Section 1

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- What is a vector?
  - A magnitude and a direction? (not all vectors in the real world can be added, so not entirely true)
  - For our course, vectors exist in vector spaces ($\mathbb{R}^2$, $\mathbb{R}^3$, ..., $\mathbb{R}^n$)
  - $\overline{v} = \langle v_1, v_2, \ldots, v_n\rangle$
  - $\mathbb{R}^1$ represents scalars, while $\mathbb{R}^2$, $\mathbb{R}^3$, ..., $\mathbb{R}^n$ are vectors
- Properties of Vectors
  - Can be added
    - $\overline{v} = \langle v_1, v_2, \ldots, v_n\rangle + \overline{w} = \langle w_1, w_2, \ldots, w_n\rangle = \langle v_1+w_1, v_2+w_2, \ldots, v_n+w_n\rangle$
    - If forming a parallelogram from the vectors, the diagonal is the sum, $\overline{v}+\overline{w}$, of two vectors
  - Can be scaled (scalar multiplication)
    - $2\overline{v} = \langle 2v_1, 2v_2, \ldots, 2v_n\rangle$
    - Magnitude is multiplied by the factor
  - Can find magnitude (length)
    - $|\overline{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
    - *Ex.* $\overline{v} = \langle 2, -3\rangle \Rightarrow |\overline{v}| = \sqrt{(2)^2 + (-3)^2)} = \sqrt{13}$
  - A vector divided by its own magnitude becomes a vector of magnitude 1 (unit vector)
    - $|\frac{\overline{v}}{|\overline{v}|}| = 1$
    - Unit vectors are dimensionless (no units)
    - A vector that is by itself of length 1 is not a unit vector
    - A unit vector is simply a direction (all unit vectors from a given point form a circle)

- Any non-zero vector is the product of its magnitude and its direction
  - $\overline{v} = |\overline{v}| \cdot \frac{\overline{v}}{|\overline{v}|}$

- Linear Combinations
  - $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s$
  - A linear combination of $\overline{v}_i$ is any sum of the form $r_1\overline{v}_1 + r_2\overline{v}_2 + \cdots + r_n\overline{v}_n$, where r_i are scalars

- Basis Vectors
  - $\mathbb{R}^n$ standard basis vectors: $\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_n \Rightarrow \begin{cases} \overline{e}_1 = \langle 1, 0, \ldots, 0\rangle \\ \overline{e}_2 = \langle 0, 1, \ldots, 0\rangle \\ \vdots \\ \overline{e}_n = \langle 0, 0, \ldots, 1\rangle \end{cases}$
  - Any vector is a linear combination of the standard basis vectors
  - $\overline{w} = \langle w_1, w_2, \ldots, w_n\rangle = w_1\overline{e}_1 + w_2\overline{e}_2 + \cdots + w_n\overline{e}_n$
  - *Ex.* $\overline{v} = \langle 2, -3\rangle = 2\langle 1, 0\rangle + -3\langle 0, 1\rangle$

- Dot Product
  - The dot product of two vectors is always a scalar
  - Geometric Definition: $\overline{v} \cdot \overline{w} = |\overline{v}||\overline{w}|\cos(\theta)$, where θ is the angle between $\overline{v}$ and $\overline{w}$
    - $\overline{v} \cdot \overline{w} = 0$ when $\theta = \frac{\pi}{2}$
    - $\overline{v} \cdot \overline{w} > 0$ when θ is acute
    - $\overline{v} \cdot \overline{w} < 0$ when θ is obtuse
  - Algebraic Definition: $\begin{cases} \overline{v} = \langle v_1, v_2, \ldots, v_n\rangle \\ \overline{w} = \langle w_1, w_2, \ldots, w_n\rangle \end{cases} \Rightarrow \overline{v} \cdot \overline{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$
    - *Ex.* $\begin{cases} \overline{v} = \langle 4, 9, 5\rangle \\ \overline{w} = \langle 4, 10, 3\rangle \end{cases} \Rightarrow \overline{v} \cdot \overline{w} = 4(4) + 9(10) + 5(3) = 121$
  - Together, the two definitions yield $\theta = \cos^{-1}\left(\frac{\overline{v} \cdot \overline{w}}{|\overline{v}||\overline{w}|}\right)$
    - *Ex.* Given $\overline{v}$ and $\overline{w}$ above, find the angle: $\cos^{-1}\left(\frac{121}{\sqrt{122}\sqrt{125}}\right) \approx .2$ rad
  - Vector Projection
    - Assuming $\overline{u}$ is a unit vector, the projection of $\overline{F}$ onto $\overline{u}$ can be found using: $\text{proj}_{\overline{u}}\overline{F} = \left(\overline{F} \cdot \overline{u}\right)\overline{u}$
    - In general, because $\overline{u} = \frac{\overline{v}}{|\overline{v}|}$, the formula becomes: $\text{proj}_{\overline{v}}\overline{F} = \left(\overline{F} \cdot \frac{\overline{v}}{|\overline{v}|}\right)\frac{\overline{v}}{|\overline{v}|} = \left(\frac{\overline{F} \cdot \overline{v}}{|\overline{v}|^2}\right)\overline{v}$

- Work
  - $\overline{F}$ is a constant vector, $\overline{d}$ represents the displacement — work is defined as $\overline{F} \cdot \overline{d}$
  - $W = \overline{F} \cdot \overline{d} = |\overline{F}||\overline{d}| \cos(\theta)$

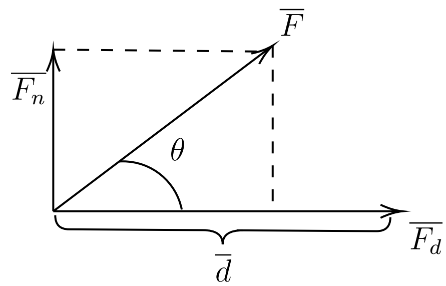

Figure 1: Diagram of Work

- Lines and Planes
  - *Ex.* Given a point in $\mathbb{R}^2$, $(x_o, y_o) = p$ and a vector $\overline{n} = \langle a, b \rangle$, find an equation of a line passing through p and $\perp$ to $\overline{n}$

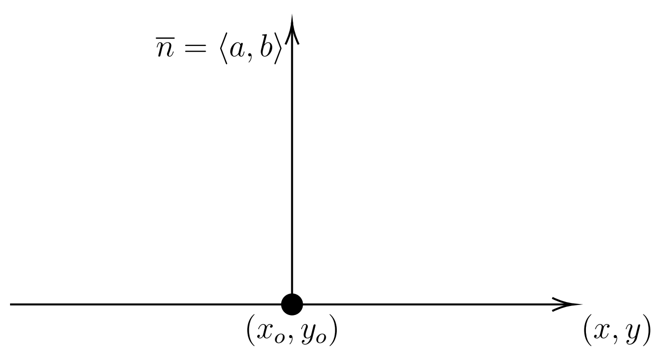

Figure 2: Finding an Equation for a Line

    * Create a vector: $\langle x - x_o, y - y_o \rangle$, then, by definition, dot product becomes: $\langle a, b \rangle \cdot \langle x - x_o, y - y_o \rangle = 0$, which yields $ax + by - ax_o - by_o = 0$, which can be simplified to $ax + by + c = 0$
  - In $\mathbb{R}^3$: $\langle a, b, c \rangle \cdot \langle x - x_o, y - y_o, z - z_o \rangle$ becomes $a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$ and then $ax + by + cz + d = 0$, this forms a plane through point p (in $\mathbb{R}^3$)
- Parametric Description of a Line

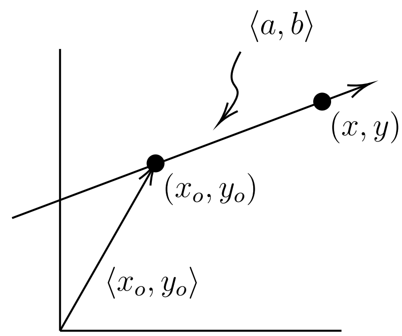

Figure 3: Parametrization

- $\begin{cases} x(t) = x_o + ta \\ y(t) = y_o + tb \end{cases}$
- In $\mathbb{R}^3$: $\begin{cases} x(t) = x_o + ta \\ y(t) = y_o + tb \\ z(t) = z_o + tc \end{cases}$
- The general equation can be summed up as $\overline{r}(t) = \overline{r}_o + t\overline{v}$, where $\overline{r}(t)$ generates a parametrized equation, $\overline{r}_o$ is a position vector, t is the parameter, and $\overline{v}$ is a vector parallel to a given equation
- To parametrize a line segment from point p to point q:
  $\overline{r}(t) = (1-t)\overline{p} + t\overline{q}$, $0 \leq t \leq 1$, or $0 \xrightarrow{t} 1$
  * *Ex.* Line segment from (2,1) to (3, -4):
    $\overline{r}(t) = (1-t)\langle 2, 1\rangle + t\langle 3, -4\rangle \Rightarrow \langle 2+t, 1-5t\rangle$