# Homework 2

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1. Per the affine connection, we may use the Christoffel Symbol to write:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[ \partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu} \right]$$

We are given the metric for polar coordinates as:

$$ds^2 = dr^2 + r^2\, d\theta^2$$

Which gives us $g^{rr} = 1$ and $g^{\theta\theta} = r^{-2}$.

(a) We can begin with what Carroll supplied:

$$\boxed{\begin{cases} \Gamma^{r}_{rr} & = 0 \\ \Gamma^{r}_{\theta\theta} & = -r \\ \Gamma^{r}_{\theta r} = \Gamma^{r}_{r\theta} & = 0 \\ \Gamma^{\theta}_{rr} & = 0 \\ \Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} & = (1/r) \\ \Gamma^{\theta}_{\theta\theta} & = 0 \end{cases}}$$

(b) We can continue to find the divergence of $V$ using the simplified formula:

$$\nabla_{\mu} V^{\mu} = \frac{1}{\sqrt{|g|}} \partial_{\mu} \left( \sqrt{|g|} V^{\mu} \right)$$

Which gives us:

$$\boxed{\nabla \cdot \mathbf{V} = \frac{1}{\sqrt{|g|}} \partial_r \left( \sqrt{|g|} V^r \right) + \frac{1}{\sqrt{|g|}} \partial_{\theta} \left( \sqrt{|g|} V^{\theta} \right)}$$

The gradient can be found as:

$$\boxed{\nabla \mathbf{V} = \frac{\partial V}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \mathbf{e}_{\theta}}$$

(c) In general, we may write:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

From here, we can expand to:

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

From (a), we know that $\Gamma^\mu_{\rho\sigma}$ is non-zero for only two combinations, $\Gamma^r_{\theta\theta}$ and $\Gamma^\theta_{r\theta}$. With this, we are able to construct two equations:

$$\frac{d^2x^r}{d\lambda^2} - r\frac{d^2x^\theta}{d\lambda^2} = 0$$
$$\frac{d^2x^\theta}{d\lambda^2} + \frac{2}{r}\frac{dx^r}{d\lambda}\frac{dx^\theta}{d\lambda} = 0$$

This gives us the equations:

$$\boxed{\frac{d^2x^r}{d\lambda^2} = r\frac{dx^\theta}{d\lambda}\frac{dx^\theta}{d\lambda}}$$
$$\boxed{\frac{d^2x^\theta}{d\lambda^2} = -\frac{2}{r}\frac{dx^r}{d\lambda}\frac{dx^\theta}{d\lambda}}$$

(d) Using the equation for a line, we may write:

$$ax + by = c$$

In polar, this would be equivalent to:

$$ar\cos(\theta) + br\sin(\theta) = c$$

We can differentiate to get:

$$(a\cos(\theta) + b\sin(\theta))\,dr = (ar\sin(\theta) - br\cos(\theta))\,d\theta$$

And then we square:

$$(a^2\cos^2(\theta) + b^2\sin^2(\theta) - 2ab\sin^2(\theta)\cos^2(\theta))\,dr^2 =$$
$$r^2(a^2\sin(\theta) + b^2\cos(\theta) + 2ab\sin(\theta)\cos(\theta))\,d\theta^2$$

Per the metric, we can write:

$$ds^2 = dr^2 + r^2d\theta^2$$

$$1 = \left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\theta}{ds}\right)^2$$

Combining our equations, we get:

$$\left[1 - r^2 d\theta^2\right] \left((a^2 \cos^2(\theta) + b^2 \sin^2(\theta) - 2ab \sin^2(\theta) \cos^2(\theta))\right) - \\ r^2 d\theta^2 \left(a^2 \sin(\theta) + b^2 \cos(\theta) + 2ab \sin(\theta) \cos(\theta))\right) = 0$$

Simplifying, we get:

$$r^2(a^2 + b^2)d\theta^2 = a^2 \cos^2(\theta) + b^2 \sin^2(\theta) - 2ab \sin^2(\theta) \cos^2(\theta)$$

From the linear equation, we may simply rewrite this as:

$$r^2(a^2 + b^2)d\theta^2 = \frac{c^2}{r^2}$$

Which gets us:

$$r^4(a^2 + b^2)d\theta^2 = c^2$$

$$r^2\sqrt{(a^2 + b^2)}d\theta = c$$

$$\boxed{r^2 d\theta = \frac{c}{\sqrt{(a^2 + b^2)}}}$$

As such, we see that $r^2\, d\theta$ equals a constant and, thus:

$$\frac{d}{d\lambda}(r^2\, d\theta) = 0 \Rightarrow ax + by = c$$

2. (a) We begin with the expression for the Christoffel Symbol:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]$$

Given the diagonal matrix, we know that these terms may be non-zero only for $\sigma = \rho$, which allows us to obtain:

$$\Gamma^t_{\mu\nu} = \frac{1}{2} g^{tt} \left[\partial_\mu g_{\nu t} + \partial_\nu g_{t\mu} - \partial_t g_{\mu\nu}\right]$$

$$\Gamma^\theta_{\mu\nu} = \frac{1}{2} g^{\theta\theta} \left[\partial_\mu g_{\nu\theta} + \partial_\nu g_{\theta\mu} - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^\phi_{\mu\nu} = \frac{1}{2} g^{\phi\phi} \left[\partial_\mu g_{\nu\phi} + \partial_\nu g_{\phi\mu} - \partial_\phi g_{\mu\nu}\right]$$

Furthermore, we can expand the diagonal nature of the metric to write:

$$\Gamma^t_{\mu\nu} = \frac{1}{2} g^{tt} \left[\partial_\mu g_{tt} + \partial_\nu g_{tt} - \partial_t g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2} g^{\theta\theta} \left[\partial_\mu g_{\theta\theta} + \partial_\nu g_{\theta\theta} - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2} g^{\phi\phi} \left[\partial_\mu g_{\phi\phi} + \partial_\nu g_{\phi\phi} - \partial_\phi g_{\mu\nu}\right]$$

This can be simplified as:

$$\Gamma^{t}_{\mu\nu} = -\frac{1}{2} \left[\partial_\mu[-1] + \partial_\nu[-1] - \partial_t g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2} \left[R^2\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2}(R^2\sin^2(\theta)) \left[\partial_\mu(R^2\sin^2(\theta)) + \partial_\nu(R^2\sin^2(\theta)) - \partial_\phi g_{\mu\nu}\right]$$

By inspection, we can tell that the $\sigma = t$ case will always be zero, since the partials of constant become zero, and the partial with respect to time of any entry in the matrix is also zero (since they are not time-dependent). This leaves us with:

$$\Gamma^{\theta}_{\mu\nu} = \frac{1}{2} \left[R^2\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\phi}_{\mu\nu} = \frac{1}{2}(R^2\sin^2(\theta)) \left[\partial_\mu(R^2\sin^2(\theta)) + \partial_\nu(R^2\sin^2(\theta)) - \partial_\phi g_{\mu\nu}\right]$$

We begin by analyzing the $\sigma = \theta$ case:

$$\Gamma^{\theta}_{tt} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{t\theta} = \Gamma^{\theta}_{\theta t} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{t\phi} = \Gamma^{\theta}_{\phi t} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\mu(R^2) + \partial_\nu(R^2) - \partial_\theta g_{\mu\nu}\right]$$

Simplifying all of these, we get:

$$\Gamma^{\theta}_{tt} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_\theta(-1)\right] = 0$$

$$\Gamma^{\theta}_{t\theta} = \Gamma^{\theta}_{\theta t} = \frac{1}{2} \left[R^{-2}\right] \left[\partial_t(R^2)\right] = 0$$

$$\Gamma^{\theta}_{t\phi} = \Gamma^{\theta}_{\phi t} = \frac{1}{2} \left[R^{-2}\right] [0] = 0$$

$$\Gamma^{\theta}_{\theta\theta} = \frac{1}{2}\left[R^{-2}\right]\left[\partial_\theta(R^2) + \partial_\theta(R^2) - \partial_\theta(R^2)\right] = 0$$

$$\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta} = \frac{1}{2}\left[R^{-2}\right]\left[\partial_\phi(R^2)\right] = 0$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}\left[R^{-2}\right]\left[\partial_\phi(R^2) + \partial_\phi(R^2) - \partial_\theta(R^2\sin^2(\theta))\right] = -\sin(\theta)\cos(\theta)$$

We continue with $\sigma = \phi$:

$$\Gamma^{\phi}_{tt} = \frac{1}{2}\left[[R\sin(\theta)]^{-2}\right]\left[\partial_\phi(-1)\right] = 0$$

$$\Gamma^{\phi}_{t\theta} = \Gamma^{\phi}_{\theta t} = \frac{1}{2}\left[[R\sin(\theta)]^{-2}\right]\left[0\right] = 0$$

$$\Gamma^{\phi}_{t\phi} = \Gamma^{\phi}_{\phi t} = \frac{1}{2}\left[[R\sin(\theta)]^{-2}\right]\left[\partial_t(R^2\sin^2(\theta))\right] = 0$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2}\left[[R\sin(\theta)]^{-2}\right]\left[-\partial_\phi(R^2)\right] = 0$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{2}\left[[R\sin(\theta)]^{-2}\right]\left[\partial_\theta(R^2\sin^2(\theta))\right] = \cot(\theta)$$

$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2}\left[[R\sin(\theta)]^{-2}\right]\left[\partial_\phi(R^2\sin^2(\theta)) + \partial_\phi(R^2\sin^2(\theta)) - \partial_\phi(R^2\sin^2(\theta))\right] = 0$$

Thus, the only non-zero terms are:

$$\boxed{\Gamma^{\phi}_{\phi\phi} = -\sin(\theta)\cos(\theta) \quad \text{and} \quad \Gamma^{\theta}_{\phi\theta} = \Gamma^{\theta}_{\theta\phi} = \cot(\theta)}$$

(b) The geodesic equations give us:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

Using the values obtained in (a), we may write:

$$\frac{d^2x^\phi}{d\lambda^2} + \Gamma^{\phi}_{\phi\theta}\frac{dx^\phi}{d\lambda}\frac{dx^\theta}{d\lambda} = 0$$
$$\frac{d^2x^\theta}{d\lambda^2} + \Gamma^{\theta}_{\phi\phi}\frac{dx^\phi}{d\lambda}\frac{dx^\phi}{d\lambda} = 0$$

And we plug in known values to get:

$$\boxed{\begin{cases} \dfrac{d^2x^\theta}{d\lambda^2} = \sin(\theta)\cos(\theta)\dfrac{dx^\phi}{d\lambda}\dfrac{dx^\phi}{d\lambda} \\[2ex] \dfrac{d^2x^\phi}{d\lambda^2} = \cot(\theta)\dfrac{dx^\phi}{d\lambda}\dfrac{dx^\theta}{d\lambda} \end{cases}}$$

3. (a) We may begin by using the values assigned to $x, y$, and $z$ to obtain expressions for $r, \theta$, and $\phi$:

$$r = \frac{\cos(\lambda)}{\sin(\theta)\cos(\phi)} = \frac{\sin(\lambda)}{\sin(\theta)\sin(\phi)}$$

Using this, we get:

$$\frac{\cos(\lambda)}{\sin(\theta)\cos(\phi)} = \frac{\sin(\lambda)}{\sin(\theta)\sin(\phi)}$$
$$\tan(\lambda) = \tan(\phi)$$
$$\lambda = \phi$$

Next, combining provided parametrization with the above, we get:

$$r\cos(\theta) = \lambda$$
$$r\sin(\theta)\sin(\phi) = \sin(\lambda)$$
$$\frac{\lambda}{\cos(\theta)} = \frac{\sin(\lambda)}{\sin(\theta)\sin(\phi)}$$

But, from what we obtained for $\lambda = \phi$, we can simplify:

$$\lambda = \frac{\cos(\theta)}{\sin(\theta)}$$
$$\theta = \cot^{-1}(\lambda)$$

Finally, from our equation for $z$, we can say:

$$r\sin(\theta)\cos(\phi) = \cos(\lambda)$$

Which gets us:

$$r = \frac{1}{\sin(\theta)}$$

Using our trigonometric simplification rules, we may see:

$$r = \sqrt{\lambda^2 + 1}$$

Combining our findings, we may write the parametrization as:

$$\boxed{\{r, \theta, \phi\} \to \{\sqrt{\lambda^2 + 1}, \cot^{-1}(\lambda), \lambda\}}$$

(b) The tangent vector to the curve may simply be found by taking the differential of the parametrizations. Let us begin by using the $x, y, z$ parametrization:

$$\frac{d}{d\lambda}\{x, y, z\} = \frac{d}{d\lambda}\{\cos(\lambda), \sin(\lambda), \lambda\}$$

$$\boxed{V^p = \{-\sin(\lambda), \cos(\lambda), 1\}}$$

Now, we use our result from (a):

$$\frac{d}{d\lambda}\{r, \theta, \phi\} = \frac{d}{d\lambda}\left\{\sqrt{\lambda^2+1}, \cot^{-1}(\lambda), \lambda\right\}$$

$$\boxed{V^p = \left\{\frac{\lambda}{\sqrt{\lambda^2+1}}, -\frac{1}{\lambda^2+1}, 1\right\}}$$

4. (a) The transformation of an arbitrary matrix may be written as:

$$T^{\mu_1{}'\ldots\mu_k{}'}_{\nu_1{}'\ldots\nu_l{}'} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k{}'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1{}'}} \cdots \frac{\partial x^{\nu_k}}{\partial x^{\nu_k{}'}} T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}$$

Given this, we may focus on our case to write the matrix as:

$$\frac{\partial x^\mu}{\partial x^{\nu'}} \to \begin{bmatrix} \partial x/\partial\chi & \partial z/\partial\chi \\ \partial x/\partial\theta & \partial z/\partial\theta \end{bmatrix}$$

Given the restriction to the $y$ plane, we may observe that $\phi \to 0$, which gives us:

$$\begin{cases} x = \sinh(\chi)\sin(\theta) \\ z = \cosh(\chi)\cos(\theta) \end{cases}$$

Taking the above partials, we get:

$$\begin{cases} \partial x/\partial\chi = \cosh(\chi)\sin(\theta) \\ \partial z/\partial\chi = \sinh(\chi)\cos(\theta) \\ \partial x/\partial\theta = \cosh(\chi)\cos(\theta) \\ \partial z/\partial\theta = -\sinh(\chi)\sin(\theta) \end{cases}$$

Using matrix notation, we finally get:

$$\boxed{\frac{\partial x^\mu}{\partial x^{\nu'}} \to \begin{bmatrix} \cosh(\chi)\sin(\theta) & \sinh(\chi)\cos(\theta) \\ \sinh(\chi)\cos(\theta) & -\cosh(\chi)\sin(\theta) \end{bmatrix}}$$

(b) To find the expression for $ds^2$, we can use the metric:

$$g_{\mu\nu} = \partial_\mu x^\mu \partial_\nu x^\nu \eta_{\nu\mu}$$

From this, we can obtain:

$$g_{\mu\nu} = g_{\mu'\nu'} \begin{bmatrix} \cosh(\chi)\sin(\theta) & \sinh(\chi)\cos(\theta) \\ \sinh(\chi)\cos(\theta) & -\cosh(\chi)\sin(\theta) \end{bmatrix} \begin{bmatrix} \cosh(\chi)\sin(\theta) & \sinh(\chi)\cos(\theta) \\ \sinh(\chi)\cos(\theta) & -\cosh(\chi)\sin(\theta) \end{bmatrix}$$

We continue to simplify and compute:

$$g_{\mu\nu} = g_{\mu'\nu'} \left( \begin{bmatrix} \cosh(\chi)\sin(\theta) & \sinh(\chi)\cos(\theta) \\ \sinh(\chi)\cos(\theta) & -\cosh(\chi)\sin(\theta) \end{bmatrix} \right)^2$$

$$g_{\mu\nu} = g_{\mu'\nu'} \begin{bmatrix} \cosh^2(\chi)\sin^2(\theta) + \sinh^2(\chi)\cos^2(\theta) & 0 \\ 0 & \cosh^2(\chi)\sin^2(\theta) + \sinh^2(\chi)\cos^2(y) \end{bmatrix}$$

Thus, we can equivalently write:

$$\boxed{ds^2 = (\cosh^2(\chi)\sin^2(\theta) + \sinh^2(\chi)\cos^2(\theta))dx^2 + (\cosh^2(\chi)\sin^2(\theta) + \sinh^2(\chi)\cos^2(\theta))d\theta^2}$$

Or, alternatively, this can be simplified as:

$$\boxed{ds^2 = (\sinh^2(\chi)\cos^2(\theta) + \cosh^2(\chi)\sin^2(\theta))(dx^2 + d\theta^2)}$$

5. We may begin by writing out the expression for Christoffel Symbols:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left[ \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right]$$

From the expression, and the fact that we have a diagonal matrix, we know that the value can be non-zero ONLY when $\sigma = \rho$. Thus, to simplify, using $\lambda$, we may write:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\lambda} \left[ \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right]$$

Furthermore, since it is given that $\mu \neq \nu \neq \lambda$, we know that the partials will be zero, as non-diagonal terms of $g$ will be used, which gives us:

$$\boxed{\Gamma^{\lambda}_{\mu\nu} = 0}$$

Now, we return to the expression; however, now $\mu = \nu$. We may see that, with this, the only partial that remains (*i.e.* depends on a diagonal entry) is the last one:

$$\Gamma^{\lambda}_{\mu\mu} = \frac{1}{2} g^{\lambda\lambda} \left[ -\partial_\lambda g_{\mu\mu} \right]$$

$$\boxed{\Gamma^{\lambda}_{\mu\mu} = -\frac{1}{2} (g_{\lambda\lambda})^{-1} \partial_\lambda g_{\mu\mu}}$$

Continuing, we may check the case when $\nu = \lambda$. We may see that, in this case, only the first partial relies on a diagonal entry, which gives us:

$$\Gamma^{\lambda}_{\mu\lambda} = \frac{1}{2} g^{\lambda\lambda} \partial_{\mu}(g_{\lambda\lambda})$$

Using the property that:

$$\partial_{\mu} \log(A) = \frac{1}{A} du_{\mu}(A)$$

We can simplify to:

$$\boxed{\Gamma^{\lambda}_{\mu\lambda} = \partial_{\mu}\left[\ln\left(\sqrt{|g_{\lambda\lambda}|}\right)\right]}$$

Finally, checking the all-$\lambda$ case, we can see that the only partial that does not go to zero is the second one:

$$\Gamma^{\lambda}_{\lambda\lambda} = \frac{1}{2} g^{\lambda\lambda} \partial_{\lambda}(g_{\lambda\lambda})$$

Similar to the third equation, we can simplify to get:

$$\boxed{\Gamma^{\lambda}_{\lambda\lambda} = \partial_{\lambda}\left[\ln\left(\sqrt{|g_{\lambda\lambda}|}\right)\right]}$$

Thus, we have shown that, for a diagonal matrix, the terms simplify as:

$$\boxed{\mu \neq \nu \neq \lambda \rightarrow \begin{cases} \Gamma^{\lambda}_{\mu\nu} & = & 0 \\ \Gamma^{\lambda}_{\mu\mu} & = & -\frac{1}{2}(g_{\lambda\lambda})^{-1}\partial_{\lambda} g_{\mu\mu} \\ \Gamma^{\lambda}_{\mu\lambda} & = & \partial_{\mu}\left[\ln\left(\sqrt{|g_{\lambda\lambda}|}\right)\right] \\ \Gamma^{\lambda}_{\mu\nu} & = & \partial_{\lambda}\left[\ln\left(\sqrt{|g_{\lambda\lambda}|}\right)\right] \end{cases}}$$