Variation of Parameters

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- The general solution of a first order linear DE is of form (1)

$$y = c_1 e^{-\int P(x)\,dx} + e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx} f(x)\,dx \tag{1}$$

- A way to attain a solution is **Variation of Parameters**.
- For a second order DE, use $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$
- Cramer's Rule: (2)

$$\begin{aligned}
u_1' &= \frac{W_1}{W} = -\frac{y_2 f(x)}{W} \\
u_2' &= \frac{W_2}{W} = -\frac{y_1 f(x)}{W} \\
W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
W_1 &= \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}
\end{aligned} \tag{2}$$

- Steps to solving a differential equation using Variation of Parameters:
  1. Solve the auxiliary equation to obtain the complimentary function
  2. Compute the Wronskian
  3. Find the other Wronskians, W_1 and W_2
  4. Integrate and solve for u_1 and u_2
  5. Form the particular solution

6. Add to the complimentary solution and add

- A simplified for for solving this is:

$$u_1(x) = -\int_{x_0}^{x} \frac{y_2(t)f(t)}{W(t)}\,dt \text{ and } u_2(x) = \int_{x_0}^{x} \frac{y_1(t)f(t)}{W(t)}\,dt \tag{3}$$

- For a Third-Order DE:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}$$