Homework 9

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November 29, 2023

1. Two long coaxial conducting cylinders have radii a and b and length L. The space in between them ($a < s < b$) is filled with a conducting material with a conductivity that varies with radius: $\sigma = k/s$ where k is a constant. Find the resistance between the two cylinders.

   First and foremost, we may write:

$$\vec{J} = \sigma\vec{E}$$

   We can find $\vec{J}$, by using:

$$\vec{J} = \frac{I}{A}$$
$$\vec{J} = \frac{I}{2\pi sL}$$

   Combining the two, we get:

$$\vec{E} = \frac{I}{2\pi s\sigma L}\hat{\mathbf{s}}$$
$$\vec{E} = \frac{I}{2\pi kL}\hat{\mathbf{s}}$$

   From here, we can integrate to find:

$$\Delta V = -\int_b^a \vec{E}\,d\vec{s}$$
$$\Delta V = \frac{I(b-a)}{2\pi kL}$$

   Since, using Ohm's law, we can find the resistance by dividing the voltage by the current, we get:

$$\boxed{R = \frac{(b-a)}{2\pi kL}}$$

2. A rectangular loop of wire with length L and width w is rotated about the y-axis as shown, starting in the xy plane at $t = 0$. There is a uniform magnetic field in the vertical direction $\vec{B} = B_o\hat{\mathbf{z}}$. The total resistance of the loop is R.

   (a) Find the EMF around the loop as a function of time.

   Given an angle θ made between the normal vector, $\hat{\mathbf{n}}$ of the loop and the magnetic field, we may write the flux as:

$$\Phi = BA\cos(\theta)$$

   We know the EMF can be defines as:

$$\varepsilon = -\frac{d\Phi}{dt}$$

   This gives us:

$$\varepsilon = BA\sin(\theta)\frac{d\theta}{dt}$$

   We know that the angle changes at a rate of ω. Furthermore, we can also write $\theta = \omega t$. This yields:

$$\varepsilon = BA\omega\sin(\omega t)$$

   Finally, the area may be described as $A = Lw$, which gives us:

$$\boxed{\varepsilon = BLw\omega\sin(\omega t)}$$

   (b) Find the current as a function of time.

   Since the EMF is essentially an induced voltage, we may find the current as:

$$I = \frac{\varepsilon}{R}$$

   This gives us:

$$\boxed{I = \frac{BLw\omega}{R}\sin(\omega t)}$$

   (c) Find the torque required to keep the loop rotating at a constant angular velocity ω

   The torque may be described as:

$$\tau = Fd$$

   The force from the magnetic field may be described as:

$$F_{\vec{B}} = BIL$$

The distance may be defined by the sine of the same angle defined by θ:

$$d = \frac{w}{2}\sin(\theta) + \frac{w}{2}\sin(\theta)$$

Thus, we combine the two to get:

$$\tau = BILw\sin(\theta)$$

Substituting the value of I found in (b), we obtain:

$$\boxed{\tau = \frac{\omega(BLw\sin(\omega t))^2}{R}}$$

(d) Find the power that the loop converts from mechanical energy to electrical energy.

This power may be described by the EMF, as the electromotive force converts mechanical to electrical energy. According to the formula for power, we may write:

$$P = \frac{V^2}{R} \rightarrow \frac{\varepsilon^2}{R}$$

This results in:

$$\boxed{P = \frac{(BLw\omega\sin(\omega t))^2}{R}}$$

3. A square loop is cut out of a thick sheet of aluminum. It is them placed so that the top portion is in a uniform magnetic field $\vec{B}$, and the loop is allowed to fall under gravity. (In the diagram, shading indicates the field region, and $\vec{B}$ points out of the page.) The field strength is 1[T].

(a) Find the terminal velocity of the loop in m/s. (You will need to look up the mass density and resistivity of aluminum. You can assign variables to the dimensions of the loop, but the value of the terminal velocity should not depend on these.)

Finding the net forces, we may write:

$$mg - BIl = ma$$

From the definition of EMF, we can write:

$$I = \frac{\varepsilon}{R} = \frac{Blv}{R}$$

This gives us:

$$mg - \frac{B^2l^2v}{R} = ma$$

Rearranging, we may write this as:

$$1 = \frac{a}{g - \dfrac{B^2l^2v}{mR}}$$

Taking $a \to \frac{dv}{dt}$, we write:

$$\int_0^t dt = \int_0^v \frac{dv}{g - \dfrac{B^2l^2v}{mR}}$$

To make this easier, we once again rearrange:

$$\int_0^t dt = \int_0^v \frac{mR}{mRg - B^2l^2v}\, dv$$

We take $u = mRg - B^2l^2v$, which gives us $du = -B^2l^2dv$. We can then plug this in to get:

$$\int_0^t dt = -\frac{mR}{B^2l^2}\int_{mRg}^{mRg-B^2l^2v} \frac{1}{u}\, du$$

This gives us:

$$t = \frac{mR}{B^2l^2}\ln\left(\frac{mRg}{mRg - B^2l^2v}\right)$$

$$\frac{B^2l^2}{mR}t = \ln\left(\frac{g}{g - \frac{B^2l^2v}{mR}}\right)$$

$$e^{\frac{B^2l^2}{mR}t} = \frac{g}{g - \frac{B^2l^2v}{mR}}$$

$$e^{-\frac{B^2l^2}{mR}t} = 1 - \frac{B^2l^2v}{mRg}$$

$$1 - e^{-\frac{B^2l^2}{mR}t} = \frac{B^2l^2v}{mRg}$$

And finally, we find the velocity as:

$$v = \frac{mRg}{B^2l^2}\left(1 - e^{-\frac{B^2l^2}{mR}t}\right)$$

To find the terminal velocity, the $m\frac{dv}{dt}$ term would be zero, which means that we find:

$$v_t = \frac{mRg}{B^2 l^2}$$

Since the magnetic field is one Tesla, we may write:

$$v_t = \frac{mRg}{l^2}$$

The resistivity of aluminum is given by:

$$R = \frac{\rho l}{A}$$

$$R = \frac{(2.82 \cdot 10^{-8})l}{A}$$

The mass density of aluminum (in kilogram per cubic meter) is given by:

$$\rho = 2710$$

Thus, the mass is given by:

$$m = 2710V = 2710Al$$

Substituting these values into our formula (note: we multiply both by 4 for each side contribution), we get:

$$v_t = \frac{16(2.82 \cdot 10^{-8} l)(2710Al)g}{Al^2}$$

Canceling out the values, we get:

$$v_t = 16(2.82 \cdot 10^{-8})(2710)g$$

$$\boxed{v_t = .0112 \left[\frac{\mathrm{m}}{\mathrm{s}}\right]}$$

(b) If the loop is released from rest, find the time (in seconds) for it to reach 90% of its terminal velocity.

Using the velocity obtained in (a), we can write:

$$.9v_t = v_t \left(1 - e^{-\frac{B^2 l^2}{mR} t}\right)$$

$$.9 = \left(1 - e^{-\frac{B^2 l^2}{mR} t}\right)$$

$$.1 = e^{-\frac{B^2 l^2}{mR} t}$$

We then take the logarithm of both sides:

$$\frac{B^2l^2}{mR}t = \ln(10)$$

$$t = \frac{mR}{B^2l^2}\ln(10)$$

$$t = 16(2.82 \cdot 10^{-8})(2710)\ln(10)$$

$$\boxed{t = .002815[\mathrm{s}]}$$

4. Consider a toroid with rectangular cross section (inner radius = a, outer radius = b, height = h, number of turns of wire = N)

   (a) What is its self-inductance (assuming the core is empty)?

   We know the formula for the self inductance is:

   $$L = \frac{\mu N^2 l}{2\pi}\ln\left(\frac{r_2}{r_1}\right)$$

   Plugging in the given values, we find:

   $$\boxed{L = \frac{\mu_o N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)}$$

   (b) What is this inductance in Henrys for $a = 0.5[\mathrm{cm}]$, $b = 1.2[\mathrm{cm}]$, $h = 3[\mathrm{mm}]$, $N = 50$

   Using our formula from (a), we get:

   $$L = \frac{(4\pi \cdot 10^{-7})(50)^2(.003)}{2\pi}\ln\left(\frac{1.2}{.5}\right)$$

   This gives us:

   $$\boxed{L = 1.313\,[\mu\mathrm{H}]}$$

   (c) Now calculate the inductance for the same geometry if the core is filled with Ferrite N41 with a relative permeability $\mu_r = 3000$

   Using our formula from part (a), but with the new magnetic permeability, we get:

   $$L = \frac{(3000)(4\pi \cdot 10^{-7})(50)^2(.003)}{2\pi}\ln\left(\frac{1.2}{.5}\right)$$

   This results in:

   $$\boxed{L = 3.94[\mathrm{mH}]}$$

5. Calculate the approximate stored energy of a medical MRI magnet. Treat is as a long solenoid of radius 0.8[m] and length 2.0[m] with a field strength of 1.5[T].

   The stored energy may be defined as:

$$U_B = \frac{VB^2}{2\mu_o}$$

   We know the volume may be defined as the cross-sectional area times the length. This gives us:

$$U_B = \frac{(2)(\pi(.8)^2)(1.5)^2}{2(4\pi \cdot 10^{-7})}$$

   This gives us:

$$\boxed{U_B = 3.6[\text{MJ}]}$$

6. Two coils are wrapped around a cylindrical form in such a way that the same flux passes through every turn of both coils. This forms a simple transformer. The primary turn has N_1 turns and the secondary has N_2 turns.

   (a) Suppose the primary coil is driven with an AC voltage $V_{in} = V_1 \cos(\omega t)$ and the secondary coil is connected to a resistor, R. Show that the two currents satisfy the relations:

$$L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = V_1 \cos(\omega t)$$
$$L_2 \frac{dI_2}{dt} + M \frac{dI_1}{dt} = -I_2 R$$

   First, we can write the fluxes through the two coils as:

$$\phi_1 = I_1 L_1 + I_2 M$$
$$\phi_1 = I_2 L_2 + I_1 M$$

   Given the driving voltage of the first coil, we know that the EMF through the coil must cancel it:

$$V_{in} - \varepsilon_1 = 0$$
$$V_1 \cos(\omega t) - \varepsilon_1 = 0$$

   We can define the EMF as:

$$\varepsilon_1 = \frac{d\phi_1}{dt}$$

This gives us:

$$V_1\cos(\omega t) - L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} = 0$$

$$\boxed{L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt} = V_1\cos(\omega t)}$$

This is the first relation. Now, we know that the sum of the voltages in the second loop must also be zero. This gives us:

$$\varepsilon_2 + I_2R = 0$$

Taking a similar approach, we differentiate to get:

$$L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} + I_2R = 0$$

$$\boxed{L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt} = -I_2R}$$

(b) With a common flux through each loop, the square of the mutual inductance is equal to the product of the two self-inductances: $M^2 = L_1L_2$. Use this to solve the equations above to find $I_1(t)$ and $I_2(t)$. (Assume I_1 has no DC component.)

Using what we know, we can rewrite the two equations as:

$$L_1L_2\frac{dI_1}{dt} + ML_2\frac{dI_2}{dt} = L_2V_1\cos(\omega t)$$

$$L_2\frac{dI_2}{dt} = -M\frac{dI_1}{dt} - I_2R$$

This allows us to combine the two into one equation:

$$M^2\frac{dI_1}{dt} + M\left(-M\frac{dI_1}{dt} - I_2R\right) = L_2V_1\cos(\omega t)$$

$$\cancel{M^2\frac{dI_1}{dt} - M^2\frac{dI_1}{dt}} - MI_2R = L_2V_1\cos(\omega t)$$

$$\boxed{I_2 = -\frac{L_2V_1\cos(\omega t)}{MR}}$$

From here, we can solve for I_1. We first find:

$$\frac{dI_2}{dt} = \frac{\omega L_2V_1\sin(\omega t)}{MR}$$

Substituting into the first equaton, we get:

$$L_1\frac{dI_1}{dt} + \frac{\omega L_2V_1\sin(\omega t)}{R} = V_1\cos(\omega t)$$

$$\frac{dI_1}{dt} = \frac{V_1\cos(\omega t)}{L_1} - \frac{\omega L_2 V_1 \sin(\omega t)}{L_1 R}$$

We then distribute the differentials and integrate:

$$I_1 = \int_0^t \frac{V_1\cos(\omega t)}{L_1} - \frac{\omega L_2 V_1 \sin(\omega t)}{L_1 R}\, dt$$

$$\boxed{I_1 = \frac{V_1}{\omega L_1}\left(\sin(\omega t) + \frac{\omega L_2 \cos(\omega t)}{R}\right)}$$

We can confirm the lack of a DC constant, as all terms are sinusoidal.

(c) Calculate the input power ($P_{in} = V_{in} I_1$) and the output power ($P_{out} = V_{out} I_2$) and show that their averages over a full cycle are equal.

The input power becomes:

$$P_{in} = (V_1\cos(\omega t))\left(\frac{V_1}{\omega L_1}\left(\sin(\omega t) + \frac{\omega L_2 \cos(\omega t)}{R}\right)\right)$$

$$\boxed{P_{in} = \frac{V_1^2}{\omega L_1}\left(\frac{1}{2}\sin(2\omega t) + \frac{\omega L_2 \cos^2(\omega t)}{R}\right)}$$

The output power becomes:

$$P_{out} = I_2(I_2 R)$$

$$P_{out} = I_2^2 R$$

$$\boxed{P_{out} = \frac{(L_2 V_1 \cos(\omega t))^2}{M^2 R}}$$

The average power over a period may be determined by taking:

$$\langle P \rangle = \frac{1}{T}\int_0^T P\, dt$$

Thus, we do this for both powers:

$$\langle P_{in} \rangle = \frac{1}{T}\int_0^T \frac{V_1^2}{\omega L_1}\left(\frac{1}{2}\sin(2\omega t) + \frac{\omega L_2 \cos^2(\omega t)}{R}\right) dt$$

$$\langle P_{in} \rangle = \frac{L_2 V_1^2}{2 L_1 R}$$

And now for the output power:

$$\langle P_{out} \rangle = \frac{1}{T}\int_0^T \frac{(L_2 V_1 \cos(\omega t))^2}{M^2 R}\, dt$$

$$\langle P_{out} \rangle = \frac{L_2^2 V_1^2}{2M^2 R}$$

From the identity, $M^2 = L_1 L_2$, which gives us:

$$\langle P_{out} \rangle = \frac{L_2 V_1^2}{2L_1 R}$$

Thus, we see:

$$\boxed{\langle P_{in} \rangle = \langle P_{out} \rangle = \frac{L_2 V_1^2}{2L_1 R}}$$