Lecture 2 — Introduction to Differential Geometry

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- Metric $\to$ measuring things ("meter")
- In differential geometry, a metric defines how we calculate distance

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -\Delta\tau^2 \qquad \text{(the metric in Minkowski space)}$$
$$= \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu$$

  - A repeated index (up and down) $\to$ sum
    * Spacetime Vector $\to Greek : 0-3$

$$\vec{x} = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \to x^\mu$$

* Vector $\to$ Latin: 1-3

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \to x^i$$

$$\eta_{00}\Delta x^0 \Delta x^0 + \eta_{01}\Delta x^0 \Delta x^1 + \eta_{02}\Delta x^0 \Delta x^2 + \eta_{10}\Delta x^1 \Delta x^0 + \eta_{11}\Delta x^1 \Delta x^1 + \cdots$$

  - Tensors, index notation:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \eta_{00}\Delta x^0 \Delta x^0 + = \eta_{11}\Delta x^1 \Delta x^1 + = \eta_{22}\Delta x^2 \Delta x^2 + = \eta_{33}\Delta x^3 \Delta x^3$$

    * Summation convention, one up and one down (order does not matter)

- Curved Space Distance
  - We know $d \neq |x_1 - x_2|$
  - $\Delta s^2 = \Delta x^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu \Rightarrow g_{\mu\nu}\Delta x^\mu \Delta x^\nu$, where $g_{\mu\nu}$ is a metric that depends on a radius R (called manifolds, encodes geometry)
  - Note: 1D or 2D analogies are embedded in 3D
  - In differential geometry, we will generally deal with:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

- Revisiting Lorentz Transformations
  - We can write a transformation in two ways:

$$\Lambda = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} \cosh(\phi) & -\sinh(\phi) & 0 & 0 \\ -\sinh(\phi) & \cosh(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  - With four-vectors:

$$\vec{x} = \begin{pmatrix} \gamma t \\ x \\ y \\ z \end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix} \gamma t - vx \\ -vt + \gamma x \\ y \\ z \end{pmatrix}$$

  - Matrix multiplication with indices becomes:

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^\nu$$

- The metric and Lorentz Transformations
  - Δs^2 is invariant under boosts ($x^\mu \rightarrow x^{\mu'}$)
  - $\Delta s^2 = \eta_{\mu\nu}\Delta x^\mu \Delta x^\nu = \eta_{\mu'\nu'}\Delta x^{\mu'} \Delta x^{\nu'} = \eta_{\mu'\nu'}\Lambda^{\mu'}_{\mu}\Delta x^\mu \Lambda^{\nu'}_{\nu}\Delta x^\nu$

- This defines Lorentz Transformations (group)

- What is a group?
  - A set $\{a, b, c, \cdots\}$
    1. Has an operation "$\cdot$"
    2. Operation is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
    3. Set is closed: if $a \cdot b = c$, c is in the group
    4. Contains an identity element: $a \cdot e = e \cdot a = a$
    5. Contains an inverse for all elements: $a \cdot a^{-1} = e$
  - Can be finite or infinite
  - Simple example: integers under addition '
  - Rotations in space are a group and can be represented by matrices with multiplication:

$$R_{z,\theta} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

    * SO(3) "special orthogonal in 3D"
  - Operations don't commute: non-abelian group
  - In 4D, spatial rotations and boosts form the Lorentz transformations (group) and translations $\rightarrow$ Poincaré group

- Vectors and Covectors
  - We already have a concept of vectors:
    * $\vec{v}$ exists at a single point in spacetime in the tangent space T_p
    * Vectors from T_p can not simply be moved to T_q
    * Example: $v^\mu = \dfrac{d}{d\lambda} x^\mu$ tangent to $x^\mu(\lambda)$
  - Vector field: one vector at each spacetime point
  - Vectors are invariant under Λ
    * Example: Wind velocity at every point in space:
      · Changing frames alters components, but not the vector itself (why we "prime" the index)
  - This can be written as:

$$\text{vector} \rightarrow A = \overbrace{A^\mu}^{\text{components}} \underbrace{\hat{e}_{(\mu)}}_{\text{basis vectors}}$$

    - $\hat{e}_{(\mu)}$ does NOT refer to dual vectors
    - (μ) is not a coordinate index

$$\text{unprimed to primed} \rightarrow \Lambda^{\nu'}_{\mu} \longleftrightarrow \Lambda^{\rho}_{\sigma'} \leftarrow \text{primed to unprimed}$$

$$\Lambda^{\nu'}_{\mu} \Lambda^{\rho}_{\sigma'} = \delta^{\rho}_{\mu} \text{ (Kronecker delta)}$$

- Dual Vectors ("One-forms," covariant vectors)
  - A map from vectors to $\mathbb{R}$
  - Ex.

$$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } w = \begin{pmatrix} d & e & f \end{pmatrix}$$

$$w(v) = \begin{pmatrix} d & e & f \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = da + eb + fc$$

    - $w(v)$ is not the dot product, though it is similar in spirit
  - Cotangent space T^*_p
  - Similar basis structure and transformations

$$w = w_\mu \hat{\theta}^{(\mu)}$$

  - where

$$\hat{\theta}^{(\nu)}\left(\hat{e}_{(\mu)}\right) = \delta^{\nu}_{\mu}$$

- The Gradient
  - Recall $\nabla\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi \Rightarrow \frac{\partial\phi}{\partial x}\hat{x} + \frac{\partial\phi}{\partial y}\hat{y} + \frac{\partial\phi}{\partial z}\hat{z}$
  - It is a dual vector

- Tensors
  - A (k, l)-rank tensor maps k dual vectors and l vectors to $\mathbb{R}$

scalar	$(0, 0)$
vector	$(1, 0)$
dual vector	$(0, 1)$
metric	$(0, 2)$

– Tensors obey "multi-linearity"

$$T(a\omega + b\eta, cV + dW) = acT(\omega, V) + adT(\omega, W) + bcT(\eta, V) + bdT(\eta, W)$$

– Tensor Product

$$T_2 = T \otimes S(\omega^(1) \cdots \omega^{(k)}, \cdots \omega^{(k+m)}, V^{(1)} \cdots V^{(l)}, V^{(l+m)})$$
$$= T(\omega^{(1)} \cdots \omega^{(k)}, V^{(1)} \cdots V^{(l)}) \times S(\omega^{(k+1)} \cdots \omega^{(k+m)}, V^{(l+1)} \cdots V^{(l+m)})$$

– Basis for a (k, l) tensor:

$$\hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{(\nu_1)} \otimes \cdots \otimes \hat{\theta}^{(\nu_l)}$$

* μ_i has D values for D dimensions ($D = 4$ for us), 4^{k+l} total basis vectors

$$T = T^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_l} \times \text{(Basis tensors)}$$

– Transformations under Λ (builds from vector transforms)

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l' = \Lambda^{\mu_1'}_{\mu_1} \cdots \Lambda^{\mu_k'}_{\mu_k} \Lambda^{\nu_1}_{\nu_1'} \cdots \Lambda^{\nu_l}_{\nu_l'} T^{\mu_1 \cdots \mu_k}_{\nu_1 \cdots \nu_l'}}$$

– T can act on a subset

- The inner product (dot product)

$$\eta(V, W) = V = \eta_{\mu\nu} V^\mu W^\nu$$

– The metric appears again!

$$\eta^{\mu\nu} \eta_{\nu\sigma} = \delta^\mu_\sigma \text{ (inverse metric)}$$

- Another famous tensor example: E&M Field Strength Tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

- Manipulating Tensors

– Contraction: $S^{\mu\rho}_\sigma = T^{\mu\nu\rho}_{\sigma\nu}$

- Indices are arbitrary, until they are set:

$$T^{\mu\nu\rho} \neq T^{\nu\mu\rho} \text{ because we compare them}$$

- The metric raises/lowers indices:
  * Recall inner product:

$$\eta_{\mu\nu} V^{\mu} U^{\nu} = V_{\nu} U^{\nu}$$

- Symmetric versus Anti-symmetric Tensors:

$$S^{\mu\nu} = S^{\nu\mu} \qquad S_{\sigma}^{\mu\nu} = S_{\mu}^{\sigma\nu}$$
$$A^{\mu\nu} = -A^{\nu\mu} \qquad A_{\sigma}^{\mu\nu} = -A_{\mu}^{\sigma\nu}$$

  * Levi-Civita Symbol (Tensor "density") is anti-symmetric

$$\tilde{\varepsilon}_{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{even perm 0123 (eg 0312)} \\ -1 & \text{odd perm 0123} \end{cases}$$