

# Lecture 4 — Manifolds and Curved Spacetime

Michael Brodskiy

Professor: J. Blazek

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- We now move from Minkowski to General Space:

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$$

- Differentiable Manifolds

- Manifold: A space (in  $n$ -dimensions) that looks locally like  $\mathbb{R}^n$  and can be constructed by smoothly stitching together these regions
- Rotations in  $\mathbb{R}^n \rightarrow$  Lie Groups are manifolds with a group structure
- To be more precise, we have a set  $M$  with a set of (all possible) charts of open subsets to  $\mathbb{R}^n$ 
  - \* Chart  $\leftrightarrow$  coordinate system
- These charts must be smooth, continuous, invertible, and differentiable
- Now we will define (co)tangent spaces on these manifolds, with metrics that map (dual) vectors to  $\mathbb{R}$

- The Equivalence Principle

- In special relativity, we had the principle that the laws of physics were the same in all inertial frames
- Einstein’s “happiest thought”: If someone falls from a roof, nothing falls in their frame
- Equivalence of inertial frames should be generalized to include gravity
- Weak Equivalence Principle (WEP)
  - \* Inertial mass = gravitational mass

$$F = m_i a \text{ (inertial)}$$

$$F = -m_g \nabla \Phi \text{ (gravitational "charge")}$$

$$m_i = m_g \text{ (WEP: Eötvös experiments, late 19th century)}$$

- \* All freely falling bodies behave the same/are indistinguishable ( $a = -\nabla\Phi$ )
- \* Define inertial trajectory as unaccelerated (subject only to gravity)
- \* In small enough regions of space-time, freely falling particles behave the same in a gravitational field or a uniformly accelerated field (physicist in a box, accelerating reference frame)
- Strong Equivalence Principle (SEP)
  - \* All laws of physics, including gravitation, look like SR
    - Einstein Equivalence Principle (EEP) plus the impact of gravitational binding energy
    - Rules out “fifth force”
- Tidal Forces
  - Causes tides on Earth
  - Locally inertial frames
- Gravitational Redshift

$$\Delta v = \frac{az}{c}$$

- Relativistic Doppler Shift:

$$\lambda_{obs} = \lambda_o \left( \frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)^{\frac{1}{2}}$$

- \* Using Taylor expansion, we may simply write this as:

$$\lambda_{obs} = 1 + \frac{v}{c}$$

- \* Bringing this together, we get:

$$\frac{\Delta\lambda}{\lambda_o} \approx \frac{\Delta v}{c} = \frac{az}{c^2}$$

- EEP says that this must be the same as a gravitational field:

$$\frac{\Delta\lambda}{\lambda_o} = \frac{a_g z}{c^2} = \frac{\Delta\Phi}{c^2}$$

- This is the time from start to end of wavelength, and can be used to compare clocks
- If we have a case where  $\Delta t_o = \lambda_o c^{-1}$  and  $\Delta t_1 = \lambda_1 c^{-1}$ , and  $\lambda_1 > \lambda_o$  then  $\Delta t_1 > \Delta t_o$ , which indicates gravitational time dilation

- Classic Tests of General Relativity

1. Precession of the perihelion of Mercury — 19<sup>th</sup> century: 43" per century discrepancy successful "post-diction" of GR (about 10% of total effect)
2. Bending of star light by sun (gravitational lensing) — GR predicts a factor of 2 larger deflection (1919 Eddington Expedition to observe the solar eclipse)
3. Gravitational Redshift — 1954: Popper measurement of a white dwarf, 1959: Pound-rebka at Jefferson lab (Harvard), 22.5m

- Vectors and Tensors on Manifolds (Curved Spacetime)

- We already saw  $V = V^\mu \hat{e}_\mu$  at point  $P$  on  $T_p$
- What is the basis?

- \* We want to define tangent vectors before we have a vector space on  $M$
- \* Instead, consider a function  $f$  and a curve  $\lambda$ . The directional derivative is:

$$\frac{d}{d\lambda} x^\mu \frac{\partial}{\partial x^\mu} f = \frac{d}{d\lambda} x^\mu \partial_\mu f \quad (\text{gradient} \cdot \text{tangent } \vec{v})$$

- \*  $f$  could have been anything, so we define the tangent vector:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \partial_\mu$$

- \*  $\{\hat{e}_\mu = \partial_\mu\}$  is the coordinate basis ("points" in the direction of  $x^\mu$ )
- \* Not orthonormal, but always well defined
- \* In this basis, things transform according to:

$$\partial_{\mu'} = \frac{\partial}{\partial x^{\mu'}} = \frac{\partial}{\partial x^\mu} \frac{\partial x^\mu}{\partial x^{\mu'}} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu$$

- \* Similarly,  $V = V^\mu \partial_\mu$  is preserved, so:

$$V^{\mu'} \partial_{\mu'} = V^\mu \partial_\mu \Rightarrow V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu$$

- General Coordinate Transform

- In flat space:  $x^{\mu'} = \Lambda_\mu^{\mu'}$

$$\frac{dx^{\mu'}}{dx^\mu} = \Lambda_\mu^{\mu'}$$

- We recover the transform of vectors
- Vector Fields:

- \*  $X$ : One vector at each point on the manifold

- \*  $X, Y$ : Both define a field that can be used to take directional derivatives of functions on  $\mu$

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \underline{\text{commutator}}$$

– Dual Vectors

- \* Recall we defined the gradient:  $df$

$$df \left( \frac{d}{d\lambda} \right) = \frac{df}{d\lambda} \quad \text{map a vector to } \mathbb{R}$$

- \* Basis for dual vectors  $dx^\mu$
- \* Gradient of the coordinate function:

$$dx^\mu(\partial_\nu) = \frac{dx^\mu}{dx^\nu} = \delta_\nu^\mu$$

$$V = V^\mu \partial_\mu$$

$$\omega = \omega_\nu dx^\nu$$

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

- We can now write the transformation of an arbitrary  $(k, l)$  tensor on a manifold:

$$T_{\nu_1' \dots \nu_l'}^{\mu_1' \dots \mu_k'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \dots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \dots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T_{\nu_1 \dots \nu_l}^{\mu_1 \dots \mu_k}$$

- \* Warning: in curved space  $\partial_\mu W_\nu$  is not a tensor; unlike in flat space, the derivative of the transform can be non-zero ( $\Lambda$  is the same everywhere)

• The Metric

- $\eta_{\mu\nu}$  in Minkowski space
- $g_{\mu\nu}$  in general curved spacetime

$$g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma \quad (\text{defines inverse})$$

- Metric really describes basically everything about a spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- \* (0,2)-tensor components, metric components, with  $ds^2$  being called the “line element” or “metric”
- \* Usually we just write  $g_{\mu\nu}$
- \* In 3D Flat Space:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

- Components and bases both change, while  $ds^2$  does not

- \* Canonical form: coordinate transform to diagonalize and normalize

$$g_{\mu\nu} = \begin{pmatrix} -1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & \\ \vdots & & -1 & & & \\ \vdots & & & 1 & & \\ \vdots & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

- \* “Signature” is -+++, etc.
- \* All positive: Euclidean or Riemannian
- \* One negative: Lorentzian or pseudo-Riemannian
- \* Any zeros? Degenerate
- \* At some point  $p$ , you can always put a metric into canonical form, and make the first derivatives vanish
  - There is always enough freedom to choose coordinates that do this
  - Second derivatives will not generally vanish
- \* In our case, choose:

$x^{\hat{\mu}}$  at  $p$  such that:

$$g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}} \quad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \quad \partial_{\hat{\rho}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \neq 0$$

- \* Note: not a coordinate system, so not a coordinate basis

$x^{\mu} \rightarrow$  locally inertial coordinates

- Locally inertial/Lorentz frame
- Do calculations in this frame, express in tensor (covariant) form — usually can just assume  $\eta_{\mu\nu}$

## • Coordinate Basis

$$V = V^{\mu} \hat{e}_{\mu}$$

$$\hat{e}_{\mu} = \{\partial_{\mu}\}$$

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

## • Tensor Density

$$\tilde{\epsilon}_{\mu_1 \dots \mu_n} = \begin{cases} +1, & \text{even perm (0123)} \\ -1, & \text{odd perm (0213)} \\ 0, & \text{other (0112)} \end{cases}$$

- Our general transformation  $\frac{\partial x^\mu}{\partial x^{\mu'}}$  is a particular case of  $M_\mu^{\mu'}$

$$\tilde{\epsilon}_{\mu'_1 \dots \mu'_n} = \left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| \tilde{\epsilon}_{\mu_1 \dots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu'_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu'_n}}$$

- We may see that there is an extra factor compared to standard tensor transform
- What about  $|g_{\mu\nu}| = g$ ?

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\hat{\mu}}} \frac{\partial x^\nu}{\partial x^{\hat{\nu}}} g_{\mu\nu}$$

- In general, for  $\tilde{t}$  that transforms with  $\left| \frac{\partial x^{\mu'}}{\partial x^{\hat{\mu}}} \right|^\omega$ , we can make a real tensor  $t = \tilde{t}|g|^{\omega/2}$ , since this will transform with:

$$\begin{aligned} \left| \frac{\partial x^\mu}{\partial x^{\hat{\mu}}} \right|^\omega \left| \frac{\partial x^{\mu'}}{\partial x^{\hat{\mu}'}} \right|^\omega &= 1 \\ \epsilon_{\mu_1 \dots \mu_n} &= \sqrt{|g|} \tilde{\epsilon}_{\mu_1 \dots \mu_n} \\ \epsilon^{\mu_1 \dots \mu_n} &= \epsilon_{\mu_1 \dots \mu_n} \cdot \text{sign}(g) \end{aligned}$$

- Differential Forms

- p-form is a  $(0, p)$  anti-symmetric tensor with:

$$\begin{aligned} A_{\mu\nu} &= -A_{\nu\mu} \\ A_{\mu\nu\sigma} &= -A_{\nu\mu\sigma} = A_{\nu\sigma\mu} \end{aligned}$$

- \* Scalars: 0-forms
- \* Dual vectors: 1-forms

- Wedge product is anti-symmetrized tensor product

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \dots \mu_p]} B_{[\mu_{p+1} \dots \mu_{p+q}]}$$

- Exterior Derivative

- A tensor, unlike the partial derivative, but only acts on forms

$$(dA)_{\mu \dots \mu_{p+1}} = (p+1) \partial_{\mu_1} A_{\mu_2 \dots \mu_{p+1}}$$

- We have already seen this operator, the gradient (of a scalar):

$$(d\phi)_\mu = \partial_\mu \phi$$

- Because partials commute:

$$d(dA) = 0$$

- Electrodynamics

$$F_{\mu\nu}$$

$$\partial_{[\mu} F_{\nu\lambda]} = 0$$

$$dF = 0$$

$$F = dA \text{ (vector potential } A_\mu)$$

- Hodge Star

- $*A$  takes  $p$ -forms to  $n - p$  forms

- Integration on Manifolds

- In general, when changing coordinates in an integral, we multiply by the Jacobian

$$x, y, z \rightarrow r, \theta, \phi$$

$$\left| \frac{\partial x^{\mu'}}{\partial x^\mu} \right| = r^2 \sin(\theta)$$

$$\int dx dy dz f(\vec{r}) \rightarrow \int r^2 \sin(\theta) dr d\theta d\phi f(\vec{r})$$

- \* This is a particular example of integration on a manifold

$$\int w(x) dx = w$$

- Represents a component  $w(x)$  with basis  $dx$
- We may write:

$$d\mu(U, V, W) \leftarrow \mathbb{R}$$

- \* This maps vectors at a point to their volume
- We generalize:

$$d^n x = dx^0 \wedge \cdots \wedge dx^{n-1}$$

- \* This is not yet a tensor because it is coordinate independent

$$\sqrt{|g|} d^n x = \sqrt{|g|} dx^0 \wedge \cdots \wedge dx^{n-1} = \sqrt{|g'|} dx^{0'} \wedge \cdots \wedge dx^{(n-1)'}$$

$$I = \int \phi(x) \sqrt{|g|} d^n x \rightarrow \int \phi(x) dx dy dz \text{ if } |g| = 1$$

- You can then evaluate as normal

- FLRW Metric (or FRW, or RW)

- Friedmann, Lemaitre, Robertson, Walker → Solution to Einstein's equations for a spatially homogenous, isotropic spacetime. Can be curved or flat. Flat FLRW:

$$dx^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

- \* Where  $a(t)$  is the scale factor
- \* For  $a(t) = t^q$ ,  $0 < q < 1$
- \*  $t = (1 - q)^{\frac{1}{1-q}}(\pm x - x_o)^{\frac{1}{1-q}}$
- \* Light not always at a  $45^\circ$  angle
- \* Singularity at  $t = 0 \rightarrow$  cosmic horizon  $\rightarrow p$  and  $s$  are completely disconnected

- Curvature, Covariant Derivatives, Geodesics

- For  $S^2$ :

$$ds^2 = \frac{R^2 dr^2}{R^2 - r^2} + r^2 d\theta^2$$

- The metric reflects the curvature; for  $R \rightarrow \infty$ , we get back to flat 2D space
- From the metric, we will derive the “connection,” which tells us the impact of curvature, including defining straight lines
- The connection ( $\Gamma_{\mu\nu}^\lambda$ , not a tensor) tells us how to compare vectors at nearby points

- Covariant Derivatives

- Recall  $\partial_\mu A_\nu$  is not a covariant because of an extra term
- External derivative  $dA = \partial_\mu A_\nu - \partial_\nu A_\mu$  is covariant but only valid on forms
- We define the covariant derivative of a vector and dual vector:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\lambda}^\nu V^\lambda$$

$$\nabla_\mu \omega^\nu = \partial_\mu \omega^\nu - \Gamma_{\mu\lambda}^\nu \omega^\lambda$$

- \* Note:  $\Gamma$  is not a tensor, do not raise or lower indices!
- \* Covariant refers to transforming like a tensor

$$\partial_{\mu'} A_{\nu'} = \left( \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \right) \left( \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\nu \right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu A_\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} A_\nu \partial_\mu \frac{\partial x^\nu}{\partial x^{\nu'}}$$

- We can determine how  $\Gamma$  must transform such that  $\nabla_\mu V^\nu$  and  $\nabla_\mu \omega_\nu$  are covariant



– Four requirements on covariant derivatives:

1. Linearity:  $\nabla(T + S) = \nabla T + \nabla S$
2. Product Rule:  $\nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S$
3. Commute with Contractions:  $\nabla_\mu(T^\lambda_{\lambda\rho}) = (\nabla T)^\lambda_{\mu\lambda\rho}$
4. Partial Derivative on Scalars:  $\nabla_\mu\phi = \partial_\mu\phi$

– For a general  $(k, l)$  tensor, we have:

$$\nabla_\sigma T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = \partial_\sigma T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \dots \mu_k}_{\nu_1 \dots \nu_l} + \dots + \Gamma^{\mu_k}_{\sigma\lambda} T^{\mu_1 \dots \lambda \dots \mu_k}_{\nu_1 \dots \nu_l} - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \dots \mu_k}_{\lambda \nu_2 \dots \nu_l} - \dots - \Gamma^\lambda_{\sigma\nu_l} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \lambda \dots \nu_l}$$

– Notation:

$$\nabla_\sigma A_\mu = A_{\mu;\sigma}$$

$$\partial_\sigma A_\mu = A_{\mu,\sigma}$$

- $\Gamma^\mu_{\sigma\lambda}$  must obey a particular (non-covariant) transform under coordinate change
- We have a lot of freedom to define  $\Gamma^\mu_{\sigma\lambda}$ , but we will impose two additional conditions (for convenience):

- \* Torsion-free:  $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0 \Rightarrow \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$
- \* Metric-compatible:  $\nabla_\alpha g_{\mu\nu} = 0$

$$\nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = 0$$

$$\nabla_\rho g^{\mu\nu} = 0$$

– Christoffel Symbol:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

## • Parallel Transport and Geodesics

– What is parallel transport?

- \* Keep the vector fixed as you “move it.” This is generally path-dependent
- \* We must do this to compare vectors
- \* Let’s be a bit more mathematical:

$$\frac{d}{d\lambda} V^\mu = 0$$

$$\frac{d}{d\lambda} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l} = 0$$

- \* Before, we had:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu}$$

\* Replace with covariant directional derivative:

$$\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda} \nabla_\mu$$

– Parallel transport is defined by:

$$\left( \frac{D}{d\lambda} T \right)_{\nu_1 \dots \nu_l}^{\mu_1 \dots \mu_k} = 0$$

- Straight Lines  $\rightarrow$  Geodesics

- Straight line is the shortest distance between two points; in flat space, the direction of your tangent vector doesn't change
- We generalize this notion: A geodesic is a path that parallel transports its own tangent vector:

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma_{\rho\sigma}^\mu \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 \quad (\text{geodesic equation})$$

– In flat space with Cartesian coordinates,  $\Gamma_{\rho\sigma}^\mu = 0$

$$\frac{d^2 x^\mu}{d\lambda^2} = 0 \quad (\text{a “straight” line})$$

- Geodesics as the “Shortest Distance”

– Consider a timelike path; recall proper time along  $x^\mu(\lambda)$ :

$$d\tau^2 = ds^2$$

$$\tau = \int \left( -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{\frac{1}{2}} d\lambda$$

- Recall (Twin Paradox) that the “shortest spacetime interval” corresponds to maximum proper time, due to the minus sign in Lorentzian spacetimes
- We want to find the stationary points of  $\tau$  (max or min) by varying  $x^\mu(\lambda)$ ; the resulting path will end up being (Note: very common method in physics, same as the Lagrangian  $\rightarrow$  action formulation, which allows the derivation of EoM):

\* We can simplify:

$$f = g_{\mu\nu} (dx^\mu/d\lambda) (dx^\nu/d\lambda)$$

\* Variation is:

$$\delta t = \int \delta \sqrt{-f} d\lambda = - \int \frac{1}{2} (-f)^{-\frac{1}{2}} \delta f d\lambda$$

\* We can simplify by taking  $\lambda = \tau$ :

$$\begin{aligned}\frac{dx^\mu}{d\tau} &= U^\mu \quad (\text{four velocity}) \\ \Rightarrow f &= g_{\mu\nu} U^\mu U^\nu = -1 \quad \text{definition of } \tau \\ \delta t &= -\frac{1}{2} \int \delta f d\tau\end{aligned}$$

– So, we can find the stationary points of:

$$I = \frac{1}{2} \int f d\tau = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

– We can use the Euler-Lagrange, or just directly calculate variation:

$$\begin{aligned}x^\mu &\rightarrow x^\mu + \delta x^\mu \\ \Rightarrow g_{\mu\nu} &\rightarrow g_{\mu\nu} + (\partial_\sigma g_{\mu\nu}) \delta x^\sigma \\ \delta I &= I_{new} - I_{old}\end{aligned}$$

– Bringing all of our terms together, we get:

$$\delta I = - \int \left[ g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\sigma d\tau$$

– Our final form takes:

$$\frac{d^2 x^\rho}{d\tau^2} + \underbrace{\frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})}_{\Gamma_{\mu\nu}^\rho} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

### • Affine Parameters

- For timelike geodesics:  $\lambda = a\tau + b$       affine parameters
- Geodesic equation is satisfied (usually just use  $\tau$ )
- Parallel transport of  $dx^\mu/d\lambda$  defines both  $x^\mu$  and  $\lambda$
- We can express timelike geodesics using  $U^\mu$  or  $p^\mu = mU^\mu$
- For null paths:  $\tau = 0$ 
  - \* Instead, we typically choose  $p^\mu = \frac{dx^\mu}{d\lambda}$  to define  $\lambda$
  - \* In either case, then  $E = -p_\mu U^\mu$

### • The Riemann Curvature Tensor

- What is curvature at some point  $p$ ?
- Flatness, in a slightly more technical sense, means:

1. Parallel transport does not depend on path
    - \* No change on a closed loop
  2. Covariant derivatives commute
  3. Parallel geodesics remain parallel
- Transporting some vector  $V^\rho$  from point  $p$  along a path, we can take, at leading order (fine in a local region) the change will be a linear function of  $V^\rho$ ,  $A^\mu$ , and  $B^\nu$ :

$$\delta V^\rho = R_{\sigma\mu\nu}^\rho V^\sigma A^\mu B^\nu$$

- Thus, we see the Riemann Tensor ( $R_{\sigma\mu\nu}^\rho$ )
- We can define the Torsion Tensor (will be zero for us) below as  $T_{\mu\nu}^\lambda$ :

$$[\nabla_\mu, \nabla_\nu]V^\rho = R_{\sigma\mu\nu}^\rho V^\sigma - T_{\mu\nu}^\lambda \nabla_\lambda V^\rho$$

- The Riemann Tensor can be defined as:

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$$

- \* A tensor, based on non-tensors
  - \* Valid for any valid connection, but we will use Christoffel
  - \* Has second derivatives of the metric; will not vanish at  $p$  for locally inertial coordinates
  - \* If the metric is constant in some coordinate system,  $R_{\sigma\mu\nu}^\rho = 0$
  - \* If  $R_{\sigma\mu\nu}^\rho = 0$ , we can always construct a coordinate system with constant metric (flat)
- Properties of the Riemann Tensor:

$$R_{\rho\sigma\mu\nu} = g_{\rho\alpha} R_{\sigma\mu\nu}^\alpha$$

- We can work in locally inertial coordinates to derive generally applicable rules (covariance)
1.  $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$
  2.  $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$
  3.  $R_{\rho\sigma\mu\nu} = R_{\nu\mu\rho\sigma}$
  4.  $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$
  5.  $R_{\rho[\sigma\mu\nu]} = 0$
  6.  $R_{[\rho\sigma\mu\nu]} = 0$
- For  $n$  dimensions,  $n^4$  possible components, but to satisfy these conditions:  $(1/12)n^2(n^2 - 1) = 20$  for  $n = 4$

$$\nabla_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} = \partial_{\hat{\lambda}} R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}}$$

- \* Cycling through first 3 indices:

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0 \quad (\text{Bianchi Identity})$$

- \* Tells us about derivatives on the surface:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b}(\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0 \quad (\text{Jacobi identity})$$

- We want to decompose  $R_{\sigma\mu\nu}^\rho$  into smaller pieces with more direct physical interpretation

$$X_{\mu\nu} = X_{(\mu\nu)} + X_{[\mu\nu]} \quad \text{general (0,2) tensor}$$

- We may define the Ricci Tensor:

$$R_{\mu\nu} = R_{\mu\lambda\nu}^\lambda$$

- \* Performs a contraction (the only independent contraction for Christoffel)

$$R_{\mu\nu} = R_{\nu\mu}$$

- The Ricci Scalar becomes:

$$R = R_\mu^\mu = g^{\mu\nu} R_{\mu\nu}$$

- This leaves us with the Weyl Tensor (Riemann Tensor with all contractions removed)

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{n-2}(g_{\rho[\mu}R_{\nu]\sigma} - g_{\sigma[\mu}R_{\nu]\rho}) + \frac{2}{(n-1)(n-2)}g_{\rho[\mu}g_{\nu]\sigma}R$$

- \* The Ricci Tensor with “removed contractions”

- We can re-express the Bianchi identity:

$$\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R \quad \text{or} \quad \nabla^\mu G_{\mu\nu} = 0$$

- \* Where  $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$  is the Einstein Tensor!

## • Geodesic Deviation

- How does curvature impact “parallel lines” → neighboring geodesics?  $T^\mu$  is tangent and  $S^\mu$  is the deviation
- We can define a relative velocity:

$$V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu$$

- And a relative acceleration to accompany it:

$$A^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu$$

- $V^\mu = 0 \Rightarrow$  initially “parallel”
- Through a bit of algebra, we get:

$$A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

- \* Applies to massive particles or photons
  - \* An observer may experience weak lensing with smaller particles
  - \* An observer may experience strong lensing with more massive particles
- Where are we?
    - Special relativity: equivalence of inertial frames and constant  $c$
    - Notion of a manifold which has a coordinate system and (co)tangent spaces
    - Curved spacetime as a manifold
    - Gravity is universal  $\rightarrow$  equivalence principle; gravity as a property of spacetime, not a force
    - The metric measures distances in spacetime ( $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ )
      - \* Invariant distance  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
    - Covariant derivatives require a connection between nearby points on the manifold ( $\Gamma \approx (d/dx)g$ )
    - Metric compatible, torsion-free connection: Christoffel
    - Connection expresses geodesics (straight lines  $\leftrightarrow$  parallel transport)
    - Parallel transport in a loop: Riemann Curvature Tensor
      - \* Ricci Tensor, Ricci Scalar, Weyl Tensor
    - Energy and Momentum in Curved Spacetime

$$U^\mu = \frac{dx^\mu}{d\tau}$$

$$p^\mu = mU^\mu$$

$$T^{\mu\nu} : p^\mu \text{ across } x^\nu$$

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}$$

- Minimal Coupling Principle
  - Law of physics in inertial, flat spacetime
  - Write it in a covariant form

- Profit! (assert that this law remains true in curved spacetime)

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} \quad \text{and} \quad \partial_\mu \nabla_\mu$$

- \* We have already been implicitly thinking this way

- Geodesic equation:

$$\partial_\mu T^{\mu\nu} \rightarrow \nabla_\mu T^{\mu\nu}$$

- Does the geodesic equation actually describe gravitational motion in the Newtonian limit?

- \* Weak field (perturbation of flat space)
- \* Slow moving ( $v \ll c$ )
- \* Constant in time

$$v \ll c \rightarrow \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau}$$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{00}^\mu \left( \frac{dt}{d\tau} \right)^2$$

$$\Gamma_{\mu\nu}^\sigma = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$

- Plugging in the Newtonian limit, we may write:

$$\frac{d^2 x^\mu}{d\tau^2} = \frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \left( \frac{dt}{d\tau} \right)^2$$

- Einstein's Equation (1907-1915)

- We want  $\nabla^2 \Phi = 4\pi G \rho$
- Trying conservation of energy and combining with the Bianchi Identity, we can get:

$$\underbrace{\nabla^\mu (R_{\mu\mu} - \frac{1}{2} g_{\mu\nu} R)}_{G_{\mu\nu}} = 0$$

- \* Where  $G_{\mu\nu}$  is the Einstein Tensor

- We can obtain:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} = k T_{\mu\nu}$$

- Contracting both sides, we can get:

$$R = -kT$$

\* This means:

$$R_{\mu\nu} = k \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

- After a series (pun intended) of Taylor expansions, and discarding many unneeded terms, and, finally, combining with Poisson’s Equation, we may write:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

\* Equivalently, we may write:

$$R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

- Quantum Gravity

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} \quad \text{quantum field}$$

- Essentially how QED came from E&M
- So far, haven’t succeeded “non-renormalizable theory”