Factors and Primes

A course book exploring factors and Prime factors in GCSE Mathematics

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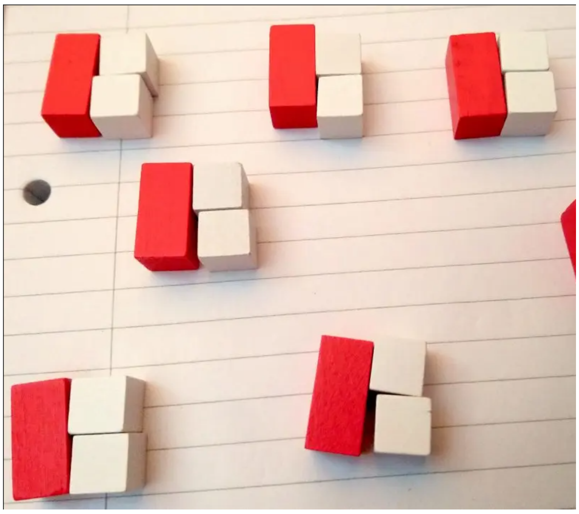


Introduction

Welcome to the course book on Factors, Multiples and Prime Factors. This course book is aimed at the middle grades in the GCSE syllabus. It's also useful if you need some practice in these areas of Maths too.
You won't need anything for this book, just a pen/pencil and yourself - there's plenty of room within for the exercises and, hopefully, your own notes and notations as we go through.

Have fun and enjoy yourself!

Contents

What is a factor?

Factors come in pairs, but factors aren't pairs...

Usually, we think of factor pairs. We're told that $3 \times 2 = 6$ and $3 \times 2 = 6$ and so when asked what are the factors of 6, for instance, our answer is usually "3 times 2, and 1 times 6". This isn't technically correct. So, let's have another think about it.

Every number (with the exception of 1) has at least 2 factors; 1 and itself. This is a fact that we know. A number is a factor of itself. And we talk of factors as single numbers. So, the factors of a number are positive integers that multiply with other positive whole numbers to make that number.

The factors of 6 are: 1, 2, 3 and 6. That is to say that 6 has four factors.

Exercise

Write the factors of the following numbers

7.
10.
15.
25.
30.
36.

Do you notice anything about the factors of square numbers and prime numbers?

Common Factors

Common factors are found between two or more numbers.

It's important that we cover all our numbers, and don't miss any factors out, so we use a systematic way of making lists of factors.

Example

Find the common factors of 36 and 54

Let's start with 36:

$1 \times 36 = 36$

$2 \times 18 = 36$

$3 \times 12 = 36$

$4 \times 9 = 36$

$5 \times$ // *36 isn't in the 5 times table*

$6 \times 6 = 36$ *We stop here, else we're just repeating ourselves - try it and see.*

The factors of 36 are **1, 2, 3, 5, 6, 9, 12, 18** and **36**

And then 54:

$1 \times 54 = 54$

$2 \times 27 = 54$

$3 \times 18 = 54$

$4 \times$ //

$6 \times 9 = 54$ *We finish here, 7 and 8 are in between 6 and 9.*

The factors of 54 are **1, 2, 3, 6, 9, 18, 27** and **54**

So, the common factors of 36 and 54 are **1, 2, 3, 6, 9** and **18** as these are the numbers that appear in both lists.

Exercise

Now it's your turn. List all the common factors of 45 and 54.

Highest Common Factors

As you may have noticed, there can be many common factors between two or more numbers.

In order to avoid long lists (can you imagine if two numbers had 10 or more common factors? Or 80?), we might just look at the Highest Common Factor of two numbers instead.

These would be useful for problems when we want to share two numbers (such as children and teachers in a school) between a single event (such as busses). We find the list of the common factors between the numbers, and then find the highest number. And that is our highest common factor.

Let's have a go at a worded question and see how this works.

Example

Colton has 16 white blocks and 8 red ones. If he wants to place them in identical groups without any blocks left over, what is the greatest number of groups Colton can make?

We want to make groups with two sets of numbers. So, find the common factors between 16 and 8.

Start with 8:

$1 \times 8 = 8$

$2 \times 4 = 8$

The factors of 8 are **1, 2, 4** and **8**

And the factors of 16 are:

$1 \times 16 = 16$

$2 \times 8 = 16$

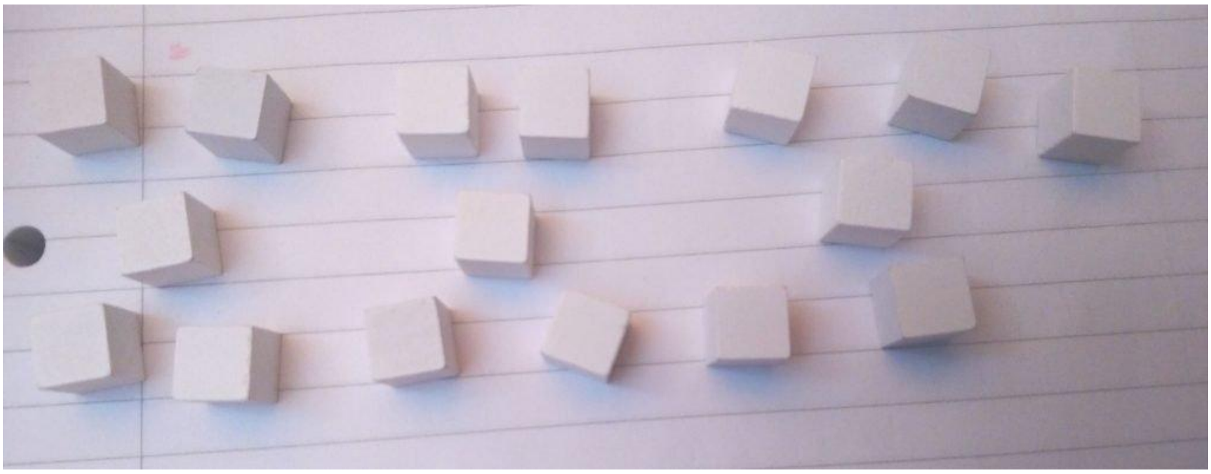
$3 \times //$

$4 \times 4 = 16$

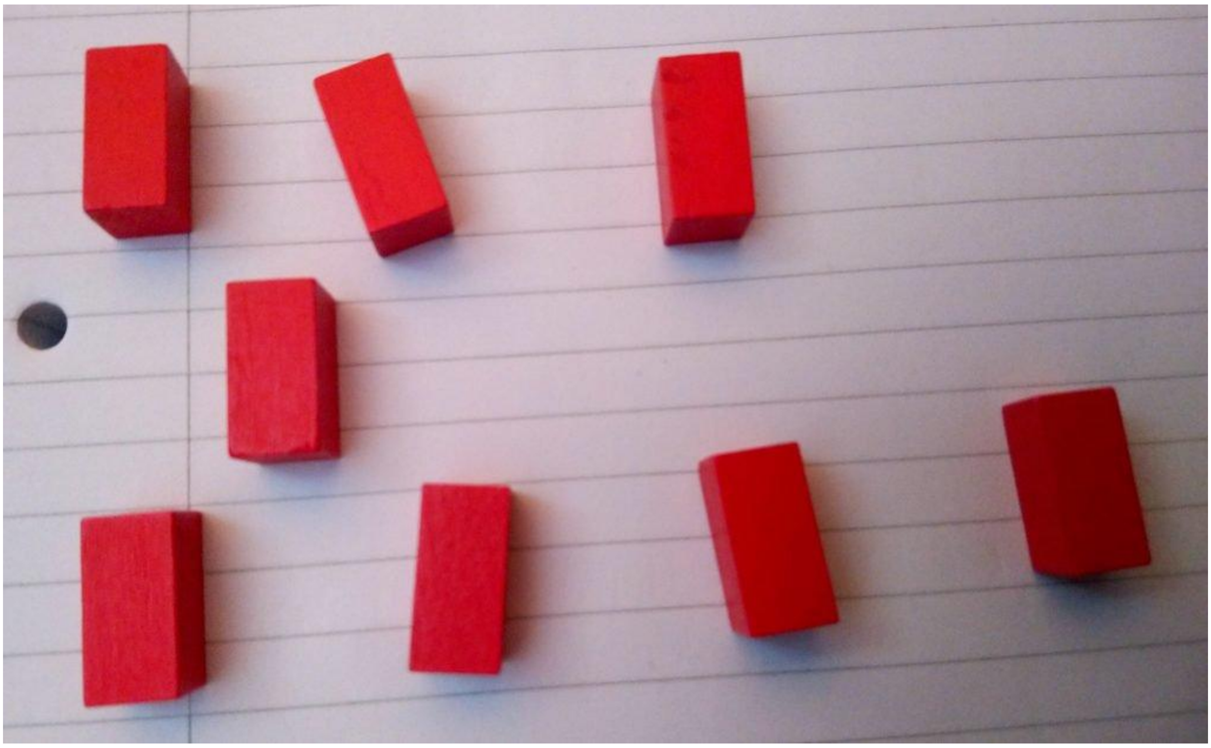
The factors of 16 are **1, 2, 4, 8** and **16**.

The highest number in the two lists is 8, and so in answer to the question: The greatest number of identical groups Colton can make without any blocks left over is 8.

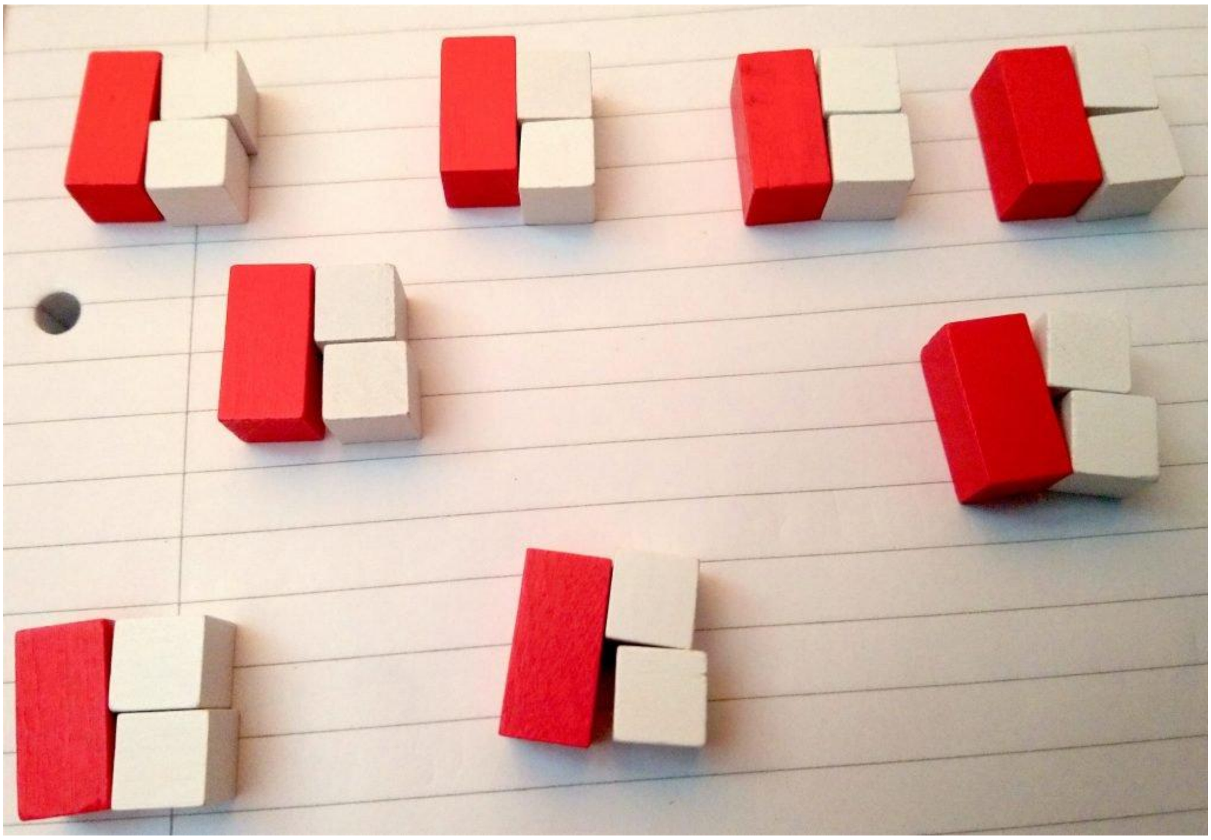
Let's see if it works using blocks



Here are our 16 white blocks



And here are our 8 red blocks



Finally, 8 identical groups of our blocks, so our answer is correct! Have a go yourself and see if you can come up with other groups. Are there any more than 8 identical groups we can make? Can you explain or see why?

Exercise

Kamal has 6 cans of regular soda and 15 cans of diet soda. He wants to create some identical refreshment tables that will operate during the American football game. He also doesn't want to have any sodas left over. What is the greatest number of refreshment tables that Kamal can stock?

(Questions to ask yourself - What is the question asking? Do I understand the question? What are the important bits of information? Is there anything else outside of this topic I might need to answer it?
Use this page for your working out if you need to! And don't be afraid to use something like blocks/pencils/etc to help you)

Multiples

We've visited factors, now it's time to visit multiples.

What operation does the word multiple sound like? If you suggested multiply, then great! You probably already know what multiples are.

Multiples are pretty much times tables for any given number. So the first few multiples of 15 for instance are: 15, 30, 45, 60, 75, 90, ...

Multiples go on and on forever - so you'll be glad to know you'll never be asked to list all the multiples of a number. You may however, be asked to list all the multiples between x and y. For instance, list all the multiples of 2 between 45 and 59: 46, 48, 50, 52, 54, 56 and 58.

Exercise

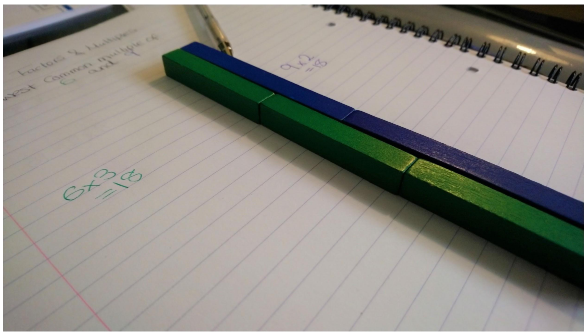
a) List all the multiples of 3 between 35 and 50

b) List all the multiples of 6 less than 25

Shall we move across to common multiples?

Lowest Common Multiples

Just like Highest Common Factors, we can find the Lowest Common Multiple between two (or more) given numbers. And remember the reason we work with the Lowest Common Multiple is because numbers go on forever - I'm not sure there's enough paper on the planet to write the highest multiples of 2 billion and 3 trillion! To find the Lowest common multiple, we - you guessed it! - make a list of the multiples of each number given and find the lowest number between the lists.



Exploring Common Multiples with cuisenaire rods

Example

Beginning at 8:30 A.M., tours of the Tate Gallery and the Science Museum begin at a tour agency. Tours for the Tate Gallery leave every 15 minutes. Tours for the Science Museum leave every 20 minutes. How often do the tours leave at the same time?

Start by listing the first few multiples of 15 and 20 (6-10 is usually a good starting point)

The multiples of 15 are:

15, 30, 45, 60, 75, 90, 105, ...

The multiples of 20 are:

20, 40, 60, 80, 100, 120, ...

There's only one number in the two lists - 60. So the tours both leave at the same time every 60 minutes, (or every hour, both would be accepted answers)

Have a go at one yourself.

Exercise

Pencils come in packages of 10. Erasers come in packages of 12. Phillip wants to purchase the smallest number of pencils and erasers so that he will have exactly 1 eraser per pencil. How many packages of pencils and erasers should Phillip buy?

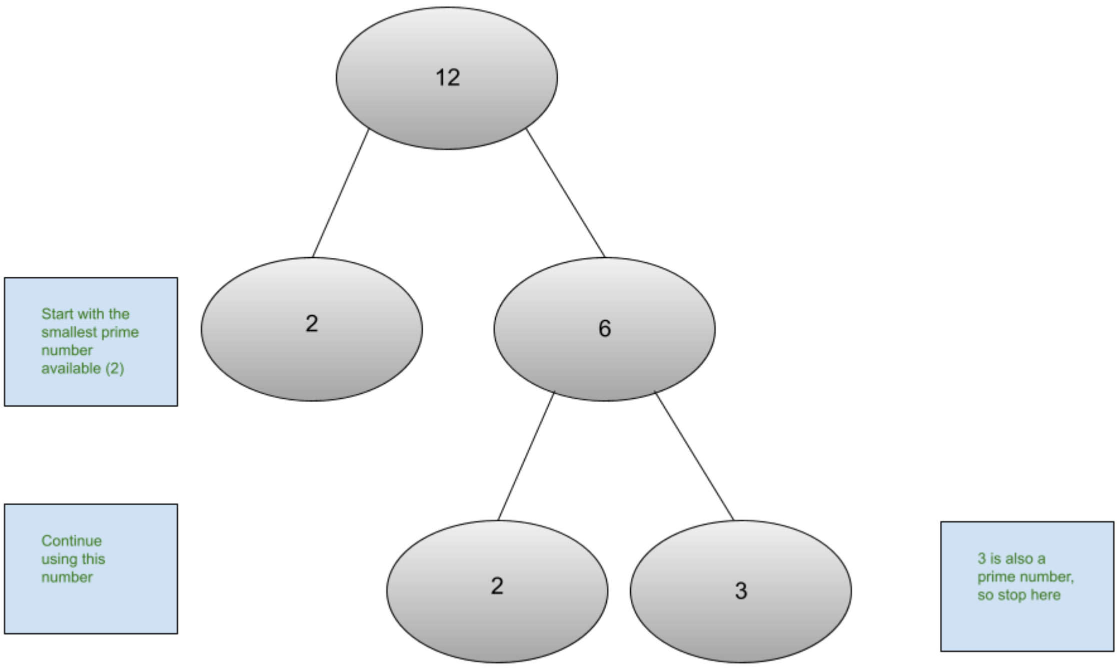
(Questions to ask yourself: Do I understand what the question is asking? Is there anything other than the topic I'll require to answer this question?)

Prime Factors

What is a Prime factor?

Let's break this down. We know what a factor is, that's a number that "makes up" a specific number. And a Prime number is a number with two factors; 1 and itself. So, when we're talking about Prime Factors, we're interested in the Primes that make up a specific number. Please remember, 1 is not a prime number. Why not? Every number has a unique prime factorisation - and what happens if we multiply by 1?

They're fairly easy to find, using a factor tree, if a little messy. Prime factors (and thus prime factorisation) crop up more on the Higher Papers, but they're interesting nonetheless, so it wouldn't hurt to have a look.
We want to find the "prime factorisation" of the number 12. A factor tree is the easiest way to find these. The following has a factor tree to help us find the prime factorisation of 12.



From here we have 3 prime numbers at the end of the branches. 2, 2, and 3. So the prime factorisation of 12 is: $2 \times 2 \times 3$ or $12 = 2^2 \times 3$.

Or if a tree doesn't work for you, you can also use a table to help: Just like with the tree, we start with the smallest available prime number, 2.

Workings	Prime factors
$12 \div 2 = 6$	**2**
$6 \div 2 = 3$	**2** **3**

This gave us the same information as the tree above; the prime factors for 12 are; **2**, **2**, and **3**, and we write this as $12 = 2^2 \times 3$

Example

1. **Write the prime factorisation for 120.**

Start with the smallest prime number there is; 2.

Workings	Prime factors
$120 \div 2 = 60$	**2**
$60 \div 2 = 30$	**2**
$30 \div 2 = 15$	**2**

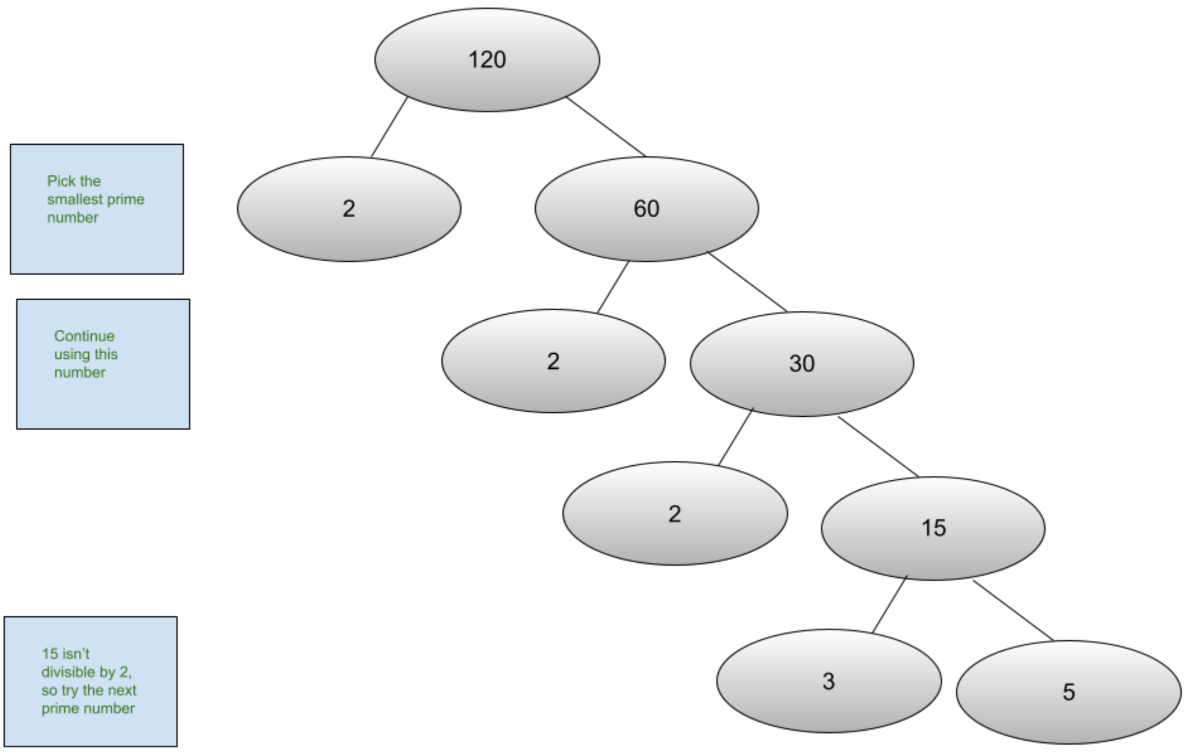
15 isn't divisible by 2, so we move to 3 - the next prime number available.

Workings	Prime factors
$120 \div 2 = 60$	**2**

$60 \div 2 = 30$	**2**
$30 \div 2 = 15$	**2**
$15 \div 3 = 5$	**3** **5**

So, the prime factorisation for 120 is $2 \times 2 \times 2 \times 3 \times 5$ or $2^3 \times 3 \times 5$

Or a factor tree would look like this



Note: You don't have to use this way exactly. Often multiples of ten are easier to work out than trying to divide by 2. For example, if you're comfortable to work with $120 = 10 \times 12$ and then able to work out the prime factorisation of 12 and 10, you might have less branches.

Exercise

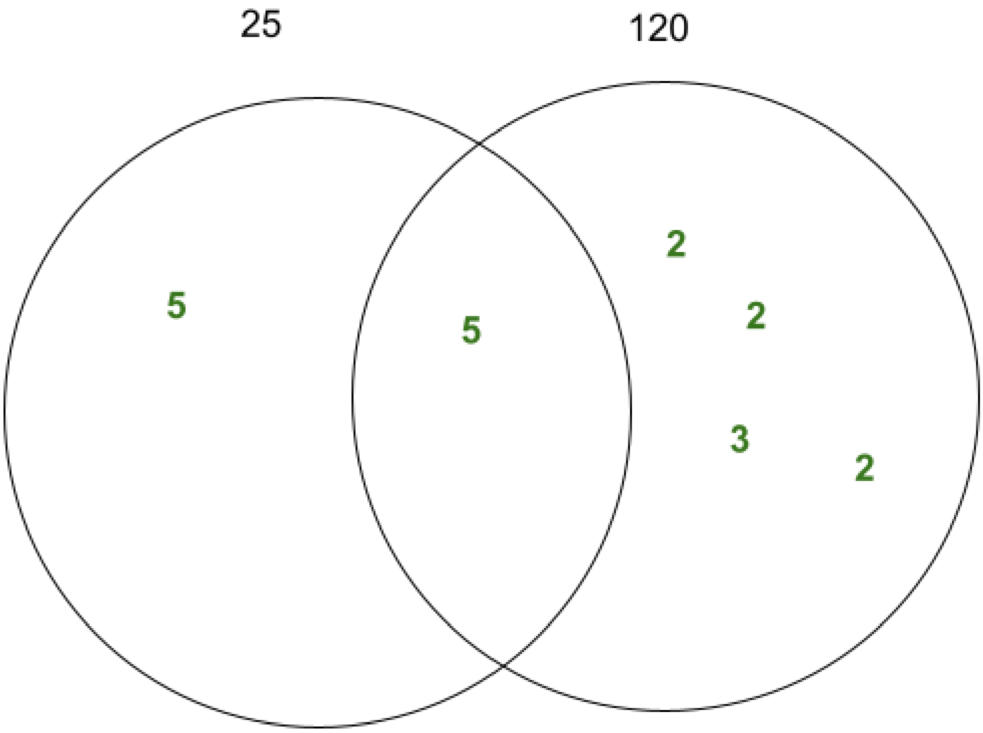
Try finding the prime factorisations for the following numbers. Use both the trees and the table to find what works best for you.
Trees are also useful to know, because every so often, they crop up on exams.

A. 220
B. 75
C. 99
D. 142

(Note: you should know your prime numbers to 100)

Prime Factors and HCF and LCM

We can also use prime factors to find the Highest Common factor or the Lowest Common Multiple of 2 or more numbers. There's the list way, or the Venn Diagram way. Let's have a look.



The prime factorisation of 25 is 5^2 and the prime factorisation of 120 is $2^3 \times 3 \times 5$. We put all these numbers into a Venn Diagram, remembering to place one of the fives in the middle (because both 25 and 120 have 5 as a prime factor).
From here, we can see that the highest common factor is 5 (it's what is shared between the two numbers) and the lowest common multiple is $2 \times 2 \times 2 \times 3 \times 5 \times 5$ - that is, we times everything together.

We could also use a list too.
The prime factors of 25: 5×5
The prime factors of 120: $2 \times 2 \times 2 \times 3 \times 5$

The Highest Common Factor is the prime numbers that are in **both** lists. In this case; **5**.
The Lowest Common Multiple is found using the prime number in **either** lists. In this case; $2 \times 2 \times 2 \times 3 \times 5 \times 5$. Easiest way to do this; write out one of the lists, then add the other, ignoring any that have already appeared in the first list.
We see that there are three 5's in total in both lists, however, the LCM only has 2. Can you see why this is the case? What would happen if we multiplied all the numbers together?

Example

Use the following information to find the Highest Common Factor and the Lowest Common Multiple of 180 and 84.

$180 = 2^2 \times 3^2 \times 5$

$84 = 2^2 \times 3 \times 7$

Let's first write it out... so we have:

$180 = 2 \times 2 \times 3 \times 3 \times 5$

$84 = 2 \times 2 \times 3 \times 7$

In order to find the HCF, we list the prime numbers that appear in **both** numbers. 2, 2 and 3 are prime factors of both numbers, so we multiply these together:

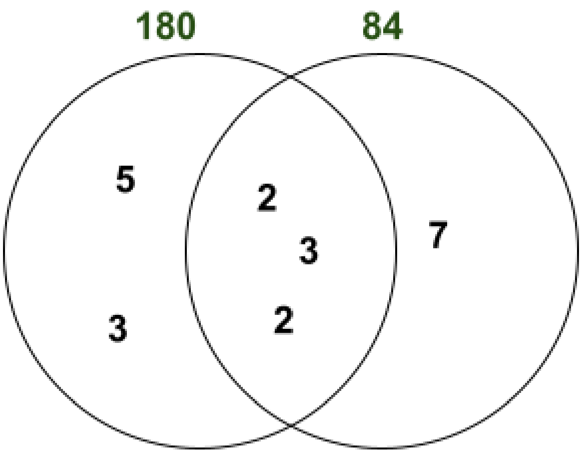
HCF: $2 \times 2 \times 3 = 12$

To find the LCM, we list the prime numbers that appear in **either** numbers. 2, 2, 3, 3, 5 and 7.

LCM: $2 \times 2 \times 3 \times 3 \times 5 \times 7 = 2^2 \times 3^2 \times 5 \times 7 = 1260$

So much easier than writing out lists of factors for big numbers, right?

And here's the same example with a Venn diagram instead.



Venn diagram for prime factors of 180 and 84

From the Venn diagram we can see that the shared factors are $2 \times 2 \times 3$ and so the highest common factor is 12. And the Lowest Common Multiple is all the numbers multiplied together $7 \times 5 \times 3^2 \times 2^2$ - same answers as before!

Exercise

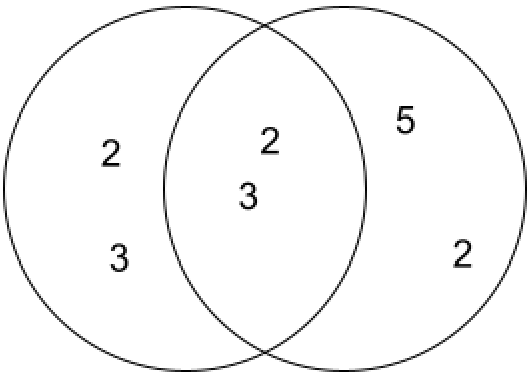
1. Use the following information to find the HCF of the numbers 190 and 45.

$$665 = 19 \times 7 \times 5$$
$$380 = 19 \times 5 \times 2^2$$

2. Use the following information to find the LCM of the numbers

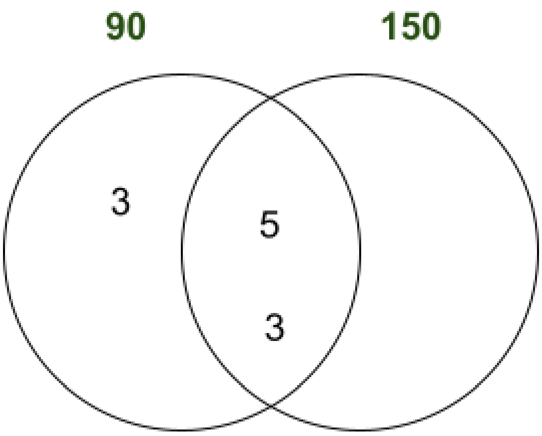
$$60 = 2^2 \times 3 \times 5$$
$$45 = 3^3 \times 5$$

3. Look at the following Venn Diagram



   a. Write the prime factorisation of two numbers in the Venn Diagram
   b. What's the highest common factor of these two numbers?
   c. What's the lowest common multiple of these two numbers?

4.
   a. Complete the following Venn Diagram



   b. Write down the Highest Common Factor of 150 and 90
   c. Find the Lowest Common Multiple of 90 and 150

End of Unit Quiz

1. Select the correct statement
   a. A multiple is a number that is multiplied to make another number
   b. All numbers have exactly 2 factors
   c. A factor is a number that is multiplied to make another number
   d. We can find the Highest Common Multiple

2. The prime factorisation of 12 is $2 \times 2 \times 3$ True/False?

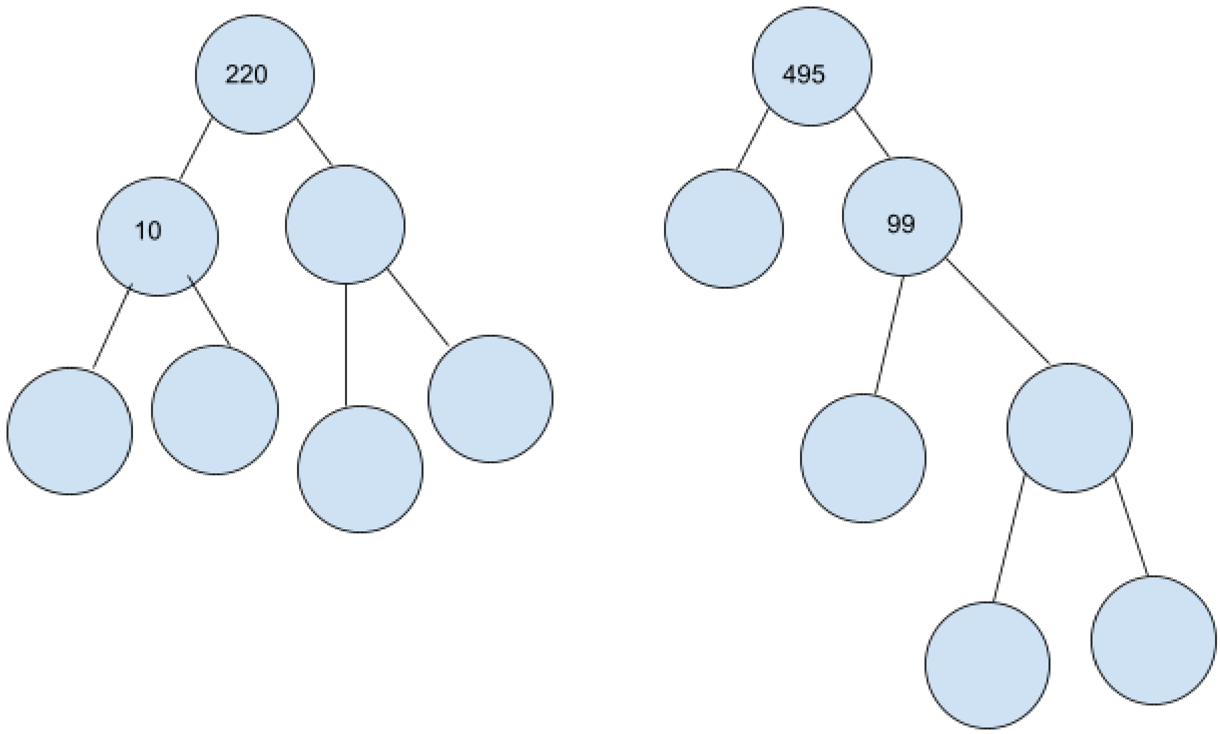
3. What's the Lowest Common Multiple of 5 and 6?

4. Find the Highest Common Factor of 6 and 10

5. Write the prime factorisation of 45

6. Express 200 as the product of powers of its prime factors

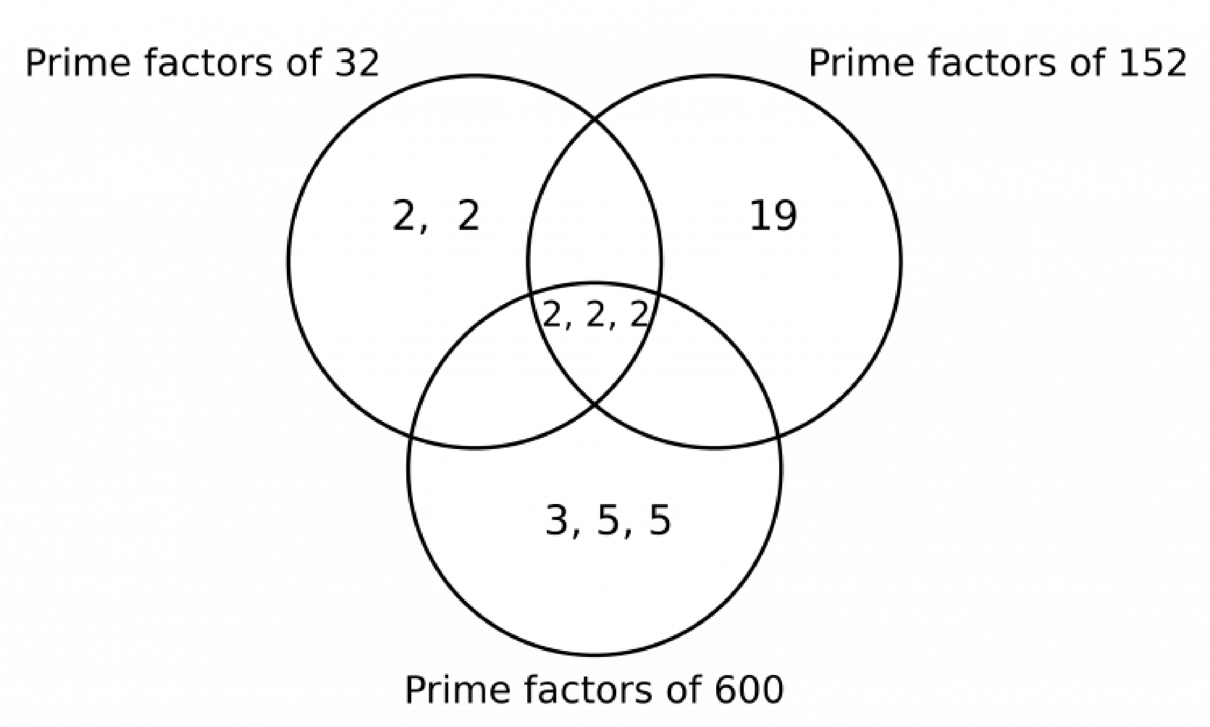
7. Complete the factor tree and find the HCF and LCM of 495 and 220.



8. Sapphire and Abe are shelving books at a public library. Sapphire shelves 5 books at a time, whereas Abe shelves 6 at a time. If they end up shelving the same number of books, what is the smallest number of books each could have shelved?

9. Select the Prime factors of 70
    a. 7
    b. 5
    c. 10
    d. 15
    e. 3
    f. 2

10. Using the given Venn Diagram, find the Lowest Common Multiple of 32, 152, and 600.



11. Kelly is organising a barbecue.
    She needs bread rolls and burgers.
    Bread rolls are sold in packs of 20.
    Burgers are sold in packs of 12.
    Kelly buys exactly the same number of bread rolls as burgers.
    What is the least number of each pack that Kelly buys?

12. Frank says "All prime numbers are odd"
    John says "1 is a prime number"

    Explain why both are incorrect.
    (Can you give an example to prove incorrect? Or what happens when we multiply by 1)

13. Express 100 as a product of its prime factors.

14. $3x^2 = 75$
    a. Find the value of x

    b. Express 75 as a product of its prime factors

15. You are given that $m = 2^3 \times 5$
   a. Calculate $10m$

   b. Write $10m$ as a product of primes

      (Think about what happens when you multiply any number by 10)

Answers

Factors

Page 4

I. 7, 1
II. 1, 2, 5, 10
III. 1, 3, 5, 15
IV. 1, 5, 25
V. 1, 2, 3, 5, 6, 10, 15, 30
VI. 1, 2. 3, 4, 6, 9, 12, 18, 36

Square numbers have an odd number of factors, primes only have 2 factors.

Page 6

I. 1, 3 and 9

Page 10

3 tables, 2 regular sodas and 5 diet sodas on each.

Multiples

Page 11

I. 36, 39, 42, 45, 48
II. 6, 12, 18, 24

Page 13

I. 6 packages of pencils and 5 packages of erasers (60 of each)

Prime Factors

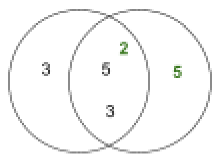
Page 17

I. $2^2 \times 5 \times 11$
II. $5^2 \times 3$
III. $3^2 \times 11$
IV. 71×2

Prime factors and HCF/LCM

Page 21

1. 19×5
2. $2^3 \times 3^2 \times 5$
3.
   a. $2^2 \times 3^2$ and $2^2 \times 3^2 \times 5$
   b. $3 \times 2 = 6$
   c. $5 \times 2^3 \times 3^2$
4.
   a.



   b. $5 \times 3 \times 2 = 30$
   c. $5^2 \times 3^2 \times 2$

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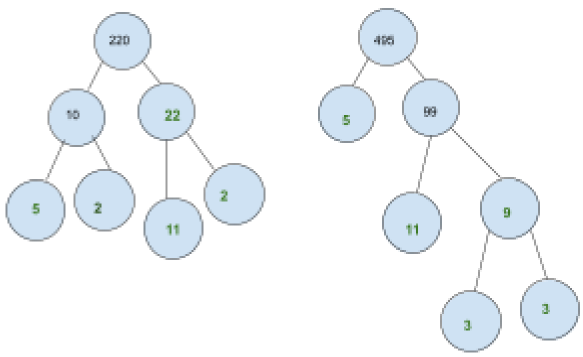
<u>**End of Unit Quiz**</u>

Page 22

1. C
2. True
3. 30
4. 2
5. $3^2 \times 5$
6. $2^3 \times 5^2$
7.



HCF: $5 \times 11 = 55$

LCM: $11 \times 5 \times 3^2 \times 2^2$

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Page 23

8. 30 books
9. 7, 5 and 2
10. $19 \times 5^2 \times 3 \times 2^5$

Page 24

11. 3 packs of bread rolls
    5 packs of burgers
12. 2 is a prime number and even
    1 is not a prime number since a number does not change if multiplied by 1. And so, there is not a unique prime factorisation for that number.
13. $5^2 \times 2^2$
14.
    a. $x = 5$
    b. 3×5^2
15.
    a. $2^3 \times 5 \times 10$
    $= 400$
    b. $2^4 \times 5^2$

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