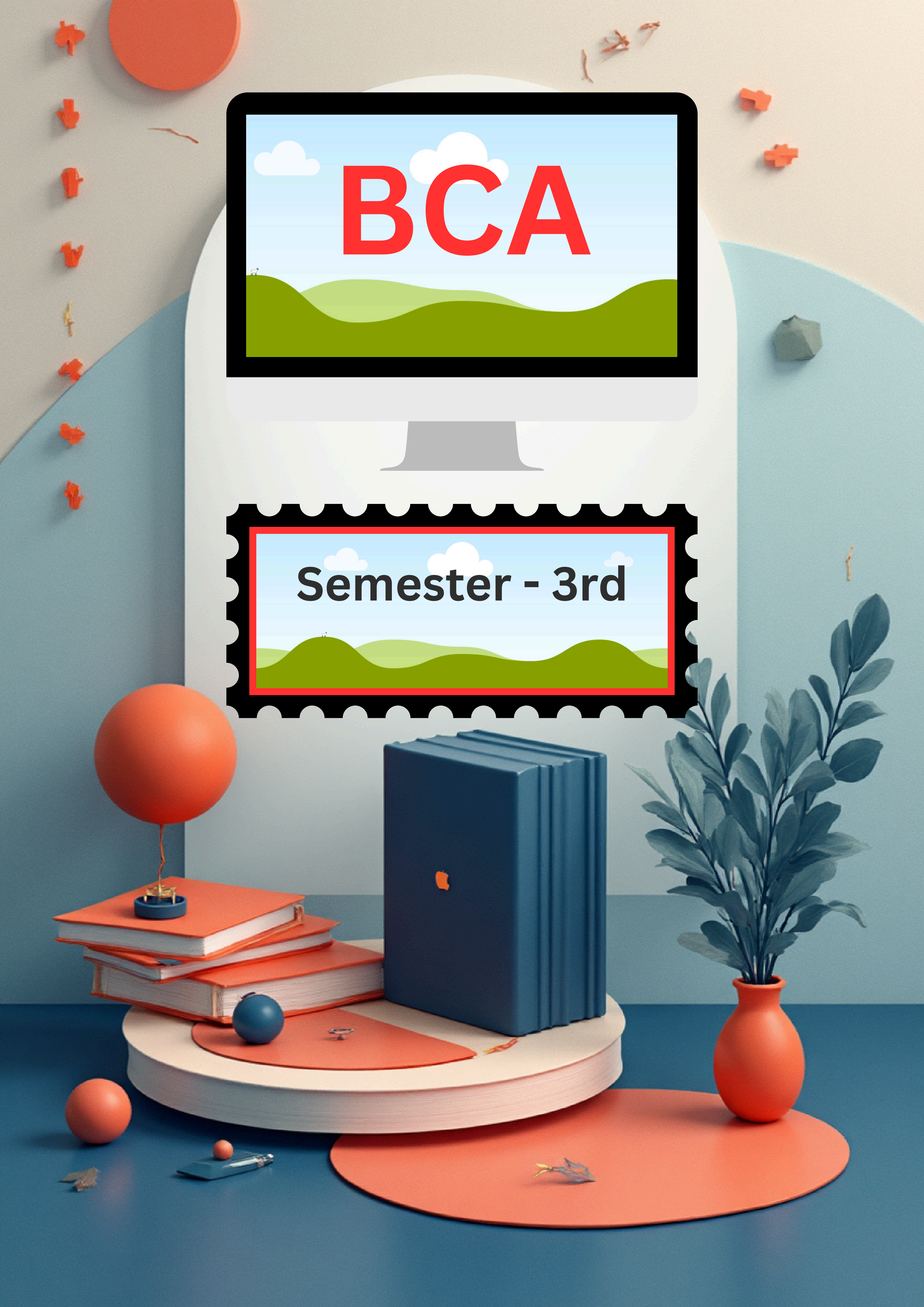
BCA
Semester - 3rd

Numerical Methodology

Notes - 1

Contents

Solution of a nonlinear algebraic ad transcendental equations:Bisection method, False position, Newton Raphson method, Iterative Method, Lin Bairstow's method.

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Introduction

In many mathematical and scientific problems, we often come across nonlinear equations — that means equations that cannot be expressed in a straight-line form like ax+b=0ax + b = 0ax+b=0.

Such equations can be algebraic (like x3−x−2=0x^3 - x - 2 = 0x3−x−2=0) or transcendental (like ex−3x=0e^x - 3x = 0ex−3x=0).

Since these equations cannot always be solved directly, we use numerical methods to find their approximate solutions.

One of the simplest and most reliable methods to solve such equations is the Bisection Method.

Concept of Bisection Method

The Bisection Method is a bracketing method.

It works on the principle of Intermediate Value Theorem, which says:

If a function f(x)f(x)f(x) is continuous between two points a and b, and f(a)) and f(b) have opposite signs, then there exists at least one root between a and b.

So, the idea is to narrow down (or bisect) the interval [a,b][a, b][a,b] repeatedly until we get a small enough interval that contains the root.

3. Step-by-Step Procedure

Let $f(x) = 0$ be the given equation.

Step 1: Choose two initial guesses

Find two points a and b such that:

$$f(a) \times f(b) < 0$$

This means the function changes its sign between a and b, hence a root lies in between.

Step 2: Find the midpoint

Compute the midpoint:

$$x_m = \frac{a+b}{2}$$

Step 3: Evaluate $f(x_m)$

Check the sign of $f(x_m)$.

- If $f(a) \times f(x_m) < 0$, the root lies between a and x_m.
  So, set $b = x_m$.
- If $f(x_m) \times f(b) < 0$, the root lies between x_m and b.
  So, set $a = x_m$.

Step 4: Repeat

Repeat the steps 2 and 3 until the difference between a and b becomes very small (i.e., the desired accuracy is achieved).

Step 5: Root approximation

The approximate root is given by:

$$x = \frac{a+b}{2}$$

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4. Formula Used

$$x_{i+1} = \frac{a_i + b_i}{2}$$

and the new interval is decided based on the sign of $f(x_{i+1})$.

5. Stopping Criteria

We can stop the iteration when either of the following conditions is true:

1. $|f(x_m)| < \text{tolerance}$, or
2. $|b - a| < \text{tolerance}$

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6. Example

Find the root of $f(x) = x^3 - x - 2 = 0$ using Bisection Method.

Step 1:
Try $f(1) = 1 - 1 - 2 = -2$
Try $f(2) = 8 - 2 - 2 = 4$
→ Since $f(1) \times f(2) < 0$, root lies between 1 and 2.

Step 2:
Midpoint $x_m = (1 + 2)/2 = 1.5$

$f(1.5) = 3.375 - 1.5 - 2 = -0.125$ (negative)

→ So, new interval is $[1.5, 2]$

Step 3:
Next midpoint $x_m = (1.5 + 2)/2 = 1.75$

$f(1.75) = 5.36 - 1.75 - 2 = 1.61$ (positive)

→ Root lies between $[1.5, 1.75]$

Step 4:
Continue bisecting until the desired accuracy is reached.

Final approximate root ≈ **1.52**

✅ **In Short (Summary):**

Step	Operation	Formula / Condition
1	Find a, b such that $f(a)f(b) < 0$	—
2	Find midpoint	$x_m = (a + b)/2$
3	Replace interval	If $f(a)f(x_m) < 0 \Rightarrow b = x_m$; else $a = x_m$
4	Repeat until small error	(
5	Approximate root	$x = (a + b)/2$

Solution of Nonlinear Algebraic and Transcendental Equations: False Position Method (Regula-Falsi Method)

1. Introduction

The False Position Method (also called Regula-Falsi Method) is another bracketing method used to find the root of nonlinear equations like f(x)=0f(x) = 0f(x)=0.

It is similar to the Bisection Method, but instead of taking the midpoint, it finds a new point using a straight line interpolation between the two ends.

This method usually converges faster than the bisection method.

2. Basic Principle

If a function f(x)f(x)f(x) is continuous between two points a and b, and f(a)f(a)f(a) and f(b)f(b)f(b) have opposite signs (i.e., f(a)f(b)<0f(a)f(b) < 0f(a)f(b)<0), then there is a root between a and b.

In the False Position Method, a chord (straight line) is drawn between the points (a,f(a))(a, f(a))(a,f(a)) and (b,f(b))(b, f(b))(b,f(b)).

The point where this line cuts the x-axis gives a better approximation of the root.

3. Formula

The formula to find the next approximation x_1 is:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

4. Step-by-Step Procedure

Step 1:

Choose two points a and b such that $f(a)f(b) < 0$.

Step 2:

Find the new point using the formula:

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Step 3:

Evaluate $f(x_1)$.

- If $f(a)f(x_1) < 0$, then the root lies between a and x_1.
  Set $b = x_1$.
- If $f(x_1)f(b) < 0$, then the root lies between x_1 and b.
  Set $a = x_1$.

Step 4:

Repeat the process until the desired accuracy is obtained.

5. Example

Find the root of $f(x) = x^3 - x - 2 = 0$.

$f(1) = -2$, $f(2) = 4$, so root lies between 1 and 2.

Now,

$$x_1 = \frac{1(4) - 2(-2)}{4 - (-2)} = \frac{4+4}{6} = 1.333$$

$$f(1.333) = (1.333)^3 - 1.333 - 2 = -0.963$$

Since $f(1.333)$ and $f(2)$ have opposite signs, new interval is $[1.333, 2]$.

Repeat the process to get the root ≈ **1.52**.

8. In Short (Summary Table)

Step	Formula / Condition
Find a, b	$f(a)f(b) < 0$
New point	$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$
Update interval	If $f(a)f(x_1) < 0$, set $b = x_1$; else $a = x_1$
Stop when	(

Solution of Nonlinear Algebraic and Transcendental Equations: Newton-Raphson Method

1. Introduction

The Newton-Raphson Method is one of the most powerful and fastest open methods used to find roots of nonlinear equations.

Unlike bisection or false position methods, it requires only one initial guess and uses derivatives of the function.

It is widely used because of its rapid convergence and simplicity in implementation.

2. Basic Idea

Suppose we have an equation: f(x)=0

We start with an initial guess x0x_0x0 near the root.

Then, by drawing a tangent to the curve y=f(x) at x0, we find where the tangent cuts the x-axis.

This intersection point becomes the next approximation x1x_1x1.

3. Formula

The Newton-Raphson formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where

- $f(x_n)$ = value of function at x_n
- $f'(x_n)$ = derivative of $f(x)$ at x_n

4. Step-by-Step Procedure

Step 1: Choose an initial guess x_0.

Step 2: Compute $f(x_0)$ and $f'(x_0)$.

Step 3: Find new approximation using

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Step 4: Repeat until desired accuracy is reached.

5. Example

Solve $f(x) = x^3 - x - 2 = 0$ using Newton-Raphson method.

$$f(x) = x^3 - x - 2, \quad f'(x) = 3x^2 - 1$$

Take $x_0 = 1.5$

Now,

$$f(1.5) = 3.375 - 1.5 - 2 = -0.125$$

$$f'(1.5) = 3(1.5)^2 - 1 = 5.75$$

$$x_1 = 1.5 - \frac{-0.125}{5.75} = 1.5 + 0.0217 = 1.5217$$

Next iteration:

$$f(1.5217) \approx 0.002, \quad f'(1.5217) \approx 5.947$$

$$x_2 = 1.5217 - \frac{0.002}{5.947} \approx 1.5214$$

Hence, the root ≈ **1.5214**.

8. In Short (Summary Table)

Step	Formula / Condition
Initial guess	x_0
Update formula	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
Stop when	(
Type	Open method (not bracketed)

Solution of Nonlinear Algebraic and Transcendental Equations: Iterative Method

1. Introduction

The Iterative Method is one of the simplest methods used to find the approximate root of an equation of the form:

f(x)=0

It works by repeatedly improving an initial guess until the value of xxx becomes close enough to the actual root.

This method is also called the Method of Successive Approximations or Fixed Point Iteration Method.

2. Basic Idea

We start with an equation $f(x) = 0$ and rearrange it as:

$$x = g(x)$$

Then we take an **initial guess** x_0 and use the formula:

$$x_{n+1} = g(x_n)$$

We keep repeating this formula until x_{n+1} and x_n are nearly the same (i.e., difference is very small).

3. Step-by-Step Procedure

Step 1:

Write the given equation $f(x) = 0$ in the form $x = g(x)$.

Step 2:

Choose an initial approximation x_0.

Step 3:

Compute next approximations using

$$x_{n+1} = g(x_n)$$

Step 4:

Continue until

$$|x_{n+1} - x_n| < \text{tolerance (e.g. 0.0001)}$$

4. Condition for Convergence

The iterative method will converge only if:

$$|g'(x)| < 1$$

near the root.

5. Example

Find the root of $f(x) = \cos x - 3x + 1 = 0$.

Rearrange:

$$x = \frac{\cos x + 1}{3}$$

Let $g(x) = \frac{\cos x+1}{3}$.

Start with $x_0 = 0.5$

$$x_1 = g(0.5) = \frac{\cos 0.5 + 1}{3} = \frac{0.877 + 1}{3} = 0.6257$$

$$x_2 = g(0.6257) = \frac{\cos 0.6257 + 1}{3} = \frac{0.806 + 1}{3} = 0.602$$

Continue until $x_{n+1} \approx x_n$.

Hence, approximate root ≈ **0.61**.

8. Summary Table

Step	Formula / Condition
Rearrange	$f(x) = 0 \Rightarrow x = g(x)$
Iteration	$x_{n+1} = g(x_n)$
Stop when	(
Converges if	(

Lin–Bairstow's Method

1. Introduction

The Lin–Bairstow Method is used to find quadratic factors and complex or real roots of polynomial equations of higher degree.

It is an improvement over Bairstow's method and works by dividing a polynomial by a quadratic factor (x2+rx+s) to find its coefficients and refine r and s until remainder terms become zero.

This method is very useful when we need all roots (real and complex) of a polynomial equation.

2. General Form of Polynomial

Consider a polynomial:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

We assume that the polynomial can be divided by a **quadratic factor**:

$$x^2 + rx + s$$

where r and s are unknowns.

3. Basic Principle

We divide $f(x)$ by $x^2 + rx + s$ to get:

$$f(x) = (x^2 + rx + s)Q(x) + (Bx + C)$$

where:

- $Q(x)$ = quotient polynomial
- $Bx + C$ = remainder

For the quadratic factor to be an **exact factor**, the remainder must be zero:

$$B = 0, \quad C = 0$$

These two equations are solved simultaneously to get new values of r and s.

4. Lin–Bairstow's Iterative Formulae

To refine the values of r and s, we use corrections Δr and Δs.

The system of linear equations is:

$$B + \Delta r D + \Delta s E = 0$$

$$C + \Delta r E + \Delta s F = 0$$

Solving these gives:

$$\Delta r = \frac{CE - BF}{DF - E^2}$$

$$\Delta s = \frac{BE - CD}{DF - E^2}$$

Then update:

$$r_{new} = r + \Delta r, \quad s_{new} = s + \Delta s$$

Repeat until B and C become very small (close to zero).

5. Steps of Lin–Bairstow Method

Step 1:
Assume initial values of r and s.

Step 2:
Use synthetic division (recurrence relations) to compute coefficients b_i, d_i, e_i.

Step 3:
Compute remainder B, C.

Step 4:
Find corrections $\Delta r, \Delta s$ using formulas.

Step 5:
Update $r, s \rightarrow r + \Delta r, s + \Delta s$.

Step 6:
Repeat until $B, C \approx 0$.

Step 7:
Then the quadratic factor $x^2 + rx + s = 0$ gives two roots.

6. Example (Conceptual)

For $f(x) = x^4 - 3x^3 + 3x^2 - x - 2 = 0$

Assume quadratic factor $x^2 + rx + s$.
After applying Lin–Bairstow iterations, we find r, s, say $r = -1.2, s = 1.0$.

Hence factor = $x^2 - 1.2x + 1.0 = 0$
→ gives two roots by quadratic formula.

Remaining factor $Q(x)$ gives the other two roots.

9. Summary Table

Step	Operation	Formula
1	Divide by quadratic	$f(x) = (x^2 + rx + s)Q(x) + (Bx + C)$
2	Set remainders zero	$B = 0, C = 0$
3	Find corrections	$\Delta r, \Delta s$ from linear equations
4	Update	$r = r + \Delta r, s = s + \Delta s$
5	Repeat until small error	$B, C \to 0$

Practice Questions

1. **Using the Bisection Method**, find the root of the equation
   $f(x) = x^3 - x - 2 = 0$
   correct up to **three decimal places**.

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2. **Find the real root** of the equation
   $f(x) = x^3 - 4x - 9 = 0$
   using the **False Position (Regula-Falsi) Method** up to **four iterations**.

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3. Solve $f(x) = \cos x - xe^x = 0$
   using the **Newton–Raphson Method**, correct up to **four decimal places**.

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4. Using the **Iterative Method**, find the root of
   $f(x) = x^3 + x - 1 = 0$
   by expressing it in the form $x = g(x)$. Perform **five iterations** starting from $x_0 = 0.5$.

5. Use the **Newton–Raphson Method** to find the **reciprocal of a number (1/N)**.
   Derive the iterative formula and compute $1/18$ correct up to **five decimal places**.

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6. Apply the **Bisection Method** to find a root of
   $f(x) = e^x - 3x = 0$
   correct up to **0.001** accuracy.

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7. Using the **False Position Method**, find a root of
   $f(x) = xe^x - 1 = 0,$
   correct up to **three iterations**.

8. Use the **Newton–Raphson Method** to find the **square root of 20**, i.e., solve $f(x) = x^2 - 20 = 0$ up to **three decimal accuracy**.

9. Find **all roots** of the polynomial
$f(x) = x^4 - 3x^3 + 3x^2 - x - 2 = 0$
using the **Lin–Bairstow Method**, correct up to **two decimal places**.

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10. Solve $f(x) = x \tan x - 1 = 0$
using the **Newton–Raphson Method**, taking $x_0 = 0.7$,
and find the root correct up to **three decimal places**.

Check the answer in the Practice Questions section on our website.


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Thank You