Homework 1

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1. Calculate:

   (a) $\vec{\nabla}\left(\frac{1}{r}\right)$

   We know $r = \sqrt{x^2+y^2+z^2}$. Converting and applying the chain rule, we get:

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\hat{x} + \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\hat{y} + \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}\hat{z}$$

$$(x^2+y^2+z^2)^{-\frac{3}{2}}(x\hat{x}+y\hat{y}=z\hat{z})$$

$$\frac{x\hat{x}+y\hat{y}+z\hat{z}}{r^3}$$

   We also know that $\vec{r} = x\hat{x}+y\hat{y}+z\hat{z}$, and that $r\hat{r} = \vec{r}$. Thus, we get:

$$\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\vec{r}}{r^3} \to \frac{\hat{r}}{r^2}}$$

   (b) $\vec{\nabla}\cdot\hat{x}$

   This implies the following:

$$v_x = 1\hat{x}, \quad v_y = 0\hat{y}, \quad v_z = 0\hat{z}$$

   Thus, we find:

$$\boxed{\vec{\nabla}\cdot\hat{x} = \frac{\partial}{\partial x}(1\hat{x}) = 0}$$

   (c) $\vec{\nabla}\cdot\hat{r}$

   This could be written as:

$$\vec{\nabla}\cdot\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}}\right)$$

Thus, we need to compute:

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

with respect to each variable. By the quotient rule, we find

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) \to \frac{\sqrt{x^2+y^2+z^2} - x^2(x^2+y^2+z^2)^{-0.5}}{x^2+y^2+z^2}$$

Converting back to r, we obtain:

$$\frac{1}{r} - \frac{x^2}{r^3}$$

By symmetry, we know that the corresponding y and z variables become:

$$\frac{1}{r} - \frac{y^2}{r^3} \quad \text{and} \quad \frac{1}{r} - \frac{z^2}{y^3}$$

Summing the results, we get:

$$\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)$$

$$\frac{3}{r} - \left(\frac{x^2+y^2+z^2}{r^3}\right)$$

$$\boxed{\vec{\nabla} \cdot \hat{r} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}}$$

(d) $\vec{\nabla} r^n$ $(n > 0)$

$$\vec{\nabla}((x^2+y^2+z^2)^{\frac{n}{2}}) \Rightarrow \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,(2x\hat{x} + 2y\hat{y} + 2z\hat{z})$$

$$n\vec{r}r^{n-2} \to \frac{n\vec{r}}{r^{2-n}}$$

Thus, we get:

$$\boxed{\frac{n\hat{r}}{r^{1-n}} \quad \text{or} \quad n\hat{r}r^{n-1}}$$

2. Calculate the divergence and curls of the following functions:

(a) $\vec{v}_a = xy\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + zy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_a = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zy)$$

$$\boxed{\mathrm{div}(\vec{v}_a) = y + z + y = 2y + z}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_a = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zy \end{vmatrix} = (z - y)\hat{x} - (0 - 0)\hat{y} + (0 - x)\hat{z}$$

$$\boxed{\mathrm{curl}(\vec{v}_a) = \langle z - y, 0, -x \rangle}$$

(b) $\vec{v}_b = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2xz\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_b = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2xz)$$

$$\boxed{\mathrm{div}(\vec{v}_b) = 0 + 2x + 2x = 4x}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_b = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2xz \end{vmatrix} = (0 - 2z)\hat{x} - (2z - 0)\hat{y} + (2y - 2y)\hat{z}$$

$$\boxed{\mathrm{curl}(\vec{v}_b) = \langle -2z, -2z, 0 \rangle}$$

(c) $\vec{v}_c = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_c = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$\boxed{\mathrm{div}(\vec{v}_c) = 0 + 0 + 0 = 0}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_c = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{x} - (y - y)\hat{y} + (z - z)\hat{z}$$

$$\boxed{\mathrm{curl}(\vec{v}_c) = \langle 0, 0, 0 \rangle}$$

3. Calculate the following. Note that some results should be vectors. It is helpful to check the dimensionality of your answers:

(a) $(\vec{r} \cdot \vec{\nabla})\vec{r}$

$$(\vec{r} \cdot \vec{\nabla}) = \frac{\partial}{\partial x}(x\hat{x}) + \frac{\partial}{\partial y}(y\hat{y}) + \frac{\partial}{\partial z}(z\hat{z}) = 1 + 1 + 1 = 3$$

$$\boxed{(\vec{r} \cdot \vec{\nabla})\vec{r} = 3\vec{r} = 3x\hat{x} + 3y\hat{y} + 3z\hat{z}}$$

(b) $(\hat{r} \cdot \vec{\nabla})r$

From Problem **1c**, we know the value of $(\hat{r} \cdot \vec{\nabla})$, which gives us:

$$\boxed{\left(\frac{2}{r}\right) r = 2}$$

(c) $(\hat{r} \cdot \vec{\nabla})\hat{r}$

Again, employing what we know from Problem **1c**, we get:

$$\boxed{\left(\frac{2}{r}\right) \hat{r} = \frac{2\hat{r}}{r}}$$

4. (a)
   (b)
5. (a)
   (b)
   (c)
   (d)
6. (a)
   (b)
7. (a)
   (b)
   (c)