that gives the taxable value T, in dollars, of the van after n years of depreciation.

Ex 4 The total investment a jeweler has in a gem-quality diamond is the price paid for the rough stone plus the amount paid to work the stone. Suppose the gem cutter earns 40 per hour. Find the function that calculates the total investment of working a stone. If a particular the stone costs \$320 what would the investment be?

Chapter 3 Section 1

Figure 3.1

A ramp on a retaining wall

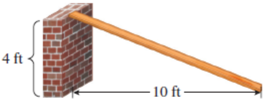


Figure 3.2

Representing the ramp line on coordinate axes

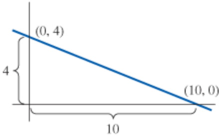


Figure 3.3

The roof of a building

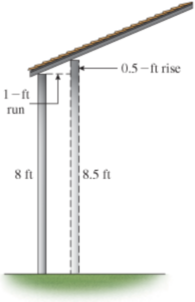


Figure 3.5

Extending the roof line to its peak

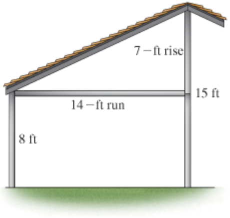


Figure 3.6

Finding the horizontal intercept

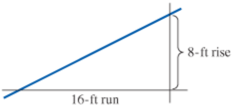


Figure 3.9

A circus tent

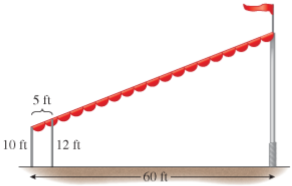
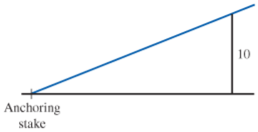


Figure 3.10

An anchor rope



Chapter 3 Section 2

Suppose the CEO of a company wants to have a dinner catered for employees. He finds that he must pay a dining hall rental fee of \$275 and an additional \$28 for each meal served.
Then the cost $C = C(n)$, in dollars, is a function of the number n of meals served. If an unexpected guest arrives for dinner, then the CEO will have to pay an additional \$28.
Suppose, for example, that the caterer, having anticipated that 35 people would attend the dinner, sent the CEO a bill for \$1255. But 48 people actually attended the dinner. What should the total price of the dinner be under these circumstances?

Ex 3.3 Oklahoma Income Tax The amount of income tax $T = T(I)$, in dollars, owed to the state of Oklahoma is a linear function of the taxable income I, in dollars, at least over a suitably restricted range of incomes.
According to the Oklahoma Income Tax table for the year 2015, a single Oklahoma resident taxpayer with a taxable income of $15,000 owes $579 in Oklahoma income tax. In functional notation, this is $T(15,000) = 579$. If the taxable income is $15,500, then the table shows a tax liability of $605.

Ex 3.3 Oklahoma Income Tax Part 1

$T = T(I)$, $T(15,000) = 579$. If the taxable income is $15,500, then the table shows a tax liability of $605.

Calculate the rate of change in T with respect to I, and explain in practical terms what it means.

Ex 3.3 Oklahoma Income Tax Part 2

$T = T(I)$, $T(15,000) = 579$. If the taxable income is \$15,500, then the table shows a tax liability of \$605.

How much does the taxpayer owe if the taxable income is \$15,350?

Ex 3.3 Oklahoma Income Tax Part 1 Function

$T = T(I)$, $T(15,000) = 579$. If the taxable income is \$15,500, then the table shows a tax liability of \$605.

Calculate the rate of change in T with respect to I, and explain in practical terms what it means.

Ex 3.3 Oklahoma Income Tax Part 2 Function

$T = T(I)$, $T(15,000) = 579$. If the taxable income is \$15,500, then the table shows a tax liability of \$605.

How much does the taxpayer owe if the taxable income is \$15,350?

If we look more carefully at the catered dinner, we can write a formula for the total cost $C = C(n)$ when there are n dinner guests:

$$\text{Cost} = \text{Cost of food} + \text{Rent}$$

Ex 3.4 Selling Jewelry at an Art Fair

Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32 each.

Ex 3.4 Selling Jewelry at an Art Fair Part 1

Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32 each.

Explain why the function that shows your net income (revenue from sales minus rental fee) as a function of the number of necklaces sold is a linear function.

Ex 3.4 Selling Jewelry at an Art Fair Part 2

Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32 each.
Write a formula for this function.

Ex 3.4 Selling Jewelry at an Art Fair Part 3

Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32 each.

Use functional notation to show your net income if you sell 25 necklaces, and then calculate that value.

Ex 3.4 Selling Jewelry at an Art Fair Part 4, 5, 6

Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32 each.

Plot

Initial value

Horizontal intercept

Getting a linear equation if you know the slope and initial value:
In Example 3.4, we were effectively told the slope, 32, of the linear function P and its initial value, -192.

Getting a linear equation if you know the slope and one data point:

Suppose that we are given the information for Example 3.4 in a different way: We are told that the price for each necklace is \$32 and that if we sell $n = 8$ necklaces, we will have a net income of $P = 64$ dollars.

Getting a linear equation from two data points:

Suppose the information in Example 3.4 were given as two data points.

For example, suppose that we make a net income of $P = 64$ dollars when we sell $n = 8$ necklaces and a net income of $P = 160$ dollars when $n = 11$ necklaces are sold.

In general, if you are given two data points and you need to find the linear function that they determine, then you proceed in two steps.
First, use the formula

$$m = \frac{\text{change in function}}{\text{change in variable}}$$

to compute the slope.
Next, use the slope you found and put either of the given data points into the equation

$$y \ = \text{Slope} \cdot x + \text{Initial value}$$

to solve for the initial value.

Ex 3.5 Changing Celsius to Fahrenheit

The temperature $F = F(C)$ in Fahrenheit is a linear function of the temperature C in Celsius.

A lab assistant placed a Fahrenheit thermometer beside a Celsius thermometer and observed the following:

When the Celsius thermometer reads 30 degrees ($C = 30$), the Fahrenheit thermometer reads 86 degrees ($F = 86$).

When the Celsius thermometer reads 40 degrees, the Fahrenheit thermometer reads 104 degrees.

Ex 3.5 Changing Celsius to Fahrenheit Part 1

The temperature $F = F(C)$ in Fahrenheit is a linear function of the temperature C in Celsius. When $C = 30$ then $F = 86$ and when $C = 40$ then $F = 104$ degrees.

Use a formula to express F as a linear function of C.

Ex 3.5 Changing Celsius to Fahrenheit Part 2

The temperature $F = F(C)$ in Fahrenheit is a linear function of the temperature C in Celsius. When $C = 30$ then $F = 86$ and when $C = 40$ then $F = 104$ degrees.

At sea level, water boils at 212 degrees Fahrenheit. What temperature in degrees Celsius makes water boil?

Ex 3.5 Changing Celsius to Fahrenheit Part 3

The temperature $F = F(C)$ in Fahrenheit is a linear function of the temperature C in Celsius. When $C = 30$ then $F = 86$ and when $C = 40$ then $F = 104$ degrees.

Explain in practical terms what the slope means in this setting.

Ex 3.2.005 If you take a brisk walk on a flat surface, you will burn about 258 calories per hour. You have just finished a hard workout that used 700 calories.

1. Find a formula that gives the total calories burned if you finish your workout with a walk of h hours.

2. How long do you need to walk at the end of your workout in order to burn a total of 1100 calories?

Ex 3.2.015 A study of average driver speed on rural highways by A. Taragin found a linear relationship between average speed S, in miles per hour, and the amount of curvature D, in degrees, of the road. On a straight road $(D = 0)$, the average speed was found to be 46.26 miles per hour. This was found to decrease by 0.746 mile per hour for each additional degree of curvature.

1. Find a linear formula relating speed to curvature.

2. Express using functional notation the speed for a road with a curvature of 10 degrees, and then calculate that value.

Chapter 3 Section 3

Information about physical and social phenomena is frequently obtained by gathering data or sampling.

Once data are gathered, an important key to further analysis is to produce a *mathematical model* describing the data.

A model is a function that

1. represents the data either exactly or approximately and
2. incorporates patterns in the data.

In many cases, such a model takes the form of a linear function.

Testing Data for Linearity

Suppose you have a set of data, how can you try to check if the data is linear?

One of the most important events in the development of modern physics was Galileo's description of how objects fall.
In about 1590, he conducted experiments in which he dropped objects and attempted to measure their downward velocities $V = V(t)$ as they fell.
Here we measure velocity V in feet per second and time t as the number of seconds after release.
If Galileo had been able to nullify air resistance, and if he had been able to measure velocity without any experimental error at all, he might have recorded the following table of values for a rock.

t in seconds	0	1	2	3	4	5
V feet per second	0	32	64	96	128	160

Recall that for a linear process the rate of change is constant. So let us look at the change in time vs the change in velocity.

t in seconds	0		1		2		3		4		5
Δt											
V feet per second	0		32		64		96		128		160
ΔV											

Note we have a constant change in time of 1 second and a constant change in velocity (ROC) of 32 feet per second. The data is **Linear!**

Now to find the linear function that describes this data.
Since it is linear we know that $V = mt + b$.
What is the slope?

What about b?

Linear Models

This linear function serves as a *mathematical model* for the experimentally gathered data, and it gives us more information than is apparent from the data table alone.

Question:

What is the velocity at time $t = 3.6$ seconds?

t in seconds	0	1	2	3	4	5
V feet per second	0	32	64	96	128	160

The key physical observation that Galileo made was that falling objects have constant acceleration, 32 feet per second per second. He did additional experiments to show that if air resistance is ignored, then this acceleration does not depend on the weight or size of the object.

The same table of values would result, and the same acceleration would be calculated, whether the experiment was done with a pebble or with a cannonball.

According to tradition, Galileo conducted some of his experiments in public, dropping objects from the top of the leaning tower of Pisa.

His observations got him into serious trouble with the authorities because they conflicted with the accepted premise of Aristotle that heavier objects would fall faster than lighter ones.

If instead of measuring velocity, we had measured the distance $D = D(t)$ that the rock fell, we would have obtained the following data.

t in seconds	0	1	2	3	4	5
D feet	0	16	64	144	256	400

Ex 3.6 Sampling Voter Registration In this hypothetical experiment, a political analyst compiled data on the number of registered voters in Payne County, Oklahoma, each year from 2011 through 2016. For each of the following possible data tables that the analyst might have obtained, determine whether the data can be modeled with a linear function. If so, find such a formula and predict the number of registered voters in Payne County in the year 2018.

1.

Date	2011	2012	2013	2014	2015	2016
Registered voters	28321	28542	29466	30381	30397	31144

2.

Date	2011	2012	2013	2014	2015	2016
Registered voters	28321	28783	29245	29707	30169	30631

Ex 3.6 Sampling Voter Registration Data Set 1 For the following possible data table, determine whether the data can be modeled with a linear function. If so, find such a formula and predict the number of registered voters in Payne County in the year 2018.

Date	2011	2012	2013	2014	2015	2016
Registered voters	28321	28542	29466	30381	30397	31144

Ex 3.6 Sampling Voter Registration Data Set 2 For the following possible data table, determine whether the data can be modeled with a linear function. If so, find such a formula and predict the number of registered voters in Payne County in the year 2018.

Date	2011	2012	2013	2014	2015	2016
Registered voters	28321	28783	29245	29707	30169	30631

Ex 3.6 Sampling Voter Registration Data Set 2 For the following possible data table, determine whether the data can be modeled with a linear function. If so, find such a formula and predict the number of registered voters in Payne County in the year 2018.

d	0	1	2	3	4	5
Date	2011	2012	2013	2014	2015	2016
Registered voters	28321	28783	29245	29707	30169	30631

Graphing Discrete Data

Calculating differences can always tell you whether data are linear, but many times it is advantageous to view such data graphically.

Lets use Desmos to graph our data:

d	R
0	28321
1	28542
2	29466
3	30381
4	30397
5	31144

d	R
0	28321
1	28783
2	29245
3	29707
4	30169
5	30631

Ex 3.7 Newton's Second Law of Motion

Newton's second law of motion shows how force on an object, measured in newtons, is related to acceleration of the object, measured in meters per second per second. The following experiment might be conducted in order to discover Newton's second law. Objects of various masses, measured in kilograms, were given an acceleration of 5 meters per second per second, and the associated forces were measured and recorded in the table below.

Mass	1	1.3	1.6	1.9	2.2
Force	5	6.5	8	9.5	11

Ex 3.7 Newton's Second Law of Motion Part 1

Mass	1	1.3	1.6	1.9	2.2
Force	5	6.5	8	9.5	11

Check differences to show that these are linear data.

Ex 3.7 Newton's Second Law of Motion Part 2

Mass	1	1.3	1.6	1.9	2.2
Force	5	6.5	8	9.5	11

Find the slope of a linear model for the data, and explain in practical terms what the slope means.

Ex 3.7 Newton's Second Law of Motion Part 3

Mass	1	1.3	1.6	1.9	2.2
Force	5	6.5	8	9.5	11

Construct a linear model for the data.

Ex 3.7 Newton's Second Law of Motion Part 4

Mass	1	1.3	1.6	1.9	2.2
Force	5	6.5	8	9.5	11

What force does your model show for an object of
mass 1.43 kilograms that is accelerating at 5 meters per second?

Ex 3.7 Newton's Second Law of Motion Part 5

Mass	1	1.3	1.6	1.9	2.2
Force	5	6.5	8	9.5	11

Make a graph showing the data, and overlay it with the graph of the linear model you made in part 3.

Chapter 3 Section 4

Linear Regression

In real life, rarely is information gathered that fits any simple formula perfectly.

In cases such as government spending, many factors influence the budget, including the political make-up of the legislature.

In the case of scientific experiments, variations may be due to *experimental error*, the inability of the data gatherer to obtain exact measurements; there may also be elements of chance involved.

Under these circumstances, it may be necessary to obtain an approximate rather than an exact mathematical model.

The Regression Line

To illustrate this idea, let's look at federal Medicare expenditures, in billions of dollars, in the United States as reported by the Centers for Medicare and Medicaid Services and recorded in the following table.

Date	2009	2010	2011	2012	2013
Expenditures in billions	498.8	520.5	546.1	569.2	586.3

Let's check for linearity.

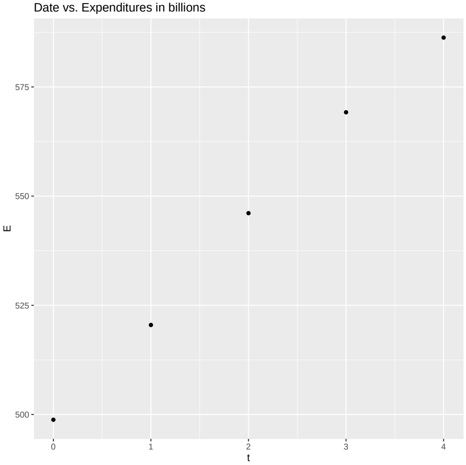
Date	2009		2010		2011		2012		2013
Expenditures in billions	498.8		520.5		546.1		569.2		586.3
ΔE		21.7		25.6		23.1		17.1	

It is not linear, but how far from linear is it?

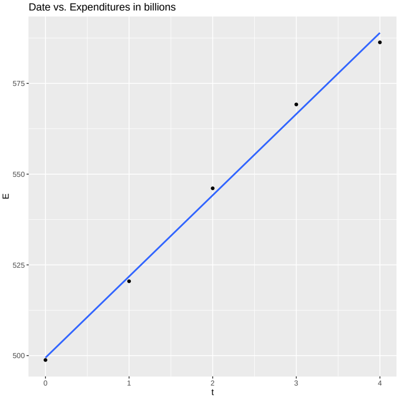
Let's alter the table by letting t be the number of years since 2009:

t	0	1	2	3	4
E	498.8	520.5	546.1	569.2	586.3

If we plot these points in a scatter plot we get:



This looks close to linear. It turns out there we can find a line that "best fits" this data. It is called the least-squares fit or linear regression line.
This is shown below:



So even though there is no line that perfectly fits the data we can find a line that best approximates the data.

Desmos graph

Let's graph this data, draw the regression line, and find it's equation. To do this we will use `https://www.desmos.com/calculator` with data:

$$\begin{array}{ll} 0, & 498.8 \\ 1, & 520.5 \\ 2, & 546.1 \\ 3, & 569.2 \\ 4, & 586.3 \end{array}$$

Desmos graph

Let's graph this data, draw the regression line, and find it's equation. To do this we will use `https://www.desmos.com/calculator` with data:

$$\begin{array}{ll} 0, & 498.8 \\ 1, & 520.5 \\ 2, & 546.1 \\ 3, & 569.2 \\ 4, & 586.3 \end{array}$$

This gives the regression line model for E as a function of t:

$$y = 22.37x + 499.44$$

$$E = 22.37t + 499.44$$

Note: *It is important to remember that even though we have written it as an equality, the equation* $y = 22.37x + 499.44$ *in fact is a model that only approximates the relationship between* t *and* E *given by the data table.*

For example, the initial value according to the linear model $y = 22.37x + 499.44$ is 499.44, whereas the entry in the table for $t\ = 0$ is $E\ = 498.8$.

We will use the equals sign as above, but you should be aware that in this setting, many people would prefer to replace it by an approximation symbol, $\approx$, or to use different letters for the regression equation. In statistics this is indicated by putting a "hat" on E, $\widehat{E}$.

Uses of the Regression Line:

Slope and Trends

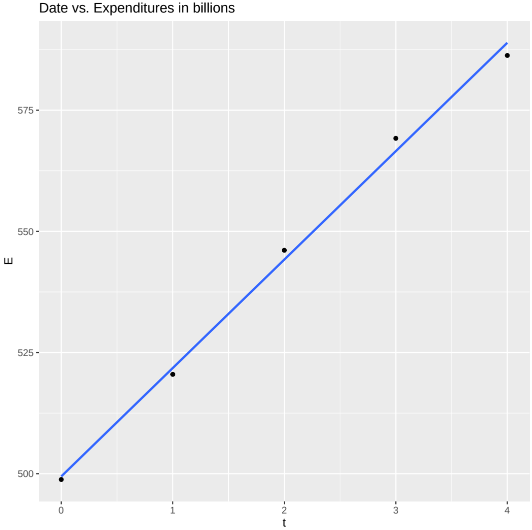
The most useful feature of the regression line is its slope, which in many cases provides the key to understanding data.

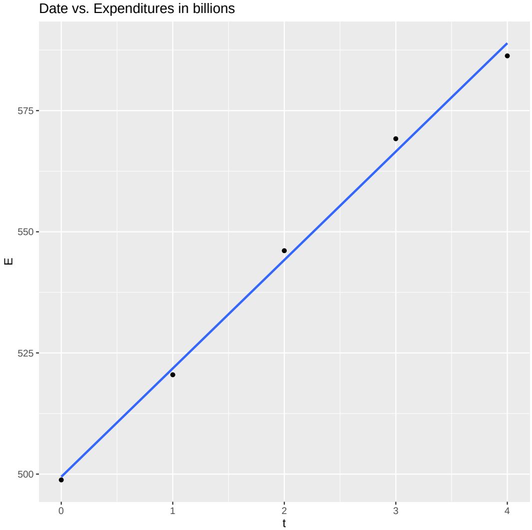
The most useful feature of the regression line is its slope, which in many cases provides the key to understanding data.

For Medicare spending, the slope, 22.37 billion dollars per year, of the regression line tells us that during the period from 2009 to 2013, Medicare spending grew by about 22.37 billion dollars per year. It tells how the data are changing.

A plot that shows the regression line with the data can be useful in analyzing trends.

For example, in the figure below, the fourth data point (corresponding to 2012) lies slightly above the regression line, whereas the fifth data point (corresponding to 2013) lies slightly below it.





Spending was definitely higher in 2013 than in 2012, but it could be argued, on the basis of the position of the data points relative to the line, that in 2012, spending was slightly ahead of the trend, whereas in 2013, it was slightly behind.

Question: What level of spending does the regression line $y = 22.37x + 499.44$ predict for 2014?

Consulting the source for our data, we find that Medicare expenditures in 2014 were in fact $E = 618.7$ billion dollars.

In this case, the projection given by the regression line was not exact but was within about 1% of the actual value.

Let's try to use the regression line to go the other way—that is, to estimate money spent before 2009.

In particular, we want to determine the level of Medicare expenditures in 2005. That is 4 years before 2009, so we want the value of E when $t = 2005 - 2009 = -4$.

Once again, we refer to the source and see that the actual expenditures for Medicare in 2005 were 339.7 billion dollars.

In this case, the value given by the regression line is a bad estimate of the real value.

The regression line is a powerful tool for analysis of certain kinds of data, but caution in its use is essential.

We saw above that the regression line gave a decent estimate for 2014 expenditures but a bad one for 2005.

In general, using the regression line to extrapolate beyond the limits of the data is risky, and the risk increases dramatically for long-range extrapolations.

What the regression line really shows is the linear trend established by the data.

Thus, for our 2014 projection, it would be appropriate to say, "If the trend established over the preceding five years had persisted, then Medicare spending in 2014 would have been about 611.29 billion dollars."

A check of the 2014 data showed that this trend did indeed persist into 2014.

An appropriate statement for our 2005 analysis might be "If the trend established from 2009 to 2013 had been valid since 2005, then expenditures in 2005 would have been about 409.96 billion dollars."

Because the actual expenditure was much less, we might proceed by gathering more data to see exactly how spending changed and then conduct historical, political, and economic investigations into why it changed.

On the other hand, the regression line can clearly show the trend of almost linear data and can appropriately be used to determine whether trends persist.

If information is available that indicates that linear trends are persisting, then the regression line can be used to make forecasts.

Question: The following table shows the amount $M = M(t)$ of money, in billions of dollars, spent by the United States on national defense. In the time row, $t = 0$ corresponds to 2004, $t = 1$ refers to 2005, and so on.

$t =$ years since 2004	0	1	2	3	4
$M =$ billons of dollars	455.8	495.3	521.8	551.3	616.1

1. Plot the data points. Does it appear that it is appropriate to approximate these data with a straight line?
2. Find the equation of the regression line for M as a function of t, and add the graph of this line to your data plot.

Question: $t = 0$ corresponds to 2004

$t =$ years since 2004	0	1	2	3	4
$M =$ billons of dollars	455.8	495.3	521.8	551.3	616.1

Part 1: Plot the points. Does it appear that it is appropriate to approximate these data with a straight line?

https://www.desmos.com/calculator

$0, \quad 455.8$
$1, \quad 495.3$
$2, \quad 521.8$
$3, \quad 551.3$
$4, \quad 616.1$

Question: $t = 0$ corresponds to 2004

$t =$ years since 2004	0	1	2	3	4
$M =$ billons of dollars	455.8	495.3	521.8	551.3	616.1

Part 2: Find the equation of the regression line for M as a function of t, and add the graph of this line to your data plot.

Question: $t = 0$ corresponds to 2004

t = years since 2004	0	1	2	3	4
M = billons of dollars	455.8	495.3	521.8	551.3	616.1

Part 3: Explain in practical terms the meaning of the slope of the regression line model we found in part 2.

Question: $t = 0$ corresponds to 2004

$t =$ years since 2004	0	1	2	3	4
$M =$ billons of dollars	455.8	495.3	521.8	551.3	616.1

Part 4: Use the regression equation to estimate military spending by the United States in 2009. The actual military expenditures in 2009 were \$661.0 billion. Did the trend established in the mid-2000s persist until 2009?

Chapter 3 Section 5

Systems of Equations

- Many physical problems can be described by a system of two equations in two unknowns, and often the desired information is found by solving the system of equations.?
- As we shall see, this involves nothing more than finding the intersection of two lines, and we already know how to do that, since we can find the intersection of any two graphs.?

Let's look at an example, but first a quick prep on concentrations for a solution.

Suppose we have 3 mL of Albuterol Sulfate Inhalation Solution at 0.083%.

How many milliliters of Albuterol Sulfate is actually in the solution?

$$0.083\% \equiv 0.00083$$

so there are

$$3\mathsf{mL} \cdot 0.00083 = 0.002\mathsf{mL}$$

A nurse has two solutions that contain different concentrations of a certain medication. One is a 12.5% concentration, and the other is a 5% concentration. How many cubic centimeters of each should she mix to obtain 20 cubic centimeters of an 8% concentration?

Let's let x represent the amount of 12.5% solution and y represent the amount of 5% solution.

Since we want 20 cc of solution we have the equation

$$x + y = 20$$

Now for the medication. The final solution consists of 20 cc at 8% so there will be

$$20 \text{ cc} \cdot 0.08 = 1.6 \text{ cc}$$

The first concentration is 12.5% and x cc so that is

$$x \text{ cc} \cdot 0.125$$

The second concentration is 5% and y cc so that is

$$y \text{ cc} \cdot 0.05$$

Dropping the units that means that our second equation is

$$0.125x + 0.05y = 1.6$$

So the system of equations we need to solve are

$$\begin{array}{rcrcr} 0.125x & + & 0.05y & = & 1.6 \\ x & + & y & = & 20 \end{array}$$

So how do we solve this?

Graphical Solutions of Systems of Equations

Let's start with one we can do by hand.

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

Going back to our concentration example:

$$\begin{array}{rcrcl} 0.125x & + & 0.05y & = & 1.6 \\ x & + & y & = & 20 \end{array}$$

Let's solve graphically in Desmos.

To show the full method, let's look at a simple example.

Example

We have \$900 to spend on the repair of a gravel drive. We want to make the repairs using a mix of coarse gravel priced at \$28 per ton and fine gravel priced at \$32 per ton. To make a good driving surface, we need three times as much fine gravel as coarse gravel. How much of each will our budget allow us to buy?

Blank

We have \$56 to spend on pizzas and drinks for a picnic. Pizzas cost \$12 each, and drinks cost \$0.50 each. Four times as many drinks as pizzas are needed. How many pizzas and how many drinks will our budget allow us to buy?

Algebraic Solutions

There are three methods of solving systems of equations:

1. Graphical
2. Substitution
3. Elimination

Substitution

Example 1S

Solve using substitution:

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

Example 2S

Solve using substitution:

$$\begin{cases} 3u + z = 0 \\ u - 2z = -7 \end{cases}$$

Example 3S

Solve using substitution:

$$\begin{cases} a - 4b = 24 \\ 3a + 6b = -72 \end{cases}$$

Elimination

Example 1E

Solve using elimination:

$$\begin{cases} x + y = 2 \\ x - 2y = -1 \end{cases}$$

Example 2E

Solve using elimination:

$$\begin{cases} 3u + z = 0 \\ u - 2z = -7 \end{cases}$$

Example 3E

Solve using elimination:

$$\begin{cases} \begin{array}{rcrcr} a & - & 4b & = & 24 \\ 3a & + & 6b & = & -72 \end{array} \end{cases}$$

For a system of two linear equations:

1. It's possible that the two lines are parallel in which case there are no solutions.
2. It's possible that the two lines are the same line in which case there are infinitely many solutions.

In class 1

Solve using substitution and then elimination and check graphically:

$$\begin{cases} \begin{array}{rcrcr} x & - & y & = & 5 \\ 2x & + & y & = & 19 \end{array} \end{cases}$$

In class 2

Solve using substitution and then elimination and check graphically:

$$\begin{cases} 3a - 2b = 24 \\ a + 4b = -6 \end{cases}$$

In class 3

Solve using substitution and then elimination and check graphically:

$$\begin{cases} 3x - 2y = 4 \\ \quad\quad\; 4y = 12 \end{cases}$$

In class 4

Solve using substitution and then elimination and check graphically:

$$\begin{cases} 2u & + & 5w & = & 36 \\ -6u & + & 2y & = & 28 \end{cases}$$

In class 5

Solve using substitution and then elimination and check graphically:

$$\begin{cases} -4s + 3t = -6 \\ 3s - 2t = 5 \end{cases}$$

Chapter 4 Section 1

Up to now we have studied linear models

$$y = mx + b$$

Recall **Ex 3.4** Selling Jewelry at an Art Fair
Suppose you pay \$192 to rent a booth for selling necklaces at an art fair. The necklaces sell for \$32 each.

When we found the function that shows the net income as a function of the number of necklaces sold it was linear. To find the income we saw that we had an initial value of \$$-192$ and for each necklace sold we increased the income by \$$32$. This resulted in the function:

$$I = 32n - 192$$

n	$I(n)$
0	-192
1	-160
2	-128
3	-96
4	-64
$\vdots$	$\vdots$

The growth is linear since we can view this as the following process:

$$\begin{array}{rcccl} I(0) & = & -192 & = & -192 \\ I(1) & = & -160 & = & -192 + 32 \\ I(2) & = & -128 & = & -192 + 32 + 32 \\ I(3) & = & -96 & = & -192 + 32 + 32 + 32 \\ I(4) & = & -64 & = & -192 + 32 + 32 + 32 + 32 \\ & \vdots & & \vdots & \end{array}$$

While many process follow this pattern, there is another pattern that is very important in the world.

Exponential Growth

“And turning to the young Brahmin, he said, 'I would like to pay you, my friend, for this miraculous gift, which has done so much to relieve my former agonies. Therefore, tell me what you desire, within the realm of what I can give, so that I can show you how grateful I can be to those deserving reward.

“Sessa appeared unaffected by the King's generous offer. His serene face indicated neither excitement nor joy nor surprise. The apparent indifference of the young Brahmin astonished the courtiers, "Magnificent lord?" the young man replied with both moderation and pride. "I desire no greater reward for the gift I have brought you than the satisfaction of knowing that I have relived the lord of Taligana of his infinite sadness. Thus I have already been rewarded, and any other prize would be excessive.”

The good king smiled somewhat disdainfully at this answer, as it suggested an indifference exceedingly rare among the normally greedy Hindus. Not able to believe in the sincerity of the young man's reply, the king insisted, "Your disdain and indifference toward material things surprises me, young man. Modesty, when excessive, is like the breeze that extinguishes the light and blinds the old man in the long darkness of the night. So that man can overcome the obstacles that life places in his path, he must subordinate his spirit to an ambition that leads him to a fixed goal. Therefore, you should not hesitate to choose a reward commensurate with the value of your gift to me. Would you like a sack of gold? Would you like a chest of jewels? What about a palace? Would you accept a province of your own to govern? Take care with your answer, and you shall have your reward, by my word of honor?"

“After what you have said, a rejection of your offer would be more disobedience than discourtesy,” Sessa replied. “Thus I will accept a reward for the game I have invented. The reward should fit your generosity. Nevertheless, I do not wish either gold or lands or palaces. I want my reward in grains of wheat.”

"Grains of wheat?” exclaimed the king, not hiding his surprise at such an astonishing request. "How could I reward you in such insignificant currency?”

“Nothing could be simpler,” Sessa explained. "You give me one grain of wheat for the first square on the board, two for the second, four for the third, eight for the fourth, and so on, doubling the amount with each square up to the sixty-fourth and last square on the board. I beg you, O King, in accordance with your magnanimous offer, to pay me in grains of wheat in the manner I have indicated.”

Not only the king but all the nobles, Brahmins, and everyone present burst out laughing at such a strange request. In fact, the young man's request amazed all those more attached to the material things of life than Sessa was. The young Brahmin, who could have acquired a province or a palace, wished only for a few grains of wheat.

"Fool!" exclaimed the king.

"Where did you learn such ridiculous disregard for wealth? The reward you request is absurd. You must know that a handful of wheat contains innumerable grains. With just a few handfuls, I could pay what you ask and more—following your formula of doubling the number of grains with each square on the board. This reward you claim would not even satisfy the hunger of the least village in my kingdom for more than a few days. But, all right, I gave my word, and I shall give you exactly what you have asked for."

“The king ordered that the most gifted mathematicians of his court be brought before him, and he bade them calculate the amount of wheat due the young Sessa. After a few hours of deep study, the wise calculators returned to the throne room to present the king with their finished calculations.

Interrupting the chess game he was playing, the king asked the mathematicians, ”How many grains of wheat must I give the young Sessa in order to comply with his request?” “Magnanimous King!” declared the wisest of the mathematicians.

"We have calculated the number of grains of wheat, and we have reached a sum that is beyond human imagination. With the greatest care, we have calculated the number of carras required to hold the appropriate quantity of wheat, and we have arrived at the following conclusion:

the wheat that you will have to give to Lahur Sessa is the equivalent of a mountain with a diameter at its base the size of the city of Taligana and a height ten times greater than that of the Himalayas.

If all the fields of India were sown with wheat, in two thousand centuries you would not harvest what you have promised young Sessa."

M. Tahan and M. Tahan, *The man who counted: a collection of mathematical adventures*, 1st American ed. New York: Norton, 1993.

Let's calculate this for ourselves. A chessboard is an 8×8 board so it has 64 squares on it.
$2^{63} = 9,223,372,036,854,775,808$

There are initially 3000 bacteria in a petri dish, and the population doubles each hour. We can calculate the number of bacteria present after each hour.

t (Hours)	$N(t)$ (Number bacteria)
0	3000
1	$3000 \cdot 2$
2	$3000 \cdot 2 \cdot 2$
3	$3000 \cdot 2 \cdot 2 \cdot 2$
4	$3000 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$\vdots$	$\vdots$

We can rewrite as

t (Hours)	$N(t)$ (Number bacteria)
0	$3000 \cdot 2^0$
1	$3000 \cdot 2^1$
2	$3000 \cdot 2^2$
3	$3000 \cdot 2^3$
4	$3000 \cdot 2^4$
$\vdots$	$\vdots$

This leads to the equation

$$N(t) = 3000 \cdot 2^t$$

We call 3000 the initial value and 2 the base.

Exponential Decay

Suppose we start with the same petri dish but this time we add an antibiotic that kills half the bacteria each hour.

t(Hours)	$N(t)$(Number bacteria)
0	3000
1	$3000 \cdot \frac{1}{2}$
2	$3000 \cdot \frac{1}{2} \cdot \frac{1}{2}$
3	$3000 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
4	$3000 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$
$\vdots$	$\vdots$

We can rewrite as

t	$N(t)$
0	$3000 \cdot \left(\frac{1}{2}\right)^0$
1	$3000 \cdot \left(\frac{1}{2}\right)^1$
2	$3000 \cdot \left(\frac{1}{2}\right)^2$
3	$3000 \cdot \left(\frac{1}{2}\right)^3$
4	$3000 \cdot \left(\frac{1}{2}\right)^4$
$\vdots$	$\vdots$

This leads to the equation

$$N(t) = 3000\left(\frac{1}{2}\right)^t$$

In general we get

$$f(t) = Pa^t$$

There are two cases (where $a > 0$):

$a > 1$ Exponential growth with growth factor a

$a < 1$ Exponential decay with decay factor a

Ex 4.1: Radioactive substances decay over time, and the rate of decay depends on the element. If, for example, there are G grams of tritium (a radioactive form of hydrogen) in a container, then, as a result of radioactive decay, 1 year later there will be $0.945G$ grams of tritium left. Suppose we begin with 50 grams of tritium.

Ex 4.1: There are G grams of tritium, then 1 year later there will be $0.945G$ grams of tritium left. Suppose we begin with 50 grams of tritium.

Part 1 Use a formula to express the number G of grams left after t years as an exponential function of t. Identify the base, or yearly decay factor, and the initial value.

Ex 4.1: There are G grams of tritium, then 1 year later there will be $0.945G$ grams of tritium left. Suppose we begin with 50 grams of tritium.
Part 2 How much tritium is left after 5 years?

Ex 4.1: There are G grams of tritium, then 1 year later there will be $0.945G$ grams of tritium left. Suppose we begin with 50 grams of tritium.
Part 3 Plot the graph of $G = G(t)$ over the first 30 years and describe in words how the amount of tritium that is present changes with time.

Ex 4.1: There are G grams of tritium, then 1 year later there will be $0.945G$ grams of tritium left. Suppose we begin with 50 grams of tritium.

Part 4 How long will it take for half of the tritium to decay?

Ex 4.2: In 1900, the world population was about 1.6 billion. On average, to find next year's population, we multiply this year's population by 1.014.

Part 1 Explain why these assumptions tell us that the world population is an exponential function of time.

Ex 4.2: In 1900, the world population was about 1.6 billion. On average, to find next year's population, we multiply this year's population by 1.014.
Part 2 Find a formula that gives the world population W, in billions, as an exponential function of t, the time in years since 1900.

Ex 4.2: In 1900, the world population was about 1.6 billion. On average, to find next year's population, we multiply this year's population by 1.014.

Part 3 Make a graph of the world population from 1900 to 2010.

Ex 4.2: In 1900, the world population was about 1.6 billion. On average, to find next year's population, we multiply this year's population by 1.014.
Part 4 According to this model, when did the world population reach 5 billion?

Unit Conversion

What if we were given a yearly growth factor but wanted a decade growth factor? From 1790 to 1860, the U.S. population grew exponentially with yearly growth factor 1.03. There are two ways to find the growth factor per decade, the long way and the short way. Let's do it the long way first.

Long $\underbrace{(1.03)(1.03)(1.03)\cdots(1.03)}_{10\text{ times}}$

Short $(1.03)^{10} \approx 1.344$

What if we were given a yearly growth factor but wanted a decade growth factor? From 1790 to 1860, the U.S. population grew exponentially with yearly growth factor 1.03. There are two ways to find the growth factor per decade, the long way and the short way. Let's do it the long way first.

Long $\underbrace{(1.03)(1.03)(1.03)\cdots(1.03)}_{10 \text{ times}}$

Short $(1.03)^{10} \approx 1.344$

So the decade growth factor is 1.344. How do we go from a decade growth rate to a yearly one? $(1.344)^{\frac{1}{10}} \approx 1.03$

Changing Units

S-17 A certain quantity has a yearly growth factor of 1.17. What is its monthly growth factor?

S-18 A certain quantity has a yearly growth factor of 1.17. What is its decade growth factor?

S-19 A certain quantity has a decade growth factor of 3.4. What is its yearly growth factor?

S-20 A certain quantity has a yearly decay factor of 0.99. What is its century decay factor?

S-21 A certain quantity has a century decay factor of 0.2. What is its yearly decay factor?

Ex 4.2: It is standard practice to give the rate at which a radioactive substance decays in terms of its *half-life*. That is the amount of time it takes for half of the substance to decay. The half-life of carbon-14 is 5770 years.

Part 1 If you start with 1 gram of carbon-14, how long will it take for only $\frac{1}{4}$ gram to remain?

Part 2 What is the yearly decay factor rounded to five decimal places?

Part 3 Assuming that we start with 25 grams of carbon-14, find a formula that gives the amount of carbon-14 left after t years.

Chapter 4 Section 2

Constant Percentage Change

- One of the most common ways we encounter exponential functions in daily life is in the form of constant percentage change.
- When a news report says that we expect 3% inflation, or that an endangered population is declining by 10% per year, or that an investment is growing by 5% each year, exponential functions are at work behind the scenes.
- Constant percentage change always gives rise to an exponential function.

Percentage Growth Rate and Growth Factor

In general if our raise is r percent as a decimal, then to find our new salary we multiply our current salary by $1+r$.
Let C denote our current salary and let N denote our new salary.
Then, our raise is given by rC.
To find our new salary N, we add this raise to our current salary:

$$N = C + rC$$

$$N = C(1+r)$$

This is an algebraic justification for the fact that a raise of r percent as a decimal multiplies our current salary by $1+r$.

If we get a 5% raise every year, then each year we multiply by 1.05. That means that our salary is an exponential function of time with a yearly growth factor 1.05. We say that our salary has a yearly percentage growth rate of 5%.
More generally, if a function shows constant percentage growth of r as a decimal, then it is an exponential function with growth factor $1 + r$.

A classic example of exponential growth is the balance of an account that earns compound interest. If the account earns 6% interest each year and interest is compounded annually, then each year, the balance is increased by 6%.
Thus, the balance is an exponential function with yearly growth factor 1.06.

Ex 4.4: A credit card holder begins the year owing \$395.00 to a bank card for credit card purchases. The bank charges 1.2% interest on the outstanding balance each month. For the purposes of this example, assume that no additional payments or charges are made and that no additional service charges are levied. Let $B = B(t)$ denote the balance of the account t months after January 1.

Ex 4.4: Owes \$395.00 to a bank that charges 1.2% interest on the outstanding balance each month. Let $B = B(t)$ denote the balance of the account t months after January 1.

Part 1 Explain why B is an exponential function of t. Identify the monthly percentage growth rate, the monthly growth factor, and the initial value. Write a formula for $B = B(t)$.

Ex 4.4: Owes \$395.00 to a bank that charges 1.2% interest on the outstanding balance each month. Let $B = B(t)$ denote the balance of the account t months after January 1.

Part 2 How much is owed after 7 months?

Ex 4.4: Owes \$395.00 to a bank that charges 1.2% interest on the outstanding balance each month. Let $B = B(t)$ denote the balance of the account t months after January 1.

Part 3 Assume that there is a limit of \$450 on the card and that the bank will demand a payment the first month this limit is exceeded. How long will it be before a payment is required?

Percentage Decay Rate and Decay Factor

Suppose a population declines by 10% this year. That means that 90% of the population remains.
Thus, to find next year's population from this year's population, we multiply by 0.90. If the population declines by 10% each year, then to find the new population, we always multiply the old population by 0.90.
That is, a 10% per year decrease indicates an exponential function with a yearly decay factor of 0.90. More generally, if a function shows a constant percentage decrease of r as a decimal, then it is an exponential function with decay factor $1–r$.

Ex 4.5: A 1000-gallon oil spill has contaminated a reservoir. The company hired to clean the reservoir can remove 10% of the remaining oil each week.

Part 1 Explain why the amount O, in gallons, of oil remaining in the reservoir is an exponential function of t, the time in weeks since the cleanup began.

Ex 4.5: A 1000-gallon oil spill has contaminated a reservoir. The company hired to clean the reservoir can remove 10% of the remaining oil each week.

Part 2 Find an exponential model for the amount of oil in the reservoir through the cleanup period.

Ex 4.5: A 1000-gallon oil spill has contaminated a reservoir. The company hired to clean the reservoir can remove 10% of the remaining oil each week.

Part 3 How long will it be until there are only 100 gallons of oil in the reservoir? Round your answer to the nearest week.

Ex 4.5: A 1000-gallon oil spill has contaminated a reservoir. The company hired to clean the reservoir can remove 10% of the remaining oil each week.

Part 4 The cleanup company charges \$7600 per week for its services. How much will it cost to remove the first 100 gallons of oil from the reservoir?

Ex 4.5: A 1000-gallon oil spill has contaminated a reservoir. The company hired to clean the reservoir can remove 10% of the remaining oil each week.
Part 5 After there are 100 gallons of oil left, how much will it cost to remove 90 more gallons of oil from the reservoir?

Percentage Change and Unit Conversion

Often we are interested in the long-term effect of percentage increases.
For example, suppose we received a 3.5% raise each year for 10 years. We want to determine the percentage increase in our wages over this decade.
To do this we first find the yearly growth factor. The yearly percentage growth rate is $r = 0.035$ as a decimal, so the yearly growth factor is $a = 1 + r = 1.035$.

Now we perform unit conversion. There are 10 years in a decade, so the decade growth factor is $1.035^{10} = 1.411$ rounded to three decimal places.

The last step is to go back to a percentage growth rate. A decade growth factor of 1.411 gives a decade percentage growth rate of $1.411–1 = 0.411$ as a decimal.

We conclude that our wages increased by 41.1% over the decade.

Ex 4.6: A recent report states that the gray wolf population that ranges through Idaho, Montana, Wyoming, Oregon, and Washington declined by 5% from 2009 to 2010. Suppose that the wolf population continues to decline at a rate of 5% per year.

Part 1 Use W_0 for the initial number of wolves, and find an exponential function that gives the wolf population W after t years.

Ex 4.6: A recent report states that the gray wolf population that ranges through Idaho, Montana, Wyoming, Oregon, and Washington declined by 5% from 2009 to 2010. Suppose that the wolf population continues to decline at a rate of 5% per year.

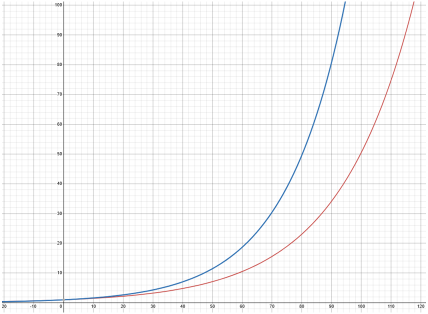
Part 2 If the 5% decline continues for 5 years, what will be the percentage decrease in the wolf population over the five-year period? Report your answer to the nearest tenth of a percent.

Exponential Functions and Daily Experience

Exponential functions are common in daily news reports, and it is important that citizens spot them and understand their significance. Economic reports are often driven by exponential functions, and inflation is a prime example.

When it is reported that the inflation rate is 3% per year, the meaning is that prices are increasing at a rate of 3% each year, and therefore prices are growing exponentially. It might be reported as good economic news that "Last year's 5% inflation rate is now down to 4%."

Although this should indeed be encouraging, the lower rate does not change the fact that we are still dealing with an exponential function whose graph will eventually get very steep.



The reduction from 5% to 4% changes the growth factor from 1.05 to 1.04 and so delays our reaching the steep part of the curve, but, as is shown in the Figure above, if the price function remains exponential in nature, then over the long term, catastrophic price increases will occur.

Short-term population growth is another phenomenon that is, in many settings, exponential in nature. It is the shape of the exponential curve, not necessarily the exact exponential formula, that causes concerns about eventual overcrowding.
Many environmental issues related to human and animal populations are inextricably tied to exponential functions.
It may be relatively inexpensive to dispose of large amounts of materials at a waste cleanup site properly, but further cleanup may be much more expensive.

That is, as we carry out the cleanup process, the amount of objectionable material remaining to be dealt with may be a decreasing exponential function.
Since certain toxic substances are dangerous even in minute quantities, it can be very expensive to reduce them to safe levels.
Once again, it is the nature of exponential decay, not the exact formula, that contributes to the astronomical expense of environmental cleanup.

Whenever phenomena are described in terms of percentage change, they may be modeled by exponential functions, and understanding how exponential functions behave is the key to understanding their true behavior.
One of the most important features of exponential phenomena is that they may change at one rate over a period of time but change at a quite different rate later on. Exponential growth may be modest for a time, but eventually the function will increase at a dramatic rate.
Similarly, exponential decay may show encouragingly rapid progress at first, but such rates of decrease cannot continue indefinitely.

Chapter 4 Section 3

Modeling Exponential Data

- We will show how to recognize exponential data and develop the appropriate tools for constructing exponential models.

Recognizing Exponential Data

The table below shows how the balance in a savings account grows over time since the initial investment.

Time in months	0	1	2	3	4
Savings Balance	\$3500.00	\$3542.00	\$3584.50	\$3627.52	\$3671.05

Let t be the time in months since the initial investment, and let B be the balance in dollars.

If the data are growing exponentially, then each month we should get the new balance by multiplying the old balance by the monthly growth factor:

$$\text{New } B = \text{Monthly growth factor } \times \text{Old } B.$$

Using division to rewrite this yields

$$\frac{\text{New } B}{\text{Old } B} = \text{Monthly growth factor}$$

Time in months	0	1	2	3	4
Savings Balance	\$3500.00	\$3542.00	\$3584.50	\$3627.52	\$3671.05

Recalling that $f(t) = Pa^t$:
$P = \$3500.00$

Thus a data table representing an exponential function should show common quotients if we divide each function entry by the one preceding it, assuming that we have evenly spaced values for the variable.
In the following table, we have calculated this quotient for each of the data entries, rounding our answers to three decimal places.

Time increment	$0 \to 1$	$1 \to 2$	$2 \to 3$	$3 \to 4$
Ratios of B	$\frac{3542}{3500} = 1.012$	$\frac{3584.50}{3542.00} = 1.012$	$\frac{3627.52}{3584.50} = 1.012$	$\frac{3671.05}{3627.52} = 1.012$

which are all the same.
From $f(t) = Pa^t$ we can see two things:

1. The ratios are always the same, therefore the data are exponential
2. The common quotient 1.012 is the monthly growth factor. Hence the monthly interest rate is $1.012 - 1 = 0.012$ or 1.2%.

Linear functions are functions that change by constant sums, so we detect linear data by looking for a common difference in function values when we use successive data points, assuming evenly spaced values for the variable.
When our data set is given in increments of 1 for the variable, this common difference is the slope of the linear model.

Exponential functions are functions that change by constant multiples, so we detect exponential data by looking for a common quotient when we use successive data points, again assuming evenly spaced values for the variable.
When our data set is given in increments of 1, this common quotient is the growth (or decay) factor of the exponential model.

Constructing an Exponential Model

In our discussion of the balance in a savings account, we saw that the data were exponential.
Since we know the growth factor, we need only one further bit of information to make an exponential model: the initial value.
But that also appears in the table as $B(0) = 3500.00$ dollars. Since our standard model of exponential growth/decay is $f(t) = Pa^t$, the exponential model for B as a function of t in months is

$$B = \text{Initial value} \cdot (\text{Monthly growth factor})^t$$

$$B = 3500.00(1.012)^t$$

In some situations, the data themselves are not exponential, but the difference from a limiting value can be modeled by an exponential function.
Underground water seepage, such as that from a toxic-waste site, can contaminate water wells many miles away.
Suppose that water seeping from a toxic-waste site is polluted with a certain contaminant at a level of 64 milligrams per liter.
Several miles away, monthly tests are made on a water well to monitor the level of this contaminant in the drinking water.

In the table below, we record the difference between the contaminant level of the waste site (64 milligrams per liter) and the contaminant level of the water well (in milligrams per liter).

Time in months	0	1	2	3	4	5
Contaminant level difference	64	45.44	32.26	22.91	16.26	11.55

Ex We record the difference between the contaminant level of the waste site (64 milligrams per liter) and the contaminant level of the water well (in milligrams per liter).

Time in months	0	1	2	3	4	5
Contaminant level difference	64	45.44	32.26	22.91	16.26	11.55

Part 1 Explain why the nature of the data suggests that an exponential model may be appropriate.

Ex We record the difference between the contaminant level of the waste site (64 milligrams per liter) and the contaminant level of the water well (in milligrams per liter).

Time in months	0	1	2	3	4	5
Contaminant level difference	64	45.44	32.26	22.91	16.26	11.55

Part 2 Test to see that the data are exponential.

Ex We record the difference between the contaminant level of the waste site (64 milligrams per liter) and the contaminant level of the water well (in milligrams per liter).

Time in months	0	1	2	3	4	5
Contaminant level difference	64	45.44	32.26	22.91	16.26	11.55

Part 3 Find an exponential model for the difference in contaminant level.

Ex We record the difference between the contaminant level of the waste site (64 milligrams per liter) and the contaminant level of the water well (in milligrams per liter).

Time in months	0	1	2	3	4	5
Contaminant level difference	64	45.44	32.26	22.91	16.26	11.55

Part 4 Use your answer in Part 3 to make a model for the contaminant level as a function of time.

Ex We record the difference between the contaminant level of the waste site (64 milligrams per liter) and the contaminant level of the water well (in milligrams per liter).

Time in months	0	1	2	3	4	5
Contaminant level difference	64	45.44	32.26	22.91	16.26	11.55

Part 5 This contaminant is considered dangerous in drinking water when it reaches a level of 57 milligrams per liter. When will this dangerous level be reached?

Growth and Decay Factor Units in Exponential Modeling

In the contaminated well example, the decay factor was the common quotient that we calculated. This will always occur when time measurements are given in 1-unit increments.
But when data are measured in different increments, adjustments to account for units must be made to get the right decay factor.
We can show what we mean by looking at the contaminated water well example again, but this time we suppose that measurements were taken every 3 months.
Under those conditions, the data table would have been as follows:

$t =$ months	0	3	6	9	12	15
$D =$ difference in contaiminat level	64	22.91	8.2	2.93	1.05	0.38

We test to see whether the data are exponential by calculating successive ratios, rounding to two decimal places.

Time increment	$0 \to 3$	$3 \to 6$	$6 \to 9$	$9 \to 12$	$12 \to 15$
Ratios of D	$\frac{22.91}{64} = 0.36$	$\frac{8.2}{22.91} = 0.36$	$\frac{2.93}{8.2} = 0.36$	$\frac{1.05}{2.93} = 0.36$	$\frac{0.38}{1.05} = 0.36$

Since the successive ratios are the same, 0.36, we conclude once again that the data are exponential.
But the common ratio 0.36 is not the monthly decay factor.
Rather, it is the 3-month decay factor.

To get the monthly decay factor, we need to change the units. Since 1 month is one-third of the 3-month recording time, we get

$$\text{Monthly decay factor} = (3\text{-month decay factor})^{1/3}$$

$$\text{Monthly decay factor} = 0.36^{1/3} = 0.71$$

The initial value is 64, so we arrive at the same exponential model, $64(0.71)^t$, as we did when we modeled the data given in 1-month intervals.

Ex 4.9: One important topic of forensic medicine is the determination of the time of death. A method that is sometimes used involves temperature. Suppose that at 6:00 P.M., a body is discovered in a basement of a building where the ambient air temperature is maintained at 72 degrees.

At the moment of death, the body temperature was 98.6 degrees, but after death, the body cools, and eventually its temperature matches the ambient air temperature. Beginning at 6:00 P.M., the body temperature is measured and the difference $D = D(t)$ between body temperature and ambient air temperature is recorded.

Ex A body is discovered in a basement of a building where the ambient air temperature is maintained at 72 degrees at 6:00 $\mathrm{P.M.}$. At the moment of death, the body temperature was 98.6

$t =$ hours since 6:00 P.M.	0	2	4	6	8
$D =$ temperature difference	12.02	8.08	5.44	3.65	2.45

Part 1 Show that the data can be modeled by an exponential function.

Ex A body is discovered in a basement of a building where the ambient air temperature is maintained at 72 degrees at 6:00 $\mathrm{P.M.}$. At the moment of death, the body temperature was 98.6

$t =$ hours since 6:00 P.M.	0	2	4	6	8
$D =$ temperature difference	12.02	8.08	5.44	3.65	2.45

Part 2 Find an exponential model for the data that shows temperature difference as a function of **hours**.

Ex A body is discovered in a basement of a building where the ambient air temperature is maintained at 72 degrees at 6:00 $\mathrm{P.M.}$. At the moment of death, the body temperature was 98.6

$t =$ hours since 6:00 P.M.	0	2	4	6	8
$D =$ temperature difference	12.02	8.08	5.44	3.65	2.45

Part 3 Find a formula for a function $T = T(t)$ that gives the temperature of the body at time t.

Ex A body is discovered in a basement of a building where the ambient air temperature is maintained at 72 degrees at 6:00 $\mathrm{P.M.}$. At the moment of death, the body temperature was 98.6

$t =$ hours since 6:00 P.M.	0	2	4	6	8
$D =$ temperature difference	12.02	8.08	5.44	3.65	2.45

Part 4 What was the time of death?

Chapter 4 Section 4

Modeling Nearly Exponential Data

- As in the case of linear data, rarely can experimentally gathered data be modeled exactly by an exponential function, but in many cases, such data can be closely approximated by an exponential model.
- This process is called exponential regression.

Exponential Regression

The following data, taken from the Information Please Almanac, show the U.S. population from 1800 to 1860, just prior to the Civil War.

Date	1800	1810	1820	1830	1840	1850	1860
Population, in millions	5.31	7.24	9.64	12.87	17.07	23.19	31.44

Let t be the time in years since 1800 and N the population in millions. The table for N as a function of t is then

t	0	10	20	30	40	50	60
N	5.31	7.24	9.64	12.87	17.07	23.19	31.44

Is the growth exponential?

Since the data table shows population growth, it is not unreasonable to suspect that the data might be exponential in nature.

If you calculate successive quotients of N, you will find that the data are not exactly exponential, but as with linear regression, it may still be appropriate to make an exponential model that approximates the data. That is the first question we want to answer:

"Is it reasonable to approximate the data with an exponential model?"

However, a caution is in order here.

It is very difficult in general to be sure that data are exponential by looking at their plot, because there are other types of data that give a similar appearance.

For the current example, it is enough to know that an exponential model is often appropriate for population growth and that a plot of the data seems to confirm this.

We use the resulting formula in the same ways in which we used the regression line formula for linear data.

For example, the growth factor 1.030 for N tells us immediately that from 1800 to 1860, the U.S. population grew at a rate of 3.0% per year, or by about 34% per decade.

We might also use the formula to estimate the population in 1870. To do this, we would put 70 in for t :

Exponential regression estimate for 1870:
$P(70) = 5.337(1.030)^{70} = 42.26$ So 42.26 million.

There is some uncertainty about the actual U.S. population in 1870. The 1870 census reported the number as 38.56 million. This figure was later revised to 39.82 million because it was thought that the southern population had been under counted.

Whether we use the original or the revised 1870 census estimate, it is clear that the U.S. population grew a good deal less than we would have expected from the exponential trend established from 1800 to 1860.

Such a discrepancy leads us to seek a historical explanation, and the most obvious culprit is the Civil War.

In fact, the death and disruption of the Civil War may have had long-lasting effects on U.S. population growth.

From 1790 to 1860, population grew at a steady rate of around 34% per decade.

From 1860 to 1870, population grew by only 27%, and this rate steadily declined to its historical low of 7.2% from 1930 to 1940, the decade of the Great Depression.

Ex 4.10 The following data table shows the colonial population from 1610 to 1670, where t is the number of years since 1610 and $C = C(t)$ is the population in thousands.
The source for the data, the Information Please Almanac, cautions that records from this period are spotty, so the numbers should be considered estimates.

t = years since 1610	0	10	20	30	40	50	60
C = population, in thousands	0.35	2.3	4.6	26.6	50.4	75.1	111.9

Ex 4.10 Colonial population from 1610 to 1670, where t is years since 1610 and $C = C(t)$ is the population in thousands.

t	0	10	20	30	40	50	60
C	0.35	2.3	4.6	26.6	50.4	75.1	111.9

Part 1 Plot the data to get an overall view of their nature. Does the plot indicate that the data might be modeled by an exponential function?

Ex 4.10 Colonial population from 1610 to 1670, where t is years since 1610 and $C = C(t)$ is the population in thousands.

t	0	10	20	30	40	50	60
C	0.35	2.3	4.6	26.6	50.4	75.1	111.9

Part 2 Use exponential regression to construct a model for C. (Round regression parameters to three decimal places.) Add the graph of the exponential model to the plot of population data.

Ex 4.10 Colonial population from 1610 to 1670, where t is years since 1610 and $C = C(t)$ is the population in thousands.

t	0	10	20	30	40	50	60
C	0.35	2.3	4.6	26.6	50.4	75.1	111.9

Part 3 Compare the population growth rate as given by exponential regression during colonial times with that in the 60 years preceding the Civil War.

Ex Another Example The following table show the number P of design patents awarded by the U.S. Patent and Trademark Office from 1950 through 2010.

$t =$ years since 1950	0	10	20	30	40	50	60
$P =$ patents	4718	2543	3214	3949	8024	17413	22799

Part 1