

# Lecture 6 — The Expanding Universe

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December 4, 2024

- Olber's Paradox: Why is the Sky Dark?
  - Absorbing Matter  $\rightarrow$  doesn't work since matter would heat up
  - Finite Size
  - Finite Time (and Finite Speed of Light)
  - Dimming of Light (“redshift”)
- Universe had some beginning (“Big Bang”) around 13.7 billion years ago
- Some units:

$$1[\text{light year}] = 9.5 \cdot 10^{15}[\text{m}]$$

$$1[\text{yr}] \approx \pi \cdot 10^7[\text{s}]$$

$$c \approx 3 \cdot 10^8 \left[ \frac{\text{m}}{\text{s}} \right]$$

$$1[\text{pc}] = 3.26[\text{light years}]$$

- If  $\theta = 1[\text{arcsec}]$ , then  $d = 1[\text{pc}]$

$$1[\text{pc}] = 2.1 \cdot 10^5[\text{AU}]$$

- The Cosmological Principle
  - Copernicus: the Sun, not the Earth, is the center of the Universe
  - Cosmological Principle: There is no center to the Universe
    - \* The Universe is statistically isotropic (same in all directions) and homogenous (same everywhere)
- Expanding Universe

- All observers see things moving away from them
- Statements are all statistical! Distinguish between structure in the universe and the geometry of the homogenous universe (about 100[Mpc] scales for homogeneity)
- We don't experience the FLRW metric
  - \* Homogenous/geometry
  - \* Structure
- Conservation of Energy Solution to Expanding Cloud:

$$\frac{1}{2}\dot{R}^2 - \frac{2GM}{R} = C$$

- \* What is the physical meaning of  $C$ ?
  - $C = 0$   
You are just at escape velocity. As  $R \rightarrow \infty$ ,  $\dot{R} = v \rightarrow 0$ . Potential and kinetic both go to zero.
  - $C > 0$   
Positive total energy. You have more than enough energy to escape.  $\dot{R} > 0$  as  $R \rightarrow \infty$
  - $C < 0$   
Negative total energy. You won't make it out to  $R = \infty$ , you will stop and turn around at some finite  $t$
- \* These cases capture an ideal universe's expansion with only real matter
- \* For a matter-only universe,  $C$  describes the spatial geometry, with zero indicating flat,  $C > 0$  indicating open (negative curvature), and  $C < 0$  indicating closed (positive curvature)
  - Note that  $\Lambda$  complicates this
- Using the first-order equation, we may write:

$$\frac{1}{2}\dot{R}^2 = \frac{4}{3}\pi G\rho R^2 + C$$

$$\left(\frac{\dot{R}}{R}\right) = \frac{8\pi G\rho}{3} + \frac{C}{R^2}$$

- At  $C = 0$ , we find  $\rho_{crit}$ :

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

Note that we define the Hubble parameter as:

$$H = \frac{\dot{R}}{R}$$

- Comoving Coordinates

- Coordinates (and distances) scale with the size of the universe:

$$x = a(t)r$$

- \*  $x$  is the “proper” coordinate
  - \*  $r$  is the “comoving” coordinate
  - \*  $a(t)$  is the scale factor, with  $a(t_o) = 1$  indicating “today”
- Ex.  $x_{12} = a(t)r_{12}$

$$\frac{dx_{12}}{dt} \quad \text{is the recession velocity}$$

- Two Things:

1. Velocity is proportional to distance (Hubble law)
2. The proportionality term is  $\frac{\dot{a}}{a} = H(t)$

- Hubble constant is approximately:

$$H_o = 70 \left[ \frac{\text{km}}{\text{sMpc}} \right]$$

- \* We define  $h$ , such that:

$$H_o = 100h \left[ \frac{\text{km}}{\text{sMpc}} \right]$$

- \* We can see that the units are actually just Hz, which gives us:

$$H_o \approx 3.24 \cdot 10^{-18} \left[ \frac{1}{\text{s}} \right]$$

- If  $a$  is constant, this gives us the doubling time or the time to go back to  $a = 0$ , which gives us the “Hubble” time, or the age of the universe
- For  $h = .7 \rightarrow 1.4 \cdot 10^{10} \left[ \frac{1}{\text{yr}} \right]$ 
  - \* Fairly accurate approximation of 14 billion years

## • FLRW Metric

- Cosmological principle tells us that spacetime (on large scales) should be homogeneous (translation invariant) and isotropic (rotation invariant)
- We seek a general form of a metric that obeys these assumptions:

$$ds^2 = \bar{c}^2 dt^2 + R^2(t) d\sigma^2$$

- We are able to derive:

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2$$

– Where:

- \*  $d\Omega^2$  is the 2-sphere metric
- \*  $k \propto R$  (Ricci scalar on 3D space)

$$k = \begin{cases} +1, & \text{Positive Curvature (3-sphere), "closed"} \\ 0, & \text{Flat} \\ -1, & \text{Negative curvature (saddle), "open"} \end{cases}$$

– Thus, we may define the metric as:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]$$

\* Where:

$$\begin{cases} \kappa > 0, & \text{closed} \\ \kappa = 0, & \text{flat} \\ \kappa < 0, & \text{open} \end{cases}$$

- Cosmological Redshift

- Like a Döppler shift, but use caution!
- $v = H_o d$ , locally  $v \ll c$
- Redshift in special relativity:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \sqrt{\frac{1 + v/c}{1 - v/c}} \approx 1 + \frac{v}{c}$$

– From this, we know:

$$\frac{\lambda_{obs}}{\lambda_{em}} = 1 + z \approx 1 + \frac{H_o d}{c}$$

– More generally, we may say (for  $a_o = \text{today}$ ):

$$1 + z = \frac{a_{obs}}{a_{em}} = a_{em}^{-1}$$

- Cosmological Redshift versus Peculiar Motions:

– For comoving coordinates:

$$v \approx H_o d = H_o |\vec{r}_2 - \vec{r}_1| / a_o = H_o |\vec{r}_2 - \vec{r}_1|$$

– For “peculiar” motions

$$v \approx H_o d + \frac{\Delta v}{c}$$

- In the FLRW metric, due to lack of simple time symmetry, energy is not conserved
  - \* There is a Killing Tensor that reflects a symmetry:

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \quad \text{for observer (comoving) } U^\mu = (1, 0, 0, 0)$$

- \* We get:

$$K^2 = K_{\mu\nu} V^\mu V^\nu$$

- Is conserved, with

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

- For a photon on a null geodesic:

$$V_\mu V^\mu = 0$$

- \* This can be simplified to:

$$v = \frac{K}{a}$$

- “Stuff” That Can fill A Universe

- “Baryons” → All standard model particles with mass (interact with light, gravity)
  - \* Non-photon force carriers
  - \* “In a box” with scale factor  $a$  scale proportionally to  $a^{-3}$
- Dark matter → Only interacts through gravity (?)
  - \* Neutrinos (?)
  - \* New particles
  - \* “Cold” dark matter (non-relativistic), scales “in a box” with scale factor  $a$  proportionally to  $a^{-3}$
- Dark energy (“cosmological constant”)
- Radiation
  - \* Photons
    - “In a box” with scale factor  $a$  scale proportionally to  $a^{-4}$
  - \* Relativistic Particles
- Black Holes (acts more or less like dark matter, but forms in other ways)
- Cosmological constant “in a box” does not scale; that is, it is a constant, so it does not change

- Curvature in the Universe

- Entropy in the Universe

- “Information”

- Formal Momenta Redshift:

- Photon redshift  $E \propto \frac{1}{a}$ ,  $\lambda \propto a$

- Energy Evolution

- We know:  $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$

- Working in a fluid’s rest frame,  $U^\mu = (1, 0, 0, 0)$ , which gives us:

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Dark Energy: For  $\Lambda$ ,  $\rho_\Lambda = \text{constant}$

- We may obtain, from a formal General Relativity derivation:

$$\rho = \rho_o a^{-3(1+w)}$$

- We may also obtain:

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} [\rho_{m,o}a^{-3} + \rho_{r,o}a^{-4} + \rho_{\Lambda,o} - 2\kappa a^{-2}]$$

\* Note that, once we have  $\Lambda$ , a closed universe won’t recollapse

- From the critical density, we write the ratio as:

$$\Omega_{i,o} = \frac{\rho_{ip}}{\rho_{crit}}$$

- This allows us to rewrite the Hubble parameter as:

$$\frac{H(a)}{H_o} = (\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2})^{\frac{1}{2}}$$

- We can then write in terms of the redshift:

$$\frac{H(z)}{H_o} = (\Omega_m (1+z)^3 + \Omega_r (1+z)^4 + \Omega_\Lambda + \Omega_\kappa (1+z)^2)^{\frac{1}{2}}$$

- Furthermore, we see:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa = 1$$

- There are several important naming conventions:

- \*  $\Omega_m = 1$ : Flat, matter-only “Einstein-deSitter Universe”
- \*  $\Omega_\kappa = 0$ : Flat
- \*  $\Omega_\Lambda = 1$ : deSitter Universe
- \*  $\Omega_\Lambda = -1$ : Anti-deSitter Universe
- \*  $\Omega_\kappa = 1$ : Empty

- How Do We Measure Distances?

- Angular Size (Object seems larger if it is closer)
- Noise (Object seems louder if it is closer)
- Light (Object seems brighter if it is closer)

- Distances:

- Comoving (radial) distance: follow a photon as it travels from a distant source

$$ds = 0 \rightarrow dt = a(t) \frac{dr}{(1 - \kappa r^2)^{\frac{1}{2}}}$$

$$\chi = \int_{t_{em}}^{t_o} \frac{dt}{a(t)} = \int_0^r \frac{dr'}{(1 - \kappa r'^2)^{\frac{1}{2}}}$$

- We will redefine the  $r$  coordinate such that  $\chi$  is always  $r$ :

$$ds^2 = -dt^2 + a^2(t) [dr^2 + S_\kappa^2(r) d\Omega^2]$$

- \* The term becomes:

$$S_\kappa(r) = \begin{cases} r, & \Omega_\kappa = 0 \quad (\text{flat}) \\ H_o^{-1}(\Omega_\kappa)^{-\frac{1}{2}} \sinh \left( H_o(\Omega_\kappa)^{\frac{1}{2}} r \right), & \Omega_\kappa > 0 \quad (\text{open}) \\ H_o^{-1}(|\Omega_\kappa|)^{-\frac{1}{2}} \sin \left( H_o(-\Omega_\kappa)^{\frac{1}{2}} r \right), & \Omega_\kappa < 0 \quad (\text{closed}) \end{cases}$$

- We may rewrite  $\chi$  in terms of  $a$  to see:

$$\chi(a) = \int_{a(t_{em})}^{a(t_o)=1} \frac{da}{a^2 H(a)}$$

- The Horizon:

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H(a)} \xrightarrow{\text{EdS Universe}} \int_0^1 \frac{da}{a^2 H_o a^{-\frac{3}{2}}} = \frac{2}{H_o}$$

- Note if we restore  $c$ :

$$\chi_{hor} = \frac{2c}{H_o}$$

- Age of the Universe

- We imagine an empty universe ( $\Omega_\kappa = 1$ ):

$$\begin{aligned}\int_0^1 da &= \int_0^{t_o} H_o dt \\ 1 &= t_o H_o \\ t_o &= \frac{1}{H_o} \quad (\text{Hubble time})\end{aligned}$$

- Similarly, for a matter-dominated universe ( $\Omega_m = 1$ ):

$$\begin{aligned}\int_0^1 a^{\frac{1}{2}} da &= \int_0^{t_o} H_o dt \\ \frac{2}{3} a^{\frac{3}{2}} &= t_o H_o \\ t_o &= \frac{2}{3H_o}\end{aligned}$$

- Luminosity Distance

- “Standard Candles”
- Luminosity is Energy per time
- Flux is the Energy per Area per time
- In normal three dimensions:

$$L = F(4\pi R^2)$$

\* This means:

$$F \propto \frac{L}{R^2}$$

- In comoving coordinates, we may write:

$$F \propto \frac{L}{\chi^2} \cdot \frac{1}{1+z} \cdot \frac{1}{1+z}$$

- We may thus find that, in a flat universe:

$$d_L = \chi(1+z)$$

- In a non-flat universe, we see:

$$d_L = S_\kappa(\chi)(1+z)$$



- Angular Diameter Distance

- Object of physical size  $l$

- \* Assume a flat universe:

$$l = d_A \theta$$

$$d_A^{flat} = \chi a = \frac{\chi}{1+z} = \frac{d_L}{(1+z)^2}$$

$$d_A = a S_\kappa(\chi)$$

- \* Importantly,  $d_A$  can increase, reach a max, and then decrease

- This means physical  $l$  takes up larger fraction of a smaller universe

- \* This is slightly more complicated in a non-flat universe, since we need to properly account for  $d\Omega$  factor

- Evidence for Dark Energy

- Until the early 1990's, people did not take  $\Lambda$  too seriously. Evidence from galaxy clusters, galaxy clustering, and CMB started to raise some questions. Universe appeared roughly flat, but  $\Omega_m < 1$

- Mapping out  $a(t)$  has been a useful way to probe the universe

- Dark Matter

- In 1781, William Herschel discovered Uranus; over the next 60 years, astronomers carefully mapped out its orbit, but it didn't quite match Newtonian theory

- Urbain Le Verrier: Showed that Uranus' orbit could be explained if there were another, more distant planet acting gravitationally

- In September 1846, he mailed a letter to a colleague at the Berlin Observatory with precise predictions

- In 1930s, Fritz Zwicky studied the Coma Cluster

- \* Galaxy Clusters: Largest gravitationally bound objects in the Universe

- Zwicky measured spectra of many galaxies and calculated the velocity dispersion

- With a quick calculation, the orbital velocity of a galaxy cluster may be expressed as:

$$M = \frac{r_c v^2}{\alpha G}$$

- MACHOs (Massive Compact Halo Objects)

- \* Planet or asteroid-sized objects: “microlensing” constraints mostly rule this out

- Axions
- Weakly-Interacting Massive Particles (WIMPs)
- Primordial Black Holes (PBHs)
- CDM: Cold Dark Matter  $\rightarrow \Lambda$ CDM model
- Can generalize  $w$ CDM such that:

$$p_{DE} = w\rho_{DE}$$

$$* \Lambda: w = -1$$

$$w(a) = w_o + w_a(a - a_o)$$

- What is Dark Energy?
  - “Cosmological Constant Problem”
  - Universe expansion is accelerating;  $\Lambda$  can do this
  - Quantum field theory says that vacuum fluctuations have an energy density
- Hot Big Bang
  - Thermal Equilibrium

$$\frac{n_1}{n_2} = e^{-(E_1 - E_2)/kT}$$

- \* State 1:  $E_1$
- \* State 2:  $E_2$

- Photons
  - Photons are bosons
  - This means they follow the Bose-Einstein distribution:
 
$$\bar{n}_i = \frac{g_i}{e^{(\epsilon_i - \mu)/kT} - 1} \Rightarrow \rho_i = \bar{n}_i \epsilon_i$$
    - \*  $g_i$  is the “degeneracy”
    - \* This expresses how many particles are in a given energy state  $i$
  - To calculate the energy density, we need to consider the “phase space”  $\{\vec{x}, \vec{p}\}$  and integrate over all states with a given  $|p|$
  - Heisenberg’s uncertainty principle:  $d^3x d^3p$  has  $\frac{d^3x d^3p}{(2\pi\hbar)^3}$  phase space elements. For photons,  $\mu = 0$ :

$$\rho = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{2p}{e^{p/kT} - 1}$$

\* We may obtain:

$$\rho(f) df = \frac{8\pi h}{c^3} \left( \frac{f^3 df}{e^{hf/kT} - 1} \right)$$

\* For photons between  $f$  and  $f + df$

– Blackbody Spectrum

\* Blackbody: Perfectly absorptive system which emits a spectrum given by photon thermal equilibrium

$$n_c(f_c) = \frac{8\pi}{c^3} \frac{f_c^2}{e^{hf_c/kT} - 1}$$

· Preserved if  $T \rightarrow T/a$

– Temperature of the universe is really defined by the distribution of particles in thermal equilibrium

- Cosmic Microwave Background

– In an expanding universe, we required:

$$\Gamma > H$$

\*  $\Gamma \propto n_2 v_1 \sigma_{12}$

\*  $H$  is the Hubble rate, which is the inverse of Hubble time

\* “Freeze-out”  $n_i$  is fixed

- Big Bang Nucleosynthesis (BBN): “The First Three Minutes”

– Fission versus Fusion

– Binding energy: Energy to break apart nucleus or, equivalently, energy to release forming the nucleus

- Matter-Antimatter Asymmetry

$$\eta = \frac{n_b}{n_\gamma} = 6 \cdot 10^{-10} \Rightarrow 6 \text{ baryons per } 10 \text{ billion photons}$$

– Standard model (mostly) treats particles and antiparticles the same

$$e^- + e^+ \leftrightarrow \gamma + \gamma$$

– Not due to freeze out: we don’t see equal amounts of anti-matter; annihilation was complete for anti-matter

- At early time,  $kT \gg 150[\text{MeV}]$  (quarks, baryons): some asymmetry was present

$$\frac{n_q - n_{q^-}}{n_q} \approx 4 \cdot 10^{-9}$$

- All of the anti-quarks will annihilate, leaving the value of  $\eta$  from above
- 2 photons per annihilation, 3 quarks per baryon
- Baryogenesis (and probably related “leptogenesis”) is one of the big open questions in cosmology and particle physics
- It may be possible within the Standard Model, but it’s not clear how. The Sakharov conditions (1967) for baryogenesis state:
  1. Baryon number violation
  2. Charge and charge parity-symmetry violation
  3. Out of thermal equilibrium
- WIMP Dark Matter and Freeze-out
  - Weakly interacting massive particles (WIMPs,  $M \geq 100[\text{GeV}]$ ) has been a very popular DM candidate. New particle could be the result of supersymmetry between fermions and bosons
- Recombination and Decoupling
  - Discusses when universe became neutral enough to (at least mostly) stop photon scattering
  - Recombination: When ionization fraction is  $1/2$
  - Decoupling: When  $H \approx \Gamma_T$ , photons no longer interact through Thomson scattering
  - Last scattering: The time (or surface) when the typically photon last scattered
  - Not really a surface, more of a narrow shell (think: analogy of looking into fog, you can see a little way in)
- We may observe the Saha solution for ionization fraction as:

$$\frac{\chi_e^2}{1 - \chi_e} = \frac{1}{n_e + n_H} \left( \frac{m_e T}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{\epsilon_0}{T}}$$

- When  $\chi_e \rightarrow 1$ , everything is still ionized
- For recombination, we want  $\chi_e \approx 1/2$
- The Saha equation assumes chemical equilibrium; when not in equilibrium, use the Boltzmann equation

- CMB Fluctuations

- We may recall that:

$$\frac{\Delta T}{T} \approx 10^{-5}$$

- Fourier versus Real Space (Called Configuration Space in Cosmology)

- Configuration space in position (function of  $x$ ), Fourier space defined with respect to wave number  $k$
- We may write (in 3D):

$$F(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}} \tilde{F}(\vec{k})$$

- Plane waves may be expressed by the wave number,  $\vec{k} = (k_x, k_y, k_z)$ 
  - \* A higher  $k$  means smaller-scale fluctuations
  - \* Plane wave pointing in  $\hat{k}$  direction with  $\lambda = \frac{2\pi}{k}$

- Correlation Functions

- Correlation functions are incredibly important in cosmology because we can predict the statistics of the initial conditions, but the initial conditions themselves are random
- A related concept is the power spectrum

- Sound Waves

- Start with an initial overdensity:  $\delta_c = \delta_b = \delta_\gamma$
- Before  $z_*$ , this creates a travelling compression wave
- At  $z_*$ , this wave has travelled a comoving distance:

$$\text{comoving sound horizon: } \eta_s^* = \int_0^{t_*} \frac{dt}{a} c_s(t)$$

$$\text{physical sound horizon: } ds = \eta_s^* a$$

- For a purely relativistic gas,  $c_s = \sqrt{\frac{1}{3}}c$  (and we drop the  $c$  in our units)

- The Sachs-Wolfe Effect (1967)

$$\frac{\delta T}{T} = \frac{1}{3} \frac{\delta\Phi}{c^2}$$

- Overdensity means  $|\Phi|$ , the gravitational redshift, increases

- The integrated Sachs-Wolfe Effect (ISW) caused by changing potentials after  $z_*$  (but due to radiation  $\rightarrow$  matter and matter  $\rightarrow \Lambda$ )
- Flatness Problem
  - The universe looks close to flat today. This means it must have been very flat early on
- Monopole Problem
  - GUT phase transition leaves “defects”  $\rightarrow$  roughly one per  $\chi_{GUT}$
- Initial conditions:
  - Why does the universe start expanding? Where do the fluctuations come from?
  - $t \rightarrow 0, a \rightarrow 0, T \rightarrow \infty$ : avoid singularity! (Quantum gravity is also relevant here)
- Inflation
  - Horizon, Conformal Time, Hubble Distance
    - \*  $\chi$  is the comoving distance
    - \* We may write:

$$\chi(a_1, a_2) = \int_{a_1}^{a_2} \frac{da}{a^2 H(a)}$$

- \* Typically, we measure from us today ( $a = 1$ )
- \* The comoving horizon today is the distance to  $a = 0$  (or  $a \rightarrow 0$ )

$$\chi_h = \chi(a = 0) = \int_0^1 \frac{da}{a^2 H(a)}$$

- \* Often, the horizon gets a new variable:

$$\eta(a) = \chi(a = 0, a)$$

- \*  $\eta$  is also sometimes called the conformal time
- \* It is monotonically increasing, perfectly good time variable
- \* We can rewrite the FRW metric:

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]$$

- \* Overall scaling of the metric: “conformal transformation”
- \* The horizon expresses global causality; how far could light have travelled since  $t = 0$ ?
- \* Our cosmic event horizon is the distance from which light could ever reach us out to  $t \rightarrow \infty$

- \* For a  $\Lambda$ CDM universe, this is not infinite
- Developed by Guth, Starobinsky, and Linde (1979-1981), originally as a way to solve the monopole problem
- $\rho_{inf}$  comes from some new field in the universe
- This is a scalar field (think electric field,  $\vec{E}$  carries energy density  $E^2$ )
- Changing  $\vec{E} \rightarrow \vec{B}$  makes total  $\rho \propto E^2 + B^2$  (the “potential” and “kinetic” energy in the field)
- Galaxy Formation
  - Dark matter and baryons collapse to form a halo
  - Baryons cool through radiation and fall to center of the halo, forming a galaxy
  - Formation looks simple from a statistical perspective (linear galaxy bias):

$$\delta_g = b\delta_m$$

- \* The overdensity in galaxies is proportional to the overdensity of dark matter
- \* Overdensity is expressed as:

$$\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}}$$

- Redshift Space Distortions

- Strength of effect given by:

$$f \equiv \frac{d \ln(D)}{d \ln(a)}$$

- \* More growth  $\rightarrow$  larger peculiar velocities

- Gravitational Lensing

- All geodesics are impacted by mass/energy density, including photons
- The path of light is bent  $\rightarrow$  gravitational lensing
- For a point source: deflection
- For an extended source (*e.g* a galaxy), you also get distortions:
  - \* Shear
  - \* Magnification
- Recall geodesic deviation

$$A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

- Regimes of Lensing (NB: All within “weak field” limit; differences depend on geometry of observation)

- Strong Lensing: Multiple images, arcs, Einstein rings, strong magnification

$$\theta \leq \theta_E$$

- Weak Lensing: Small distortions, must be statistically studied

$$\theta > \theta_E$$

- Microlensing: Unresolved strong lensing (*e.g.* by an exoplanet)

- Lensing Derivations

- Assuming that the curvature is small (weak field), we can write the metric perturbatively:

$$g_{\mu\nu} = g_{\mu\nu}^{(o)} + h_{\mu\nu}$$

- Initially, we can assume that  $g_{\mu\nu}^{(o)} = \eta_{\mu\nu}$ ; however, for cosmology, we need to remember that  $g_{\mu\nu}^{(o)}$  is actually the homogenous FLRW metric
- Decomposing the perturbation, we get:

$$\begin{cases} h_{oo} = -2\Phi, & \Phi, \Psi : \text{scalar perturbations} \\ h_{oi} = w_i, & w_i : \text{vector perturbations} \\ h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}, & s_{ij} : \text{tensor perturbations} \end{cases}$$

- Cranking through some algebra and plugging into Einstein’s equation, we get:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$$

- \* From this, we find:

$$\nabla^2\Psi = 4\pi G\rho$$

$$w_i = s_{ij} = 0$$

$$\Psi = \Phi$$

- \* The last expression is an important General Relativity result, valid only when  $\text{Tr}(T_{ij})=0$

- We may recall that, in the “Newtonian limit” only the  $h_{oo}$  was important for geodesics; now we deal with photons, so we need both time and space perturbations. Considering a null geodesic, we write:

$$x^\mu(\lambda) = x^{(o)\mu}(\lambda) + x^{(1)\mu}(\lambda)$$



– We define:

$$k^\mu = \frac{dx^{(o)\mu}}{d\lambda} \quad \text{and} \quad l^\mu = \frac{dx^{(1)\mu}}{d\lambda}$$

– The null geodesic expression becomes:

$$(\eta_{\mu\nu} + h_{\mu\nu})(k^\mu + l^\mu)(k^\nu + l^\nu) = 0$$

– This can be evaluated to:

$$\begin{aligned} \frac{dl^\mu}{d\lambda} &= -\Gamma_{\rho\sigma}^\mu k^\rho k^\sigma \\ \text{Time: } \frac{dl^o}{d\lambda} &= -2k(\vec{k} \cdot \vec{\nabla}\Phi) \\ \text{Space: } \frac{d\vec{l}}{d\lambda} &= -2k^2 \vec{\nabla}_\perp \Phi \end{aligned}$$

– We want to find the deflection angle,  $\hat{\alpha}$ , which can be expressed as:

$$\frac{\Delta \vec{l}}{k} = \Delta \theta \equiv \hat{\alpha}$$

– Thus, we can get:

$$\Delta \vec{l} = \int \frac{d\vec{l}}{d\lambda} d\lambda = -2k^2 \int \vec{\nabla}_\perp \Phi d\lambda$$

– We can thus write  $\hat{\alpha}$  as:

$$\hat{\alpha} = 2 \int \vec{\nabla}_\perp \Phi ds$$

– We obtain the lens equation as:

$$\vec{\beta} = \vec{\theta} - \vec{\alpha} = \vec{\theta} - \frac{D_{LS}}{D_S} \hat{\alpha}$$

– For a point mass, we substitute to get:

$$\beta = \theta - \frac{D_{LS}}{D_L D_S} \frac{4GM}{\theta}$$

\* Considering perfect lens-source alignment ( $\beta = 0$ ), we get the Einstein angle:

$$\theta_E = \sqrt{\frac{4GM D_{LS}}{D_L D_S}}$$

\* From here, we get the Einstein radius:

$$R_E = D_L \theta_E = \sqrt{\frac{4GM D_L D_{LS}}{D_S}}$$

\* For imperfect lens-source alignment, we get:

$$\theta^2 - \beta\theta - \theta_E^2 = 0$$

$$\theta = \frac{1}{2} \left[ \beta \pm \sqrt{\beta^2 - 4\theta_E^2} \right]$$

\* When  $\beta \gg \theta_E$ , we have:

$$\theta = \frac{1}{2} \left[ \beta \pm \beta \left( 1 + \frac{2\theta_E^2}{\beta^2} \right) \right]$$

\* This gets us:

$$\theta_+ = \beta + \frac{\theta_E^2}{\beta}$$

$$\theta_- = -\frac{\theta_E^2}{\beta} \rightarrow 0$$

- Lensing Potential

$$\psi(\vec{\theta}) = \frac{2D_L D_{LS}}{D_L D_S} \int \Phi(D_L \vec{\theta}, s) ds$$

– We may write:

$$\vec{\alpha} = \vec{\nabla}_{\theta} \psi = D_L \vec{\nabla}_{\perp} \psi$$

– We define the convergence as:

$$\kappa(\vec{\theta}) = \frac{1}{2} \nabla_{\theta}^2 \psi = \frac{D_L D_{LS}}{D_s} \int \nabla^2 \Phi ds$$

– By Poisson's equation, this is simply the integral of the overdensity, so  $\kappa$  is the lens-geometry-weighted mass density at some position  $\theta$  on the sky

$$\kappa(\vec{\theta}) = \frac{D_L D_{LS}}{D_s} \int \delta ds$$

– In terms of  $\kappa$ , the deflection angle can be written:

$$\vec{\alpha}(\vec{\theta}) = \frac{1}{\pi} \int \kappa(\vec{\theta}') \left[ \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \right] d^2 \theta'$$

- The 2D lensing transformation can be described as a matrix:

$$A_{ij} \equiv \frac{\partial \beta_i}{\partial \theta_j} = \delta_{ij} - \frac{\partial \alpha_i}{\partial \theta_j} = \delta_{ij} - \frac{\partial^2 \psi}{\partial \theta_i \partial \theta_j} \equiv \delta_{ij} - \psi_{ij}$$

- We may describe:

- \* Convergence:  $\kappa = \frac{1}{2}(\psi_{11} + \psi_{22})$  (isotropic focusing)
- \* Shear:  $\gamma_1 = \frac{1}{2}(\psi_{11} - \psi_{22}) = \gamma \cos(2\phi)$  and  $\gamma_2 = \psi_{12} = \psi_{21}$

$$A = \begin{bmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{bmatrix}$$

- $\mu = A^{-1}$  is the magnification tensor
- Surface brightness is conserved, so total observed flux is:

$$F_{obs} = \mu F_{true}$$

- For  $\mu > 1$ , a given patch of sky will appear larger, and for  $\mu < 1$  it will appear smaller

- Technical Terms

- “Cosmic Shear” — Correlations in shapes due to weak lensing
- Galaxy-Galaxy Lensing — The lensing of galaxies by galaxies (essentially determining the mass of galaxy based on its lensing contribution to a different galaxy)
- CMB Lensing — CMB photons are also affected by shear and lensing, so the CMB map could be used
- Strong Lensing — Mass mapping, time delays, dark matter constraints

- Gravitational Waves

- Ripples in spacetime
- “Gauge Transformations”
- Using the same “chirp” from above, we may write:

$$h_{\mu\nu}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & s_{ij} & \\ 0 & & & \end{pmatrix}$$

- \* This is the transverse-traceless gauge (TT)
- \* Often referred to by a box:  $\square$
- This results in a wave equation

- \* Note that this obeys the same generic solution set:

$$f(x, t) = Ae^{i(kx - \omega t)}$$

- \* This plane wave follows:

$$k = \frac{\omega}{c_s} = \frac{2\pi}{\lambda c_s}$$

- \* Can add multiple plane waves to match boundary/initial conditions
- \* We can note:

$$k\Delta x = \omega\Delta t$$

- Gravitational waves have quadrupolar symmetry (180° rotation invariant), unlike electromagnetic waves, which are dipoles (360° rotation invariant)
- For gravitational radiation, this dipole scenario doesn't work, as there is only one gravitational "charge" (universality of gravity)
- We don't have a quantized theory of gravity (proven) yet, but we know it has a spin-2 particle
- We could have derived Einstein's equations by taking the particle physics perspective and writing down  $\mathcal{L}$  including a new spin-2 particle  $h_{\mu\nu}$
- String theory naturally gives rise to 2 spin-2 degrees of freedom and spin-0 degrees of freedom

- Production and detection of gravitational waves

- This is in general a complex problem: perturbation theory breaks down, and numerical techniques are needed
- Once you have the form of the wave, the propagation is easily expressed as a sum of plane waves
- Gravitational waves are produced by changing quadrupole moment of mass distribution:

$$\bar{h}_{ij}(t, \vec{x}) = \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2}$$

- \* Where:

$$I_{ij}(t) = \int y^i y^j T^{oo}(t, y) d^3y$$

- \* Is the quadrupole of mass/energy distribution
- Using Newtonian orbits (which, technically, are not correct, especially as objects are close to merging, i.e. strong gravity), but good enough for order of magnitude:

$$\Omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{4R^3}}$$

$$\bar{h}_{ij}(t, \vec{x}) = \frac{8GM}{r} \Omega^2 R^2 \begin{pmatrix} -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ -\sin(2\Omega t) & \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Binary Pulsar: Hulse and Taylor (1974 discovery, 1993 Nobel Prize)
  - Pulsar: a spinning neutron star
  - Two neutron stars, one of which is a pulsar, which pulses 17 times per second
  - Precise timing allows us to track the orbit with very good precision
- Gravitational Wave Sources (in order of decreasing frequency)
  - Mergers of approximately stellar mass black holes, neutron stars
  - Supernovae
  - Supermassive black hole mergers
  - Inflation, early universe phase transitions
- Gravitational Wave Detection
  - LIGO: Laser Interferometer Gravitational-Wave Observatory
  - Pulsar Timing Arrays
  - CMB Polarization