Homogeneous Linear Systems

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- The general solution of the homogeneous system $\mathbf{X}' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \mathbf{X}$ is (1)

$$\mathbf{X} = c_1 \mathbf{X}_1 + c_2 \mathbf{X}_2 = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} \tag{1}$$

- Because the solution vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ have the form (2), where k_1, k_2, λ_1, and λ_2 are constants, we are prompted to ask whether we can always find a solution of the form (3) for the general homogeneous linear first-order system $\mathbf{X}' = \mathbf{A}\mathbf{X}$, where $\mathbf{A}$ is an nxn matrix of constants.

$$\mathbf{x}_i = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{\lambda_i t},\ i = 1, 2 \tag{2}$$

$$\mathbf{X} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} e^{\lambda t} = \mathbf{K} e^{\lambda t} \tag{3}$$

- Given the above, $\mathbf{X}' = \mathbf{K}\lambda e^{\lambda t}$, so the system becomes $\mathbf{K}\lambda e^{\lambda t} = \mathbf{A}\mathbf{K}e^{\lambda t}$. After dividing $e^{\lambda t}$ and rearranging, we obtain $\mathbf{A}\mathbf{K} - \lambda\mathbf{K} = 0$, giving us $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0$

- To figure out whether solutions exist for $(\mathbf{A} - \lambda\mathbf{I})\mathbf{K} = 0$, we take $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

- The polynomial equation λ is called the characteristic equation of the matrix $\mathbf{A}$; its solutions are the eigenvalues of $\mathbf{A}$ A solution $\mathbf{K} \neq 0$ of the above corresponding to an eigenvalue, λ is called an eigenvector of $\mathbf{A}$. A solution of the homogeneous system is then $\mathbf{X} = \mathbf{K}e^{\lambda t}$

- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ has n distinct real eigenvalues of the coefficient matrix $\mathbf{A}$ of the homogeneous system and let $\mathbf{K}_1, \mathbf{K}_2, \ldots, \mathbf{K}_n$ be the corresponding eigenvectors. Then the general solution of the system on the interval $(-\infty, \infty)$ is given by (4)

$$\mathbf{X} = c_1 \mathbf{K}_1 e^{\lambda_1 t} + c_2 \mathbf{K}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{K}_n e^{\lambda_n t} \tag{4}$$

- For the two parametrized functions of t, they can be represented together in the xy-plane, or the phase plane. For different values of the arbitrary constants, each new curve is called a trajectory.

- A collection of trajectories is a phase portrait.

- When both lambdas are positive, it is an attractor, whereas, when both lambdas are negative, it is a repeller

- If there is only one eigenvalue corresponding to the eigenvalue λ_1 of multiplicity m, then m linearly independent solutions of the form (5) exist, where $\mathbf{K}_{ij}$ are column vectors, and can always be found

$$\begin{aligned} \mathbf{X}_1 &= \mathbf{K}_{11}e^{\lambda_1 t} \\ \mathbf{X}_2 &= \mathbf{K}_{21}te^{\lambda_1 t} + K_{22}e^{\lambda_1 t} \\ &\vdots \\ \mathbf{X}_m &= \mathbf{K}_{m1}\frac{t^{m-1}}{(m-1)!}e^{\lambda_1 t} + \mathbf{K}_{m2}\frac{t^{m-2}}{(m-2)!}e^{\lambda_1 t} + \cdots + \mathbf{K}_{mm}e^{\lambda_1 t} \end{aligned} \tag{5}$$

- If λ_1 is an eigenvalue of multiplicity two and that there is only one eigenvector associated with this value. A second solution can be found of the form (6)

$$\mathbf{X}_2 = \mathbf{K}te^{\lambda_1 t} + \mathbf{P}e^{\lambda_1 t} \tag{6}$$

- To confirm (6), we can substitute into the system $\mathbf{X}' = \mathbf{AX}$:

$$(\mathbf{AK} - \lambda_1\mathbf{K})te^{\lambda_1 t} + (\mathbf{AP} + \lambda_1\mathbf{P} - \mathbf{K})e^{\lambda_1 t} \tag{7}$$