

Potentials and Fields

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- The energy density of a field may be written as:

$$U = \frac{1}{2} \left(\epsilon_o \vec{E}^2 + \frac{1}{\mu_o} \vec{B}^2 \right)$$

- The Poynting Vector may be defined as:

$$\vec{S} = \frac{1}{\mu_o} (\vec{E} \times \vec{B})$$

- For some volume τ , we can write the total energy in it using:

$$\mathcal{U}_{tot} = \int_V U d\tau$$

- The Poynting Vector represents the flow of energy out of the surface

$$\vec{\nabla} \cdot \vec{S} = \frac{1}{\mu_o} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \frac{1}{\mu_o} (\vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \vec{E} \cdot (\vec{\nabla} \times \vec{B}))$$

$$\vec{\nabla} \cdot \vec{S} = -\frac{\partial}{\partial t} \left(\frac{\epsilon_o \vec{E}^2}{2} + \frac{\vec{B}^2}{2\mu_o} \right)$$

- This can be simplified to attain:

$$\vec{\nabla} \cdot \vec{S} + \frac{\partial U}{\partial t} = 0$$

- A similar equation can be seen in conservation of charge:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

- Thus, we have obtained a conservation of energy formula

- The Maxwell Stress Tensor (\overleftrightarrow{T})

$$d\vec{F} = \overleftrightarrow{T} \cdot d\vec{a}$$

- T_{ij} is the force per area in the j -direction of an area oriented in the i -direction

$$\oint_S \overleftrightarrow{T} \cdot d\vec{a} = \vec{F}_{ext}$$

$$\oint_V (\vec{\nabla} \cdot \overleftrightarrow{T}) \cdot d\tau = \vec{F}_{ext}$$

$$- T_{ij} = \varepsilon_o (E_i E_j - \frac{1}{2} \delta_{ij} E^2) + \frac{1}{\mu_o} (B_i B_j - \frac{1}{2} \delta_{ij} B^2)$$

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \varepsilon_o \left((\vec{\nabla} \cdot \vec{E}) \vec{E} + (\vec{E} \cdot \vec{\nabla}) \vec{E} - \frac{1}{2} \vec{\nabla} E^2 \right) + \frac{1}{\mu_o} \left((\vec{\nabla} \cdot \vec{B}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{B} - \frac{1}{2} \vec{\nabla} B^2 \right)$$

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \varepsilon_o \left((\vec{\nabla} \cdot \vec{E}) \vec{E} - \vec{E} \times (\vec{\nabla} \times \vec{E}) \right) + \frac{1}{\mu_o} \left((\vec{\nabla} \cdot \vec{B}) \vec{B} - \vec{B} \times (\vec{\nabla} \times \vec{B}) \right)$$

- Here, we see all four of Maxwell's laws. This lets us rewrite:

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \underbrace{\rho \vec{E} + \mu_o \vec{J} \times \vec{B}}_{\text{Force per Volume, } \vec{f}} + \underbrace{\varepsilon_o \vec{E} \times \left(\frac{\partial \vec{B}}{\partial t} \right) + \varepsilon_o \vec{B} \times \left(\frac{\partial \vec{E}}{\partial t} \right)}_{\frac{\partial}{\partial t} (\varepsilon_o \vec{E} \times \vec{B})}$$

- We can ultimately simplify to get:

$$\vec{\nabla} \cdot \overleftrightarrow{T} = \vec{f} - \frac{\partial}{\partial t} \mu_o \varepsilon_o \vec{S}$$

- We can rewrite this as:

$$\frac{d\vec{p}}{dt} = F_{ext} - \frac{\partial}{\partial t} \int \varepsilon_o \mu_o \vec{S} d\tau$$

- The momentum density may be written as:

$$\vec{g} = \frac{\vec{p}}{V} = \varepsilon_o \mu_o \vec{S} = \varepsilon_o \vec{E} \times \vec{B}$$

- This can be summed up as:

$$-\vec{\nabla} \cdot \overleftrightarrow{T} + \frac{\partial \vec{g}}{\partial t} = 0$$

- Which is the continuity and conservation of momentum

- Radiation Pressure

- For Absorption:

$$P = \vec{g}c \cdot \hat{\mathbf{n}}$$

- For Reflection:

$$P = 2\vec{g}c \cdot \hat{\mathbf{n}}$$

- The time-averaged pressure may be defined as:

$$\langle P \rangle = \frac{1}{T} \int_0^T P \cdot dt$$

- Angular Momentum

- The angular momentum may be defined as:

$$\vec{L} = \vec{r} \times \vec{p}$$

- To find the angular momentum density, we may write:

$$\vec{l} = \vec{p} \times \vec{g}$$

- The torque on the inner cylinder of a coaxial cable may be written as:

$$N = R_1 q E_\phi = \frac{q\varepsilon}{2\pi} = -\frac{q}{2\pi} \frac{d\phi_1}{dt}$$

- The change in angular momentum may be written as:

$$\Delta L_1 = \frac{qB_i R_1^2}{2}$$

- For the outer cylinder, the change would be the same, except with $R_1 \rightarrow R_2$

$$\Delta L_{tot} = \frac{qB_i}{2} (R_1^2 - R_2^2)$$

$$\vec{l} = \frac{qB_i r}{2\pi s \Delta z} \hat{\theta}$$

$$l_z = \frac{qB_i}{2\pi \Delta z}$$

- We can see that angular momentum is conserved if and only if L_{EM} is included