Homework 7

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1. A hollow sphere of radius R centered at the origin is covered with a uniform surface charge σ and is rotating about the z-axis with angular frequency ω. A uniform external magnetic field is oriented in the y-direction: $\vec{B} = B_o\hat{\mathbf{y}}$

   (a) Find the total force on the sphere

   Let us first define the dipole moment of the sphere. To do this, we must first define the current:

$$I = \frac{1}{T}\left[\sigma(2\pi R\sin(\theta))R\,d\theta\right]$$
$$I = \sigma\omega R^2\sin(\theta)\,d\theta$$

   This gives us the dipole moment as:

$$d\vec{m} = IA\hat{\mathbf{z}}$$
$$d\vec{m} = \sigma\omega R^2\sin(\theta)\,d\theta(\pi R^2\sin^2(\theta))\hat{\mathbf{z}}$$
$$d\vec{m} = \pi\sigma\omega R^4\sin^3(\theta)\,d\theta\hat{\mathbf{z}}$$

   From here, we can find:

$$\vec{m} = \pi\sigma\omega R^4\int\sin^3(\theta)\,d\theta\hat{\mathbf{z}}$$
$$\vec{m} = \pi\sigma\omega R^4\int_0^\pi\sin^3(\theta)\,d\theta\hat{\mathbf{z}}$$
$$\vec{m} = \frac{4}{3}\pi\sigma\omega R^4\hat{\mathbf{z}}$$

   Now, we can write the force as:

$$\vec{F} = (\vec{m}\cdot\vec{\nabla})\vec{B}$$

   This gives us:

$$\vec{F} = \left(\frac{4}{3}\pi\sigma\omega R^4 \hat{\mathbf{z}} \cdot \vec{\nabla}\right) \vec{B}$$

$$\vec{F} = \left(\frac{4}{3}\pi\sigma\omega R^4 (\hat{\mathbf{z}} \cdot \vec{\nabla})\right) \vec{B}$$

$$\vec{F} = \left(\frac{4}{3}\pi\sigma\omega R^4\right) \underbrace{\frac{\partial}{\partial z}(B_o\hat{\mathbf{y}})}_{0}$$

Thus, we see that the net force is zero:

$$\boxed{\vec{F} = 0}$$

(b) Find the total torque on the sphere

Using the dipole defined in (a), we can write:

$$\vec{N} = \vec{m} \times \vec{B}$$

$$\vec{N} = \left(\frac{4}{3}\pi\sigma\omega R^4 \hat{\mathbf{z}}\right) \times (B_o\hat{\mathbf{y}})$$

$$\vec{N} = \frac{4}{3}\pi\sigma\omega B_o R^4 (\hat{\mathbf{z}} \times \hat{\mathbf{y}})$$

$$\boxed{\vec{N} = -\frac{4}{3}\pi\sigma\omega B_o R^4 \hat{\mathbf{x}}}$$

(c) Generalize the result of (b) to find the torque for a uniform magnetic field in an arbitrary direction, $\vec{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$

Given the general case, we may write:

$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & 1 \\ B_x & B_y & B_z \end{vmatrix} = (-B_y\hat{\mathbf{x}} + B_x\hat{\mathbf{y}})$$

Thus, we can substitute $-B_o\hat{\mathbf{x}}$ from (b) with the above result to get:

$$\boxed{\vec{N} = \frac{4}{3}\pi\sigma\omega R^4 \left(B_x\hat{\mathbf{y}} - B_y\hat{\mathbf{x}}\right)}$$

2. Calculate the magnetic field $\vec{B}(x, y)$ in the positive quadrant of the x-y plane due to a current coming on the y-axis from $y = +\infty$, turning 90° at the origin, and exiting along the x-axis to $x = +\infty$

We know that the magnetic field at a point of distance r from a wire can be defined as:

$$\vec{B} = \frac{\mu_o I}{2\pi r}$$

Thus, we can define the magnetic field produced by the y current, with direction determined through the right hand rule, as:

$$\vec{B}_y = \frac{\mu_o I}{2\pi y}\hat{\mathbf{z}}$$

Using the same method, we can write the x current magnetic field as:

$$\vec{B}_x = \frac{\mu_o I}{2\pi x}\hat{\mathbf{z}}$$

We can simply sum the magnetic field to find:

$$\boxed{\vec{B}(x,y) = \frac{\mu_o I}{2\pi}\left(\frac{1}{x} + \frac{1}{y}\right)\hat{\mathbf{z}}}$$

3. A long cylindrical conductor has a uniform current density $\vec{J}$ oriented along the axis.

   (a) Find the strength of the magnetic field as a function of s (the perpendicular distance from the z-axis) for $s < a$.

   Using Ampère's Law, we can write:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I$$

   Applying the given parameters, we find:

$$\oint \vec{B} \cdot d\vec{l} = \mu_o \vec{J}(\pi s^2)$$

$$\vec{B} \oint d\vec{l} = \mu_o \vec{J}(\pi s^2)$$

$$\vec{B}(2\pi s) = \mu_o \vec{J}(\pi s^2)$$

$$\boxed{\vec{B} = \frac{\mu_o \vec{J} s}{2}}$$

   (b) Consider a charged particle with charge q and momentum p that passes through the cylinder in part (a) with an initial velocity parallel to the axis. As a function of the distance s of the particle from the axis, find the angle of deflection after passing through a short distance Δz. Considered as lens for such charged particles, what it the focal length of this segment of the conductor as a function of $\vec{J}, p, q, \Delta z$? (Such lenses are actually used in particle accelerators). Assume that $\Delta z << R$, where R is the radius of curvature of the particle in the magnetic field, so a small angle approximation is valid for the deflection

   We can approximate this situation to centripetal rotation, which gives us:

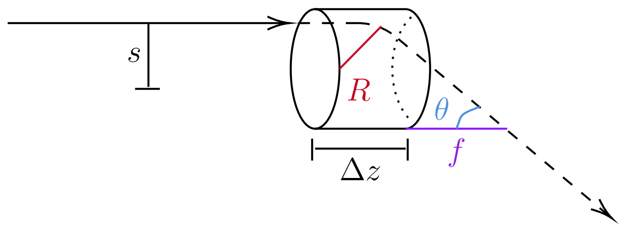

Figure 1: Set up of Particle Motion

$$\frac{mv^2}{R} = qvB$$
$$R = \frac{mv}{qB}$$
$$R = \frac{p}{qB}$$

Using Figure 1 from above, we can write:

$$\sin(\theta) = \frac{\Delta z}{R}$$

Using a small angle approximation, we write:

$$\theta = \frac{\Delta z}{R}$$
$$\theta = \frac{qB\Delta z}{p}$$

We then take B from (a):

$$\theta = \frac{\mu_o q \vec{J} s \Delta z}{2p}$$

Finally, we can define:

$$\tan(\theta) \approx \theta = \frac{s}{f}$$

Rearranging, we get:

$$f = \frac{s}{\theta}$$
$$\boxed{f = \frac{2p}{\mu_o q \vec{J} \Delta z}}$$

(c) Find the current density $\vec{J}$ needed to focus particles of charge $e = 1.6 \cdot 10^{19}$[C] and momentum $p = 75\left[\frac{\text{GeV}}{c}\right]$ with a focal length of $f = 20$[m] for $\Delta z = 0.50$[m]

Using the formula from (b), we can write:

$$\vec{J} = \frac{2\left(\frac{(75\cdot 10^9)(1.6\cdot 10^{-19})}{3\cdot 10^8}\right)}{(4\pi \cdot 10^{-7})(1.6 \cdot 10^{-9})(20)(.5)}$$

$$\boxed{\vec{J} = 3.98 \cdot 10^7 \left[\frac{\text{A}}{\text{m}^2}\right]}$$

4. A double solenoid has two co-axial coils radii a and b, with n_a and n_b turns per unit length, and with currents I_a and I_b flowing in opposite directions. Find:

For the following problems, we can assume ideal solenoids, with current present only inside of the solenoid. The magnetic field may be defined as:

$$\vec{B} = \mu_o n I$$

as was determined in class. Furthermore, let us define the axis of the solenoids as $\hat{\mathbf{x}}$, and assume that I_a flows such that the magnetic field produced is in the $+\hat{\mathbf{x}}$ direction.

(a) the magnetic field for the region inside the first coil ($s < a$)

Inside the first coil, there are two currents, I_a and I_b. Thus, we can simply sum the magnetic fields from each to determine:

$$\boxed{\vec{B}_{s<a} = \mu_o \left(n_a I_a - n_b I_b\right) \hat{\mathbf{x}}}$$

(b) the magnetic field for the region between he two coils ($a < s < b$)

Since s is now outside of the first solenoid, the only field present is from the second solenoid. This gives us:

$$\boxed{\vec{B}_{a<s<b} = -\mu_o n_b I_b \hat{\mathbf{x}}}$$

(c) the magnetic field for the region inside outside both coils ($s > b$)

Outside of both solenoids, there would be no field present. Thus, we may simply write:

$$\boxed{\vec{B}_{s>b} = 0}$$

(d) What ratio of currents would be required to have $\vec{B} = 0$ for $s < a$?

For the field to be zero, we find from (a):

$$\mu_o n_a I_a = \mu_o n_b I_b$$

Which results in a ratio of:

$$\boxed{\frac{I_a}{I_b} = \frac{n_b}{n_a}}$$