Homework 4

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1. Consider an infinite grounded conducting plane bent at a 90° angle between the yz and xz planes as shown, with a charge placed at $x = 4a$, $y = a$. Use appropriate image charge(s) to find an expression for the potential $V(x, y, z)$ in the region $x > 0$, $y > 0$.

   First, we know that the image charges assume the following layout:

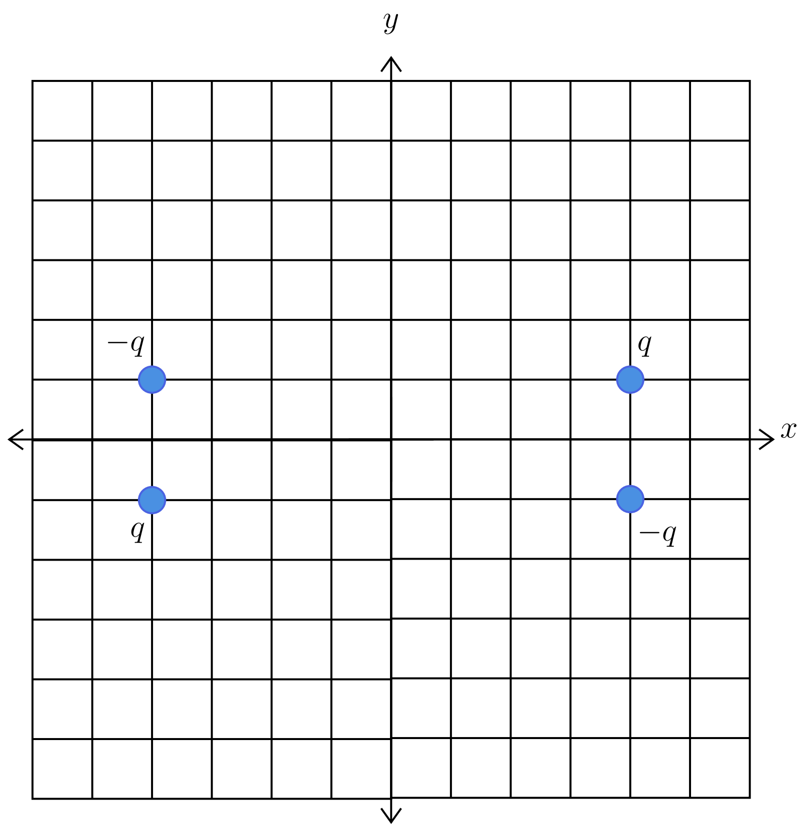

Figure 1: The Layout of Image Charges

By the principle of superposition, we know that the voltage at a point will simply be the sum of all voltages. Thus, we can assign each charge, starting with the top left, a distance $d_1 - d_4$. We can find the following:

$$\begin{aligned}
d_1 &= \sqrt{(x+4a)^2 + (y-a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 + 8ax - 2ay} \\
d_2 &= \sqrt{(x-4a)^2 + (y-a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 - 8ax - 2ay} \\
d_3 &= \sqrt{(x+4a)^2 + (y+a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 + 8ax + 2ay} \\
d_4 &= \sqrt{(x-4a)^2 + (y+a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 - 8ax + 2ay}
\end{aligned}$$

We can then write the voltage as:

$$\begin{aligned}
V &= V_1 + V_2 + V_3 + V_4 \\
&= \frac{q}{4\pi\varepsilon_o}\left[-\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} - \frac{1}{d_4}\right]
\end{aligned}$$

This gives us the final expression:

$$\begin{aligned}
V(x,y,z) = \frac{q}{4\pi\varepsilon_o}\Bigg[&\frac{1}{\sqrt{(x-4a)^2 + (y-a)^2 + z^2}} + \frac{1}{\sqrt{(x+4a)^2 + (y+a)^2 + z^2}} \\
&- \frac{1}{\sqrt{(x+4a)^2 + (y-a)^2 + z^2}} - \frac{1}{\sqrt{(x-4a)^2 + (y+a)^2 + z^2}}\Bigg]
\end{aligned}$$

2. The boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$ is held at a constant potential $+V_0$ and the other, from $y = a/2$ to $y = a$ is held at a constant potential of V_0 . Solve for the potential $V(x, y, z)$ inside the slot. Feel free to use the relevant results from Example 3.3 or from lecture as a starting point.

   We can first write the Laplace equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

   Given the boundary conditions, we may write:

$$V(x,y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

   We find the value of c_n by rearranging:

$$c_n = \frac{2}{a}\left[\int_0^{\frac{a}{2}} V_o \sin\left(\frac{n\pi y}{a}\right)\, dy - \int_{\frac{a}{2}}^{a} V_o \sin\left(\frac{n\pi y}{a}\right)\, dy\right]$$

$$c_n = \frac{2V_o a}{an\pi}\left[\left(-\cos\left(\frac{n\pi y}{a}\right)\Big|_0^{\frac{a}{2}}\right) + \left(\cos\left(\frac{n\pi y}{a}\right)\Big|_{\frac{a}{2}}^{a}\right)\right]$$

$$c_n = \frac{2V_o}{n\pi}\left[1 + \cos(n\pi) - 2\cos\left(\frac{n\pi}{2}\right)\right]$$

From this, we can see that $c_n = 0$ for any odd values of n. Furthermore, if n is a multiple of 4, $c_n = 0$ as well. Thus, we can see that c_n is non-zero only for $n = 2, 6, 10, \ldots$ for which:

$$c_n = \frac{8V_o}{n\pi}$$

We can now substitute into our previous equation to obtain:

$$\boxed{V(x, y) = \frac{8V_o}{\pi} \sum_{n=2,6,10,\ldots}^{\infty} \frac{e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}{n}}$$

3. Consider a long (semi-infinite) rectangular conducting pipe oriented V_0 parallel to the z-axis, with dimensions $a \times b$ in the xy-plane. The pipe itself is grounded, and the rectangle at the closed end is at a constant potential V_0 . Find an expression for the potential everywhere inside the pipe (for $z > 0$).

   For this problem, we must apply a three dimensional Laplace equation, with boundary conditions:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \Rightarrow \begin{cases} V = 0, & \left[\begin{array}{l} x = 0 \\ x = a \\ y = 0 \\ y = b \end{array}\right. \\ V = V_o, & z = 0 \\ V \to 0, & z \to \infty \end{cases}$$

   We can then divide the equation by $V(x, y, z)$ to obtain:

$$\underbrace{\frac{1}{V_x}\frac{\partial^2 V_x}{\partial x^2}}_{-A^2} + \underbrace{\frac{1}{V_y}\frac{\partial^2 V_y}{\partial y^2}}_{-B^2} + \underbrace{\frac{1}{V_z}\frac{\partial^2 V_z}{\partial z^2}}_{C^2} = 0$$

   Note: the z^2 term will be positive to guarantee at least one exponentially decaying solution. This gives us:

$$C^2 = A^2 + B^2$$

$$\frac{\partial^2 V_x}{\partial x^2} = -A^2 V_x \qquad \frac{\partial^2 V_y}{\partial y^2} = -B^2 V_y \qquad \frac{\partial^2 V_z}{\partial z^2} = (A^2 + B^2) V_z$$

Given this form, we know the solutions will be of form:

$$V_x = P\sin(Ax) + Q\cos(Ax)$$
$$V_y = R\sin(Bx) + S\cos(Bx)$$
$$V_z = Te^{\sqrt{A^2+B^2}z} + Ue^{-\sqrt{A^2+B^2}z}$$

To simplify, we can now apply some of our boundary conditions from above. Let us first apply the last condition ($V_z \to 0$ as $z \to \infty$). This gives us $S = 0$:

$$V_x = P\sin(Ax) + Q\cos(Ax)$$
$$V_y = R\sin(Bx) + S\cos(Bx)$$
$$V_z = Ue^{-\sqrt{A^2+B^2}z}$$

Now, by conditions one and three, we know that, when $V = 0$, $x = 0$ and $y = 0$, giving $Q = 0$ and $T = 0$:

$$V_x = P\sin(Ax)$$
$$V_y = R\sin(Bx)$$
$$V_z = Ue^{-\sqrt{A^2+B^2}z}$$

From conditions two and four, we know that, when $V = 0$, $x = a$ and $y = b$, which gives us:

$$V_x = P\sin\left(\frac{m\pi x}{a}\right)$$
$$V_y = R\sin\left(\frac{n\pi y}{b}\right)$$
$$V_z = Ue^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}$$

Thus, we get $V(x, y, z)$:

$$V(x, y, z) = PRU\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}$$

We can assume PRU is some constant, which will be expressed as M:

$$V(x, y, z) = M\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}$$

We can find all values of M by summing and replacing M with M_{mn}:

$$V(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}$$

Applying the final boundary condition, or $V = V_o$ when $z = 0$, we can obtain:

$$V_o = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Now, we multiply both sides by $\sin\left(\frac{m'\pi x}{a}\right)$ and $\sin\left(\frac{n'\pi y}{b}\right)$ and integrate to get:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M_{mn}\int_0^a\int_0^b \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)\,dx\,dy \\ = \int_0^a\int_0^b V_o\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)\,dx\,dy\end{aligned}$$

Then we get:

$$\begin{aligned}M_{mn}\frac{a}{2}\delta_{mm'}\frac{b}{2}\delta_{nn'} &= \int_0^a\int_0^b V_o\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)\,dx\,dy \\ M_{m'n'} &= \frac{4}{ab}\int_0^a\int_0^b V_o\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)\,dx\,dy\end{aligned}$$

We can replace all of the m' and n' by m and n again, since we effectively removed all of the m and n's from the equation:

$$M_{mn} = \frac{4V_o}{ab}\int_0^a\int_0^b \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\,dx\,dy$$

Analyzing the equations, we can see that, when m or n is odd, $M_{mn} = 0$, and, if m and n are both even, then:

$$M_{mn} = \frac{16V_o}{\pi^2 mn}$$

Thus, the final solution, for $z > 0$, becomes:

$$\boxed{V(x,y,z) = \frac{16V_o}{\pi^2}\sum_{m,n=1,3,5,\ldots}^{\infty}\frac{1}{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}}$$

4. Consider an empty spherical shell of charge of radius R where the potential on the surface is given by $V(R,\theta) = V_o\sin^2(\theta)$.

   Hint: Express $\sin^2(\theta)$ as a polynomial function of $\cos(\theta)$.

(a) Find $V(r,\theta)$ inside the shell.

The potential inside and outside can be expressed as:

$$\sum_{l=0}^{\infty} a_l r^l P_l(\cos(\theta)) \qquad r < R$$

$$\sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos(\theta)) \qquad r \geq R$$

Given that $V_o \sin^2(\theta) \to [1-\cos^2(\theta)]$, we can use Legendre polynomials:

$$\begin{aligned} V(R,\theta) &= V_o \left(P_0(\cos(\theta)) - \frac{2P_2(\cos(\theta)) + P_0(\cos(\theta))}{3} \right) \\ &= \frac{2V_o}{3} \left(P_0(\cos(\theta)) - P_2(\cos(\theta)) \right) \end{aligned}$$

Written in polynomial expansion form, we find:

$$a_0 R^0 P_0(\cos(\theta)) + a_2 R^2 P_2(\cos(\theta)) = \frac{2V_0}{3} \left(P_0(\cos(\theta)) - P_2(\cos(\theta)) \right)$$

Thus, we see:

$$a_0 = \frac{2V_0}{3} \qquad\qquad a_2 = -\frac{2V_0}{3R^2}$$

Finally, we find that the potential within the shell is:

$$\boxed{V(r,\theta) = \frac{2V_0}{3} - \frac{V_0 r^2}{3R^2}[3\cos^2(\theta) - 1]}$$

(b) Find $\vec{E}(R,\theta)$ just inside the shell.

To find the electric field, we can apply:

$$\vec{E} = -\vec{\nabla} V$$

This becomes:

$$\vec{\nabla} \left[\frac{2V_0}{3} - \frac{V_0 r^2}{3R^2}[3\cos^2(\theta) - 1] \right] = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$-\frac{\partial}{\partial r}\left[\frac{2V_0}{3} - \frac{V_0 r^2}{3R^2}[3\cos^2(\theta) - 1] \right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{2V_0}{3} - \frac{V_0 r^2}{3R^2}[3\cos^2(\theta) - 1] \right]\hat{\theta}$$

Finally, we get:

$$\vec{E}(r,\theta) = \frac{2V_0 r}{3R^2}[3\cos^2(\theta) - 1]\hat{\mathbf{r}} - \frac{V_0 r}{R^2}\sin(2\theta)\hat{\theta}$$

Upon plugging in R, we obtain:

$$\boxed{\vec{E}(R,\theta) = \frac{2V_0}{3R}[3\cos^2(\theta) - 1]\hat{\mathbf{r}} - \frac{V_0}{R}\sin(2\theta)\hat{\theta}}$$

(c) Find $V(r,\theta)$ out of the shell.
Using a similar process to (a), we find:

$$\frac{b_0}{R}P_0(\cos(\theta)) + \frac{b_2}{R^3}P_2(\cos(\theta)) = \frac{2V_0}{3}\left(P_0(\cos(\theta)) - P_2(\cos(\theta))\right)$$

This gives us:

$$b_0 = \frac{2V_0 R}{3} \qquad\qquad b_2 = -\frac{2V_0 R^3}{3}$$

And finally, we end up with:

$$\boxed{V(r,\theta) = \frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3}[3\cos^2(\theta) - 1]}$$

(d) Find $\vec{E}(R,\theta)$ just outside the shell.
Similarly to (b), we can find the electric field outside the shell using:

$$\vec{E} = -\vec{\nabla}V$$

This gives us:

$$-\vec{\nabla}\left[\frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3}[3\cos^2(\theta) - 1]\right]$$

$$-\frac{\partial}{\partial r}\left[\frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3}[3\cos^2(\theta) - 1]\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3}[3\cos^2(\theta) - 1]\right]\hat{\theta}$$

Finally, we obtain:

$$\vec{E}(R,\theta) = \left(\frac{2V_0 R}{3r^2} - \frac{V_0 R^3}{r^4}[3\cos^2(\theta) - 1]\right)\hat{\mathbf{r}} - \frac{V_0 R^3}{r^4}\sin(2\theta)\hat{\theta}$$

Evaluating at $r = R$, we get:

$$\boxed{\vec{E}(R,\theta) = \left(\frac{2V_0}{3R} - \frac{V_0}{R}[3\cos^2(\theta) - 1]\right)\hat{\mathbf{r}} - \frac{V_0}{R}\sin(2\theta)\hat{\theta}}$$

(e) Find $\sigma(R,\theta)$ on the shell. [answer: $\sigma = \frac{V_o \varepsilon_o}{3R}(7 - 15\cos^2(\theta))$]
At the surface, we can assume that $r = R$, and use the following formula:

$$\frac{\sigma}{\varepsilon_o} = (E_{r,out} - E_{r,in})$$

This gives us:

$$\sigma = \frac{V_0\varepsilon_o}{R}\left(\left[\frac{2}{3} - 3\cos^2(\theta) + 1\right] - \left[2\cos^2(\theta) - \frac{2}{3}\right]\right)$$

$$\sigma = \frac{V_0\varepsilon_o}{R}\left(\frac{7}{3} - 5\cos^2(\theta)\right)$$

By factoring the one-third, we can finally obtain:

$$\sigma = \frac{V_0\varepsilon_o}{3R}\left(7 - 15\cos^2(\theta)\right)$$

5. An empty spherical shell of radius R has potential V_0 on the upper hemisphere and V_0 on the lower hemisphere

   (a) Calculate the first two non-zero terms of the expression for the potential outside of the sphere to obtain an approximate expression for $V(r,\theta)$ in this region.

   First and foremost, we know the expression for the voltage outside may be written as:

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta))$$

   The constant, B_l, may be calculated using:

$$B_l = \frac{(2l+1)}{2} R^{l+1} V_0 \left[\int_0^{\frac{\pi}{2}} P_l(\cos(\theta))\sin(\theta)\,d\theta - \int_{\frac{\pi}{2}}^{\pi} P_l(\cos(\theta))\sin(\theta)\,d\theta\right]$$

   For $l = 0$, we get:

$$B_0 = \frac{V_0 R}{2}\left[0\right] = 0$$

   For $l = 1$, we get:

$$B_1 = \frac{3V_0R^2}{2}\left[\int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{2}\,d\theta - \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(2\theta)}{2}\,d\theta\right]$$

$$B_1 = \frac{3V_0R^2}{2}$$

   For $l = 2$, we get:

$$B_2 = \frac{5V_0R^3}{2}\left[\int_0^{\pi} \frac{3\cos^2(\theta)\sin(\theta) - \sin(\theta)}{2}\,d\theta\right] = 0$$

   For $l = 3$, we get:

$$B_3 = \frac{7V_0R^4}{2}\left[\int_0^\pi \frac{5\cos^3(\theta)\sin(\theta) - 3\cos(\theta)\sin(\theta)}{2}\,d\theta\right] = -\frac{7V_0R^4}{8}$$

We now plug these coefficients into the sum to get:

$$\boxed{V(r,\theta) = \frac{3V_0R^2}{2r^2}\cos(\theta) - \frac{7V_0R^4}{16r^4}\left(5\cos^3(\theta) - 3\cos(\theta)\right)}$$

(b) From this approximate expression, compute the value of $V(R,\theta)$ (on the surface of the shell) for $\theta = 0, \theta = \pi/4$, and $\theta = 3\pi/4$ compare the results with the exact values at those locations

First and foremost, we know $r = R$, which gives us:

$$V(R,\theta) = \frac{3}{2}V_0\cos(\theta) - \frac{7}{16}V_0\left(5\cos^3(\theta) - 3\cos(\theta)\right)$$

We can then check:

- At $\theta = 0$:

$$V(R,0) = \frac{3}{2}V_0 - \frac{7}{16}V_0(2)$$

$$\boxed{V(R,0) = \frac{5}{8}V_0}$$

  Since this is in the upper hemisphere, it makes sense that it should be positive; however, it is not equal to simply V_0 as expected. Given that this is an approximation with only two terms, this difference is logical.

- At $\theta = \frac{\pi}{4}$:

$$\begin{aligned} V\left(R,\frac{\pi}{4}\right) &= \frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0\left(\frac{5(2)^{\frac{3}{2}}}{8} - \frac{3\sqrt{2}}{2}\right) \\ &= \frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0\left(\frac{5\sqrt{2}}{4} - \frac{3\sqrt{2}}{2}\right) \\ &= \frac{3\sqrt{2}}{4}V_0 + \frac{7\sqrt{2}}{64}V_0 \end{aligned}$$

$$\boxed{V\left(R,\frac{\pi}{4}\right) = \frac{55\sqrt{2}}{64}V_0}$$

  Still in the upper hemisphere, the value is still positive as expected; however, the factor is a bit above one (approximately 1.215), which means that this approximation is a bit greater than the true value, V_0.

- At $\theta = \frac{3\pi}{4}$

$$
\begin{aligned}
V\left(R, \frac{3\pi}{4}\right) &= -\frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0\left(-\frac{5(2)^{\frac{3}{2}}}{8} + \frac{3\sqrt{2}}{2}\right) \\
&= -\frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0\left(-\frac{5\sqrt{2}}{4} + \frac{3\sqrt{2}}{2}\right) \\
&= -\frac{3\sqrt{2}}{4}V_0 - \frac{7\sqrt{2}}{64}V_0
\end{aligned}
$$

$$
\boxed{V\left(R, \frac{3\pi}{4}\right) = -\frac{55\sqrt{2}}{64}V_0}
$$

Now in the lower hemisphere, we can see that the value is now negative. Similar to the π-fourths angle, we find a value slightly above (in magnitude) than we expected, as it should be $-V_0$