Lecture 5 — Einstein's Equations and Schwarzschild

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- We may begin with Einstein's equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

  - Note that the tensors form second order non-linear differential equations

- Symmetric Tensor ($n = 4$ dimensions)
  - $\frac{n^2-n}{n} + n = \frac{n^2+n}{2} = 10$ degrees of freedom
  - General relativity sees diffeomorphism invariance, so four of the degrees of freedom are removed (since it isn't $x^\mu \to x^{\mu'}$)
  - Solving these differential equations is extremely complex, so we will make some assumptions to simplify analysis:
    * Boundary conditions and initial conditions
    * Limits
    * Simplify through symmetry

- Symmetric General Relativity
  - Spherical symmetry and static
  - Homogenous and isotropic: FLRW universe/cosmology
  - $T_{\mu\nu} = 0$, small perturbations are gravitational waves
  - In Newtonian mechanics, with three masses M_1, M_2, and much smaller M_3, we may write: $\Phi_{M_3} = \Phi_{M_1} + \Phi_{M_2}$
    * In General Relativity, $g_{\mu\nu}$ depends on M_1 and M_2, as well as the binding energy between

- Schwarzschild

– Only vacuum solution with spherical symmetry

– We assume a spherical system that is static

– We use Minkowski space, with spherical coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = -dt^2 + dr^2 + r^2 \underbrace{\left(d\theta^2 + \sin^2(\theta)\, d\theta^2\right)}_{d\Omega^2}$$

* We may rescale this with functions of r:

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + C(r) r^2\, d\Omega^2$$

* Furthermore, we define $r \to \sqrt{C(r)}$

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + r^2\, d\Omega^2$$
$$ds^2 = -e^{2\alpha(r)}\, dt^2 + e^{\beta(r)}\, dr^2 + r^2\, d\Omega^2$$

* For the diagonal metric, we way find $\Gamma^{\phi}_{r\phi} = (1/r)$, then continuing Christoffel calculations, using Riemann, and then contracting to Ricci, we find:

$$R_{tt} = e^{2(\alpha-\beta)} \left[\partial_r^2 \alpha + (\partial_r \alpha)^2 - \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \alpha\right]$$

$$R_{rr} = -\partial_r^2 \alpha - (\partial_r \alpha)^2 + \partial_r \alpha \partial_r \beta + \frac{2}{r} \partial_r \beta$$

$$R_{\theta\theta} = e^{-2\beta} \left[r(\partial_r \beta - \partial_r \alpha) - 1\right] + 1$$

$$R_{\phi\phi} = \sin^2(\theta) R_{\theta\theta}$$

* We want a $T_{\mu\nu} = 0$ solution, which implies $G_{\mu\nu} = 0$, which then implies $R_{\mu\nu} = 0$
  · This is known as "Ricci Flat" (not really flat)

* We move terms around to find:

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0$$
$$\frac{2}{r} \left(\partial_r \alpha + \partial_r \beta\right) = 0$$
$$\alpha = -\beta$$

* Taking:

$$R_{\theta\theta} = 0 \quad \text{and} \quad -e^{2\alpha} \left[2r \partial_r \alpha + 1\right] + 1 = 0$$

* We get:

$$e^{2\alpha} \left[2r \partial_r \alpha + 1\right] = \partial_r (r e^{2\alpha})$$

* We define $A(r) = e^{2\alpha}$ and $y(r) = rA(r)$, which gives:

$$y = r + C \Rightarrow A(r) = 1 + \frac{c}{r}$$

$$A(r) = 1 - \frac{R_s}{r}$$

* Where R_s is the Schwarzschild radius, which allows us to write:

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2\, d\Omega$$

- Black Holes
  - Using Newtonian $g_{tt} = -(1 + 2\Phi)$, we get:

$$R_s = -2\Phi$$

  - With a point mass:

$$\Phi = -\frac{GM}{r}$$

  - Thus, $R_s = 2GM$
  - Schwarzschild Properties:
    1. $M \to 0, \quad g_{MV} \to \eta_{M\nu}$
    2. $r \to \infty, \quad g_{M\nu} \to \eta_{M\nu}$
    3. $r = 0, \quad \frac{R_s}{r} \to \infty$
    4. $r = R_s, \quad \left(1 - \frac{R_s}{r}\right)^{-1} \to \infty$
  - For Black Holes, light cones are deformed by null geodesics. We may derive:

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}$$

    * As $r \to \infty$ back to $45°$

- Anthropic Principle