Section 2

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Contents

1 Partial Derivatives

- The slope of $f(x, y)$ depends on the direction in the xy-plane
  - The slope in the x-direction is called the partial derivative of f with respect to x
  - The slope in the y-direction is called the partial derivative of f with respect to y
  - Notation: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ or f_x, f_y
  - For Second Derivatives: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$ or f_{xx}, f_{xy}, f_{yx}, f_{yy}

If f, f_x, f_y, and f_{xy} are defined in a small disc around (x_o, y_o) and f_{yx} is continuous, then:

$$f_{xy} = f_{yx}$$

in that disc

- The gradient of f
  - Given $f(x_1, x_2, \ldots, x_n)$, the gradient of f, $\overline{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$
  - It can be computed at a point: $\overline{\nabla} f(p) = \left\langle \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right\rangle$
  - $\overline{\nabla} f \approx f'(x_o)\Delta x$

2 Linear Approximation, Tangent Planes, and the Differential

- In calculus I, the linear approximation is given by: $f(x) \approx f(a) + f'(a)(x - a)$
- In calculus III, the approximation uses the gradient: $\Delta f \approx \overline{\nabla} f(p) \cdot \Delta \overline{x}$
- *Ex. in* $\mathbb{R}^2$ $z = f(x, y)$, $p = (a, b)$:

  $\Delta f \approx \overline{\nabla} f(a, b) \cdot \langle x - a, y - b \rangle \Rightarrow f(x, y) - f(a, b) \Rightarrow f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Thus:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We find the linearization of $f(x, y)$ near (a, b) (in $\mathbb{R}^2$)

- Linearization of $f(x, y)$ near (a, b) is denoted by $L_f(\overline{x}, \overline{p})$, where p is the vector $\langle a, b \rangle$

- *Ex. Given a cylinder of radius $r = .5$ and a height of $h = 1$, estimate the change in volume when the radius is increased by $.1$ and the height is decreased by $.1$*

$$V = \pi r^2 h \Rightarrow \Delta V \approx 2\pi(.5)(1)(r-.5) + \pi(.5)^2(h-1) = \pi(r-.5) + .25\pi(h-1) \approx$$

$$-.75\pi + \pi r + .25\pi h \Rightarrow -.75\pi + \pi(.6) + .25\pi(.9) = .075\pi$$

- The graph of $z = L_f(\overline{x}, \overline{p})$ is called the tangent set to f at p
- Differentials
  - $df = \overline{\nabla} f \cdot d\overline{x}$
  - $df = f_x dx + f_y dy + f_z dz$
  - $df = \dfrac{\partial f}{\partial x_1} dx_1 + \dfrac{\partial f}{\partial x_2} dx_2 + \cdots + \dfrac{\partial f}{\partial x_n} dx_n$
- Relative Differentials
  - $\dfrac{df}{f}$
  - Think of this as a stencil for relative error $\left(\dfrac{\Delta f}{f}\right)$

3 Differentiation Rules

- Linearity of differentiation

  1. $\nabla(af \pm bg) = a\nabla f \pm b\nabla g$

- Product rule

  2. $\nabla(fg)(p) = \nabla f(p) g(p) = \nabla g(p) f(p)$

- Quotient rule

  3. $\nabla\left(\dfrac{f}{g}\right)\Big|_p = \dfrac{g(p)\nabla f(p) - f(p)\nabla g(p)}{g^2(p)}$

- Power rule

  4. $\nabla f^{\alpha}(p) = \alpha f^{\alpha-1}(p) \nabla f(p)$

- Chain rule

  5. $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \cdots + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt} \Rightarrow \nabla f \cdot \dfrac{d\overline{x}}{dt}$

- *Ex.* A particle is moving through space. At $t = 2$ seconds, the particle is at $(3, 4, 7)$, and is moving with velocity $\langle -2, 1, 5 \rangle$ meters per second. Suppose that there is also an electric potential in space, given by $\phi(x, y, z) = xy - z^2$ volts. Find the instantaneous rate of change, with respect to time t, of the electric potential at the particle's position at $t = 2$ seconds.

$$\left.\frac{d\phi}{dt}\right|_{t=2} = \nabla\phi(\langle 3,4,7\rangle) \cdot \left.\frac{d\overline{p}}{dt}\right|_{t=2} = \nabla\phi(\langle 3,4,7\rangle) \cdot v(2) \Rightarrow$$

$\nabla\phi = \langle y, x, -2z\rangle(\langle 3,4,7\rangle) = \langle 4, 3, -14\rangle \Rightarrow \langle 4, 3, -14\rangle \cdot \langle -2, 1, 5\rangle = -75$ volts per second

4 The Directional Derivative

- Directions live in the input of the function
- Same is true for ∇f
- What is rate of change of $z = f(x, y)$ in the direction $\overline{u}$?
  - Fix point p in xy-plane
  - Choose direction ($\overline{u}$-direction) in xy-plane
  - $D_{\overline{u}} f(p) = \nabla f(p) \cdot \overline{u}$
    * This is the derivative of f at p in the direction of $\overline{u}$
- Alternative: Fix $f(x, y)$ and point p, but $\overline{u}$ varies
- What is $\overline{u}$ in which $f(x, y)$ changes in the fastest possible way?
  - When the angle between the direction and ∇f is $\theta = 0$ (because $\cos(\theta)$) is greatest at this angle
    * Smallest occurs in opposite directions (when $\theta = \pi$) because $\cos(\theta)$ is the largest possible negative
    * Equals zero when $\overline{u} \perp \nabla f(p)$ $\left(i.e.\ \theta = \dfrac{\pi}{2}\right)$
  - Thus, $D_{\overline{u}} f(p)$ is largest possible in direction of $\nabla f(p)$
  - The largest rate is the magnitude of the gradient vector ($|\nabla f(p)|$)
    * The largest rate is the negative magnitude, $-|\nabla f(p)|$
  - This is all assuming that $\nabla f(p) \neq 0$

5 Level Curves and Gradient Vectors

- Given $z = f(x, y)$
  - Pick a constant, c
  - Plug it in for z, such that $f(x, y) = c$
  - The level curve of $f(x, y)$ corresponding to c is formed
- *Ex.* $z = x^2 + y^2$
  - At $c = -1$, there is nothing to draw because there are no real solutions
  - At $c = 0$, there is one point at the origin $(0, 0)$
  - At $c = 1$, we obtain a circle of radius one
  - At $c = 4$, we obtain a circle of radius two
- The level curves corresponding to different values of c can not intersect