Section 4

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Contents

1 Vector Fields

- A vector field in $\mathbb{R}^n$ is an assignment that goes from $\mathbb{R}^n \to \mathbb{R}^n$, where the former is viewed as a bunch of points, and the latter is a bunch of vectors
  - Two examples: a force field (*e.g.* gravity, electrostatic, magnetic) or a velocity field (*e.g.* fluid mechanics)
- *Ex.* A gravitational field, with two masses, one fixed (M), and one floating (m)
  - The pull = magnitude · direction $\to G\dfrac{Mm}{|\overline{d}|^2} \cdot \dfrac{\overline{d}}{|\overline{d}|} = G\dfrac{Mm}{|\overline{d}|^3}\overline{d}$
- *Ex.* Gradient vector fields
  - $f(\overline{x}) \to \nabla f = \left\langle \dfrac{\delta f}{\delta x_1}, \dfrac{\delta f}{\delta x_2}, \cdots, \dfrac{\delta f}{\delta x_n} \right\rangle$
  - Not every vector field can be realized as a gradient vector field of some function
  - The 'del' operator is as follows: $\nabla := \left\langle \dfrac{\delta}{\delta x_1}, \dfrac{\delta}{\delta x_2}, \cdots, \dfrac{\delta}{\delta x_n} \right\rangle$
  - Apply ∇ to a function f to obtain a gradient vector field (use dot product)
  - $\nabla \cdot F = \dfrac{\delta F_1}{\delta x_1} + \dfrac{\delta F_2}{\delta x_2} + \cdots + \dfrac{\delta F_n}{\delta x_n} = \text{div}(F)$
    - This is the divergence of F
  - $\nabla \times F$ describes the curl of F (how the vector field curls in three dimensions)

Using the ∇ operator:

input	output	significance
function f	∇f	gradient of f (a vector field)
vector field of f	$\nabla \cdot f$	divergence of f (a function)
$n = 3$ vector field of f	$\nabla \times f$	curl of f (a vector field)

- To view 2D vector fields as 3D vector fields, convert $F = \langle P(x,y), Q(x,y)\rangle \to \langle P(x,y), Q(x,y), 0\rangle$
- The curl of a two dimensional vector field, converted to three dimensions, is $\text{curl}(F) = Q_x - P_y$

2 Line Integrals

- Can find Work, W, done by a force (a.k.a. a vector field) on an object along an oriented curve, C

- $W = \int_C \overline{F} \cdot dr$ — This is the line integral of $\overline{F}$ along the oriented curve C
- To compute the line integral, parametrize C, and reduce it to a Calculus II integral
  - $d\overline{r} = \overline{r}'(t)\,dt$
  - $\int_a^b \overline{F}(\overline{r}(t)) \cdot \overline{r}'(t)\,dt$
- Conservative vector fields are independent of path (taking any path from one point to another will always yield the same value)
- Alternative notation: $\int_C P\,dx + Q\,dy + R\,dz \to \int_C P \cdot x'(t)\,dt + Q \cdot y'(t)\,dt + R \cdot z'(t)\,dt$

3 Conservative Vector Fields

- A vector field $\overline{F}$ is said to be conservative if $\overline{F} = \nabla f$ for some function f
- The fundamental theorem of line integrals: Supposed $\overline{F}$ is conservative with potential f. $\overline{F} = \nabla f$. Let C be an oriented curve starting at A and ending at B, then: $\int_C \overline{F} \cdot d\overline{r} = f(B) - f(A)$
- Any constant vector field is conservative
- Before doing a line integral, check if $\overline{F}$ is conservative
- $\overline{F}$ is conservative if and only if curl($\overline{F}$)=0

4 Green's Theorem

- Let R be a region in $\mathbb{R}^2$ bounded by a simple, closed, oriented, counter-clockwise, piecewise-regular curve δR. Let $\overline{F} = \langle P(x,y), Q(x,y) \rangle$ be a continuous, differentiable vector field on an open set containing R, then:

$$\int_{\delta R} \overline{F}\,d\overline{r} = \iint_R Q_x - P_y\,dA$$

  - Simple: The curve can not fold in on itself
  - Closed: The curve starts and ends at the same point

- Because, in $\mathbb{R}^2$, $Q_x - P_y = \text{curl}(\overline{F}) = \nabla \times \overline{F}$, the integral can be rewritten as $\int_{\delta R} \overline{F}\,d\overline{r} = \iint_R \delta\overline{F}\,dA$

5 Flux Through a Surface

- To find the volume of fluid flowing through a surface, M, per unit time, a normal unit vector, $\overline{n}$ is needed; $\overline{n}$ is supposed to vary continuously through M; when such a choice for $\overline{n}$ is possible, M is said to be orientable
- Volume $= (\overline{V} \cdot \overline{n})\, dS$
- Flux of V through $M = \iint_M (\overline{V} \cdot \overline{n})\, dS$
  - $\overline{n} = \dfrac{\overline{r}_u \times \overline{r}_v}{|\overline{r}_u \times \overline{r}_v|}$
  - Remark: If $\overline{n}$ produces an incorrect orientation, flip the sign
- The final formula becomes $\displaystyle\iint_{D(u,v)} V(\overline{r}(u,v)) \cdot (\overline{r}_u \times \overline{r}_v)\, du\, dv$

6 The Divergence Theorem

- This theorem computes the flux of a vector field through the boundary of a solid
- Theorem: The flux through E equals the flux through δE

$$\iint_{\delta E} (\overline{F} \cdot \overline{n})\, dS = \iiint_E \operatorname{div}(\overline{F})\, dV$$

- Where $\overline{n}$ should point outward from E
- If $\operatorname{div}(\overline{F}) = 0$, $\overline{F}$ is incompressible
- If $\operatorname{curl}(\overline{F}) = 0$, $\overline{F}$ is irrotational (no rotation)
- Symmetry $\Rightarrow$ Conservation Laws (Noether's theorem)

7 Stokes' Theorem

- $\displaystyle\int_{\delta M} \overline{F} = \int_M \delta\overline{F}$
- "Warped Green's Theorem"
- (Oriented) Surface M, $\overline{n}$ with boundary δM, with compatible orientation
  - Compatible orientation means that, if walking along the boundary, the surface would stay on the left

- $\overline{F}$ vector field
  - $\displaystyle\int_{\delta M} \overline{F}\, d\overline{r} = \iint_M (\text{curl}(\overline{F}) \cdot \overline{n})\, dS = \iint_M (Q_x - P_y)\, dA$
- Consequence: The flux of the curl($\overline{F}$) through a closed surface (*i.e.* no boundary) is always 0