Exact Equations

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- Recall that the **differential** is defined as:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- $f(x, y) = c$ is said to be a solution of the differential equation:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- A differential expression $M(x, y)\, dx + N(x, y)\, dy$ is an exact differential if it corresponds to the differential of some function $f(x, y)$.
- $M(x, y)\, dx + N(x, y)\, dy$ is said to be an exact equation if the expression on the left-hand side is an exact differential
- If $M(x, y)$ and $N(x, y)$ are continuous and have continuous first partial derivatives, then, to be an exact differential:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- If the equality above holds true, then a solution exists for a function f for which:

$$\frac{\partial f}{\partial x} = M(x, y)$$

- We can find f by integrating $M(x, y)$ with respect to x while holding y constant:

$$f(x, y) = \int M(x, y)\, dx + g(y) \text{ where } g(y) \text{ represents the "constant" of integration}$$

- This can be confirmed when we differentiate with respect to x:

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\int M(x,y)\,dx + \frac{\partial g(y)}{\partial x}$$
$$\frac{\partial f(x,y)}{\partial x} = M(x,y)$$

- And when we differentiate with respect to y:

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + \frac{\partial g(y)}{\partial y}$$
$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y)$$
$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

- The integrating factor technique may be used for exact equations as well:

$$\mu_x N - \mu_y M = (M_y - N_x)\mu$$

- For $\mu_x = d\mu/dx, \mu_y = 0$, then:

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$$
$$\therefore \mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

- If, conversely $\mu_y = d\mu/dy, \mu_x = 0$, then:

$$\frac{d\mu}{dy} = \frac{N_x - M_y}{M}\mu$$
$$\therefore \mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$

- This method may be used on non-exact DEs to make them exact