Plane-Wave Propagation

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- We will have problems about fields from sources radiating
  - In a source-free region, there are no electric and magnetic fields. That would make Maxwell's equations:

$$\nabla \cdot \tilde{E} = 0$$
$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$$
$$\nabla \cdot \tilde{H} = 0$$
$$\nabla \times \tilde{H} = \tilde{J} + j\omega\varepsilon\tilde{E}$$

  - In a region with a source, Maxwell's equations may be written as:

$$\nabla \cdot \tilde{E} = \frac{\rho_V}{\varepsilon}$$
$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$$
$$\nabla \cdot \tilde{H} = 0$$
$$\nabla \times \tilde{H} = \tilde{J}_{cond} + j\omega\varepsilon\tilde{E}$$

  - There are two components that contribute to current density:

$$\tilde{J} = \tilde{J}_{impressed} + \tilde{J}_{cond}$$

    * Impressed is from a source, and conductive is an intrinsic property

$$\tilde{J} = \sigma\tilde{E}$$

  - The homogenous form of Maxwell's equations can thus be written as:

$$\nabla \cdot \tilde{E} = 0$$
$$\nabla \times \tilde{E} = -j\omega\mu\tilde{H}$$
$$\nabla \cdot \tilde{H} = 0$$
$$\nabla \times \tilde{H} = (\sigma + j\omega\varepsilon)\tilde{E}$$

- Far from sources, fields propagate like a plane wave (the circle becomes so large, it can be approximated as a line)
- The $\tilde{J}$ conduction component is known as drift
  - If the conductivity is non-zero, we can see from the equation above that:

$$\nabla \times \tilde{H} = j\omega \left(\varepsilon - \frac{j\sigma}{\omega} \right) \tilde{E}$$

$$\varepsilon_c = \varepsilon - \frac{j\sigma}{\omega}$$

    * If $\frac{\sigma}{\omega} << \varepsilon$, then the material is an insulator, and:

$$\varepsilon_c = \varepsilon$$

    * If $\frac{\sigma}{\omega} >> \varepsilon$, the material is conductive; Note: this means that a good conductor depends on the angular frequency
    * We can determine that:

$$\tan(\theta) = \frac{\sigma}{\omega\varepsilon}$$

    * If the tangent is approximately 0, the conductivity is negligible
- Wave Equations
  - From manipulating the first non-zero equation, we get:

$$\nabla \times \nabla \times \vec{E} - \omega^2 \mu \varepsilon_c \tilde{E} = 0$$

  - We can also get:

$$\nabla^2 \vec{E} - \omega \mu \varepsilon_c \tilde{E} = 0$$

  - In lossy media, $\sigma \neq 0$ and ε_c is a complex value. We assign $\gamma = -\omega^2 \mu \varepsilon_c$, and can now write:

$$(\nabla^2 - \gamma^2)\tilde{E} = 0$$
$$(\nabla^2 - \gamma^2)\tilde{H} = 0$$

  - For lossless media, $\sigma = 0$, and ε_c is purely real
  - We can rewrite the equation as:

$$(\nabla^2 + k^2)\tilde{E}(r) = 0, \text{ where } r = (x, y, z)$$

    * k is the wave number, $\omega\sqrt{\mu\varepsilon}$
- Mapping from source to field is known as radiation

- A long distance from the source, the spherical wavefront may be approximated to a line

- Using one of our equations from above, and plugging in the value of $\nabla \times \tilde{H}$, we can obtain:

$$\nabla^2 \tilde{E} + \omega^2 \mu \varepsilon \tilde{E} - j\omega\mu\sigma \tilde{E} = 0$$

- For plane waves, we end up with two important equations:

$$\tilde{H} = \frac{1}{\eta} \hat{k} \times \tilde{E}$$

$$\tilde{E} = -\eta \hat{k} \times \tilde{E}$$

  - Where $\tilde{E}$ and $\tilde{H}$ are plane waves, $\hat{k}$ is the direction of propagation, and $\eta = \sqrt{\frac{\mu}{\varepsilon}}$. This means we know:

$$\tilde{E} \cdot \hat{k} = 0$$

$$\tilde{H} \cdot \hat{k} = 0$$

- Polarization

  - Occurs when two components or more components of a wave are in different phases

- Power Density

$$S = E \times H \quad \text{(time-dependent)}$$

  - Poynting Vector

$$P = \oint_A dS(\hat{\mathbf{n}} \cdot S) \quad \text{(in Watts)}$$

  - For time-harmonic, single frequency:

$$\tilde{S} = \frac{1}{2} Re \left\{ \tilde{E} \times \tilde{H} \right\} \quad \text{(phasors)}$$

  - The average power density in circuits is similar to this:

$$P_{avg} = \frac{1}{2} Re \left\{ \tilde{V} \cdot \tilde{I} \right\}$$

- The plane-wave average is:

$$\tilde{S}_{avg} = \frac{1}{2}\frac{|\tilde{E}|^2}{\eta}\hat{\mathbf{z}} \quad \text{(in Watts per square meter)}$$

  where $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

- In circuits, the expressions are similar:

$$P_{avg} = \frac{1}{2} Re\left\{\frac{|\tilde{V}|^2}{z_L}\right\}$$

  * If z_L is real, we can simply write:

$$P_{avg} = \frac{1}{2}\frac{|\tilde{V}|^2}{z_L}$$

  * For non-real z_L, we can write:

$$P_{avg} = \frac{1}{2}\frac{|\tilde{V}|^2}{|z_L|}\cos(\theta)$$

- In a lossy medium, we can find:

$$\tilde{S}_{avg} = \frac{1}{2}|\tilde{E}|^2 Re\left\{\frac{1}{\eta}\right\} = \tilde{S}_{avg}\frac{1}{2}|E_0|^2 e^{-2\alpha z}\hat{\mathbf{z}}\frac{\cos(\theta)}{|\eta|}$$

- Important Formulas

  - Some critical formulas for use with plane-waves are:

$$\alpha = \omega\left[\frac{\mu\varepsilon\prime}{2}\left[\sqrt{1+\left(\frac{\varepsilon\prime\prime}{\varepsilon\prime}\right)^2} - 1\right]\right]^{\frac{1}{2}}$$

$$\beta = \omega\left[\frac{\mu\varepsilon\prime}{2}\left[\sqrt{1+\left(\frac{\varepsilon\prime\prime}{\varepsilon\prime}\right)^2} + 1\right]\right]^{\frac{1}{2}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon\prime}}\left(1 - j\frac{\varepsilon\prime\prime}{\varepsilon\prime}\right)^{-\frac{1}{2}}$$

  - If $\alpha = \beta$, then we are dealing with a good conductor