Homework 2

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1. Per the affine connection, we may use the Christoffel Symbol to write:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\alpha\rho}\left[\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right]$$

We are given the metric for polar coordinates as:

$$ds^2 = dr^2 + r^2\, d\theta^2$$

Which gives us $g^{rr} = 1$ and $g^{\theta\theta} = r^{-2}$.

(a) We can begin with what Carroll supplied:

$$\boxed{\begin{cases} \Gamma^r_{rr} & = 0 \\ \Gamma^r_{\theta\theta} & = -r \\ \Gamma^r_{\theta r} = \Gamma^r_{r\theta} & = 0 \\ \Gamma^\theta_{rr} & = 0 \\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} & = (1/r) \\ \Gamma^\theta_{\theta\theta} & = 0 \end{cases}}$$

(b) We can continue to find the divergence of V using the simplified formula:

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}V^\mu\right)$$

Which gives us:

$$\boxed{\nabla\cdot\mathbf{V} = \frac{1}{\sqrt{|g|}}\partial_r\left(\sqrt{|g|}V^r\right) + \frac{1}{\sqrt{|g|}}\partial_\theta\left(\sqrt{|g|}V^\theta\right)}$$

The gradient can be found as:

$$\boxed{\nabla\mathbf{V} = \frac{\partial V}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{e}_\theta}$$

(c) In general, we may write:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

From here, we can expand to:

$$\frac{d^2x^\rho}{d\tau^2} + \frac{1}{2}g^{\rho\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

From (a), we know that $\Gamma^\mu_{\rho\sigma}$ is non-zero for only two combinations, $\Gamma^r_{\theta\theta}$ and $\Gamma^\theta_{r\theta}$. With this, we are able to construct two equations:

$$\frac{d^2x^r}{d\lambda^2} - r\frac{d^2x^\theta}{d\lambda^2} = 0$$
$$\frac{d^2x^\theta}{d\lambda^2} + \frac{2}{r}\frac{dx^r}{d\lambda}\frac{dx^\theta}{d\lambda} = 0$$

This gives us the equations:

$$\boxed{\frac{d^2x^r}{d\lambda^2} = r\frac{d^2x^\theta}{d\lambda^2}}$$
$$\boxed{\frac{d^2x^\theta}{d\lambda^2} = -\frac{2}{r}\frac{dx^r}{d\lambda}\frac{dx^\theta}{d\lambda}}$$

(d) Using the equation for a line, we may write:

$$ax + by = c$$

In polar, this would be equivalent to:

$$ar\cos(\theta) + br\sin(\theta) = c$$

We can differentiate to get:

$$(a\cos(\theta) + b\sin(\theta))\,dr = (ar\sin(\theta) - br\cos(\theta))\,d\theta$$
$$dr = \frac{(ar\sin(\theta) - br\cos(\theta))}{(a\cos(\theta) + b\sin(\theta))}\,d\theta$$

Plugging this into our metric, we get:

$$ds^2 = \left(\frac{(ar\sin(\theta) - br\cos(\theta))}{(a\cos(\theta) + b\sin(\theta))}\,d\theta\right)^2 + r^2\,d\theta$$

2. (a)

(b)

3. (a)
   (b)
4. (a)
   (b)
5.