Lecture 6 — The Expanding Universe

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- Olber's Paradox: Why is the Sky Dark?
  - Absorbing Matter $\rightarrow$ doesn't work since matter would heat up
  - Finite Size
  - Finite Time (and Finite Speed of Light)
  - Dimming of Light ("redshift")
- Universe had some beginning ("Big Bang") around 13.7 billion years ago
- Some units:

$$1[\text{light year}] = 9.5 \cdot 10^{15}[\text{m}]$$
$$1[\text{yr}] \approx \pi \cdot 10^{7}[\text{s}]$$
$$c \approx 3 \cdot 10^{8}\left[\frac{\text{m}}{\text{s}}\right]$$
$$1[\text{pc}] = 3.26[\text{light years}]$$

  - If $\theta = 1[\text{arcsec}]$, then $d = 1[\text{pc}]$

$$1[\text{pc}] = 2.1 \cdot 10^{5}[\text{AU}]$$

- The Cosmological Principle
  - Copernicus: the Sun, not the Earth, is the center of the Universe
  - Cosmological Principle: There is no center to the Universe
    - The Universe is statistically isotropic (same in all directions) and homogenous (same everywhere)
- Expanding Universe

- All observers see things moving away from them
- Statements are all statistical! Distinguish between structure in the universe and the geometry of the homogenous universe (about 100[Mpc] scales for homogeneity)
- We don't experience the FLRW metric
  * Homogenous/geometry
  * Structure
- Conservation of Energy Solution to Expanding Cloud:

$$\frac{1}{2}\dot{R}^2 - \frac{2GM}{R} = C$$

  * What is the physical meaning of C?
    · $C = 0$
    You are just at escape velocity. As $R \to \infty$, $\dot{R} = v \to 0$. Potential and kinetic both go to zero.
    · $C > 0$
    Positive total energy. You have more than enough energy to escape. $\dot{R} > 0$ as $R \to \infty$
    · $C < 0$
    Negative total energy. You won't make it out to $R = \infty$, you will stop and turn around at some finite t
  * These cases capture an ideal universe's expansion with only real matter
  * For a matter-only universe, C describes the spatial geometry, with zero indicating flat, $C > 0$ indicating open (negative curvature), and $C < 0$ indicating closed (positive curvature)
    · Note that Λ complicates this
- Using the first-order equation, we may write:

$$\frac{1}{2}\dot{R}^2 = \frac{4}{3}\pi G\rho R^2 + C$$
$$\left(\frac{\dot{R}}{R}\right) = \frac{8\pi G\rho}{3} + C$$

- At $C = 0$, we find ρ_{crit}:

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

  Note that we define the Hubble parameter as:

$$H = \frac{\dot{R}}{R}$$

- Comoving Coordinates

- Coordinates (and distances) scale with the size of the universe:

$$x = a(t)r$$

  * x is the "proper" coordinate
  * r is the "comoving" coordinate
  * $a(t)$ is the scale factor, with $a(t_o) = 1$ indicating "today"
- Ex. $x_{12} = a(t)r_{12}$

$$\frac{dx_{12}}{dt} \quad \text{is the recession velocity}$$

- Two Things:
  1. Velocity is proportional to distance (Hubble law)
  2. The proportionality term is $\frac{\dot{a}}{a} = H(t)$
- Hubble constant is approximately:

$$H_o = 70 \left[\frac{\text{km}}{\text{sMpc}}\right]$$

  * We define h, such that:

$$H_o = 100h \left[\frac{\text{km}}{\text{sMpc}}\right]$$

  * We can see that the units are actually just Hz, which gives us:

$$H_o \approx 3.24 \cdot 10^{-18} \left[\frac{1}{\text{s}}\right]$$

- If a is constant, this gives us the doubling time or the time to go back to $a = 0$, which gives us the "Hubble" time, or the age of the universe
- For $h = .7 \to 1.4 \cdot 10^{10} \left[\frac{1}{\text{yr}}\right]$
  * Fairly accurate approximation of 14 billion years

- FLRW Metric
  - Cosmological principle tells us that spacetime (on large scales) should be homogenous (translation invariant) and isotropic (rotation invariant)
  - We seek a general form of a metric that obeys these assumptions:

$$ds^2 = \bar{c}^2\, dt^2 + R^2(t)\, d\sigma^2$$

  - We are able to derive:

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2\, d\Omega^2$$

- Where:
  - $d\Omega^2$ is the 2-sphere metric
  - $k \propto R$ (Ricci scalar on 3D space)

$$k = \begin{cases} +1, & \text{Positive Curvature (3-sphere), "closed"} \\ 0, & \text{Flat} \\ -1, & \text{Negative curvature (saddle), "open"} \end{cases}$$

- Thus, we may define the metric as:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\, d\Omega^2\right]$$

  - Where:

$$\begin{cases} \kappa > 0, & \text{closed} \\ \kappa = 0, & \text{flat} \\ \kappa < 0, & \text{open} \end{cases}$$

- Cosmological Redshift
  - Like a Döppler shift, but use caution!
  - $v = H_o d$, locally $v << c$
  - Redshift in special relativity:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \sqrt{\frac{1+v/c}{1-v/c}} \approx 1 + \frac{v}{c}$$

  - From this, we know:

$$\frac{\lambda_{obs}}{\lambda_{em}} = 1 + z \approx 1 + \frac{H_o d}{c}$$

  - More generally, we may say (for a_o = today):

$$1 + z = \frac{a_{obs}}{a_{em}} = a_{em}^{-1}$$

- Cosmological Redshift versus Peculiar Motions:
  - For comoving coordinates:

$$v \approx H_o d = H_o|\vec{r}_2 - \vec{r}_1|/a_o = H_o|\vec{r}_2 - \vec{r}_1|$$

  - For “peculiar” motions

$$v \approx H_o d + \frac{\Delta v}{c}$$

- In the FLRW metric, due to lack of simple time symmetry, energy is not conserved
  - There is a Killing Tensor that reflects a symmetry:

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \quad \text{for observer (comoving) } U^\mu = (1,0,0,0)$$

  - We get:

$$K^2 = K_{\mu\nu} V^\mu V^\nu$$

    - Is conserved, with

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

    - For a photon on a null geodesic:

$$V_\mu V^\mu = 0$$

  - This can be simplified to:

$$v = \frac{K}{a}$$