

Section 4

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1 Vector Fields

- A vector field in \mathbb{R}^n is an assignment that goes from $\mathbb{R}^n \rightarrow \mathbb{R}^n$, where the former is viewed as a bunch of points, and the latter is a bunch of vectors
 - Two examples: a force field (*e.g.* gravity, electrostatic, magnetic) or a velocity field (*e.g.* fluid mechanics)
- *Ex.* A gravitational field, with two masses, one fixed (M), and one floating (m)
 - The pull = magnitude \cdot direction $\rightarrow G \frac{Mm}{|\bar{d}|^2} \cdot \frac{\bar{d}}{|\bar{d}|} = G \frac{Mm}{|\bar{d}|^3} \bar{d}$
- *Ex.* Gradient vector fields
 - $f(\bar{x}) \rightarrow \nabla f = \left\langle \frac{\delta f}{\delta x_1}, \frac{\delta f}{\delta x_2}, \dots, \frac{\delta f}{\delta x_n} \right\rangle$
 - Not every vector field can be realized as a gradient vector field of some function
 - The ‘del’ operator is as follows: $\nabla := \left\langle \frac{\delta}{\delta x_1}, \frac{\delta}{\delta x_2}, \dots, \frac{\delta}{\delta x_n} \right\rangle$
 - Apply ∇ to a function f to obtain a gradient vector field (use dot product)
 - $\nabla \cdot F = \frac{\delta F_1}{\delta x_1} + \frac{\delta F_2}{\delta x_2} + \dots + \frac{\delta F_n}{\delta x_n} = \text{div}(F)$
 - * This is the divergence of F
 - $\nabla \times F$ describes the curl of F (how the vector field curls in three dimensions)

Using the ∇ operator:

input	output	significance
function f	∇f	gradient of f (a vector field)
vector field of f	$\nabla \cdot f$	divergence of f (a function)
$n = 3$ vector field of f	$\nabla \times f$	curl of f (a vector field)

- To view 2D vector fields as 3D vector fields, convert $F = \langle P(x, y), Q(x, y) \rangle \rightarrow \langle P(x, y), Q(x, y), 0 \rangle$
- The curl of a two dimensional vector field, converted to three dimensions, is $\text{curl}(F) = Q_x - P_y$

2 Line Integrals

- Can find Work, W , done by a force (a.k.a. a vector field) on an object along an oriented curve, C

- $W = \int_C \overline{F} \cdot d\vec{r}$ — This is the line integral of \overline{F} along the oriented curve C
- To compute the line integral, parametrize C , and reduce it to a Calculus II integral
 - $d\vec{r} = \vec{r}'(t) dt$
 - $\int_a^b \overline{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$
- Conservative vector fields are independent of path (taking any path from one point to another will always yield the same value)
- Alternative notation: $\int_C P dx + Q dy + R dz \rightarrow \int_C P \cdot x'(t) dt + Q \cdot y'(t) dt + R \cdot z'(t) dt$

3 Conservative Vector Fields

- A vector field \overline{F} is said to be conservative if $\overline{F} = \nabla f$ for some function f
- The fundamental theorem of line integrals: Supposed \overline{F} is conservative with potential f . $\overline{F} = \nabla f$. Let C be an oriented curve starting at A and ending at B , then:

$$\int_C \overline{F} \cdot d\vec{r} = f(B) - f(A)$$
- Any constant vector field is conservative
- Before doing a line integral, check if \overline{F} is conservative
- \overline{F} is conservative if and only if $\text{curl}(\overline{F})=0$

4 Green's Theorem

- Let R be a region in \mathbb{R}^2 bounded by a simple, closed, oriented, counter-clockwise, piecewise-regular curve δR . Let $\overline{F} = \langle P(x, y), Q(x, y) \rangle$ be a continuous, differentiable vector field on an open set containing R , then:

$$\int_{\delta R} \overline{F} d\vec{r} = \iint_R Q_x - P_y dA$$

- Simple: The curve can not fold in on itself
- Closed: The curve starts and ends at the same point

- Because, in \mathbb{R}^2 , $Q_x - P_y = \text{curl}(\overline{F}) = \nabla \times \overline{F}$, the integral can be rewritten as $\int_{\delta R} \overline{F} d\vec{r} = \iint_R \delta \overline{F} dA$

5 Flux Through a Surface

- To find the volume of fluid flowing through a surface, M , per unit time, a normal unit vector, \bar{n} is needed; \bar{n} is supposed to vary continuously through M ; when such a choice for \bar{n} is possible, M is said to be orientable
- Volume = $(\bar{V} \cdot \bar{n}) dS$
- Flux of V through $M = \iint_M (\bar{V} \cdot \bar{n}) dS$
 - $\bar{n} = \frac{\bar{r}_u \times \bar{r}_v}{|\bar{r}_u \times \bar{r}_v|}$
 - Remark: If \bar{n} produces an incorrect orientation, flip the sign
- The final formula becomes $\iint_{D(u,v)} V(\bar{r}(u,v)) \cdot (\bar{r}_u \times \bar{r}_v) du dv$

6 The Divergence Theorem

- This theorem computes the flux of a vector field through the boundary of a solid
- Theorem: The flux through E equals the flux through δE

$$\iint_{\delta E} (\bar{F} \cdot \bar{n}) dS = \iiint_E \text{div}(\bar{F}) dV$$

- Where \bar{n} should point outward from E
- If $\text{div}(\bar{F}) = 0$, \bar{F} is incompressible
- If $\text{curl}(\bar{F}) = 0$, \bar{F} is irrotational (no rotation)
- Symmetry \Rightarrow Conservation Laws (Noether's theorem)