Potentials

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$$\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc}}{\varepsilon_o} \Leftrightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_o}$$

This means that, in a region with no charge, $\vec{\nabla}\cdot\vec{E} = 0$, which also means $\nabla^2 V = 0$ (the Laplacian) Rectangular Coordinates:

$$\nabla^2 V = 0 \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x,y,z) = X(x)Y(y)Z(z) \Rightarrow \frac{\partial^2 X}{\partial x^2}YZ + \frac{\partial^2 Y}{\partial y^2}XZ + \frac{\partial^2 Z}{\partial z^2}XY = 0$$

If we divide by the respective functions, we get:

$$\frac{\partial^2}{\partial x^2}\frac{1}{X} + \frac{\partial^2}{\partial y^2}\frac{1}{Y} + \frac{\partial^2}{\partial z^2}\frac{1}{Z} = 0$$

$$\frac{d^2X}{dx^2} = c_1 X(x) \Rightarrow c_1 > 0 :$$

$$X(x) = \begin{cases} Ae^{\pm kx} & c_1 = k^2 \\ A\sin(kx) & c_1 = -k^2 \\ A\cos(kx) & c_1 = -k^2 \end{cases}$$

- – Repeating this for each variable, we find

$$\frac{d^2Y}{dy^2} = c_2 Y(y) \qquad \frac{d^2Z}{dz^2} = c_2 Z(z)$$

$$c_1 + c_2 + c_3 = 0$$

- Fourier Inversion

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\, dy = \frac{a}{2}\delta_{nm}$$

This is used to define the terms for Fourier analysis

- A semi-infinite square tube, with the four walls grounded:
  - The boundary conditions are (with $V = 0$) given by:

$$\begin{cases} x = 0, & 0 < y < b \\ x = a, & 0 < y < b \\ y = 0, & 0 < x < a \\ y = b, & 0 < x < a \end{cases}$$

  - This would mean:

$$X = \sin\left(\frac{n\pi x}{a}\right) \qquad Y \propto \sin\left(\frac{m\pi y}{b}\right) \qquad Z = e^{-k_z z}$$

$$V = \sum B_{n,m} e^{-k_z z} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

  - Because we know $c_1 + c_2 + c_3 = 0$, we know:

$$-\left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right) + k_z^2 = 0$$

$$k_z = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$