

Homework 4

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1. Consider an infinite grounded conducting plane bent at a 90° angle between the yz and xz planes as shown, with a charge placed at $x = 4a$, $y = a$. Use appropriate image charge(s) to find an expression for the potential $V(x, y, z)$ in the region $x > 0$, $y > 0$.

First, we know that the image charges assume the following layout:

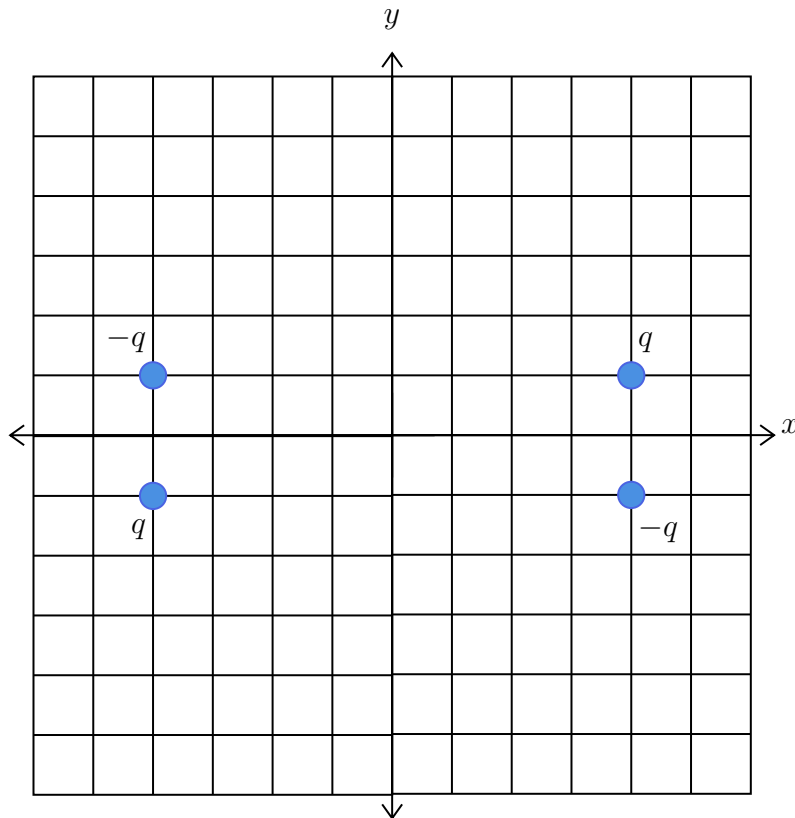


Figure 1: The Layout of Image Charges

By the principle of superposition, we know that the voltage at a point will simply be the sum of all voltages. Thus, we can assign each charge, starting with the top left, a distance $d_1 - d_4$. We can find the following:

$$\begin{aligned}d_1 &= \sqrt{(x+4a)^2 + (y-a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 + 8ax - 2ay} \\d_2 &= \sqrt{(x-4a)^2 + (y-a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 - 8ax - 2ay} \\d_3 &= \sqrt{(x+4a)^2 + (y+a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 + 8ax + 2ay} \\d_4 &= \sqrt{(x-4a)^2 + (y+a)^2 + z^2} = \sqrt{x^2 + y^2 + z^2 + 17a^2 - 8ax + 2ay}\end{aligned}$$

We can then write the voltage as:

$$\begin{aligned}V &= V_1 + V_2 + V_3 + V_4 \\&= \frac{q}{4\pi\epsilon_o} \left[-\frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3} - \frac{1}{d_4} \right]\end{aligned}$$

This gives us the final expression:

$$\begin{aligned}V(x, y, z) &= \frac{q}{4\pi\epsilon_o} \left[\frac{1}{\sqrt{(x-4a)^2 + (y-a)^2 + z^2}} + \frac{1}{\sqrt{(x+4a)^2 + (y+a)^2 + z^2}} \right. \\&\quad \left. - \frac{1}{\sqrt{(x+4a)^2 + (y-a)^2 + z^2}} - \frac{1}{\sqrt{(x-4a)^2 + (y+a)^2 + z^2}} \right]\end{aligned}$$

2. The boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$ is held at a constant potential $+V_0$ and the other, from $y = a/2$ to $y = a$ is held at a constant potential of V_0 . Solve for the potential $V(x, y, z)$ inside the slot. Feel free to use the relevant results from Example 3.3 or from lecture as a starting point.

We can first write the Laplace equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Given the boundary conditions, we may write:

$$V(x, y) = \sum_{n=1}^{\infty} c_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

We find the value of c_n by rearranging:

$$c_n = \frac{2}{a} \left[\int_0^{\frac{a}{2}} V_o \sin\left(\frac{n\pi y}{a}\right) dy - \int_{\frac{a}{2}}^a V_o \sin\left(\frac{n\pi y}{a}\right) dy \right]$$

$$c_n = \frac{2V_o a}{an\pi} \left[\left(-\cos\left(\frac{n\pi y}{a}\right) \Big|_0^{\frac{a}{2}} \right) + \left(\cos\left(\frac{n\pi y}{a}\right) \Big|_{\frac{a}{2}}^a \right) \right]$$

$$c_n = \frac{2V_o}{n\pi} \left[1 + \cos(n\pi) - 2\cos\left(\frac{n\pi}{2}\right) \right]$$

From this, we can see that $c_n = 0$ for any odd values of n . Furthermore, if n is a multiple of 4, $c_n = 0$ as well. Thus, we can see that c_n is non-zero only for $n = 2, 6, 10, \dots$ for which:

$$c_n = \frac{8V_o}{n\pi}$$

We can now substitute into our previous equation to obtain:

$$V(x, y) = \frac{8V_o}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{n\pi x}{a} \sin\left(\frac{n\pi y}{a}\right)}}{n}$$

3. Consider a long (semi-infinite) rectangular conducting pipe oriented V_0 parallel to the z -axis, with dimensions $a \times b$ in the xy -plane. The pipe itself is grounded, and the rectangle at the closed end is at a constant potential V_0 . Find an expression for the potential everywhere inside the pipe (for $z > 0$).

For this problem, we must apply a three dimensional Laplace equation, with boundary conditions:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \Rightarrow \begin{cases} V = 0, & \begin{cases} x = 0 \\ x = a \\ y = 0 \\ y = b \end{cases} \\ V = V_o, & z = 0 \\ V \rightarrow 0, & z \rightarrow \infty \end{cases}$$

We can then divide the equation by $V(x, y, z)$ to obtain:

$$\underbrace{\frac{1}{V_x} \frac{\partial^2 V_x}{\partial x^2}}_{-A^2} + \underbrace{\frac{1}{V_y} \frac{\partial^2 V_y}{\partial y^2}}_{-B^2} + \underbrace{\frac{1}{V_z} \frac{\partial^2 V_z}{\partial z^2}}_{C^2} = 0$$

Note: the z^2 term will be positive to guarantee at least one exponentially decaying solution. This gives us:

$$C^2 = A^2 + B^2$$

$$\frac{\partial^2 V_x}{\partial x^2} = -A^2 V_x \quad \frac{\partial^2 V_y}{\partial y^2} = -B^2 V_y \quad \frac{\partial^2 V_z}{\partial z^2} = (A^2 + B^2) V_z$$

Given this form, we know the solutions will be of form:

$$\begin{aligned}V_x &= P \sin(Ax) + Q \cos(Ax) \\V_y &= R \sin(Bx) + S \cos(Bx) \\V_z &= T e^{\sqrt{A^2+B^2}z} + U e^{-\sqrt{A^2+B^2}z}\end{aligned}$$

To simplify, we can now apply some of our boundary conditions from above. Let us first apply the last condition ($V_z \rightarrow 0$ as $z \rightarrow \infty$). This gives us $S = 0$:

$$\begin{aligned}V_x &= P \sin(Ax) + Q \cos(Ax) \\V_y &= R \sin(Bx) + S \cos(Bx) \\V_z &= U e^{-\sqrt{A^2+B^2}z}\end{aligned}$$

Now, by conditions one and three, we know that, when $V = 0$, $x = 0$ and $y = 0$, giving $Q = 0$ and $T = 0$:

$$\begin{aligned}V_x &= P \sin(Ax) \\V_y &= R \sin(Bx) \\V_z &= U e^{-\sqrt{A^2+B^2}z}\end{aligned}$$

From conditions two and four, we know that, when $V = 0$, $x = a$ and $y = b$, which gives us:

$$\begin{aligned}V_x &= P \sin\left(\frac{m\pi x}{a}\right) \\V_y &= R \sin\left(\frac{n\pi y}{b}\right) \\V_z &= U e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}\end{aligned}$$

Thus, we get $V(x, y, z)$:

$$V(x, y, z) = PRU \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}$$

We can assume PRU is some constant, which will be expressed as M :

$$V(x, y, z) = M \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}z}$$

We can find all values of M by summing and replacing M with M_{mn} :

$$V(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}z}$$

Applying the final boundary condition, or $V = V_o$ when $z = 0$, we can obtain:

$$V_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

Now, we multiply both sides by $\sin\left(\frac{m'\pi x}{a}\right)$ and $\sin\left(\frac{n'\pi y}{b}\right)$ and integrate to get:

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn} \int_0^a \int_0^b \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{n'\pi y}{b}\right) dx dy \\ = \int_0^a \int_0^b V_o \sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{n'\pi y}{b}\right) dx dy \end{aligned}$$

Then we get:

$$\begin{aligned} M_{mn} \frac{a}{2} \delta_{mm'} \frac{b}{2} \delta_{nn'} &= \int_0^a \int_0^b V_o \sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{n'\pi y}{b}\right) dx dy \\ M_{m'n'} &= \frac{4}{ab} \int_0^a \int_0^b V_o \sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{n'\pi y}{b}\right) dx dy \end{aligned}$$

We can replace all of the m' and n' by m and n again, since we effectively removed all of the m and n 's from the equation:

$$M_{mn} = \frac{4V_o}{ab} \int_0^a \int_0^b \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) dx dy$$

Analyzing the equations, we can see that, when m or n is odd, $M_{mn} = 0$, and, if m and n are both even, then:

$$M_{mn} = \frac{16V_o}{\pi^2 mn}$$

Thus, the final solution, for $z > 0$, becomes:

$$V(x, y, z) = \frac{16V_o}{\pi^2} \sum_{m,n=1,3,5,\dots}^{\infty} \frac{1}{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}z}$$

4. Consider an empty spherical shell of charge of radius R where the potential on the surface is given by $V(R, \theta) = V_o \sin^2(\theta)$.

Hint: Express $\sin^2(\theta)$ as a polynomial function of $\cos(\theta)$.

(a) Find $V(r, \theta)$ inside the shell.

The potential inside and outside can be expressed as:

$$\begin{aligned} \sum_{l=0}^{\infty} a_l r^l P_l(\cos(\theta)) & \quad r < R \\ \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos(\theta)) & \quad r \geq R \end{aligned}$$

Given that $V_o \sin^2(\theta) \rightarrow [1 - \cos^2(\theta)]$, we can use Legendre polynomials:

$$\begin{aligned} V(R, \theta) &= V_o \left(P_0(\cos(\theta)) - \frac{2P_2(\cos(\theta)) + P_0(\cos(\theta))}{3} \right) \\ &= \frac{2V_o}{3} (P_0(\cos(\theta)) - P_2(\cos(\theta))) \end{aligned}$$

Written in polynomial expansion form, we find:

$$a_0 R^0 P_0(\cos(\theta)) + a_2 R^2 P_2(\cos(\theta)) = \frac{2V_o}{3} (P_0(\cos(\theta)) - P_2(\cos(\theta)))$$

Thus, we see:

$$a_0 = \frac{2V_o}{3} \quad a_2 = -\frac{2V_o}{3R^2}$$

Finally, we find that the potential within the shell is:

$$\boxed{V(r, \theta) = \frac{2V_o}{3} - \frac{V_o r^2}{3R^2} [3 \cos^2(\theta) - 1]}$$

(b) Find $\vec{E}(R, \theta)$ just inside the shell.

To find the electric field, we can apply:

$$\vec{E} = -\vec{\nabla} V$$

This becomes:

$$\begin{aligned} \vec{\nabla} \left[\frac{2V_o}{3} - \frac{V_o r^2}{3R^2} [3 \cos^2(\theta) - 1] \right] &= -\frac{\partial V}{\partial r} \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} \\ -\frac{\partial}{\partial r} \left[\frac{2V_o}{3} - \frac{V_o r^2}{3R^2} [3 \cos^2(\theta) - 1] \right] \hat{\mathbf{r}} &- \frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{2V_o}{3} - \frac{V_o r^2}{3R^2} [3 \cos^2(\theta) - 1] \right] \hat{\theta} \end{aligned}$$

Finally, we get:

$$\boxed{\vec{E}(R, \theta) = \frac{2V_o R}{3R^2} [3 \cos^2(\theta) - 1] \hat{\mathbf{r}} - \frac{V_o R}{R^2} \sin(2\theta) \hat{\theta}}$$

(c) Find $V(r, \theta)$ out of the shell.

Using a similar process to (a), we find:

$$\frac{b_0}{R} P_0(\cos(\theta)) + \frac{b_2}{R^3} P_2(\cos(\theta)) = \frac{2V_0}{3} (P_0(\cos(\theta)) - P_2(\cos(\theta)))$$

This gives us:

$$b_0 = \frac{2V_0 R}{3} \quad b_2 = -\frac{2V_0 R^3}{3}$$

And finally, we end up with:

$$V(r, \theta) = \frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3} [3 \cos^2(\theta) - 1]$$

(d) Find $\vec{E}(R, \theta)$ just outside the shell.

Similarly to (b), we can find the electric field outside the shell using:

$$\vec{E} = -\vec{\nabla} V$$

This gives us:

$$-\vec{\nabla} \left[\frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3} [3 \cos^2(\theta) - 1] \right] - \frac{\partial}{\partial r} \left[\frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3} [3 \cos^2(\theta) - 1] \right] \hat{\mathbf{r}} - \frac{1}{r} \frac{\partial}{\partial \theta} \left[\frac{2V_0 R}{3r} - \frac{V_0 R^3}{3r^3} [3 \cos^2(\theta) - 1] \right] \hat{\theta}$$

Finally, we obtain:

$$\vec{E}(R, \theta) = \left(\frac{2V_0 R}{3r^2} - \frac{V_0 R^3}{r^4} [3 \cos^2(\theta) - 1] \right) \hat{\mathbf{r}} - \frac{V_0 R^3}{r^4} \sin(2\theta) \hat{\theta}$$

(e) Find $\sigma(R, \theta)$ on the shell. [answer: $\sigma = \frac{V_0 \epsilon_o}{3R} (7 - 15 \cos^2(\theta))$]

At the surface, we can assume that $r = R$, and use the following formula:

$$\frac{\sigma}{\epsilon_o} = (E_{r,out} - E_{r,in})$$

This gives us:

$$\sigma = \frac{V_0 \epsilon_o}{R} \left(\left[\frac{2}{3} - 3 \cos^2(\theta) + 1 \right] - \left[2 \cos^2(\theta) - \frac{2}{3} \right] \right) \\ \sigma = \frac{V_0 \epsilon_o}{R} \left(\frac{7}{3} - 5 \cos^2(\theta) \right)$$

By factoring the one-third, we can finally obtain:

$$\sigma = \frac{V_0 \epsilon_o}{3R} (7 - 15 \cos^2(\theta))$$

5. An empty spherical shell of radius R has potential V_0 on the upper hemisphere and V_0 on the lower hemisphere

- (a) Calculate the first two non-zero terms of the expression for the potential outside of the sphere to obtain an approximate expression for $V(r, \theta)$ in this region.

First and foremost, we know the expression for the voltage outside may be written as:

$$\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta))$$

The constant, B_l , may be calculated using:

$$B_l = \frac{(2l+1)}{2} R^{l+1} V_0 \left[\int_0^{\frac{\pi}{2}} P_l(\cos(\theta)) \sin(\theta) d\theta - \int_{\frac{\pi}{2}}^{\pi} P_l(\cos(\theta)) \sin(\theta) d\theta \right]$$

For $l = 0$, we get:

$$B_0 = \frac{V_0 R}{2} [0] = 0$$

For $l = 1$, we get:

$$B_1 = \frac{3V_0 R^2}{2} \left[\int_0^{\frac{\pi}{2}} \frac{\sin(2\theta)}{2} d\theta - \int_{\frac{\pi}{2}}^{\pi} \frac{\sin(2\theta)}{2} d\theta \right]$$

$$B_1 = \frac{3V_0 R^2}{2}$$

For $l = 2$, we get:

$$B_2 = \frac{5V_0 R^3}{2} \left[\int_0^{\pi} \frac{3 \cos^2(\theta) \sin(\theta) - \sin(\theta)}{2} d\theta \right] = 0$$

For $l = 3$, we get:

$$B_3 = \frac{7V_0 R^4}{2} \left[\int_0^{\pi} \frac{5 \cos^3(\theta) \sin(\theta) - 3 \cos(\theta) \sin(\theta)}{2} d\theta \right] = -\frac{7V_0 R^4}{8}$$

We now plug these coefficients into the sum to get:

$$V(r, \theta) = \frac{3V_0 R^2}{2r^2} \cos(\theta) - \frac{7V_0 R^4}{16r^4} (5 \cos^3(\theta) - 3 \cos(\theta))$$

- (b) From this approximate expression, compute the value of $V(R, \theta)$ (on the surface of the shell) for $\theta = 0$, $\theta = \pi/4$, and $\theta = 3\pi/4$ compare the results with the exact values at those locations

First and foremost, we know $r = R$, which gives us:

$$V(R, \theta) = \frac{3}{2}V_0 \cos(\theta) - \frac{7}{16}V_0 (5 \cos^3(\theta) - 3 \cos(\theta))$$

We can then check:

- At $\theta = 0$:

$$V(R, 0) = \frac{3}{2}V_0 - \frac{7}{16}V_0(2)$$

$$\boxed{V(R, 0) = \frac{5}{8}V_0}$$

- At $\theta = \frac{\pi}{4}$:

$$\begin{aligned} V\left(R, \frac{\pi}{4}\right) &= \frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0 \left(\frac{5(2)^{\frac{3}{2}}}{8} - \frac{3\sqrt{2}}{2}\right) \\ &= \frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0 \left(\frac{5\sqrt{2}}{4} - \frac{3\sqrt{2}}{2}\right) \\ &= \frac{3\sqrt{2}}{4}V_0 + \frac{7\sqrt{2}}{64}V_0 \end{aligned}$$

$$\boxed{V\left(R, \frac{\pi}{4}\right) = \frac{55\sqrt{2}}{64}V_0}$$

- At $\theta = \frac{3\pi}{4}$:

$$\begin{aligned} V\left(R, \frac{3\pi}{4}\right) &= -\frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0 \left(-\frac{5(2)^{\frac{3}{2}}}{8} + \frac{3\sqrt{2}}{2}\right) \\ &= -\frac{3\sqrt{2}}{4}V_0 - \frac{7}{16}V_0 \left(-\frac{5\sqrt{2}}{4} + \frac{3\sqrt{2}}{2}\right) \\ &= -\frac{3\sqrt{2}}{4}V_0 - \frac{7\sqrt{2}}{64}V_0 \end{aligned}$$

$$\boxed{V\left(R, \frac{3\pi}{4}\right) = -\frac{55\sqrt{2}}{64}V_0}$$