Operational Properties II

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- The derivative of a transform is: $\mathcal{L}\left\{t^n f(t)\right\} = (-1)^n \frac{d^n}{ds^n} F(s)$
- A convolution (f convolves g) is defined as (1)

$$f * g = \int_0^t f(\tau) g(t-\tau)\, d\tau \tag{1}$$

- For all examples, the Laplace transform of a convolution, or $\mathcal{L}\{f * g\}$, we obtain (2)

$$\mathcal{L}\left\{f * g\right\} = \mathcal{L}\left\{f(t)\right\} \cdot \mathcal{L}\left\{g(t)\right\} = F(s)G(s) \tag{2}$$

- Inversely, the inverse Laplace transform would mean (3)

$$f * g = \mathcal{L}^{-1} F(s)G(s) \tag{3}$$

- When $g(t) = 1$ and $\mathcal{L}\left\{g(t)\right\} = G(s) = \frac{1}{s}$, the convolution theorem implies that the Laplace transform of the integral of f is (4)

$$\begin{gathered} \mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s} \\ \int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \end{gathered} \tag{4}$$

- The Volterra integral equation for $f(t)$ is (5)

$$f(t) = g(t) + \int_0^t f(\tau) h(t-\tau)\, d\tau \tag{5}$$

- For LRC circuits, we could set up an integral equation to solve. This would look like (6), which is known as an integrodifferential equation

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t) \tag{6}$$

- Green's Function Redux states that, given a differential equation, $y'' + ay' + by = f(t)$, with $y(0) = 0, y'(0) = 0$, we attain (7)

$$Y(s) = \frac{F(s)}{s^2 + as + b} \tag{7}$$

- From there, we can use convolution, as $\mathcal{L}^{-1}\left\{\frac{1}{s^2+as+b}\right\} = g(t)$ and $\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$
- The transform of a periodic function becomes (8)

$$\mathcal{L}\left\{f(t)\right\} = \frac{1}{1 - e^{-sT}}\int_0^T e^{-st} f(t)\,dt \tag{8}$$

Example:

$$\text{Given } E(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \le t < 2 \end{cases}, \text{ find } \mathcal{L}\left\{E(t)\right\}$$

$$\mathcal{L}\left\{E(t)\right\} = \frac{1}{1 - e^{-2s}}\int_0^2 e^{-st} E(t)\,dt \tag{9}$$