

# Operational Properties II

Michael Brodskiy

Professor: Meetal Shah

November 16, 2020

- The derivative of a transform is:  $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$
- A convolution (f convolves g) is defined as (1)

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau \quad (1)$$

- For all examples, the Laplace transform of a convolution, or  $\mathcal{L}\{f * g\}$ , we obtain (2)

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} = F(s)G(s) \quad (2)$$

- Inversely, the inverse Laplace transform would mean (3)

$$f * g = \mathcal{L}^{-1} F(s)G(s) \quad (3)$$

- When  $g(t) = 1$  and  $\mathcal{L}\{g(t)\} = G(s) = \frac{1}{s}$ , the convolution theorem implies that the Laplace transform of the integral of  $f$  is (4)

$$\begin{aligned} \mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} &= \frac{F(s)}{s} \\ \int_0^t f(\tau) d\tau &= \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \end{aligned} \quad (4)$$

- The Volterra integral equation for  $f(t)$  is (5)

$$f(t) = g(t) + \int_0^t f(\tau) h(t - \tau) d\tau \quad (5)$$

- For LRC circuits, we could set up an integral equation to solve. This would look like (6), which is known as an integrodifferential equation

$$L \frac{di}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = E(t) \quad (6)$$

- Green's Function Redux states that, given a differential equation,  $y'' + ay' + by = f(t)$ , with  $y(0) = 0, y'(0) = 0$ , we attain (7)

$$Y(s) = \frac{F(s)}{s^2 + as + b} \quad (7)$$

- From there, we can use convolution, as  $\mathcal{L}^{-1} \left\{ \frac{1}{s^2 + as + b} \right\} = g(t)$  and  $\mathcal{L}^{-1} \{F(s)\} = f(t)$
- The transform of a periodic function becomes (8)

$$\mathcal{L} \{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt \quad (8)$$

*Example:*

$$\text{Given } E(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \end{cases}, \text{ find } \mathcal{L} \{E(t)\}$$

$$\mathcal{L} \{E(t)\} = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} E(t) dt \quad (9)$$