

Lecture 5 — Einstein's Equations and Schwarzschild

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- We may begin with Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

- Note that the tensors form second order non-linear differential equations

- Symmetric Tensor ($n = 4$ dimensions)

- $\frac{n^2-n}{2} + n = \frac{n^2+n}{2} = 10$ degrees of freedom
- General relativity sees diffeomorphism invariance, so four of the degrees of freedom are removed (since it isn't $x^\mu \rightarrow x^{\mu'}$)
- Solving these differential equations is extremely complex, so we will make some assumptions to simplify analysis:
 - * Boundary conditions and initial conditions
 - * Limits
 - * Simplify through symmetry

- Symmetric General Relativity

- Spherical symmetry and static
- Homogenous and isotropic: FLRW universe/cosmology
- $T_{\mu\nu} = 0$, small perturbations are gravitational waves
- In Newtonian mechanics, with three masses M_1 , M_2 , and much smaller M_3 , we may write: $\Phi_{M_3} = \Phi_{M_1} + \Phi_{M_2}$
 - * In General Relativity, $g_{\mu\nu}$ depends on M_1 and M_2 , as well as the binding energy between

- Schwarzschild

- Only vacuum solution with spherical symmetry
- We assume a spherical system that is static
- We use Minkowski space, with spherical coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2 \underbrace{(d\theta^2 + \sin^2(\theta) d\theta^2)}_{d\Omega^2}$$

- * We may rescale this with functions of r :

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + C(r) r^2 d\Omega^2$$

- * Furthermore, we define $r \rightarrow \sqrt{C(r)}$

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 d\Omega^2$$

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{\beta(r)} dr^2 + r^2 d\Omega^2$$

- * For the diagonal metric, we may find $\Gamma_{r\phi}^\phi = (1/r)$, then continuing Christoffel calculations, using Riemann, and then contracting to Ricci, we find:

$$R_{tt} = e^{2(\alpha-\beta)} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha \right]$$

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta$$

$$R_{\theta\theta} = e^{-2\beta} [r(\partial_r \beta - \partial_r \alpha) - 1] + 1$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}$$

- * We want a $T_{\mu\nu} = 0$ solution, which implies $G_{\mu\nu} = 0$, which then implies $R_{\mu\nu} = 0$
 - This is known as “Ricci Flat” (not really flat)
- * We move terms around to find:

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0$$

$$\frac{2}{r} (\partial_r \alpha + \partial_r \beta) = 0$$

$$\alpha = -\beta$$

- * Taking:

$$R_{\theta\theta} = 0 \quad \text{and} \quad -e^{2\alpha} [2r \partial_r \alpha + 1] + 1 = 0$$

- * We get:

$$e^{2\alpha} [2r \partial_r \alpha + 1] = \partial_r (r e^{2\alpha})$$

* We define $A(r) = e^{2\alpha}$ and $y(r) = rA(r)$, which gives:

$$y = r + C \Rightarrow A(r) = 1 + \frac{C}{r}$$

$$A(r) = 1 - \frac{R_s}{r}$$

* Where R_s is the Schwarzschild radius, which allows us to write:

$$ds^2 = - \left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega$$

- Black Holes

- Using Newtonian $g_{tt} = -(1 + 2\Phi)$, we get:

$$R_s = -2\Phi$$

- With a point mass:

$$\Phi = -\frac{GM}{r}$$

- Thus, $R_s = 2GM$

- Schwarzschild Properties:

1. $M \rightarrow 0, \quad g_{MV} \rightarrow \eta_{M\nu}$

2. $r \rightarrow \infty, \quad g_{M\nu} \rightarrow \eta_{M\nu}$

3. $r = 0, \quad \frac{R_s}{r} \rightarrow \infty$

4. $r = R_s, \quad \left(1 - \frac{R_s}{r}\right)^{-1} \rightarrow \infty$

- For Black Holes, light cones are deformed by null geodesics. We may derive:

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}$$

* As $r \rightarrow \infty$ back to 45°

- Anthropic Principle

- Observations made about the universe are implicitly biased as a result of the fact that observations can only be made where the possibility of intelligence life exists

- Gravitational Bending of Spacetime (Back-Reaction Term):

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 + \frac{L^2}{2r^2} - \frac{\epsilon GM}{r} - \underbrace{\frac{GML^2}{r^3}}_{\text{New Term}} + \frac{1}{2}\epsilon = \frac{1}{2}E^3$$

- Defining $V(r)$:

$$\frac{L^2}{2r^2} - \frac{\epsilon GM}{r} - \frac{GML^2}{r^3} + \frac{1}{2}\epsilon$$

- * The first term is known as the angular momentum “barrier”

- We may write:

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 + V(r) = \epsilon$$

- Which lets us determine that, for circular orbit, $V'(r) = 0$, and for a stable circular orbit $V''(r) = 0$
- Using Newtonian mechanics, we may see:

$$\frac{d}{dr} \left(\frac{L^2}{2r^2} - \frac{GM}{r} \right) = -\frac{L^2}{r^3} + \frac{GM}{r^2}$$

- * Rearranging terms, we come to the familiar formula:

$$v = \sqrt{\frac{GM}{r}}$$

- We may perform similar calculations with General Relativity, but the new term adds a twist:
 - * 2 solutions instead of 1 for massive particles
 - * Not always stable
- For a massless particle ($v = c$), $\epsilon = 0$:

$$\begin{aligned} V(r) &= \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \\ V'(r) = 0 &\Rightarrow -\frac{L^2}{r^3} + \frac{3GML^2}{r^4} = 0 = 3GM \\ V''(r) &= \frac{3L^2}{r^4} - \frac{12GML^2}{r^5} \end{aligned}$$

- * Always negative! Always unstable!

- For massive particles, $\epsilon = 1$

$$\begin{aligned} V(r) &= \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} \\ V'(r) = 0 &\Rightarrow GMr^2 + L^2r + 3GML^2 = 0 \rightarrow r_L = \frac{L^2 \pm \sqrt{L^4 - 4(GM)(3GML^2)}}{2GM} \end{aligned}$$

- * We may observe:

1. For large L : $\frac{L^2}{GM}$ is stable (goes to Newtonian), and $3GM$ is unstable (the massless case)
 2. For small L : $L = \sqrt{12}GM$ provides smallest circular orbit, $r_c = L^2/2GM = 6GM = 3R_w$; No circular orbits for smaller L (particle goes to $r = 0$ (I.S.C.O))
- This new $1/r^3$ term reflects the non-linear nature of general relativity (back-reaction), which is important for small r
 - For eccentricity e , we may write:

$$x_1 = \frac{3G^2M^2}{L^2} \left[\left(1 + \frac{1}{2}e^2\right) + e\phi \sin(\phi) - \frac{1}{6}e^2 \cos(2\phi) \right]$$

- * Term 1: The overall shift to larger x (smaller r)
 - * Term 2: Accumulates $\phi : 0, 2\pi, 4\pi, 6\pi \dots$
 - * Term 3: Oscillations around elliptical orbit
 - * Note that terms 1 and 3 are small
- Thus, we have:

$$x = 1 + e \cos[(1 - \alpha)\phi] \text{ with } \alpha = \frac{3G^2M^2}{L^2}$$

- * With a period slightly less than 2π
 - * For each orbit, $\Delta\phi = 2\pi\alpha = 6\pi \frac{G^2M^2}{L^2}$
- Precession of Mercury:

$$\Delta\phi_{mer} = 5.01 \cdot 10^{-7} \left[\frac{\text{rad}}{\text{orbit}} \right] = .103'' \left[\frac{1}{\text{orbit}} \right]$$

• Gravitational Redshift

- Consider an observer at rest $U^i = 0$:

$$U_\mu U^\mu = -1 \text{ for a time-like observer}$$

$$\Rightarrow U^o = \left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}}$$

- For a photon traveling on a null geodesic, observer (U^μ) sees energy: $-\frac{dx^\nu}{d\lambda} U^\mu g_{\mu\nu} = -p_\mu U^\mu$
- * This defines λ for a null geodesic
- * Observed frequency:

$$f = \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} \frac{dt}{d\lambda}$$

- * Recall the conserved quantities for geodesics:

$$E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\lambda}$$

$$f \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} = \text{a constant}$$

- * When emitted at r_1 and observed at r_2 , we see:

$$\frac{f_2}{f_1} = \left(\frac{1 - \frac{2GM}{r_1}}{1 - \frac{2GM}{r_2}} \right)^{\frac{1}{2}}$$

- Redshift for $r_1 < r_2 \Rightarrow f_2 < f_1$
- Blueshift for $r_1 > r_2 \Rightarrow f_2 > f_1$

– Eddington-Finkelstein Coordinates:

- * Using $v = t + r^*$ and $u = t - r^*$ (where v is constant ingoing null geodesics, and u is constant outgoing null geodesics), we get:

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dv^2 + (dv dr + dr dv) + r^2 d\Omega^2$$

- * This gives: $\det(g_{\mu\nu}) = -r^4 \sin^2(\theta)$
- * Null geodesics:

$$ds^2 = 0 = - \left(1 - \frac{2GM}{r}\right) dv^2 + 2 dv dr$$

$$dv = 0 \text{ or } dv \left(1 - \frac{2GM}{r}\right) = 2 dr$$

$$\frac{dv}{dr} = 0 \text{ or } \frac{dv}{dr} = 2 \left(1 - \frac{2GM}{r}\right)^{-1}$$

– Event Horizon — Surface past which particles can never escape to infinity

- * Global rather than local feature of spacetime

• Kruskal Coordinates Introduce Distinct Regions of Black Holes:

- Region I: Our Universe
- Region II: Black Hole
- Region III: White Hole
- Region IV: Separate Universe (connected via wormhole, closes before anything can pass through)

• Physical Black Holes

- This assumes perfect spherical symmetry over an infinite time

- Not the real universe; we lack this symmetry, so Einstein did not believe black holes actually existed
- Black holes do form at some finite time, but they do form!
- Observational Evidence
 - Black holes as “central engines”
 - * Active galactic nuclei
 - * Quasars, blazars, etc, ...
 - Our galactic center
 - * 2020 Nobel Prize: Genzel and Ghez
 - * Mapping out star orbits near galaxy’s center (Sag A*) with $M_{BH} \approx 4 \cdot 10^6 [M_\odot]$
 - Nearby galaxy center (M87) with $M_{BH} \approx 6 \cdot 10^9 [M_\odot]$
 - * Image using combined telescopes (mm/radio)
 - * “Event Horizon Telescope”
- Theoretical Evidence
 - Penrose and Hawking: Singularity theorems (2020 Nobel Prize for Penrose)
 - * Singularities can and will form once collapse reaches a certain point
 - * Trapped surfaces
 - No physical pressure able to stop collapse
- Black Hole Theorems
 - No-Hair Theorem: Stationary (Stable) Black Holes can be described with only:
 1. Mass (Schwarzschild)
 2. Charge (Reissner-Nordström)
 3. Spin (Kerr)
 - Cosmic Censorship Conjecture: Naked singularities (no horizon) do not form generically in gravitational collapse (e.g. no “white holes”)
 - Hawking’s Area Theorem: The area of an event horizon will not decrease
 - * Quantum Mechanics complicates this picture: Hawking Radiation
 - * Information loss paradox: quantum gravity?