

# Section 2

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# 1 Partial Derivatives

- The slope of  $f(x, y)$  depends on the direction in the  $xy$ -plane
  - The slope in the  $x$ -direction is called the partial derivative of  $f$  with respect to  $x$
  - The slope in the  $y$ -direction is called the partial derivative of  $f$  with respect to  $y$
  - Notation:  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$  or  $f_x$ ,  $f_y$
  - For Second Derivatives:  $\frac{\partial^2 f}{\partial x^2}$ ,  $\frac{\partial^2 f}{\partial y^2}$ ,  $\frac{\partial^2 f}{\partial y \partial x}$ ,  $\frac{\partial^2 f}{\partial x \partial y}$  or  $f_{xx}$ ,  $f_{xy}$ ,  $f_{yx}$ ,  $f_{yy}$

If  $f$ ,  $f_x$ ,  $f_y$ , and  $f_{xy}$  are defined in a small disc around  $(x_o, y_o)$  and  $f_{yx}$  is continuous, then:

$$f_{xy} = f_{yx}$$

in that disc

- The gradient of  $f$ 
  - Given  $f(x_1, x_2, \dots, x_n)$ , the gradient of  $f$ ,  $\bar{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$
  - It can be computed at a point:  $\bar{\nabla} f(p) = \left\langle \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \dots, \frac{\partial f}{\partial x_n}(p) \right\rangle$
  - $\bar{\nabla} f \approx f'(x_o) \Delta x$

# 2 Linear Approximation, Tangent Planes, and the Differential

- In calculus I, the linear approximation is given by:  $f(x) \approx f(a) + f'(a)(x - a)$
- In calculus III, the approximation uses the gradient:  $\Delta f \approx \bar{\nabla} f(p) \cdot \Delta \bar{x}$
- Ex. in  $\mathbb{R}^2$   $z = f(x, y)$ ,  $p = (a, b)$ :

$$\Delta f \approx \bar{\nabla} f(a, b) \cdot \langle x - a, y - b \rangle \Rightarrow f(x, y) - f(a, b) \Rightarrow f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Thus:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We find the linearization of  $f(x, y)$  near  $(a, b)$  (in  $\mathbb{R}^2$ )

- Linearization of  $f(x, y)$  near  $(a, b)$  is denoted by  $L_f(\bar{x}, \bar{p})$ , where  $p$  is the vector  $\langle a, b \rangle$

- *Ex. Given a cylinder of radius  $r = .5$  and a height of  $h = 1$ , estimate the change in volume when the radius is increased by  $.1$  and the height is decreased by  $.1$*

$$V = \pi r^2 h \Rightarrow \Delta V \approx 2\pi(.5)(1)(r - .5) + \pi(.5)^2(h - 1) = \pi(r - .5) + .25\pi(h - 1) \approx$$

$$-.75\pi + \pi r + .25\pi h \Rightarrow -.75\pi + \pi(.6) + .25\pi(.9) = .075\pi$$

- The graph of  $z = L_f(\bar{x}, \bar{p})$  is called the tangent set to  $f$  at  $p$
- Differentials

$$- df = \bar{\nabla} f \cdot d\bar{x}$$

$$- df = f_x dx + f_y dy + f_z dz$$

$$- df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

- Relative Differentials

$$- \frac{df}{f}$$

$$- \text{Think of this as a stencil for relative error } \left( \frac{\Delta f}{f} \right)$$

### 3 Differentiation Rules

- Linearity of differentiation

$$1. \nabla(af \pm bg) = a\nabla f \pm b\nabla g$$

- Product rule

$$2. \nabla(fg)(p) = \nabla f(p)g(p) = \nabla g(p)f(p)$$

- Quotient rule

$$3. \nabla \left( \frac{f}{g} \right) \Big|_p = \frac{g(p)\nabla f(p) - f(p)\nabla g(p)}{g^2(p)}$$

- Power rule

$$4. \nabla f^\alpha(p) = \alpha f^{\alpha-1}(p)\nabla f(p)$$

- Chain rule

$$5. \frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \cdots + \frac{\partial f}{\partial z} \frac{dz}{dt} \Rightarrow \nabla f \cdot \frac{d\bar{x}}{dt}$$

- *Ex.* A particle is moving through space. At  $t = 2$  seconds, the particle is at  $(3, 4, 7)$ , and is moving with velocity  $\langle -2, 1, 5 \rangle$  meters per second. Suppose that there is also an electric potential in space, given by  $\phi(x, y, z) = xy - z^2$  volts. Find the instantaneous rate of change, with respect to time  $t$ , of the electric potential at the particle's position at  $t = 2$  seconds.

$$\left. \frac{d\phi}{dt} \right|_{t=2} = \nabla \phi(\langle 3, 4, 7 \rangle) \cdot \left. \frac{d\bar{p}}{dt} \right|_{t=2} = \nabla \phi(\langle 3, 4, 7 \rangle) \cdot v(2) \Rightarrow$$

$$\nabla \phi = \langle y, x, -2z \rangle(\langle 3, 4, 7 \rangle) = \langle 4, 3, -14 \rangle \Rightarrow \langle 4, 3, -14 \rangle \cdot \langle -2, 1, 5 \rangle = -75 \text{ volts per second}$$

## 4 The Directional Derivative

- Directions live in the input of the function
- Same is true for  $\nabla f$
- What is rate of change of  $z = f(x, y)$  in the direction  $\bar{u}$ ?
  - Fix point  $p$  in  $xy$ -plane
  - Choose direction ( $\bar{u}$ -direction) in  $xy$ -plane
  - $D_{\bar{u}}f(p) = \nabla f(p) \cdot \bar{u}$ 
    - \* This is the derivative of  $f$  at  $p$  in the direction of  $\bar{u}$
- Alternative: Fix  $f(x, y)$  and point  $p$ , but  $\bar{u}$  varies
- What is  $\bar{u}$  in which  $f(x, y)$  changes in the fastest possible way?
  - When the angle between the direction and  $\nabla f$  is  $\theta = 0$  (because  $\cos(\theta)$ ) is greatest at this angle
    - \* Smallest occurs in opposite directions (when  $\theta = \pi$ ) because  $\cos(\theta)$  is the largest possible negative
    - \* Equals zero when  $\bar{u} \perp \nabla f(p)$  (*i.e.*  $\theta = \frac{\pi}{2}$ )
  - Thus,  $D_{\bar{u}}f(p)$  is largest possible in direction of  $\nabla f(p)$
  - The largest rate is the magnitude of the gradient vector ( $|\nabla f(p)|$ )
    - \* The largest rate is the negative magnitude,  $-|\nabla f(p)|$
  - This is all assuming that  $\nabla f(p) \neq 0$

## 5 Level Sets and Gradient Vectors

- Given  $z = f(x, y)$ 
  - Pick a constant,  $c$
  - Plug it in for  $z$ , such that  $f(x, y) = c$
  - The level curve of  $f(x, y)$  corresponding to  $c$  is formed
- *Ex.*  $z = x^2 + y^2$ 
  - At  $c = -1$ , there is nothing to draw because there are no real solutions
  - At  $c = 0$ , there is one point at the origin  $(0, 0)$
  - At  $c = 1$ , we obtain a circle of radius one
  - At  $c = 4$ , we obtain a circle of radius two
- The level curves corresponding to different values of  $c$  can not intersect
- The gradient vector  $\nabla f(p)$  is always perpendicular ( $\perp$ ) to the level curve passing through  $p$
- Implicit functions do not expressly define one variable in terms of different variables (*Ex.*  $x^2 + y^2 = 1$ ), while explicit functions do (*Ex.*  $z = \cos(y) + e^x$ )
- To find the tangent line function, use the formula  $\nabla f(p) \cdot \langle X - p \rangle = 0$

## 6 Parametrizing Surfaces

- A line in  $\mathbb{R}^3$ :  $\begin{cases} x = x_o + at \\ y = y_o + bt \\ z = z_o + ct \end{cases}$
- A circle in  $\mathbb{R}^2$ :  $\begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases}$ , where  $r$  is the radius
- Parametrizing a surface
  - Parametrization should be differentiable
  - Using  $u$  and  $v$  creates  $\bar{r}(u, v)$ , so that  $\bar{r}_u \times \bar{r}_v \neq 0$
  - $\bar{r}_u$  should not be parallel to  $\bar{r}_v$
  - A parametrization satisfying this property is said to be regular
  - *Ex.* Parametrize the cone with equation  $z^2 = x^2 + y^2$ :  $\begin{cases} x = u \cos(v) \\ y = u \sin(v) \\ z = u \end{cases}$
- The equation of a tangent plane  $a\bar{r}_u(u_o, v_o) + b\bar{r}_v(u_o, v_o)$

## 7 Local Extrema

- In Calc I, critical points occurred where  $f'(x) = 0$  or undefined with the first derivative test, or depending on concavity for the second derivative test
- The second derivative test was as follows:
  - If  $f''(p) > 0$ ,  $p$  is a local min
  - If  $f''(p) < 0$ ,  $p$  is a local max
  - If  $f''(p) = 0$ ,  $p$  is neither
- In Calc III, critical points occur where  $\nabla f = 0$  or undefined, but the first derivative test is not practical
- The second derivative test is now done as:
  - Hessian Matrix:  $\begin{bmatrix} f_{xx}(p) & f_{yx}(p) \\ f_{xy}(p) & f_{yy}(p) \end{bmatrix}$ , then find the determinant:
  - $D = f_{xx}(p)f_{yy}(p) - f_{xy}^2(p)$
  - A critical point  $p$  is non-degenerate if  $D \neq 0$
  - Assuming  $p$  is non-degenerate:
    - \* If  $D > 0$  and  $f_{xx}(p) > 0$ , then  $f$  has a local minimum at  $p$
    - \* If  $D > 0$  and  $f_{xx}(p) < 0$ , then  $f$  has a local maximum at  $p$
    - \* If  $D < 0$ , then  $p$  is a saddle point of  $f$
- Saddle points occur where there are critical points, but a local extreme value is not contained

## 8 Optimization

- Optimization involves finding the largest or smallest value of a function
  - Can be constrained or unconstrained
  - Unconstrained assumes there are no limits on the inputs of a function
  - Constrained contains limits on inputs
- The Unconstrained Case:
  1. Find critical points in the interior of the domain, and find values of the function at those points
  2. Work out the boundaries