Linear Models – Boundary-Value Problems

Michael Brodskiy

Professor: Meetal Shah

October 28, 2020

- In the absence of any load on the beam (including its weight), a curve joining the centroids of all its cross sections is a straight line called the axis of symmetry
- When a load is applied on a beam, the line connecting all centroids is called the deflection curve or elastic curve.o
- An important formula relates the bending moment, $M(x)$ at a point x on the beam is related to the load per unit length $w(x)$ (1)

$$\frac{d^2M}{dx^2} = w(x) \tag{1}$$

- The product EI is called the flexural rigidity of the beam (2)

$$M(x) = EI\kappa \tag{2}$$

- From Calculus III, the curvature is $\kappa = \frac{y''}{[1+(y')^2]^{\frac{3}{2}}}$
- When the deflection $y(x)$ is small, the slope $y' \approx 0$ and so $\kappa \approx y''$, so formula (2) is $M(x) = EIy''$ This means that the equation may be written as (3)

$$EI\frac{d^4y}{dx^4} = w(x) \tag{3}$$

- Boundary conditions depend on how the ends of the beam are structured. Embedded or clamped at one end and free at the other means the beam is stuck at one point. Some things we can assume in such a case at the embedded end is:
  1. $y(0) = 0$ because there is no deflection
  2. $y'(0) = 0$ because the deflection curve is tangent to the x-axis (the slope is zero)
  3. At the free end: $y''(L) = 0$
  4. $y'''(L) = 0$

- There are three possible conditions for beams: embedded at both ends, embedded at one end, and simply supported at both ends

Embedded	$y = 0$	$y' = 0$
Free	$y'' = 0$	$y''' = 0$
Simply supported	$y = 0$	$y'' = 0$

- The numbers $\lambda_n = \frac{n^2\pi^2}{L^2}$, where $n = 1, 2, 3 \ldots$, for which a boundary-value problem possesses nontrivial solutions are known as eigenvalues
- Nontrivial solutions that depend on these values of λ_n, $y_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right)$ or simply $y_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ are called eigenfunctions
- The formula for a buckling, vertical column, where P is the downward force is (4)

$$EI\frac{d^2y}{dx^2} + Py = 0 \tag{4}$$

- For vertical beams, $\lambda = \frac{P}{EI}$
- The first value for which the vertical beam will buckle, or the Euler load, is $P_1 = \frac{\pi^2 EI}{L}$, and is known as the first buckling node
- The simple linear second-order differential equation defined in (5) occurs over and over again in mathematics.

$$y'' + \lambda y = 0 \tag{5}$$

- In a spinning string, the total force at a point is:

$$F = T\sin\theta_2 - T\sin\theta_1 \tag{6}$$

- When angles θ_1 and θ_2 are small, $\sin\theta_2 \approx \tan\theta_2$. Therefore, the tangent at a certain point is approximately the derivative, meaning that the derivative at the two points is approximately equal to the sin of the angle, or (7)

$$F \approx T[y'(x + \Delta x) - y(x)] \tag{7}$$

- Where ρ is the unit density per length, and $a_c = r\omega^2$, and the height being $r = y$, then, with a small Δx, the equation for force becomes (8)

$$F \approx -(\rho\Delta x)y\omega^2 \tag{8}$$

- Therefore, when Δx is close to zero, the equation finally becomes (9)

$$T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0 \tag{9}$$