Linear Models – Initial-Value Problems

Michael Brodskiy
Professor: Meetal Shah

October 26, 2020

- This section will focus on several linear dynamical systems, modeled by second-order differential equations
- The function g is variously called the driving function, forcing function, or input of the system
- A solution $y(t)$ of the differential equation on an interval I containing $t = 0$ that satisfies the initial conditions is called the response or output of the system
- A Spring/Mass System:

  1. Newton's Second Law: When a mass m is attached to the lower end of a spring of negligible mass, it stretches the spring by an amount s and attains an equilibrium position or rest position at which its weight W is balanced by the restoring force ks of the spring. Hooke's Law is shown in (1)

$$F = -kx \tag{1}$$

- When the spring is in free motion, or when no external forces act on the system, Newton's second law gives (2)

$$m\frac{d^2x}{dt^2} = -k(x+s) + mg = -kx + mg - ks = -kx \tag{2}$$

- For simple harmonic motion, the differential equation looks like (3), where $\omega^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{3}$$

- The general solution for this type of equation is (4)

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t \tag{4}$$

- The period of motion can be found using $T = \frac{2\pi}{\omega}$. The frequency of motion is the inverse of the period, $f = \frac{\omega}{2\pi}$
- The number $\omega = \sqrt{\frac{k}{m}}$ is the circular frequency, in radians per second
- Functions in form (3) are problematic, though, because it is difficult to determine the amplitude. It can be rewritten in form (5) using $A = \sqrt{c_1^2 + c_2^2}$, and ϕ is a phase angle which may be found using $\tan\phi = \frac{c_1}{c_2}$

$$x(t) = A\sin(\omega t + \phi) \tag{5}$$

- Double Spring Systems:
  1. Springs in parallel:
     (a) The effective spring constant is $k_{eff} = k_1 + k_2$
     (b) Once k_{eff} is found, the whole process is the same as a single-springed system
  2. Springs in series:
     (a) $-k_{eff}(x_1 + x_2) = -k_1 x_1 = -k_2 x_2$, because the force exerted on each spring is the same
     (b) Simplifying this, we get $k_{eff} = \frac{k_1 k_2}{k_1 + k_2}$
- Systems with Variable Spring Constants
  1. Aging Spring Function is shown in (6)

$$K(t) = ke^{-\alpha t} \tag{6}$$

  2. In an environment where the temperature is rapidly decreasing, $K(t) = kt$, and Airy's differential equation can be used to model this: (7)

$$mx'' + ktx = 0 \tag{7}$$

- In damped motion, the object has a retarding force act on it
- For damped motion, the object follows the differential equation (8), and can be rewritten as (9)

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} \tag{8}$$

$$\begin{gathered} \frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \\ \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \\ 2\lambda = \frac{\beta}{m} \text{ and } \omega^2 = \frac{k}{m} \end{gathered} \tag{9}$$

- Therefore, for the differential equation shown in equation (9), the terms of the complementary solution can be found using (10)

$$
\begin{aligned}
\text{Case One (Overdamped): } & \lambda^2-\omega^2>0 \\
& m_{1,2}=-\lambda\pm\sqrt{\lambda^2-\omega^2} \\
& x(t)=e^{-\lambda t}\left(c_1e^{\sqrt{\lambda^2-\omega^2}t}+c_2e^{-\sqrt{\lambda^2-\omega^2}}\right) \\
\text{Case Two (Critically Damped): } & \lambda^2-\omega^2=0 \\
& m_{1,2}=0 \\
& x(t)=e^{-\lambda t}(c_1+c_2t) \\
\text{Case Three (Underdamped): } & \lambda^2-\omega^2<0 \\
& m_{1,2}=-\lambda\pm\sqrt{\omega^2-\lambda^2}i \\
& x(t)=e^{-\lambda t}\left(c_1\cos\sqrt{\omega^2-\lambda^2}t+c_2\sin\sqrt{\omega^2-\lambda^2}t\right)
\end{aligned} \tag{10}
$$

- For such problems, it is important to remember $g\approx 32\left[\frac{ft}{s}\right]$
- The term $Ae^{-\lambda t}$ is called the damped amplitude of vibrations. The quasi period and the quasi frequency are defined as $\frac{2\pi}{\sqrt{\omega^2-\lambda^2}}$ and $\frac{\sqrt{\omega^2-\lambda^2}}{2\pi}$
- In a system with driven or forced motion, $f(t)$ is defined as a function representing a force at a time. The differential equation is then modified to be (11)

$$\frac{d^2x}{dt^2}=-2\lambda\frac{dx}{dt}-\omega^2x+f(t) \tag{11}$$

- The complementary function is called the transient solution, while the particular function is called the steady-state solution
- Driven motion without damping motion can be written in form (12)

$$\frac{d^2x}{dt^2}+\omega^2x=f(t) \tag{12}$$

- When $\gamma=\omega$, the function reaches what is known as pure resonance
- Series Circuits in Analogue − LRC-Series Circuits may be modelled by a second-order differential equation (13)

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t) \tag{13}$$

- Inductor (L) − Units of henries (h). Voltage drop across: $L\frac{di}{dt}$
- Resistor (R) − Units of ohms (ω). Voltage drop across: iR

- Capacitor (C) $-$ Units of farads (f). Voltage drop across: $q\frac{1}{C}$
- If the $E(t)$ function equals zero, the electrical vibrations of the circuit are saidd to be free.
- The circuit is...

  Overdamped if $R^2 - \frac{4L}{C} > 0$

  Critically damped if $R^2 - \frac{4L}{C} = 0$

  Underdamped if $R^2 - \frac{4L}{C} < 0$
- When the equation is underdamped and $q(0) = q_0$, the charge of the capacitor oscillates as it decays.
- When $E(t) = 0$ and $R = 0$, the circuit is undamped and the circuit is in a simple harmonic state.
- When there is an impressed voltage $E(t)$ on the circuit the electrical vibrations are said to be forced.
- In the case when $R \neq 0$, the complementary function $q_c(t)$ is called a transient solution.
- If $E(t)$ is periodic or a constant, then the particular solution $q_p(t)$ is a steady-state solution.