All-Pairs Shortest Paths

Introduction

It aims to figure out the shortest path from each vertex v to every other u. Storing all the paths explicitly can be very memory expensive indeed, as we need one spanning tree for each vertex. This is often impractical regarding memory consumption, so these are generally considered as all pairs-shortest distance problems, which aim to find just the distance from each to each node to another. We usually want the output in tabular form: the entry in u's row and v's column should be the weight of the shortest path from u to v.

Three approaches for improvement:

Algorithm	Cost
Matrix Multiplication	O (V^3 logV)
Floyd-Warshall	O (V^3)
Johnson O	(V^2 log?V+VE)

Unlike the single-source algorithms, which assume an adjacency list representation of the graph, most of the algorithm uses an adjacency matrix representation. (Johnson's Algorithm for sparse graphs uses adjacency lists.) The input is a n x n matrix W representing the edge weights of an n-vertex directed graph G = (V, E). That is, W = (w_{ij}), where

$$w_{ij} = \begin{cases} 0 & \text{if} \quad i = j, \\ w(i,j) & \text{if} \quad i \neq j \text{ and } (i,j) \in E, \\ \infty & \text{if} \quad i \neq j \text{ and } (i,j) \notin E. \end{cases}$$

Floyd-Warshall Algorithm

Let the vertices of G be V = {1, 2........n} and consider a subset {1, 2........k} of vertices for some k. For any pair of vertices i, j ∈ V, considered all paths from i to j whose intermediate vertices are all drawn from {1, 2.......k}, and let p be a minimum weight path from amongst them. The Floyd-Warshall algorithm exploits a link between path p and shortest paths from i to j with all intermediate vertices in the set {1, 2.......k-1}. The link depends on whether or not k is an intermediate vertex of path p.

If k is not an intermediate vertex of path p, then all intermediate vertices of path p are in the set {1, 2........k-1}. Thus, the shortest path from vertex i to vertex j with all intermediate vertices in the set {1, 2.......k-1} is also the shortest path i to j with all intermediate vertices in the set {1, 2.......k}.

If k is an intermediate vertex of path p, then we break p down into i → k → j.

Let $d_{ij}^{(k)}$ be the weight of the shortest path from vertex i to vertex j with all intermediate vertices in the set {1, 2.......k}.

A recursive definition is given by

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if} \quad k=0 \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \text{if} \quad k \geq 1 \end{cases}$$

FLOYD - WARSHALL (W)

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1. n ← rows [W].
2. D⁰ ← W
3. for k ← 1 to n
4. do for i ← 1 to n
5. do for j ← 1 to n
6. do dij(k) ← min (dij(k-1),dik(k-1)+dkj(k-1) )
7. return D(n)
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The strategy adopted by the Floyd-Warshall algorithm is **Dynamic Programming**. The running time of the Floyd-Warshall algorithm is determined by the triply nested for loops of lines 3-6. Each execution of line 6 takes O (1) time. The algorithm thus runs in time θ(n^3).

Example: Apply Floyd-Warshall algorithm for constructing the shortest path. Show that matrices $D^{(k)}$ and $\pi^{(k)}$ computed by the Floyd-Warshall algorithm for the graph.

Solution:

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$$\pi_{ij}^{(k)} = \begin{cases} \pi_{ij}^{(k-1)} \text{ if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \\ \pi_{kj}^{(k-1)} \text{ if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \end{cases}$$

Step (i) When k = 0

$$D^{(0)} = \begin{matrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & -5 & \infty \\ 2 & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{matrix} \qquad \pi^{(0)} = \begin{matrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 3 & NIL \\ 4 & NIL & NIL & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{matrix}$$

Step (ii) When k =1

$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

$d_{14}^{(1)}$= min ($d_{14}^{(0)}$, $d_{11}^{(0)}$ +$d_{14}^{(0)}$)

$d_{14}^{(1)}$= min (∞ , 0 + ∞) = ∞

$d_{15}^{(1)}$ = min ($d_{15}^{(0)}$, $d_{11}^{(0)}$ +$d_{15}^{(0)}$)

$d_{15}^{(1)}$ = min (-4, 0 + -4) = -4

$d_{21}^{(1)}$ = min ($d_{21}^{(0)}$, $d_{21}^{(0)}$ +$d_{11}^{(0)}$)

$d_{21}^{(1)}$ = min (∞,∞ + 0) = ∞

$d_{23}^{(1)}$ = min ($d_{23}^{(0)}$, $d_{21}^{(0)}$ +$d_{13}^{(0)}$)

$d_{23}^{(1)}$ = min ((∞,∞ + 8) = ∞

$d_{31}^{(1)}$ = min ($d_{31}^{(0)}$, $d_{31}^{(0)}$ +$d_{11}^{(0)}$)

$d_{31}^{(1)}$ = min (∞,∞ + 0) = ∞

$d_{35}^{(1)}$ = min ($d_{35}^{(0)}$, $d_{31}^{(0)}$ +$d_{15}^{(0)}$)

$d_{35}^{(1)}$ = min (∞,∞ + (-4)) = ∞

$d_{42}^{(1)}$ = min ($d_{42}^{(0)}$, $d_{41}^{(0)}$ +$d_{12}^{(0)}$)

$d_{42}^{(1)}$ = min (∞, 2 + 3) = 5

$d_{43}^{(1)}$ = min ($d_{43}^{(0)}$, $d_{41}^{(0)}$ +$d_{13}^{(0)}$)

$d_{43}^{(1)}$ = min (∞, 2 + 8) = 10

$d_{45}^{(1)}$ = min ($d_{45}^{(0)}$, $d_{41}^{(0)}$ +$d_{15}^{(0)}$)

$d_{45}^{(1)}$ = min (∞, 2 + (-4)) = -2

$d_{51}^{(1)}$ = min ($d_{51}^{(0)}$, $d_{51}^{(0)}$ +$d_{11}^{(0)}$)

$d_{51}^{(1)}$ = min (∞,∞ + 0) = ∞

$$D_{ij}^{(1)} = \begin{matrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & -5 & \infty \\ 2 & 5 & 10 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{matrix} \qquad \pi^{(1)} = \begin{matrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 3 & NIL \\ 4 & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{matrix}$$

Step (iii) When k = 2

$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

$d_{14}^{(2)} = \min(d_{14}^{(1)}, d_{12}^{(1)} + d_{24}^{(1)})$

$d_{14}^{(2)} = \min(\infty, 3 + 1) = 4$

$d_{21}^{(2)} = \min(d_{21}^{(1)}, d_{22}^{(1)} + d_{21}^{(1)})$

$d_{21}^{(2)} = \min(\infty , 0 + \infty) = \infty$

$d_{34}^{(2)} = \min(d_{34}^{(1)}, d_{32}^{(1)} + d_{24}^{(1)})$

$d_{34}^{(2)} = \min(-5, 4 + 1) = -5$

$d_{35}^{(2)} = \min(d_{35}^{(1)}, d_{32}^{(1)} + d_{25}^{(1)})$

$d_{35}^{(2)} = \min(\infty, 4 + 7) = 11$

$d_{43}^{(2)} = \min(d_{43}^{(1)}, d_{42}^{(1)} + d_{23}^{(1)})$

$d_{43}^{(2)} = \min(10, 5 + \infty) = 10$

$$D_{ij}^{(2)} = \begin{matrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & -5 & 11 \\ 2 & 5 & 10 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{matrix} \qquad \pi^{(2)} = \begin{matrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 3 & 2 \\ 4 & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{matrix}$$

Step (iv) When k = 3

$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

$d_{14}^{(3)} = \min(d_{14}^{(2)}, d_{13}^{(2)} + d_{34}^{(2)})$

$d_{14}^{(3)} = \min(4, 8 + (-5)) = 3$

$$D_{ij}^{(3)} = \begin{matrix} 0 & 3 & 8 & 3 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & -5 & 11 \\ 2 & 5 & 10 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{matrix} \qquad \pi^{(3)} = \begin{matrix} NIL & 1 & 1 & 3 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 3 & 2 \\ 4 & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{matrix}$$

Step (v) When k = 4

$d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

$d_{21}^{(4)} = \min(d_{21}^{(3)}, d_{24}^{(3)} + d_{41}^{(3)})$

$d_{21}^{(4)} = \min(\infty, 1 + 2) = 3$

$d_{23}^{(4)} = \min(d_{23}^{(3)}, d_{24}^{(3)} + d_{43}^{(3)})$

$d_{23}^{(4)} = \min(\infty, 1 + 10) = 11$

$d_{25}^{(4)} = \min(d_{25}^{(3)}, d_{24}^{(3)} + d_{45}^{(3)})$

$d_{25}^{(4)} = \min(7, 1 + (-2)) = -1$

$d_{31}^{(4)} = \min(d_{31}^{(3)}, d_{34}^{(3)} + d_{41}^{(3)})$

$d_{31}^{(4)} = \min(\infty, -5 + 2) = -3$

$d_{32}^{(4)} = \min(d_{32}^{(3)}, d_{34}^{(3)} + d_{42}^{(3)})$

$d_{32}^{(4)} = \min(4, -5 + 5) = 0$

$d_{51}^{(4)} = \min(d_{51}^{(3)}, d_{54}^{(3)} + d_{41}^{(3)})$

$d_{51}^{(4)} = \min(\infty, 6 + 2) = 8$

$d_{52}^{(4)} = \min(d_{52}^{(3)}, d_{54}^{(3)} + d_{42}^{(3)})$

$d_{52}^{(4)} = \min(\infty, 6 + 5) = 11$

$d_{53}^{(4)} = \min(d_{53}^{(3)}, d_{54}^{(3)} + d_{43}^{(3)})$

$d_{53}^{(4)} = \min(\infty, 6 + 10) = 16$

$$D_{ij}^{(4)} = \begin{matrix} 0 & 3 & 8 & 3 & -4 \\ 3 & 0 & 11 & 1 & -1 \\ -3 & 0 & 0 & -5 & -7 \\ 2 & 5 & 10 & 0 & -2 \\ 8 & 11 & 16 & 6 & 0 \end{matrix} \qquad \pi^{(4)} = \begin{matrix} \text{NIL} & 1 & 1 & 3 & 1 \\ 4 & \text{NIL} & 4 & 2 & 2 \\ 4 & 4 & \text{NIL} & 3 & 4 \\ 4 & 1 & 1 & \text{NIL} & 1 \\ 4 & 4 & 4 & 5 & \text{NIL} \end{matrix}$$

Step (vi) When k = 5

$$d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$$

$$d_{25}^{(5)} = \min\left(d_{25}^{(4)}, d_{25}^{(4)} + d_{55}^{(3)}\right)$$

$$d_{25}^{(5)} = \min(-1, -1 + 0) = -1$$

$$d_{23}^{(5)} = \min\left(d_{23}^{(4)}, d_{25}^{(4)} + d_{53}^{(3)}\right)$$

$$d_{23}^{(5)} = \min(11, -1 + 16) = 11$$

$$d_{35}^{(5)} = \min\left(d_{35}^{(4)}, d_{35}^{(4)} + d_{55}^{(3)}\right)$$

$$d_{35}^{(5)} = \min(-7, -7 + 0) = -7$$

$$D_{ij}^{(5)} = \begin{array}{ccccc} 0 & 3 & 8 & 3 & -4 \\ 3 & 0 & 11 & 1 & -1 \\ -3 & 0 & 0 & -5 & -7 \\ 2 & 5 & 10 & 0 & -2 \\ 8 & 11 & 16 & 6 & 0 \end{array} \qquad \pi^{(5)} = \begin{array}{ccccc} \text{NIL} & 1 & 1 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 4 \\ 4 & 4 & \text{NIL} & 3 & 4 \\ 4 & 1 & 1 & \text{NIL} & 1 \\ 4 & 4 & 4 & 5 & \text{NIL} \end{array}$$

TRANSITIVE- CLOSURE (G)

n ← |V[G]|
for i ← 1 to n
do for j ← 1 to n
do if i = j or (i, j) ∈ E [G]
the $t_{ij}^{(0)}$ ← 1
else $t_{ij}^{(0)}$ ← 0
for k ← 1 to n
do for i ← 1 to n
do for j ← 1 to n
dod $_{ij}^{(k)}$ ← $t_{ij}^{(k-1)} \vee (t_{ik}^{(k-1)} \wedge t_{kj}^{(k-1)}$
Return $T^{(n)}$.