Lecture 6 — The Expanding Universe

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- Olber's Paradox: Why is the Sky Dark?
  - Absorbing Matter $\rightarrow$ doesn't work since matter would heat up
  - Finite Size
  - Finite Time (and Finite Speed of Light)
  - Dimming of Light ("redshift")
- Universe had some beginning ("Big Bang") around 13.7 billion years ago
- Some units:

$$1[\text{light year}] = 9.5 \cdot 10^{15}[\text{m}]$$
$$1[\text{yr}] \approx \pi \cdot 10^{7}[\text{s}]$$
$$c \approx 3 \cdot 10^{8}\left[\frac{\text{m}}{\text{s}}\right]$$
$$1[\text{pc}] = 3.26[\text{light years}]$$

  - If $\theta = 1[\text{arcsec}]$, then $d = 1[\text{pc}]$

$$1[\text{pc}] = 2.1 \cdot 10^{5}[\text{AU}]$$

- The Cosmological Principle
  - Copernicus: the Sun, not the Earth, is the center of the Universe
  - Cosmological Principle: There is no center to the Universe
    * The Universe is statistically isotropic (same in all directions) and homogenous (same everywhere)
- Expanding Universe

- All observers see things moving away from them
- Statements are all statistical! Distinguish between structure in the universe and the geometry of the homogenous universe (about 100[Mpc] scales for homogeneity)
- We don't experience the FLRW metric
  * Homogenous/geometry
  * Structure
- Conservation of Energy Solution to Expanding Cloud:

$$\frac{1}{2}\dot{R}^2 - \frac{2GM}{R} = C$$

  * What is the physical meaning of C?
    - $C = 0$
      You are just at escape velocity. As $R \to \infty$, $\dot{R} = v \to 0$. Potential and kinetic both go to zero.
    - $C > 0$
      Positive total energy. You have more than enough energy to escape. $\dot{R} > 0$ as $R \to \infty$
    - $C < 0$
      Negative total energy. You won't make it out to $R = \infty$, you will stop and turn around at some finite t
  * These cases capture an ideal universe's expansion with only real matter
  * For a matter-only universe, C describes the spatial geometry, with zero indicating flat, $C > 0$ indicating open (negative curvature), and $C < 0$ indicating closed (positive curvature)
    - Note that Λ complicates this
- Using the first-order equation, we may write:

$$\frac{1}{2}\dot{R}^2 = \frac{4}{3}\pi G\rho R^2 + C$$
$$\left(\frac{\dot{R}}{R}\right) = \frac{8\pi G\rho}{3} + C$$

- At $C = 0$, we find ρ_{crit}:

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

  Note that we define the Hubble parameter as:

$$H = \frac{\dot{R}}{R}$$

- Comoving Coordinates

- Coordinates (and distances) scale with the size of the universe:

$$x = a(t)r$$

  * x is the "proper" coordinate
  * r is the "comoving" coordinate
  * $a(t)$ is the scale factor, with $a(t_o) = 1$ indicating "today"

- Ex. $x_{12} = a(t)r_{12}$

$$\frac{dx_{12}}{dt} \quad \text{is the recession velocity}$$

- Two Things:
  1. Velocity is proportional to distance (Hubble law)
  2. The proportionality term is $\dfrac{\dot{a}}{a} = H(t)$

- Hubble constant is approximately:

$$H_o = 70 \left[\frac{\text{km}}{\text{sMpc}}\right]$$

  * We define h, such that:

$$H_o = 100h \left[\frac{\text{km}}{\text{sMpc}}\right]$$

  * We can see that the units are actually just Hz, which gives us:

$$H_o \approx 3.24 \cdot 10^{-18} \left[\frac{1}{\text{s}}\right]$$

- If a is constant, this gives us the doubling time or the time to go back to $a = 0$, which gives us the "Hubble" time, or the age of the universe
- For $h = .7 \to 1.4 \cdot 10^{10} \left[\frac{1}{\text{yr}}\right]$
  * Fairly accurate approximation of 14 billion years

- FLRW Metric
  - Cosmological principle tells us that spacetime (on large scales) should be homogenous (translation invariant) and isotropic (rotation invariant)
  - We seek a general form of a metric that obeys these assumptions:

$$ds^2 = \bar{c}^2\, dt^2 + R^2(t)\, d\sigma^2$$

  - We are able to derive:

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2\, d\Omega^2$$

- Where:
  - $d\Omega^2$ is the 2-sphere metric
  - $k \propto R$ (Ricci scalar on 3D space)

$$k = \begin{cases} +1, & \text{Positive Curvature (3-sphere), "closed"} \\ 0, & \text{Flat} \\ -1, & \text{Negative curvature (saddle), "open"} \end{cases}$$

- Thus, we may define the metric as:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2\, d\Omega^2\right]$$

  - Where:

$$\begin{cases} \kappa > 0, & \text{closed} \\ \kappa = 0, & \text{flat} \\ \kappa < 0, & \text{open} \end{cases}$$

- Cosmological Redshift
  - Like a Döppler shift, but use caution!
  - $v = H_o d$, locally $v << c$
  - Redshift in special relativity:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \sqrt{\frac{1+v/c}{1-v/c}} \approx 1 + \frac{v}{c}$$

  - From this, we know:

$$\frac{\lambda_{obs}}{\lambda_{em}} = 1 + z \approx 1 + \frac{H_o d}{c}$$

  - More generally, we may say (for a_o = today):

$$1 + z = \frac{a_{obs}}{a_{em}} = a_{em}^{-1}$$

- Cosmological Redshift versus Peculiar Motions:
  - For comoving coordinates:

$$v \approx H_o d = H_o|\vec{r}_2 - \vec{r}_1|/a_o = H_o|\vec{r}_2 - \vec{r}_1|$$

  - For "peculiar" motions

$$v \approx H_o d + \frac{\Delta v}{c}$$

- In the FLRW metric, due to lack of simple time symmetry, energy is not conserved
    * There is a Killing Tensor that reflects a symmetry:

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \quad \text{for observer (comoving) } U^\mu = (1,0,0,0)$$

    * We get:

$$K^2 = K_{\mu\nu}V^\mu V^\nu$$

        · Is conserved, with

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

        · For a photon on a null geodesic:

$$V_\mu V^\mu = 0$$

    * This can be simplified to:

$$v = \frac{K}{a}$$

- "Stuff" That Can fill A Universe
  - "Baryons" → All standard model particles with mass (interact with light, gravity)
    * Non-photon force carriers
    * "In a box" with scale factor a scale proportionally to a^{-3}
  - Dark matter → Only interacts through gravity (?)
    * Neutrinos (?)
    * New particles
    * "Cold" dark matter (non-relativistic), scales "in a box" with scale factor a proportionally to a^{-3}
  - Dark energy ("cosmological constant")
  - Radiation
    * Photons
        · "In a box" with scale factor a scale proportionally to a^{-4}
    * Relativistic Particles
  - Black Holes (acts more or less like dark matter, but forms in other ways)
  - Cosmological constant "in a box" does not scale; that is, it is a constant, so it does not change

- Curvature in the Universe

- Entropy in the Universe

  - “Information”

- Formal Momenta Redshift:
  - Photon redshift $E \propto \frac{1}{a}$, $\lambda \propto a$

- Energy Evolution
  - We know: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$
  - Working in a fluid’s rest frame, $U^\mu = (1, 0, 0, 0)$, which gives us:

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  - Dark Energy: For Λ, $\rho_\Lambda =$ constant
  - We may obtain, from a formal General Relativity derivation:

$$\rho = \rho_o a^{-3(1+w)}$$

  - We may also obtain:

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left[\rho_{m,o}a^{-3} + \rho_{r,o}a^{-4} + \rho_{\lambda,o} - 2\kappa a^{-2}\right]$$

    * Note that, once we have Λ, a closed universe won’t recollapse
  - From the critical density, we write the ratio as:

$$\Omega_{i,o} = \frac{\rho_{ip}}{\rho_{crit}}$$

  - This allows us to rewrite the Hubble parameter as:

$$\frac{H(a)}{H_o} = \left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}\right)^{\frac{1}{2}}$$

  - We can then write in terms of the redshift:

$$\frac{H(z)}{H_o} = \left(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_\Lambda + \Omega_\kappa(1+z)^2\right)^{\frac{1}{2}}$$

  - Furthermore, we see:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa = 1$$

  - There are several important naming conventions:

- $\Omega_m = 1$: Flat, matter-only "Einstein-deSitter Universe"
- $\Omega_\kappa = 0$: Flat
- $\Omega_\Lambda = 1$: deSitter Universe
- $\Omega_\Lambda = -1$: Anti-deSitter Universe
- $\Omega_\kappa = 1$: Empty

- How Do We Measure Distances?
  - Angular Size (Object seems larger if it is closer)
  - Noise (Object seems louder if it is closer)
  - Light (Object seems brighter if it is closer)

- Distances:
  - Comoving (radial) distance: follow a photon as it travels from a distant source

$$ds = 0 \rightarrow dt = a(t)\frac{dr}{(1-\kappa r^2)^{\frac{1}{2}}}$$

$$\chi = \int_{t_{em}}^{t_o} \frac{dt}{a(t)} = \int_0^r \frac{dr'}{(1-\kappa r'^2)^{\frac{1}{2}}}$$

  - We will redefine the r coordinate such that χ is always r:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_k^2(r)d\Omega^2\right]$$

    - The term becomes:

$$S_\kappa(r) = \begin{cases} r, & \Omega_\kappa = 0 \quad \text{(flat)} \\ H_o^{-1}(\Omega_\kappa)^{-\frac{1}{2}} \sinh\left(H_o(\Omega_\kappa)^{\frac{1}{2}} r\right), & \Omega_\kappa > 0 \quad \text{(open)} \\ H_o^{-1}(|\Omega_\kappa|)^{-\frac{1}{2}} \sin\left(H_o(-\Omega_\kappa)^{\frac{1}{2}} r\right), & \Omega_\kappa < 0 \quad \text{(closed)} \end{cases}$$

  - We may rewrite χ in terms of a to see:

$$\chi(a) = \int_{a(t_{em})}^{a(t_o)=1} \frac{da}{a^2 H(a)}$$

- The Horizon:

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H(a)} \xrightarrow{\text{EdS Universe}} \int_0^1 \frac{da}{a^2 H_o a^{-\frac{3}{2}}} = \frac{2}{H_o}$$

  - Note if we restore c:

$$\chi_{hor} = \frac{2c}{H_o}$$

- Age of the Universe
  - We imagine an empty universe ($\Omega_\kappa = 1$):

$$\int_0^1 da = \int_0^{t_o} H_o\, dt$$
$$1 = t_o H_o$$
$$t_o = \frac{1}{H_o} \qquad \text{(Hubble time)}$$

  - Similarly, for a matter-dominated universe ($\Omega_m = 1$):

$$\int_0^1 a^{\frac{1}{2}}\, da = \int_0^{t_o} H_o\, dt$$
$$\frac{2}{3} a^{\frac{3}{2}} = t_o H_o$$
$$t_o = \frac{2}{3H_o}$$

- Luminosity Distance
  - "Standard Candles"
  - Luminosity is Energy per time
  - Flux is the Energy per Area per time
  - In normal three dimensions:

$$L = F(4\pi R^2)$$

    * This means:

$$F \propto \frac{L}{R^2}$$

  - In comoving coordinates, we may write:

$$F \propto \frac{L}{\chi^2} \cdot \frac{1}{1+z} \cdot \frac{1}{1+z}$$

  - We may thus find that, in a flat universe:

$$d_L = \chi(1+z)$$

  - In a non-flat universe, we see:

$$d_L = S_\kappa(\chi)(1+z)$$

- Angular Diameter Distance
  - Object of physical size l
    - Assume a flat universe:

$$l = d_A \theta$$
$$d_A^{flat} = \chi a = \frac{\chi}{1+z} = \frac{d_L}{(1+z)^2}$$
$$d_A = a S_\kappa(\chi)$$

    - Importantly, d_A can increase, reach a max, and then decrease
      - This means physical l takes up larger fraction of a smaller universe
    - This is slightly more complicated in a non-flat universe, since we need to properly account for $d\Omega$ factor

- Evidence for Dark Energy
  - Until the early 1990's, people did not take Λ too seriously. Evidence from galaxy clusters, galaxy clustering, and CMB started to raise some questions. Universe appeared roughly flat, but $\Omega_m < 1$
  - Mapping out $a(t)$ has been a useful way to probe the universe

- Dark Matter
  - In 1781, William Herschel discovered Uranus; over the next 60 years, astronomers carefully mapped out it orbit, but it didn't quite match Newtonian theory
  - Urbain Le Verrier: Showed that Uranus' orbit could be explained if there were another, more distant planet acting gravitationally
  - In September 1846, he mailed a letter to a colleague at the Berlin Observatory with precise predictions
  - In 1930s, Fritz Zwicky studied the Coma Cluster
    - Galaxy Clusters: Largest gravitationally bound objects in the Universe
  - Zwicky measured spectra of many galaxies and calculated the velocity dispersion