Cosmology Review

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- The Cosmological Principle
  - The universe is homogenous and isotropic (statistically the same everywhere, and in every direction)
  - Not formally true, but it is *statistically* true
  - Lets us make assumptions when solving the Einstein equation
  - Also, lets us arrive at the FLRW Metric (the only consistent solution to Einstein equations)

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]$$

    * For $\kappa > 0$, we have a closed universe
    * For $\kappa = 0$, we have a flat universe
    * For $\kappa < 0$, we have an open universe
    * Our universe is near flat, so we usually don't need to consider this; however, it is good to know that the FLRW metric can be used with non-flat universes
    * Spacetime is curved in 4-dimensions
  - Analogy to Newtonian Cosmology: Expanding Sphere
    * Given a sphere with some density ρ expanding at velocity v, if the density and velocity are balanced, the sphere stops at $t \to \infty$ (in a flat universe with a critical density)
    * If ρ is larger, this corresponds to a closed universe that will turn around
    * If ρ is smaller, this corresponds to an open universe, which will keep expanding
  - Once we introduce dark energy (Λ), the Newtonian analogy breaks down
  - $a(t)$ is referred to as the scale factor, which captures the expansion of the universe (relates comoving coordinates with physical coordinates)

$$x = a(t) \cdot r$$

    * Where x is the physical (proper) coordinate, and r is the comoving coordinate

  - Thus, we arrive at the Hubble expansion law:

$$v_{12} = \frac{\dot{a}}{a} \cdot x_{12} \equiv H(a) \cdot x_{12}$$
$$H(a) = \frac{\dot{a}}{a}$$

  - "Today" corresponds to $a = 1$, where:

$$H(a = 1) = H(t = t_o) = H_o$$

  - Using this to solve the Einstein equations, we arrive at the Friedmann equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{2\kappa}{a^2}$$

- Energy Density Evolution

  - We model energy density in the homogenous universe as the sum of various perfect fluids

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & g_{ij}p \end{pmatrix}$$

  - In the case of zero pressure (dust) we get:

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$$

  - We treat dark matter and normal matter as dust, since $v << c$
  - The density of such components scales as:

$$\rho = \rho_o a^{-3(1+w)}$$

    * Where w is the equation of state, $p = w\rho$
    * $w = 0$ for dust ($\rho = \rho_o a^{-3}$)
    * $w = 1/3$ for radiation ($\rho = \rho_o a^{-4}$)
    * $w = -1$ for Λ ($\rho = \rho_o$)

  - Redshifting of radiation

$$E_{photon}(a) = \frac{E_o}{a}$$

* Formally, this comes from the redshift of momenta

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(a) = \frac{8\pi G}{3}\left(\rho_{m,o}a^{-3} + \rho_{r,o}a^{-4} + \rho_{\Lambda,o} - 2\kappa a^{-2}\right)$$

* If we want $\kappa = 0$, today $a = 1$, which gives us:

$$H^2(a = 1) = H_o^2 = \frac{8\pi G}{3}\rho_{tot}$$

  · Thus, the critical density (the density required today for there to be no curvature) is defined as:

$$\rho_{crit} = \frac{3H_o^2}{8\pi G}$$

* Remember, the scale factor and redshift are related by:

$$a = \frac{1}{1+z}$$

* We define:

$$\Omega = \frac{\rho}{\rho_{crit}}$$

* This lets us define:

$$\frac{H(a)}{H_o} = \sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}}$$

* Incorporating the equation of state for dark energy:

$$\frac{H(a)}{H_o} = \sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_{DE} a^{-3(1+w)} + \Omega_\kappa a^{-2}}$$

* Also, we may write:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa = 1$$
$$\Omega_m + \Omega_r + \Omega_{DE} + \Omega_\kappa = 1$$

– Important Universe Types:

* $\Omega_\kappa = 0 \rightarrow$ flat
* $\Omega_m = 1 \rightarrow$ Einstein-deSitter (flat, matter only)
* $\Omega_\Lambda = 1 \rightarrow$ deSitter
* $\Omega_\kappa = 1 \rightarrow$ empty

• Distances

– Comoving radial distance

$$\chi = \int_{t_{em}}^{t_o} \frac{dt}{a(t)} = \int_o^r \frac{dr}{\sqrt{1-\kappa r^2}}$$

* For various κ:
  * $\kappa > 0$ (closed):

$$\chi = \kappa^{-\frac{1}{2}} \sin\left(\kappa^{\frac{1}{2}} r\right)$$

  * $\kappa < 0$ (open):

$$\chi = \kappa^{-\frac{1}{2}} \sinh\left(\kappa^{\frac{1}{2}} r\right)$$

  * $\kappa = 0$ (flat):

$$\chi = r$$

* We redefine the metric with $S_\kappa(r)$ to relate the above, so we get:

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + S_\kappa^2(r) d\Omega^2\right]$$

  * Note that, for a flat universe, this S function equals r

- We may obtain:

$$\chi(a) = \int_{a_{em}}^{a_o=1} \frac{da}{a^2 H(a)} \to \chi(z)$$

- The horizon distance may be written as:

$$\chi_{hor} = \int_{a=0}^{1} \frac{da}{a^2 H(a)}$$

  * Note that for an Einstein-deSitter universe, we get:

$$\chi_{hor} = \frac{2}{H_o} \to \frac{2c}{H_o}$$

- We can calculate the age of the universe using:

$$\frac{\dot{a}}{a} = H_o \sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}}$$

- We can break $\dot{a}$ into its differential form, and multiply the dt over to solve for the age of the universe
  * For example, in an empty universe ($\Omega_\kappa = 1$), we see:

$$\frac{da}{dt} = aH_o(a^{-2})$$
$$\int_0^1 a\, da = \int_0^t {}_oH_o\, dt$$
$$1 = H_o t_o$$
$$t_o = \frac{1}{H_o}$$

    * In a matter-only universe, we see:

$$t_o = \frac{2}{3H_o}$$

- Luminosity Distance
  - Relate intrinsic luminosity to observed luminosity:

$$F \approx \frac{L}{d_L^2}$$

  - We derived:

$$d_L = \chi(1+z) = \chi/a \quad \text{(flat universe)}$$
$$d_L = \chi S_k(1+z) \quad \text{(in general)}$$

  - The angular diameter distance can be written as:

$$\theta \approx l/d_A$$
$$d_A = \frac{\chi}{1+z} = \frac{d_L}{(1+z)^2} \quad \text{(flat universe)}$$
$$d_A = \frac{S_k \chi}{1+z} \quad \text{(in general)}$$

  - Luminosity distance used to discover Dark Energy:

$$SNI_a \rightarrow \text{"standard candles"}$$

    * For Ω_Λ: d_L is larger $\Rightarrow$ objects are fainter

- Dark Matter
  - Zwicky: velocities in clusters were too large
  - Rubin: rotation of galaxies were too large
  - Both of the above indicate some kind of missing mass $\rightarrow$ dark matter
  - What is dark matter?
    * We don't actually know
    * Came up with weakly-interacting massive particles (WIMPs) as a possible candidate
    * Could also be axions
    * Neutrinos (new "sterile" neutrinos)
    * Black holes?
    * We combine this together to determine the concordance model: ΛCDM

- $\cdot$ Dark energy (Λ) with cold dark matter
- $\cdot$ Could also generalize to some dark energy with $w < -1/3$

- Hot Big Bang and Thermal Processes
  - Universe started off very hot and dense
    - Expansion makes it less dense and cooler
  - Photons have a blackbody spectrum with temperature T

$$T \propto \frac{1}{a}$$

  - Processes start in equilibrium
  - As the universe expands, interaction rate (Γ) drops
    - When $\Gamma \leq H$, we experience freeze out (matter can not find a partner to interact with, and remains as is)
  - The temperature today is approximately $T_o = 2.726$[K] (microwaves)
  - Kinetic equilibrium $\rightarrow$ particles follow the DE, FD distribution
    - Can be assumed as always true
  - Chemical equilibrium $\rightarrow 1 + 2 \leftrightarrow 3 + 4$ is in equilibrium
  - Our assumptions are:
    1. Kinetic equilibrium
    2. $E\mu > T$ (ignore quantum effects ± 1)
  - The Boltzmann Equation is:

$$a^{-3}\frac{d}{dt}(n_1 a^3) = n_1^{(o)} n_2^{(o)} < \sigma_v > \left\{ \frac{n_3 n_4}{n_3^{(o)} n_4^{(o)}} - \frac{n_1 n_2}{n_1^{(o)} n_2^{(o)}} \right\}$$

    - Where:

$$n_i^{(o)} = \begin{cases} g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{-m_i/T}, & m_i >> T \text{ (non-relativistic)} \\ g_i \left(\frac{T^3}{\pi^2}\right), & m_i << T \text{ (relativistic)} \end{cases}$$

  - Chemical equilibrium occurs when:

$$\frac{n_1 n_2}{n_1^{(o)} n_2^{(o)}} = \frac{n_3 n_4}{n_3^{(o)} n_4^{(o)}}$$

    - Freeze out occurs when:

$$n_2^{(o)} < \sigma_v > << H$$

– We assume l_{sm} remains in chemical equilibrium, $n_1 = n_2 = n_x$, and $n_3 = n_4 = n_l = n_l^{(o)}$ to get:

$$a^{-3}\frac{d}{dt}(n_x a^3) =< \sigma_v > \left\{(n_x^{(o)})^2 - n_x^2\right\}$$

– Recombination:

* Particle one is an electron (e^-), particle two is a proton (p^+), particle three is a hydrogen (H), and particle four is radiation (γ):

$$e^- + p^+ \leftrightarrow H + \gamma$$

· This gives us:

$$\frac{n_e n_p}{n_e^{(o)} n_p^{(o)}} = \frac{n_H n_\gamma}{n_H^{(o)} n_\gamma^{(o)}}$$

$$\frac{n_e n_p}{n_H} = \frac{n_e^{(o)} n_p^{(o)}}{n_H^{(o)}}$$

· Per our non-relativistic particle formula, we may obtain:

$$\chi_e = \frac{n_e}{n_e + n_H} \rightarrow \frac{\chi_e^2}{1 - \chi_e} = \left(\frac{1}{n_e + n_H}\right)\left(\frac{m_e T}{2\pi}\right)^{\frac{3}{2}} e^{-m_e/T}$$