

# Electrostatics

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September 20, 2023

- In this section, we focus on electrostatics
  - Not doing (for now):
    - \* Magnetic field
    - \* Forces on moving charges
    - \* Finite of propagation
- Coulomb's Law
  - Given a source charge,  $q$ , and a test charge,  $Q$ , with  $\vec{R}$  as the difference between their positions ( $\vec{r} - \vec{r}'$ ), we can generate Coulomb's Law:

$$F = \frac{qQ\hat{\mathbf{R}}}{4\pi\epsilon_o R^2}$$

- \*  $\epsilon_o$  is known as the permittivity of free space
- \*  $\epsilon_o = 8.85 \cdot 10^{-12} \left[ \frac{\text{C}}{\text{Nm}^2} \right]$
- A Coulomb is defined as an Ampère per second

- Superposition
  - A force per charge ( $q$ ) can be calculated and then summed to find the total force on a test charge ( $Q$ )

$$\vec{F} = \sum_n \frac{Q}{4\pi\epsilon_o} \frac{q_n \hat{\mathbf{R}}_n}{R_n^2}$$

$$\vec{F} = Q\vec{E} \Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_o} \sum_n \frac{q_n \hat{\mathbf{R}}_n}{R_n^2}$$

- Continuous

$$q_n \rightarrow dq = \rho(\vec{r}) d\tau$$

$$\vec{E} = \frac{1}{4\pi\epsilon_o} \int \frac{1}{R^2} \hat{\mathbf{R}} dq = \frac{1}{4\pi\epsilon_o} \int \frac{\rho(\vec{r}) d\tau'}{R^2} \hat{\mathbf{R}}$$

- For various shapes:

- Volume:  $dq = \rho d\tau$
- Line:  $dq = \lambda dl$
- Surface:  $dq = \sigma da$

- Electric Potential —  $V$  (volts)

$$\vec{E} = -\vec{\nabla}V \iff V_b - V_a = - \int_a^b \vec{E} \cdot d\vec{l}$$

- Note:  $V$  is a scalar function
- For a charge  $q$  and some reference radius,  $r$ :

$$V(r) - V_{ref} = - \int_{r_{ref}}^r \vec{E} \cdot d\vec{r}$$

- This yields

$$V(r) = \frac{q}{4\pi\epsilon_o r}$$

- With multiple charges:

$$V(r) = \frac{1}{4\pi\epsilon_o} \sum_n \frac{q_n}{R_n}$$

- Taylor Series Expansions

- We can write the expression  $\frac{1}{x-L}$ , where  $\frac{L}{x} \ll 1$ , as:

$$\frac{1}{x-L} = \frac{1}{x} \left(1 - \frac{L}{x}\right)^{-1} \approx \frac{1}{x} + \frac{L}{x^2} + \dots$$

This means:

$$\frac{1}{x-L} - \frac{1}{x} \approx \frac{L}{x^2}$$

- Coulomb's Law with Gauss's Theorem

- Develops into Gauss’s Law

$$\oint_S \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \cdot d\tau$$

$$\oint \vec{E} \cdot d\vec{a} = \sum \underbrace{\int_V \frac{q_n}{\epsilon_o} \delta^3(R_n)}_{\text{enc charge over } \epsilon_o}$$

$$\oint \vec{E} \cdot d\vec{a} = \frac{q_{enc}}{\epsilon_o}$$

- Gauss’s Law:

- \* Exploit symmetry
- \* Large or small distance to approximate symmetry

- Potential Energy

- The potential required to bring in a new charge to a configuration can be calculated using:

$$\sum_j q_j \sum_{i < j} \frac{q_i}{4\pi\epsilon_o R_{ij}} = \sum_j \sum_{i < j} \frac{q_i q_j}{4\pi\epsilon_o R_{ij}}$$

$$W_{tot} = \frac{1}{2} \sum_j \sum_{i \neq j} \frac{q_i q_j}{4\pi\epsilon_o R_{ij}}$$

when  $i = j$  this is “self-energy” of a particle

$$W = \frac{1}{2} \int \rho V d\tau$$

Using the definition  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}$ , we get:

$$W = \frac{1}{2} \oint \epsilon_o V \vec{E} \cdot d\tau$$

$$W = \frac{1}{2} \oint \epsilon_o V \vec{E} \cdot d\vec{a} - \underbrace{\frac{1}{2} \int \epsilon_o \vec{E} \cdot \vec{E} d\tau}_{\int \frac{\epsilon_o E^2}{2} d\tau}$$

$$W = \frac{1}{2} \int \frac{\epsilon_o E^2}{2} d\tau$$

- The term of integration is known as the energy density,  $u = \frac{\epsilon_o E^2}{2}$ 
  - \* This means electric fields themselves carry energy

- Conductors

- $\vec{E} = 0$  inside the material of a conductor (if there is a cavity, there may be a field)