Greedy algorithms
Huffman Schedule

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Huffman encoding

Encoding

- Consider problem of designing a binary code where each symbol is represented by a unique binary string.
- Suppose that the alphabet consists of four symbols (A, B, C, D).
- Fixed-length code requires 2 bits per symbol (00, 01, 10, 11).
- Given the frequencies for each symbol, using ***variable*-length** code can **compress** data considerably by giving **frequent** symbols **short** codewords and **infrequent** symbols **long** codewords.

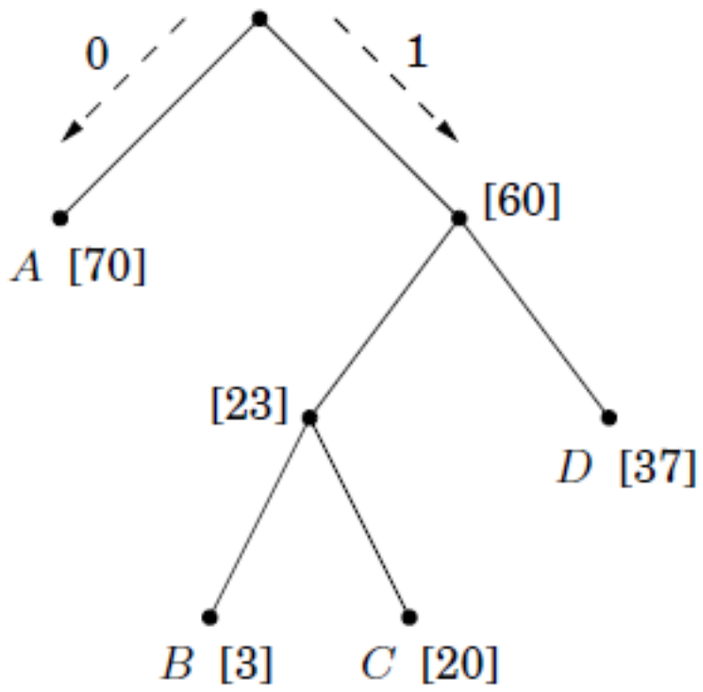
Symbol	Frequency
A	70 million
B	3 million
C	20 million
D	37 million

Symbol	Codeword
A	0
B	100
C	101
D	11

Prefix-free encoding

- If codewords are {0, 01, 11, 001}, the decoding of strings like 001 is **ambiguous**.
- ***Prefix-free*** encoding - no codeword can be a prefix of another codeword.
- Decoding process can be represented with a binary tree whose leaves are the given characters.
- An **optimal code** is represented by a ***full*** **binary tree**.

Symbol	Codeword
A	0
B	100
C	101
D	11



Cost of tree

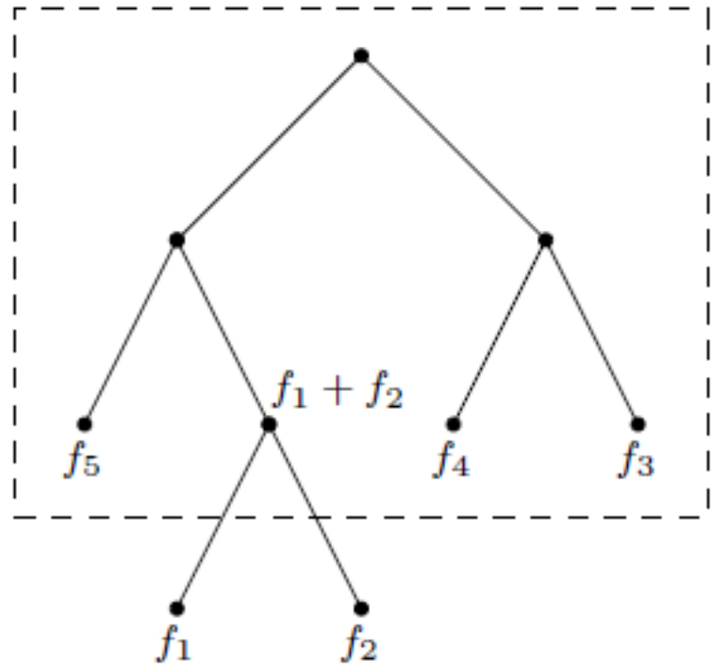
- How do we find the optimal coding tree, given the frequencies $f_1, f_2, \ldots, f_n$ of n symbols?
- (1) We want a tree which minimizes the overall length of the encoding,

$$\text{cost of tree} = \sum_{i=1}^{n} f_i \cdot (\text{depth of } i\text{th symbol in tree})$$

- (The number of bits required for a symbol is its depth in the tree.)
- Define the frequency of an *internal* node as the sum of the frequencies of its descendant leaves - the number of times the internal node is visited during decoding.
- (2) So the total number of bits = cost of tree = the sum of the frequencies of all leaves and internal nodes except the root.

Greedy algorithm

- By (1), the two symbols with the **smallest frequencies** must be at the bottom of the optimal tree, as children of the lowest internal node.
- (this internal node has two children since the tree is *full)*.
- Otherwise, swapping these two symbols with whatever is lowest in the tree would improve the encoding.
- Assume *f1* and *f2* are the two smallest frequencies.
- By (2), any tree in which *f1* ,*f2* are sibling-leaves has cost *f1* + *f2* plus the cost for a tree with *n*-1 leaves of frequencies (*f1* + *f2*) , *f3, f4, ... fn*



```
procedure Huffman(f)
Input:  An array f[1···n] of frequencies
Output:  An encoding tree with n leaves

let H be a priority queue of integers, ordered by f
for i = 1 to n:  insert(H, i)
for k = n+1 to 2n−1:
  i = deletemin(H),  j = deletemin(H)
  create a node numbered k with children i, j
  f[k] = f[i] + f[j]
  insert(H, k)
```

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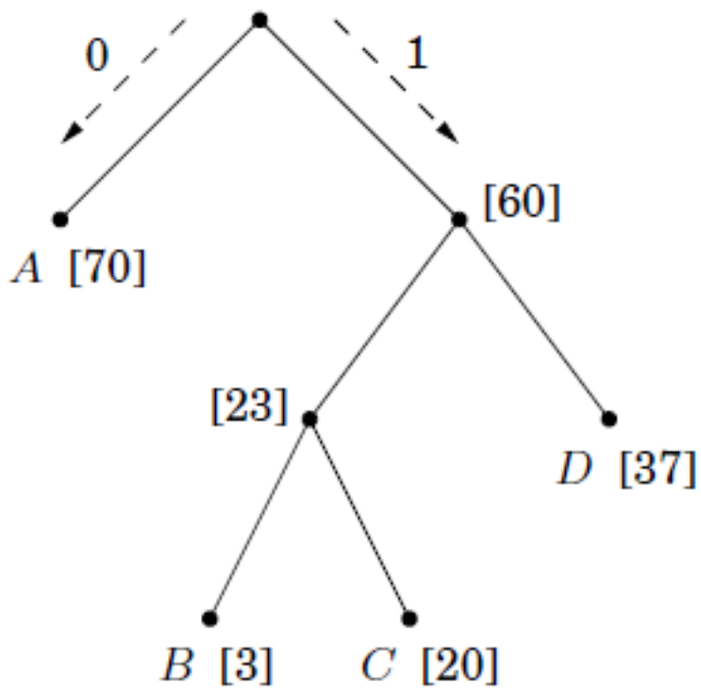
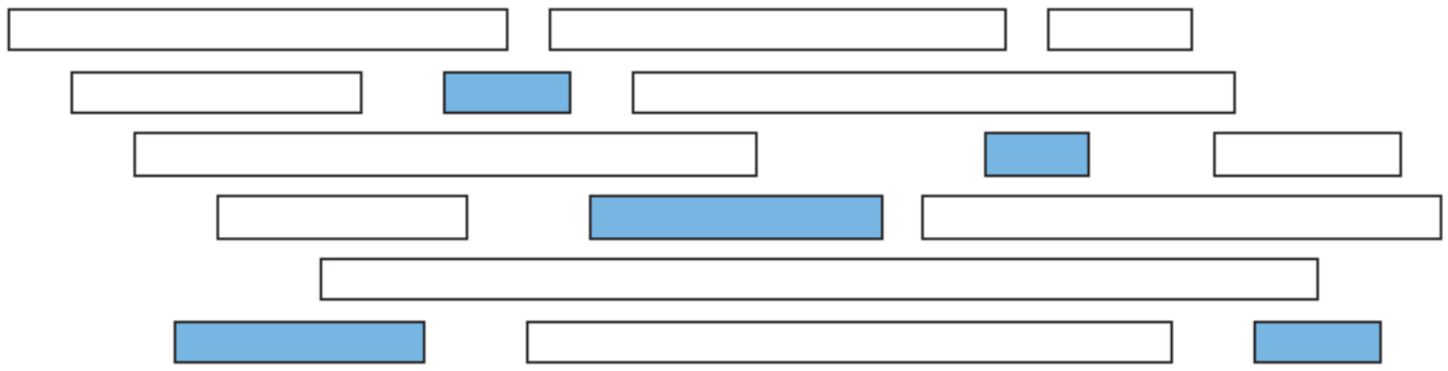


Scheduling problem

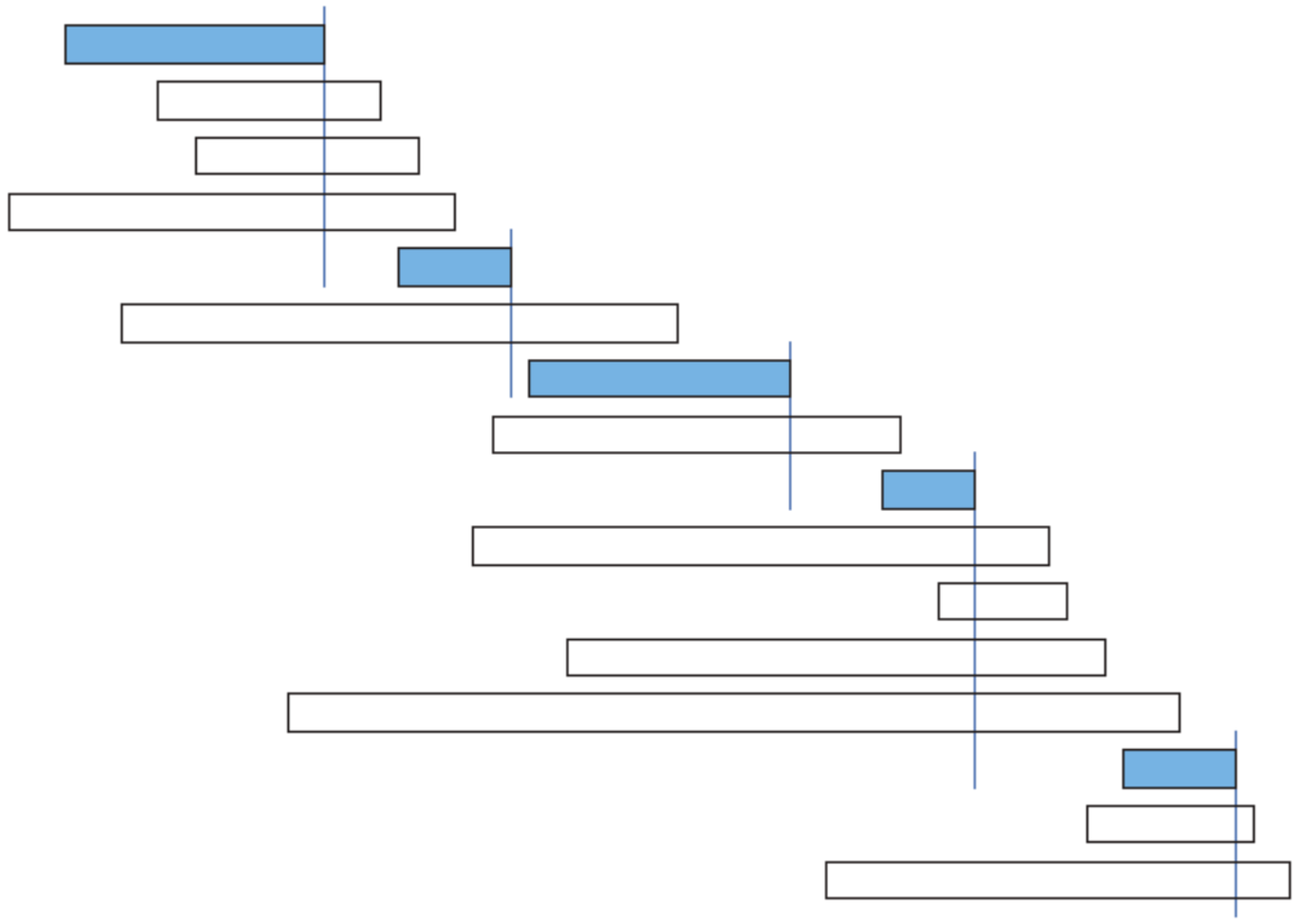
- Suppose you are given two arrays $S[1..\ n]$ *and* $F[1..\ n]$ listing the start and finish times of each class.
- Your task is to choose the largest possible subset $X \in \{1, 2,.., n\}$ so that for any pair $i, j \in X$, either $S[i] > F[j]$ *or* $S[j] > F[i]$.



A maximal conflict-free schedule for a set of classes.

Greedy algorithm

- **Sort** classes **by finish times**.
- Scan classes in the sorted order and choose the next class that **does not conflict with** the latest class so far.



The same classes sorted by finish times and the greedy schedule.

```
GreedySchedule(S[1..n], F[1..n]):
  sort F and permute S to match
  count ← 1
  X[count] ← 1
  for i ← 2 to n
      if S[i] > F[X[count]]
          count ← count + 1
          X[count] ← i
  return X[1..count]
```

Correctness

- To prove that this greedy algorithm actually gives us a maximal conflict-free schedule, we use an ***exchange argument***.
- Note that there can be maximal conflict-free schedules **different** from the output of the algorithm.
- We only claim that the **at least one** of the maximal schedules is the one that the greedy algorithm produces.

Correctness

- **Lemma : At least one maximal conflict-free schedule includes the class that finishes first.**
- Proof:
- Let *f* be the class that finishes first.
- Suppose we have a maximal conflict-free schedule *X* that does not include *f* .
- Let *g* be the first class in *X* to finish.
- Since *f* finishes before *g* does, *f* cannot conflict with any class in the set *S* -{*g*}.
- Thus, the schedule $X' = X \cup \{f\} - \{g\}$ is also conflict-free.
- Since *X'* has the same size as *X*, it is also maximal.

Correctness

- **Theorem : The greedy schedule is an optimal schedule.**
- Proof:
- Let *f* be the class that finishes first, and let *L* be the subset of classes the start after *f* finishes.
- The previous lemma implies that some optimal schedule contains *f* , so the best schedule that contains *f* is an optimal schedule.
- The best schedule that includes *f* must contain an optimal schedule for the classes that do not conflict with *f* , that is, an optimal schedule for *L*.
- The greedy algorithm chooses *f* and then, by the inductive hypothesis, computes an optimal schedule of classes from *L*.

Proving correctness of greedy algorithms

- *An inductive exchange argument.*

- Assume that there is an optimal solution that is different from the greedy solution.

- Find the ‘first’ difference between the two solutions.

- Argue that we can exchange the optimal choice for the greedy choice without degrading the solution.

- This argument implies by induction that there is an optimal solution that contains the entire greedy solution.