Homework 1

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1. Calculate:

   (a) $\vec{\nabla}\left(\frac{1}{r}\right)$
   We know $r = \sqrt{x^2+y^2+z^2}$. Converting and applying the chain rule, we get:

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\hat{\mathbf{z}}$$
$$-(x^2+y^2+z^2)^{-\frac{3}{2}}(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}=z\hat{\mathbf{z}})$$
$$\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{r^3}$$

   We also know that $\vec{r} = x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$, and that $r\hat{\mathbf{r}} = \vec{r}$. Thus, we get:

$$\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3} \to -\frac{\hat{\mathbf{r}}}{r^2}}$$

   (b) $\vec{\nabla}\cdot\hat{\mathbf{x}}$
   This implies the following:

$$v_x = 1\hat{\mathbf{x}}, \quad v_y = 0\hat{\mathbf{y}}, \quad v_z = 0\hat{\mathbf{z}}$$

   Thus, we find:

$$\boxed{\vec{\nabla}\cdot\hat{\mathbf{x}} = \frac{\partial}{\partial x}(1\hat{\mathbf{x}}) = 0}$$

   (c) $\vec{\nabla}\cdot\hat{\mathbf{r}}$
   This could be written as:

$$\vec{\nabla}\cdot\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}\right)$$

Thus, we need to compute:

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

with respect to each variable. By the quotient rule, we find

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) \to \frac{\sqrt{x^2+y^2+z^2} - x^2(x^2+y^2+z^2)^{-0.5}}{x^2+y^2+z^2}$$

Converting back to r, we obtain:

$$\frac{1}{r} - \frac{x^2}{r^3}$$

By symmetry, we know that the corresponding y and z variables become:

$$\frac{1}{r} - \frac{y^2}{r^3} \quad \text{and} \quad \frac{1}{r} - \frac{z^2}{y^3}$$

Summing the results, we get:

$$\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)$$

$$\frac{3}{r} - \left(\frac{x^2+y^2+z^2}{r^3}\right)$$

$$\boxed{\vec{\nabla} \cdot \hat{\mathbf{r}} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}}$$

(d) $\vec{\nabla} r^n$ $(n > 0)$

$$\vec{\nabla}((x^2+y^2+z^2)^{\frac{n}{2}}) \Rightarrow \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,(2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}})$$

$$n\vec{r}r^{n-2} \to \frac{n\vec{r}}{r^{2-n}}$$

Thus, we get:

$$\boxed{\frac{n\hat{\mathbf{r}}}{r^{1-n}} \quad \text{or} \quad n\hat{\mathbf{r}}r^{n-1}}$$

2. Calculate the divergence and curls of the following functions:

(a) $\vec{v}_a = xy\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + zy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_a = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zy)$$

$$\boxed{\mathrm{div}(\vec{v}_a) = y + z + y = 2y + z}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zy \end{vmatrix} = (z - y)\hat{\mathbf{x}} - (0 - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}}$$

$$\boxed{\mathrm{curl}(\vec{v}_a) = \langle z - y, 0, -x \rangle}$$

(b) $\vec{v}_b = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2xz\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_b = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2xz)$$

$$\boxed{\mathrm{div}(\vec{v}_b) = 0 + 2x + 2x = 4x}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2xz \end{vmatrix} = (0 - 2z)\hat{\mathbf{x}} - (2z - 0)\hat{\mathbf{y}} + (2y - 2y)\hat{\mathbf{z}}$$

$$\boxed{\mathrm{curl}(\vec{v}_b) = \langle -2z, -2z, 0 \rangle}$$

(c) $\vec{v}_c = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_c = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$\boxed{\mathrm{div}(\vec{v}_c) = 0 + 0 + 0 = 0}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{\mathbf{x}} - (y - y)\hat{\mathbf{y}} + (z - z)\hat{\mathbf{z}}$$

$$\boxed{\mathrm{curl}(\vec{v}_c) = \langle 0, 0, 0 \rangle}$$

3. Calculate the following. Note that some results should be vectors. It is helpful to check the dimensionality of your answers:

(a) $(\vec{r}\cdot\vec{\nabla})\vec{r}$

$$\left(\vec{r}\cdot\vec{\nabla}\right)\vec{r} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$

$$\left(x\frac{\partial}{\partial x}(x\hat{\mathbf{x}}) + y\frac{\partial}{\partial y}(y\hat{\mathbf{y}}) + z\frac{\partial}{\partial z}(z\hat{\mathbf{z}})\right) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

$$\boxed{(\vec{r}\cdot\vec{\nabla})\vec{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} = \vec{r}}$$

(b) $(\hat{\mathbf{r}}\cdot\vec{\nabla})r$
Within the parentheses, the problem is similar, except that we divide by r because $r\hat{\mathbf{r}} = \vec{r}$:

$$\left(\hat{\mathbf{r}}\cdot\vec{\nabla}\right)r = \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)\left(\sqrt{x^2+y^2+z^2}\right)$$

$$\left(x\frac{\partial}{\partial x}(r) + y\frac{\partial}{\partial y}(r) + z\frac{\partial}{\partial z}(r)\right) = \frac{2x}{2r}\frac{x}{r} + \frac{2y}{2r}\frac{y}{r} + \frac{2z}{2r}\frac{z}{r}$$

$$\boxed{(\hat{\mathbf{r}}\cdot\vec{\nabla})r = \frac{x^2+y^2+z^2}{r^2} = 1}$$

(c) $(\hat{\mathbf{r}}\cdot\vec{\nabla})\hat{\mathbf{r}}$

$$\left(\hat{\mathbf{r}}\cdot\vec{\nabla}\right)\hat{\mathbf{r}} = \left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)(\hat{\mathbf{r}})$$

$$\left(\frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} + \frac{z}{r}\frac{\partial}{\partial z}\right)\left(\frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}\right) \Rightarrow$$

$$\frac{x}{r}\left(\frac{y^2+z^2}{r^3}\right)\hat{\mathbf{x}} + \frac{y}{r}\left(\frac{x^2+z^2}{r^3}\right)\hat{\mathbf{y}} + \frac{z}{r}\left(\frac{x^2+y^2}{r^3}\right)\hat{\mathbf{z}}$$

$$(\hat{\mathbf{r}}\cdot\vec{\nabla})\hat{\mathbf{r}} = \left(\frac{xy^2+xz^2}{(x^2+y^2+z^2)^2}\right)\hat{\mathbf{x}} + \left(\frac{yx^2+yz^2}{(x^2+y^2+z^2)^2}\right)\hat{\mathbf{y}} + \left(\frac{zx^2+zy^2}{(x^2+y^2+z^2)^2}\right)\hat{\mathbf{z}}$$

4. Consider a vector field $\vec{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ and a cube with vertices at (0,0,0), (0,2,0), (2,0,0), (0,0,2), etc. In the text (Example 1.7), the flux was calculated through the top five faces of the cube, and found to total to 20.

(a) Find the upward flux through the bottom surface, i.e. the square surface in the xy-plane bounded by the points (0,0,0), (2,0,0), (2,2,0), (0,2,0).
According to Gauss's Theorem, we know:

$$\iiint \vec{\nabla}\cdot\vec{N}\,d\tau = \iint \vec{v}\cdot d\vec{a}$$

Furthermore, because the surface is in the xy plane, and the flux we want to find is in the $\hat{\mathbf{z}}$ direction, we know:

$$d\vec{a} = dx\,dy\,\hat{\mathbf{z}}$$

Combining the two, and drawing from the flux in the $\hat{\mathbf{z}}$ direction, we obtain:

$$\int_0^2 \int_0^2 (yz^2 - 3y)\,dx\,dy\Big|_{z=0} \Rightarrow \int_0^2 \int_0^2 (-3y)\,dx\,dy$$

$$\boxed{2\int_0^2 (-3y)\,dy = -3y^2\Big|_0^2 = -12}$$

(b) Both surfaces have the same boundary. In this case, does the flux depend on the surface, or just the boundary? Explain.

In this case, the flux depends on the surface, not just the boundary. If we take, for example, the solved cube with the same boundary — that is, the one bounded by $0 \leq x, y, z \leq 2$, the integration would be consequently different. The boundary itself is the same; however, the flux is different. We can see this because of the value given to us from the problem:

$$-12 \neq 20$$

Thus, we see that, despite having the same boundary, the two surfaces yield a different flux.

5. Consider a vector field $\vec{v} = r\cos(\theta)\hat{\mathbf{r}} + r\sin(\theta)\hat{\theta} + r\sin(\theta)\cos(\theta)\hat{\phi}$

(a) Calculate the outward flux of $\vec{v}$ through a closed hemispherical surface of radius R shown in the figure[1]

First and foremost, we know:

$$d\vec{a} = \hat{\mathbf{r}}(r^2\sin(\theta))\,d\theta\,d\phi$$

We also know the bounds of θ and ϕ:

$$\begin{cases} 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \phi \leq 2\pi \end{cases}$$

This yields:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^3 \sin(\theta)\cos(\theta)\,d\theta\,d\phi\Big|_{r=R} \Rightarrow 2\pi R^3 \underbrace{\int_0^{\frac{\pi}{2}} \sin(\theta)\cos(\theta)\,d\theta}_{\frac{1}{2}} = \pi R^3$$

Thus, the top part contributes a flux of πR^3. The bottom circle is constrained as follows:

[1] figure omitted in this document

$$d\vec{a} = \hat{\theta} r \, dr \, d\phi$$

$$\begin{cases} \theta = \frac{\pi}{2} \\ 0 \leq \phi \leq 2\pi \end{cases}$$

This yields:

$$\int_0^{2\pi} \int_0^R r^2 \sin(\theta) \, dr \, d\phi \Big|_{\theta = \frac{\pi}{2}} \Rightarrow 2\pi \int_0^R r^2 \, dr = 2\pi \frac{R^3}{3}$$

Summing the two together we get the flux as:

$$\boxed{\frac{5\pi}{3} R^3}$$

(b) Calculate the divergence of $\vec{v}$

$$\frac{1}{r^2} \frac{\partial}{\partial r}(r^3 \cos(\theta)) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta}(r \sin^2(\theta)) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \phi}(r \sin(\theta) \cos(\theta)) \Rightarrow$$

$$3 \cos(\theta) + 2 \cos(\theta) + 0 = \boxed{5 \cos(\theta)}$$

(c) Check the divergence theorem by comparing the flux integral from (a) with the volume integral of the divergence.

For the hemisphere itself, we know the following:

$$\begin{cases} 0 \leq r \leq R \\ 0 \leq \theta \leq \frac{\pi}{2} \\ 0 \leq \phi \leq 2\pi \end{cases}$$

Then taking the boundaries defined above, we obtain the following integral expression:

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^R (5 \cos(\theta))(r^2 \sin(\theta)) \, dr \, d\theta \, d\phi \Rightarrow$$

$$2\pi \int_0^{\frac{\pi}{2}} \int_0^R (5r^2 \cos(\theta) \sin(\theta)) \, dr \, d\theta \Rightarrow$$

$$\frac{10\pi R^3}{3} \int_0^{\frac{\pi}{2}} \sin(\theta) \cos(\theta) \, d\theta \Rightarrow$$

$$\frac{1}{2} \left(\frac{10\pi R^3}{3} \right) = \boxed{\frac{5\pi}{3} R^3}$$

Thus, the volume integral and flux integral are both $\frac{5\pi}{3} R^3$

6. Start with these three expressions that relate rectangular coordinates to cylindrical coordinates: $x = s \cos(\phi), y = s \sin(\phi), z = z$

(a) Derive the expressions for the unit vectors $\hat{\mathbf{s}}, \hat{\phi}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$

$$\hat{\mathbf{s}} = \frac{\frac{\partial}{\partial s}(s\cos(\phi))\hat{\mathbf{x}} + \frac{\partial}{\partial s}(s\sin(\phi))\hat{\mathbf{y}} + \frac{\partial}{\partial s}(z)\hat{\mathbf{z}}}{\sqrt{\cos^2(\phi)+\sin^2(\phi)}} = \cos(\phi)\hat{\mathbf{x}} + \sin(\phi)\hat{\mathbf{y}}$$
$$\hat{\phi} = \frac{\frac{\partial}{\partial \phi}(s\cos(\phi))\hat{\mathbf{x}} + \frac{\partial}{\partial \phi}(s\sin(\phi))\hat{\mathbf{y}} + \frac{\partial}{\partial \phi}(z)\hat{\mathbf{z}}}{\sqrt{s^2\sin^2(\phi)+s^2\cos^2(\phi)}} = -\sin(\phi)\hat{\mathbf{x}} + \cos(\phi)\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \frac{\frac{\partial}{\partial z}(s\cos(\phi))\hat{\mathbf{x}} + \frac{\partial}{\partial z}(s\sin(\phi))\hat{\mathbf{y}} + \frac{\partial}{\partial z}(z)\hat{\mathbf{z}}}{\sqrt{0^2+0^2+1^1}} = \hat{\mathbf{z}}$$

Thus, we see the values as follows:

$$\boxed{\begin{cases} \hat{\mathbf{s}} &= \cos(\phi)\hat{\mathbf{x}} + \sin(\phi)\hat{\mathbf{y}} \\ \hat{\phi} &= -\sin(\phi)\hat{\mathbf{x}} + \cos(\phi)\hat{\mathbf{y}} \\ \hat{\mathbf{z}} &= \hat{\mathbf{z}} \end{cases}}$$

(b) Confirm that $\hat{\mathbf{s}}, \hat{\phi}$, and $\hat{\mathbf{z}}$ are mutually orthogonal and are normalized to unity

We can check for orthogonality by applying the dot product; because $\hat{\mathbf{s}}$ and $\hat{\phi}$ both only have components in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ directions, they are orthogonal to the z component. As such, we need only check $\hat{\mathbf{s}}$ against $\hat{\phi}$:

$$\hat{\mathbf{s}} \cdot \hat{\phi} = (\cos(\phi))(-\sin(\phi)) + (\sin(\phi))(\cos(\phi)) = 0$$

Thus, they are mutually orthogonal. To check for normalization, we must see whether the magnitude of each is equal to 1:

$$\hat{\mathbf{s}} = \sqrt{\cos^2(\phi)+\sin^2(\phi)} = 1$$
$$\hat{\phi} = \sqrt{\sin^2(\phi)+\cos^2(\phi)} = 1$$
$$\hat{\mathbf{z}} = \sqrt{1^2} = 1$$

$$\boxed{\text{As such, } \hat{\mathbf{s}},\ \hat{\phi}, \text{ and } \hat{\mathbf{z}} \text{ are mutually orthogonal and normalized to unity}}$$

In this manner, they make up a coordinate system.

7. Consider two vector functions $\vec{v} = x^2\hat{\mathbf{z}}$ and $\vec{w} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

(a) calculate the divergence and curl of each

- $\text{div}(x^2\hat{\mathbf{z}})$

$$\boxed{\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) = 0}$$

- $\text{curl}(x^2\hat{\mathbf{z}})$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = (0-0)\hat{\mathbf{x}} - (2x-0)\hat{\mathbf{y}} + (0-0)\hat{\mathbf{z}} = \boxed{-2x\hat{\mathbf{y}}}$$

- $\text{div}(\vec{w} = \vec{r})$

$$\boxed{\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3}$$

- $\text{curl}(\vec{w} = \vec{r})$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\hat{\mathbf{x}} - (0-0)\hat{\mathbf{y}} + (0-0)\hat{\mathbf{z}} = \boxed{0}$$

(b) Can $\vec{v}$ or $\vec{w}$ be written as the gradient of a scalar function $g(x, y, z)$? If either case is possible, find such a function g. The answer should be unique aside from an arbitrary constant.

- For $\vec{v} = x^2\hat{\mathbf{z}}$:
  From the textbook, we know that, if a vector function has a non-zero curl, it can not be written as the gradient of a scalar function. Thus, this is not possible for $\vec{v}$.
- For $\vec{w} = \vec{r}$:

$$\int x\,dx = \frac{x^2}{2} + f(y) + g(z) \Rightarrow \frac{\partial}{\partial y}\left(\frac{x^2}{2} + f(y) + g(z)\right) = f'(y)$$

$$f'(y) = y \Rightarrow \int f'(y)\,dy = \frac{x^2}{2} + \frac{y^2}{2} + g(z) \Rightarrow \frac{\partial}{\partial z}\left(\frac{1}{2}\left(x^2 + y^2 + g(z)\right)\right) = g'(z)$$

$$g'(z) = z \Rightarrow \int g'(z)\,dz = \frac{z^2}{2} + c$$

Thus, the final expression for $g(x, y, z)$ becomes:

$$\boxed{g_{\vec{w}}(x, y, z) = \frac{1}{2}\left(x^2 + y^2 + z^2\right) + c}$$

(c) Can $\vec{v}$ or $\vec{w}$ be written as the curl of a vector function $\vec{u}(x, y, z)$? If either case is possible, find such a function $\vec{u}$. The answer is not unique, but it is sufficient to find one function that gives the desired curl.

i. For $\vec{v} = x^2\hat{\mathbf{z}}$:
   Per the definition of curl, we know:

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} \rightarrow \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\hat{\mathbf{z}}$$

This implies:

$$\begin{cases} \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} & = & 0 \\ \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} & = & 0 \\ \\ \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} & = & x^2 \end{cases}$$

Let us assume that $u_z = 0$, since both terms equal zero. We can then take $u_x = 0$ to try and find a function:

$$\int \partial u_y = \int x^2 \, \partial x$$

Solving, we obtain the vector function $\vec{u}$ as:

$$\vec{u} = 0\hat{\mathbf{x}} + \left(\frac{x^3}{3} + c\right)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}$$

We then need to check the curl once more to confirm this works for $\vec{u}$ (using $c = 0$):

$$\begin{cases} \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} & = & 0 & \checkmark \\ \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} & = & 0 & \checkmark \\ \\ \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} & = & x^2 & \checkmark \end{cases}$$

As such, a possible solution for $\vec{u}$ is:

$$\boxed{\vec{u} = 0\hat{\mathbf{x}} + \left(\frac{x^3}{3}\right)\hat{\mathbf{y}} + 0\hat{\mathbf{z}}}$$

ii. For $\vec{w} = \vec{r}$:
Again, per the definition of curl, we find:

$$\begin{cases} \dfrac{\partial u_z}{\partial y} - \dfrac{\partial u_y}{\partial z} & = & x \\ \\ \dfrac{\partial u_x}{\partial z} - \dfrac{\partial u_z}{\partial x} & = & y \\ \\ \dfrac{\partial u_y}{\partial x} - \dfrac{\partial u_x}{\partial y} & = & z \end{cases}$$

We can see from this that, to be a possible solution, $\frac{\partial u_z}{\partial y}$ needs to take the form cx, where c is a constant. This would mean $\frac{\partial u_y}{\partial z}$ would have to take the form dx, where $d = c-1$. This would mean u_z would be of the form $u_z = cxy$ and $u_y = dxz$.
Continuing down, this would mean that $\frac{\partial u_z}{\partial x}$ would be of the form cy, meaning that $\frac{\partial u_x}{\partial z}$ would be of the form ey, where $c = e - 1$. This leaves us with:

$$\begin{cases} u_x = eyz \\ u_y = dxz \\ u_z = cxy \end{cases}$$

This can be written in terms of c:

$$\begin{cases} u_x = (c+1)yz \\ u_y = (c-1)xz \\ u_z = cxy \end{cases}$$

Incorportating this into the last equation, we get:

$$(c-1)z - (c+1)z = z$$

This can never be correct, as, for any constant c, the left side would always produce $-2z$. Thus, because of this "round-about" result, $\vec{v}$ can not be represented as the curl of a function $\vec{u}$[2]

[2] Also, this is not possible by the theorem from the textbook stating that, with a non-zero divergence, the function can not be written as the curl of a vector function.