Homework 5

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1. Four point charges are located a distance a from the origin on the y and z axes as shown. Find the approximate expression for the electric **potential** far from the charges. Use spherical coordinates and retain the only the **first** non-vanishing terms in the multipole expansion. [Hint: consider breaking down the distribution into a superposition of individual dipoles.]

   We can write the dipole moment from the configuration of charges:

$$\begin{aligned} p &= -2q(-a_y) + 3q(a_z) - 2q(a_y) + q(-a_z) \\ &= \cancel{2qaa_y} + 3qaa_z \cancel{-2qa(a_y)} - aqa_z \\ &= 2qa_z \end{aligned}$$

   We know that:

$$V_{mon}(r) = \frac{Q}{4\pi\varepsilon_o r^2} = 0$$
$$V_{dip}(r) = \frac{\vec{p}\hat{\mathbf{r}}}{4\pi\varepsilon_o r^2}$$

   Thus, we may write the dipole as:

$$V_{dip}(r) = \frac{2qa_z\hat{\mathbf{r}}}{4\pi\varepsilon_o r^2}$$

   This can finally be written in spherical coordinates using:

$$\boxed{V_{dip}(r,\theta) \approx \frac{qa\cos(\theta)}{2\pi\varepsilon_o r^2}}$$

2. Three point charges are located a distance a from the origin on the y and z axes as shown. Find the approximate expression for the electric **field** far from the charges. Use spherical coordinates and retain the **first two** non-vanishing terms in the multipole expansion.

   We begin by working from the lowest order up, which is the monopole. First we find the aggregate charge:

$$Q = -q + q - q = -q$$

   This can simply be plugged into the formula:

$$V_{mon}(r) = -\frac{q}{4\pi\varepsilon_o r}$$

   We then find the dipole contribution, beginning with the moment:

$$\begin{aligned} p &= -q(-a_y) + q(a_z) - q(a_y) \\ &= \cancel{qa_y} + qa_z \cancel{-qa_y} \\ &= qa_z \end{aligned}$$

   We then use the formula:

$$V_{dip}(r) = \frac{\vec{p}\hat{\mathbf{r}}}{4\pi\varepsilon_o r^2}$$

$$V_{dip}(r) = \frac{qa_z\hat{\mathbf{r}}}{4\pi\varepsilon_o r^2}$$

$$V_{dip}(r,\theta) = \frac{qa\cos(\theta)}{4\pi\varepsilon_o r^2}$$

   Summing the two contributions, we get:

$$V(r,\theta) \approx -\frac{q}{4\pi\varepsilon_o r} + \frac{qa\cos(\theta)}{4\pi\varepsilon_o r^2}$$

   We now use the formula:

$$\vec{E} = -\vec{\nabla}V$$

$$\vec{E} = \vec{\nabla}\left(\frac{q}{4\pi\varepsilon_o r} - \frac{qa\cos(\theta)}{4\pi\varepsilon_o r^2}\right)$$

$$\vec{E} = \frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_o r} - \frac{qa\cos(\theta)}{4\pi\varepsilon_o r^2}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(-\frac{qa\cos(\theta)}{4\pi\varepsilon_o r^2}\right)$$

$$\vec{E} = \left(-\frac{q}{4\pi\varepsilon_o r^2} + \frac{qa\cos(\theta)}{2\pi\varepsilon_o r^3}\right)\hat{\mathbf{r}} + \left(\frac{qa\sin(\theta)}{4\pi\varepsilon_o r^3}\right)\hat{\theta}$$

Finally, this yields:

$$\boxed{\vec{E}(r,\theta) = \frac{q}{2\pi\varepsilon_o r^2}\left[\left(-\frac{1}{2} + \frac{a\cos(\theta)}{r}\right)\hat{\mathbf{r}} + \left(\frac{a\sin(\theta)}{2r}\right)\hat{\theta}\right]}$$

3. For the charged spherical shell in Problem 4 of Assignment 4 (the one with $V(R,\theta) = V_o \sin^2(\theta)$ and $\sigma = \frac{V_o\varepsilon_o}{3R}(7 - 15\cos^2(\theta))$, find the monopole and dipole moments.

   First, we compute the monopole moment. This is done by first finding the total charge:

$$q_{tot} = \int \sigma(R,\theta)\, da$$
$$da' = 2\pi R^2 \sin(\theta)\, d\theta$$

   We can then compute the integral:

$$q = \int_0^\pi \frac{V_o\varepsilon_o}{3R}[7 - 15\cos^2(\theta)](2\pi R^2\sin(\theta))\, d\theta$$
$$q = \frac{2V_o\varepsilon_o\pi R}{3}\int_0^\pi 7\sin(\theta) - 15\sin(\theta)\cos^2(\theta)\, d\theta$$
$$q = \frac{2V_o\varepsilon_o\pi R}{3}\int_0^\pi -8\sin(\theta) + 15\sin^3(\theta)\, d\theta$$

   Using a numerical solver, we obtain:

$$q = \frac{2V_o\varepsilon_o\pi R}{3}\left(-7\cos(\theta) + 5\cos^3(\theta)\right)\Big|_0^\pi$$
$$q = \frac{4V_o\varepsilon_o\pi R}{3} - \left(-\frac{4V_o\varepsilon_o\pi R}{3}\right)$$

   And finally, we find:

$$\boxed{q_{tot} = \frac{8V_o\varepsilon_o\pi R}{3}}$$

   Now, we find the dipole moment. The integral set-up becomes very similar, except that:

$$\vec{p} = p\hat{\mathbf{z}} = \int \vec{r}\hat{\mathbf{z}}\sigma\, da$$

   This can be converted to:

$$p\hat{\mathbf{z}} = \int_0^\pi \frac{V_o\varepsilon_o}{3R}[7 - 15\cos^2(\theta)](2\pi R^2\sin(\theta))(R\cos(\theta))\, d\theta$$

$$= \frac{2V_o\varepsilon_o R^2\pi}{3}\int_0^{\pi}[7-15\cos^2(\theta)](\sin(\theta))(\cos(\theta))\,d\theta$$
$$= \frac{V_o\varepsilon_o R^2\pi}{3}\int_0^{\pi} 7\sin(2\theta)-15\sin(2\theta)\cos^2(\theta)\,d\theta$$

Again implementing a numerical solver, we obtain:

$$= \frac{V_o\varepsilon_o R^2\pi}{3}\left(-\frac{7}{2}\cos(2\theta)+\frac{15}{2}\cos^4(\theta)\right)\Big|_0^{\pi}$$
$$= \frac{V_o\varepsilon_o R^2\pi}{3}\left(-\frac{7}{2}+\frac{15}{2}-\left[-\frac{7}{2}+\frac{15}{2}\right]\right)$$
$$= 0$$

Thus, we see:

$$\boxed{\begin{cases} \vec{p}_{mon} = \dfrac{8V_o\varepsilon_o\pi R}{3} \\ \vec{p}_{dip} = 0 \end{cases}}$$

4. A thin rod on the z-axis goes from $z=-a$ to $z=+a$ and carries a linear charge density of $\lambda(z)$. Find the leading term in the multipole expansion for:

   (a) $\lambda(z) = \lambda_o\cos\left(\frac{\pi z}{a}\right)$

   We can write the formula for the multipole expansion as:

$$V = \frac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}\frac{P_n(\cos(\theta))}{r^{n+1}}\int_{-a}^{a} z^n\lambda(z)\,dz$$

   For the $n=0$ case, we can write:

$$V = \frac{1}{4\pi\varepsilon_o r}\int_{-a}^{a}\lambda(z)\,dz$$
$$V = \frac{\lambda_0}{4\pi\varepsilon_o r}\int_{-a}^{a}\cos\left(\frac{\pi z}{a}\right)\,dz$$

   From this, we can see that the integral expression would evaluate to zero, meaning we have to try the next term. At $n=1$, we get:

$$V = \frac{\cos(\theta)}{4\pi\varepsilon_o r^2}\int_{-a}^{a} z\lambda(z)\,dz$$
$$V = \frac{\lambda_o\cos(\theta)}{4\pi\varepsilon_o r^2}\int_{-a}^{a} z\cos\left(\frac{\pi z}{a}\right)\,dz$$

$$V = \frac{\lambda_o \cos(\theta)}{4\pi\varepsilon_o r^2}\left(\underbrace{\cancel{\frac{a}{\pi} z \sin\left(\frac{\pi z}{a}\right)}}_{0} + \frac{a^2}{\pi^2}\cos\left(\frac{\pi z}{a}\right)\right)\Big|_{-a}^{a}$$

$$V = \frac{\lambda_o \cos(\theta)}{4\pi\varepsilon_o r^2}\left(\frac{a^2}{\pi^2}\cos\left(\frac{\pi z}{a}\right)\right)\Big|_{-a}^{a}$$

Once again, we see that the integral expression evaluates to zero. Thus, we move up another order to $n = 2$:

$$V_{quad}(r,\theta) = \frac{1}{4\pi\varepsilon_o}\frac{3\cos^2(\theta) - 1}{2r^3}\int_{-a}^{a} z^2 \lambda(z)\, dz$$

$$V_{quad}(r,\theta) = \frac{\lambda_o}{4\pi\varepsilon_o}\frac{3\cos^2(\theta) - 1}{2r^3}\int_{-a}^{a} z^2 \cos\left(\frac{\pi z}{a}\right)\, dz$$

Using a numerical solver, we calculate the integral:

$$V_{quad}(r,\theta) = \frac{\lambda_o}{4\pi\varepsilon_o}\frac{3\cos^2(\theta) - 1}{2r^3}\underbrace{\left(\frac{z^2 a}{\pi}\sin\left(\frac{\pi z}{a}\right) + \frac{2a^2 z}{\pi^2}\cos\left(\frac{\pi z}{a}\right) - \frac{2a^3}{\pi^2}\sin\left(\frac{\pi z}{a}\right)\right)\Big|_{-a}^{a}}_{-\frac{4a^3}{\pi^2}}$$

Thus, the leading term becomes:

$$\boxed{V_{quad}(r,\theta) = -\frac{a^3 \lambda_o \left(3\cos^2(\theta) - 1\right)}{2\pi^3 \varepsilon_o r^3}}$$

(b) $\lambda(z) = \lambda_1 \cos\left(\frac{\pi z}{2a}\right)$
where λ_o and λ_1 are constants.
For $\lambda(z) = \lambda_1 \cos\left(\frac{\pi z}{2a}\right)$, we can begin by finding the first term, according to the formula from (a), at $n = 0$:

$$V_{mon} = \frac{1}{4\pi\varepsilon_o}\frac{1}{r}\int_{-a}^{a} \lambda(z)\, dz$$

$$V_{mon} = \frac{1}{4\pi\varepsilon_o}\frac{\lambda_1}{r}\int_{-a}^{a} \cos\left(\frac{\pi z}{2a}\right)\, dz$$

We can then evaluate the integral:

$$\int_{-a}^{a} \cos\left(\frac{\pi z}{2a}\right)\, dz = \frac{2a}{\pi}\sin\left(\frac{\pi z}{2a}\right)\Big|_{-a}^{a}$$

$$= \frac{4a}{\pi}$$

Which then becomes:

$$\frac{\lambda_1}{4\pi\varepsilon_o r}\left(\frac{4a}{\pi}\right)$$

And finally:

$$\boxed{V_{mon}(r) = \frac{a\lambda_1}{\pi^2\varepsilon_o r}}$$