Homework 2

Michael Brodskiy
Professor: D. Wood

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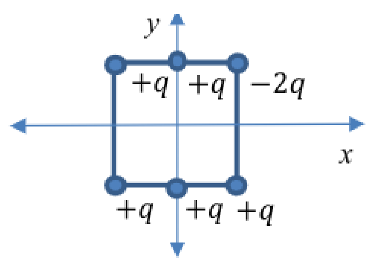
1. Six charges are arranged along the sides and corners of a square with sides of length L as shown. Calculate the magnitude and direction of the electric field at the origin. Use symmetry and superposition to make the calculation simple.



We know, by definition, that $\vec{F} = \vec{E}q$. Using the concepts we know about force, we know the following charges cancel out each other, as they are symmetric about the test charge at the origin:

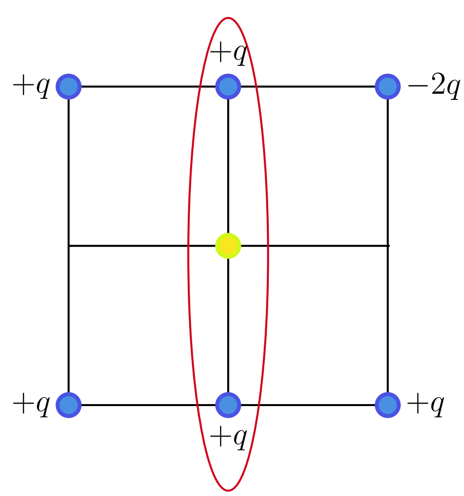


Figure 1: The Opposite Forces Negate Each Other

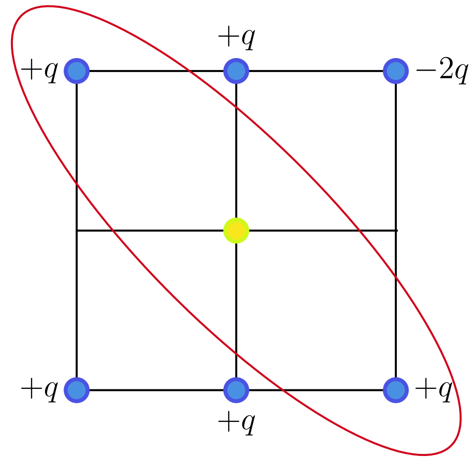


Figure 2: The Opposite Forces Negate Each Other

Thus, we need only consider the effects of these charges:

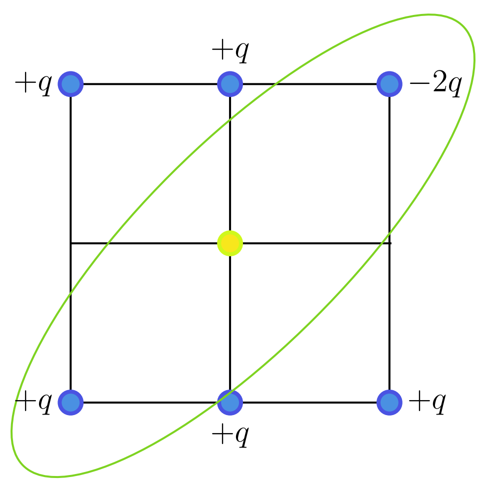


Figure 3: These Charges Remain Relevant

Using superposition, we know that the two charges can be summed, and they produce a horizontal force proportional to $3q$ at an angle of $\frac{\pi}{4}$ radians with respect to the x-axis. Decomposing this, we know the force can be expressed, with Q as the test charge, as:

$$E_Q = \frac{(3q)}{4\pi\varepsilon_o R^2}\cos\left(\frac{\pi}{4}\right)\hat{\mathbf{x}} + \frac{(3q)}{4\pi\varepsilon_o R^2}\sin\left(\frac{\pi}{4}\right)\hat{\mathbf{y}}$$

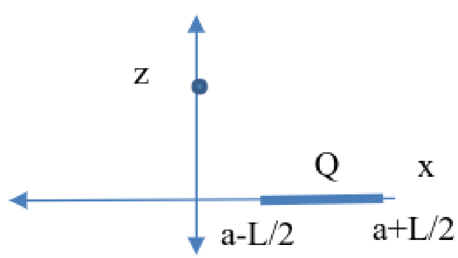
Additionally, we know that $R = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2} = \sqrt{\frac{L^2}{2}} = \frac{L}{\sqrt{2}}$. This gives us:

$$\boxed{E_Q = \frac{3\sqrt{2}q}{4\pi\varepsilon_o L^2}\hat{\mathbf{x}} + \frac{3\sqrt{2}q}{4\pi\varepsilon_o L^2}\hat{\mathbf{y}}}$$

It can also be said that the field, in the direction of the $-2q$ charge, is:

$$\boxed{E_Q = \frac{3q}{2\pi\varepsilon_o L^2}\hat{\mathbf{-2q}}}$$

2. A uniformly charged rod of length L and charge q is placed along the x-axis with its center at $x = a$. Find the x-component of the electric field at a point on the z axis. (Hint: use R as the variable of integration.) Check your expression in the following limit: $z = 0$ and $a >> L$.



We know the rod is length L, with charge Q. This means the linear charge density can be defined as:

$$\lambda = \frac{Q}{L}$$

Furthermore, we can refer to the angle between the test charge and point on the rod as θ, measured in the counter-clockwise direction, and the distance from said point on the rod to the test charge can be called R. This yields us the following expression for the x-axis:

$$E_x = -\int \frac{\sin(\theta)}{4\pi\varepsilon_o R^2}\, dQ$$

We can express $dq \to \lambda\, dx$, which gives us:

$$E_x = -\int \frac{\lambda \sin(\theta)}{4\pi\varepsilon_o R^2}\, dx$$

Then, if we were to assume x as the horizontal distance from the test charge to the rod, and z as the vertical distance from the test charge to the rod, we may see that:

$$\tan(\theta) = \frac{x}{z}$$
$$z\tan(\theta) = x$$
$$z\sec^2(\theta)\,d\theta = dx$$

Taking R into account, we can further simplify our calculation:

$$\frac{dx}{R^2} = \frac{z\,d\theta}{R^2\cos^2(\theta)}$$

Again referring to our set up, we know $\cos(\theta) = \frac{z}{R}$:

$$dx = \frac{d\theta}{z}$$

Finally, we can use this:

$$E_x = -\frac{\lambda}{4\pi\varepsilon_o z}\int_{\theta_{a-\frac{L}{2}}}^{\theta_{a+\frac{L}{2}}} \sin(\theta)\,d\theta$$
$$\frac{\lambda}{4\pi\varepsilon_o z}\left[\cos(\theta)\right]\Big|_{\theta_{a-\frac{L}{2}}}^{\theta_{a+\frac{L}{2}}}$$

We can once again refer to the set-up, finding that:

$$\begin{cases} \cos\left(\theta_{a-\frac{L}{2}}\right) = \frac{z}{\sqrt{z^2+(a-L/2)^2}} \\ \cos\left(\theta_{a+\frac{L}{2}}\right) = \frac{z}{\sqrt{z^2+(a+L/2)^2}} \end{cases}$$

Substituting this into our final expression, we get:

$$\boxed{\frac{\lambda}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{z^2+(a+L/2)^2}} - \frac{1}{\sqrt{z^2+(a-L/2)^2}}\right)}$$

$z = 0$ CASE:

$$\frac{\lambda}{4\pi\varepsilon_o}\left(\frac{1}{a+L/2} - \frac{1}{a-L/2}\right)$$
$$\frac{\lambda}{4\pi\varepsilon_o}\left(\frac{a-L/2-(a+L/2)}{a^2-(L/2)^2}\right)$$

$$-\frac{\lambda L}{(4\pi\varepsilon_o)(a^2-(L/2)^2)}$$
$$-\frac{Q}{(4\pi\varepsilon_o)(a^2-(L/2)^2)}$$

$a >> L$ CASE:

$$\frac{\lambda}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{z^2+(a+L/2)^2}}-\frac{1}{\sqrt{z^2+(a-L/2)^2}}\right); \quad a \pm L \to 0$$
$$\frac{\lambda}{4\pi\varepsilon_o}\left(\cancel{\frac{1}{\sqrt{z^2+a^2}}-\frac{1}{\sqrt{z^2+a^2}}}\right)=0$$

This makes sense, as, logically, if the rod were centered at a point very far away from the test charge, there would be no significant electric field.

3. Calculate the electric potential on the z-axis due to a uniformly charged annulus in the xy-plane centered at the origin with inner radius a and outer radius b. Then find the electric field from the gradient of the potential.

   Let us assume the annulus holds a charge of q, with an area A. This defines the charge density as:

$$\sigma = \frac{q}{A}$$

   This gives us:

$$dq = \sigma dA$$

   If we assume that i and j are the horizontal and vertical distances, respectively, to the test charge from any point on the annulus, and $r = \sqrt{i^2+j^2}$ is the radial distance from the test charge to any point, then we can express the above as:

$$dq = \sigma(2\pi i\, di)$$

   We can then define the voltage as:

$$V = \frac{q}{4\pi\varepsilon_o r}$$
$$dV = \frac{dq}{4\pi\varepsilon_o r}$$
$$dV = \frac{\sigma i\, di}{2\varepsilon_o\sqrt{i^2+j^2}}$$

   We can then integrate to find the voltage expression:

$$V = \frac{\sigma}{2\varepsilon_o}\int_a^b \frac{i}{\sqrt{i^2+j^2}}\,di$$

Using u substitution, with $u = i^2 + j^2$, we get:

$$V = \frac{\sigma}{\varepsilon_o}\int_{a^2+j^2}^{b^2+j^2} \frac{1}{\sqrt{u}}\,du$$

$$V = \frac{\sigma}{\varepsilon_o}\left(\frac{\sqrt{u}}{2}\right)\Big|_{a^2+j^2}^{b^2+j^2}$$

$$V = \frac{\sigma}{2\varepsilon_o}\left(\sqrt{j^2+b^2} - \sqrt{j^2+a^2}\right)$$

We can then find the electric field using the gradient formula:

$$\vec{E} = -\vec{\nabla}V$$

$$\vec{E} = -\vec{\nabla}\left(\frac{\sigma}{2\varepsilon_o}\left(\sqrt{j^2+b^2} - \sqrt{j^2+a^2}\right)\right)$$

$$\boxed{\vec{E} = \langle 0, 0, \frac{\sigma}{4\varepsilon_o}\left(\frac{j}{\sqrt{j^2+a^2}} - \frac{j}{\sqrt{j^2+b^2}}\right)\rangle}$$

Because j simply indicates the z direction, and the vector above is with respect to the i, j, k vectors, we can rewrite this as:

$$\boxed{\vec{E} = \frac{\sigma}{4\varepsilon_o}\left(\frac{z}{\sqrt{z^2+a^2}} - \frac{z}{\sqrt{z^2+b^2}}\right)\hat{\mathbf{k}}}$$

4. Consider an infinitely long uniformly-charged solid cylinder of radius a and charge per unit volume ρ surrounded by a coaxial cylindrical shell of radius b and charge per unit area of σ. Take the axis of the cylinders as the z-axis.

   (a) Calculate the electric field everywhere in space

   (b) Also calculate the potential as a function of the distance from the axis, taking the potential to be zero on the z-axis.

5. The electric field for two charged concentric spherical shells is given by

$$\begin{cases} 0, & r < a \\ \hat{\mathbf{r}}A_1/r^2, & a < r < b \\ \hat{\mathbf{r}}A_2/r^2, & r > b \end{cases}$$

   Where $A_1 = 5\times10^6\left[\frac{\mathrm{N\,m^2}}{\mathrm{C}}\right]$, $A_2 = -3\times10^6\left[\frac{\mathrm{N\,m^2}}{\mathrm{C}}\right]$, $a = .25[\mathrm{m}]$, and $b = .45[\mathrm{m}]$. Find the surface charge densities σ_a and σ_b on the two shells.