Vector Calculus

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- A vector is defined by:
  - Transformation under rotation

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \mathbb{R}_{xx} & \mathbb{R}_{yy} \\ \mathbb{R}_{yx} & \mathbb{R}_{yy} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$
$$\mathbb{R} = \begin{pmatrix} \cos(\delta\phi) & -\sin(\delta\phi) \\ \sin(\delta\phi) & \cos(\delta\phi) \end{pmatrix}$$

  - Examples include electric fields, magnetic fields, momentum, displacement, etc.
- Scalars
  - Invariant under rotation
  - Examples include charge, mass, electric potential, energy, etc.
- Tensors (rank 2)
  - $\mathbb{R}$ above is an example
- Differential Operators
  - Gradient $\longrightarrow \vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$
    * Must operate on something to be useful
    * Ex. $\vec{E} = -\vec{\nabla}V(x, y, z)$
- Maxwell's Equations in a Vacuum (in SI units)

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_o}\rho$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

- SI units for E&M: Coulomb, Volt, Tesla, Ampere

- Force between two objects
  - In SI:

$$\vec{F}_{12} = \frac{q_1 q_2 (\widehat{r_1 - r_2})}{4\pi\epsilon_o r_{12}^2}$$

  - In CGS:

$$F = \frac{q_1 q_2}{r^2}$$

- Cross Products

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

  - Not cumulative:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

  - Distributive:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

  - Not associative:

$$(\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$$

- Unit Vectors:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

- Gradient of a scalar field
  - If T is a scalar field, then:

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

  - Ex. $T = y^2 z$

$$\vec{\nabla} T = 0\hat{x} + (2yz)\hat{y} + (y^2)\hat{z}$$

– Ex. $T = r^3 = (x^2 + y^2 + z^2)^{\frac{3}{2}}$

$$\vec{\nabla} T = \left(\frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}}(2x)\right)\hat{x} + \left(\frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}}(2y)\right)\hat{y}$$
$$+ \left(\frac{3}{2}(x^2 + y^2 + z^2)^{\frac{1}{2}}(2z)\right)\hat{z}$$
$$\vec{\nabla} T = 3r(x\hat{x} + y\hat{y} + z\hat{z}) = 3r\vec{r} = 3r^2\hat{r}$$
$$\vec{\nabla}(r^3) = 3r^2(\vec{\nabla} r)$$

Thus, we see:

$$\vec{\nabla} r = \hat{r}$$

Think in terms of dimensionality.