# Section 4

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November 10, 2022

# Contents

# 1 Vector Fields

- A vector field in $\mathbb{R}^n$ is an assignment that goes from $\mathbb{R}^n \to \mathbb{R}^n$, where the former is viewed as a bunch of points, and the latter is a bunch of vectors
  - Two examples: a force field (*e.g.* gravity, electrostatic, magnetic) or a velocity field (*e.g.* fluid mechanics)
- *Ex.* A gravitational field, with two masses, one fixed ($M$), and one floating ($m$)
  - The pull = magnitude $\cdot$ direction $\to G\frac{Mm}{|\overline{d}|^2} \cdot \frac{\overline{d}}{|\overline{d}|} = G\frac{Mm}{|\overline{d}|^3}\overline{d}$
- *Ex.* Gradient vector fields
  - $f(\overline{x}) \to \nabla f = \left\langle \frac{\delta f}{\delta x_1}, \frac{\delta f}{\delta x_2}, \cdots, \frac{\delta f}{\delta x_n} \right\rangle$
  - Not every vector field can be realized as a gradient vector field of some function
  - The 'del' operator is as follows: $\nabla := \left\langle \frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}, \cdots, \frac{\delta}{\delta x_n} \right\rangle$
  - Apply $\nabla$ to a function $f$ to obtain a gradient vector field (use dot product)
  - $\nabla \cdot F = \frac{\delta F_1}{\delta x_1} + \frac{\delta F_2}{\delta x_2} + \cdots + \frac{\delta F_n}{\delta x_n} = \text{div}(F)$
    - This is the divergence of $F$
  - $\nabla \times F$ describes the curl of $F$ (how the vector field curls in three dimensions)

Using the $\nabla$ operator:

| input | output | significance |
|---|---|---|
| function $f$ | $\nabla f$ | gradient of $f$ (a vector field) |
| vector field of $f$ | $\nabla \cdot f$ | divergence of $f$ (a function) |
| $n = 3$ vector field of $f$ | $\nabla \times f$ | curl of $f$ (a vector field) |

- To view 2D vector fields as 3D vector fields, convert $F = \langle P(x,y), Q(x,y) \rangle \to \langle P(x,y), Q(x,y), 0 \rangle$
- The curl of a two dimensional vector field, converted to three dimensions, is $\text{curl}(F) = Q_x - P_y$

# 2 Line Integrals

- Can find Work, $W$, done by a force (a.k.a. a vector field) on an object along an oriented curve, $C$

- $W = \int_C \overline{F} \cdot dr$ — This is the line integral of $\overline{F}$ along the oriented curve $C$
- To compute the line integral, parametrize $C$, and reduce it to a Calculus II integral
  - $d\overline{r} = \overline{r}'(t)\,dt$
  - $\int_a^b \overline{F}(\overline{r}(t)) \cdot \overline{r}'(t)\,dt$
- Conservative vector fields are independent of path (taking any path from one point to another will always yield the same value)