Lecture 4 — Manifolds and Curved Spacetime

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- We now move from Minkowski to General Space:

$$\eta_{\mu\nu} \to g_{\mu\nu}$$

- Differentiable Manifolds
  - Manifold: A space (in n-dimensions) that looks locally like $\mathbb{R}^n$ and can be constructed by smoothly stitching together these regions
  - Rotations in $\mathbb{R}^n$ $\to$ Lie Groups are manifolds with a group structure
  - To be more precise, we have a set M with a set of (all possible) charts of open subsets to $\mathbb{R}^n$
    - Chart $\leftrightarrow$ coordinate system
  - These charts must be smooth, continuous, invertible, and differentiable
  - Now we will define (co)tangent spaces on these manifolds, with metrics that map (dual) vectors to $\mathbb{R}$
- The Equivalence Principle
  - In special relativity, we had the principle that the laws of physics were the same in all inertial frames
  - Einstein's "happiest though": If someone falls from a roof, nothing falls in their frame
  - Equivalence of inertial frames should be generalized to include gravity
  - Weak Equivalence Principle (WEP)
    - Inertial mass = gravitational mass

$$F = m_i a \text{ (inertial)}$$
$$F = -m_g \nabla\Phi \text{ (gravitational "charge")}$$
$$m_i = m_g \text{ (WEP: Eötuös experiments, late 19th century)}$$

        * All freely falling bodies behave the same/are indistinguishable ($a = -\nabla\Phi$)
        * Define inertial trajectory as unaccelerated (subject only to gravity)
        * In small enough regions of space-time, freely falling particles behave the same in a gravitational field or a uniformly accelerated field (physicist in a box, accelerating reference frame)
    - Strong Equivalence Principle (SEP)
        * All laws of physics, including gravitation, look like SR
            · Einstein Equivalence Principle (EEP) plus the impact of gravitational binding energy
            · Rules out "fifth force"

- Tidal Forces
    - Causes tides on Earth
    - Locally inertial frames

- Gravitational Redshift

$$\Delta v = \frac{az}{c}$$

    - Relativistic Doppler Shift:

$$\lambda_{obs} = \lambda_o \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)^{\frac{1}{2}}$$

        * Using Taylor expansion, we may simply write this as:

$$\lambda_{obs} = 1 + \frac{v}{c}$$

        * Bringing this together, we get:

$$\frac{\Delta\lambda}{\lambda_o} \approx \frac{\Delta v}{c} = \frac{az}{c^2}$$

    - EEP says that this must be the same as a gravitational field:

$$\frac{\Delta\lambda}{\lambda_o} = \frac{a_g z}{c^2} = \frac{\Delta\Phi}{c^2}$$

    - This is the time from start to end of wavelength, and can be used to compare clocks
    - If we have a case where $\Delta t_o = \lambda_o c^{-1}$ and $\Delta t_1 = \lambda_1 c^{-1}$, and $\lambda_1 > \lambda_o$ then $\Delta t_1 > \Delta t_o$, which indicates gravitational time dilation

- Classic Tests of General Relativity

  1. Precession of the perihelion of Mercury — 19^{th} century: 43" per century discrepancy successful "post-diction" of GR (about 10% of total effect)
  2. Bending of star light by sun (gravitational lensing) — GR predicts a factor of 2 larger deflection (1919 Eddington Expedition to observe the solar eclipse)
  3. Gravitational Redshift — 1954: Popper measurement of a white dwarf, 1959: Pound-rebka at Jefferson lab (Harvard), 22.5m

- Vectors and Tensors on Manifolds (Curved Spacetime)

  – We already saw $V = V^\mu \hat{e}_\mu$ at point P on T_p
  – What is the basis?
    * We want to define tangent vectors before we have a vector space on M
    * Instead, consider a function f and a curve λ. The directional derivative is:

$$\frac{d}{d\lambda}x^\mu \frac{\partial}{\partial x^\mu} f = \frac{d}{d\lambda}x^\mu \partial_\mu f \quad \text{(gradient} \cdot \text{tangent } \vec{v}\text{)}$$

    * f could have been anything, so we define the tangent vector:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

    * $\{\hat{e}_\mu = \partial_\mu\}$ is the coordinate basis ("points" in the direction of x^μ)
    * Not orthonormal, but always well defined
    * In this basis, things transform according to:

$$\partial_{\mu\prime} = \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu\prime}} = \frac{\partial x^\mu}{\partial x^{\mu\prime}}\partial_\mu$$

    * Similarly, $V = V^\mu \partial_\mu$ is preserved, so:

$$V^{\mu\prime}\partial_{\mu\prime} = V^\mu \partial_\mu \Rightarrow V^{\mu\prime} = \frac{\partial x^{\mu\prime}}{\partial x^\mu} V^\mu$$

- General Coordinate Transform

  – In flat space: $x^{\mu\prime} = \Lambda^{\mu\mu}_{\mu}$

$$\frac{dx^{\mu\prime}}{dx^\mu} = \Lambda^{\mu}_{\mu\prime}$$

  – We recover the transform of vectors
  – Vector Fields:
    * X: One vector at each point on the manifold

- X, Y: Both define a field that can be used to take directional derivatives of functions on μ

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad \underline{\text{commutator}}$$

- Dual Vectors
  - Recall we defined the gradient: df

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} \quad \text{map a vector to } \mathbb{R}$$

  - Basis for dual vectors dx^μ
  - Gradient of the coordinate function:

$$dx^\mu(\partial_\nu) = \frac{dx^\mu}{dx^\nu} = \delta^\mu_\nu$$
$$V = V^\mu \partial_\mu$$
$$\omega = \omega_\nu dx^\nu$$
$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

- We can now write the transformation of an arbitrary (k, l) tensor on a manifold:

$$T^{\mu_1{}' \ldots \mu_k{}'}_{\nu_1{}' \ldots \nu_l{}'} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k{}'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1{}'}} \cdots \frac{\partial x^{\nu_k}}{\partial x^{\nu_k{}'}} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l}$$

  - Warning: in curved space $\partial_\mu W_\nu$ is not a tensor; unlike in flat space, the derivative of the transform can be non-zero (Λ is the same everywhere)

- The Metric
  - $\eta_{\mu\nu}$ in Minkowski space
  - $g_{\mu\nu}$ in general curved spacetime

$$g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu \quad \text{(defines inverse)}$$

  - Metric really describes basically everything about a spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

    - (0,2)-tensor components, metric components, with ds^2 being called the "line element" or "metric"
    - Usually we just write $g_{\mu\nu}$
    - In 3D Flat Space:

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

      - Components and bases both change, while ds^2 does not

* Canonical form: coordinate transform to diagonalize and normalize

$$g_{\mu\nu} = \begin{pmatrix} -1 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & \\ \vdots & & -1 & & & \\ \vdots & & & 1 & & \\ \vdots & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

* "Signature" is -+++, etc.
* All positive: Euclidean or Riemannian
* One negative: Lorentzian of pseudo-Riemannian
* Any zeros? Degenerate
* At some point p, you can always put a metric into canonical form, and make the first derivatives vanish
  * There is always enough freedom to choose coordinates that do this
  * Second derivatives will not generally vanish
* In our case, choose:

$$x^{\hat{\mu}} \text{ at } p \text{ such that:}$$

$$g_{\hat{\mu}\hat{\nu}}(p) = \eta_{\hat{\mu}\hat{\nu}} \qquad \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \qquad \partial_{\hat{\rho}} \partial_{\hat{\sigma}} g_{\hat{\mu}\hat{\nu}} \neq 0$$

* Note: not a coordinate system, so not a coordinate basis

$$x^\mu \rightarrow \text{ locally inertial coordinates}$$

  * Locally inertial/Lorentz frame
  * Do calculations in this frame, express in tensor (covariant) form — usually can just assume $\eta_{\mu\nu}$

- Coordinate Basis

$$V = V^\mu \hat{e}_\mu$$
$$\hat{e}_\mu = \{\partial_\mu\}$$
$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

- Tensor Density

$$\tilde{\epsilon}_{\mu_1 \ldots \mu_n} = \begin{cases} +1, & \text{even perm (0123)} \\ -1, & \text{odd perm (0213)} \\ 0, & \text{other (0112)} \end{cases}$$

– Our general transformation $\frac{\partial x^{\mu}}{\partial x^{\mu'}}$ is a particular case of $M_{\mu}^{\mu'}$

$$\tilde{\epsilon}_{\mu_1' \ldots \mu_2'} = \left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| \tilde{\epsilon}_{\mu_1 \ldots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

– We may see that there is an extra factor compared to standard tensor transform

– What about $|g_{\mu\nu}| = g$?

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\hat{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\hat{\nu}}} g_{\mu\nu}$$

– In general, for $\tilde{t}$ that transforms with $\left|\frac{\partial x^{\mu'}}{\partial x^{\hat{\mu}}}\right|^{\omega}$, we can make a real tensor $t = \tilde{t}|g|^{\omega/2}$, since this will transform with:

$$\left|\frac{\partial x^{\mu}}{\partial x^{\hat{\mu'}}}\right|^{\omega} \left|\frac{\partial x^{\mu'}}{\partial x^{\hat{\mu}}}\right|^{\omega} = 1$$

$$\epsilon_{\mu_1 \ldots \mu_n} = \sqrt{|g|} \tilde{\epsilon}_{\mu_1 \ldots \mu_n}$$

$$\epsilon^{\mu_1 \ldots \mu_n} = \epsilon_{\mu_1 \ldots \mu_n} \cdot \operatorname{sign}(g)$$

- Differential Forms

  – p-form is a $(0, p)$ anti-symmetric tensor with:

$$A_{\mu\nu} = -A_{\nu\mu}$$

$$A_{\mu\nu\sigma} = -A_{\nu\mu\sigma} = A_{\nu\sigma\mu}$$

    * Scalars: 0-forms
    * Dual vectors: 1-forms

  – Wedge product is anti-symmetrized tensor product

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p]} B_{[\mu_{p+1} \ldots \mu_{p+q}]}$$

- Exterior Derivative

  – A tensor, unlike the partial derivative, but only acts on forms

$$(dA)_{\mu \ldots \mu_{p+1}} = (p+1) \partial_{\mu_1} A_{\mu_2 \ldots \mu_{p+1}}$$

  – We have already seen this operator, the gradient (of a scalar):

$$(d\phi)_{\mu} = \partial_{\mu} \phi$$

- Because partials commute:

$$d(dA) = 0$$

- Electrodynamics

$$F_{\mu\nu}$$
$$\partial_{[\mu}F_{\nu\lambda]} = 0$$
$$dF = 0$$
$$F = dA \text{ (vector potential } A_\mu)$$

- Hodge Star
  - $*A$ takes p-forms to $n-p$ forms
- Integration on Manifolds
  - In general, when changing coordinates in an integral, we multiply by the Jacobian

$$x, y, z \to r, \theta, \phi$$
$$\left|\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right| = r^2 \sin(\theta)$$
$$\int dx\, dy\, dz\, f(\vec{r}) \to \int r^2 \sin(\theta)\, dr\, d\theta\, d\phi\, f(\vec{r})$$

    * This is a particular example of integration on a manifold

$$\int w(x)\, dx = w$$

  - Represents a component $w(x)$ with basis dx
  - We may write:

$$d\mu\,(U, V, W) \leftarrow \mathbb{R}$$

    * This maps vectors at a point to their volume
  - We generalize:

$$d^n x = dx^o \wedge \cdots \wedge dx^{n-1}$$

    * This is not yet a tensor because it is coordinate independent

$$\sqrt{|g|} d^n x = \sqrt{|g|} dx^o \wedge \cdots \wedge dx^{n-1} = \sqrt{|g'|} dx^{o'} \wedge \cdots \wedge dx^{(n-1)'}$$
$$I = \int \phi(x) \sqrt{|g|}\, d^n x \to \int \phi(x)\, dx\, dy\, dz \text{ if } |g| = 1$$

- You can then evaluate as normal

- FLRW Metric (or FRW, or RW)
  - Friedmann, Lemaitre, Robertson, Walker $\rightarrow$ Solution to Einstein's equations for a spatially homogenous, isotropic spacetime. Can be curved or flat. Flat FLRW:

$$dx^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

    * Where $a(t)$ is the scale factor
    * For $a(t) = t^q$, $0 < q < 1$
    * $t = (1-q)^{\frac{1}{1-q}}(\pm x - x_o)^{\frac{1}{1-q}}$
    * Light not always at a $45°$ angle
    * Singularity at $t = 0 \rightarrow$ cosmic horizon $\rightarrow$ p and s are completely disconnected

- Curvature, Covariant Derivatives, Geodesics
  - For S^2:

$$ds^2 = \frac{R^2 dr}{R^2 - r^2} + r^2 d\theta^2$$

  - The metric reflects the curvature; for $R \rightarrow \infty$, we get back to flat 2D space
  - From the metric, we will derive the "connection," which tells us the impact of curvature, including defining straight lines
  - The connection ($\Gamma^{\lambda}_{\mu\nu}$, not a tensor) tells us how to compare vectors at nearby points

- Covariant Derivatives
  - Recall $\partial_\mu A_\nu$ is not a covariant because of an extra term
  - External derivative $dA = \partial_\mu A_\nu - \partial_\nu A_\mu$ is covariant but only valid on forms
  - We define the covariant derivative of a vector and dual vector:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda$$
$$\nabla_\mu \omega^\nu = \partial_\mu \omega^\nu - \Gamma^{\nu}_{\mu\lambda} \omega^\lambda$$

    * Note: Γ is not a tensor, do no raise or lower indices!
    * Covariant refers to transforming like a tensor

$$\partial_{\mu'} A_{\nu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\partial_\mu\right)\left(\frac{\partial x^\nu}{\partial x^{\nu'}}\partial_\nu\right) = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\partial_\mu A_\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} A_\nu \partial_\mu \frac{\partial x^\nu}{\partial x^{\nu'}}$$

  - We can detemine how Γ must transform such that $\nabla_\mu V^\nu$ and $\nabla_\mu \omega_\nu$ are covariant

– Four requirements on covariant derivatives:

1. Linearity: $\nabla(T+S) = \nabla T + \nabla S$
2. Product Rule: $\nabla(T \otimes S) = \nabla T \otimes S + T \otimes \nabla S$
3. Commute with Contractions: $\nabla_\mu (T^\lambda_{\lambda\rho}) = (\nabla T)^{\lambda}_{\mu\lambda\rho}$
4. Partial Derivative on Scalars: $\nabla_\mu \phi = \partial_\mu \phi$

– For a general (k,l) tensor, we have:

$$\nabla_\sigma T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} = \partial_{\sigma T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l}} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} + \cdots + \Gamma^{\mu_k}_{\sigma\lambda} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} - \cdots - \Gamma^{\lambda}_{\sigma\nu_l} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l}$$

– Notation:

$$\nabla_\sigma A_\mu = A_{\mu;\sigma}$$
$$\partial_\sigma A_\mu = A_{\mu,\sigma}$$

– $\Gamma^\mu_{\sigma\lambda}$ must obey a particular (non-covariant) transform under coordinate change

– We have a lot of freedom to define $\Gamma^\mu_{\sigma\lambda}$, but we will impose two additional conditions (for convenience):

* Torsion-free: $T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 0 \Rightarrow \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$
* Metric-compatible: $\nabla_\propto g_{\mu\nu} = 0$

$$\nabla_\lambda \epsilon_{\mu\nu\rho\sigma} = 0$$
$$\nabla_\rho g^{\mu\nu} = 0$$

– Christoffel Symbol:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right)$$

• Parallel Transport and Geodesics

– What is parallel transport?

* Keep the vetor fixed as you "move it." This is generally path-dependent
* We must do this to compare vectors
* Let's be a bit more mathematical:

$$\frac{d}{d\lambda} V^\mu = 0$$
$$\frac{d}{d\lambda} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} = 0$$

* Before, we had:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu}$$

- Replace with covariant directional derivative:

$$\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu$$

- Parallel transport is defined by:

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l} = 0$$

- Straight Lines → Geodesics
  - Straight line is the shortest distance between two points; in flat space, the direction of your tangent vector doesn't change
  - We generalize this notion: A geodesic is a path that parallel transports its own tangent vector:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0 \qquad \text{(geodesic equation)}$$

  - In flat space with Cartesian coordinates, $\Gamma^\mu_{\rho\sigma} = 0$

$$\frac{d^2x^\mu}{d\lambda^2} = 0 \qquad \text{(a ``straight'' line)}$$

- Geodesics as the "Shortest Distance"
  - Consider a timelike path; recall proper time along $x^\mu(\lambda)$:

$$d\tau^2 = ds^2$$

$$\tau = \int \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{\frac{1}{2}} d\lambda$$

  - Recall (Twin Paradox) that the "shortest spacetime interval" corresponds to maximum proper time, due to the minus sign in Lorentzian spacetimes
  - We want to find the stationary points of τ (max or min) by varying $x^\mu(\lambda)$; the resulting path will end up being (Note: very common method in physics, same as the Lagrangian → action formulation, which allows the derivation of EoM):
    - We can simplify:

$$f = g_{\mu\nu}(dx^\mu/d\lambda)(dx^\nu/d\lambda)$$

    - Variation is:

$$\delta t = \int \delta\sqrt{-f}\, d\lambda = -\int \frac{1}{2}(-f)^{-\frac{1}{2}}\, \delta f\, d\lambda$$

* We can simplify by taking $\lambda = \tau$:

$$\frac{dx^\mu}{d\tau} = U^\mu \quad \text{(four velocity)}$$
$$\Rightarrow f = g_{\mu\nu}U^\mu U^\nu = -1 \quad \text{definition of } \tau$$
$$\delta t = -\frac{1}{2}\int \delta f \, d\tau$$

- So, we can find the stationary points of:

$$I = \frac{1}{2}\int f \, d\tau = \frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau d\tau}$$

- We can use the Euler-Lagrange, or just directly calculate variation:

$$x^\mu \to x^\mu + \delta x^\mu$$
$$\Rightarrow g_{\mu\nu} \to g_{\mu\nu} + (\partial_\sigma g_{\mu\nu})\delta x^\sigma$$
$$\delta I = I_{new} - I_{old}$$

- Bringing all of our terms together, we get:

$$\delta I = -\int \left[g_{\mu\sigma}\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]\delta x^\sigma \, d\tau$$

- Our final form takes:

$$\frac{d^2x^\rho}{d\tau^2} + \underbrace{\frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})\frac{dx^\mu}{d\tau}}_{\Gamma^\rho_{\mu\nu}}\frac{dx^\nu}{d\tau} = 0$$

- Affine Parameters
  - For timelike geodesics: $\lambda = a\tau + b$ affine parameters
  - Geodesic equation is satisfied (usually just use τ)
  - Parallel transport of $dx^\mu/d\lambda$ defines both x^μ and λ
  - We can express timelike geodesics using U^μ or $p^\mu = mU^\mu$
  - For null paths: $\tau = 0$
    * Instead, we typically choose $p^\mu = \frac{dx^\mu}{d\lambda}$ to define λ
    * In either case, then $E = -p_\mu U^\mu$
- The Riemann Curvature Tensor
  - What is curvature at some point p?
  - Flatness, in a slightly more technical sense, means:

1. Parallel transport does not depend on path
    * No change on a closed loop
2. Covariant derivatives commute
3. Parallel geodesics remain parallel