Preliminary Theory – Linear Equations

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- Boundary-value Problem (1):

$$\begin{aligned} \text{Solve: } & a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \\ \text{Subject to: } & y(a) = y_0, y(b) = y_1 \end{aligned} \tag{1}$$

- $y(a) = y_0$ and $y(b) = y_1$ are called boundary values
- If the function purely of x (on the right side) in a linear differential equation is equal to zero, that function is called homogeneous. When the term is not zero, it is called nonhomogeneous.
- The symbol D is called the differential operator. Ex. (2)

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(D_y) = D^2y \tag{2}$$

- In addition to this, we can define an **nth-order differential operator** or polynomial operator (or linear operator)to be (3)

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta L(g(x)) \tag{3}$$

- An example which combines both aforementioned operators: Ex (4)

$$\begin{aligned} y'' + 5y' + 6y &= 5x - 3 \\ D^2y + 5D_y + 6y &= 5x - 3 \\ (D^2 + 5D + 6)y &= 5x - 3 \\ L(y) &= 5x - 3 \end{aligned} \tag{4}$$

- Many combinations of solutions may be found if $y_1, y_2 \ldots, y_k$ are solutions and are combined using a linear combination (5)

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x) \tag{5}$$

- A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous differential equation is also a solution

- A homogeneous linear differential equation always possesses the trivial solution $y = 0$

- A set of functions $f_1(x), f_2(x) \ldots, f_n(x)$ is said to be linearly dependent on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero such that:

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

- For every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be linearly independent.

- The Wronskian: (6)

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & \ldots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \ldots & f_n^{n-1} \end{vmatrix} \tag{6}$$

- If the Wronskian does not equal zero, the solutions are said to be linearly independent

- Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation on an interval I is said to be a fundamental set of solutions on the interval

- The general solution of the equation on the interval is (7)

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \tag{7}$$

- The general solution of the nonhomogeneous equation is (8)

$$\begin{aligned} y &= \text{complementary function} + \text{particular solution} \\ &= y_c + y_p \end{aligned} \tag{8}$$