

# Section 1

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# 1 $\mathbb{R}^n$ as a Vector Space

- What is a vector?
  - A magnitude and a direction? (not all vectors in the real world can be added, so not entirely true)
  - For our course, vectors exist in vector spaces ( $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$ )
  - $\bar{v} = \langle v_1, v_2, \dots, v_n \rangle$
  - $\mathbb{R}^1$  represents scalars, while  $\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$  are vectors
- Properties of Vectors
  - Can be added
    - \*  $\bar{v} = \langle v_1, v_2, \dots, v_n \rangle + \bar{w} = \langle w_1, w_2, \dots, w_n \rangle = \langle v_1 + w_1, v_2 + w_2, \dots, v_n + w_n \rangle$
    - \* If forming a parallelogram from the vectors, the diagonal is the sum,  $\bar{v} + \bar{w}$ , of two vectors
  - Can be scaled (scalar multiplication)
    - \*  $2\bar{v} = \langle 2v_1, 2v_2, \dots, 2v_n \rangle$
    - \* Magnitude is multiplied by the factor
  - Can find magnitude (length)
    - \*  $|\bar{v}| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$
    - \* *Ex.*  $\bar{v} = \langle 2, -3 \rangle \Rightarrow |\bar{v}| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$
  - A vector divided by its own magnitude becomes a vector of magnitude 1 (unit vector)
    - \*  $|\frac{\bar{v}}{|\bar{v}|}| = 1$
    - \* Unit vectors are dimensionless (no units)
    - \* A vector that is by itself of length 1 is not a unit vector
    - \* A unit vector is simply a direction (all unit vectors from a given point form a circle)
  - Any non-zero vector is the product of its magnitude and its direction
    - \*  $\bar{v} = |\bar{v}| \cdot \frac{\bar{v}}{|\bar{v}|}$
- Linear Combinations
  - $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_s$
  - A linear combination of  $\bar{v}_i$  is any sum of the form  $r_1\bar{v}_1 + r_2\bar{v}_2 + \dots + r_n\bar{v}_n$ , where  $r_i$  are scalars
- Basis Vectors

- $\mathbb{R}^n$  standard basis vectors:  $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n \Rightarrow \begin{cases} \bar{e}_1 = \langle 1, 0, \dots, 0 \rangle \\ \bar{e}_2 = \langle 0, 1, \dots, 0 \rangle \\ \vdots \\ \bar{e}_n = \langle 0, 0, \dots, 1 \rangle \end{cases}$
- Any vector is a linear combination of the standard basis vectors
- $\bar{w} = \langle w_1, w_2, \dots, w_n \rangle = w_1\bar{e}_1 + w_2\bar{e}_2 + \dots + w_n\bar{e}_n$
- *Ex.*  $\bar{v} = \langle 2, -3 \rangle = 2\langle 1, 0 \rangle + -3\langle 0, 1 \rangle$

- Dot Product

- The dot product of two vectors is always a scalar
- Geometric Definition:  $\bar{v} \cdot \bar{w} = |\bar{v}||\bar{w}| \cos(\theta)$ , where  $\theta$  is the angle between  $\bar{v}$  and  $\bar{w}$ 
  - \*  $\bar{v} \cdot \bar{w} = 0$  when  $\theta = \frac{\pi}{2}$
  - \*  $\bar{v} \cdot \bar{w} > 0$  when  $\theta$  is acute
  - \*  $\bar{v} \cdot \bar{w} < 0$  when  $\theta$  is obtuse
- Algebraic Definition:  $\begin{cases} \bar{v} = \langle v_1, v_2, \dots, v_n \rangle \\ \bar{w} = \langle w_1, w_2, \dots, w_n \rangle \end{cases} \Rightarrow \bar{v} \cdot \bar{w} = v_1w_1 + v_2w_2 + \dots + v_nw_n$ 
  - \* *Ex.*  $\begin{cases} \bar{v} = \langle 4, 9, 5 \rangle \\ \bar{w} = \langle 4, 10, 3 \rangle \end{cases} \Rightarrow \bar{v} \cdot \bar{w} = 4(4) + 9(10) + 5(3) = 121$
- Together, the two definitions yield  $\theta = \cos^{-1} \left( \frac{\bar{v} \cdot \bar{w}}{|\bar{v}||\bar{w}|} \right)$ 
  - \* *Ex.* Given  $\bar{v}$  and  $\bar{w}$  above, find the angle:  $\cos^{-1} \left( \frac{121}{\sqrt{122}\sqrt{125}} \right) \approx .2 \text{ rad}$
- Vector Projection
  - \* Assuming  $\bar{u}$  is a unit vector, the projection of  $\bar{F}$  onto  $\bar{u}$  can be found using:  
 $\text{proj}_{\bar{u}} \bar{F} = (\bar{F} \cdot \bar{u}) \bar{u}$
  - \* In general, because  $\bar{u} = \frac{\bar{v}}{|\bar{v}|}$ , the formula becomes:  $\text{proj}_{\bar{v}} \bar{F} = \left( \bar{F} \cdot \frac{\bar{v}}{|\bar{v}|} \right) \frac{\bar{v}}{|\bar{v}|} = \left( \frac{\bar{F} \cdot \bar{v}}{|\bar{v}|^2} \right) \bar{v}$

- Work

- $\vec{F}$  is a constant vector,  $\vec{d}$  represents the displacement — work is defined as  $\vec{F} \cdot \vec{d}$
- $W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}|\cos(\theta)$

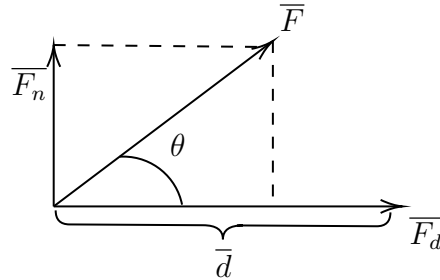


Figure 1: Diagram of Work

## 2 Lines, Planes, and Hyperplanes

- Lines and Planes

- Ex. Given a point in  $\mathbb{R}^2$ ,  $(x_o, y_o) = p$  and a vector  $\vec{n} = \langle a, b \rangle$ , find an equation of a line passing through  $p$  and  $\perp$  to  $\vec{n}$

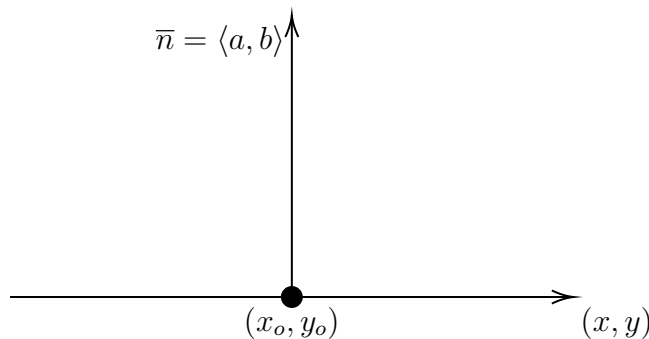


Figure 2: Finding an Equation for a Line

- \* Create a vector:  $\langle x - x_o, y - y_o \rangle$ , then, by definition, dot product becomes:  
 $\langle a, b \rangle \cdot \langle x - x_o, y - y_o \rangle = 0$ , which yields  $ax + by - ax_o - by_o = 0$ , which can be simplified to  $ax + by + c = 0$

- In  $\mathbb{R}^3$ :  $\langle a, b, c \rangle \cdot \langle x - x_o, y - y_o, z - z_o \rangle$  becomes  $a(x - x_o) + b(y - y_o) + c(z - z_o) = 0$  and then  $ax + by + cz + d = 0$ , this forms a plane through point  $p$  (in  $\mathbb{R}^3$ )

- Parametric Description of a Line

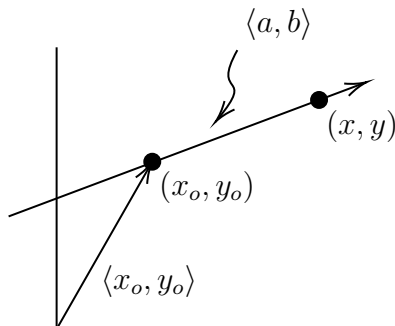


Figure 3: Parametrization

- $\begin{cases} x(t) = x_o + ta \\ y(t) = y_o + tb \end{cases}$
- In  $\mathbb{R}^3$ :  $\begin{cases} x(t) = x_o + ta \\ y(t) = y_o + tb \\ z(t) = z_o + tc \end{cases}$
- The general equation can be summed up as  $\bar{r}(t) = \bar{r}_o + t\bar{v}$ , where  $\bar{r}(t)$  generates a parametrized equation,  $\bar{r}_o$  is a position vector,  $t$  is the parameter, and  $\bar{v}$  is a vector parallel to a given equation
- To parametrize a line segment from point  $p$  to point  $q$ :  
 $\bar{r}(t) = (1 - t)\bar{p} + t\bar{q}$ ,  $0 \leq t \leq 1$ , or  $0 \xrightarrow{t} 1$ 
  - \* *Ex.* Line segment from (2,1) to (3, -4):  
 $\bar{r}(t) = (1 - t)\langle 2, 1 \rangle + t\langle 3, -4 \rangle \Rightarrow \langle 2 + t, 1 - 5t \rangle$

- Parametric Planes

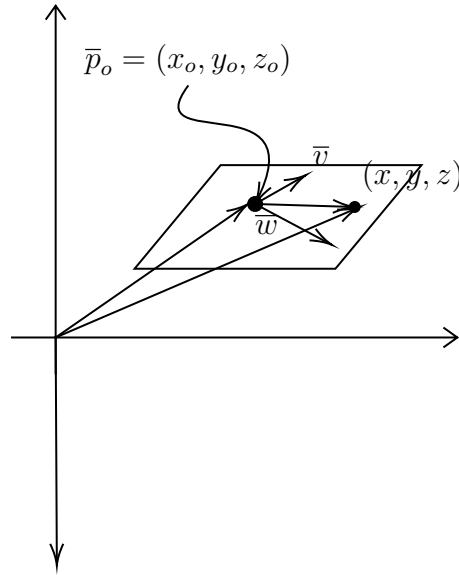


Figure 4: Parametrization of a Plane

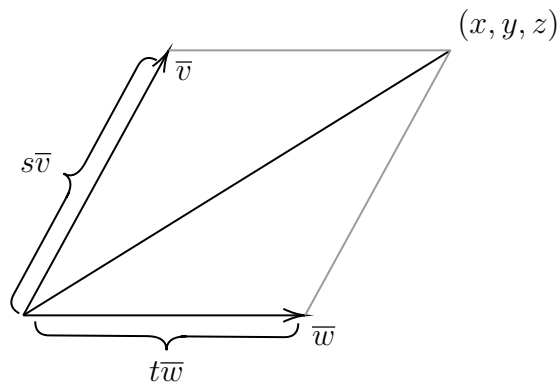


Figure 5: Specific View of Parametrization of a Plane

$$- \bar{r} = \bar{p}_o + s\bar{v} + t\bar{w}, -\infty \xrightarrow{s,t} \infty$$

\*  $\bar{v}$  and  $\bar{w}$  must not be parallel to each other

### 3 The Cross Product

- Cross Product

- $\bar{v} \times \bar{w}$  produces a vector in  $\mathbb{R}^3$
- Geometric Definition:
  - \* Magnitude:  $|\bar{v}||\bar{w}| \sin(\theta)$ ,  $0 \leq \theta \leq \pi$
  - \* Direction:  $\bar{v} \times \bar{w}$  is  $\perp$  to both  $\bar{v}$  and  $\bar{w}$ 
    - Direction is uniquely determined by the right-hand rule
  - \*  $\bar{v} \parallel \bar{w} \Leftrightarrow \bar{v} \times \bar{w} = 0$
  - \*  $|\bar{v} \times \bar{w}|$  = the area of a parallelogram formed by  $\bar{v}$  and  $\bar{w}$
- Algebraic Definition:
  - \* A matrix is a rectangular array of numbers
  - \* Determinants:
    - $\det([c]) = c$
    - $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$
    - $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a(ei - fh) - b(di - fg) + c(dh - eg)$
  - \*  $\begin{cases} \bar{v} = \langle v_1, v_2, v_3 \rangle \\ \bar{w} = \langle w_1, w_2, w_3 \rangle \end{cases} \Rightarrow$ 

$$\bar{v} \times \bar{w} = \begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \langle v_2 w_3 - v_3 w_2, -v_1 w_3 - v_3 w_1, v_1 w_2 - v_2 w_1 \rangle$$

## 4 Functions of a Single Variable

- $f'(t) = \lim_{h \rightarrow 0} \left( \frac{f(t+h) - f(t)}{h} \right)$
- Ex.  $f(t) = \langle t^2, t^3 \rangle \Rightarrow f'(t) = \langle 2t, 3t^2 \rangle$ 
  - Distance is found with arc length

$$\text{dist} = \lim_{\Delta t \rightarrow 0} \sum |\vec{p}'(t)| \Delta t = \int_a^b s(t) dt \Rightarrow \int_0^1 \sqrt{(2t)^2 + (3t^2)^2} dt = \frac{1}{27} (13^{1.5} - 8)$$

- If  $p(t)$  is position, then  $\begin{cases} p'(t) = \text{velocity} \\ p''(t) = \text{acceleration} \end{cases}$
- Ex. Given  $\bar{p}(t) = \langle \cos(t), \sin(t), t \rangle$ , find velocity and acceleration:

$$\bar{v}(t) = \bar{p}'(t) = \langle -\sin(t), \cos(t), 1 \rangle \Rightarrow \bar{a}(t) = \bar{p}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

- Speed =  $|\bar{v}(t)|$

$$|\bar{v}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1} = \sqrt{2}$$



## 5 Multivariable Functions

- Linear Functions —  $L(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n$ , where  $a_i$  is a fixed number
- Affine Linear Functions —  $L(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \dots + a_nx_n + b$ , where  $a_i$  and  $b$  are fixed