Potentials and Fields

Michael Brodskiy
Professor: D. Wood

November 27, 2023

- The energy density of a field may be written as:

$$U = \frac{1}{2}\left(\varepsilon_o \vec{E}^2 + \frac{1}{\mu_o}\vec{B}^2\right)$$

- The Poynting Vector may be defined as:

$$\vec{S} = \frac{1}{\mu_o}(\vec{E} \times \vec{B})$$

- For some volume τ, we can write the total energy in it using:

$$\mathcal{U}_{tot} = \int_V U \, d\tau$$

- The Poynting Vector represents the flow of energy out of the surface

$$\vec{\nabla} \cdot \vec{S} = \frac{1}{\mu_o}\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \frac{1}{\mu_o}\left(\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E}(\vec{\nabla} \times \vec{B})\right)$$
$$\vec{\nabla} \cdot \vec{S} = -\frac{\partial}{\partial t}\left(\frac{\varepsilon_o \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_o}\right)$$

  - This can be simplified to attain:

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial U}{\partial t} = 0$$

  - A similar equation can be seen in conservation of charge:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

  - Thus, we have obtained a conservation of energy formula

- The Maxwell Stress Tensor ($\overleftrightarrow{T}$)

$$d\vec{F} = \overleftrightarrow{T} \cdot d\vec{a}$$

  - T_{ij} is the force per area in the j-direction of an area oriented in the i-direction

$$\oint_S \overleftrightarrow{T} \cdot d\vec{a} = \vec{F}_{ext}$$

$$\oint_V (\vec{\nabla} \cdot \overleftrightarrow{T}) \cdot d\tau = \vec{F}_{ext}$$

  - $T_{ij} = \varepsilon_o \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_o}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \varepsilon_o \left((\vec{\nabla} \cdot \vec{E})\vec{E} + (\vec{E} \cdot \vec{\nabla})\vec{E} - \frac{1}{2}\vec{\nabla}\vec{E}^2\right) + \frac{1}{\mu_o}\left((\vec{\nabla} \cdot \vec{B})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{B} - \frac{1}{2}\vec{\nabla}\vec{B}^2\right)$$

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \varepsilon_o \left((\vec{\nabla} \cdot \vec{E})\vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E})\right) + \frac{1}{\mu_o}\left((\vec{\nabla} \cdot \vec{B})\vec{B} - \vec{B} \times (\vec{\nabla} \times \vec{B})\right)$$

  - Here, we see all four of Maxwell's laws. This lets us rewrite:

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \underbrace{\rho\vec{E} + \mu_o \vec{J} \times \vec{B}}_{\text{Force per Volume, } \vec{f}} + \underbrace{\varepsilon_o \vec{E} \times \left(\frac{\partial \vec{B}}{\partial t}\right) + \varepsilon_o \vec{B} \times \left(\frac{\partial \vec{E}}{\partial t}\right)}_{\frac{\partial}{\partial t}(\varepsilon_o \vec{E} \times \vec{B})}$$

  - We can ultimately simplify to get:

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \vec{f} - \frac{\partial}{\partial t}\mu_o \varepsilon_o \vec{S}$$

  - We can rewrite this as:

$$\frac{d\vec{p}}{dt} = F_{ext} - \frac{\partial}{\partial t}\int \varepsilon_o \mu_o \vec{S}\, d\tau$$

  - The momentum density may be written as:

$$\vec{g} = \frac{\vec{p}}{V} = \varepsilon_o \mu_o \vec{S} = \varepsilon_o \vec{E} \times \vec{B}$$

  - This can be summed up as:

$$-\vec{\nabla} \cdot \overleftrightarrow{T} + \frac{\partial \vec{g}}{\partial t} = 0$$

  - Which is the continuity and conservation of momentum

- Radiation Pressure

- For Absorption:

$$P = \vec{g}c \cdot \hat{\mathbf{n}}$$

- For Reflection:

$$P = 2\vec{g}c \cdot \hat{\mathbf{n}}$$

- The time-averaged pressure may be defined as:

$$\langle P \rangle = \frac{1}{T}\int_0^T P \cdot dt$$

- Angular Momentum
  - The angular momentum may be defined as:

  $$\vec{L} = \vec{r} \times \vec{p}$$

  - To find the angular momentum density, we may write:

  $$\vec{l} = \vec{p} \times \vec{g}$$

  - The torque on the inner cylinder of a coaxial cable may be written as:

  $$N = R_1 q E_\phi = \frac{q\varepsilon}{2\pi} = -\frac{q}{2\pi}\frac{d\phi_1}{dt}$$

  - The change in angular momentum may be written as:

  $$\Delta L_1 = \frac{qB_i R_1^2}{2}$$

  - For the outer cylinder, the change would be the same, except with $R_1 \to R_2$

  $$\Delta L_{tot} = \frac{qB_i}{2}(R_1^2 - R_2^2)$$
  $$\vec{l} = \frac{qB_i r}{2\pi s \Delta z}\hat{\theta}$$
  $$l_z = \frac{qB_i}{2\pi \Delta z}$$

  - We can see that angular momentum is conserved if and only if L_{EM} is included

- Relativity
  - Motion perpendicular to $\vec{E}$-field increases strength by a factor γ[1]

[1]Note: $\gamma = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$

- Creates a $\vec{B}$-field
- Motion parallel to $\vec{E}$-field does not change the $\vec{E}$-field, but produces a $\vec{B}$-field

- 4-Vectors

$$X^\mu = \mathbb{X} = (x_o, x_{1,2}, x_3)$$

  - Where $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_o = ct$
  - We can rewrite this for momentum as:

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

  - To switch reference frames, we can write:

$$p^\mu = \sum_{\nu=0}^{3} \lambda^\mu_\nu p^\nu$$

- 4-Vectors in Electricity and Magnetism
  - $J^\mu = (J_o, \vec{J})$, where $J_o = c\rho$
  - $A^\mu = (A_o, \vec{A})$, where $A_o = \frac{V}{c}$

- For a tensor transformation, we may write:

$$F^{\mu,\nu} = \sum_{\sigma,\rho=0}^{3} \lambda^\mu_\sigma F^{\sigma,\rho} \lambda^\nu_\rho$$

- Electric and magnetic fields are invariant (so is $E^2 - c^2B^2$ — that is to say, the magnitudes are also invariant)