

Magnetostatics

Michael Brodskiy

Professor: D. Wood

October 19, 2023

- Magnetostatics:

$$\vec{F}_{\text{lorentz}} = q\vec{v} \times \vec{B}$$

- This is “static” in the sense of steady flow of magnetic field
- The units of \vec{B} are Teslas [T]
- In the simple case where $\vec{B} = B_o\hat{z}$ and $v_z = 0$, the force could be described as:

$$|\vec{F}| = qvB$$

- With inward direction. We can then write:

$$qvB = \frac{mv^2}{R}$$

$$qB = \frac{mv}{R}$$

$$qB = \frac{p}{R}$$

$$R = \frac{p}{qB}$$

- The frequency of rotation may be written:

$$\omega = \frac{qB}{m}$$

- Total force may be written as:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} = q(\vec{E} + \vec{v} \times \vec{B})$$

- * Special Case: $\vec{F} = 0$

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

- * Given $\vec{E} = E_o \hat{\mathbf{z}}$ and $\vec{B} = B_o \hat{\mathbf{x}}$, we would get:

$$E_o \hat{\mathbf{z}} + v_z B_o \hat{\mathbf{y}} - v_y B_o \hat{\mathbf{z}} = 0$$

- * From this, we get $v_z = 0$ and $v_y = \frac{E_o}{B_o}$
 - * This can be used to construct a velocity selector
 - * In the event that $\vec{E} \parallel \vec{B}$, an expand or contracting helix about the fields as a pole would be constructed
- We now go back to a general case: $\vec{E} \perp \vec{B}$, $\vec{F}_{net} \neq 0$, $\vec{E} = E \hat{\mathbf{z}}$, and $\vec{B} = B \hat{\mathbf{x}}$

$$\frac{1}{m} \frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

- * The solutions for this would look as follows:

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{E}{B} t + c_3$$

$$z(t) = c_2 \cos(\omega t) + c_1 \sin(\omega t) + c_4$$

$$\omega = \frac{qB}{m}$$

- * This shape is known as a cycloid
- * Special case: starting from rest at (0,0); this would give us:

$$c_1 = 0 \quad c_2 = -\frac{E}{\omega B} \quad c_3 = -c_1 \quad c_4 = -c_2$$

$$y(t) = -\frac{E}{\omega B} \sin(\omega t) + \frac{E}{B} t = R(\omega t - \sin(\omega t))$$

$$z(t) = R(1 - \cos(\omega t))$$

$$\vec{v}_{cent} = \left(\frac{\vec{E} \times \vec{B}}{B^2} \right)$$

- * Work:

$$dW = \vec{F} \cdot d\vec{l} = (q\vec{v} \times \vec{B}) \cdot \vec{v} dt = q dt (\vec{v} \times \vec{B}) \cdot \vec{v} = 0$$

- * Critical note: magnetic fields do no work
- * Magnetic fields may induce electric fields to do work, but do not do work themselves

• Continuous Systems

- Given a wire in space, with a bit of charge, dq , moving with velocity \vec{v} shaped in rectangle, and placed in a magnetic field $\vec{B} = \frac{A}{z} \hat{\mathbf{x}}$:

$$dq \vec{v} = I d\vec{l}$$

$$d\vec{F} = I d\vec{l} \times \vec{B}$$

$$\vec{F}_{mag} = \int I d\vec{l} \times \vec{B}$$

– Solving would give us:

$$\begin{aligned} F_{tot} &= -IL \left(\frac{A}{a} \right) \hat{\mathbf{z}} + \hat{\mathbf{y}} \int_a^b \frac{A}{z} dz + \hat{\mathbf{z}} IL \left(\frac{A}{b} \right) - \hat{\mathbf{y}} I \int_a^b \frac{A}{z} dz \\ &= -\hat{\mathbf{z}} ILA \left(\frac{1}{a} - \frac{1}{b} \right) \end{aligned}$$

– There are several types of densities:

$$\text{Linear: } I d\vec{l}$$

$$\text{Surface: } \vec{K} da$$

$$\text{Bulk: } \vec{J} d\tau$$

- The following is an important continuity equation:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

– This is another form of charge conservation

- The Biot-Savart law is akin to Coulomb’s law, but for magnetism:

$$d\vec{B} = \frac{\mu_o I d\vec{l} \times \hat{\mathbf{R}}}{4\pi R^2}$$

– Not the expression can be modified:

$$I d\vec{l} \Longleftrightarrow \vec{K} da \Longleftrightarrow \vec{J} d\tau$$

– The magnetic permeability of free space is:

$$\mu_o = 4\pi \cdot 10^{-7} \left[\frac{\text{N}}{\text{A}^2} \right]$$

* This term “defines the amp”, and the current then “defines the coulomb”

* Also note: the magnetic field is defined in Newtons per amp-meter

- Ampère’s Law

$$\vec{\nabla} \times \vec{B} = \mu_o J \Longleftrightarrow \vec{B} = \frac{\mu_o}{4\pi} \int \frac{\vec{J} \times \hat{\mathbf{R}}}{R^2} d\tau$$

– Notice, we can then use Stokes’ Theorem to write:

$$\oint \vec{B} dl = \int_V (\vec{\nabla} \times \vec{B}) d\vec{a} = \int \mu_o J^2 \cdot d\vec{a} = \mu_o I_{enc}$$