Lecture 4 — Manifolds and Curved Spacetime

Michael Brodskiy
Professor: J. Blazek

September 18, 2024

- We now move from Minkowski to General Space:

$$\eta_{\mu\nu} \to g_{\mu\nu}$$

- Differentiable Manifolds
  - Manifold: A space (in n-dimensions) that looks locally like $\mathbb{R}^n$ and can be constructed by smoothly stitching together these regions
  - Rotations in $\mathbb{R}^n \to$ Lie Groups are manifolds with a group structure
  - To be more precise, we have a set M with a set of (all possible) charts of open subsets to $\mathbb{R}^n$
    * Chart $\leftrightarrow$ coordinate system
  - These charts must be smooth, continuous, invertible, and differentiable
  - Now we will define (co)tangent spaces on these manifolds, with metrics that map (dual) vectors to $\mathbb{R}$

- The Equivalence Principle
  - In special relativity, we had the principle that the laws of physics were the same in all inertial frames
  - Einstein's "happiest though": If someone falls from a roof, nothing falls in their frame
  - Equivalence of inertial frames should be generalized to include gravity
  - Weak Equivalence Principle (WEP)
    * Inertial mass = gravitational mass

$$F = m_i a \text{ (inertial)}$$
$$F = -m_g \nabla\Phi \text{ (gravitational "charge")}$$
$$m_i = m_g \text{ (WEP: Eötuös experiments, late 19th century)}$$

        * All freely falling bodies behave the same/are indistinguishable ($a = -\nabla\Phi$)
        * Define inertial trajectory as unaccelerated (subject only to gravity)
        * In small enough regions of space-time, freely falling particles behave the same in a gravitational field or a uniformly accelerated field (physicist in a box, accelerating reference frame)
    - Strong Equivalence Principle (SEP)
        * All laws of physics, including gravitation, look like SR
            · Einstein Equivalence Principle (EEP) plus the impact of gravitational binding energy
            · Rules out "fifth force"

- Tidal Forces
    - Causes tides on Earth
    - Locally inertial frames

- Gravitational Redshift

$$\Delta v = \frac{az}{c}$$

    - Relativistic Doppler Shift:

$$\lambda_{obs} = \lambda_o \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)^{\frac{1}{2}}$$

        * Using Taylor expansion, we may simply write this as:

$$\lambda_{obs} = 1 + \frac{v}{c}$$

        * Bringing this together, we get:

$$\frac{\Delta\lambda}{\lambda_o} \approx \frac{\Delta v}{c} = \frac{az}{c^2}$$

    - EEP says that this must be the same as a gravitational field:

$$\frac{\Delta\lambda}{\lambda_o} = \frac{a_g z}{c^2} = \frac{\Delta\Phi}{c^2}$$

    - This is the time from start to end of wavelength, and can be used to compare clocks
    - If we have a case where $\Delta t_o = \lambda_o c^{-1}$ and $\Delta t_1 = \lambda_1 c^{-1}$, and $\lambda_1 > \lambda_o$ then $\Delta t_1 > \Delta t_o$, which indicates gravitational time dilation

- Classic Tests of General Relativity

    1. Precession of the perihelion of Mercury — 19^{th} century: 43" per century discrepancy successful "post-diction" of GR (about 10% of total effect)
    2. Bending of star light by sun (gravitational lensing) — GR predicts a factor of 2 larger deflection (1919 Eddington Expedition to observe the solar eclipse)
    3. Gravitational Redshift — 1954: Popper measurement of a white dwarf, 1959: Pound-rebka at Jefferson lab (Harvard), 22.5m

- Vectors and Tensors on Manifolds (Curved Spacetime)

    - We already saw $V = V^\mu \hat{e}_\mu$ at point P on T_p
    - What is the basis?
        - We want to define tangent vectors before we have a vector space on M
        - Instead, consider a function f and a curve λ. The directional derivative is:

$$\frac{d}{d\lambda}x^\mu \frac{\partial}{\partial x^\mu} f = \frac{d}{d\lambda}x^\mu \partial_\mu f \qquad \text{(gradient} \cdot \text{tangent } \vec{v}\text{)}$$

        - f could have been anything, so we define the tangent vector:

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu$$

        - $\{\hat{e}_\mu = \partial_\mu\}$ is the coordinate basis ("points" in the direction of x^μ)
        - Not orthonormal, but always well defined
        - In this basis, things transform according to:

$$\partial_{\mu\prime} = \frac{\partial}{\partial x^{\mu\prime}} = \frac{\partial}{\partial x^\mu}\frac{\partial x^\mu}{\partial x^{\mu\prime}} = \frac{\partial x^\mu}{\partial x^{\mu\prime}}\partial_\mu$$

        - Similarly, $V = V^\mu \partial_\mu$ is preserved, so:

$$V^{\mu\prime}\partial_{\mu\prime} = V^\mu \partial_\mu \Rightarrow V^{\mu\prime} = \frac{\partial x^{\mu\prime}}{\partial x^\mu} V^\mu$$

- General Coordinate Transform

    - In flat space: $x^{\mu\prime} = \Lambda^{\mu\mu}_{\mu}$

$$\frac{dx^{\mu\prime}}{dx^\mu} = \Lambda^{\mu}_{\mu\prime}$$

    - We recover the transform of vectors
    - Vector Fields:
        - X: One vector at each point on the manifold

  * X, Y: Both define a field that can be used to take directional derivatives of functions on μ

$$[X,Y](f) = X(Y(f)) - Y(X(f)) \quad \underline{\text{commutator}}$$

- Dual Vectors
  * Recall we defined the gradient: df

$$df\left(\frac{d}{d\lambda}\right) = \frac{df}{d\lambda} \quad \text{map a vector to } \mathbb{R}$$

  * Basis for dual vectors dx^μ
  * Gradient of the coordinate function:

$$dx^\mu(\partial_\nu) = \frac{dx^\mu}{dx^\nu} = \delta^\mu_\nu$$
$$V = V^\mu \partial_\mu$$
$$\omega = \omega_\nu dx^\nu$$
$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu$$

- We can now write the transformation of an arbitrary (k,l) tensor on a manifold:

$$T^{\mu_1{}' \ldots \mu_k{}'}_{\nu_1{}' \ldots \nu_l{}'} = \frac{\partial x^{\mu_1{}'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k{}'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1{}'}} \cdots \frac{\partial x^{\nu_k}}{\partial x^{\nu_k{}'}} T^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l}$$

  * Warning: in curved space $\partial_\mu W_\nu$ is not a tensor; unlike in flat space, the derivative of the transform can be non-zero (Λ is the same everywhere)

- The Metric
  - $\eta_{\mu\nu}$ in Minkowski space
  - $g_{\mu\nu}$ in general curved spacetime

$$g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu \quad \text{(defines inverse)}$$

  - Metric really describes basically everything about a spacetime

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

    * (0,2)-tensor components, metric components, with ds^2 being called the "line element" or "metric"