Operational Properties

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- If $\mathcal{L}\{f(t)\} = F(s)$ and a is any real number, then:

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \tag{1}$$

- This theorem works both ways, as:

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\} \tag{2}$$

Example:

$$\begin{aligned} &\mathcal{L}^{-1}\left\{\frac{\frac{s}{2}+\frac{5}{3}}{s^2+4s+6}\right\} \\ &= \frac{1}{2}\left(\frac{s+2}{(s+2)^2+2}\right) + \frac{2}{3}\left(\frac{1}{(s+2)^2+2}\right) \\ s \to s+2 &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{2}{3\sqrt{2}}\mathcal{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} \\ &= \frac{1}{2}e^{-2t}\cos\sqrt{2}t + \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t \end{aligned} \tag{3}$$

- The unit step function, $\mathcal{U}(t-a) = \begin{cases} 0 & 0 \le t < a \\ 1 & t \ge a \end{cases}$, may be used to represent functions.

- For example, given function: $f(t) = \begin{cases} g(t) & 0 \le t \le a \\ h(t) & t \ge a \end{cases}$, one may represent it using step functions in the way expressed in (4). As such, the function is $h(t)$ when $t \ge a$, and $g(t)$ when $0 \le t < a$.

$$f(t) = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) \tag{4}$$

- The Second Translation Theorem holds that: $\mathcal{L}\left\{f(t-a)\mathcal{U}(t-a)\right\} = e^{-as}F(s)$
- The Inverse of the Second Translation Theorem holds that: $f(t-a)\mathcal{U}(t-a) = \mathcal{L}^{-1}\left\{e^{-as}F(s)\right\}$
- A better way of figuring out $\mathcal{L}\left\{g(t)\mathcal{U}(t-a)\right\}$ is the following:

$$\begin{aligned}\int_a^\infty e^{-st}g(t)\,dt &= \int_0^\infty e^{-s(u+a)}g(u+a)\,du \\ &= e^{-as}\mathcal{L}\left\{g(t+a)\right\}\end{aligned} \tag{5}$$

- Now, the Laplace Transform may be applied to one of the previous sections, where $EI\frac{d^4y}{dx^4} = w(x)$, where E is Young's modulus of elasticity and I is a moment of inertia of a cross section of the beam
- Given $w(x) = \begin{cases} w_0\left(1 - \frac{2}{L}x\right) & 0 < x < \frac{L}{2} \\ 0 & \frac{L}{2} < x < L \end{cases}$ means that the beam is embedded at both ends, with force applied only to the left, meaning that $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, and $y'(L) = 0$