

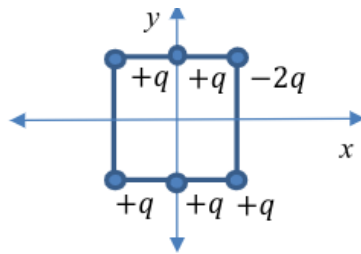
Homework 2

Michael Brodskiy

Professor: D. Wood

September 28, 2023

1. Six charges are arranged along the sides and corners of a square with sides of length L as shown. Calculate the magnitude and direction of the electric field at the origin. Use symmetry and superposition to make the calculation simple.



We know, by definition, that $\vec{F} = \vec{E}q$. Using the concepts we know about force, we know the following charges cancel out each other, as they are symmetric about the test charge at the origin:

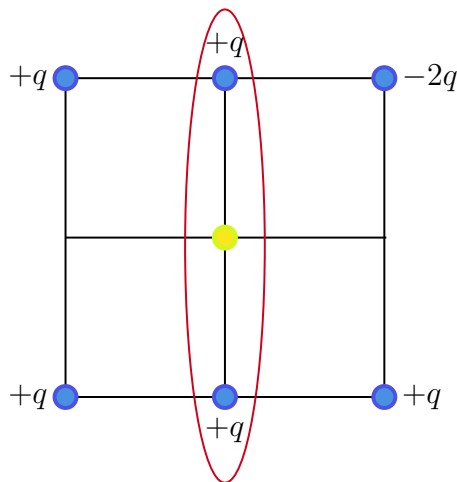


Figure 1: The Opposite Forces Negate Each Other

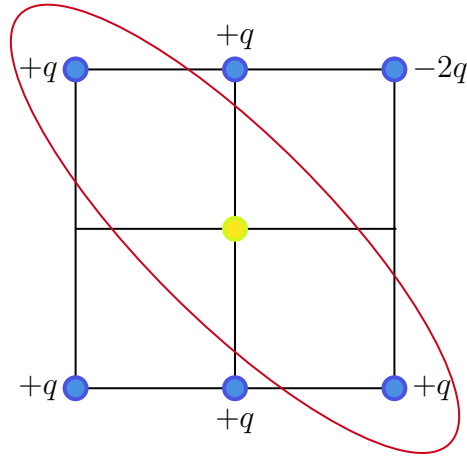


Figure 2: The Opposite Forces Negate Each Other

Thus, we need only consider the effects of these charges:

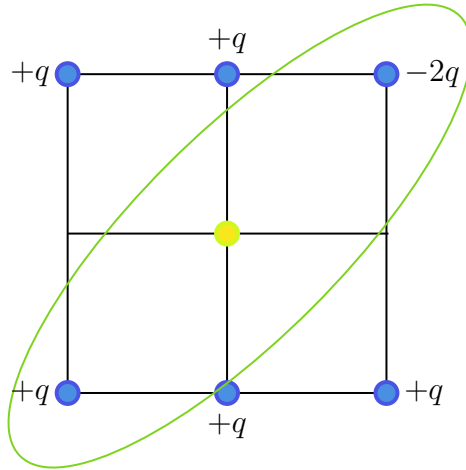


Figure 3: These Charges Remain Relevant

Using superposition, we know that the two charges can be summed, and they produce a horizontal force proportional to $3q$ at an angle of $\frac{\pi}{4}$ radians with respect to the x -axis. Decomposing this, we know the force can be expressed, with Q as the test charge, as:

$$E_Q = \frac{(3q)}{4\pi\epsilon_o R^2} \cos\left(\frac{\pi}{4}\right) \hat{\mathbf{x}} + \frac{(3q)}{4\pi\epsilon_o R^2} \sin\left(\frac{\pi}{4}\right) \hat{\mathbf{y}}$$

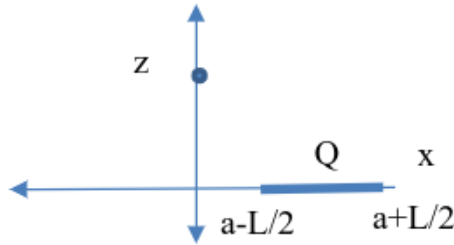
Additionally, we know that $R = \sqrt{\left(\frac{L}{2}\right)^2 + \left(\frac{L}{2}\right)^2} = \sqrt{\frac{L^2}{2}} = \frac{L}{\sqrt{2}}$. This gives us:

$$E_Q = \frac{3\sqrt{2}q}{4\pi\epsilon_o L^2} \hat{\mathbf{x}} + \frac{3\sqrt{2}q}{4\pi\epsilon_o L^2} \hat{\mathbf{y}}$$

It can also be said that the field, in the direction of the $-2q$ charge, is:

$$E_Q = \frac{3q}{2\pi\epsilon_o L^2} - \hat{\mathbf{2q}}$$

2. A uniformly charged rod of length L and charge q is placed along the x -axis with its center at $x = a$. Find the x -component of the electric field at a point on the z axis. (Hint: use R as the variable of integration.) Check your expression in the following limit: $z = 0$ and $a \gg L$.



We know the rod is length L , with charge Q . This means the linear charge density can be defined as:

$$\lambda = \frac{Q}{L}$$

Furthermore, we can refer to the angle between the test charge and point on the rod as θ , and the distance from said point on the rod to the test charge can be called R . This yields us the following expression for the x -axis:

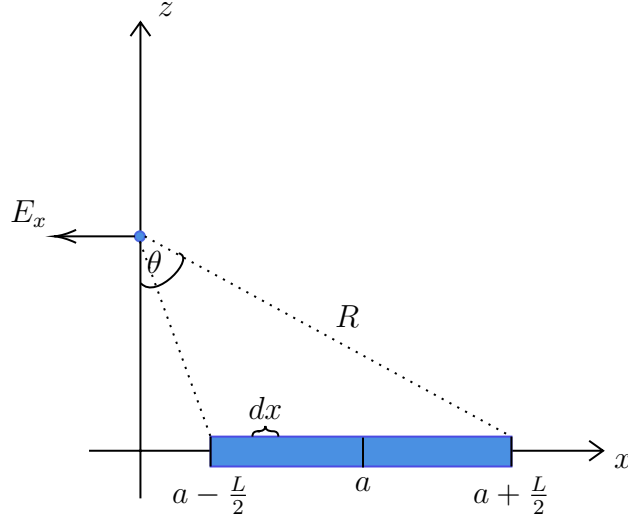


Figure 4: Supporting Diagram — Problem 2

$$E_x = - \int \frac{\sin(\theta)}{4\pi\epsilon_o R^2} dQ$$

We can express $dq \rightarrow \lambda dx$, which gives us:

$$E_x = - \int \frac{\lambda \sin(\theta)}{4\pi\epsilon_o R^2} dx$$

Then, if we were to assume x as the horizontal distance from the test charge to the rod, and z as the vertical distance from the test charge to the rod, we may see that:

$$\begin{aligned} \tan(\theta) &= \frac{x}{z} \\ z \tan(\theta) &= x \\ z \sec^2(\theta) d\theta &= dx \end{aligned}$$

Taking R into account, we can further simplify our calculation:

$$\frac{dx}{R^2} = \frac{z d\theta}{R^2 \cos^2(\theta)}$$

Again referring to our set up, we know $\cos(\theta) = \frac{z}{R}$:

$$\frac{dx}{R^2} = \frac{d\theta}{z}$$

Finally, we can use this:

$$E_x = -\frac{\lambda}{4\pi\epsilon_o z} \int_{\theta_{a-\frac{L}{2}}}^{\theta_{a+\frac{L}{2}}} \sin(\theta) d\theta$$

$$\frac{\lambda}{4\pi\epsilon_o z} [\cos(\theta)] \Big|_{\theta_{a-\frac{L}{2}}}^{\theta_{a+\frac{L}{2}}}$$

We can once again refer to the set-up, finding that:

$$\begin{cases} \cos(\theta_{a-\frac{L}{2}}) = \frac{z}{\sqrt{z^2 + (a-L/2)^2}} \\ \cos(\theta_{a+\frac{L}{2}}) = \frac{z}{\sqrt{z^2 + (a+L/2)^2}} \end{cases}$$

Substituting this into our final expression, we get:

$$\boxed{\frac{\lambda}{4\pi\epsilon_o} \left(\frac{1}{\sqrt{z^2 + (a+L/2)^2}} - \frac{1}{\sqrt{z^2 + (a-L/2)^2}} \right)}$$

$z = 0$ CASE:

$$\begin{aligned} & \frac{\lambda}{4\pi\epsilon_o} \left(\frac{1}{a+L/2} - \frac{1}{a-L/2} \right) \\ & \frac{\lambda}{4\pi\epsilon_o} \left(\frac{a-L/2 - (a+L/2)}{a^2 - (L/2)^2} \right) \\ & - \frac{\lambda L}{(4\pi\epsilon_o)(a^2 - (L/2)^2)} \\ & - \frac{Q}{(4\pi\epsilon_o)(a^2 - (L/2)^2)} \end{aligned}$$

$a \gg L$ CASE:

$$\frac{\lambda}{4\pi\epsilon_o} \left(\frac{1}{\sqrt{z^2 + (a+L/2)^2}} - \frac{1}{\sqrt{z^2 + (a-L/2)^2}} \right); \quad a \pm L \rightarrow 0$$

$$\frac{\lambda}{4\pi\epsilon_o} \left(\frac{1}{\sqrt{z^2 + a^2}} - \frac{1}{\sqrt{z^2 + a^2}} \right) = 0$$

This makes sense, as, logically, if the rod were centered at a point very far away from the test charge, there would be no significant electric field.

3. Calculate the electric potential on the z -axis due to a uniformly charged annulus in the xy -plane centered at the origin with inner radius a and outer radius b . Then find the electric field from the gradient of the potential.

Let us assume the annulus holds a charge of q , with an area A . This defines the charge density as:

$$\sigma = \frac{q}{A}$$

This gives us:

$$dq = \sigma dA$$

If we assume that i and j are the horizontal and vertical distances, respectively, to the test charge from any point on the annulus, and $r = \sqrt{i^2 + j^2}$ is the radial distance from the test charge to any point, then we can express the above as:

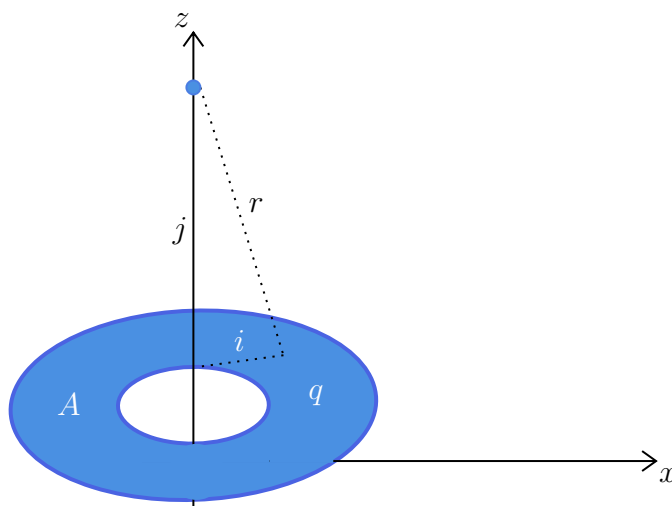


Figure 5: Supporting Diagram — Problem 3

$$dq = \sigma(2\pi i \, di)$$

We can then define the voltage as:

$$V = \frac{q}{4\pi\epsilon_0 r}$$

$$dV = \frac{dq}{4\pi\epsilon_0 r}$$

$$dV = \frac{\sigma i \, di}{2\varepsilon_o \sqrt{i^2 + j^2}}$$

We can then integrate to find the voltage expression:

$$V = \frac{\sigma}{2\varepsilon_o} \int_a^b \frac{i}{\sqrt{i^2 + j^2}} di$$

Using u substitution, with $u = i^2 + j^2$, we get:

$$\begin{aligned} V &= \frac{\sigma}{\varepsilon_o} \int_{a^2+j^2}^{b^2+j^2} \frac{1}{\sqrt{u}} du \\ V &= \frac{\sigma}{\varepsilon_o} \left(\frac{\sqrt{u}}{2} \right) \Big|_{a^2+j^2}^{b^2+j^2} \\ V &= \frac{\sigma}{2\varepsilon_o} \left(\sqrt{j^2 + b^2} - \sqrt{j^2 + a^2} \right) \end{aligned}$$

We can then find the electric field using the gradient formula:

$$\begin{aligned} \vec{E} &= -\vec{\nabla} V \\ \vec{E} &= -\vec{\nabla} \left(\frac{\sigma}{2\varepsilon_o} \left(\sqrt{j^2 + b^2} - \sqrt{j^2 + a^2} \right) \right) \\ \vec{E} &= \left\langle 0, 0, \frac{\sigma}{2\varepsilon_o} \left(\frac{j}{\sqrt{j^2 + a^2}} - \frac{j}{\sqrt{j^2 + b^2}} \right) \right\rangle \end{aligned}$$

Because j simply indicates the z direction, and the vector above is with respect to the i, j, k vectors, we can rewrite this as:

$$\vec{E} = \frac{\sigma}{2\varepsilon_o} \left(\frac{z}{\sqrt{z^2 + a^2}} - \frac{z}{\sqrt{z^2 + b^2}} \right) \hat{\mathbf{k}}$$

4. Consider an infinitely long uniformly-charged solid cylinder of radius a and charge per unit volume ρ surrounded by a coaxial cylindrical shell of radius b and charge per unit area of σ . Take the axis of the cylinders as the z -axis.

(a) Calculate the electric field everywhere in space

We must calculate the electric field in three different cases:

- i. The radius $r < a$ (that is, in the cylinder)

First, we must find the electric field inside of the cylinder, since we can not assume it is a cylinder. We do this with the following set up:

$$\int E(r) \cdot dA = \frac{q_{enc}}{\varepsilon_o}$$

Since the electric field is a function of r , it can be removed from the integral:

$$E \int dA = \frac{q_{enc}}{\epsilon_o}$$

The area integral, with l being the length of the cylinder, gives us:

$$E(2\pi rl) = \frac{q_{enc}}{\epsilon_o}$$

The charge enclosed is simply the charge density of the cylinder is the charge density multiplied by the volume:

$$E(2\pi rl) = \frac{\rho(4\pi r^2 l)}{\epsilon_o}$$

We then divide over to get:

$$E = \frac{2\rho r}{\epsilon_o}$$

- ii. The radius $a < r < b$ (that is, in between the cylinder and shell)
We can find the electric field in a similar process to the above:

$$E \int dA = \frac{q_{enc}}{\epsilon_o}$$

$$E(2\pi rl) = \frac{\rho(4\pi a^2 l)}{\epsilon_o}$$

$$E = \frac{2\rho a^2}{r\epsilon_o}$$

- iii. The radius $r > b$ (that is, outside of the coaxial cable)
Again, we repeat a similar process:

$$E \int dA = \frac{q_{enc}}{\epsilon_o}$$

$$E(2\pi rl) = \frac{\rho(4\pi a^2 l) + \sigma(2\pi bl)}{\epsilon_o}$$

$$E = \frac{2\rho a^2 + b\sigma}{r\epsilon_o}$$

- (b) Also calculate the potential as a function of the distance from the axis, taking the potential to be zero on the z -axis.

To find the voltage, we must integrate the three above cases:

- i. $r < a$

$$V = - \int \frac{2\rho r}{\epsilon_o} dr$$

$$\begin{aligned}
&= -\frac{2\rho}{\varepsilon_o} \int r \, dr \\
&= -\frac{r^2 \rho}{\varepsilon_o} + V_{ref}
\end{aligned}$$

ii. $a < r < b$

$$\begin{aligned}
V &= - \int \frac{2\rho a^2}{r\varepsilon_o} \, dr \\
&= -\frac{2\rho a^2}{\varepsilon_o} \int r^{-1} \, dr \\
&= -\frac{2\rho a^2}{\varepsilon_o} \ln(r) + V_{ref}
\end{aligned}$$

iii. $r > b$

$$\begin{aligned}
V &= - \int \frac{2\rho a^2 + b\sigma}{r\varepsilon_o} \, dr \\
&= -\frac{2\rho a^2 + b\sigma}{\varepsilon_o} \int r^{-1} \, dr \\
&= -\frac{2\rho a^2 + b\sigma}{\varepsilon_o} \ln(r) + V_{ref}
\end{aligned}$$

5. The electric field for two charged concentric spherical shells is given by

$$\begin{cases} 0, & r < a \\ \hat{\mathbf{r}}A_1/r^2, & a < r < b \\ \hat{\mathbf{r}}A_2/r^2, & r > b \end{cases}$$

Where $A_1 = 5 \times 10^6 \left[\frac{\text{Nm}^2}{\text{C}} \right]$, $A_2 = -3 \times 10^6 \left[\frac{\text{Nm}^2}{\text{C}} \right]$, $a = .25[\text{m}]$, and $b = .45[\text{m}]$. Find the surface charge densities σ_a and σ_b on the two shells.

First and foremost, we know:

$$\sigma_a = \frac{4q_a}{\pi} \quad \text{and} \quad \sigma_b = \frac{q_b}{4\pi(.45)^2}$$

To find the enclosed charge, we can use the definition:

$$\vec{\nabla} \cdot \vec{E} = \frac{q_{enc}}{\varepsilon_o}$$

This gives us:

$$\frac{q_{enc}}{\varepsilon_o} = \begin{cases} 0, & r < a \\ -2A_1/r^3, & a < r < b \\ -2A_2/r^3, & r > b \end{cases}$$

$$q_{enc} = \begin{cases} 0, & r < a \\ -2A_1\varepsilon_o/r^3, & a < r < b \\ -2A_2\varepsilon_o/r^3, & r > b \end{cases}$$

To get the respective charges, we need to do the following:

$$q_a = \frac{-2A_1\varepsilon_o}{(.25)^3} = -5.664 \cdot 10^{-3}$$

$$q_b = \frac{-2A_2\varepsilon_o}{(.45)^3} - \frac{2A_1\varepsilon_o}{(.25)^3} = -5.0813 \cdot 10^{-3}$$

We can then find the charge densities:

$$\sigma_a = \frac{4 \cdot |(-5.664 \cdot 10^{-3})|}{\pi} = 6.4697 \cdot 10^{-3}$$

$$\sigma_b = \frac{|(-5.0813 \cdot 10^{-3})|}{4\pi(.45)^2} = 1.9968 \cdot 10^{-3}$$