Homogeneous Linear Systems

Michael Brodskiy
Professor: Meetal Shah

November 23, 2020

- The general solution of the homogeneous system $\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$ is (1)

$$\mathbf{X} = c_1\mathbf{X}_1 + c_2\mathbf{X}_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} \tag{1}$$

- Because the solution vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ have the form (2), where k_1, k_2, λ_1, and λ_2 are constants, we are prompted to ask whether we can always find a solution of the form (3) for the general homogeneous linear first-order system $\mathbf{X}' = \mathbf{AX}$, where $\mathbf{A}$ is an nxn matrix of constants.

$$\mathbf{x}_i = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda_i t},\ i = 1, 2 \tag{2}$$

$$\mathbf{X} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K}e^{\lambda t} \tag{3}$$

- Given the above, $\mathbf{X}' = \mathbf{K}\lambda e^{\lambda t}$, so the system becomes $\mathbf{K}\lambda e^{\lambda t} = \mathbf{AK}e^{\lambda t}$. After dividing $e^{\lambda t}$ and rearranging, we obtain $\mathbf{AK} - \lambda\mathbf{K} = 0$, giving us $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0$

- To figure out whether solutions exist for $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0$, we take $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

- The polynomial equation λ is called the characteristic equation of the matrix $\mathbf{A}$; its solutions are the eigenvalues of $\mathbf{A}$ A solution $\mathbf{K} \neq 0$ of the above corresponding to an eigenvalue, λ is called an eigenvector of $\mathbf{A}$. A solution of the homogeneous system is then $\mathbf{X} = \mathbf{K}e^{\lambda t}$

- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ has n distinct real eigenvalues of the coefficient matrix $\mathbf{A}$ of the homogeneous system and let $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ be the corresponding eigenvectors. Then the general solution of the system on the interval $(-\infty, \infty)$ is given by (4)

$$\mathbf{X} = c_1\mathbf{K}_1 e^{\lambda_1 t} + c_2\mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{K}_n e^{\lambda_n t} \tag{4}$$

- For the two parametrized functions of t, they can be represented together in the xy-plane, or the phase plane. For different values of the arbitrary constants, each new curve is called a trajectory.

- A collection of trajectories is a phase portrait.

- When both lambdas are positive, it is an attractor, whereas, when both lambdas are negative, it is a repeller

- If there is only one eigenvalue corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form (5) exist, where $\mathbf{K}_{ij}$ are column vectors, and can always be found

$$\begin{aligned}
\mathbf{X}_1 &= \mathbf{K}_{11}e^{\lambda_1 t} \\
\mathbf{X}_2 &= \mathbf{K}_{21}te^{\lambda_1 t} + K_{22}e^{\lambda_1 t} \\
&\vdots \\
\mathbf{X}_m &= \mathbf{K}_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + \mathbf{K}_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \cdots + \mathbf{K}_{mm}e^{\lambda_1 t}
\end{aligned} \tag{5}$$

- If λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form (6)

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t} \tag{6}$$

- To confirm (6), we can substitute into the system $\mathbf{X}' = \mathbf{AX}$:

$$(\mathbf{AK} - \lambda_1\mathbf{K})te^{\lambda_1 t} + (\mathbf{AP} + \lambda_1\mathbf{P} - \mathbf{K})e^{\lambda_1 t} \tag{7}$$

- With an eigenvalue of multiplicity three, we obtain (8)

$$\mathbf{X}_3 = \mathbf{K}\frac{t^2}{2}e^{\lambda_1 t} + \mathbf{P}te^{\lambda_1 t} + \mathbf{Q}e^{\lambda_1 t} \tag{8}$$

- The matrices in (8) are represented by $\mathbf{K} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$, $\mathbf{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$, and $\mathbf{Q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$

- To determine $\mathbf{K}$, $\mathbf{P}$, and $\mathbf{Q}$, we apply (9)

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = 0(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{P} = \mathbf{K}(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{Q} = \mathbf{P} \tag{9}$$

- The Solution Process:

1. Determine $\mathbf{A}$ ($\mathbf{X}' = \mathbf{A}\mathbf{X}$)
2. Plug into the equation $\mathbf{A} - \lambda_1\mathbf{I}$, where $\mathbf{I}$ is the identity matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3. Take the determinant of the matrix resulting from $\mathbf{A} - \lambda_1\mathbf{I}$, and set equal to zero. The solution(s) are λ_1, λ_2, $\ldots$, λ_n, respectively.
4. Now, find a system of equations using the various λ_n values, and the formula $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{K} = 0$ to find $\mathbf{K}$. Plug in simple values for k_1, k_2, and k_n to find values that work as solutions for the systems.
5. If the eigenvalues have multiplicity greater than one, solve using the various formulas given in (8) and (9). If the multiplicity is equal to one, solve for each eigenvalue.

- If $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, where $\beta > 0$, then solutions are (10)

$$\mathbf{K}_1 e^{\lambda_1 t} \text{ and } \bar{\mathbf{K}}_1 e^{\bar{\lambda}_1 t} \tag{10}$$

- Furthermore, given $\mathbf{B}_1$ and $\mathbf{B}_2$ are column vectors, linearly independent solutions with complex eigenvalues are (11)

$$\begin{aligned} \mathbf{X}_1 &= [\mathbf{B}_1 \cos\beta t - \mathbf{B}_2 \sin\beta t]e^{\alpha t} \\ \mathbf{X}_2 &= [\mathbf{B}_2 \cos\beta t + \mathbf{B}_1 \sin\beta t]e^{\alpha t} \end{aligned} \tag{11}$$