Section 2

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Contents

1 Partial Derivatives

- The slope of $f(x, y)$ depends on the direction in the xy-plane
  - The slope in the x-direction is called the partial derivative of f with respect to x
  - The slope in the y-direction is called the partial derivative of f with respect to y
  - Notation: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ or f_x, f_y
  - For Second Derivatives: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$ or f_{xx}, f_{xy}, f_{yx}, f_{yy}

If f, f_x, f_y, and f_{xy} are defined in a small disc around (x_o, y_o) and f_{yx} is continuous, then:
$$f_{xy} = f_{yx}$$
in that disc

- The gradient of f
  - Given $f(x_1, x_2, \ldots, x_n)$, the gradient of f, $\overline{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$
  - It can be computed at a point: $\overline{\nabla} f(p) = \left\langle \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right\rangle$
  - $\overline{\nabla} f \approx f'(x_o)\Delta x$

2 Linear Approximation, Tangent Planes, and the Differential

- In calculus I, the linear approximation is given by: $f(x) \approx f(a) + f'(a)(x - a)$
- In calculus III, the approximation uses the gradient: $\Delta f \approx \overline{\nabla} f(p) \cdot \Delta \overline{x}$
- *Ex. in* $\mathbb{R}^2$ $z = f(x, y)$, $p = (a, b)$:

  $\Delta f \approx \overline{\nabla} f(a, b) \cdot \langle x - a, y - b \rangle \Rightarrow f(x, y) - f(a, b) \Rightarrow f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Thus:
$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$
We find the linearization of $f(x, y)$ near (a, b) (in $\mathbb{R}^2$)

- Linearization of $f(x, y)$ near (a, b) is denoted by $L_f(\overline{x}, \overline{p})$, where p is the vector $\langle a, b \rangle$

- *Ex. Given a cylinder of radius $r = .5$ and a height of $h = 1$, estimate the change in volume when the radius is increased by .1 and the height is decreased by .1*

$$V = \pi r^2 h \Rightarrow \Delta V \approx 2\pi(.5)(1)(r - .5) + \pi(.5)^2(h - 1) = \pi(r - .5) + .25\pi(h - 1) \approx$$

$$-.75\pi + \pi r + .25\pi h \Rightarrow -.75\pi + \pi(.6) + .25\pi(.9) = .075\pi$$

- The graph of $z = L_f(\overline{x}, \overline{p})$ is called the tangent set to f at p
- Differentials
  - $df = \overline{\nabla} f \cdot d\overline{x}$
  - $df = f_x dx + f_y dy + f_z dz$
  - $df = \dfrac{\partial f}{\partial x_1} dx_1 + \dfrac{\partial f}{\partial x_2} dx_2 + \cdots + \dfrac{\partial f}{\partial x_n} dx_n$
- Relative Differentials
  - $\dfrac{df}{f}$
  - Think of this as a stencil for relative error $\left(\dfrac{\Delta f}{f}\right)$

3 Differentiation Rules

- Linearity of differentiation

  1. $\nabla(af \pm bg) = a\nabla f \pm b\nabla g$

- Product rule

  2. $\nabla(fg)(p) = \nabla f(p)g(p) = \nabla g(p)f(p)$

- Quotient rule

  3. $\nabla\left(\dfrac{f}{g}\right)\Big|_p = \dfrac{g(p)\nabla f(p) - f(p)\nabla g(p)}{g^2(p)}$

- Power rule

  4. $\nabla f^\alpha(p) = \alpha f^{\alpha-1}(p)\nabla f(p)$

- Chain rule

  5. $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \cdots + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt} \Rightarrow \nabla f \cdot \dfrac{d\overline{x}}{dt}$

- *Ex.* A particle is moving through space. At $t = 2$ seconds, the particle is at $(3, 4, 7)$, and is moving with velocity $\langle -2, 1, 5\rangle$ meters per second. Suppose that there is also an electric potential in space, given by $\phi(x, y, z) = xy - z^2$ volts. Find the instantaneous rate of change, with respect to time t, of the electric potential at the particle's position at $t = 2$ seconds.

$$\frac{d\phi}{dt}\bigg|_{t=2} = \nabla\phi(\langle 3,4,7\rangle) \cdot \frac{d\overline{p}}{dt}\bigg|_{t=2} = \nabla\phi(\langle 3,4,7\rangle) \cdot v(2) \Rightarrow$$

$$\nabla\phi = \langle y, x, -2z\rangle(\langle 3,4,7\rangle) = \langle 4, 3, -14\rangle \Rightarrow \langle 4, 3, -14\rangle \cdot \langle -2, 1, 5\rangle = -75 \text{ volts per second}$$

4 The Directional Derivative

- Directions live in the input of the function
- Same is true for ∇f
- What is rate of change of $z = f(x, y)$ in the direction $\overline{u}$?
  - Fix point p in xy-plane
  - Choose direction ($\overline{u}$-direction) in xy-plane
  - $D_{\overline{u}}f(p) = \nabla f(p) \cdot \overline{u}$
    * This is the derivative of f at p in the direction of $\overline{u}$
- Alternative: Fix $f(x, y)$ and point p, but $\overline{u}$ varies
- What is $\overline{u}$ in which $f(x, y)$ changes in the fastest possible way?
  - When the angle between the direction and ∇f is $\theta = 0$ (because $\cos(\theta)$) is greatest at this angle
    * Smallest occurs in opposite directions (when $\theta = \pi$) because $\cos(\theta)$ is the largest possible negative
    * Equals zero when $\overline{u} \perp \nabla f(p)$ $\left(i.e.\ \theta = \dfrac{\pi}{2}\right)$
  - Thus, $D_{\overline{u}}f(p)$ is largest possible in direction of $\nabla f(p)$
  - The largest rate is the magnitude of the gradient vector ($|\nabla f(p)|$)
    * The largest rate is the negative magnitude, $-|\nabla f(p)|$
  - This is all assuming that $\nabla f(p) \neq 0$

5 Level Sets and Gradient Vectors

- Given $z = f(x, y)$
  - Pick a constant, c
  - Plug it in for z, such that $f(x, y) = c$
  - The level curve of $f(x, y)$ corresponding to c is formed
- *Ex.* $z = x^2 + y^2$
  - At $c = -1$, there is nothing to draw because there are no real solutions
  - At $c = 0$, there is one point at the origin $(0, 0)$
  - At $c = 1$, we obtain a circle of radius one
  - At $c = 4$, we obtain a circle of radius two
- The level curves corresponding to different values of c can not intersect
- The gradient vector $\overline{\nabla} f(p)$ is always perpendicular ($\perp$) to the level curve passing through p
- Implicit functions do not expressly define one variable in terms of different variables (*Ex.* $x^2 + y^2 = 1$), while explicit functions do (*Ex.* $z = \cos(y) + e^x$)
- To find the tangent line function, use the formula $\overline{\nabla} f(p) \cdot \langle X - p \rangle = 0$

6 Parametrizing Surfaces

- A line in $\mathbb{R}^3$: $\begin{cases} x = x_o + at \\ y = y_o + bt \\ z = z_o + ct \end{cases}$
- A circle in $\mathbb{R}^2$: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$, where r is the radius
- Parametrizing a surface
  - Parametrization should be differentiable
  - Using u and v creates $\overline{r}(u, v)$, so that $\overline{r}_u \times \overline{r}_v \neq 0$
  - $\overline{r}_u$ should not be parallel to $\overline{r}_v$
  - A parametrization satisfying this property is said to be regular
  - *Ex.* Parametrize the cone with equation $z^2 = x^2 + y^2$: $\begin{cases} x = u\cos(v) \\ y = u\sin(v) \\ u \end{cases}$
- The equation of a tangent plane $a\overline{r}_u(u_o, v_o) + b\overline{r}_v(u_o, v_o)$

7 Local Extrema

- In Calc I, critical points occurred where $f'(x) = 0$ or undefined with the first derivative test, or depending on concavity for the second derivative test
- The second derivative test was as follows:
  - If $f''(p) > 0$, p is a local min
  - If $f''(p) < 0$, p is a local max
  - If $f''(p) = 0$, p is neither
- In Calc III, critical points occur where $\nabla f = 0$ or undefined, but the first derivative test is not practical
- The second derivative test is now done as:
  - Hessian Matrix: $\begin{bmatrix} f_{xx}(p) & f_{yx}(p) \\ f_{xy}(p) & f_{yy}(p) \end{bmatrix}$, then find the determinant:
  - $D = f_{xx}(p)f_{yy}(p) - f_{xy}^2(p)$
  - A critical point p is non-degenerate if $D \neq 0$
  - Assuming p is non-degenerate:
    - If $D > 0$ and $f_{xx}(p) > 0$, then f has a local minimum at p
    - If $D > 0$ and $f_{xx}(p) < 0$, then f has a local maximum at p
    - If $D < 0$, then p is a saddle point of f
- Saddle points occur where there are critical points, but a local extreme value is not contained

8 Optimization

- Optimization involves finding the largest or smallest value of a function
  - Can be constrained or unconstrained
  - Unconstrained assumes there are no limits on the inputs of a function
  - Constrained contains limits on inputs
- The Unconstrained Case:
  1. Find critical points in the interior of the domain, and find values of the function at those points
  2. Work out the boundaries

- Steps to Solve

  1. Determine critical points in the interior of the domain. Compute and tabulate the values of function at these points
  2. Proceed to the boundary, and find all critical points interior of the boundary; compute and tabulate
  3. Keep going