Homework 1

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1. Calculate:

   (a) $\vec{\nabla}\left(\frac{1}{r}\right)$

   We know $r = \sqrt{x^2+y^2+z^2}$. Converting and applying the chain rule, we get:

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\hat{\mathbf{x}}+\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\hat{\mathbf{y}}+\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}\hat{\mathbf{z}}$$

$$(x^2+y^2+z^2)^{-\frac{3}{2}}(x\hat{\mathbf{x}}+y\hat{\mathbf{y}}=z\hat{\mathbf{z}})$$

$$\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{r^3}$$

   We also know that $\vec{r} = x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$, and that $r\hat{\mathbf{r}} = \vec{r}$. Thus, we get:

$$\boxed{\vec{\nabla}\left(\frac{1}{r}\right) = \frac{\vec{r}}{r^3} \to \frac{\hat{\mathbf{r}}}{r^2}}$$

   (b) $\vec{\nabla}\cdot\hat{\mathbf{x}}$

   This implies the following:

$$v_x = 1\hat{\mathbf{x}}, \quad v_y = 0\hat{\mathbf{y}}, \quad v_z = 0\hat{\mathbf{z}}$$

   Thus, we find:

$$\boxed{\vec{\nabla}\cdot\hat{\mathbf{x}} = \frac{\partial}{\partial x}(1\hat{\mathbf{x}}) = 0}$$

   (c) $\vec{\nabla}\cdot\hat{\mathbf{r}}$

   This could be written as:

$$\vec{\nabla}\cdot\left(\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}}{\sqrt{x^2+y^2+z^2}}\right)$$

Thus, we need to compute:

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)$$

with respect to each variable. By the quotient rule, we find

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right) \to \frac{\sqrt{x^2+y^2+z^2} - x^2(x^2+y^2+z^2)^{-0.5}}{x^2+y^2+z^2}$$

Converting back to r, we obtain:

$$\frac{1}{r} - \frac{x^2}{r^3}$$

By symmetry, we know that the corresponding y and z variables become:

$$\frac{1}{r} - \frac{y^2}{r^3} \quad \text{and} \quad \frac{1}{r} - \frac{z^2}{y^3}$$

Summing the results, we get:

$$\left(\frac{1}{r} - \frac{x^2}{r^3}\right) + \left(\frac{1}{r} - \frac{y^2}{r^3}\right) + \left(\frac{1}{r} - \frac{z^2}{r^3}\right)$$

$$\frac{3}{r} - \left(\frac{x^2+y^2+z^2}{r^3}\right)$$

$$\boxed{\vec{\nabla}\cdot\hat{\mathbf{r}} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}}$$

(d) $\vec{\nabla} r^n$ $(n > 0)$

$$\vec{\nabla}((x^2+y^2+z^2)^{\frac{n}{2}}) \Rightarrow \frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,(2x\hat{\mathbf{x}} + 2y\hat{\mathbf{y}} + 2z\hat{\mathbf{z}})$$

$$n\vec{r}r^{n-2} \to \frac{n\vec{r}}{r^{2-n}}$$

Thus, we get:

$$\boxed{\frac{n\hat{\mathbf{r}}}{r^{1-n}} \quad \text{or} \quad n\hat{\mathbf{r}}r^{n-1}}$$

2. Calculate the divergence and curls of the following functions:

(a) $\vec{v}_a = xy\hat{\mathbf{x}} + yz\hat{\mathbf{y}} + zy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_a = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zy)$$

$$\boxed{\mathrm{div}(\vec{v}_a) = y + z + y = 2y + z}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zy \end{vmatrix} = (z - y)\hat{\mathbf{x}} - (0 - 0)\hat{\mathbf{y}} + (0 - x)\hat{\mathbf{z}}$$

$$\boxed{\mathrm{curl}(\vec{v}_a) = \langle z - y, 0, -x \rangle}$$

(b) $\vec{v}_b = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2xz\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_b = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2xz)$$

$$\boxed{\mathrm{div}(\vec{v}_b) = 0 + 2x + 2x = 4x}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 2xz \end{vmatrix} = (0 - 2z)\hat{\mathbf{x}} - (2z - 0)\hat{\mathbf{y}} + (2y - 2y)\hat{\mathbf{z}}$$

$$\boxed{\mathrm{curl}(\vec{v}_b) = \langle -2z, -2z, 0 \rangle}$$

(c) $\vec{v}_c = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$

- Divergence:

$$\vec{\nabla} \cdot \vec{v}_c = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$\boxed{\mathrm{div}(\vec{v}_c) = 0 + 0 + 0 = 0}$$

- Curl:

$$\vec{\nabla} \times \vec{v}_c = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\hat{\mathbf{x}} - (y - y)\hat{\mathbf{y}} + (z - z)\hat{\mathbf{z}}$$

$$\boxed{\mathrm{curl}(\vec{v}_c) = \langle 0, 0, 0 \rangle}$$

3. Calculate the following. Note that some results should be vectors. It is helpful to check the dimensionality of your answers:

(a) $(\vec{r}\cdot\vec{\nabla})\vec{r}$

$$(\vec{r}\cdot\vec{\nabla}) = \frac{\partial}{\partial x}(x\hat{\mathbf{x}}) + \frac{\partial}{\partial y}(y\hat{\mathbf{y}}) + \frac{\partial}{\partial z}(z\hat{\mathbf{z}}) = 1+1+1 = 3$$

$$\boxed{(\vec{r}\cdot\vec{\nabla})\vec{r} = 3\vec{r} = 3x\hat{\mathbf{x}} + 3y\hat{\mathbf{y}} + 3z\hat{\mathbf{z}}}$$

(b) $(\hat{\mathbf{r}}\cdot\vec{\nabla})r$

From Problem **1c**, we know the value of $(\hat{\mathbf{r}}\cdot\vec{\nabla})$, which gives us:

$$\boxed{\left(\frac{2}{r}\right) r = 2}$$

(c) $(\hat{\mathbf{r}}\cdot\vec{\nabla})\hat{\mathbf{r}}$

Again, employing what we know from Problem **1c**, we get:

$$\boxed{\left(\frac{2}{r}\right)\hat{\mathbf{r}} = \frac{2\hat{\mathbf{r}}}{r}}$$

4. Consider a vector field $\vec{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$ and a cube with vertices at (0,0,0), (0,2,0), (2,0,0), (0,0,2), etc. In the text (Example 1.7), the flux was calculated through the top five faces of the cube, and found to total to 20.

(a) Find the upward flux through the bottom surface, i.e. the square surface in the xy-plane bounded by the points (0,0,0), (2,0,0), (2,2,0), (0,2,0).

According to Green's Theorem, we know:

$$\iiint \vec{\nabla}\cdot\vec{N}\,d\tau = \iint \vec{v}\cdot d\vec{a}$$

Furthermore, because the surface is in the xy plane, and the flux we want to find is in the $\hat{\mathbf{z}}$ direction, we know:

$$d\vec{a} = dx\,dy\,\hat{\mathbf{z}}$$

Combining the two, and drawing from the flux in the $\hat{\mathbf{z}}$ direction, we obtain:

$$\int_0^2\int_0^2 (yz^2 - 3y)\,dx\,dy\Big|_{z=0} \Rightarrow \int_0^2\int_0^2 (3y)\,dx\,dy$$

$$\boxed{2\int_0^2 (3y)\,dy = 3y^2\Big|_0^2 = 12}$$

(b) Both surfaces have the same boundary. In this case, does the flux depend on the surface, or just the boundary? Explain.

5. Consider a vector field $\vec{v} = r\cos(\theta)\hat{\mathbf{r}} + r\sin(\theta)\hat{\theta} + r\sin(\theta)\cos(\theta)\hat{\phi}$

   (a) Calculate the outward flux of $\vec{v}$ through a closed hemispherical surface of radius R shown in the figure[1]

   First and foremost, we know:

$$d\vec{a} = \hat{\mathbf{r}}\, d\theta\, d\phi$$

   We also know the bounds of θ and ϕ:

$$\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{2} \end{cases}$$

   This yields:

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} r\cos(\theta)\, d\theta\, d\phi \Big|_{r=R} \Rightarrow \frac{\pi R}{2} \int_0^{2\pi} \cos(\theta)\, d\theta = 0$$

   (b) Calculate the divergence of $\vec{v}$

$$\frac{\partial}{\partial r}(r\cos(\theta)) + \frac{\partial}{\partial \theta}(r\sin(\theta)) + \frac{\partial}{\partial \phi}(r\sin(\theta)\cos(\theta)) = \cos(\theta) + r\cos(\theta)$$

   (c) Check the divergence theorem by comparing the flux integral from (a) with the volume integral of the divergence.

   For the hemisphere itself, we know the following:

$$\begin{cases} 0 \leq r \leq R \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \frac{\pi}{2} \end{cases}$$

   Then taking the boundaries defined above, we obtain the following integral expression:

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{R} (\cos(\theta) + r\cos(\theta))(r^2\sin(\theta))\, dr\, d\theta\, d\phi \Rightarrow$$
$$\frac{\pi}{2} \int_0^{2\pi} \int_0^{R} (r^2\sin(\theta)\cos(\theta) + r^3\sin(\theta)\cos(\theta))\, dr\, d\theta \Rightarrow$$
$$\frac{\pi}{2} \int_0^{2\pi} \frac{R^3}{3}\sin(\theta)\cos(\theta) + \frac{R^4}{4}\sin(\theta)\cos(\theta)\, d\theta \Rightarrow$$
$$\frac{\pi}{2}\left(\frac{4R^3 + 3R^4}{12}\right) \underbrace{\int_0^{2\pi} \sin(\theta)\cos(\theta)\, d\theta}_{0} = 0$$

   Then we need to find the flux through the bottom circle:

[1] figure omitted in this document

6. (a)
   (b)

7. Consider two vector functions $\vec{v} = x^2\hat{\mathbf{z}}$ and $\vec{w} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

   (a) calculate the divergence and curl of each

   - div($x^2\hat{\mathbf{z}}$)

   $$\boxed{\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) = 0}$$

   - curl($x^2\hat{\mathbf{z}}$)

   $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} = (0-0)\hat{\mathbf{x}} - (2x-0)\hat{\mathbf{y}} + (0-0)\hat{\mathbf{z}} = \boxed{-2x\hat{\mathbf{y}}}$$

   - div($\vec{w} = \vec{r}$)

   $$\boxed{\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3}$$

   - curl($\vec{w} = \vec{r}$)

   $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)\hat{\mathbf{x}} - (0-0)\hat{\mathbf{y}} + (0-0)\hat{\mathbf{z}} = \boxed{0}$$

   (b)
   (c)