Homework 4

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1. We may express the comoving distance function as (note that the numerator should contain c, but since we eliminated the speed of light, we instead get the below equation):

$$\chi(z) = \int_0^z \frac{dz'}{H(z')}$$

Furthermore, the Hubble function may be expressed as a function of parameter densities and the redshift, z, such that:

$$H(z) = H_o\sqrt{\Omega_r(1+z)^4 + \Omega_m(1+z)^3 + \Omega_\kappa(1+z)^2 + \Omega_\Lambda}$$

Combining the two equations, we may obtain:

$$\boxed{\chi(z) = \int_0^z \frac{dz'}{H_o\sqrt{\Omega_r(1+z')^4 + \Omega_m(1+z')^3 + \Omega_\kappa(1+z')^2 + \Omega_\Lambda}}}$$

Such a formula is logical since it contains the various densities of the universe, as well as being an expression in terms of redshift.

2. First and foremost, we know that we may write:

$$\frac{\dot{a}}{a} = H_o\left[\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}\right]^{\frac{1}{2}}$$

   (a) For the Einstein-de Sitter universe, we use the equation above to get:

$$\frac{da}{dt} = a^{-.5}H_o$$

   From this, we can solve:

$$\int a^{.5}\,da = \int H_o\,dt$$

$$\frac{2}{3}a^{1.5} = H_o t$$

This gives us the scale factor as:

$$\boxed{a(t) = \frac{3}{2}\left(H_o t\right)^{\frac{2}{3}}}$$

To find the age of the universe, we take $a \to 1$ and $t \to t_o$ to get:

$$\boxed{t_o = \frac{2}{3H_o}}$$

Finally, we find the comoving horizon distance as (note that restoring c would return this value to the numerator):

$$\chi_{hor} = \int_0^1 \frac{da}{a^{.5}H_o}$$

$$\chi_{hor} = \frac{2\sqrt{a}}{H_o}\bigg|_0^1$$

$$\boxed{\chi_{hor} = \frac{2}{H_o}}$$

(b) For a radiation-dominated universe, we get:

$$\frac{da}{dt} = \frac{H_o}{a}$$

$$\int a\,da = \int H_o\,dt$$

$$\frac{1}{2}a^2 = H_o t$$

$$\boxed{a(t) = \sqrt{2H_o t}}$$

From here, we can find the age as:

$$1 = \sqrt{2H_o t_o}$$

$$\boxed{t_o = \frac{1}{2H_o}}$$

Finally, we find the horizon distance:

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H(a)}$$

$$\chi_{hor} = \int_0^1 \frac{1}{H_o}\,da$$

$$\boxed{\chi_{hor} = \frac{1}{H_o}}$$

(c) For an empty universe, we obtain:

$$\frac{da}{dt} = H_o$$
$$\int da = \int H_o\, dt$$
$$\boxed{a(t) = H_o t}$$

We continue to get the age:

$$1 = H_o t_o$$
$$\boxed{t_o = \frac{1}{H_o}}$$

Note that this corresponds to Hubble time. Finally, we get the horizon distance:

$$\chi_{hor} = \int_0^{t_o} \frac{dt}{a(t)}$$
$$\chi_{hor} = \int_0^{t_o} (H_o t)^{-1}\, dt$$
$$\chi_{hor} = H_o \ln(t)\Big|_0^{t_o}$$
$$\chi_{hor} \to \infty$$

(d) For a de Sitter universe, we get:

$$\frac{da}{dt} = aH_o$$
$$\int a^{-1}\, da = \int H_o\, dt$$
$$\ln(a) = H_o t$$
$$\boxed{a(t) = e^{H_o t}}$$

Now, we can find the age as:

$$t_o = \int_0^1 \frac{da}{aH(a)}$$
$$t_o = \int_0^1 \frac{da}{aH_o}$$

$$t_o = H_o \ln(a)\Big|_0^1$$
$$t_o \to \infty$$

Finally, we find the horizon distance:

$$\chi_{hor} = \int_0^{t_o} \frac{dt}{a(t)}$$
$$\chi_{hor} = \int_0^{t_o} e^{-H_o t}\, dt$$
$$\chi_{hor} = -\frac{e^{-H_o t}}{H_o}\Big|_0^\infty$$
$$\boxed{\chi_{hor} = \frac{1}{H_o}}$$

(e) For a flat, dark-energy dominated (all matter is dark energy) universe, we get:

$$\frac{da}{dt} = aH_o\sqrt{a^{-3[1+(-1.1)]}}$$
$$\frac{da}{dt} = a^{1.15}H_o$$
$$\int a^{-1.15}\, da = \int H_o\, dt$$
$$-\frac{a^{-.15}}{.15} = H_o t$$
$$a^{-.15} = -.15H_o t$$
$$\boxed{a(t) = (-.15H_o t)^{-6.6\bar{6}}}$$

From here, we find the age:

$$t_o = \int_0^1 \frac{da}{a^{1.15}H_o}$$
$$\boxed{t_o \to \infty}$$

Finally, we get the horizon distance:

$$\chi_{hor} = \int_0^1 \frac{da}{a^{2.15}H_o}$$
$$\chi_{hor} = -\frac{1}{1.15a^{1.15}H_o}\Big|_0^1$$
$$\boxed{\chi_{hor} \to \infty}$$

We may observe that this universe's parameters approach infinity, which may signify a "big rip."

(f) We now check a standard ΛCDM. We analyze two cases: with and without radiation. Thus, we find the scale factors:

$$\frac{da}{dt} = aH_o\sqrt{.31a^{-3} + .69}$$

$$\frac{da}{dt} = aH_o\sqrt{(.31 - 9 \cdot 10^{-5})a^{-3} + (9 \cdot 10^{-5})a^{-4} + .69}$$

These can be rearranged to:

$$\int \left(a\sqrt{.31a^{-3} + .69}\right)^{-1} da = \int H_o\, dt$$

$$\int \left(a\sqrt{(.31 - 9 \cdot 10^{-5})a^{-3} + (9 \cdot 10^{-5})a^{-4} + .69}\right)^{-1} da = \int H_o\, dt$$

Given the complex nature of this integration, I will leave these expressions in terms of integration:

$$\boxed{\int \left(a\sqrt{.31a^{-3} + .69}\right)^{-1} da = H_o t}$$

$$\boxed{\int \left(a\sqrt{(.31 - 9 \cdot 10^{-5})a^{-3} + (9 \cdot 10^{-5})a^{-4} + .69}\right)^{-1} da = H_o t}$$

We may obtain the age of the universe by writing:

$$t_o = \int_0^1 \frac{da}{aH(a)}$$

$$t_o = \int_0^1 \frac{da}{aH_o\sqrt{.31a^{-3} + .69}}$$

Using a numerical solver, we get:

$$\boxed{t_o = \frac{.955277}{H_o}}$$

We then solve for the comoving horizon distance:

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H(a)}$$

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H_o\sqrt{.31a^{-3} + .69}}$$

Once again using a solver, we get:

$$\boxed{\chi_{hor} = \frac{3.26134}{H_o}}$$

If we incorporate the radiation, we can recalculate our values to yield:

$$t_o = \int_0^1 \frac{da}{aH_o\sqrt{(.31 - 9 \cdot 10^{-5})a^{-3} + (9 \cdot 10^{-5})a^{-4} + .69}}$$

$$\chi_{hor} = \int_0^1 \frac{da}{a^2H_o\sqrt{(.31 - 9 \cdot 10^{-5})a^{-3} + (9 \cdot 10^{-5})a^{-4} + .69}}$$

This yields:

$$\boxed{t_o = \frac{.954973}{H_o}}$$

$$\boxed{\chi_{hor} = \frac{3.20122}{H_o}}$$

Note that we may observe that the universe with radiation is a bit younger as a result of the radiation, as we would expect.

3. Given that we are trying to find the point at which the two equal, we obtain:

$$\Omega_{m,o} = a^3\Omega_{\Lambda,o}$$

$$a_{m,\Lambda} = \sqrt[3]{\frac{\Omega_{m,o}}{\Omega_{\Lambda,o}}}$$

This gives us:

$$a_{m,\Lambda} = \sqrt[3]{\frac{.31}{.69}}$$

$$\boxed{a_{m,\Lambda} = .7659}$$

From here, we know that:

$$z = \frac{1}{a_{m,\lambda}} - 1$$

$$z = \frac{1}{.7659} - 1$$

$$\boxed{z = .3057}$$

Redshift is .3057 when densities in matter and Λ are equivalent. Proceeding to find the matter-radiation equality point, we get:

$$\Omega_{r,o} = a\Omega_{m,o}$$

$$a_{m,r} = \frac{\Omega_{r,o}}{\Omega_{m,o}}$$

This gives us:

$$a = \frac{9 \cdot 10^{-5}}{.31}$$

$$\boxed{a_{m,r} = 2.9032 \cdot 10^{-4}}$$

From here, we get:

$$z = (2.9032 \cdot 10^{-4})^{-1} - 1$$

$$z = 3444.5 - 1$$

$$\boxed{z = 3443.5}$$

Redshift is 3443.5 when densities in matter and radiation are equivalent.

4. Let us begin by defining the Hubble radius as:

$$R_H = \frac{1}{aH(a)}$$

We differentiate both signs to get:

$$\dot{R}_H = \frac{d}{dt}\left[\frac{1}{aH(a)}\right]$$

We know that the Hubble parameter is defined by:

$$H(a) = \frac{\dot{a}}{a}$$

And thus, we get:

$$aH(a) = \dot{a}$$

$$\frac{d}{dt}[aH(a)] = \ddot{a}$$

Combining this with the differential equation above gives us:

$$\dot{R}_H = -\frac{\ddot{a}}{[aH(a)]^2}$$

We may observe that, because $\ddot{a}$ is strictly positive, and the denominator can not be negative since it is real and squared, the derivative, due to the negative sign, must always be less than zero. This can be express as:

$$\boxed{\dot{R}_H = -\frac{\ddot{a}}{[aH(a)]^2} < 0}$$

And, therefore, the Hubble radius is strictly decreasing for periods of accelerating expansion. This indicates that the distance over which light may act is contracting, since the universe expands at an increasing rate and regions of spacetime move away from one another at a velocity greater than that of light. Using the second Friedmann equation, we know:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

With the relationship that $p = \omega\rho$, we may observe that the smallest (in magnitude) such ω is:

$$3\omega\rho < -\rho$$

$$\boxed{\omega < -\frac{1}{3}}$$

5. Per the Friedmann equation at turnaround, we know that $\dot{a} \to 0$, which gives us:

$$H_o^2 \left[\Omega_m a^{-3} + \Omega_k a^{-2}\right] = 0$$

Rearranging in terms of the scale factor, we find:

$$a = -\frac{\Omega_m}{\Omega_k}$$

$$a = -\frac{1.2}{-.2}$$

$$\boxed{a_{turn} = 6}$$

We can find the time by integrating the expression with respect to a:

$$\frac{da}{dt} = aH_o\sqrt{1.2a^{-3} - .2a^{-2}}$$

$$t = \int_0^{a_{turn}} \frac{da}{aH_o\sqrt{1.2a^{-3} - .2a^{-2}}}$$

$$t = \frac{1}{H_o}\int_0^6 \frac{da}{\sqrt{1.2a^{-1} - .2}}$$

$$t = \frac{21.0744}{H_o}$$

We need to convert H_o to suitable units; we know that, in inverse seconds, this is: $H_o = 2.27 \cdot 10^{-18}[\text{s}^{-1}]$. Thus, we get:4

$$t = \frac{21.0744}{2.27 \cdot 10^{-18}}$$

$$\boxed{t = 9.2838 \cdot 10^{18}[\text{s}]}$$

Note that this is a long time, as it is equivalent to $2.9439 \cdot 10^{11}$ years. To show that a matter and cosmological universe will not recollapse, we return to the Friedmann equation. We know that a universe will begin collapsing when $\dot{a}$ goes from positive to negative. Thus, we may write:

$$\dot{a} = aH_o\sqrt{\frac{1}{a^3} + \Omega_\Lambda}$$

$$\dot{a} = H_o\sqrt{\frac{1}{a} + a^2\Omega_\Lambda}$$

We may observe that, due to the square root, this expression will never be negative. We may find a critical point at:

$$\frac{1}{a} = -a^2\Omega_\Lambda$$

However, for this to be true, Ω_Λ would need to be less than zero. Thus, <u>a closed universe with matter and cosmological constant will not recollapse</u>

6. We know that:

$$\rho_{crit} = \frac{3H_o^2}{8\pi G}$$

In standard units, $H_o = 70\left[\frac{\text{km}}{\text{sMpc}}\right]$, becomes:

$$H_o = 2.27 \cdot 10^{-18}\left[\frac{1}{\text{s}}\right]$$

Combining this and other known equations in our expression above, we get:

$$\rho_{crit} = \frac{3(2.27 \cdot 10^{-18})^2}{8\pi(6.674 \cdot 10^{-11})}$$

$$\boxed{\rho_{crit} = 8.55 \cdot 10^{-27}\left[\frac{\text{kg}}{\text{m}^3}\right]}$$

We know that a solar mass may be expressed as $1[M_\odot] \approx 1.989 \cdot 10^{30}[\text{kg}]$. Furthermore, astronomical units may be written as: $1[au] \approx 1.496 \cdot 10^{11}[\text{m}]$. This gives us:

$$\rho_{crit} = \left(\frac{8.55 \cdot 3.3481}{1.989}\right) \cdot 10^{-27} \cdot 10^{33} \cdot 10^{-30}$$

$$\boxed{\rho_{crit} = 1.4392 \cdot 10^{-23} \left[\frac{\mathrm{M}_\odot}{\mathrm{au}^3}\right]}$$

Now, we find the value in terms of proton mass. We know that a proton mass is: $m_+ \approx 1.67 \cdot 10^{-27}$[kg]. As such, we get:

$$\rho_{crit} = \left(\frac{8.55}{1.67}\right) \cdot 10^{27} \cdot 10^{-27}$$

$$\boxed{\rho_{crit} = 5.1198 \left[\frac{\mathrm{m_p}}{\mathrm{m}^3}\right]}$$

Note that the densities are small, indicating that space itself is quite empty.

7. We may begin by finding the mass in the universe, in terms of a density function (assuming spherical shape). This gives us:

$$m(r) = \int_0^r 4\pi r^2 \rho(r)\, dr$$

We assume that the density may expressed in some terms such that $\rho(r) = r^\alpha$. This gives us:

$$m(r) = 4\pi \int_0^r r^{2+\alpha} \rho(r)\, dr$$

$$m(r) = \frac{4\pi r^{3+\alpha}}{3+\alpha}$$

Now, we want to relate gravitation to velocity. This can be done by setting the equations for gravitational force and centripetal acceleration equal to each other. This yields:

$$\frac{Gm(r)m_2}{r^2} = \frac{m_2 v(r)^2}{r}$$

$$m(r) = \frac{rv(r)^2}{G}$$

It is given that $v(r)$ does not actually depend on r in a flat galaxy, and we can thus conclude that the mass is proportional to r^1. Thus, we may say:

$$r^{3+\alpha} = r^1$$

$$3 + \alpha = 1$$

$$\boxed{\alpha = -2}$$

As such, we know that the density profile must be of some form such that:

$$\boxed{\rho(r) \propto r^{-2}}$$