

Section 2

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1 Partial Derivatives

- The slope of $f(x, y)$ depends on the direction in the xy -plane
 - The slope in the x -direction is called the partial derivative of f with respect to x
 - The slope in the y -direction is called the partial derivative of f with respect to y
 - Notation: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ or f_x , f_y
 - For Second Derivatives: $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial y \partial x}$, $\frac{\partial^2 f}{\partial x \partial y}$ or f_{xx} , f_{xy} , f_{yx} , f_{yy}

If f , f_x , f_y , and f_{xy} are defined in a small disc around (x_o, y_o) and f_{yx} is continuous, then:

$$f_{xy} = f_{yx} \\ \text{in that disc}$$

- The gradient of f
 - Given $f(x_1, x_2, \dots, x_n)$, the gradient of f , $\bar{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$
 - It can be computed at a point: $\bar{\nabla} f(p) = \left\langle \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right\rangle$
 - $\bar{\nabla} f \approx f'(x_o) \Delta x$

2 Linear Approximation, Tangent Planes, and the Differential

- In calculus I, the linear approximation is given by: $f(x) \approx f(a) + f'(a)(x - a)$
- In calculus III, the approximation uses the gradient: $\Delta f \approx \bar{\nabla} f(p) \cdot \Delta \bar{x}$
- Ex. in \mathbb{R}^2 $z = f(x, y)$, $p = (a, b)$:

$$\Delta f \approx \bar{\nabla} f(a, b) \cdot \langle x - a, y - b \rangle \Rightarrow f(x, y) - f(a, b) \Rightarrow f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Thus:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We find the linearization of $f(x, y)$ near (a, b) (in \mathbb{R}^2)

- Linearization of $f(x, y)$ near (a, b) is denoted by $L_f(\bar{x}, \bar{p})$, where p is the vector $\langle a, b \rangle$

- *Ex. Given a cylinder of radius $r = .5$ and a height of $h = 1$, estimate the change in volume when the radius is increased by $.1$ and the height is decreased by $.1$*

$$V = \pi r^2 h \Rightarrow \Delta V \approx 2\pi(.5)(1)(r - .5) + \pi(.5)^2(h - 1) = \pi(r - .5) + .25\pi(h - 1) \approx$$

$$-.75\pi + \pi r + .25\pi h \Rightarrow -.75\pi + \pi(.6) + .25\pi(.9) = .075\pi$$

- The graph of $z = L_f(\bar{x}, \bar{p})$ is called the tangent set to f at p
- Differentials

$$- df = \bar{\nabla} f \cdot d\bar{x}$$

$$- df = f_x dx + f_y dy + f_z dz$$

$$- df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Relative Differentials

$$- \frac{df}{f}$$

$$- \text{Think of this as a stencil for relative error } \left(\frac{\Delta f}{f} \right)$$

3 Differentiation Rules

- Linearity of differentiation

$$1. \nabla(af \pm bg) = a\nabla f \pm b\nabla g$$

- Product rule

$$2. \nabla(fg)(p) = \nabla f(p)g(p) = \nabla g(p)f(p)$$

- Quotient rule

$$3. \nabla \left(\frac{f}{g} \right) \Big|_p = \frac{g(p)\nabla f(p) - f(p)\nabla g(p)}{g^2(p)}$$

- Power rule

$$4. \nabla f^\alpha(p) = \alpha f^{\alpha-1}(p)\nabla f(p)$$

- Chain rule

$$5. \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \cdots + \frac{\partial f}{\partial z} \frac{dz}{dt} \Rightarrow \nabla f \cdot \frac{d\bar{x}}{dt}$$

- *Ex.* A particle is moving through space. At $t = 2$ seconds, the particle is at $(3, 4, 7)$, and is moving with velocity $\langle -2, 1, 5 \rangle$ meters per second. Suppose that there is also an electric potential in space, given by $\phi(x, y, z) = xy - z^2$ volts. Find the instantaneous rate of change, with respect to time t , of the electric potential at the particle's position at $t = 2$ seconds.

$$\left. \frac{d\phi}{dt} \right|_{t=2} = \nabla \phi(\langle 3, 4, 7 \rangle) \cdot \left. \frac{d\bar{p}}{dt} \right|_{t=2} = \nabla \phi(\langle 3, 4, 7 \rangle) \cdot v(2) \Rightarrow$$

$$\nabla \phi = \langle y, x, -2z \rangle(\langle 3, 4, 7 \rangle) = \langle 4, 3, -14 \rangle \Rightarrow \langle 4, 3, -14 \rangle \cdot \langle -2, 1, 5 \rangle = -75 \text{ volts per second}$$

4 The Directional Derivative

- Directions live in the input of the function
- Same is true for ∇f
- What is rate of change of $z = f(x, y)$ in the direction \bar{u} ?
 - Fix point p in xy -plane
 - Choose direction (\bar{u} -direction) in xy -plane
 - $D_{\bar{u}}f(p) = \nabla f(p) \cdot \bar{u}$
 - * This is the derivative of f at p in the direction of \bar{u}
- Alternative: Fix $f(x, y)$ and point p , but \bar{u} varies
- What is \bar{u} in which $f(x, y)$ changes in the fastest possible way?
 - When the angle between the direction and ∇f is $\theta = 0$ (because $\cos(\theta)$) is greatest at this angle
 - * Smallest occurs in opposite directions (when $\theta = \pi$) because $\cos(\theta)$ is the largest possible negative
 - * Equals zero when $\bar{u} \perp \nabla f(p)$ (*i.e.* $\theta = \frac{\pi}{2}$)
 - Thus, $D_{\bar{u}}f(p)$ is largest possible in direction of $\nabla f(p)$
 - The largest rate is the magnitude of the gradient vector ($|\nabla f(p)|$)
 - * The largest rate is the negative magnitude, $-|\nabla f(p)|$
 - This is all assuming that $\nabla f(p) \neq 0$

5 Level Sets and Gradient Vectors

- Given $z = f(x, y)$
 - Pick a constant, c
 - Plug it in for z , such that $f(x, y) = c$
 - The level curve of $f(x, y)$ corresponding to c is formed
- *Ex.* $z = x^2 + y^2$
 - At $c = -1$, there is nothing to draw because there are no real solutions
 - At $c = 0$, there is one point at the origin $(0, 0)$
 - At $c = 1$, we obtain a circle of radius one
 - At $c = 4$, we obtain a circle of radius two
- The level curves corresponding to different values of c can not intersect
- The gradient vector $\nabla f(p)$ is always perpendicular (\perp) to the level curve passing through p
- Implicit functions do not expressly define one variable in terms of different variables (*Ex.* $x^2 + y^2 = 1$), while explicit functions do (*Ex.* $z = \cos(y) + e^x$)
- To find the tangent line function, use the formula $\nabla f(p) \cdot \langle X - p \rangle = 0$

6 Parametrizing Surfaces

- A line in \mathbb{R}^3 : $\begin{cases} x = x_o + at \\ y = y_o + bt \\ z = z_o + ct \end{cases}$
- A circle in \mathbb{R}^2 : $\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases}$, where r is the radius
- Parametrizing a surface
 - Parametrization should be differentiable
 - Using u and v creates $\bar{r}(u, v)$, so that $\bar{r}_u \times \bar{r}_v \neq 0$
 - \bar{r}_u should not be parallel to \bar{r}_v
 - A parametrization satisfying this property is said to be regular
 - *Ex.* Parametrize the cone with equation $z^2 = x^2 + y^2$: $\begin{cases} x = u \cos(v) \\ y = u \sin(v) \\ z = u \end{cases}$
- The equation of a tangent plane $a\bar{r}_u(u_o, v_o) + b\bar{r}_v(u_o, v_o)$

7 Local Extrema

- In Calc I, critical points occurred where $f'(x) = 0$ or undefined with the first derivative test, or depending on concavity for the second derivative test
- The second derivative test was as follows:
 - If $f''(p) > 0$, p is a local min
 - If $f''(p) < 0$, p is a local max
 - If $f''(p) = 0$, p is neither
- In Calc III, critical points occur where $\nabla f = 0$ or undefined, but the first derivative test is not practical
- The second derivative test is now done as:
 - Hessian Matrix: $\begin{bmatrix} f_{xx}(p) & f_{yx}(p) \\ f_{xy}(p) & f_{yy}(p) \end{bmatrix}$, then find the determinant:
 - $D = f_{xx}(p)f_{yy}(p) - f_{xy}^2(p)$
 - A critical point p is non-degenerate if $D \neq 0$
 - Assuming p is non-degenerate:
 - * If $D > 0$ and $f_{xx}(p) > 0$, then f has a local minimum at p
 - * If $D > 0$ and $f_{xx}(p) < 0$, then f has a local maximum at p
 - * If $D < 0$, then p is a saddle point of f
- Saddle points occur where there are critical points, but a local extreme value is not contained