# Homework 6


Michael Brodskiy

Professor: J. Blazek


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1. (a) First and foremost, we can eliminate the pressure contribution, since we are assuming a case of the ΛCDM universe. As such, this universe has low thermal pressure, which can be approximated to zero.

   (b) We can begin by decomposing the components of the fluid equations into "perturbation form" as follows:

$$\rho = \rho_o + \delta\rho$$
$$\vec{v} = \vec{v}_o + \delta\vec{v}$$
$$\Phi = \Phi_o + \delta\Phi$$

   Using standard convention, we take $\delta\Phi \to \Phi$. This allows us to rewrite the equations as

$$\begin{cases} \frac{D(\vec{v}_o+\delta\vec{v})}{Dt} & = -\nabla\Phi \\ \frac{D(\rho_o+\delta\rho)}{Dt} & = -(\rho_o+\delta\rho)\nabla\cdot(\vec{v}_o+\delta\vec{v}) \\ \nabla^2\Phi & = 4\pi G(\rho_o+\delta\rho) \end{cases}$$

   And finally we linearize (removing zeroth-order terms):

$$\boxed{\begin{cases} \dfrac{d(\delta\vec{v})}{dt} & = -\nabla\Phi \\ \\ \dfrac{d(\delta)}{dt} & = -\nabla\cdot(\delta\vec{v}) \\ \\ \nabla^2\Phi & = 4\pi G(\delta\rho) \end{cases}}$$

   (c) Incorporating the background velocity ($\vec{v}_o = H\vec{x}$), we may write:

$$\boxed{\begin{cases} \dfrac{d(\delta\vec{v})}{dt} + 2H\delta\vec{v} &= -\nabla\Phi \\ \\ \dfrac{d(\delta)}{dt} &= -\nabla\cdot(\delta\vec{v}) \\ \\ \nabla^2\Phi &= 4\pi G(\delta\rho) \end{cases}}$$

We may see that this contributes a damping term proportional to twice the Hubble expansion.

(d) To transition to comoving coordinates, we may use the following relationships:

$$\vec{x} = a\vec{r}$$

The peculiar velocity:

$$\delta\vec{v} = a\vec{u}$$

And the gradient:

$$\nabla_c = \frac{1}{a}\nabla$$

Incorporating this into the above, we get:

$$\begin{cases} a\dfrac{d\vec{u}}{dt} + 2aH\vec{u} &= -\nabla_c\Phi \\ \\ \dfrac{d(\delta)}{dt} &= -\nabla_c\cdot(\vec{u}) \\ \\ \nabla_c^2\Phi &= 4\pi G\bar{\rho}a^2\delta \end{cases}$$

We can then simplify using dot notation to get the equations in terms of comoving coordinates:

$$\boxed{\begin{cases} \dot{\vec{u}} + 2H\vec{u} &= -a^{-2}\nabla_c\Phi \\ \dot{\delta} &= -\nabla_c\cdot\vec{u} \\ \nabla_c^2\Phi &= 4\pi G\bar{\rho}a^2\delta \end{cases}}$$

(e) Taking the divergence of the first equation, we get:

$$\nabla_c\cdot\dot{\vec{u}} + 2H\nabla_c\cdot\vec{u} = -\frac{1}{a^2}\nabla_c^2\Phi$$

We may observe that this can be combined with the third equation to get:

$$\nabla_c \cdot \dot{\vec{u}} + 2H\nabla_c \cdot \vec{u} = -4\pi G\bar{\rho}\delta$$

We then take the time derivative of the second equation to write:

$$\ddot{\delta} = -\nabla_c \cdot \dot{\vec{u}}$$
$$\nabla_c \cdot \dot{\vec{u}} = -\ddot{\delta}$$

We then plug this and the undifferentiated form of the second equation into the first and third combined equation to write:

$$-\ddot{\delta} - 2H\dot{\delta} = -4\pi G\bar{\rho}\delta$$

We distribute the negative sign to get:

$$\boxed{\ddot{\delta} + 2H\dot{\delta} = 4\pi G\bar{\rho}\delta}$$

(f) We know that the mean matter density can be written as:

$$\bar{\rho}(a) = \rho_{crit}\Omega_m(a)$$

Furthermore, we know that the critical density is:

$$\rho_{crit} = \frac{3H_O^2}{8\pi G}$$

Combining this with part (e), we get:

$$\boxed{\ddot{\delta} + 2H\dot{\delta} = \frac{3H_o^2\Omega_m(a)\delta}{2}}$$

(g)
- Matter Domination
  In this case, we may see that:

  $$\ddot{\delta} + 2H\dot{\delta} = \frac{3}{2}H_o^2\delta$$

  We know that:

  $$H(a) = \frac{\dot{a}}{a} = H_o\sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}}$$

  Taking the purely matter component, we may write:

  $$H(a) = H_o\sqrt{a^{-3}}$$

  This gives us:

  $$\frac{d^2\delta}{dt^2} + 2\sqrt{a^{-3}}\frac{d\delta}{dt} = \frac{3}{2}H_o\delta$$

We can then write this as:

$$\frac{d^2\delta}{dt^2} + \frac{4}{3t}\frac{d\delta}{dt} - \frac{2}{3t^2}\delta =$$

We can determine that, since $\delta \propto t^{2/3}$ and $a \propto t^{2/3}$ then:

$$\boxed{\delta \propto a}$$

- Radiation Domination
  We may observe that $H(a) = H_o/a^2$ and that $\Omega_m = 0$, which gives us:

  $$\ddot{\delta} + \frac{2H_o}{a^2}\dot{\delta} = 0$$

  We can expand to write:

  $$\frac{d^2\delta}{dt^2} = -\frac{2H_o}{a^2}\frac{d\delta}{dt}$$
  $$d\delta = -\frac{2H_o}{a^2}\,dt$$
  $$\int d\delta = -\frac{2H_o}{a^2}\int dt$$

  And finally, we get:

  $$\boxed{\delta = -\frac{2H_o t_o}{a^2}}$$

- $\Lambda$ Domination
  We may observe that $H(a) = H_o$, and that $\Omega_m = 0$, which gives us:

  $$\ddot{\delta} + 2H_o\dot{\delta} = 0$$

  We expand to write:

  $$\frac{d^2\delta}{dt^2} + 2H_o\frac{d\delta}{dt} = 0$$
  $$\frac{d^2\delta}{dt^2} = -2H_o\frac{d\delta}{dt}$$
  $$\int d\delta = \int -2H_o\,dt$$

  Finally, this gets us:

  $$\boxed{\delta = -2H_o t_o}$$

  We see that this term is constant.

(h)
- Matter Domination
  We may observe that, during this period, $\Phi$ remains constant

- Radiation and $\Lambda$ Domination
  We may observe that, as $\delta$ is either slowing or constant, $\Phi$ decays

(i) Based on the results from (h), we may conclude that, in a matter-dominated region, the photon would remain at the same energy, since the gravitational potential doesn't change; however, the photon would gain energy (experience the ISW effect) in a radiation or $\Lambda$ dominated universe, since the gravitational potential would decay, meaning that the decrease in potential would be gained by the photon.

2. We first use the Born approximation to find the perpendicular acceleration:

$$a_\perp = \frac{GM}{r^2}\cos(\theta)$$

This acceleration results in the deflection of the light ray. From here, we may define the angle $\hat{\alpha}$ as the integral of the perpendicular acceleration. We first define:

$$r^2 = \varepsilon^2 + z^2$$

And then:

$$\cos(\theta) = \frac{\varepsilon}{\sqrt{\varepsilon^2 + z^2}}$$

This allows us to write:

$$\hat{\alpha} = \int_{-\infty}^{\infty} \frac{GM\varepsilon}{(\varepsilon^2 + z^2)^{\frac{3}{2}}}\, dz$$

We integrate to obtain:

$$\boxed{\hat{\alpha} = \frac{2GM}{\varepsilon}}$$

We may observe that the General Relativity case predicts a deflection angle that is twice that of the Newtonian prediction.

3. First and foremost, we know that gravitational lensing results in two effects: first, the magnification of luminosity, which results in observed luminosity $\mu L$ with magnification factor $\mu$ and intrinsic luminosity $L$; second, the apparent area of the sky is magnified by the same factor $\mu$, which results in the density of galaxies being decreased by a factor $\mu^{-1}$. Given that $n(L)$ corresponds to the number density of galaxies, we may write:

$$n(L) \rightarrow n_{app}(L_{app})$$

$$n_{app}(L_{app}) \propto \frac{1}{\mu}\left(\frac{\mu}{L_{app}}\right)^{\alpha}$$

We may simplify to get:

$$n_{app}(L_{app}) \propto \left(\frac{\mu^{\alpha-1}}{L_{app}^{\alpha}}\right)$$

Since we are to assume $\alpha > 1$, the magnification factor, $\mu^{\alpha-1}$ must be increasing, which indicates that the number of galaxies with the given apparent luminosity detected increases as a result of the magnification. As such, magnification as a result of gravitational lensing in the foreground increases the quantity of detected galaxies for $\alpha > 1$.

4. We may begin by calculating the luminosity distance as:

$$d_L = \chi(1+z)$$

For a ΛCDM universe, we know that $\chi$ may be obtained using:

$$\chi = \int_0^z \frac{dz'}{H_o\sqrt{.31(1+z')^3 + .69}}$$

Since the redshift is given, we get:

$$\chi = \int_0^{.01} \frac{dz'}{H_o\sqrt{.31(1+z')^3 + .69}}$$

Entering this into a numerical solver, we may obtain:

$$\chi = \frac{.009977}{H_o}$$

Which ultimately gives us:

$$d_L = \frac{.009977(1+.01)c}{70}$$

$$\boxed{d_L \approx 1.3224 \cdot 10^{21}[\text{km}]}$$

Since we are given the time delay as $\Delta t = 1.7[\text{s}]$, we can obtain the difference in speed as:

$$\Delta v = \frac{d_L}{\Delta t}$$

This gives us:

$$\Delta v = \frac{1.3224 \cdot 10^{21}}{1.7}$$

$$\boxed{\Delta v = 7.779 \cdot 10^{20} \left[\frac{\text{km}}{\text{s}}\right]}$$

Thus, we see that the upper limit of the order-of-magnitude difference between the speed of light and the speed of gravitational waves is on the order of:

$$10^{20} \left[\frac{\text{km}}{\text{s}}\right]$$