Homework 1

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1. Calculate:

   (a) $\vec{\nabla}\left(\frac{1}{r}\right)$

   We know $r = \sqrt{x^2 + y^2 + z^2}$. Converting and applying the chain rule, we get:

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\hat{x} + \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\hat{y} + \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\hat{z}$$
$$(x^2+y^2+z^2)^{-\frac{3}{2}}(x\hat{x}+y\hat{y}=z\hat{z})$$
$$\frac{x\hat{x}+y\hat{y}+z\hat{z}}{r^3}$$

   We also know that $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$, and that $r\hat{r} = \vec{r}$. Thus, we get:

$$\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\vec{r}}{r^3} \to \frac{\hat{r}}{r^2}}$$

   (b) $\vec{\nabla}\hat{x}$

   This could be written as:

$$\hat{x}\frac{\partial}{\partial x}\hat{x} + \hat{y}\frac{\partial}{\partial y}\hat{x} + \hat{z}\frac{\partial}{\partial z}\hat{x}$$

   (c)

   (d) $\vec{\nabla}r^n$ $(n > 0)$

$$\vec{\nabla}((x^2+y^2+z^2)^{\frac{n}{2}}) \Rightarrow \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}(2x\hat{x}+2y\hat{y}+2z\hat{z})$$
$$n\vec{r}r^{n-2} \to \frac{n\vec{r}}{r^{2-n}}$$

   Thus, we get:

$$\boxed{\frac{n\hat{r}}{r^{1-n}} \quad \text{or} \quad n\hat{r}r^{n-1}}$$

2. Calculate the divergence and curls of the following functions:

   (a) $\vec{v}_a = xy\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + zy\hat{\mathbf{z}}$

   - Divergence:

$$\vec{\nabla} \cdot \vec{v}_a = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zy)$$
$$\text{div}(\vec{v}_a) = y + z + y = 2y + z$$

   - Curl:

$$\vec{\nabla} \times \vec{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zy \end{vmatrix} = (z - y)\hat{x} - (0 - 0)\hat{y} + (0 - x)\hat{z}$$
$$\text{curl}(\vec{v}_a) = \langle z - y, 0, -x \rangle$$

   (b) $\vec{v}_b = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2xz\hat{\mathbf{z}}$

   - Divergence:

$$\vec{\nabla} \cdot \vec{v}_b = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2xz)$$
$$\text{div}(\vec{v}_b) = 0 + 2x + 2x = 4x$$

   - Curl:

$$\vec{\nabla} \times \vec{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2xz \end{vmatrix} = (0 - 2z)\hat{x} - (2z - 0)\hat{y} + (2y - 2y)\hat{z}$$
$$\text{curl}(\vec{v}_b) = \langle -2z, -2z, 0 \rangle$$

   (c) $\vec{v}_c = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$

   - Divergence:

$$\vec{\nabla} \cdot \vec{v}_c = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$
$$\text{div}(\vec{v}_c) = 0 + 0 + 0 = 0$$

   - Curl:

$$\vec{\nabla} \times \vec{v}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{x} - (y - y)\hat{y} + (z - z)\hat{z}$$
$$\text{curl}(\vec{v}_c) = \langle 0, 0, 0 \rangle$$

3. (a)

(b)

(c)

4. (a)

   (b)

5. (a)

   (b)

   (c)

   (d)

6. (a)

   (b)

7. (a)

   (b)

   (c)