

Section 1

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- What is a vector?
 - A magnitude and a direction? (not all vectors in the real world can be added, so not entirely true)
 - For our course, vectors exist in vector spaces ($\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$)
 - $\bar{v} = \langle v_1, v_2, \dots, v_n \rangle$
 - \mathbb{R}^1 represents scalars, while $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$ are vectors
- Properties of Vectors
 - Can be added
 - * $\bar{v} = \langle v_1, v_2, \dots, v_n \rangle + \bar{w} = \langle w_1, w_2, \dots, w_n \rangle = \langle v_1 + w_1, v_2 + w_2, \dots, v_n + w_n \rangle$
 - * If forming a parallelogram from the vectors, the diagonal is the sum, $\bar{v} + \bar{w}$, of two vectors
 - Can be scaled (scalar multiplication)
 - * $2\bar{v} = \langle 2v_1, 2v_2, \dots, 2v_n \rangle$
 - * Magnitude is multiplied by the factor
 - Can find magnitude (length)
 - * $|\bar{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$
 - * *Ex.* $\bar{v} = \langle 2, -3 \rangle \Rightarrow |\bar{v}| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$
 - A vector divided by its own magnitude becomes a vector of magnitude 1 (unit vector)
 - * $|\frac{\bar{v}}{|\bar{v}|}| = 1$
 - * Unit vectors are dimensionless (no units)
 - * A vector that is by itself of length 1 is not a unit vector
 - * A unit vector is simply a direction (all unit vectors from a given point form a circle)

- Any non-zero vector is the product of its magnitude and its direction

$$* \bar{v} = |\bar{v}| \cdot \frac{\bar{v}}{|\bar{v}|}$$

- Linear Combinations

- $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_s$

- A linear combination of \bar{v}_i is any sum of the form $r_1\bar{v}_1 + r_2\bar{v}_2 + \dots + r_n\bar{v}_n$, where r_i are scalars

- Basis Vectors

$$- \mathbb{R}^n \text{ standard basis vectors: } \bar{e}_1, \bar{e}_2, \dots, \bar{e}_n \Rightarrow \begin{cases} \bar{e}_1 = \langle 1, 0, \dots, 0 \rangle \\ \bar{e}_2 = \langle 0, 1, \dots, 0 \rangle \\ \vdots \\ \bar{e}_n = \langle 0, 0, \dots, 1 \rangle \end{cases}$$

- Any vector is a linear combination of the standard basis vectors

$$- \bar{w} = \langle w_1, w_2, \dots, w_n \rangle = w_1\bar{e}_1 + w_2\bar{e}_2 + \dots + w_n\bar{e}_n$$

$$- \text{Ex. } \bar{v} = \langle 2, -3 \rangle = 2\langle 1, 0 \rangle + -3\langle 0, 1 \rangle$$

- Dot Product

- The dot product of two vectors is always a scalar

- Geometric Definition: $\bar{v} \cdot \bar{w} = |\bar{v}||\bar{w}|\cos(\theta)$, where θ is the angle between \bar{v} and \bar{w}

$$* \bar{v} \cdot \bar{w} = 0 \text{ when } \theta = \frac{\pi}{2}$$

$$* \bar{v} \cdot \bar{w} > 0 \text{ when } \theta \text{ is acute}$$

$$* \bar{v} \cdot \bar{w} < 0 \text{ when } \theta \text{ is obtuse}$$

$$- \text{Algebraic Definition: } \begin{cases} \bar{v} = \langle v_1, v_2, \dots, v_n \rangle \\ \bar{w} = \langle w_1, w_2, \dots, w_n \rangle \end{cases} \Rightarrow \bar{v} \cdot \bar{w} = v_1w_1 + v_2w_2 + \dots + v_nw_n$$

$$* \text{Ex. } \begin{cases} \bar{v} = \langle 4, 9, 5 \rangle \\ \bar{w} = \langle 4, 10, 3 \rangle \end{cases} \Rightarrow \bar{v} \cdot \bar{w} = 4(4) + 9(10) + 5(3) = 121$$

- Together, the two definitions yield $\theta = \cos^{-1} \left(\frac{\bar{v} \cdot \bar{w}}{|\bar{v}||\bar{w}|} \right)$

$$* \text{Ex. Given } \bar{v} \text{ and } \bar{w} \text{ above, find the angle: } \cos^{-1} \left(\frac{121}{\sqrt{122}\sqrt{125}} \right) \approx .2 \text{ rad}$$

- Vector Projection

$$* \text{ Assuming } \bar{u} \text{ is a unit vector, the projection of } \bar{F} \text{ onto } \bar{u} \text{ can be found using: } \text{proj}_{\bar{u}} \bar{F} = (\bar{F} \cdot \bar{u}) \bar{u}$$

$$* \text{ In general, because } \bar{u} = \frac{\bar{v}}{|\bar{v}|}, \text{ the formula becomes: } \text{proj}_{\bar{v}} \bar{F} = \left(\bar{F} \cdot \frac{\bar{v}}{|\bar{v}|} \right) \frac{\bar{v}}{|\bar{v}|} = \left(\frac{\bar{F} \cdot \bar{v}}{|\bar{v}|^2} \right) \bar{v}$$

- Work

- \vec{F} is a constant vector, \vec{d} represents the displacement — work is defined as $\vec{F} \cdot \vec{d}$
- $W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}|\cos(\theta)$

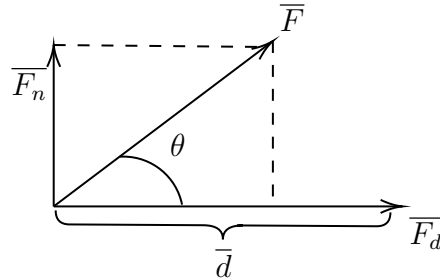


Figure 1: Diagram of Work

- Lines and Planes

- *Ex.* Given a point in \mathbb{R}^2 , $(x_o, y_o) = p$ and a vector $\vec{n} = \langle a, b \rangle$, find an equation of a line passing through p and \perp to \vec{n}

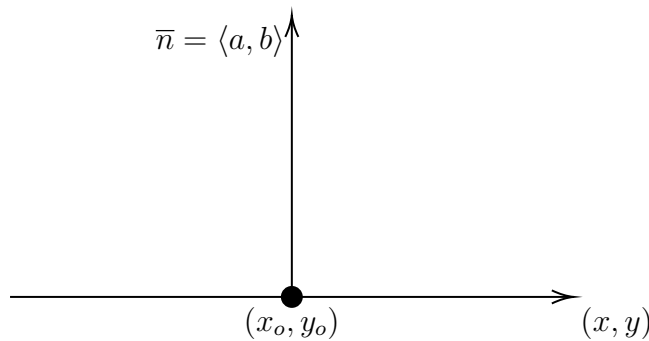


Figure 2: Finding an Equation for a Line

- * Create a vector: $\langle x - x_o, y - y_o \rangle$, then, by definition, dot product becomes:
 $\langle a, b \rangle \cdot \langle x - x_o, y - y_o \rangle = 0$, which yields $ax + by - ax_o - by_o = 0$, which can be simplified to $ax + by + c = 0$
- In \mathbb{R}^3 : $\langle a, b, c \rangle \cdot \langle x - x_o, y - y_o, z - z_o \rangle$ becomes $a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$ and then $ax + by + cz + d = 0$, this forms a plane through point p (in \mathbb{R}^3)

- Parametric Description of a Line

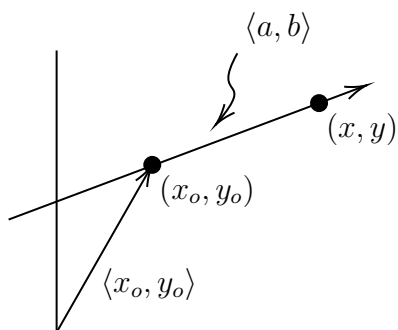


Figure 3: Parametrization

$$- \begin{cases} x = x_o + ta \\ y = y_o + tb \end{cases}$$