Homework 7 for probability & statistics

1. Study the following knowledge points carefully, and then copy them to your homework answer sheet, and remember them:

(1). Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a population X. Then,

the sample mean: $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$;

the sample variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right)$;

$$\hat{S}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right);$$

the kth moment about origin of the sample: $A_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$.

the kth moment about central of the sample: $B_k = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^k$.

(2). Given a random sample $X_1, X_2, \ldots, X_n$ from a **standard normal** population $X \sim N(0,1)$. Then the statistic $\chi^2(n) = X_1^2 + X_2^2 + \cdots + X_n^2$ has a chi-square distribution with n degrees of freedom .

(3). Given two **independent** random variables $X \sim N(0,1)$ and $Y \sim \chi^2(n)$. The random variable $t = \frac{X}{\sqrt{Y/n}}$ has a probability distribution called t distribution with n degrees of freedom, denoted by $t \sim t(n)$.

(4). Given two **independent** chi-squared random variables $U \sim \chi^2(n_1)$ and $V \sim \chi^2(n_2)$. The random variable $F = \frac{U/n_1}{V/n_2}$ has a probability distribution called F distribution with n_1, n_2 degrees of freedom, denoted by $F \sim F(n_1, n_2)$.

(5). Given a random sample $X_1, X_2, \ldots, X_n$ from a normal population $X \sim N(\mu, \sigma^2)$. Then,

① $\frac{X_i-\mu}{\sigma} \sim N(0,1)$ ② $\overline{X} \sim N(\mu, \sigma^2/n)$; ③ $\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$;

③ $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$; ④ $\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$;

⑤ $\overline{X}$ and S^2 are independent;

(6). If random variables $X_1, X_2, \ldots, X_n$ are independent, and have normal distributions as $X_i \sim N(\mu_i, \sigma_i^2)$. Then $\frac{X_i - \mu_i}{\sigma_i} \sim N(0,1)$, $c_1X_1 + c_2X_2 + \cdots + c_nX_n \sim N\left(\sum_{i=1}^{n} c_i\mu_i, \sum_{i=1}^{n} c_i^2\sigma_i^2\right)$.

Particularly, if $X_i \sim N(\mu, \sigma^2)$, then $X_1 - X_2 \sim N(0, 2\sigma^2)$, which can be standardized as: $\frac{X_1 - X_2}{\sqrt{2}\sigma} \sim N(0,1)$.