Final Exam

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1. (a) We know that, with a Hertzian dipole antenna centered at the origin, we may express the phasor as (as per eq. 9.44a):

$$\boxed{\tilde{E}_\theta = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jkR}}{R}\right)}$$

   (b) We know that the resistance as perceive by the transmission line may be written as:

$$R_{in} = R_{rad} + R_{loss}$$

A half-wave dipole means that $R_{loss} << R_{rad}$, which allows us to approximate:

$$R_{in} \approx R_{rad}$$

Using equation 9.33a, we may write:

$$R_{in} \approx \frac{2P_{rad}}{I_o^2}$$

Using our answer from (d), we write:

$$R_{in} \approx 2(36.57)$$
$$R_{in} = 73.14[\Omega]$$

As stated in the textbook, for a half-wave dipole, the impedance is approximately $42[\Omega]$, which yields:

$$\boxed{Z_{in} = 73.14 + 42j[\Omega]}$$

Note: the complex term can go to zero with a very slight change in dipole length, so we may simply write:

$$\boxed{Z_{in} = 73.14[\Omega]}$$

(c) To match the impedance with no reflection, we want the reflection coefficient to be 0. Thus, we may write the reflection coefficient as:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Since we want no reflection, $\Gamma \to 0$, which gives us:

$$Z_L - Z_0 = 0$$

Thus, to match with no reflection, we want to create a system such that $Z_L = Z_0$, which would mean:

$$\boxed{Z_0 = 73.14 + 42j[\Omega]}$$

Or, if we drop the complex term:

$$\boxed{Z_0 = 73.14[\Omega]}$$

(d) We know the radiated power may be found by integrating the real part of:

$$dP_{rad} = R^2 S(R, \theta, \phi)\, d\omega$$

Which is the Poynting vector. Thus, we write:

$$P_{rad} = \int\int \frac{R^2}{2\eta_o} \tilde{E}_\theta^2 \, d\Omega$$

This gives us:

$$P_{rad} = \frac{15 I_o^2}{\pi} \int\int \left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)} \right]^2 d\Omega$$

$$P_{rad} = \frac{15 I_o^2}{\pi} \int\int \left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)} \right]^2 \sin(\theta)\, d\theta\, d\phi$$

$$P_{rad} = \frac{15 I_o^2}{\pi} \int_0^{2\pi} \int_0^{\pi} \left[\frac{\cos^2\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)} \right] d\theta\, d\phi$$

$$P_{rad} = \frac{15 I_o^2}{\pi} (2\pi)(1.219)$$

Substituting our value for I_o, we get:

$$\boxed{P_{rad} = 36.57[\text{W}]}$$

(e) We want the equation to shift such that $R\prime$ is the new radius, which depends on R, θ, and h. We know the parametrization of spherical coordinates may be written as:

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$

We can substitute R from (a) such that $R \to \sqrt{x^z + y^2 + z^2}$:

$$\tilde{E}_\theta = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jk\sqrt{x^2+y^2+z^2}}}{\sqrt{x^2+y^2+z^2}}\right)$$

We know that the shift in x will lead to $x \to x - h$:

$$\tilde{E}_\theta = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jk\sqrt{(x-h)^2+y^2+z^2}}}{\sqrt{(x-h)^2+y^2+z^2}}\right)$$

Combining the above with the parametrization, we get:

$$\tilde{E}_\theta = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jk\sqrt{(R\sin(\theta)\cos(\phi)-h)^2+(R\sin(\theta)\sin(\phi))^2+(R\cos(\theta))^2}}}{\sqrt{(R\sin(\theta)\cos(\phi)-h)^2+(R\sin(\theta)\sin(\phi))^2+(R\cos(\theta))^2}}\right)$$

We can simplify this a bit to get:

$$\boxed{\tilde{E}_\theta = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jk\sqrt{R^2-2hR\sin(\theta)\cos(\phi)+h^2}}}{\sqrt{R^2-2hR\sin(\theta)\cos(\phi)+h^2}}\right)}$$

(f) The PEC will produce some reflection such that we can write:

$$\tilde{E}_\theta^i + \tilde{E}_\theta^r = \tilde{E}_\theta^t$$

The reflected term, $\tilde{E}_\theta^r$, can be defined similar to how we defined the h-shifted antenna, except that we add the h (picture this as a two-antenna array, with both h away):

$$\tilde{E}_\theta^r = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jk\sqrt{R^2+2hR\sin(\theta)\cos(\phi)+h^2}}}{\sqrt{R^2+2hR\sin(\theta)\cos(\phi)+h^2}}\right)$$

For ease of reading, let us assign two variables such that:

$$a = \sqrt{R^2 - 2hR\sin(\theta)\cos(\phi) + h^2}$$

and

$$b = \sqrt{R^2 + 2hR\sin(\theta)\cos(\phi) + h^2}$$

From here, the total electric field may be computed as:

$$\boxed{\tilde{E}_\theta^t = 60jI_o \left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jka}}{a} - \frac{e^{-jkb}}{b}\right)}$$

(g) The plots generated look as follows:

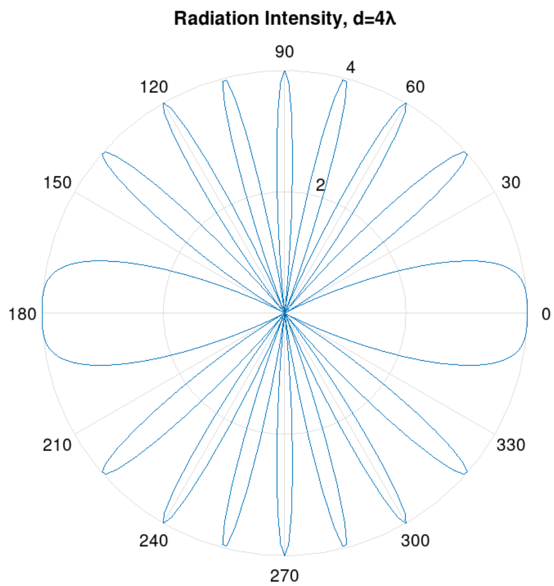

Figure 1: Figure for $h = 4\lambda$

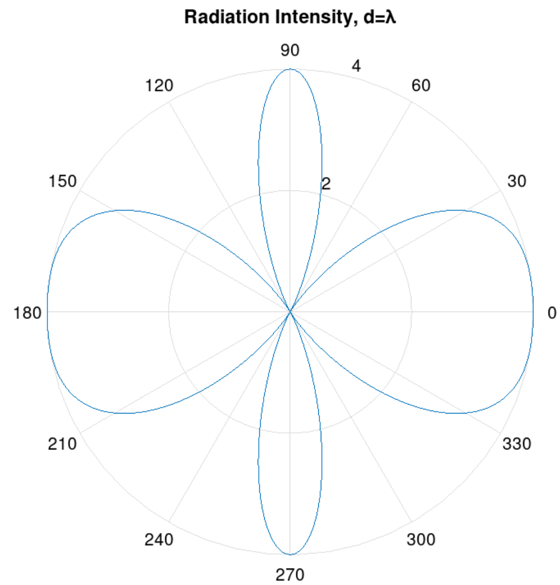

Figure 2: Figure for $h = \lambda$

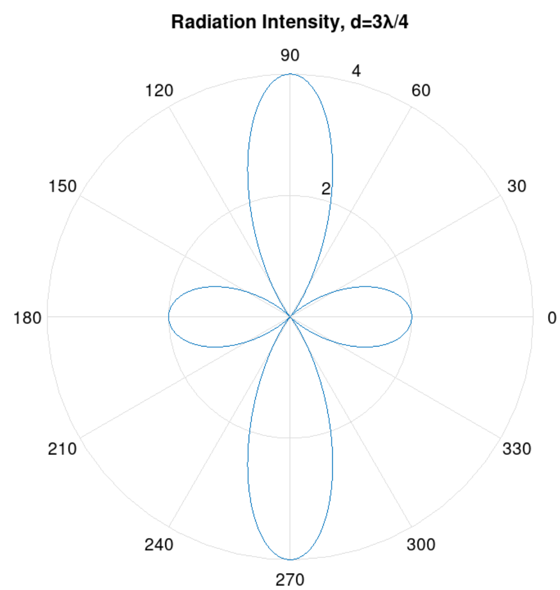

Figure 3: Figure for $h = .75\lambda$

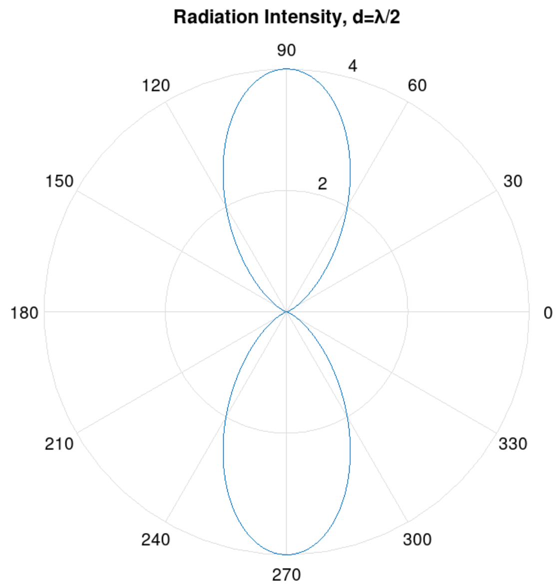

Figure 4: Figure for $h = .5\lambda$

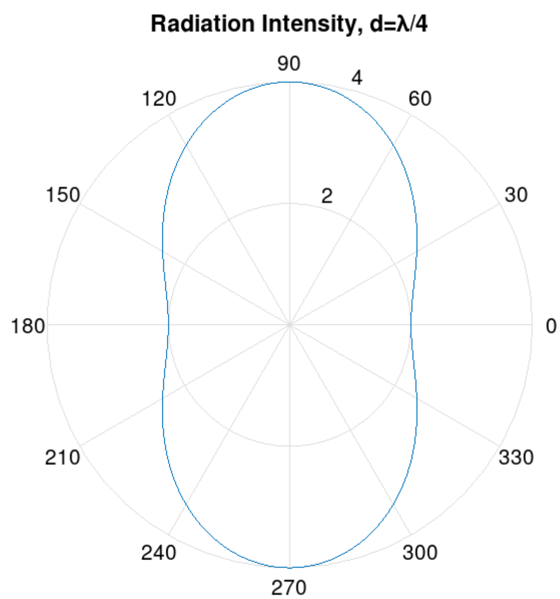

Figure 5: Figure for $h = .25\lambda$

(h) The PEC will produce some reflection such that we can write:

$$\tilde{E}^i_\theta + \tilde{E}^r_\theta = \tilde{E}^t_\theta$$

This time, we can imagine a reflected antenna shifted -4λ and $-h$. Thus, we can get:

$$c \to \sqrt{(R\sin(\theta)\cos(\phi) - h)^2 + (R\sin(\theta)\sin(\phi))^2 + (R\cos(\theta) - 4\lambda)^2}$$
$$c \to \sqrt{R^2 - 2hR\sin(\theta)\cos(\phi) - 8\lambda R\cos(\theta) + h^2 + 16\lambda^2}$$

$$\tilde{E}_\theta^r = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jkc}}{c}\right)$$

The total electric fields thus becomes:

$$\tilde{E}_\theta^t = 60jI_o\left[\frac{\cos\left(\frac{\pi}{2}\cos(\theta)\right)}{\sin(\theta)}\right]\left(\frac{e^{-jka}}{a} + \frac{e^{-jkc}}{c}\right)$$

(i) See Plots Below

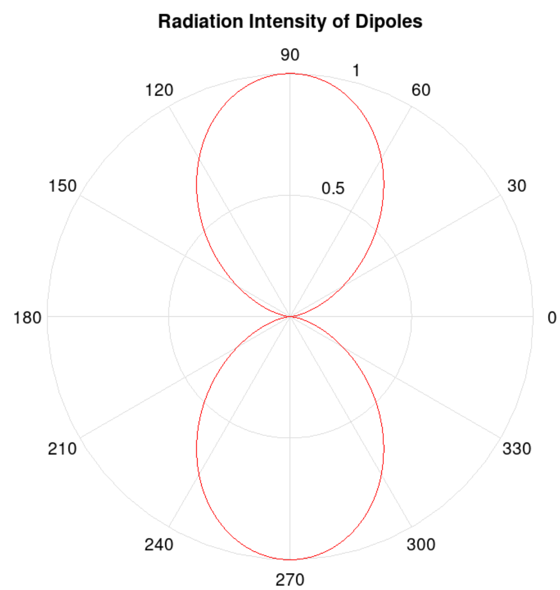

Figure 6: Radiation Intensity Plots

2. We know that, for an array consisting of two antennas, we may write the array factor as:

$$AF_2 = 4\cos^2\left[\frac{\pi}{\lambda}d\cos(\theta) + \frac{\phi}{2}\right]$$

with β as the wave number.

(a) Given that there is no phase difference, we may write (using the above equation):

$$AF_2 = 4\cos^2\left[\frac{\pi d}{\lambda}\cos(\theta)\right]$$

We are asked to find an angle θ such that there is a maximum at:

$$\theta = 30$$

Plugging this into our equation, we may write:

$$AF_2 = 4\cos^2\left(\frac{\pi d\sqrt{3}}{2\lambda}\right)$$

We know that the cos term will be at a maximum when the inside is an integer (n) multiple of π. Thus, we get:

$$\frac{\pi d\sqrt{3}}{2\lambda} = \pi n$$

As such, we can find the d/λ ratio for maximum at $\theta = 30$ as:

$$\boxed{\frac{d}{\lambda} = \frac{2n}{\sqrt{3}}}$$

(b) Here, we have a similar set up as that of part (a), except that we want the array factor to be null (0). Thus, we can use the following set up:

$$\frac{\pi d\sqrt{3}}{\lambda} = (2n-1)\pi$$

$$\boxed{\frac{d}{\lambda} = \frac{2n-1}{\sqrt{3}}}$$

(c) We may adjust our approach to part (a), and write:

$$AF_2 = 4\cos^2\left[\frac{\pi}{\lambda}d\cos(\theta) + \frac{\phi}{2}\right]$$

$$AF_2 = 4\cos^2\left[d\frac{\pi\sqrt{3}}{2\lambda} + \frac{\pi}{2}\right]$$

$$AF_2 = 4\cos^2\left[\frac{\pi d\sqrt{3}}{2\lambda} + \frac{\pi}{2}\right]$$

Again, to maximize the cos term, we write:

$$\frac{\pi d\sqrt{3}}{2\lambda} + \frac{\pi}{2} = n\pi$$

We then rearrange:

$$\frac{\pi d\sqrt{3}}{\lambda} + 1 = 2n$$

Finally, we get:

$$\boxed{\frac{d}{\lambda} = \frac{2n-1}{\sqrt{3}}}$$

(d) Once again, to make the array null, we find:

$$\frac{\pi d\sqrt{3}}{\lambda} + \pi = (2n+1)\pi$$

We rearrange:

$$\frac{d\sqrt{3}}{\lambda} + 1 = (2n+1)$$

This gives us:

$$\boxed{\frac{d}{\lambda} = \frac{2n}{\sqrt{3}}}$$

(e) We plot the array factors at the maximization points, with the assumption $n = 1$. Thus, we obtain:

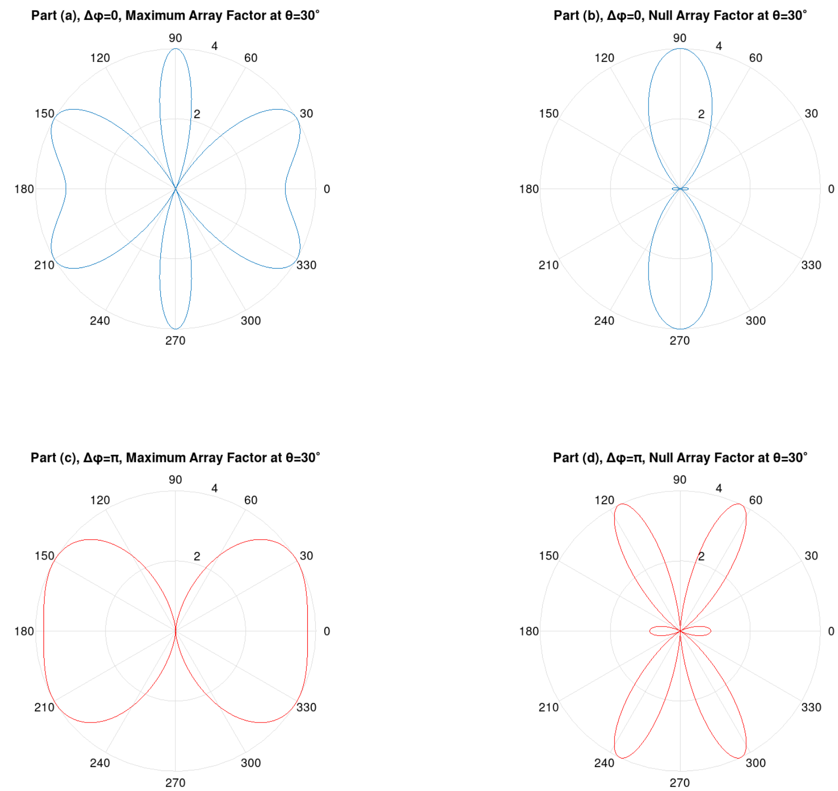

Figure 7: Plots of Array Factors

As expected, we can see plots 1 and 3 maximized at $30°$ and plots 2 and 4 at 0.

3. (a) We can find the parallel polarization reflection coefficient using the formula:

$$\Gamma_{\parallel} = \frac{\eta_2 \cos(\theta_t) - \eta_1 \cos(\theta_i)}{\eta_2 \cos(\theta_i) + \eta_1 \cos(\theta_i)}$$

We can first find the transmitted angle using Snell's Law:

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$

$$\theta_t = \sin^{-1}\left(\frac{n_1 \sin(\theta_i)}{n_1}\right)$$

$$\theta_t = \sin^{-1}\left(\frac{(1)\sin(50)}{\sqrt{4}}\right)$$

$$\theta_t = 22.52$$

We apply this information to our formula:

$$\Gamma_{\|} = \frac{\frac{1}{\sqrt{4}}\cos(22.52) - (1)\cos(50)}{\frac{1}{\sqrt{4}}\cos(22.52) + (1)\cos(50)}$$

$$\boxed{\Gamma_{\|} = -.1638}$$

(b) We know that the transmission coefficient is related to the reflection coefficient for parallel polarization by:

$$\tau_{\|} = (1 + \Gamma_{\|})\frac{\cos(\theta_i)}{\cos(\theta_t)}$$

This gives us:

$$\tau_{\|} = (1 - .1638)\frac{\cos(50)}{\cos(22.52)}$$

$$\boxed{\tau_{\|} = .5819}$$

(c) For perpendicular polarization, we know that the reflection coefficient may be expressed as:

$$\Gamma_{\perp} = \frac{\eta_2\cos(\theta_i) - \eta_1\cos(\theta_t)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

This yields:

$$\Gamma_{\perp} = \frac{\frac{1}{2}\cos(50) - (1)\cos(22.52)}{\frac{1}{2}\cos(50) + (1)\cos(22.52)}$$

$$\boxed{\Gamma_{\perp} = -.4838}$$

(d) We know that the transmission coefficient relation to the perpendicular reflection coefficient is:

$$\tau_{\perp} = (1 + \Gamma_{\perp})$$

This gives us:

$$\tau_{\perp} = (1 - .4838)$$

$$\boxed{\tau_{\perp} = .5162}$$

(e) The Brewster angle can be determined using:

$$\theta_B = \tan^{-1}\left(\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right)$$

This gives us:

$$\theta_{B\|} = \tan^{-1}\left(\sqrt{4}\right)$$
$$\theta_{B\|} = \tan^{-1}(2)$$
$$\boxed{\theta_{B\|} = 63.435^\circ}$$

(f) We know that, for a nonmagnetic material (as the one described in the problem), the Brewster angle exists only for the parallel polarization (thus the use of $\theta_{B\|}$). A beam entering a medium at a this angle of incidence would fully transmit its parallel component (only the perpendicular component is reflected). This can be explained in the diagram below:

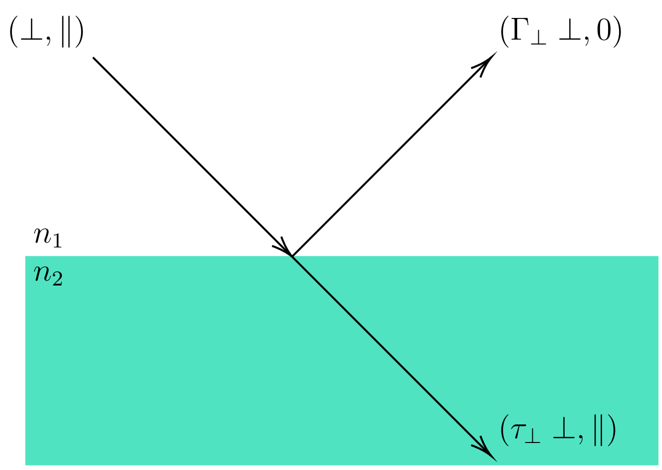

Figure 8: Brewster Angle Scenario

Given a coordinate set $(\perp, \|)$ representing each components strength, we can see that for the incident wave and transmitted wave, the perpendicular component experiences reflection. On the other hand, there is no parallel reflection.

(g) We now assume that the same scenario occurs, but with the Brewster angle. This gives us:

$$n_1 \sin(\theta_i) = n_2 \sin(\theta_t)$$
$$\theta_t = \sin^{-1}\left(\frac{n_1 \sin(\theta_i)}{n_2}\right)$$

$$\theta_t = \sin^{-1}\left(\frac{\sin(63.435)}{2}\right)$$

$$\boxed{\theta_t = 26.565°}$$

From here, we can find the perpendicular reflection coefficient:

$$\Gamma_\perp = \frac{\eta_2\cos(\theta_i) - \eta_1\cos(\theta_t)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

$$\Gamma_\perp = \frac{.5\cos(63.435) - \cos(26.565)}{.5\cos(63.435) + \cos(26.565)}$$

$$\boxed{\Gamma_\perp = -.6}$$

Note that the above result makes sense, as we expect the perpendicular component to be reflected. The parallel reflection becomes:

$$\Gamma_\parallel = \frac{\eta_2\cos(\theta_t) - \eta_1\cos(\theta_i)}{\eta_2\cos(\theta_t) + \eta_1\cos(\theta_i)}$$

$$\Gamma_\parallel = \frac{.5\cos(26.565) - \cos(63.435)}{.5\cos(26.565) + \cos(63.435)}$$

$$\boxed{\Gamma_\parallel = 0}$$

The perpendicular transmission may be expressed as:

$$\tau_\perp = (1 + \Gamma_\perp)$$

$$\boxed{\tau_\perp = .4}$$

Again, this makes sense as the perpendicular component will be reflected. The parallel transmission is:

$$\tau_\parallel = (1 + \Gamma_\parallel)\frac{\cos(\theta_i)}{\cos(\theta_t)}$$

$$\tau_\parallel = (1 + 0)\frac{\cos(63.435)}{\cos(26.565)}$$

$$\boxed{\tau_\parallel = .5}$$

(h) We can measure the Brewster angle using a similar lab set up to the one picture in 9 below:

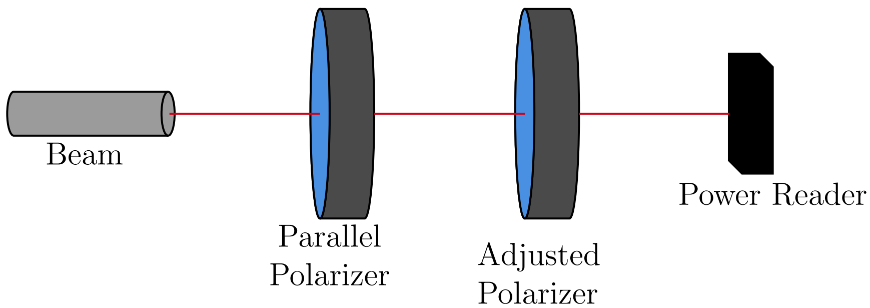

Figure 9: Simplified Lab Set-up

A beam would be focused through a parallel polarizer and then a material with an adjustable angle (the ability to rotate to different angles). As this beam passes through the parallel polarizer, only the parallel component will be passed through. The parallel-only beam would then pass through the adjustable material, and, subsequently, to the power reader. The power value is recorded with respect to the angle of the adjustable material, and then the angle of the material is modified (preferably in uniform steps). By definition, the transmitted parallel component of the beam would be maximized at the Brewster angle. Thus, as the power readings are taken, the Brewster angle is the angle that corresponds to the maximum power reading.

4. (a) Given by the fact that field $\tilde{E}_i$ is directed in the $\hat{\mathbf{y}}$ direction, while being influenced by x and z components, we know that the wave is affected by <u>perpendicular</u> polarization.

(b) Given $3x + 4z$, we may write:

$$\theta_i = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\boxed{\theta_i = 36.87°}$$

(c) This may be written in the time domain as:

$$\tilde{E}_i = \hat{\mathbf{y}}20e^{-j(3x+4z)}$$

$$E_i = 20\cos(\omega t - (3x + 4z))\hat{\mathbf{y}}$$

We can find the angular frequency using the formula:

$$\omega = ck$$

This gives us:

$$\omega = c\sqrt{3^2 + 4^2}$$
$$\omega = 5c$$
$$\omega = 1.5 \cdot 10^9 \left[\frac{\text{rad}}{\text{s}}\right]$$

Thus, we get:

$$\boxed{E_i = 20\cos\big((1.5 \cdot 10^9)t - (3x + 4z)\big)\hat{\mathbf{y}}}$$

(d) The average power density is defined by the formula:

$$S_{avg} = \frac{|E|^2}{2\eta}$$

Since the incident wave is traveling in air, we may write:

$$\eta = \eta_o = 376.819[\Omega]$$

Since, from the time domain, we know the magnitude of the wave, we may write:

$$S_{avg} = \frac{(20)^2}{2 \cdot 376.819}$$
$$\boxed{S_{avg} = .531 \left[\frac{\text{W}}{\text{m}^2}\right]}$$

(e) Since the wave is perpendicularly polarized, know that the magnitude of the reflected wave can be defined using:

$$|E^r| = \Gamma|E^i|$$

The reflection coefficient may be defined as:

$$\Gamma_\perp = \frac{\eta_2\cos(\theta_i) - \eta_1\cos(\theta_t)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

Now, we need to find the transmitted angle:

$$\theta_t = \sin^{-1}\left(\frac{n_1\sin(\theta_i)}{n_2}\right)$$
$$\theta_t = \sin^{-1}\left(\frac{\sin(36.87)}{2}\right)$$
$$\theta_t = 17.48^\circ$$

From here, we return to the reflection coefficient:

$$\Gamma_{\perp} = \frac{.5\cos(36.87) - \cos(17.48)}{.5\cos(36.87) + \cos(17.48)}$$
$$\Gamma_{\perp} = -.409$$

This means that the magnitude becomes:

$$|E^r| = |-.409 \cdot 20| = 8.18$$

This gives us:

$$S_{avg} = \frac{|8.18|^2}{2 \cdot 376.819}$$
$$\boxed{S_{avg} = .0888 \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]}$$