Section 2

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1 Partial Derivatives

- The slope of $f(x, y)$ depends on the direction in the xy-plane
  - The slope in the x-direction is called the partial derivative of f with respect to x
  - The slope in the y-direction is called the partial derivative of f with respect to y
  - Notation: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ or f_x, f_y
  - For Second Derivatives: $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$ or $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

If f, f_x, f_y, and f_{xy} are defined in a small disc around (x_o, y_o) and f_{yx} is continuous, then:
$f_{xy} = f_{yx}$
in that disc

- The gradient of f
  - Given $f(x_1, x_2, \ldots, x_n)$, the gradient of f, $\overline{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$
  - It can be computed at a point: $\overline{\nabla} f(p) = \left\langle \frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_n}(p) \right\rangle$
  - $\overline{\nabla} f \approx f'(x_o)\Delta x$

2 Linear Approximation, Tangent Planes, and the Differential

- In calculus I, the linear approximation is given by: $f(x) \approx f(a) + f'(a)(x - a)$
- In calculus III, the approximation uses the gradient: $\Delta f \approx \overline{\nabla} f(p) \cdot \Delta \overline{x}$
- *Ex. in* $\mathbb{R}^2$ $z = f(x, y)$, $p = (a, b)$:

  $\Delta f \approx \overline{\nabla} f(a, b) \cdot \langle x - a, y - b\rangle \Rightarrow f(x, y) - f(a, b) \Rightarrow f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Thus:

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We find the linearization of $f(x, y)$ near (a, b) (in $\mathbb{R}^2$)

- Linearization of $f(x, y)$ near (a, b) is denoted by $L_f(\overline{x}, \overline{p})$, where p is the vector $\langle a, b\rangle$

- *Ex. Given a cylinder of radius $r = .5$ and a height of $h = 1$, estimate the change in volume when the radius is increased by .1 and the height is decreased by .1*

$$V = \pi r^2 h \Rightarrow \Delta V \approx 2\pi(.5)(1)(r - .5) + \pi(.5)^2(h-1) = \pi(r - .5) + .25\pi(h-1) \approx$$

$$-.75\pi + \pi r + .25\pi h \Rightarrow -.75\pi + \pi(.6) + .25\pi(.9) = .075\pi$$

- The graph of $z = L_f(\overline{x}, \overline{p})$ is called the tangent set to f at p
- Differentials
  - $df = \overline{\nabla} f \cdot d\overline{x}$
  - $df = f_x dx + f_y dy + f_z dz$
  - $df = \dfrac{\partial f}{\partial x_1} dx_1 + \dfrac{\partial f}{\partial x_2} dx_2 + \cdots + \dfrac{\partial f}{\partial x_n} dx_n$
- Relative Differentials
  - $\dfrac{df}{f}$
  - Think of this as a stencil for relative error $\left(\dfrac{\Delta f}{f}\right)$