

Homework 7

Michael Brodskiy

Professor: D. Wood

November 9, 2023

1. A hollow sphere of radius R centered at the origin is covered with a uniform surface charge σ and is rotating about the z -axis with angular frequency ω . A uniform external magnetic field is oriented in the y -direction: $\vec{B} = B_o \hat{y}$

- (a) Find the total force on the sphere

Let us first define the dipole moment of the sphere. To do this, we must first define the current:

$$I = \frac{1}{T} [\sigma(2\pi R \sin(\theta)) R d\theta]$$
$$I = \sigma \omega R^2 \sin(\theta) d\theta$$

This gives us the dipole moment as:

$$d\vec{m} = I A \hat{z}$$
$$d\vec{m} = \sigma \omega R^2 \sin(\theta) d\theta (\pi R^2 \sin^2(\theta)) \hat{z}$$
$$d\vec{m} = \pi \sigma \omega R^4 \sin^3(\theta) d\theta \hat{z}$$

From here, we can find:

$$\vec{m} = \pi \sigma \omega R^4 \int \sin^3(\theta) d\theta \hat{z}$$
$$\vec{m} = \pi \sigma \omega R^4 \int_0^\pi \sin^3(\theta) d\theta \hat{z}$$
$$\vec{m} = \frac{4}{3} \pi \sigma \omega R^4 \hat{z}$$

Now, we can write the force as:

$$\vec{F} = (\vec{m} \cdot \vec{\nabla}) \vec{B}$$

This gives us:

$$\vec{F} = \left(\frac{4}{3} \pi \sigma \omega R^4 \hat{\mathbf{z}} \cdot \vec{\nabla} \right) \vec{B}$$

$$\vec{F} = \left(\frac{4}{3} \pi \sigma \omega R^4 (\hat{\mathbf{z}} \cdot \vec{\nabla}) \right) \vec{B}$$

$$\vec{F} = \left(\frac{4}{3} \pi \sigma \omega R^4 \right) \underbrace{\frac{\partial}{\partial z} (B_o \hat{\mathbf{y}})}_0$$

Thus, we see that the net force is zero:

$$\boxed{\vec{F} = 0}$$

(b) Find the total torque on the sphere

Using the dipole defined in (a), we can write:

$$\vec{N} = \vec{m} \times \vec{B}$$

$$\vec{N} = \left(\frac{4}{3} \pi \sigma \omega R^4 \hat{\mathbf{z}} \right) \times (B_o \hat{\mathbf{y}})$$

$$\vec{N} = \frac{4}{3} \pi \sigma \omega B_o R^4 (\hat{\mathbf{z}} \times \hat{\mathbf{y}})$$

$$\boxed{\vec{N} = -\frac{4}{3} \pi \sigma \omega B_o R^4 \hat{\mathbf{x}}}$$

(c) Generalize the result of (b) to find the torque for a uniform magnetic field in an arbitrary direction, $\vec{B} = B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}$

Given the general case, we may write:

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ B_x & B_y & B_z \end{vmatrix} = (-B_y \hat{\mathbf{x}} + B_x \hat{\mathbf{y}})$$

Thus, we can substitute $-B_o \hat{\mathbf{x}}$ from (b) with the above result to get:

$$\boxed{\vec{N} = \frac{4}{3} \pi \sigma \omega R^4 (B_x \hat{\mathbf{y}} - B_y \hat{\mathbf{x}})}$$

2. Calculate the magnetic field $\vec{B}(x, y)$ in the positive quadrant of the x - y plane due to a current coming on the y -axis from $y = +\infty$, turning 90° at the origin, and exiting along the x -axis to $x = +\infty$

We know that the magnetic field at a point of distance r from a wire can be defined as:

$$\vec{B} = \frac{\mu_o I}{2\pi r}$$

Thus, we can define the magnetic field produced by the y current, with direction determined through the right hand rule, as:

$$\vec{B}_y = \frac{\mu_o I}{2\pi y} \hat{z}$$

Using the same method, we can write the x current magnetic field as:

$$\vec{B}_x = \frac{\mu_o I}{2\pi x} \hat{z}$$

We can simply sum the magnetic field to find:

$$\boxed{\vec{B}(x, y) = \frac{\mu_o I}{2\pi} \left(\frac{1}{x} + \frac{1}{y} \right) \hat{z}}$$

3. A long cylindrical conductor has a uniform current density \vec{J} oriented along the axis.
- (a) Find the strength of the magnetic field as a function of s (the perpendicular distance from the z -axis) for $s < a$.

Using Ampère's Law, we can write:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I$$

Applying the given parameters, we find:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \vec{J}(\pi s^2)$$

$$\vec{B} \oint d\vec{l} = \mu_o \vec{J}(\pi s^2)$$

$$\vec{B}(2\pi s) = \mu_o \vec{J}(\pi s^2)$$

$$\boxed{\vec{B} = \frac{\mu_o \vec{J} s}{2}}$$

- (b) Consider a charged particle with charge q and momentum p that passes through the cylinder in part (a) with an initial velocity parallel to the axis. As a function of the distance s of the particle from the axis, find the angle of deflection after passing through a short distance Δz . Considered as lens for such charged particles, what is the focal length of this segment of the conductor as a function of $\vec{J}, p, q, \Delta z$? (Such lenses are actually used in particle accelerators). Assume that $\Delta z \ll R$, where R is the radius of curvature of the particle in the magnetic field, so a small angle approximation is valid for the deflection
- (c) Find the current density \vec{J} needed to focus particles of charge $e = 1.6 \cdot 10^{19}[\text{C}]$ and momentum $p = 75 \left[\frac{\text{GeV}}{c} \right]$ with a focal length of $f = 20[\text{m}]$ for $\Delta z = 0.50[\text{m}]$

4. A double solenoid has two co-axial coils radii a and b , with n_a and n_b turns per unit length, and with currents I_a and I_b flowing in opposite directions. Find:

For the following problems, we can assume ideal solenoids, with current present only inside of the solenoid. The magnetic field may be defined as:

$$\vec{B} = \mu_o n I$$

as was determined in class. Furthermore, let us define the axis of the solenoids as \hat{x} , and assume that I_a flows such that the magnetic field produced is in the $+\hat{x}$ direction.

- (a) the magnetic field for the region inside the first coil ($s < a$)

Inside the first coil, there are two currents, I_a and I_b . Thus, we can simply sum the magnetic fields from each to determine:

$$\vec{B}_{s < a} = \mu_o (n_a I_a - n_b I_b) \hat{x}$$

- (b) the magnetic field for the region between the two coils ($a < s < b$)

Since s is now outside of the first solenoid, the only field present is from the second solenoid. This gives us:

$$\vec{B}_{a < s < b} = -\mu_o n_b I_b \hat{x}$$

- (c) the magnetic field for the region outside both coils ($s > b$)

Outside of both solenoids, there would be no field present. Thus, we may simply write:

$$\vec{B}_{s > b} = 0$$

- (d) What ratio of currents would be required to have $\vec{B} = 0$ for $s < a$?

For the field to be zero, we find from (a):

$$\mu_o n_a I_a = \mu_o n_b I_b$$

Which results in a ratio of:

$$\frac{I_a}{I_b} = \frac{n_b}{n_a}$$