

# Homework 6

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1. A point charge of charge  $q$  is located a distance  $d$  from a neutral atom with polarizability  $\alpha$ . The field from the point charge will induce a dipole moment in the atom, resulting in a force between the two objects. Find the magnitude of the force and indicate if it is attractive or repulsive.

We know that the electric field due to the point charge may be expressed as:

$$\vec{E}_q = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

The induced dipole moment may be defined as:

$$\vec{p} = \alpha \vec{E}_q = \frac{\alpha q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

The electric field as a result of the dipole then becomes:

$$\vec{E}_{\vec{p}} = \frac{\vec{p}}{4\pi\epsilon_0 d^3}$$

Since we know  $d \rightarrow r$ , we can simplify:

$$\vec{E}_{\vec{p}} = \frac{1}{4\pi\epsilon_0 r^3} \left( \frac{\alpha q}{4\pi\epsilon_0 r^2} \right)$$

Since we know  $\vec{F} = \vec{E}q$ , we can find the force as:

$$\vec{F}_{\vec{p}} = \frac{1}{4\pi\epsilon_0 r^3} \left( \frac{\alpha q}{4\pi\epsilon_0 r^2} \right) q$$

This can be simplified to:

$$\boxed{\vec{F}_{\vec{p}} = \frac{\alpha q^2}{(4\pi\epsilon_0)^2 r^5}}$$

Because the sign opposes  $q$ , we know this is an attractive force

2. A parallel plate capacitor has two metal plates and is filled with two different linear dielectrics. Each dielectric has thickness  $d/2$ . The lower dielectric (A) has a dielectric constant  $\epsilon_r = 3$  and the upper one (B) has dielectric constant  $\epsilon_r = 5$ . The upper plate as a charge density of  $+\sigma$  and the lower plate as a charge density of  $-\sigma$

- (a) Find the electric displacement  $\vec{D}$  in the dielectrics

We can define the electric field inside the region as twice the electric field due to a single sheet, which gives us:

$$\begin{aligned}\vec{E} &= \vec{E}_1 + \vec{E}_2 \\ \vec{E} &= \frac{\sigma}{2\epsilon_1} + \frac{\sigma}{2\epsilon_2} \\ \vec{E} &= \frac{\sigma}{2\epsilon_o\epsilon_{r1}} + \frac{\sigma}{2\epsilon_o\epsilon_{r2}} \\ \vec{E} &= \frac{\sigma}{6\epsilon_o} + \frac{\sigma}{10\epsilon_o} \\ \vec{E} &= \frac{4\sigma}{15\epsilon_o}\end{aligned}$$

We know that, by definition,  $\vec{D} = \epsilon_r\epsilon_o\vec{E}$ , which then gives us:

$$\begin{aligned}\vec{D}_A &= (3\epsilon_o)\vec{E} \\ \boxed{\vec{D}_A &= \frac{4\sigma}{5}} \\ \vec{D}_B &= (5\epsilon_o)\vec{E} \\ \boxed{\vec{D}_B &= \frac{4\sigma}{3}}\end{aligned}$$

- (b) Find the electric field  $\vec{E}$  in each dielectric

The electric field can be defined as above by:

$$\vec{E} = \frac{\sigma}{\epsilon_r\epsilon_o}$$

Thus, for the first slab we get:

$$\begin{aligned}\vec{E}_A &\rightarrow \epsilon_r = 3 \\ \boxed{\vec{E}_A &= \frac{\sigma}{3\epsilon_o}} \\ \vec{E}_B &\rightarrow \epsilon_r = 5 \\ \boxed{\vec{E}_B &= \frac{\sigma}{5\epsilon_o}}\end{aligned}$$

- (c) Find the potential difference between the plates

The potential can be defined as:

$$V = \vec{E} \cdot d$$

Which we can then use to write:

$$V = \left( \frac{\sigma}{3\epsilon_o} + \frac{\sigma}{5\epsilon_o} \right) \frac{d}{2}$$

This evaluates to:

$$\boxed{V = \frac{4\sigma d}{15\epsilon_o}}$$

- (d) Find the location and value of all of the bound charge

Given that this is a plate capacitor, we know  $\rho_b = 0$ , and all of the charge resides on the surface. We can find the surface bound charge by taking the vertical direction as  $+\hat{\mathbf{x}}$ . We first find the polarizability:

$$\vec{P} = \epsilon_o \chi_e \vec{E}$$

$$\vec{P} = \epsilon_o (\epsilon_r - 1) \vec{E}$$

$$\vec{P}_A = \frac{2\sigma}{3}$$

$$\vec{P}_B = \frac{4\sigma}{5}$$

We can then find the charge in three locations, between the top plate and  $B$ , between  $A$  and  $B$ , and between the bottom plate and  $A$ . This gives us:

$$\boxed{\sigma_B = \frac{4\sigma}{5} \cdot \hat{\mathbf{x}} \rightarrow \frac{4\sigma}{5}}$$

$$\boxed{\sigma_{AB} = \frac{4\sigma}{5} \cdot -\hat{\mathbf{x}} + \frac{2\sigma}{3} \cdot \hat{\mathbf{x}} \rightarrow -\frac{2\sigma}{15}}$$

$$\boxed{\sigma_A = \frac{2\sigma}{3} \cdot -\hat{\mathbf{x}} \rightarrow -\frac{2\sigma}{3}}$$

Summing up the bound charge gives us  $\sigma_b = 0$

3. Calculate the minimum possible volume for a 1[F] capacitor that can withstand 2.5[V] without breaking down. Assume that the geometry and plate separation can be optimized and that the thickness of the conducting plates is negligible.

- (a) Assume the dielectric is air (dielectric strength = 3[MV/m], dielectric constant = 1)

In this case, we know  $V = 2.5[\text{V}]$ ,  $\vec{E} = 3 \cdot 10^6[\text{V/m}]$ ,  $C = 1[\text{F}]$ , and  $\varepsilon_r = 0$ . We can then write:

$$\begin{aligned}\vec{E} &= \frac{V}{d} \\ d &= \frac{V}{\vec{E}} \\ d &= \frac{2.5}{3 \cdot 10^6} \\ d &= 8.33\bar{3} \cdot 10^{-7}[\text{m}]\end{aligned}$$

We know:

$$\begin{aligned}C &= \frac{\varepsilon_o \varepsilon_r A}{d} \\ A &= \frac{dC}{\varepsilon_o \varepsilon_r} \\ A &= \frac{8.33\bar{3} \cdot 10^{-7}(1)}{8.85 \cdot 10^{-12}} \\ A &= 9.416 \cdot 10^4[\text{m}^2]\end{aligned}$$

Now, we can find the volume:

$$\begin{aligned}V &= A \cdot d \\ V &= 9.416 \cdot 10^4 \cdot 8.33\bar{3} \cdot 10^{-7} \\ V &= .0785[\text{m}^3]\end{aligned}$$

- (b) Assume the dielectric is strontium titanate (dielectric strength = 8[MV/m], dielectric constant = 233)

We can use a similar approach as that from (a):

$$\begin{aligned}d &= \frac{V}{\vec{E}} \\ d &= \frac{2.5}{8 \cdot 10^6} \\ d &= 3.125 \cdot 10^{-7}[\text{m}]\end{aligned}$$

And then:

$$A = \frac{Cd}{\varepsilon_o \varepsilon_r}$$

$$A = \frac{(1)(3.125 \cdot 10^{-7})}{8.85 \cdot 10^{-12} \cdot 233}$$

$$A = 151.55[\text{m}^2]$$

Finally, we get:

$$V = Ad$$

$$V = (3.125 \cdot 10^{-7}) (151.55)$$

$$V = 4.925 \cdot 10^{-5}[\text{m}^3]$$

Hint: Consider stored energy per unit volume

4. Two long coaxial cylindrical metal tubes (inner radius  $a$  and outer radius  $b$ ) stand vertically in a tank of dielectric oil (susceptibility  $\chi_e$ , mass density  $\rho_m$ ). The inner cylinder is maintained at a potential  $V$  and the outer one is grounded. To what height  $h$  does the oil rise, in the space between the tubes?

We know that for a long coaxial cable, with charge density  $\lambda$ , the electric field can be defined as:

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_o r} \hat{\mathbf{r}}$$

Thus, for the air portion, the voltage would be:

$$V = \int_a^b \frac{\lambda}{2\pi\epsilon_o r} dr$$

$$V(r) = \frac{\lambda}{2\pi\epsilon_o} \ln\left(\frac{b}{a}\right)$$

For the oil portion, we can write:

$$\vec{E} = \frac{\lambda'}{2\pi\epsilon r} \hat{\mathbf{r}}$$

Which then gives:

$$V(r) = \frac{\lambda'}{2\pi\epsilon} \ln\left(\frac{b}{a}\right)$$

We then set the two voltages equal to obtain:

$$\lambda' = \frac{\lambda\epsilon}{\epsilon_o}$$

We know the total charge in the cable with total length  $l$  is:

$$Q = \lambda(l - h) + \lambda h$$

We can substitute our value of  $\lambda$  to get:

$$Q = \lambda(l - h) + \frac{\lambda \varepsilon h}{\varepsilon_o}$$

Since we know  $\varepsilon = \varepsilon_o \varepsilon_r$ , we can write:

$$Q = \lambda(l - h) + \lambda \varepsilon_r h$$

$$Q = \lambda l + \lambda h(\varepsilon_r - 1)$$

Furthermore, since  $(\varepsilon_r - 1) = \chi_e$ , we get:

$$Q = \lambda(l + h\chi_e)$$

We then use the definition of capacitance to find:

$$C = \frac{Q}{\Delta V}$$

This gives us:

$$C = \frac{\lambda(l + h\chi_e)(2\pi\varepsilon_o)}{\lambda \ln\left(\frac{b}{a}\right)}$$

$$C = \frac{(l + h\chi_e)(2\pi\varepsilon_o)}{\ln\left(\frac{b}{a}\right)}$$

Since we know the energy of a capacitor is defined as:

$$U = \frac{1}{2}CV^2$$

We can find the upward force using:

$$F = \frac{1}{2}V^2 \frac{\partial C}{\partial h}$$

This is defined as:

$$\frac{\partial}{\partial h} \left( \frac{(l + h\chi_e)(2\pi\varepsilon_o)}{\ln\left(\frac{b}{a}\right)} \right) \rightarrow \frac{2\chi_e\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}$$

This gives us an upward force of:

$$F = \frac{\chi_e \pi \epsilon_o V^2}{\ln\left(\frac{b}{a}\right)}$$

Now, we need to find the gravitational force (which acts downward) on the oil. This can be done using the density of the oil:

$$m = \rho_m V$$

The volume of the area with oil present may be defined as the difference of volumes with radii  $a$  and  $b$ :

$$V = \pi(b^2 - a^2)h$$

$$m = \rho_m \pi (b^2 - a^2) h$$

The gravitational force is known to be  $F = mg$ , which gives us:

$$F = \rho_m \pi (b^2 - a^2) hg$$

We want the oil to be in equilibrium, so the upward force must be equal to the downward force. This yields:

$$\rho_m \pi (b^2 - a^2) hg = \frac{\chi_e \pi \epsilon_o V^2}{\ln\left(\frac{b}{a}\right)}$$

We then rearrange to obtain the height as:

$$h = \frac{\chi_e \epsilon_o V^2}{g \rho_m (b^2 - a^2) \ln\left(\frac{b}{a}\right)}$$

5. A point charge with charge  $q$  is fixed at the center of a sphere of radius  $R$  made of a linear dielectric material with susceptibility  $\chi_e$ . Find:

- (a) The electric field outside the sphere

The electric field outside the sphere can be defined as normal, using Gauss's law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_o}$$

$$\vec{E} \oint d\vec{a} = \frac{q_{enc}}{\epsilon_o}$$

$$\vec{E}_{r>R} = \frac{q}{4\pi\epsilon_o r^2} \hat{\mathbf{r}}$$

(b) The electric field in the sphere

Within the material, we need to define the electric field with respect to  $\vec{D}$ :

$$\oint \vec{D} \cdot d\vec{a} = q_{enc}$$

$$\vec{D} \oint d\vec{a} = q_{enc}$$

$$\vec{D} = \frac{q}{4\pi r^2} \hat{\mathbf{r}}$$

Furthermore, we know  $\vec{D} = \varepsilon_r \vec{E} = \varepsilon_o(1 + \chi_e) \vec{E}$ , which yields:

$$\boxed{\vec{E}_{r < R} = \frac{q}{4\pi\varepsilon_o(1 + \chi_e)r^2} \hat{\mathbf{r}}}$$

(c) The bound volume charged density  $\rho_b$

The polarization may be defined as:

$$\vec{P} = \varepsilon_o \chi_e \vec{E}_{r < R}$$

We can find the bound volume charge density by taking the negative gradient of the polarization:

$$-\vec{\nabla}(\vec{P}) = \vec{\nabla} \left( \frac{q\chi_e}{4\pi(1 + \chi_e)r^2} \hat{\mathbf{r}} \right)$$

$$\rho_b = -\frac{q\chi_e}{4\pi(1 + \chi_e)} \vec{\nabla} \left( \frac{\hat{\mathbf{r}}}{r^2} \right)$$

We know:

$$\vec{\nabla} \left( \frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi\delta^2(\vec{r})$$

Thus, we end up with:

$$\boxed{\rho_b = -\frac{q\chi_e}{(1 + \chi_e)} \delta^3(\vec{r})}$$

(d) The bound surface charge density  $\sigma_b$  on the outer surface

The surface charge density can be found once again using the polarization, except with  $r = R$ :

$$\sigma_b = \vec{P}_{r=R} \cdot \hat{\mathbf{r}}$$

This gives us:



$$\sigma_b = \frac{q\chi_e}{4\pi(1 + \chi_e)R^2}$$

Interesting question (not for credit): The dielectric sphere itself must be neutral, so where is the missing charge?

Using our result from part (d), we can see:

$$q_{surf} = \sigma_b \cdot A = \sigma_b(4\pi R^2)$$

$$q_{surf} = \frac{q\chi_e}{(1 + \chi_e)}$$

Thus, we see that, to compensate, there must be charge located at the center.