Section 1

Michael Brodskiy
Professor: A. Martsinkovsky

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1 $\mathbb{R}^n$ as a Vector Space

- What is a vector?
  - A magnitude and a direction? (not all vectors in the real world can be added, so not entirely true)
  - For our course, vectors exist in vector spaces ($\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$, $\mathbb{R}^n$)
  - $\overline{v} = \langle v_1, v_2, \ldots, v_n \rangle$
  - $\mathbb{R}^1$ represents scalars, while $\mathbb{R}^2$, $\mathbb{R}^3$, $\ldots$, $\mathbb{R}^n$ are vectors
- Properties of Vectors
  - Can be added
    * $\overline{v} = \langle v_1, v_2, \ldots, v_n \rangle + \overline{w} = \langle w_1, w_2, \ldots, w_n \rangle = \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle$
    * If forming a parallelogram from the vectors, the diagonal is the sum, $\overline{v} + \overline{w}$, of two vectors
  - Can be scaled (scalar multiplication)
    * $2\overline{v} = \langle 2v_1, 2v_2, \ldots, 2v_n \rangle$
    * Magnitude is multiplied by the factor
  - Can find magnitude (length)
    * $|\overline{v}| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
    * *Ex.* $\overline{v} = \langle 2, -3 \rangle \Rightarrow |\overline{v}| = \sqrt{(2)^2 + (-3)^2)} = \sqrt{13}$
  - A vector divided by its own magnitude becomes a vector of magnitude 1 (unit vector)
    * $|\frac{\overline{v}}{|\overline{v}|}| = 1$
    * Unit vectors are dimensionless (no units)
    * A vector that is by itself of length 1 is not a unit vector
    * A unit vector is simply a direction (all unit vectors from a given point form a circle)
  - Any non-zero vector is the product of its magnitude and its direction
    * $\overline{v} = |\overline{v}| \cdot \frac{\overline{v}}{|\overline{v}|}$
- Linear Combinations
  - $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_s$
  - A linear combination of $\overline{v}_i$ is any sum of the form $r_1\overline{v}_1 + r_2\overline{v}_2 + \cdots + r_n\overline{v}_n$, where r_i are scalars
- Basis Vectors

- $\mathbb{R}^n$ standard basis vectors: $\overline{e}_1,\ \overline{e}_2,\ \ldots,\ \overline{e}_n \Rightarrow \begin{cases} \overline{e}_1 = \langle 1, 0, \ldots, 0 \rangle \\ \overline{e}_2 = \langle 0, 1, \ldots, 0 \rangle \\ \quad\vdots \\ \overline{e}_n = \langle 0, 0, \ldots, 1 \rangle \end{cases}$
- Any vector is a linear combination of the standard basis vectors
- $\overline{w} = \langle w_1,\ w_2,\ \ldots,\ w_n \rangle = w_1\overline{e}_1 + w_2\overline{e}_2 + \cdots + w_n\overline{e}_n$
- *Ex.* $\overline{v} = \langle 2, -3 \rangle = 2\langle 1, 0 \rangle + -3\langle 0, 1 \rangle$

- Dot Product
  - The dot product of two vectors is always a scalar
  - Geometric Definition: $\overline{v} \cdot \overline{w} = |\overline{v}||\overline{w}| \cos(\theta)$, where θ is the angle between $\overline{v}$ and $\overline{w}$
    * $\overline{v} \cdot \overline{w} = 0$ when $\theta = \frac{\pi}{2}$
    * $\overline{v} \cdot \overline{w} > 0$ when θ is acute
    * $\overline{v} \cdot \overline{w} < 0$ when θ is obtuse
  - Algebraic Definition: $\begin{cases} \overline{v} = \langle v_1,\ v_2,\ \ldots,\ v_n \rangle \\ \overline{w} = \langle w_1,\ w_2,\ \ldots,\ w_n \rangle \end{cases} \Rightarrow \overline{v} \cdot \overline{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$
    * *Ex.* $\begin{cases} \overline{v} = \langle 4, 9, 5 \rangle \\ \overline{w} = \langle 4, 10, 3 \rangle \end{cases} \Rightarrow \overline{v} \cdot \overline{w} = 4(4) + 9(10) + 5(3) = 121$
  - Together, the two definitions yield $\theta = \cos^{-1}\left(\frac{\overline{v} \cdot \overline{w}}{|\overline{v}||\overline{w}|}\right)$
    * *Ex.* Given $\overline{v}$ and $\overline{w}$ above, find the angle: $\cos^{-1}\left(\frac{121}{\sqrt{122}\sqrt{125}}\right) \approx .2$ rad
  - Vector Projection
    * Assuming $\overline{u}$ is a unit vector, the projection of $\overline{F}$ onto $\overline{u}$ can be found using: $\text{proj}_{\overline{u}}\overline{F} = \left(\overline{F} \cdot \overline{u}\right)\overline{u}$
    * In general, because $\overline{u} = \frac{\overline{v}}{|\overline{v}|}$, the formula becomes: $\text{proj}_{\overline{v}}\overline{F} = \left(\overline{F} \cdot \frac{\overline{v}}{|\overline{v}|}\right)\frac{\overline{v}}{|\overline{v}|} = \left(\frac{\overline{F} \cdot \overline{v}}{|\overline{v}|^2}\right)\overline{v}$

- Work
  - $\overline{F}$ is a constant vector, $\overline{d}$ represents the displacement — work is defined as $\overline{F} \cdot \overline{d}$
  - $W = \overline{F} \cdot \overline{d} = |\overline{F}||\overline{d}| \cos(\theta)$

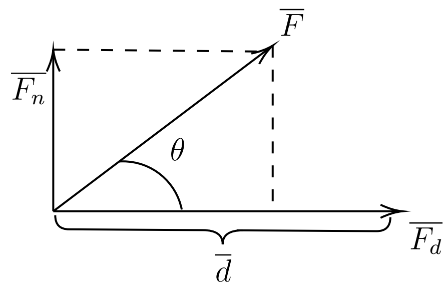

Figure 1: Diagram of Work

2 Lines, Planes, and Hyperplanes

- Lines and Planes
  - *Ex.* Given a point in $\mathbb{R}^2$, $(x_o, y_o) = p$ and a vector $\overline{n} = \langle a, b\rangle$, find an equation of a line passing through p and $\perp$ to $\overline{n}$

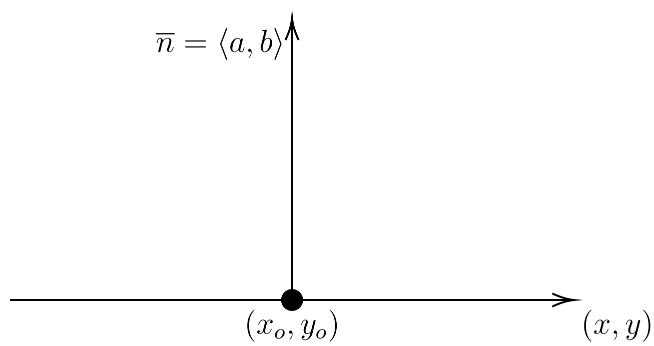

Figure 2: Finding an Equation for a Line

    * Create a vector: $\langle x - x_o, y - y_o\rangle$, then, by definition, dot product becomes: $\langle a, b\rangle \cdot \langle x - x_o, y - y_o\rangle = 0$, which yields $ax + by - ax_o - by_o = 0$, which can be simplified to $ax + by + c = 0$

- In $\mathbb{R}^3$: $\langle a,b,c\rangle \cdot \langle x-x_o, y-y_o, z-z_o\rangle$ becomes $a(x-x_o)+b(y-y_o)+c(z-z_o)=0$ and then $ax+by+cz+d=0$, this forms a plane through point p (in $\mathbb{R}^3$)

- Parametric Description of a Line

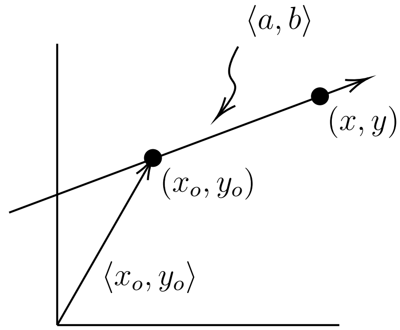

Figure 3: Parametrization

  - $\begin{cases} x(t) = x_o + ta \\ y(t) = y_o + tb \end{cases}$
  - In $\mathbb{R}^3$: $\begin{cases} x(t) = x_o + ta \\ y(t) = y_o + tb \\ z(t) = z_o + tc \end{cases}$
  - The general equation can be summed up as $\overline{r}(t) = \overline{r}_o + t\overline{v}$, where $\overline{r}(t)$ generates a parametrized equation, $\overline{r}_o$ is a position vector, t is the parameter, and $\overline{v}$ is a vector parallel to a given equation
  - To parametrize a line segment from point p to point q:
    $\overline{r}(t) = (1-t)\overline{p} + t\overline{q}$, $0 \le t \le 1$, or $0 \xrightarrow{t} 1$
    - *Ex.* Line segment from (2,1) to (3, -4):
      $\overline{r}(t) = (1-t)\langle 2,1\rangle + t\langle 3,-4\rangle \Rightarrow \langle 2+t, 1-5t\rangle$

- Parametric Planes

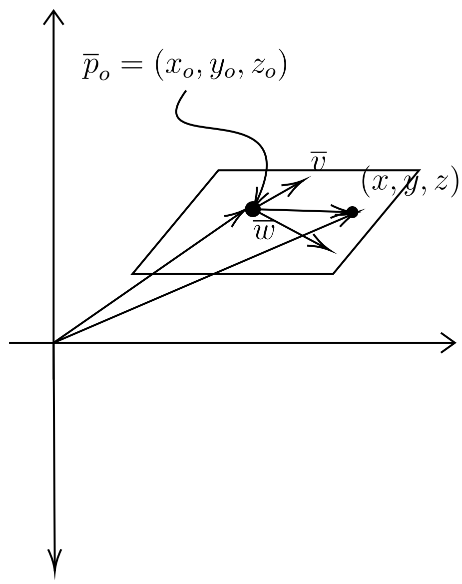

Figure 4: Parametrization of a Plane

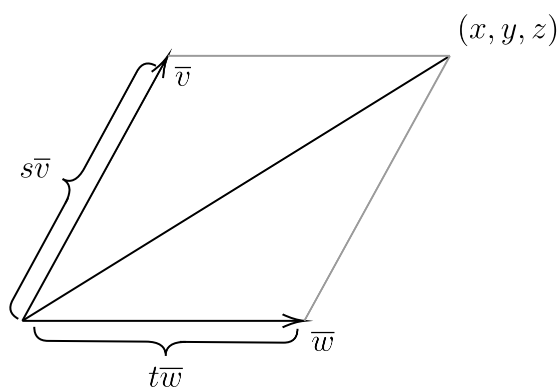

Figure 5: Specific View of Parametrization of a Plane

  - $\overline{r} = \overline{p}_o + s\overline{v} + t\overline{w}$, $-\infty \xrightarrow{s,t} \infty$
    - $\overline{v}$ and $\overline{w}$ must <u>not</u> be parallel to each other

3 The Cross Product

- Cross Product

- $\overline{v} \times \overline{w}$ produces a vector in $\mathbb{R}^3$
- Geometric Definition:
  - Magnitude: $|\overline{v}||\overline{w}|\sin(\theta)$, $0 \leq \theta \leq \pi$
  - Direction: $\overline{v} \times \overline{w}$ is $\perp$ to both $\overline{v}$ and $\overline{w}$
    - Direction is uniquely determined by the right-hand rule
  - $\overline{v} \parallel \overline{w} \Leftrightarrow \overline{v} \times \overline{w} = 0$
  - $|\overline{v} \times \overline{w}|$ = the area of a parallelogram formed by $\overline{v}$ and $\overline{w}$
- Algebraic Definition:
  - A matrix is a rectangular array of numbers
  - Determinants:
    - $\det([c]) = c$
    - $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$
    - $\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a(ei - fh) - b(di - fg) + c(dh - eg)$
  - $\begin{cases} \overline{v} = \langle v_1, v_2, v_3 \rangle \\ \overline{w} = \langle w_1, w_2, w_3 \rangle \end{cases} \Rightarrow$
    $\overline{v} \times \overline{w} = \begin{bmatrix} \overline{i} & \overline{j} & \overline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \langle v_2w_3 - v_3w_2, -v_1w_3 - v_3w_1, v_1w_2 - v_2w_1 \rangle$

4 Functions of a Single Variable

- $f'(t) = \lim_{h \to 0} \left(\frac{f(t+h) - f(t)}{h} \right)$
- *Ex.* $f(t) = \langle t^2, t^3 \rangle \Rightarrow f'(t) = \langle 2t, 3t^2 \rangle$
  - Distance is found with arc length

$$\text{dist} = \lim_{\Delta t \to 0} \sum |\overline{p}'(t)| \, \Delta t = \int_a^b s(t)\, dt \Rightarrow \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \frac{1}{27}\left(13^{1.5} - 8\right)$$

- If $p(t)$ is position, then $\begin{cases} p'(t) = & \text{velocity} \\ p''(t) = & \text{acceleration} \end{cases}$
- *Ex.* Given $\overline{p}(t) = \langle \cos(t), \sin(t), t \rangle$, find velocity and acceleration:

$$\overline{v}(t) = \overline{p}'(t) = \langle -\sin(t), \cos(t), 1 \rangle \Rightarrow \overline{a}(t) = \overline{p}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$$

- Speed = $|\overline{v}(t)|$

$$|\overline{v}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1 = \sqrt{2}}$$