Electromagnetic Waves

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- Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_o}$$
$$\vec{\nabla}\cdot\vec{B}=-2$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B}=\mu_o\vec{J}+\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

  – The second equation indicates no magnetic monopoles exist
  – Where $J_d=\oint\vec{B}\cdot d\vec{l}=\mu_oI$
  – Electric fields diverge with strength proportional to charge
  – Magnetic fields do not diverge
  – Curling electric field opposes change in magnetic field

$$\vec{F}=q(\vec{E}+\vec{v}\vec{B})$$

  – Describes the Lorentz force, which indicates how charges move in fields
  – A key term, the displacement current ($\vec{J_d}$), was added by Maxwell

$$\vec{J_d}=\varepsilon_o\frac{\partial\vec{E}}{\partial t}=\frac{\partial\vec{D}}{\partial t}$$
$$I_d=\int\vec{J_d}\,d\vec{a}$$

  – Within a capacitor, the displacement current may be defined by:

$$I_d=\varepsilon_o\frac{d\vec{E}}{dt}A$$

- In matter, the equivalent to $\vec{J}_d$ is polarization current

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{J}_p = -\frac{\partial \rho_b}{\partial t}$$

- Thus, we can rewrite some of the equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\varepsilon_o} + \frac{-1}{\varepsilon_o}(\vec{\nabla} \cdot \vec{P}) \rightarrow \vec{\nabla} \cdot \vec{D} = \rho_f$$
$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}_f + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t} + \mu_o \frac{\partial \vec{P}}{\partial t} + \mu_o(\vec{\nabla} \times \vec{m}) \rightarrow \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

- Electromagnetic Waves

$$\vec{\nabla}(\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_o \varepsilon_o \frac{\partial}{\partial t}\left(\frac{\partial \vec{E}}{\partial t}\right)$$
$$\vec{\nabla}^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

- Applying our knowledge of the wave equation, we may write:

$$v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c$$

- We obtain the same result:

$$\vec{\nabla}^2 \vec{E} = c^2 \frac{\partial^2 \vec{E}}{\partial t^2}$$
$$\vec{\nabla}^2 \vec{B} = c^2 \frac{\partial^2 \vec{B}}{\partial t^2}$$

- Magnetic and electric fields propagate at the speed of light, perpendicular to each other, to create light
- The direction of the wave is $\vec{E} \times \vec{B}$

- In a vacuum with no sources, we may write:

$$\vec{\nabla} \cdot \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

- Which gives:

$$\vec{\nabla}^2 \vec{E} = \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{B} = \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2}$$

- We can write our wave equations as:

$$\vec{E} = \vec{E}_o \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

$$\vec{B} = \vec{B}_o \cos\left(\vec{k} \cdot \vec{r} - \omega t\right)$$

  - Where $\vec{k}$ is the wave number with units $[\mathrm{m}^{-1}]$
  - This gives us:

$$v = \frac{\omega}{k} = \sqrt{\frac{1}{\mu_o \varepsilon_o}}$$

  - Using the first two Maxwell's equations, we can see that no component of the electric or magnetic field may contribute in the direction of propagation

$$\vec{E} \times \vec{B} = \frac{E^2}{c} \vec{k}$$