Potentials

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$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_o} \Leftrightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_o}$$

This means that, in a region with no charge, $\vec{\nabla} \cdot \vec{E} = 0$, which also means $\nabla^2 V = 0$ (the Laplacian) Rectangular Coordinates:

$$\nabla^2 V = 0 \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x, y, z) = X(x)Y(y)Z(z) \Rightarrow \frac{\partial^2 X}{\partial x^2}YZ + \frac{\partial^2 Y}{\partial y^2}XZ + \frac{\partial^2 Z}{\partial z^2}XY = 0$$

If we divide by the respective functions, we get:

$$\frac{\partial^2}{\partial x^2}\frac{1}{X} + \frac{\partial^2}{\partial y^2}\frac{1}{Y} + \frac{\partial^2}{\partial z^2}\frac{1}{Z} = 0$$

$$\frac{d^2 X}{dx^2} = c_1 X(x) \Rightarrow c_1 > 0 :$$

$$X(x) = \begin{cases} Ae^{\pm kx} & c_1 = k^2 \\ A\sin(kx) & c_1 = -k^2 \\ A\cos(kx) & c_1 = -k^2 \end{cases}$$

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  - Repeating this for each variable, we find

$$\frac{d^2 Y}{dy^2} = c_2 Y(y) \qquad \frac{d^2 Z}{dz^2} = c_2 Z(z)$$

$$c_1 + c_2 + c_3 = 0$$

- Fourier Inversion

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi y}{a}\right) dy = \frac{a}{2}\delta_{nm}$$

This is used to define the terms for Fourier analysis

- A semi-infinite square tube, with the four walls grounded:

  - The boundary conditions are (with $V = 0$) given by:

$$\begin{cases} x = 0, & 0 < y < b \\ x = a, & 0 < y < b \\ y = 0, & 0 < x < a \\ y = b, & 0 < x < a \end{cases}$$

  - This would mean:

$$X = \sin\left(\frac{n\pi x}{a}\right) \qquad Y \propto \sin\left(\frac{m\pi y}{b}\right) \qquad Z = e^{-k_z z}$$

$$V = \sum B_{n,m} e^{-k_z z} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

  - Because we know $c_1 + c_2 + c_3 = 0$, we know:

$$-\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right) + k_z^2 = 0$$

$$k_z = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

- Poisson's Equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o}$$

  - When the charge density is zero, we get Laplace's Equation:

$$\nabla^2 V = 0$$

- Given a box in three dimensions with lengths a, b, c, we know:

$$c_1 + c_2 + c_3 = 0 \longrightarrow \begin{cases} c > 0, & \text{Exponential} \\ c < 0, & \text{Sinusoidal} \end{cases}$$

  - We know the voltage is of the form

$$V = f(z) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{a}\right)$$

  - The function, $f(z)$, must then be of the form:

$$f(z) = Ae^{kz} + Be^{-kz}$$

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– If $V = 0$, then $z = 0$, so:

$$A + B = 0 \rightarrow A = -B$$
$$f(z) = Ae^{kz} - Ae^{-kz}$$
$$= A(e^{kz} - e^{-kz})$$
$$= 2A\sinh(kz)$$

– Thus, we can set up the function as:

$$V(x, y, z) = \sum_{m,n} c_{m,n} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sinh(kz)$$

• Spherical Coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r^2}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial V}{\partial\theta}\right) = 0$$

– In Quantum Mechanics, we have spherical harmonics:

$$Y_{lm}(\theta, \phi) = P_l(\cos(\theta))f_{ml}(\phi)$$

– For our purposes, we can find:

$$V(r, \theta) = \sum_{l=0,1,\dots} \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_e(\cos(\theta))$$

– A typical problem for this would be:

  * Given some potential, $V(\theta)$ on a spherical surface, find V_{in} and V_{out}

$$R_e(r) = A_l r^l + \frac{B_l}{r^{l+1}}$$
$$V_{in} = A_l r^l P_l(\cos(\theta))$$
$$V_{out} = \frac{B_l}{r^{l+1}} P_l(\cos(\theta))$$

  * We know the following boundary conditions:

$$r \rightarrow \infty, V \rightarrow 0$$

  * We can integrate the function:

$$\int_{-1}^{1} V(R, \theta) P_l(\cos(\theta)) d(\cos(\theta)) = \frac{2R^l A_l}{2l + 1}$$

– Example: Given $V(R, \theta) = E_o R \cos(\theta) = E_o R P_1(\cos(\theta))$

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∗ This gives us:

$$\begin{cases} \text{Inside:} & V = A_1 r^2 \cos(\theta) \\ \text{Outside:} & V = \frac{B_1}{r^2} P_1(\cos(\theta)) \end{cases}$$