Potentials

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$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_o} \Leftrightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$$

This means that, in a region with no charge, $\vec{\nabla} \cdot \vec{E} = 0$, which also means $\nabla^2 V = 0$ (the Laplacian) Rectangular Coordinates:

$$\nabla^2 V = 0 \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x,y,z) = X(x)Y(y)Z(z) \Rightarrow \frac{\partial^2 X}{\partial x^2}YZ + \frac{\partial^2 Y}{\partial y^2}XZ + \frac{\partial^2 Z}{\partial z^2}XY = 0$$

If we divide by the respective functions, we get:

$$\frac{\partial^2}{\partial x^2}\frac{1}{X} + \frac{\partial^2}{\partial y^2}\frac{1}{Y} + \frac{\partial^2}{\partial z^2}\frac{1}{Z} = 0$$

$$\frac{d^2 X}{dx^2} = c_1 X(x) \Rightarrow c_1 > 0 :$$

$$X(x) = \begin{cases} Ae^{\pm kx} & c_1 = k^2 \\ A\sin(kx) & c_1 = -k^2 \\ A\cos(kx) & c_1 = -k^2 \end{cases}$$

- – Repeating this for each variable, we find

$$\frac{d^2 Y}{dy^2} = c_2 Y(y) \qquad \frac{d^2 Z}{dz^2} = c_2 Z(z)$$

$$c_1 + c_2 + c_3 = 0$$

- Fourier Inversion

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\, dy = \frac{a}{2}\delta_{nm}$$

This is used to define the terms for Fourier analysis

- A semi-infinite square tube, with the four walls grounded:
  - The boundary conditions are (with $V = 0$) given by:

$$\begin{cases} x = 0, & 0 < y < b \\ x = a, & 0 < y < b \\ y = 0, & 0 < x < a \\ y = b, & 0 < x < a \end{cases}$$

  - This would mean:

$$X = \sin\left(\frac{n\pi x}{a}\right) \qquad Y \propto \sin\left(\frac{m\pi y}{b}\right) \qquad Z = e^{-k_z z}$$
$$V = \sum B_{n,m} e^{-k_z z} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

  - Because we know $c_1 + c_2 + c_3 = 0$, we know:

$$-\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right) + k_z^2 = 0$$
$$k_z = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

- Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o}$$

  - When the charge density is zero, we get Laplace's Equation:

$$\nabla^2 V = 0$$

- Given a box in three dimensions with lengths a, b, c, we know:

$$c_1 + c_2 + c_3 = 0 \longrightarrow \begin{cases} c > 0, & \text{Exponential} \\ c < 0, & \text{Sinusoidal} \end{cases}$$

  - We know the voltage is of the form

$$V = f(z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$

  - The function, $f(z)$, must then be of the form:

$$f(z) = Ae^{kz} + Be^{-kz}$$

- If $V = 0$, then $z = 0$, so:

$$\begin{aligned} A + B = 0 \rightarrow A &= -B \\ f(z) &= Ae^{kz} - Ae^{-kz} \\ &= A(e^{kz} - e^{-kz}) \\ &= 2A\sinh(kz) \end{aligned}$$

- Thus, we can set up the function as:

$$V(x, y, z) = \sum_{m,n} c_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(kz)$$

• Spherical Coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r^2}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial V}{\partial \theta}\right) = 0$$

- In Quantum Mechanics, we have spherical harmonics:

$$Y_{lm}(\theta, \phi) = P_l(\cos(\theta)) f_{ml}(\phi)$$

- For our purposes, we can find:

$$V(r, \theta) = \sum_{l=0,1,\ldots} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_e(\cos(\theta))$$

- A typical problem for this would be:
  * Given some potential, $V(\theta)$ on a spherical surface, find V_{in} and V_{out}

$$\begin{aligned} R_e(r) &= A_l r^l + \frac{B_l}{r^{l+1}} \\ V_{in} &= A_l r^l P_l(\cos(\theta)) \\ V_{out} &= \frac{B_l}{r^{l+1}} P_l(\cos(\theta)) \end{aligned}$$

  * We know the following boundary conditions:

$$r \rightarrow \infty, V \rightarrow 0$$

  * We can integrate the function:

$$\int_{-1}^{1} V(R, \theta) P_l(\cos(\theta)) d(\cos(\theta)) = \frac{2R^l A_l}{2l + 1}$$

- Example: Given $V(R, \theta) = E_o R\cos(\theta) = E_o R P_1(\cos(\theta))$

* This gives us:

$$\begin{cases} \text{Inside:} & V = A_1 r^2 \cos(\theta) \\ \text{Outside:} & V = \frac{B_1}{r^2} P_1(\cos(\theta)) \end{cases}$$

- For a sphere
  - We find $E_o R \cos(\theta)$:

$$\begin{cases} \text{Inside:} & V = E_o r \cos(\theta) = E_o z \\ \text{Outside:} & V = \dfrac{E_o R^3}{r^2}(\cos(\theta)) \end{cases}$$

  - We know:

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{\partial V}{\partial z}\hat{\mathbf{z}} = -E_o \hat{\mathbf{z}} = E_o(-\cos(\theta)\hat{\mathbf{r}} + \sin(\theta)\hat{\theta})$$
$$\vec{E}_{out} = -\vec{\nabla} V_{out} = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} = -\frac{1}{r}\frac{\partial V_{out}}{\partial \theta} = \frac{E_o R^3}{r^3}(2\cos(\theta)\hat{\mathbf{r}} + \sin(\theta)\hat{\theta})$$

  - To confirm our calculation, we can take the partial derivative with respect to the angle:

$$E_{in_\theta}(R) = E_o \sin(\theta)$$
$$E_{out_\theta}(R) = E_o \cos(\theta)$$

  - We can then use Gauss's law:

$$\frac{\sigma A}{\varepsilon_o} = A(E_{out_r} - E_{in_r})$$
$$\sigma = \varepsilon_o(E_{out}(R,\theta) - E_{in}(R,\theta))$$
$$\sigma = 3\varepsilon_o(E_o \cos(\theta))$$

  - This implies there is a positive charge at the north pole, a negative charge at the south pole, and no charge at the equator
  - To find the total charge we can do the following:

$$Q = \int_0^\pi R^2 \sin(\theta)\, d\theta (3\varepsilon_o E_o \cos(\theta)) = 0$$

- Poles:
  - We can determine that:

$$\begin{cases} \text{Monopole:} & V \propto \frac{1}{r} \\ \text{Dipole:} & V \propto \frac{1}{r^2} \\ \text{Quadrupole:} & V \propto \frac{1}{r^3} \\ \text{Octopole:} & V \propto \frac{1}{r^4} \end{cases}$$

- The arrangement increases in dimension each time we go up an order; that, is, the shape is a single point charge for monopole, two equal but opposite charges along a line for dipole, a square with aggregate charge of zero for quadrupole, and a cube with aggregate charge zero for octopole