

Linear Models — Initial-Value Problems

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- This section will focus on several linear dynamical systems, modeled by second-order differential equations
- The function g is variously called the driving function, forcing function, or input of the system
- A solution $y(t)$ of the differential equation on an interval I containing $t = 0$ that satisfies the initial conditions is called the response or output of the system
- A Spring/Mass System:
 1. Newton's Second Law: When a mass m is attached to the lower end of a spring of negligible mass, it stretches the spring by an amount s and attains an equilibrium position or rest position at which its weight W is balanced by the restoring force ks of the spring. Hooke's Law is shown in (1)

$$F = -kx \tag{1}$$

- When the spring is in free motion, or when no external forces act on the system, Newton's second law gives (2)

$$m \frac{d^2x}{dt^2} = -k(x + s) + mg = -kx + mg - ks = -kx \tag{2}$$

- For simple harmonic motion, the differential equation looks like (3), where $\omega^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{3}$$

- The general solution for this type of equation is (4)

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{4}$$

- The period of motion can be found using $T = \frac{2\pi}{\omega}$. The frequency of motion is the inverse of the period, $f = \frac{\omega}{2\pi}$
- The number $\omega = \sqrt{\frac{k}{m}}$ is the circular frequency, in radians per second
- Functions in form (3) are problematic, though, because it is difficult to determine the amplitude. It can be rewritten in form (5) using $A = \sqrt{c_1^2 + c_2^2}$, and ϕ is a phase angle which may be found using $\tan \phi = \frac{c_1}{c_2}$

$$x(t) = A \sin(\omega t + \phi) \quad (5)$$

- Double Spring Systems:

1. Springs in parallel:

- (a) The effective spring constant is $k_{eff} = k_1 + k_2$
- (b) Once k_{eff} is found, the whole process is the same as a single-springed system

2. Springs in series:

- (a) $-k_{eff}(x_1 + x_2) = -k_1x_1 = -k_2x_2$, because the force exerted on each spring is the same
- (b) Simplifying this, we get $k_{eff} = \frac{k_1k_2}{k_1+k_2}$

- Systems with Variable Spring Constants

1. Aging Spring Function is shown in (6)

$$K(t) = ke^{-\alpha t} \quad (6)$$

2. In an environment where the temperature is rapidly decreasing, $K(t) = kt$, and Airy's differential equation can be used to model this: (7)

$$mx'' + ktx = 0 \quad (7)$$

- In damped motion, the object has a retarding force act on it
- For damped motion, the object follows the differential equation (8), and can be rewritten as (9)

$$m \frac{d^2x}{dt^2} = -kx - \beta \frac{dx}{dt} \quad (8)$$

$$\begin{aligned} \frac{d^2x}{dt^2} + \frac{\beta}{m} \frac{dx}{dt} + \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2x &= 0 \\ 2\lambda &= \frac{\beta}{m} \text{ and } \omega^2 = \frac{k}{m} \end{aligned} \quad (9)$$