Section 3

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Contents

1 Partial Antiderivatives

- *Ex.* $\dfrac{\delta f}{\delta x} = 3x^2 - 5y^2 \Rightarrow f = x^3 - 5y^2x + g(y)$

$$\int_y^{y^2} 3x^2 - 5y^2\,dx \Rightarrow y^6 - 5y^4 - (y^3 - 5y^3) = y^6 - 5y^4 + 4y^3$$

$$\int_0^2 \int_y^{y^2} 3x^2 - 5y^2\,dx\,dy \Rightarrow y^6 - 5y^4 - (y^3 - 5y^3) = \int_0^2 (y^6 - 5y^4 + 4y^3)\,dy =$$

$$\frac{1}{7}(2)^7 - (2)^5 + (2)^4 = \frac{128}{7} - 16 = \frac{16}{7}$$

- *Ex.* $\displaystyle\int_1^3 \int_0^{\sin(x)} \frac{1+2y}{\sin(x)}\,dy\,dx =$

$$\frac{1}{\sin(x)}\left(\sin(x) + (\sin(x))^2\right) = \int_1^3 1 + \sin(x)\,dx = x - \cos(x) =$$

$$3 - \cos(3) - 1 + \cos(1) = 2 + \cos(1) - \cos(3)$$

- *Ex.* $\displaystyle\int_0^2 \int_y^1 \int_z^{yz} 8xyz\,dx\,dz\,dy =$

$$yz((4(yz)^2 - 4z^2) = \int_0^2 \int_y^1 4y^3z^3 - 4yz^3\,dz\,dy = (y^3 - y) - (y^7 - y^4) =$$

$$\int_0^2 -y^7 + y^4 + y^3 - y\,dy = -\frac{1}{8}(2)^7 + \frac{1}{5}(2)^5 + \frac{1}{4}(2)^4 - \frac{1}{2}(2)^2 = -16 + \frac{32}{5} + 4 - 2 = -\frac{38}{5}$$

- *Ex.* $\dfrac{\delta f}{\delta x} = 3x^2 - 5y^2,\ \dfrac{\delta f}{\delta y} = -10xy + 8y^3,\ f = ?$

$$\int \frac{\delta f}{\delta x}\,dx = x^3 - 5xy^2 + g(y) = \frac{\delta f}{\delta y} = -10xy + g'(y) \Rightarrow g'(y) = 8y^3 \Rightarrow f(x,y) = x^3 - 5xy^2 + 2y^4 + c$$

2 Integration in $\mathbb{R}^2$

- Double Integral
  - $\displaystyle\iint_R f(x,y)\,dA$
- Fubini's Theorem: Utilize iterated integration to calculate multiple-integration

- $\iint_R f(x,y)\,dA \longrightarrow \int_a^b \int_c^d f(x,y)\,dy\,dx$

- For Type I Regions:
  - $\int_a^b \int_{p(x)}^{q(x)} f(x,y)\,dy\,dx$
  - Occurs when y is bounded by functions of x and x is bounded by vertical lines $(x = c)$
- For Type II Regions:
  - $\int_c^d \int_{r(x)}^{s(x)} f(x,y)\,dx\,dy$
  - Occurs when x is bounded by functions of y and y is bounded by horizontal lines $(y = c)$
- A function can be Type I, Type II, both, or neither
- Regions can be broken down into parts to make calculations easier:
  - Given R and two subregions, R' and R'', the integral becomes:
    * $\iint_R f\,dA = \iint_{R'} f\,dA + \iint_{R''} f\,dA$
- Remark: If $f(x,y) = 1$, then:
  - $\iint_R dA =$ Area of R

3 Integration with Polar Coordinates

- Basic Properties $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ x^2 + y^2 = r^2 \end{cases}$
- $dA = r\,dr\,d\theta$
- $\iint_R f(x,y)\,dA = \iint_R f(r\cos(\theta), r\sin(\theta))r\,dr\,d\theta$

4 Integration in $\mathbb{R}^3$

- $\iiint_S f(x, y, z)\ dV$, where $S \to \begin{cases} p(x,y) \leq z \leq q(x,y) \\ u(x) \leq y \leq v(x) \\ a \leq x \leq b \end{cases}$
- This can be expressed as: $\int_a^b \int_{u(x)}^{v(x)} \int_{p(x,y)}^{q(x,y)} f(x, y, z)\ dz\, dy\, dx$

5 Volume

- $\iiint_S f(x, y, z)\, dV \Rightarrow f(x, y, z) = 1 \Rightarrow \iiint_S dV$ gives the volumes of solid S

6 Cylindrical and Spherical Coordinates

- For cylindrical coordinates, refer to polar:
  - $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases}$, and $dV = r\, dr\, d\theta\, dz$
  - $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, and $-\infty < z < \infty$
- Spherical coordinates use variables ρ, ϕ, and θ, where ρ is the distance to the origin, ϕ is the vertical deviation (with respect to the positive z-axis), and θ is the horizontal deviation (or polar angle)
  - $\begin{cases} x = \rho\sin(\phi)\cos(\theta) \\ y = \rho\sin(\phi)\sin(\theta) \\ z = \rho\cos(\phi) \end{cases}$, and $dV = \rho^2\sin(\phi)\, d\rho\, d\phi\, d\theta$
  - It is important to remember the property $x^2 + y^2 + z^2 = \rho^2$
  - $0 \leq \rho < \infty$, $0 \leq \theta \leq 2\pi$, and $0 \leq \phi < \pi$

7 Density and Mass

- Mass $= \begin{cases} \int_a^b \delta(x)\, dx \\ \iint_R \delta(x, y)\, dA \\ \iiint_S \delta(x, y, z)\, dV \end{cases}$

8 Surfaces and Area

- Break the area into sub-rectangles that are parametrized. The area of one sub-rectangle (in $\mathbb{R}^2$)/parallelogram (in $\mathbb{R}^3$) is $|\overline{r}_u \times \overline{r}_v|\, du\, dv$
- Summing all of the sub-rectangles together, we get: Area $= \displaystyle\iint_D |\overline{r}_u \times \overline{r}_v|\, du\, dv$
- To parametrize $f(x, y)$, simply choose: $\left\{\begin{array}{c} \left.\begin{array}{c} x = x \\ y = y \end{array}\right\}\text{parameters} \\ z = f(x, y) \end{array}\right.$
  - This would make $\left\{\begin{array}{l} \overline{r}_x = \langle 1, 0, f_x \rangle \\ \overline{r}_y = \langle 0, 1, f_y \rangle \end{array}\right.$
  - $|\overline{r}_x \times \overline{r}_y| = |\langle -f_x, -f_y, 1 \rangle| = \displaystyle\iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1\, dx\, dy}$
- If $S = S_1 + S_2 + \cdots + S_n$ is smooth, then Area(S) = Area(S_1) + Area(S_2) $+ \cdots +$ Area(S_n)