# Electrostatics

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- In this section, we focus on electrostatics
  - Not doing (for now):
    * Magnetic field
    * Forces on moving charges
    * Finite of propagation
- Coulomb's Law
  - Given a source charge, $q$, and a test charge, $Q$, with $\vec{R}$ as the difference between their positions ($\vec{r} - \vec{r}'$), we can generate Coulomb's Law:

$$F = \frac{qQ\hat{\mathbf{R}}}{4\pi\varepsilon_o R^2}$$

    * $\varepsilon_o$ is known as the permittivity of free space
    * $\varepsilon_o = 8.85 \cdot 10^{-12} \left[\frac{\mathrm{C}}{\mathrm{N\,m^2}}\right]$
  - A Coulomb is defined as an Ampère per second
- Superposition
  - A force per charge ($q$) can be calculated and then summed to find the total force on a test charge ($Q$)

$$\vec{F} = \sum_n \frac{Q}{4\pi\varepsilon_o} \frac{q_n\hat{\mathbf{R}}_n}{R_n^2}$$

$$\vec{F} = Q\vec{E} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_o} \sum_n \frac{q_n\hat{\mathbf{R}}_n}{R_n^2}$$

- Continuous

$$q_n \rightarrow dq = \rho(\vec{r\prime})\, d\tau$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \int \frac{1}{R^2} \hat{\mathbf{R}}\, dq = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r\prime})\, d\tau'}{R^2} \hat{\mathbf{R}}$$

- For various shapes:
  - Volume: $dq = \rho\, d\tau$
  - Line: $dq = \lambda\, dl$
  - Surface: $dq = \ da$
- Electric Potential — $V$ (volts)

$$\vec{E} = -\vec{\nabla} V \Longleftrightarrow V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{l}$$

  - Note: $V$ is a scalar function
  - For a charge $q$ and some reference radius, $r$:

$$V(r) - V_{ref} = -\int_{r_{ref}}^{r} \vec{E} \cdot d\vec{r\prime}$$

  - This yields

$$V(r) = \frac{q}{4\pi\varepsilon_o r}$$

  - With multiple charges:

$$V(r) = \frac{1}{4\pi\varepsilon_o} \sum_n \frac{q_n}{R_n}$$

- Taylor Series Expansions
  - We can write the expression $\frac{1}{x-L}$, where $\frac{L}{x} << 1$, as:

$$\frac{1}{x-L} = \frac{1}{x}\left(1 - \frac{L}{x}\right)^{-1} \approx \frac{1}{x} + \frac{L}{x^2} + \dots$$

    This means:

$$\frac{1}{x-L} - \frac{1}{x} \approx \frac{L}{x^2}$$

- Coulomb's Law with Gauss's Theorem

- Develops into Gauss's Law

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \cdot d\tau$$

$$\oint \vec{E} \cdot d\vec{a} = \sum \underbrace{\int_V \frac{q_n}{\varepsilon_o} \delta^3(R_n)}_{\text{enc charge over } \varepsilon_o}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_o}$$

- Gauss's Law:
  - Exploit symmetry
  - Large or small distance to approximate symmetry

- Potential Energy
  - The potential required to bring in a new charge to a configuration can be calculated using:

$$\sum_j q_j \sum_{i<j} \frac{q_i}{4\pi\varepsilon_o R_{ij}} = \sum_j \sum_{i<j} \frac{q_i q_j}{4\pi\varepsilon_o R_{ij}}$$

$$W_{tot} = \frac{1}{2} \sum_j \sum_{i \neq j} \frac{q_i q_j}{4\pi\varepsilon_o R_{ij}}$$

    when $i = j$ this is "self-energy" of a particle

$$W = \frac{1}{2} \int \rho V \, d\tau$$

    Using the definition $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$, we get:

$$W = \frac{1}{2} \oint \varepsilon_o V \vec{E} \cdot d\tau$$

$$W = \frac{1}{2} \oint \varepsilon_o V \vec{E} \cdot d\vec{a} \underbrace{- \frac{1}{2} \int \varepsilon_o \vec{E} \vec{W} \cdot d\tau}_{\int \frac{\varepsilon_o E^2}{2} d\tau}$$

$$W = \frac{1}{2} \int \frac{\varepsilon_o E^2}{2} d\tau$$

  - The term of integration is known as the energy density, $u = \dfrac{\varepsilon_o E^2}{2}$
    - This means electric fields themselves carry energy

- Conductors
  - $\vec{E} = 0$ inside the material of a conductor (if there is a cavity, there may be a field)
- Capacitance

$$C = \frac{Q}{\Delta V}$$

  - Given the geometry of capacitors, work can be found as:

$$\text{Energy: } dW = V\,dq = \frac{q}{C}\,dq = VC\,dV \Rightarrow$$
$$W = \int_0^V CV'\,dV' = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

  - Units of capacitance are farads, which are equal to coulombs per volt

$$C = \frac{\varepsilon_o A}{d}$$