

Homework 1

Michael Brodskiy

Professor: D. Wood

September 11, 2023

1. Calculate:

(a) $\vec{\nabla} \left(\frac{1}{r} \right)$

We know $r = \sqrt{x^2 + y^2 + z^2}$. Converting and applying the chain rule, we get:

$$\begin{aligned} \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \hat{\mathbf{x}} + \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \hat{\mathbf{y}} + \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{-\frac{1}{2}} \hat{\mathbf{z}} \\ (x^2 + y^2 + z^2)^{-\frac{3}{2}} (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) \\ \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3} \end{aligned}$$

We also know that $\vec{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, and that $r\hat{\mathbf{r}} = \vec{r}$. Thus, we get:

$$\boxed{\vec{\nabla} \left(\frac{1}{r} \right) = \frac{\vec{r}}{r^3} \rightarrow \frac{\hat{\mathbf{r}}}{r^2}}$$

(b) $\vec{\nabla} \cdot \hat{\mathbf{x}}$

This implies the following:

$$v_x = 1\hat{\mathbf{x}}, \quad v_y = 0\hat{\mathbf{y}}, \quad v_z = 0\hat{\mathbf{z}}$$

Thus, we find:

$$\boxed{\vec{\nabla} \cdot \hat{\mathbf{x}} = \frac{\partial}{\partial x} (1\hat{\mathbf{x}}) = 0}$$

(c) $\vec{\nabla} \cdot \hat{\mathbf{r}}$

This could be written as:

$$\vec{\nabla} \cdot \left(\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} \right)$$

Thus, we need to compute:

$$\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right)$$

with respect to each variable. By the quotient rule, we find

$$\frac{\partial}{\partial x} \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right) \rightarrow \frac{\sqrt{x^2 + y^2 + z^2} - x^2(x^2 + y^2 + z^2)^{-0.5}}{x^2 + y^2 + z^2}$$

Converting back to r , we obtain:

$$\frac{1}{r} - \frac{x^2}{r^3}$$

By symmetry, we know that the corresponding y and z variables become:

$$\frac{1}{r} - \frac{y^2}{r^3} \quad \text{and} \quad \frac{1}{r} - \frac{z^2}{r^3}$$

Summing the results, we get:

$$\begin{aligned} & \left(\frac{1}{r} - \frac{x^2}{r^3} \right) + \left(\frac{1}{r} - \frac{y^2}{r^3} \right) + \left(\frac{1}{r} - \frac{z^2}{r^3} \right) \\ & \frac{3}{r} - \left(\frac{x^2 + y^2 + z^2}{r^3} \right) \\ & \boxed{\vec{\nabla} \cdot \hat{\mathbf{r}} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}} \end{aligned}$$

(d) $\vec{\nabla} r^n \ (n > 0)$

$$\begin{aligned} \vec{\nabla}((x^2 + y^2 + z^2)^{\frac{n}{2}}) & \Rightarrow \frac{n}{2}(x^2 + y^2 + z^2)^{\frac{n}{2}-1} (2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}}) \\ n\vec{r}r^{n-2} & \rightarrow \frac{n\vec{r}}{r^{2-n}} \end{aligned}$$

Thus, we get:

$$\boxed{\frac{n\hat{\mathbf{r}}}{r^{1-n}} \quad \text{or} \quad n\hat{\mathbf{r}}r^{n-1}}$$

2. Calculate the divergence and curls of the following functions:

(a) $\vec{v}_a = xy\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + zy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_a = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zy)$$

$$\boxed{\operatorname{div}(\vec{v}_a) = y + z + y = 2y + z}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zy \end{vmatrix} = (z - y)\hat{\mathbf{x}} - (0 - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}}$$

$$\boxed{\operatorname{curl}(\vec{v}_a) = \langle z - y, 0, -x \rangle}$$

(b) $\vec{v}_b = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2xz\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_b = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2xz)$$

$$\boxed{\operatorname{div}(\vec{v}_b) = 0 + 2x + 2x = 4x}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2xz \end{vmatrix} = (0 - 2z)\hat{\mathbf{x}} - (2z - 0)\hat{\mathbf{y}} + (2y - 2y)\hat{\mathbf{z}}$$

$$\boxed{\operatorname{curl}(\vec{v}_b) = \langle -2z, -2z, 0 \rangle}$$

(c) $\vec{v}_c = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_c = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$\boxed{\operatorname{div}(\vec{v}_c) = 0 + 0 + 0 = 0}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{\mathbf{x}} - (y - y)\hat{\mathbf{y}} + (z - z)\hat{\mathbf{z}}$$

$$\boxed{\operatorname{curl}(\vec{v}_c) = \langle 0, 0, 0 \rangle}$$

3. Calculate the following. Note that some results should be vectors. It is helpful to check the dimensionality of your answers:

(a) $(\vec{r} \cdot \vec{\nabla})\vec{r}$

$$(\vec{r} \cdot \vec{\nabla}) = \frac{\partial}{\partial x}(x\hat{\mathbf{x}}) + \frac{\partial}{\partial y}(y\hat{\mathbf{y}}) + \frac{\partial}{\partial z}(z\hat{\mathbf{z}}) = 1 + 1 + 1 = 3$$

$$\boxed{(\vec{r} \cdot \vec{\nabla})\vec{r} = 3\vec{r} = 3x\hat{\mathbf{x}} + 3y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}}$$

(b) $(\hat{\mathbf{r}} \cdot \vec{\nabla})r$

From Problem **1c**, we know the value of $(\hat{\mathbf{r}} \cdot \vec{\nabla})$, which gives us:

$$\boxed{\left(\frac{2}{r}\right)r = 2}$$

(c) $(\hat{\mathbf{r}} \cdot \vec{\nabla})\hat{\mathbf{r}}$

Again, employing what we know from Problem **1c**, we get:

$$\boxed{\left(\frac{2}{r}\right)\hat{\mathbf{r}} = \frac{2\hat{\mathbf{r}}}{r}}$$

4. Consider a vector field $\vec{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ and a cube with vertices at $(0,0,0)$, $(0,2,0)$, $(2,0,0)$, $(0,0,2)$, etc. In the text (Example 1.7), the flux was calculated through the top five faces of the cube, and found to total to 20.

- (a) Find the upward flux through the bottom surface, i.e. the square surface in the xy -plane bounded by the points $(0,0,0)$, $(2,0,0)$, $(2,2,0)$, $(0,2,0)$.

According to the Stoke's Theorem, we must first find the curl of \vec{v} :

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz & x+2 & yz^2-3y \end{vmatrix} = (z^2-3-0)\hat{\mathbf{x}} - (0-2x)\hat{\mathbf{y}} + (1-0)\hat{\mathbf{z}}$$

$$\text{curl}(\vec{v}) = \langle z^2-3, 2x, 1 \rangle$$

Furthermore, because the surface is in the xy plane, and the flux we want to find is in the $\hat{\mathbf{z}}$ direction, we know:

$$d\vec{a} = dx dy \hat{\mathbf{z}}$$

(b)

5. (a)

(b)

(c)

(d)

6. (a)

(b)

7. Consider two vector functions $\vec{v} = x^2\hat{\mathbf{z}}$ and $\vec{w} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

(a) calculate the divergence and curl of each

- $\text{div}(x^2\hat{\mathbf{z}})$

$$\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) = 0$$

- $\text{curl}(x^2\hat{\mathbf{z}})$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = (0 - 0)\hat{\mathbf{x}} - (2x - 0)\hat{\mathbf{y}} + (0 - 0)\hat{\mathbf{z}} = \boxed{-2x\hat{\mathbf{y}}}$$

- $\text{div}(\vec{w} = \vec{r})$

$$\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

- $\text{curl}(\vec{w} = \vec{r})$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0 - 0)\hat{\mathbf{x}} - (0 - 0)\hat{\mathbf{y}} + (0 - 0)\hat{\mathbf{z}} = \boxed{0}$$

(b)

(c)