Lecture 3 — Energy and Momentum

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- In general, the trace of a matrix g is the dimensionality of manifold described by g:

$$\mathrm{Tr}(g) = \dim(\mathrm{manifold}_g)$$

- Results from special relativity:

$$E^2 = (mc^2)^2 + (pc)^2 \to m^2 + p^2$$
$$p = mv\gamma \Rightarrow E = m\gamma$$

- In proper covariant notation, the four velocity of a particle on $x^\mu(t)$ is:

$$U^\mu = \frac{dx^\mu}{d\tau} \to \eta_{\mu\nu}U^\mu u^\nu = -1 \text{ (due to definition of } \tau)$$

  - This makes the four momentum:

$$p^\mu = mU^\mu$$
$$P^o = E$$
$$P^i = \vec{p} \to \begin{pmatrix} E \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}$$
$$p_\mu p^{mu} = -m^2$$

  - Thus, we may write:

$$E^2 = m^2 + p^2$$

– With a Λ:

$$p^{\mu'} = \begin{pmatrix} \gamma m \\ mv\gamma \\ 0 \\ 0 \end{pmatrix} \text{ corresponding to } (-v)!$$

- Force in Spacetime

$$f^{\mu} = m\frac{d^2}{d\tau^2}x^{\mu}(\tau) = \frac{d}{d\tau}p^{\mu}(\tau)$$

  – For electromagnetics, we may write:

$$f^{\mu} = qU^{\lambda}F^{\mu}_{\lambda} \text{ (the power of symmetry)}$$

- Particle to Fluid
  – From individual particles, we get fluid elements with: ρ, P, viscosity,...
- Energy-Momentum Tensor (Stress-Energy Tensor)
  – We want to derive $T^{\mu\nu}$, or the flux of p^{μ} across x^{ν} surface (Recall $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}$)
  – Since time is now a dimension, there is a flux for particles moving solely in time
  – Imagine two cases:
    * At Rest
      · Particles at rest have a flux in time $= x^o$
      · $T^{oo} = \rho = mn$, where m is the mass and n is the number density
      · $T^{\mu'\nu'} = \Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}T^{\alpha\beta}$
    * In a boosted frame
      · There is now a flux across x and x'
      · Particles are no longer "at rest"
- Dust: Particles at rest with respect to each other
  – At rest:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \rho U^{\mu}U^{\nu}$$

- Perfect fluid (with density ρ and pressure p), the fluid is isotropic (same in all directions)

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}$$

  * Pressure is used because energy and mass are then interchangeable
  * This kinetic energy really makes a box more "massive"
    · Imagine protons (1% quark mass)

- Equation of State

$$w = \frac{p}{\rho}$$

  - Will appear in cosmology
  - $w = 0$ for matter ("dust") $\rightarrow$ non-relative
  - $w = 1/3$ for photons (CMB)
  - $w = -1$ (?) for dark energy $T^{\mu\nu}_\Lambda = -\rho_{vac}\eta^{\mu\nu}$ (absolute energy is important)

- Conservation of Energy-Momentum

$$\partial_\mu T^{\mu\nu} = 0$$

  - For $\nu = 0$ conservation of energy
  - For $\nu = k = 1, 2, 3$ conservation of momentum

- Classical Field Theory

$$\text{Action: } S = \int dt \, \underbrace{L(q, \dot{q})}_{\text{Lagrangian}}$$

  - $L = K - V$ (kinetic minus potential energy)
  - Equations of motion:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial q}\right) = 0$$

  - Move to a field:

$$q \rightarrow \left\{\Phi^i(x^\mu)\right\}$$

    * Quantize Φ to particles

$$S = \int d^4 \times \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

- Integrate the Lagrangian density
- Our field will be the metric of spacetime
- $\mathcal{L}$ and symmetry are powerful tools