

Lecture 6 — The Expanding Universe

Michael Brodskiy

Professor: J. Blazek

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- Olber's Paradox: Why is the Sky Dark?
 - Absorbing Matter \rightarrow doesn't work since matter would heat up
 - Finite Size
 - Finite Time (and Finite Speed of Light)
 - Dimming of Light (“redshift”)
- Universe had some beginning (“Big Bang”) around 13.7 billion years ago
- Some units:

$$1[\text{light year}] = 9.5 \cdot 10^{15}[\text{m}]$$

$$1[\text{yr}] \approx \pi \cdot 10^7[\text{s}]$$

$$c \approx 3 \cdot 10^8 \left[\frac{\text{m}}{\text{s}} \right]$$

$$1[\text{pc}] = 3.26[\text{light years}]$$

- If $\theta = 1[\text{arcsec}]$, then $d = 1[\text{pc}]$

$$1[\text{pc}] = 2.1 \cdot 10^5[\text{AU}]$$

- The Cosmological Principle
 - Copernicus: the Sun, not the Earth, is the center of the Universe
 - Cosmological Principle: There is no center to the Universe
 - * The Universe is statistically isotropic (same in all directions) and homogenous (same everywhere)
- Expanding Universe

- All observers see things moving away from them
- Statements are all statistical! Distinguish between structure in the universe and the geometry of the homogenous universe (about 100[Mpc] scales for homogeneity)
- We don't experience the FLRW metric
 - * Homogenous/geometry
 - * Structure
- Conservation of Energy Solution to Expanding Cloud:

$$\frac{1}{2}\dot{R}^2 - \frac{2GM}{R} = C$$

- * What is the physical meaning of C ?
 - $C = 0$
You are just at escape velocity. As $R \rightarrow \infty$, $\dot{R} = v \rightarrow 0$. Potential and kinetic both go to zero.
 - $C > 0$
Positive total energy. You have more than enough energy to escape. $\dot{R} > 0$ as $R \rightarrow \infty$
 - $C < 0$
Negative total energy. You won't make it out to $R = \infty$, you will stop and turn around at some finite t
- * These cases capture an ideal universe's expansion with only real matter
- * For a matter-only universe, C describes the spatial geometry, with zero indicating flat, $C > 0$ indicating open (negative curvature), and $C < 0$ indicating closed (positive curvature)
 - Note that Λ complicates this
- Using the first-order equation, we may write:

$$\begin{aligned}\frac{1}{2}\dot{R}^2 &= \frac{4}{3}\pi G\rho R^2 + C \\ \left(\frac{\dot{R}}{R}\right) &= \frac{8\pi G\rho}{3} + C\end{aligned}$$

- At $C = 0$, we find ρ_{crit} :

$$\rho_{crit} = \frac{3H^2}{8\pi G}$$

Note that we define the Hubble parameter as:

$$H = \frac{\dot{R}}{R}$$

- Comoving Coordinates

- Coordinates (and distances) scale with the size of the universe:

$$x = a(t)r$$

- * x is the “proper” coordinate
 - * r is the “comoving” coordinate
 - * $a(t)$ is the scale factor, with $a(t_o) = 1$ indicating “today”
- Ex. $x_{12} = a(t)r_{12}$

$$\frac{dx_{12}}{dt} \quad \text{is the recession velocity}$$

- Two Things:
 1. Velocity is proportional to distance (Hubble law)
 2. The proportionality term is $\frac{\dot{a}}{a} = H(t)$
- Hubble constant is approximately:

$$H_o = 70 \left[\frac{\text{km}}{\text{sMpc}} \right]$$

- * We define h , such that:

$$H_o = 100h \left[\frac{\text{km}}{\text{sMpc}} \right]$$

- * We can see that the units are actually just Hz, which gives us:

$$H_o \approx 3.24 \cdot 10^{-18} \left[\frac{1}{\text{s}} \right]$$

- If a is constant, this gives us the doubling time or the time to go back to $a = 0$, which gives us the “Hubble” time, or the age of the universe
- For $h = .7 \rightarrow 1.4 \cdot 10^{10} \left[\frac{1}{\text{yr}} \right]$
 - * Fairly accurate approximation of 14 billion years

- FLRW Metric

- Cosmological principle tells us that spacetime (on large scales) should be homogeneous (translation invariant) and isotropic (rotation invariant)
- We seek a general form of a metric that obeys these assumptions:

$$ds^2 = \bar{c}^2 dt^2 + R^2(t) d\sigma^2$$

- We are able to derive:

$$d\sigma^2 = \frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2 d\Omega^2$$

– Where:

- * $d\Omega^2$ is the 2-sphere metric
- * $k \propto R$ (Ricci scalar on 3D space)

$$k = \begin{cases} +1, & \text{Positive Curvature (3-sphere), "closed"} \\ 0, & \text{Flat} \\ -1, & \text{Negative curvature (saddle), "open"} \end{cases}$$

– Thus, we may define the metric as:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right]$$

* Where:

$$\begin{cases} \kappa > 0, & \text{closed} \\ \kappa = 0, & \text{flat} \\ \kappa < 0, & \text{open} \end{cases}$$

- Cosmological Redshift

- Like a Döppler shift, but use caution!
- $v = H_o d$, locally $v \ll c$
- Redshift in special relativity:

$$\frac{\lambda_{obs}}{\lambda_{em}} = \sqrt{\frac{1 + v/c}{1 - v/c}} \approx 1 + \frac{v}{c}$$

– From this, we know:

$$\frac{\lambda_{obs}}{\lambda_{em}} = 1 + z \approx 1 + \frac{H_o d}{c}$$

– More generally, we may say (for $a_o = \text{today}$):

$$1 + z = \frac{a_{obs}}{a_{em}} = a_{em}^{-1}$$

- Cosmological Redshift versus Peculiar Motions:

– For comoving coordinates:

$$v \approx H_o d = H_o |\vec{r}_2 - \vec{r}_1| / a_o = H_o |\vec{r}_2 - \vec{r}_1|$$

– For “peculiar” motions

$$v \approx H_o d + \frac{\Delta v}{c}$$

- In the FLRW metric, due to lack of simple time symmetry, energy is not conserved
 - * There is a Killing Tensor that reflects a symmetry:

$$K_{\mu\nu} = a^2(g_{\mu\nu} + U_\mu U_\nu) \quad \text{for observer (comoving) } U^\mu = (1, 0, 0, 0)$$

- * We get:

$$K^2 = K_{\mu\nu} V^\mu V^\nu$$

- Is conserved, with

$$V^\mu = \frac{dx^\mu}{d\lambda}$$

- For a photon on a null geodesic:

$$V_\mu V^\mu = 0$$

- * This can be simplified to:

$$v = \frac{K}{a}$$

- “Stuff” That Can fill A Universe

- “Baryons” → All standard model particles with mass (interact with light, gravity)
 - * Non-photon force carriers
 - * “In a box” with scale factor a scale proportionally to a^{-3}
- Dark matter → Only interacts through gravity (?)
 - * Neutrinos (?)
 - * New particles
 - * “Cold” dark matter (non-relativistic), scales “in a box” with scale factor a proportionally to a^{-3}
- Dark energy (“cosmological constant”)
- Radiation
 - * Photons
 - “In a box” with scale factor a scale proportionally to a^{-4}
 - * Relativistic Particles
- Black Holes (acts more or less like dark matter, but forms in other ways)
- Cosmological constant “in a box” does not scale; that is, it is a constant, so it does not change

- Curvature in the Universe

- Entropy in the Universe

- “Information”

- Formal Momenta Redshift:

- Photon redshift $E \propto \frac{1}{a}$, $\lambda \propto a$

- Energy Evolution

- We know: $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$

- Working in a fluid’s rest frame, $U^\mu = (1, 0, 0, 0)$, which gives us:

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Dark Energy: For Λ , $\rho_\Lambda = \text{constant}$

- We may obtain, from a formal General Relativity derivation:

$$\rho = \rho_o a^{-3(1+w)}$$

- We may also obtain:

$$H^2(a) = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} [\rho_{m,o} a^{-3} + \rho_{r,o} a^{-4} + \rho_{\Lambda,o} - 2\kappa a^{-2}]$$

* Note that, once we have Λ , a closed universe won’t recollapse

- From the critical density, we write the ratio as:

$$\Omega_{i,o} = \frac{\rho_{ip}}{\rho_{crit}}$$

- This allows us to rewrite the Hubble parameter as:

$$\frac{H(a)}{H_o} = (\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2})^{\frac{1}{2}}$$

- We can then write in terms of the redshift:

$$\frac{H(z)}{H_o} = (\Omega_m (1+z)^3 + \Omega_r (1+z)^4 + \Omega_\Lambda + \Omega_\kappa (1+z)^2)^{\frac{1}{2}}$$

- Furthermore, we see:

$$\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_\kappa = 1$$

- There are several important naming conventions:

- * $\Omega_m = 1$: Flat, matter-only “Einstein-deSitter Universe”
- * $\Omega_\kappa = 0$: Flat
- * $\Omega_\Lambda = 1$: deSitter Universe
- * $\Omega_\Lambda = -1$: Anti-deSitter Universe
- * $\Omega_\kappa = 1$: Empty

- How Do We Measure Distances?

- Angular Size (Object seems larger if it is closer)
- Noise (Object seems louder if it is closer)
- Light (Object seems brighter if it is closer)

- Distances:

- Comoving (radial) distance: follow a photon as it travels from a distant source

$$ds = 0 \rightarrow dt = a(t) \frac{dr}{(1 - \kappa r^2)^{\frac{1}{2}}}$$

$$\chi = \int_{t_{em}}^{t_o} \frac{dt}{a(t)} = \int_0^r \frac{dr'}{(1 - \kappa r'^2)^{\frac{1}{2}}}$$

- We will redefine the r coordinate such that χ is always r :

$$ds^2 = -dt^2 + a^2(t) [dr^2 + S_\kappa^2(r) d\Omega^2]$$

- * The term becomes:

$$S_\kappa(r) = \begin{cases} r, & \Omega_\kappa = 0 \quad (\text{flat}) \\ H_o^{-1}(\Omega_\kappa)^{-\frac{1}{2}} \sinh \left(H_o(\Omega_\kappa)^{\frac{1}{2}} r \right), & \Omega_\kappa > 0 \quad (\text{open}) \\ H_o^{-1}(|\Omega_\kappa|)^{-\frac{1}{2}} \sin \left(H_o(-\Omega_\kappa)^{\frac{1}{2}} r \right), & \Omega_\kappa < 0 \quad (\text{closed}) \end{cases}$$

- We may rewrite χ in terms of a to see:

$$\chi(a) = \int_{a(t_{em})}^{a(t_o)=1} \frac{da}{a^2 H(a)}$$

- The Horizon:

$$\chi_{hor} = \int_0^1 \frac{da}{a^2 H(a)} \xrightarrow{\text{EdS Universe}} \int_0^1 \frac{da}{a^2 H_o a^{-\frac{3}{2}}} = \frac{2}{H_o}$$

- Note if we restore c :

$$\chi_{hor} = \frac{2c}{H_o}$$

- Age of the Universe

- We imagine an empty universe ($\Omega_\kappa = 1$):

$$\begin{aligned}\int_0^1 da &= \int_0^{t_o} H_o dt \\ 1 &= t_o H_o \\ t_o &= \frac{1}{H_o} \quad (\text{Hubble time})\end{aligned}$$

- Similarly, for a matter-dominated universe ($\Omega_m = 1$):

$$\begin{aligned}\int_0^1 a^{\frac{1}{2}} da &= \int_0^{t_o} H_o dt \\ \frac{2}{3} a^{\frac{3}{2}} &= t_o H_o \\ t_o &= \frac{2}{3H_o}\end{aligned}$$

- Luminosity Distance

- “Standard Candles”
- Luminosity is Energy per time
- Flux is the Energy per Area per time
- In normal three dimensions:

$$L = F(4\pi R^2)$$

* This means:

$$F \propto \frac{L}{R^2}$$

- In comoving coordinates, we may write:

$$F \propto \frac{L}{\chi^2} \cdot \frac{1}{1+z} \cdot \frac{1}{1+z}$$

- We may thus find that, in a flat universe:

$$d_L = \chi(1+z)$$

- In a non-flat universe, we see:

$$d_L = S_\kappa(\chi)(1+z)$$

- Angular Diameter Distance

- Object of physical size l

- * Assume a flat universe:

$$l = d_A \theta$$

$$d_A^{flat} = \chi a = \frac{\chi}{1+z} = \frac{d_L}{(1+z)^2}$$

$$d_A = a S_\kappa(\chi)$$

- * Importantly, d_A can increase, reach a max, and then decrease

- This means physical l takes up larger fraction of a smaller universe

- * This is slightly more complicated in a non-flat universe, since we need to properly account for $d\Omega$ factor

- Evidence for Dark Energy

- Until the early 1990's, people did not take Λ too seriously. Evidence from galaxy clusters, galaxy clustering, and CMB started to raise some questions. Universe appeared roughly flat, but $\Omega_m < 1$

- Mapping out $a(t)$ has been a useful way to probe the universe

- Dark Matter

- In 1781, William Herschel discovered Uranus; over the next 60 years, astronomers carefully mapped out its orbit, but it didn't quite match Newtonian theory

- Urbain Le Verrier: Showed that Uranus' orbit could be explained if there were another, more distant planet acting gravitationally

- In September 1846, he mailed a letter to a colleague at the Berlin Observatory with precise predictions

- In 1930s, Fritz Zwicky studied the Coma Cluster

- * Galaxy Clusters: Largest gravitationally bound objects in the Universe

- Zwicky measured spectra of many galaxies and calculated the velocity dispersion

- With a quick calculation, the orbital velocity of a galaxy cluster may be expressed as:

$$M = \frac{r_c v^2}{\alpha G}$$

- MACHOs (Massive Compact Halo Objects)

- * Planet or asteroid-sized objects: "microlensing" constraints mostly rule this out

- Axions
- Weakly-Interacting Massive Particles (WIMPs)
- Primordial Black Holes (PBHs)
- CDM: Cold Dark Matter $\rightarrow \Lambda$ CDM model
- Can generalize w CDM such that:

$$p_{DE} = w\rho_{DE}$$

$$* \Lambda: w = -1$$

$$w(a) = w_o + w_a(a - a_o)$$

- What is Dark Energy?
 - “Cosmological Constant Problem”
 - Universe expansion is accelerating; Λ can do this
 - Quantum field theory says that vacuum fluctuations have an energy density
- Hot Big Bang
 - Thermal Equilibrium

$$\frac{n_1}{n_2} = e^{-(E_1 - E_2)/kT}$$

- * State 1: E_1
- * State 2: E_2

- Photons
 - Photons are bosons
 - This means they follow the Bose-Einstein distribution:

$$\bar{n}_i = \frac{g_i}{e^{(\epsilon_i - \mu)/kT} - 1} \Rightarrow \rho_i = \bar{n}_i \epsilon_i$$
 - * g_i is the “degeneracy”
 - * This expresses how many particles are in a given energy state i
 - To calculate the energy density, we need to consider the “phase space” $\{\vec{x}, \vec{p}\}$ and integrate over all states with a given $|p|$
 - Heisenberg’s uncertainty principle: $d^3x d^3p$ has $\frac{d^3x d^3p}{(2\pi\hbar)^3}$ phase space elements. For photons, $\mu = 0$:

$$\rho = \int \frac{d^3p}{(2\pi\hbar)^3} \frac{2p}{e^{p/kT} - 1}$$

- * We may obtain:

$$\rho(f) df = \frac{8\pi h}{c^3} \left(\frac{f^3 df}{e^{hf/kT} - 1} \right)$$

- * For photons between f and $f + df$

- Blackbody Spectrum

- * Blackbody: Perfectly absorptive system which emits a spectrum given by photon thermal equilibrium

$$n_c(f_c) = \frac{8\pi}{c^3} \frac{f_c^2}{e^{hf_c/akT} - 1}$$

- Preserved if $T \rightarrow T/a$

- Temperature of the universe is really defined by the distribution of particles in thermal equilibrium

- Cosmic Microwave Background

- In an expanding universe, we required:

$$\Gamma > H$$

- * $\Gamma \propto n_2 v_1 \sigma_{12}$

- * H is the Hubble rate, which is the inverse of Hubble time

- * “Freeze-out” n_i is fixed

- Big Bang Nucleosynthesis: “The First Three Minutes”

- Fission versus Fusion

- Binding energy: Energy to break apart nucleus or, equivalentl, energy tou release forming the nucleus