Vector Calculus

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- A vector is defined by:
  - Transformation under rotation

$$\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = \begin{pmatrix} \mathbb{R}_{xx} & \mathbb{R}_{yy} \\ \mathbb{R}_{yx} & \mathbb{R}_{yy} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$\mathbb{R} = \begin{pmatrix} \cos(\delta\phi) & -\sin(\delta\phi) \\ \sin(\delta\phi) & \cos(\delta\phi) \end{pmatrix}$$

  - Examples include electric fields, magnetic fields, momentum, displacement, etc.
- Scalars
  - Invariant under rotation
  - Examples include charge, mass, electric potential, energy, etc.
- Tensors (rank 2)
  - $\mathbb{R}$ above is an example
- Differential Operators
  - Gradient $\longrightarrow \vec{\nabla} = \mathbf{hatx}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$
    * Must operate on something to be useful
    * Ex. $\vec{E} = -\vec{\nabla}V(x, y, z)$
- Maxwell's Equations in a Vacuum (in SI units)

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_o}\rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

- SI units for E&M: Coulomb, Volt, Tesla, Ampere

- Force between two objects
  - In SI:

$$\vec{F}_{12} = \frac{q_1 q_2 (\widehat{r_1 - r_2})}{4\pi\epsilon_o r_{12}^2}$$

  - In CGS:

$$F = \frac{q_1 q_2}{r^2}$$

- Cross Products

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

  - Not cumulative:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

  - Distributive:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

  - Not associative:

$$(\vec{A} \times \vec{B}) \times \vec{C} \neq \vec{A} \times (\vec{B} \times \vec{C})$$

- Unit Vectors:

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

- Gradient of a scalar field
  - If T is a scalar field, then:

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

  - Ex. $T = y^2 z$

$$\vec{\nabla} T = 0\hat{x} + (2yz)\hat{y} + (y^2)\hat{z}$$

- Ex. $T = r^3 = (x^2+y^2+z^2)^{\frac{3}{2}}$

$$\vec{\nabla}T = \left(\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2x)\right)\hat{x} + \left(\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2y)\right)\hat{y}$$
$$+ \left(\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}(2z)\right)\hat{z}$$
$$\vec{\nabla}T = 3r(x\hat{x}+y\hat{y}+z\hat{z}) = 3r\vec{r} = 3r^2\hat{r}$$
$$\vec{\nabla}(r^3) = 3r^2(\vec{\nabla}r)$$

  Thus, we see:

$$\vec{\nabla}r = \hat{r}$$

  Think in terms of dimensionality.

- Product Rule
  - In One Dimension:

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$$

  - Three Dimensions:

$$\vec{\nabla}(fg) = (\vec{\nabla}f)g + f(\vec{\nabla}g) = g\vec{\nabla}f + f\vec{\nabla}g$$

    where f, g are scalar functions of x, y, z
  - Where a is constant:

$$\vec{\nabla}(af) = a\vec{\nabla}(f)$$

- Chain Rule
  - In One Dimension:

$$\frac{d}{dx}(f(g(x))) = f'(g(x))\frac{dg}{dx} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x}$$

  - Three Dimensions:

$$\vec{\nabla}(f(g(x,y,z))) = \frac{\partial f}{\partial g}\vec{\nabla}g$$

  - Example:

$$f(g) = g^3,\ g = r = \sqrt{x^2+y^2+z^2}$$
$$\frac{\partial f}{\partial g}\vec{\nabla}g = 3g^2\vec{\nabla}(r) = 3r^2\hat{r}$$

- Divergence (where $\vec{v}$ is a vector function)

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

  - Example of positive divergence (where $\vec{v} = x\hat{\mathbf{x}}$). Looking at the graph of the vector field and taking a sample volume, there is more going "out" than "in," which indicates that the divergence is greater than 0

$$\vec{\nabla} \cdot \vec{v} = 1 + 0 + 0 = 1$$

  - Zero divergence would mean the same quantity "out" as "in," like when $\vec{v}$ is a constant in any direction
  - Negative divergence

$$\vec{v} = \frac{\hat{\mathbf{r}}}{r^3}$$

$$\vec{\nabla} \cdot \frac{\hat{\mathbf{r}}}{r^3} = \vec{\nabla} \cdot \frac{\vec{r}}{r^4} = (\vec{\nabla} \cdot \vec{r})\frac{1}{r^4} + \vec{r}\left(\vec{\nabla}\frac{1}{r^4}\right) = \frac{3}{r^4} + \vec{r}\left(-\frac{4\hat{\mathbf{r}}}{r^5}\right) = -\frac{1}{r^4}$$ [1]

- Curl

$$\vec{\nabla} \times \vec{v} = \mathrm{curl}(\vec{v}) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

  - Product rule for curl:
    * Scalar times vector:

$$\vec{\nabla} \times (f(x,y,z)\vec{A}(x,y,z)) = \vec{\nabla} f \times A + f(\vec{\nabla} \times \vec{A})$$

    * Vector times vector:

$$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$$

- The Laplacian
  - $\nabla^2 T = \vec{\nabla} \cdot \vec{\nabla} T = \dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} + \dfrac{\partial^2 T}{\partial z^2}$
  - $\nabla^2 \vec{v} = \nabla^2 v_x \hat{\mathbf{x}} + \nabla^2 v_y \hat{\mathbf{y}} + \nabla^2 v_z \hat{\mathbf{z}}$

- The Fundamental Theorem of Calculus

[1] Keep in mind, $r\hat{\mathbf{r}} = \vec{r}$, and $\vec{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

- In one dimension

$$\int_a^b f(x)\,dx = f(b) - f(a)$$

- In three dimensions:

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}T \cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$

  - Where $d\vec{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$
  - This means that the above formula is independent of path; that is:

$$\oint \vec{\nabla}T \cdot d\vec{l} = \int_{\vec{a}}^{\vec{b}} \vec{\nabla}T \cdot d\vec{l}$$

- Fundamental Theorem for Divergence

$$\underbrace{\int \vec{\nabla} \cdot \vec{N} d\tau}_{\text{volume}} = \underbrace{\oint \vec{v} \cdot d\vec{a}}_{\text{surface}}$$

  - $d\tau$ refers to the differential volume element (that is, $dx\,dy\,dz$, $r^2\sin(\theta)\,dr\,d\theta\,d\phi$, etc.)

- Fundamental Theorem for Curl (Stoke's Theorem)

$$\underbrace{\oint \vec{v} \cdot d\vec{l}}_{\text{boundary}} = \underbrace{\int (\vec{\nabla} \times \vec{v}) \cdot d\vec{a}}_{\text{surface}}$$

Object	Boundary
Line	End Points
Open Surface	Perimeter
Volume	Surface

- Spherical Coordinates

  - In spherical coordinates, we obtain the following transformation:

$$\begin{cases} x = r\sin(\theta)\cos(\phi) \\ y = r\sin(\theta)\sin(\phi) \\ z = r\cos(\theta) \end{cases}$$

- Whereas $d\vec{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$ in rectangular, in spherical coordinates, $d\vec{l} = dr\hat{\mathbf{r}} + r\,d\theta\hat{\theta} + r\sin(\theta)\,d\phi\hat{\phi}$
- This makes $d\tau = r^2\sin(\theta)\,dr\,d\theta\,d\phi$
- The gradient becomes:

$$\vec{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin(\theta)}\frac{\partial}{\partial\phi}$$

- The divergence becomes:

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial\theta}(\sin(\theta)v_\theta) + \frac{1}{r\sin(\theta)}\frac{\partial v_\phi}{\partial\phi}$$

- The Laplacian becomes:

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 T}{\partial\phi^2}$$

- Delta Functions
  - In one dimension:

$$\int_b^c \delta(x-a) = \begin{cases} 1, & b<a<c \\ 0, & \text{otherwise} \end{cases}$$

  - In three dimensions:

$$\int_{\text{vol}} \delta^3(\vec{r}-\vec{a})\,dx\,dy\,dz = \begin{cases} 1, & \vec{a} \text{ in volume} \\ 0, & \text{otherwise} \end{cases}$$

  - Note:

$$\vec{\nabla}\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 0, \text{ for } r \neq 0$$