Transmission Lines

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September 21, 2023

- Transmission lines connect inputs to loads
  - l is the length of the transmission lines
  - If l is not much smaller than λ, we need detailed analysis
  - If l is comparable to λ, then we can not use the lumped parameter model
  - We can, however, partition the transmission lines into segments where $l << \lambda$, then we can apply Kirchoff's circuit laws to each subdivided segment
  - For an imperfect dielectric, there is some loss
  - Some per unit-length properties:
    1. Resistance per unit length: R' (ohm per meter)
    $$\boxed{R = R'\Delta z}$$
    2. Inductance per unit length: L' (Henry per meter)
    $$\boxed{L = L'\Delta z}$$
    3. Capacitance per unit length: C' (Farad per meter)
    $$\boxed{C = C'\Delta z}$$
    4. Conductance per unit length: G' (Siemens per meter)
    $$\boxed{G = G'\Delta z}$$
- Using this, we obtain the Helmholtz Equation:
$$\boxed{\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0}$$
  - Where $\gamma = \sqrt{(j\omega C' + G')(j\omega L' + R')}$

- For a Unidimensional Wave Equation:
  - The characteristic equation becomes: $m^2 - \gamma^2 = 0 \to m = \pm\gamma$
  - The solutions to the above differential equation are superpositions of $\{e^{\gamma z}, e^{-\gamma z}\}$
  - $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$, where:
    * R' is the resistance per unit length
    * L' is the impedance per unit length
    * $j = \sqrt{-1}$
    * $\gamma = R_e\gamma + I_m\gamma$
    * $R_e\gamma = \alpha$, the attenuation constant
    * $I_m\gamma = \beta$, the phase constant (sometimes, notation used is $k = \beta$, like the wave number)

$$\widetilde{V}(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z}$$

  - We know from the definitions above that $\gamma = \alpha + j\beta$
  - For $\alpha > 0$, we are dealing with a (passive) lossy material ($\alpha = 0$) is a loss less material
  - For $\alpha < 0$, we are dealing with a gainy material
  - The wave coming from the source is known as the "incident" wave, and the wave combing from the load is known as "reflected"

$$-\frac{dV}{dz} = (R' + j\omega L')\widetilde{I}(z)$$

  - Solving this by incorporation the equation for $\widetilde{V}$ above, we obtain:

$$\widetilde{I} = \left(V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}\right)\left(\frac{\gamma}{R' + j\omega L'}\right)$$

  - We then define $z_o = \dfrac{R' + j\omega L}{\gamma}$ as our characteristic impedance:

$$\widetilde{I} = \frac{1}{z_o}\left(V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}\right)$$

  - There are thus two unknowns: V_o^+, which depends on source, and V_o^-, which is the reflected wave amplitude (depends on load)
    * The incident wave (in phase domain): $\widetilde{V} = V_o^+ e^{-\alpha z} e^{-j\beta z}$
    * In time domain, this becomes: $v(z,t) = |V_o^+| e^{-\alpha z}\cos(\omega t - \beta z + \phi_+)$[1]

[1] β represents the wave number

- For a lossless line ($R' = 0, G' = 0$, and a pure inductor and capacitor)

$$\gamma = \sqrt{(jC'\omega)(jL'\omega)} \rightarrow \beta = \omega\sqrt{L'C'} = \frac{\omega}{U_{ph}}$$

$$U_{ph} = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\mu\varepsilon}} = c \text{ in dielectric}$$

- The reflection coefficient (Γ) is given by:

$$\Gamma = \frac{V_o^-}{V_o^+}$$

$$V_{load} = V_o^+(1+\Gamma)$$

$$I_{load} = \frac{V_o^+}{z_o}(1-\Gamma)$$

- The normalized load impedance: $\hat{z_L} = \frac{z_L}{z_o}$

$$\Gamma = \frac{z_l - z_o}{z_l + z_o} \rightarrow \Gamma = \frac{\hat{z_L} - 1}{\hat{z_L} + 1}$$

- Special Cases:
  * Short Circuit:

$$\Gamma_{sc} = -1$$

  * Open Circuit:

$$\Gamma_{oc} = 1$$

    · Reactive load, no real absorption

- A Phase-Shifted Γ would look as follows:

$$\Gamma_l = \left(\frac{z_L - z_o}{z_L + z_o}\right) e^{-j(2\beta l)}$$

- Standing Waves
  - $\tilde{V}(d) = V_o^+ e^{-j\beta d} + V_o^+ \Gamma e^{j\beta d}$

$$|\tilde{V}(d)| = |V_o^+ \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta d - \theta_r)\right]^{\frac{1}{2}}$$

    * This fluctuates between a minimum (V_{min}) and maximum (V_{max}) value
    * Maximum/minimum possible value is determined by the cos term and is $\pm 2|\Gamma$
  - From the formula for Γ, we see that $\Gamma = 0$ when the load impedance matches the internal impedance
    * This is known as load matching

- Thus, when $\Gamma = 0$, we know $|\tilde{V}(d)| = |V_0^+|$
- The standing wave ratio is defined as follows:

$$SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

- This measures the matching level of load to the line; ideally, $\Gamma = 0$, and $SWR = 1$; worst case, $|\Gamma| = 1$, and $SWR = \infty$
- In reference to the maximums and minimums, these occur when $2\beta d - \theta_r = 2n\pi$ and $2\beta d - \theta_r = (2n+1)\pi$, respectively. This results in:

$$\begin{cases} d_{max} = \dfrac{2n\pi + \theta_r}{2\beta} \\ \\ d_{min} = \dfrac{(2n+1)\pi + \theta_r}{\beta} \end{cases}$$

- Power Flow in Transmission Line
  - Average Power

$$\langle P_i \rangle = \frac{|V_o|^2}{2Z_o} [\mathrm{W}]$$