Lecture 6 — The Expanding Universe

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- Olber's Paradox: Why is the Sky Dark?
  - Absorbing Matter $\to$ doesn't work since matter would heat up
  - Finite Size
  - Finite Time (and Finite Speed of Light)
  - Dimming of Light ("redshift")
- Universe had some beginning ("Big Bang") around 13.7 billion years ago
- Some units:

$$1[\text{light year}] = 9.5 \cdot 10^{15}[\text{m}]$$
$$1[\text{yr}] \approx \pi \cdot 10^{7}[\text{s}]$$
$$c \approx 3 \cdot 10^{8}\left[\frac{\text{m}}{\text{s}}\right]$$
$$1[\text{pc}] = 3.26[\text{light years}]$$

  - If $\theta = 1[\text{arcsec}]$, then $d = 1[\text{pc}]$

$$1[\text{pc}] = 2.1 \cdot 10^{5}[\text{AU}]$$

- The Cosmological Principle
  - Copernicus: the Sun, not the Earth, is the center of the Universe
  - Cosmological Principle: There is no center to the Universe
    - The Universe is statistically isotropic (same in all directions) and homogenous (same everywhere)
- Expanding Universe

- All observers see things moving away from them
- Statements are all statistical! Distinguish between structure in the universe and the geometry of the homogenous universe (about 100[Mpc] scales for homogeneity)
- We don't experience the FLRW metric
  * Homogenous/geometry
  * Structure
- Conservation of Energy Solution to Expanding Cloud:

$$\frac{1}{2}\dot{R}^2 - \frac{2GM}{R} = C$$

  * What is the physical meaning of C?
    - $C = 0$
      You are just at escape velocity. As $R \to \infty$, $\dot{R} = v \to 0$. Potential and kinetic both go to zero.
    - $C > 0$
      Positive total energy. You have more than enough energy to escape. $\dot{R} > 0$ as $R \to \infty$
    - $C < 0$
      Negative total energy. You won't make it out to $R = \infty$, you will stop and turn around at some finite t
  * These cases capture an ideal universe's expansion with only real matter
  * For a matter-only universe, C describes the spatial geometry, with zero indicating flat, $C > 0$ indicating open (negative curvature), and $C < 0$ indicating closed (positive curvature)
    - Note that Λ complicates this
- Using the first-order equation, we may write:

$$\frac{1}{2}\dot{R}^2 = \frac{4}{3}\pi G\rho R^2 + C$$
$$\left(\frac{\dot{R}}{R}\right) = \frac{8\pi G\rho}{3} + C$$

- At $C = 0$, we find ρ_{crit}:

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

  Note that we define the Hubble parameter as:

$$H = \frac{\dot{R}}{R}$$

- Comoving Coordinates

- Coordinates (and distances) scale with the size of the universe:

$$x = a(t)r$$

  - x is the "proper" coordinate
  - r is the "comoving" coordinate
  - $a(t)$ is the scale factor, with $a(t_o) = 1$ indicating "today"

- Ex. $x_{12} = a(t)r_{12}$

$$\frac{dx_{12}}{dt} \quad \text{is the recession velocity}$$

- Two Things:
  1. Velocity is proportional to distance (Hubble law)
  2. The proportionality term is $\dfrac{\dot{a}}{a} = H(t)$

- Hubble constant is approximately:

$$H_o = 70 \left[\frac{\text{km}}{\text{sMpc}}\right]$$

  - We define h, such that:

$$H_o = 100h \left[\frac{\text{km}}{\text{sMpc}}\right]$$

  - We can see that the units are actually just Hz, which gives us:

$$H_o \approx 3.24 \cdot 10^{-18} \left[\frac{1}{\text{s}}\right]$$

- If a is constant, this gives us the doubling time or the time to go back to $a = 0$, which gives us the "Hubble" time, or the age of the universe
- For $h = .7 \rightarrow 1.4 \cdot 10^{10} \left[\frac{1}{\text{yr}}\right]$
  - Fairly accurate approximation of 14 billion years

- FLRW Metric
  - Cosmological principle tells us that spacetime (on large scales) should be homogenous (translation invariant) and isotropic (rotation invariant)
  - We seek a general form of a metric that obeys these assumptions:

$$ds^2 = \bar{c}^2\, dt^2 + R^2(t)\, d\sigma^2$$

  - We are able to derive:

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2\, d\Omega^2$$

- Where:
  - $d\Omega^2$ is the 2-sphere metric
  - $k \propto R$ (Ricci scalar on 3D space)

$$k = \begin{cases} +1, & \text{Positive Curvature (3-sphere), "closed"} \\ 0, & \text{Flat} \\ -1, & \text{Negative curvature (saddle), "open"} \end{cases}$$

- Thus, we may define the metric as:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\, d\Omega^2\right]$$

  - Where:

$$\begin{cases} \kappa > 0, & \text{closed} \\ \kappa = 0, & \text{flat} \\ \kappa < 0, & \text{open} \end{cases}$$

- Cosmological Redshift
  - Like a Döppler shift, but use caution!
  - $v = H_o d$, locally $v << c$
  - Redshift in special relativity:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \sqrt{\frac{1+v/c}{1-v/c}} \approx 1 + \frac{v}{c}$$

  - From this, we know:

$$\frac{\lambda_{obs}}{\lambda_{em}} = 1 + z \approx 1 + \frac{H_o d}{c}$$

  - More generally, we may say (for a_o = today):

$$1 + z = \frac{a_{obs}}{a_{em}} = a_{em}^{-1}$$

- Cosmological Redshift versus Peculiar Motions:
  - For comoving coordinates:

$$v \approx H_o d = H_o|\vec{r}_2 - \vec{r}_1|/a_o = H_o|\vec{r}_2 - \vec{r}_1|$$

  - For "peculiar" motions

$$v \approx H_o d + \frac{\Delta v}{c}$$

- In the FLRW metric, due to lack of simple time symmetry, energy is not conserved
    - There is a Killing Tensor that reflects a symmetry:

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \quad \text{for observer (comoving) } U^\mu = (1,0,0,0)$$

    - We get:

$$K^2 = K_{\mu\nu} V^\mu V^\nu$$

        - Is conserved, with

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

        - For a photon on a null geodesic:

$$V_\mu V^\mu = 0$$

    - This can be simplified to:

$$v = \frac{K}{a}$$

- "Stuff" That Can fill A Universe
  - "Baryons" $\rightarrow$ All standard model particles with mass (interact with light, gravity)
    - Non-photon force carriers
    - "In a box" with scale factor a scale proportionally to a^{-3}
  - Dark matter $\rightarrow$ Only interacts through gravity (?)
    - Neutrinos (?)
    - New particles
    - "Cold" dark matter (non-relativistic), scales "in a box" with scale factor a proportionally to a^{-3}
  - Dark energy ("cosmological constant")
  - Radiation
    - Photons
      - "In a box" with scale factor a scale proportionally to a^{-4}
    - Relativistic Particles
  - Black Holes (acts more or less like dark matter, but forms in other ways)
  - Cosmological constant "in a box" does not scale; that is, it is a constant, so it does not change
- Curvature in the Universe
- Entropy in the Universe

- "Information"

- Formal Momenta Redshift:
  - Photon redshift $E \propto \frac{1}{a}$, $\lambda \propto a$

- Energy Evolution
  - We know: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$
  - Working in a fluid's rest frame, $U^\mu = (1, 0, 0, 0)$, which gives us:

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Dark Energy: For Λ, $\rho_\Lambda = \text{constant}$
  - We may obtain, from a formal General Relativity derivation:

$$\rho = \rho_o a^{-3(1+w)}$$

  - We may also obtain:

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\rho_{m,o}a^{-3} + \rho_{r,o}a^{-4} + \rho_{\lambda,o} - 2\kappa a^{-2}\right]$$

    * Note that, once we have Λ, a closed universe won't recollapse
  - From the critical density, we write the ratio as:

$$\Omega_{i,o} = \frac{\rho_{ip}}{\rho_{crit}}$$

  - This allows us to rewrite the Hubble parameter as:

$$\frac{H(a)}{H_o} = \left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}\right)^{\frac{1}{2}}$$

  - We can then write in terms of the redshift:

$$\frac{H(z)}{H_o} = \left(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda + \Omega_\kappa(1+z)^2\right)^{\frac{1}{2}}$$

  - Furthermore, we see:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa = 1$$

  - There are several important naming conventions:

    * $\Omega_m = 1$: Flat, matter-only "Einstein-deSitter Universe"
    * $\Omega_\kappa = 0$: Flat
    * $\Omega_\Lambda = 1$: deSitter Universe
    * $\Omega_\Lambda = -1$: Anti-deSitter Universe
    * $\Omega_\kappa = 1$: Empty

- How Do We Measure Distances?
  - Angular Size (Object seems larger if it is closer)
  - Noise (Object seems louder if it is closer)
  - Light (Object seems brighter if it is closer)

- Distances:
  - Comoving (radial) distance: follow a photon as it travels from a distant source

$$ds = 0 \rightarrow dt = a(t)\frac{dr}{(1-\kappa r^2)^{\frac{1}{2}}}$$

$$\chi = \int_{t_{em}}^{t_o} \frac{dt}{a(t)} = \int_0^r \frac{dr'}{(1-\kappa r'^2)^{\frac{1}{2}}}$$

  - We will redefine the r coordinate such that χ is always r:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right]$$

    * The term becomes:

$$S_\kappa(r) = \begin{cases} r, & \Omega_\kappa = 0 \quad \text{(flat)} \\ H_o^{-1}(\Omega_\kappa)^{-\frac{1}{2}}\sinh\left(H_o(\Omega_\kappa)^{\frac{1}{2}}r\right), & \Omega_\kappa > 0 \quad \text{(open)} \\ H_o^{-1}(|\Omega_\kappa|)^{-\frac{1}{2}}\sin\left(H_o(-\Omega_\kappa)^{\frac{1}{2}}r\right), & \Omega_\kappa < 0 \quad \text{(closed)} \end{cases}$$

  - We may rewrite χ in terms of a to see:

$$\chi(a) = \int_{a(t_{em})}^{a(t_o)=1} \frac{da}{a^2 H(a)}$$

- The Horizon:

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H(a)} \xrightarrow{\text{EdS Universe}} \int_0^1 \frac{da}{a^2 H_o a^{-\frac{3}{2}}} = \frac{2}{H_o}$$

  - Note if we restore c:

$$\chi_{hor} = \frac{2c}{H_o}$$

- Age of the Universe
  - We imagine an empty universe ($\Omega_\kappa = 1$):

$$\int_0^1 da = \int_0^{t_o} H_o \, dt$$
$$1 = t_o H_o$$
$$t_o = \frac{1}{H_o} \quad \text{(Hubble time)}$$

  - Similarly, for a matter-dominated universe ($\Omega_m = 1$):

$$\int_0^1 a^{\frac{1}{2}} \, da = \int_0^{t_o} H_o \, dt$$
$$\frac{2}{3} a^{\frac{3}{2}} = t_o H_o$$
$$t_o = \frac{2}{3H_o}$$

- Luminosity Distance
  - "Standard Candles"
  - Luminosity is Energy per time
  - Flux is the Energy per Area per time
  - In normal three dimensions:

$$L = F(4\pi R^2)$$

    * This means:

$$F \propto \frac{L}{R^2}$$

  - In comoving coordinates, we may write:

$$F \propto \frac{L}{\chi^2} \cdot \frac{1}{1+z} \cdot \frac{1}{1+z}$$

  - We may thus find that, in a flat universe:

$$d_L = \chi(1+z)$$

  - In a non-flat universe, we see:

$$d_L = S_\kappa(\chi)(1+z)$$

- Angular Diameter Distance
  - Object of physical size l
    - Assume a flat universe:

$$l = d_A\theta$$
$$d_A^{flat} = \chi a = \frac{\chi}{1+z} = \frac{d_L}{(1+z)^2}$$
$$d_A = aS_\kappa(\chi)$$

    - Importantly, d_A can increase, reach a max, and then decrease
      - This means physical l takes up larger fraction of a smaller universe
    - This is slightly more complicated in a non-flat universe, since we need to properly account for $d\Omega$ factor
- Evidence for Dark Energy
  - Until the early 1990's, people did not take Λ too seriously. Evidence from galaxy clusters, galaxy clustering, and CMB started to raise some questions. Universe appeared roughly flat, but $\Omega_m < 1$
  - Mapping out $a(t)$ has been a useful way to probe the universe
- Dark Matter
  - In 1781, William Herschel discovered Uranus; over the next 60 years, astronomers carefully mapped out it orbit, but it didn't quite match Newtonian theory
  - Urbain Le Verrier: Showed that Uranus' orbit could be explained if there were another, more distant planet acting gravitationally
  - In September 1846, he mailed a letter to a colleague at the Berlin Observatory with precise predictions
  - In 1930s, Fritz Zwicky studied the Coma Cluster
    - Galaxy Clusters: Largest gravitationally bound objects in the Universe
  - Zwicky measured spectra of many galaxies and calculated the velocity dispersion
  - With a quick calculation, the orbital velocity of a galazy cluster may be expressed as:

$$M = \frac{r_c v^2}{\alpha G}$$

  - MACHOs (Massive Compact Halo Objects)
    - Planet or asteroid-sized objects: "microlensing" constraints mostly rule this out

- Axions
- Weakly-Interacting Massive Particles (WIMPs)
- Primordial Black Holes (PBHs)
- CDM: Cold Dark Matter → ΛCDM model
- Can generalize wCDM such that:

$$p_{DE} = w\rho_{DE}$$

  - Λ: $w = -1$

$$w(a) = w_o + w_a(a - a_o)$$

- What is Dark Energy?
  - "Cosmological Constant Problem"
  - Universe expansion is accelerating; Λ can do this
  - Quantum field theory says that vacuum fluctuations have an energy density
- Hot Big Bang
  - Thermal Equilibrium

$$\frac{n_1}{n_2} = e^{-(E_1 - E_2)/kT}$$

    - State 1: E_1
    - State 2: E_2
- Photons
  - Photons are bosons
  - This means they follow the Bose-Einstein distribution:

$$\bar{n}_i = \frac{g_i}{e^{(\epsilon_i - \mu)/kT} - 1} \Rightarrow \rho_i = \bar{n}_i \epsilon_i$$

    - g_i is the "degeneracy"
    - This expresses how many particles are in a given energy state i
  - To calculate the energy density, we need to consider the "phase space" $\{\vec{x}, \vec{p}\}$ and integrate over all states with a given $|p|$
  - Heisenberg's uncertainty principle: $d^3x d^3p$ has $\frac{d^3x d^3p}{(2\pi\hbar)^3}$ phase space elements. For photons, $\mu = 0$:

$$\rho = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{2p}{e^{p/kT} - 1}$$

    * We may obtain:

$$\rho(f)\,df = \frac{8\pi h}{c^3}\left(\frac{f^3\,df}{e^{hf/kT}-1}\right)$$

    * For photons between f and $f + df$

  - Blackbody Spectrum
    * Blackbody: Perfectly absorptive system which emits a spectrum given by photon thermal equilibrium

$$n_c(f_c) = \frac{8\pi}{c^3}\frac{f_c^2}{e^{hf_c/akT}-1}$$

      · Preserved if $T \to T/a$

  - Temperature of the universe is really defined by the distribution of particles in thermal equilibrium

- Cosmic Microwave Background
  - In an expanding universe, we required:

$$\Gamma > H$$

    * $\Gamma \propto n_2 v_1 \sigma_{12}$
    * H is the Hubble rate, which is the inverse of Hubble time
    * "Freeze-out" n_i is fixed

- Big Bang Nucleosynthesis (BBN): "The First Three Minutes"
  - Fission versus Fusion
  - Binding energy: Energy to break apart nucleus or, equivalently, energy to release forming the nucleus

- Matter-Antimatter Asymmetry

$$\eta = \frac{n_b}{n_\gamma} = 6 \cdot 10^{-10} \Rightarrow 6 \text{ baryons per 10 billion photons}$$

  - Standard model (mostly) treats particles and antiparticles the same

$$e^- + e^+ \leftrightarrow \gamma + \gamma$$

  - Not due to freeze out: we don't see equal amounts of anti-matter; annihilation was complete for anti-matter

  - At early time, $kT >> 150$[MeV] (quarks, baryons): some asymmetry was present

$$\frac{n_q - n_{q^-}}{n_q} \approx 4 \cdot 10^{-9}$$

  - All of the anti-quarks will annihilate, leaving the value of η from above
  - 2 photons per annihilation, 3 quarks per baryon

- Baryogenesis (and probably related "leptogenesis") is one of the big open questions in cosmology and particle physics

- It may be possible within the Standard Model, but it's not clear how. The Sakharov conditions (1967) for baryogenesis state:

  1. Baryon number violation
  2. Charge and charge parity-symmetry violation
  3. Out of thermal equilibrium

- WIMP Dark Matter and Freeze-out

  - Weakly interacting massive particles (WIMPs, $M \geq 100$[GeV]) has been a very popular DM candidate. New particle could be the result of supersymmetry between fermions and bosons

- Recombination and Decoupling

  - Discusses when universe became neutral enough to (at least mostly) stop photon scattering
  - Recombination: When ionization fraction is 1/2
  - Decoupling: When $H \approx \Gamma_T$, photons no longer interact through Thomson scattering
  - Last scattering: THe time (or surface) when the typicaly photon last scattered
  - Not really a surface, more of a narrow shell (think: analogy of looking into fog, you can see a little way in)

- We may observe the Saha solution for ionization fraction as:

$$\frac{\chi_e^2}{1-\chi_e} = \frac{1}{n_e + n_H} \left(\frac{m_e T}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{\epsilon_o}{T}}$$

  - When $\chi_e \to 1$, everything is still ionized
  - For recombination, we want $\chi_e \approx 1/2$
  - The Saha equation assumes chemical equilibrium; when not in equilibrium, use the Boltzmann equation

- CMB Fluctuations
  - We may recall that:

$$\frac{\Delta T}{T} \approx 10^{-5}$$

- Fourier versus Real Space (Called Configuration Space in Cosmology)
  - Configuration space in position (function of x), Fourier space defined with respect to wave number k
  - We may write (in 3D):

$$F(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}} \tilde{F}(\vec{k})$$

  - Plane waves may be expressed by the wave number, $\vec{k} = (k_x, k_y, k_z)$
    * A higher k means smaller-scale fluctuations
    * Plane wave pointing in $\hat{k}$ direction with $\lambda = \frac{2\pi}{k}$

- Correlation Functions
  - Correlation functions are incredibly important in cosmology because we can predict the statistics of the initial conditions, but the initial conditions themselves are random
  - A related concept is the power spectrum

- Sound Waves
  - Start with an initial overdenstiy: $\delta_c = \delta_b = \delta_\gamma$
  - Before z_*, this creates a travelling compression wave
  - At z_*, this wave has travelled a comoving distance:

$$\text{comoving sound horizon: } \eta_s^* = \int_0^{t_*} \frac{dt}{a} c_s(t)$$
$$\text{physical sound horizon: } ds = \eta_s^* a$$

  - For a purely relativistic gas, $c_s = \sqrt{\frac{1}{3}}c$ (and we drop the c in our units)

- The Sachs-Wolfe Effect (1967)

$$\frac{\delta T}{T} = \frac{1}{3}\frac{\delta \Phi}{c^2}$$

  - Overdensity means $|\Phi|$, the gravitational redshift, increases

- The integrated Sachs-Wolfe Effect (ISW) caused by changing potentials after z_* (bot due to radiation $\to$ matter and matter $\to \Lambda$)

- Flatness Problem
  - The universe looks close to flat today. This means it must have been very flat early on

- Monopole Problem
  - GUT phase transition leaves "defects" $\to$ roughyl oneper χ_{GUT}

- Initial conditions:
  - Why does the universe start expanding? Where do the fluctuations come from?
  - $t \to 0$, $a \to 0$, $T \to \infty$: avoid singularity! (Quantum gravity is also relevant here)

- Inflation
  - Horizon, Conformal Time, Hubble Distance
    * χ is the comoving distance
    * We may write:

$$\chi(a_1, a_2) = \int_{a_1}^{a_2} \frac{da}{a^2 H(a)}$$

    * Typically, we measure from us today ($a = 1$)
    * The comoving horizon today is the distance to $a = 0$ (or $a \to 0$)

$$\chi_h = \chi(a = 0) = \int_0^1 \frac{da}{a^2 H(a)}$$

    * Often, the horizon gets a new variable:

$$\eta(a) = \chi(a = 0, a)$$

    * η is also sometimes called the conformal time
    * It is monotonically increasing, perfectly good time variable
    * We can rewrite the FRW metric:

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \frac{dr^2}{1 - \kappa r^2} + r^2\, d\Omega^2\right]$$

    * Overall scaling of the metric: "conformal transformation"
    * The horizon expresses global causality; how far could light have travelled since $t = 0$?
    * Our cosmic event horizon is the distance from which light could ever readh us out to $t \to \infty$

        - For a ΛCDM universe, this is <u>not</u> infinite
    - Developed by Guth, Starobinsky, and Linde (1979-1981), originally as a way to solve the monopole problem
    - ρ_{inf} comes from some new field in the universe
    - This is a scalar field (think electric field, $\vec{E}$ carries energy density E^2)
    - Changing $\vec{E} \to \vec{B}$ makes total $\rho \propto E^2 + B^2$ (the "potential" and "kinetic" energy in the field)

- Galaxy Formation
    - Dark matter and baryons collapse to form a halo
    - Baryons cool through radiation and fall to center of the halo, forming a galazy
    - Formatiom looks simple from a statistical perspective (linear galaxy bias):

$$\delta_g = b\delta_m$$

        - The overdensity in galaxies is proportional to the overdensity of dark matter
        - Overdensity is expressed as:

$$\delta = \frac{\rho - \bar{\rho}}{\rho}$$

- Redshift Space Distortions
    - Strength of effect given by:

$$f \equiv \frac{d\ln(D)}{d\ln(a)}$$

        - More growth $\to$ larger peculiar velocities

- Gravitational Lensing
    - All geodesics are impacted by mass/energy density, including photons
    - The path of light is bent $\to$ gravitational lensing
    - For a point source: deflection
    - For an extended source (*e.g* a galaxy), you also get distortions:
        - Shear
        - Magnification
    - Recall geodesic deviation

$$A^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

- Regimes of Lensing (NB: All within "weak field" limit; differences depend on geometry of observation)
  - Strong Lensing: Multiple images, arcs, Einstein rings, strong magnification
    $$\theta \leq \theta_E$$
  - Weak Lensing: Small distrotions, must be statistically studied
    $$\theta > \theta_E$$
  - Microlensing: Unresolved strong lensing (*e.g.* by an exoplanet)
- Lensing Derivations
  - Assuming that the curvature is small (weak field), we can write the metric perturbatively:
    $$g_{\mu\nu} = g^{(o)}_{\mu\nu} + h_{\mu\nu}$$
  - Initially, we can assume that $g^{(o)}_{\mu\nu} = \eta_{\mu\nu}$; however, for cosmology, we need to remember that $g^{(o)}_{\mu\nu}$ is actually the homogenous FLRW metric
  - Decomposing the perturbation, we get:
    $$\begin{cases} h_{oo} = -2\Phi, & \Phi, \Psi: \text{ scalar perturbations} \\ h_{oi} = w_i, & w_i: \text{ vector perturbations} \\ h_{ij} = 2s_{ij} - 2\Psi\delta_{ij}, & s_{ij}: \text{ tensor perturbations} \end{cases}$$
  - Cranking through some algebra and plugging into Einstein's equation, we get:
    $$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$
    * From this, we find:
      $$\nabla^2\Psi = 4\pi G\rho$$
      $$w_i = s_{ij} = 0$$
      $$\Psi = \Phi$$
    * The last expression is an important General Relativity result, valid only when $\text{Tr}(T_{ij})$=0
  - We may recall that, in the "Newtonian limit" only the h_{oo} was important for geodesics; now we deal with photons, so we need both time and space perturbations. Considering a null geodesic, we write:
    $$x^\mu(\lambda) = x^{(o)\mu}(\lambda) + x^{(1)\mu}(\lambda)$$

- We define:

$$k^\mu = \frac{dx^{(o)\mu}}{d\lambda} \quad \text{and} \quad l^\mu = \frac{dx^{(1)\mu}}{d\lambda}$$

- The null geodesic expression becomes:

$$(\eta_{\mu\nu} + h_{\mu\nu})(k^\mu + l^\mu)(k^\nu + l^\nu) = 0$$

- This can be evaluated to:

$$\frac{dl^\mu}{d\lambda} = -\Gamma^\mu_{\rho\sigma} k^\rho k^\sigma$$

$$\text{Time:} \quad \frac{dl^o}{d\lambda} = -2k(\vec{k} \cdot \vec{\nabla}\Phi)$$

$$\text{Space:} \quad \frac{d\vec{l}}{d\lambda} = -2k^2 \vec{\nabla}_\perp \Phi$$

- We want to find the deflection angle, $\hat{\alpha}$, which can be expressde as:

$$\frac{\Delta \vec{l}}{k} = \Delta\theta \equiv \hat{\alpha}$$

- Thus, we can get:

$$\Delta \vec{l} = \int \frac{d\vec{l}}{d\lambda} \, d\lambda = -2k^2 \int \vec{\nabla}_\perp \Phi \, d\lambda$$

- We can thus write $\hat{\alpha}$ as:

$$\hat{\alpha} = 2 \int \vec{\nabla}_\perp \Phi \, ds$$

- We obtain the lens equation as:

$$\vec{\beta} = \vec{\theta} - \vec{\alpha} = \vec{\theta} - \frac{D_{LS}}{D_S} \hat{\alpha}$$

- For a point mass, we substitute to get:

$$\beta = \theta - \frac{D_{LS}}{D_L D_S} \frac{4GM}{\theta}$$

  * Considering perfect lens-source alignment ($\beta = 0$), we get the Einstein angle:

$$\theta_E = \sqrt{\frac{4GM D_{LS}}{D_L D_S}}$$

* From here, we get the Einstein radius:

$$R_E = D_L \theta_E = \sqrt{\frac{4GMD_L D_{LS}}{D_S}}$$

* For imperfect lens-source alignment, we get:

$$\theta^2 - \beta\theta - \theta_E^{=}0$$

$$\theta = \frac{1}{2}\left[\beta \pm \sqrt{\beta^2 - 4\theta_E^2}\right]$$

* When $\beta >> \theta_E$, we have:

$$\theta = \frac{1}{2}\left[\beta \pm \beta\left(1 + \frac{2\theta_E}{\beta}\right)\right]$$

* This gets us:

$$\theta_+ = \beta + \frac{\theta_E^2}{\beta}$$

$$\theta_- = -\frac{\theta_E^2}{\beta} \to 0$$