

Homework 5

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1. Four point charges are located a distance a from the origin on the y and z axes as shown. Find the approximate expression for the electric **potential** far from the charges. Use spherical coordinates and retain the only the **first** non-vanishing terms in the multipole expansion. [Hint: consider breaking down the distribution into a superposition of individual dipoles.]

We can write the dipole moment from the configuration of charges:

$$\begin{aligned} p &= -2q(-a_y) + 3q(a_z) - 2q(a_y) + q(-a_z) \\ &= \cancel{2qa a_y} + 3qa a_z - \cancel{2qa(a_y)} - aqa_z \\ &= 2qa_z \end{aligned}$$

We know that:

$$\begin{aligned} V_{mon}(r) &= \frac{Q}{4\pi\epsilon_o r^2} = 0 \\ V_{dip}(r) &= \frac{\vec{p}\vec{r}}{4\pi\epsilon_o r^2} \end{aligned}$$

Thus, we may write the dipole as:

$$V_{dip}(r) = \frac{2qa_z \hat{r}}{4\pi\epsilon_o r^2}$$

This can finally be written in spherical coordinates using:

$$\boxed{V_{dip}(r, \theta) \approx \frac{qa \cos(\theta)}{2\pi\epsilon_o r^2}}$$

2. Three point charges are located a distance a from the origin on the y and z axes as shown. Find the approximate expression for the electric **field** far from the charges. Use spherical coordinates and retain the **first two** non-vanishing terms in the multipole expansion.

We begin by working from the lowest order up, which is the monopole. First we find the aggregate charge:

$$Q = -q + q - q = -q$$

This can simply be plugged into the formula:

$$V_{mon}(r) = -\frac{q}{4\pi\epsilon_0 r}$$

We then find the dipole contribution, beginning with the moment:

$$\begin{aligned} p &= -q(-a_y) + q(a_z) - q(a_y) \\ &= \cancel{qa_y} + qa_z - \cancel{qa_y} \\ &= qa_z \end{aligned}$$

We then use the formula:

$$\begin{aligned} V_{dip}(r) &= \frac{\vec{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \\ V_{dip}(r) &= \frac{qa_z \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \\ V_{dip}(r, \theta) &= \frac{qa \cos(\theta)}{4\pi\epsilon_0 r^2} \end{aligned}$$

Summing the two contributions, we get:

$$V(r, \theta) \approx -\frac{q}{4\pi\epsilon_0 r} + \frac{qa \cos(\theta)}{4\pi\epsilon_0 r^2}$$

We now use the formula:

$$\begin{aligned} \vec{E} &= -\vec{\nabla} V \\ \vec{E} &= \vec{\nabla} \left(\frac{q}{4\pi\epsilon_0 r} - \frac{qa \cos(\theta)}{4\pi\epsilon_0 r^2} \right) \\ \vec{E} &= \frac{\partial}{\partial r} \left(\frac{q}{4\pi\epsilon_0 r} - \frac{qa \cos(\theta)}{4\pi\epsilon_0 r^2} \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left(-\frac{qa \cos(\theta)}{4\pi\epsilon_0 r^2} \right) \\ \vec{E} &= \left(-\frac{q}{4\pi\epsilon_0 r^2} + \frac{qa \cos(\theta)}{2\pi\epsilon_0 r^3} \right) \hat{\mathbf{r}} + \left(\frac{qa \sin(\theta)}{4\pi\epsilon_0 r^3} \right) \hat{\theta} \end{aligned}$$

Finally, this yields:

$$\boxed{\vec{E}(r, \theta) = \frac{q}{2\pi\epsilon_o r^2} \left[\left(-\frac{1}{2} + \frac{a \cos(\theta)}{r} \right) \hat{\mathbf{r}} + \left(\frac{a \sin(\theta)}{2r} \right) \hat{\boldsymbol{\theta}} \right]}$$

3. For the charged spherical shell in Problem 4 of Assignment 4 (the one with $V(R, \theta) = V_o \sin^2(\theta)$ and $\sigma = \frac{V_o \epsilon_o}{3R} (7 - 15 \cos^2(\theta))$, find the monopole and dipole moments.
4. A thin rod on the z -axis goes from $z = -a$ to $z = +a$ and carries a linear charge density of $\lambda(z)$. Find the leading term in the multipole expansion for:

(a) $\lambda(z) = \lambda_o \cos\left(\frac{\pi z}{a}\right)$

We can write the formula for the multipole expansion as:

$$V = \frac{1}{4\pi\epsilon_o} \sum_{n=0}^{\infty} \frac{P_n(\cos(\theta))}{r^{n+1}} \int_{-a}^a z^n \lambda(z) dz$$

For the $n = 0$ case, we can write:

$$V = \frac{1}{4\pi\epsilon_o r} \int_{-a}^a \lambda(z) dz$$

$$V = \frac{\lambda_o}{4\pi\epsilon_o r} \int_{-a}^a \cos\left(\frac{\pi z}{a}\right) dz$$

From this, we can see that the integral expression would evaluate to zero, meaning we have to try the next term. At $n = 1$, we get:

$$V = \frac{\cos(\theta)}{4\pi\epsilon_o r^2} \int_{-a}^a z \lambda(z) dz$$

$$V = \frac{\lambda_o \cos(\theta)}{4\pi\epsilon_o r^2} \int_{-a}^a z \cos\left(\frac{\pi z}{a}\right) dz$$

$$V = \frac{\lambda_o \cos(\theta)}{4\pi\epsilon_o r^2} \left(\underbrace{\frac{a}{\pi} z \sin\left(\frac{\pi z}{a}\right)}_0 + \frac{a^2}{\pi^2} \cos\left(\frac{\pi z}{a}\right) \right) \Big|_{-a}^a$$

$$V = \frac{\lambda_o \cos(\theta)}{4\pi\epsilon_o r^2} \left(\frac{a^2}{\pi^2} \cos\left(\frac{\pi z}{a}\right) \right) \Big|_{-a}^a$$

Once again, we see that the integral expression evaluates to zero. Thus, we move up another order to $n = 2$:

$$V_{quad}(r, \theta) = \frac{1}{4\pi\epsilon_o} \frac{3 \cos^2(\theta) - 1}{2r^3} \int_{-a}^a z^2 \lambda(z) dz$$

$$V_{quad}(r, \theta) = \frac{\lambda_o}{4\pi\epsilon_o} \frac{3 \cos^2(\theta) - 1}{2r^3} \int_{-a}^a z^2 \cos\left(\frac{\pi z}{a}\right) dz$$

Using a numerical solver, we calculate the integral:

$$V_{quad}(r, \theta) = \frac{\lambda_o}{4\pi\epsilon_o} \frac{3\cos^2(\theta) - 1}{2r^3} \underbrace{\left(\frac{z^2 a}{\pi} \sin\left(\frac{\pi z}{a}\right) + \frac{2a^2 z}{\pi^2} \cos\left(\frac{\pi z}{a}\right) - \frac{2a^3}{\pi^2} \sin\left(\frac{\pi z}{a}\right) \right)}_{-\frac{4a^3}{\pi^2}} \Big|_{-a}^a$$

Thus, the leading term becomes:

$$V_{quad}(r, \theta) = -\frac{a^3 \lambda_o (3\cos^2(\theta) - 1)}{2\pi^3 \epsilon_o r^3}$$

(b) $\lambda(z) = \lambda_1 \cos\left(\frac{\pi z}{2a}\right)$

where λ_o and λ_1 are constants.

For $\lambda(z) = \lambda_1 \cos\left(\frac{\pi z}{2a}\right)$, we can begin by finding the first term, according to the formula from (a), at $n = 0$:

$$V_{mon} = \frac{1}{4\pi\epsilon_o} \frac{1}{r} \int_{-a}^a \lambda(z) dz$$

$$V_{mon} = \frac{1}{4\pi\epsilon_o} \frac{\lambda_1}{r} \int_{-a}^a \cos\left(\frac{\pi z}{2a}\right) dz$$

We can then evaluate the integral:

$$\int_{-a}^a \cos\left(\frac{\pi z}{2a}\right) dz = \frac{2a}{\pi} \sin\left(\frac{\pi z}{2a}\right) \Big|_{-a}^a$$

$$= \frac{4a}{\pi}$$

Which then becomes:

$$\frac{\lambda_1}{4\pi\epsilon_o r} \left(\frac{4a}{\pi} \right)$$

And finally:

$$V_{mon}(r) = \frac{a\lambda_1}{\pi^2 \epsilon_o r}$$