Magnetostatics

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- Magnetostatics:

$$\vec{F}_{lorentz} = q\vec{v} \times \vec{B}$$

  - This is "static" in the sense of steady flow of magnetic field
  - The units of $\vec{B}$ are Teslas [T]
  - In the simple case where $\vec{B} = B_o\hat{\mathbf{z}}$ and $v_z = 0$, the force could be described as:

$$|\vec{F}| = qvB$$

  - With inward direction. We can then write:

$$qvB = \frac{mv^2}{R}$$
$$qB = \frac{mv}{R}$$
$$qB = \frac{p}{R}$$
$$R = \frac{p}{qB}$$

  - The frequency of rotation may be written:

$$\omega = \frac{qB}{m}$$

  - Total force may be written as:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} = q(\vec{E} + \vec{v} \times \vec{B})$$

    - Special Case: $\vec{F} = 0$

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

    * Given $\vec{E} = E_o\hat{\mathbf{z}}$ and $\vec{B} = B_o\hat{\mathbf{x}}$, we would get:

$$E_o\hat{\mathbf{z}} + v_z B_o\hat{\mathbf{y}} - v_y B_o\hat{\mathbf{z}} = 0$$

    * From this, we get $v_z = 0$ and $v_y = \frac{E_o}{B_o}$
    * This can be used to construct a velocity selector
    * In the event that $\vec{E} \parallel \vec{B}$, an expand or contracting helix about the fields as a pole would be constructed
  - We now go back to a general case: $\vec{E} \perp \vec{B}$, $\vec{F}_{net} \neq 0$, $\vec{E} = E\hat{\mathbf{z}}$, and $\vec{B} = B\hat{\mathbf{x}}$

$$\frac{1}{m}\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

    * The solutions for this would look as follows:

$$y(t) = c_1\cos(\omega t) + c_2\sin(\omega t) + \frac{E}{B}t + c_3$$
$$z(t) = c_2\cos(\omega t) + c_1\sin(\omega t) + c_4$$
$$\omega = \frac{qB}{m}$$

    * This shape is known as a cycloid
    * Special case: starting from rest at (0,0); this would give us:

$$c_1 = 0 \qquad c_2 = -\frac{E}{\omega B} \qquad c_3 = -c_1 \qquad c_4 = -c_2$$
$$y(t) = -\frac{E}{\omega B}\sin(\omega t) + \frac{E}{B}t = R(\omega t - \sin(\omega t))$$
$$z(t) = R(1 - \cos(\omega t))$$
$$\vec{v}_{cent} = \left(\frac{\vec{E} \times \vec{B}}{B^2}\right)$$

    * Work:

$$dW = \vec{F} \cdot d\vec{l} = (q\vec{v} \times \vec{B}) \cdot \vec{v}\,dt = q\,dt(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$$

    * Critical note: magnetic fields <u>do no work</u>
    * Magnetic fields may induce electric fields to do work, but <u>do not do work themselves</u>

- Continuous Systems

  - Given a wire in space, with a bit of charge, dq, moving with velocity $\vec{v}$ shaped in rectangle, and placed in a magnetic field $\vec{B} = \frac{A}{z}\hat{\mathbf{x}}$:

$$dq\,\vec{v} = I\,d\vec{l}$$
$$d\vec{F} = I\,d\vec{l} \times \vec{B}$$
$$\vec{F}_{mag} = \int I\,d\vec{l} \times \vec{B}$$

- Solving would give us:

$$F_{tot} = -IL\left(\frac{A}{a}\right)\hat{\mathbf{z}} + \hat{\mathbf{y}}\int_a^b \frac{A}{z}\,dz + \hat{\mathbf{z}}IL\left(\frac{A}{b}\right) - \hat{\mathbf{y}}I\int_a^b \frac{A}{z}\,dz$$
$$= -\hat{\mathbf{z}}ILA\left(\frac{1}{a} - \frac{1}{b}\right)$$

- There are several types of densities:

$$\text{Linear:}\quad I\,d\vec{l}$$
$$\text{Surface:}\quad \vec{K}\,da$$
$$\text{Bulk:}\quad \vec{J}\,d\tau$$

- The following is an important continuity equation:

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

  - This is another form of charge conservation

- The Biot-Savart law is akin to Coulomb's law, but for magnetism:

$$d\vec{B} = \frac{\mu_o I d\vec{l}\times\hat{\mathbf{R}}}{4\pi R^2}$$

  - Not the expression can be modified:

$$I\,d\vec{l} \Longleftrightarrow \vec{K}\,da \Longleftrightarrow \vec{J}\,d\tau$$

  - The magnetic permeability of free space is:

$$\mu_o = 4\pi\cdot 10^{-7}\left[\frac{\mathrm{N}}{\mathrm{A}^2}\right]$$

    * This term "defines the amp", and the current then "defines the coulomb"
    * Also note: the magnetic field is defined in Newtons per amp-meter

- Ampère's Law

$$\vec{\nabla}\times\vec{B} = \mu_o J \Longleftrightarrow \vec{B} = \frac{\mu_o}{4\pi}\int\frac{\vec{J}\times\hat{\mathbf{R}}\,d\tau}{R^2}$$

  - Notice, we can then use Stokes' Theorem to write:

$$\oint \vec{B}\,dl = \int_V (\vec{\nabla}\times\vec{B})\,d\vec{a} = \int \mu_o J^2\cdot\,d\vec{a} = \mu_o I_{enc}$$

- Vector Potential
  - We already know:
  $$\vec{E} = -\vec{\nabla}V$$
  - We can derive a magnetic counterpart as:
  $$\vec{B} = \vec{\nabla} \times \vec{A}$$
  - By definition, we know:
  $$\vec{\nabla} \times (\vec{A} + \vec{\nabla}t) = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times \vec{\nabla}t}_{0}$$
    - This is known as "choice of gauge" — we can expand upon this using:
    $$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot (\vec{A} + \vec{\nabla}t) = \vec{\nabla} \cdot \vec{A} + \nabla^2 t$$
    $$\vec{\nabla} \cdot \vec{A} = 0 \rightarrow \nabla^2 t = -\vec{\nabla} \cdot \vec{A}\prime$$
    - This is known as the "Coulomb gauge"
  - If we assume $\vec{\nabla} = 0$ and use Ampère's Law, we get:
  $$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$
  $$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_o \vec{J}$$
    - This then becomes
    $$\vec{\nabla}\underbrace{(\vec{\nabla} \cdot \vec{A})}_{0} - \nabla^2 \vec{A} = \mu_o \vec{J}$$
    $$\nabla^2 \vec{A} = -\mu_o \vec{J}$$
  - From the definition in electrostatics, we can obtain a definition for magnetostatics:
  $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(\vec{r}\prime)}{R} \, d\tau'$$
  $$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}(\vec{r}\prime)}{R} \, d\tau'$$
- Multipole Expansion
$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}}{R} \, d\tau$$

- Where

$$\frac{1}{R} = \frac{1}{r}\sum_n \left(\frac{r'}{r}\right)^n P_n(\cos(\alpha))$$

- This gives us:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r^n P_n(\cos(\alpha))\vec{J}\,d\tau$$

- We can see that magnetic monopoles in current loops do not exist (Note: this does not mean that magnetic monopoles don't exist at all)
- Note: another difference in magnetism is that integration occurs over a vector (vector current, vector current density, etc.), and there is no "dot" component

- Magnetic Dipole Moments:

$$\vec{m} = I\vec{a}$$

  - For a dipole field, we know:

$$\vec{B} = \vec{\nabla}\times\vec{A}_1 \quad \text{and} \quad \vec{m} = m\hat{\mathbf{z}}$$

  - This gives us:

$$\vec{A}(\vec{r}) = \frac{\mu_o\vec{m}\times\hat{\mathbf{r}}}{4\pi r^2} = \frac{\mu_o m\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{4\pi r^2}$$

  - We can then take:

$$\vec{B} = \vec{\nabla}\times\vec{A}_1 = \frac{\mu_o m}{4\pi r^3}(2\cos(\theta)\hat{\mathbf{r}} + \sin(\theta)\hat{\theta})$$