Lecture 5 — Einstein's Equations and Schwarzschild

Michael Brodskiy
Professor: J. Blazek

October 9, 2024

- We may begin with Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

  - Note that the tensors form second order non-linear differential equations

- Symmetric Tensor ($n = 4$ dimensions)

  - $\frac{n^2-n}{n} + n = \frac{n^2+n}{2} = 10$ degrees of freedom
  - General relativity sees diffeomorphism invariance, so four of the degrees of freedom are removed (since it isn't $x^\mu \to x^{\mu'}$)
  - Solving these differential equations is extremely complex, so we will make some assumptions to simplify analysis:
    * Boundary conditions and initial conditions
    * Limits
    * Simplify through symmetry

- Symmetric General Relativity

  - Spherical symmetry and static
  - Homogenous and isotropic: FLRW universe/cosmology
  - $T_{\mu\nu} = 0$, small perturbations are gravitational waves
  - In Newtonian mechanics, with three masses M_1, M_2, and much smaller M_3, we may write: $\Phi_{M_3} = \Phi_{M_1} + \Phi_{M_2}$
    * In General Relativity, $g_{\mu\nu}$ depends on M_1 and M_2, as well as the binding energy between

- Schwarzschild

– Only vacuum solution with spherical symmetry
– We assume a spherical system that is static
– We use Minkowski space, with spherical coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2 \underbrace{\left(d\theta^2 + \sin^2(\theta)\, d\theta^2\right)}_{d\Omega^2}$$

  * We may rescale this with functions of r:

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + C(r) r^2\, d\Omega^2$$

  * Furthermore, we define $r \to \sqrt{C(r)}$

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + r^2\, d\Omega^2$$
$$ds^2 = -e^{2\alpha(r)}\, dt^2 + e^{\beta(r)}\, dr^2 + r^2\, d\Omega^2$$

  * For the diagonal metric, we way find $\Gamma^{\phi}_{r\phi} = (1/r)$, then continuing Christoffel calculations, using Riemann, and then contracting to Ricci, we find:

$$R_{tt} = e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right]$$

$$R_{rr} = -\partial_r^2\alpha - (\partial_r\alpha)^2 + \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\beta$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}$$

  * We want a $T_{\mu\nu} = 0$ solution, which implies $G_{\mu\nu} = 0$, which then implies $R_{\mu\nu} = 0$
    · This is known as "Ricci Flat" (not really flat)
  * We move terms around to find:

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0$$
$$\frac{2}{r}\left(\partial_r\alpha + \partial_r\beta\right) = 0$$
$$\alpha = -\beta$$

  * Taking:

$$R_{\theta\theta} = 0 \quad \text{and} \quad -e^{2\alpha}\left[2r\partial_r\alpha + 1\right] + 1 = 0$$

  * We get:

$$e^{2\alpha}\left[2r\partial_r\alpha + 1\right] = \partial_r(re^{2\alpha})$$

* We define $A(r) = e^{2\alpha}$ and $y(r) = rA(r)$, which gives:

$$y = r + C \Rightarrow A(r) = 1 + \frac{c}{r}$$

$$A(r) = 1 - \frac{R_s}{r}$$

* Where R_s is the Schwarzschild radius, which allows us to write:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2\, d\Omega$$

- Black Holes
  - Using Newtonian $g_{tt} = -(1 + 2\Phi)$, we get:

$$R_s = -2\Phi$$

  - With a point mass:

$$\Phi = -\frac{GM}{r}$$

  - Thus, $R_s = 2GM$
  - Schwarzschild Properties:
    1. $M \to 0, \quad g_{MV} \to \eta_{M\nu}$
    2. $r \to \infty, \quad g_{M\nu} \to \eta_{M\nu}$
    3. $r = 0, \quad \dfrac{R_s}{r} \to \infty$
    4. $r = R_s, \quad \left(1 - \frac{R_s}{r}\right)^{-1} \to \infty$
  - For Black Holes, light cones are deformed by null geodesics. We may derive:

$$\frac{dt}{dr} = \pm \left(1 - \frac{2GM}{r}\right)^{-1}$$

    * As $r \to \infty$ back to $45°$

- Anthropic Principle
  - Observations made about the universe are implictly biased as a result of the fact that observations can only be made where the possibility of intelligence life exists

- Gravitational Bending of Spacetime (Back-Reaction Term):

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{2r^2} - \frac{\epsilon GM}{r} \underbrace{-\frac{GML^2}{r^3}}_{\text{New Term}} + \frac{1}{2}\epsilon = \frac{1}{2}E^3$$

- Defining $V(r)$:

$$\frac{L^2}{2r^2} - \frac{\epsilon GM}{r} - \frac{GML^2}{r^3} + \frac{1}{2}\epsilon$$

  * The first term is known as the angular momentum "barrier"

- We may write:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \epsilon$$

- Which lets us determine that, for circular orbit, $V'(r) = 0$, and for a stable circular orbit $V''(r) = 0$
- Using Newtonian mechanics, we may see:

$$\frac{d}{dr}\left(\frac{L^2}{2r^2} - \frac{GM}{r}\right) = -\frac{L^2}{r^3} + \frac{GM}{r^2}$$

  * Rearranging terms, we come to the familiar formula:

$$v = \sqrt{\frac{GM}{r}}$$

- We may perform similar calculations with General Relativity, but the new term adds a twist:
  * 2 solutions instead of 1 for massive particles
  * Not always stable
- For a massless particle ($v = c$), $\epsilon = 0$:

$$V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$$V'(r) = 0 \Rightarrow -\frac{L^2}{r^3} + \frac{3GML^2}{r^4} = 0 = 3GM$$

$$V''(r) = \frac{3L^2}{r^4} - \frac{12GML^2}{r^5}$$

  * Always negative! Always unstable!

- For massive particles, $\epsilon = 1$

$$V(r) = \frac{1}{2} - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$$

$$V'(r) = 0 \Rightarrow GMr^2 + L^2r + 3GML^2 = 0 \rightarrow r_L = \frac{L^2 \pm \sqrt{L^4 - 4(GM)(3GML^2)}}{2GM}$$

  * We may observe:

1. For large L: $\dfrac{L^2}{GM}$ is stable (goes to Newtonian), and $3GM$ is unstable (the massless case)
2. For small L: $L = \sqrt{12}GM$ provides smallest circular orbit, $r_c = L^2/2GM = 6GM = 3R_w$; No circular orbits for smaller L (particle goes to $r = 0$ (I.S.C.O))

– This new $1/r^3$ term reflects the non-linear nature of general relativity (back-reaction), which is important for small r