Orthogonal Functions

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- Properties of inner (dot) product:
  1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
  2. $\langle k\mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{v}, \mathbf{u} \rangle$, where k is a scalar
  3. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if $\mathbf{u} = 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ if $\mathbf{u} > 0$
  4. $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$

- The inner product of two functions f_1 and f_2 on an interval $[a, b]$ is a number given by (1)

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx \tag{1}$$

- Two functions f_1 and f_2 are orthogonal if (2) is true

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x)\, dx = 0 \tag{2}$$

- A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is said to be orthogonal on an interval $[a, b]$ if (3) and $m \neq n$

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0 \tag{3}$$

- The square norm of a function ϕ_n is $||\phi_n(x)||^2 = (\phi_n, \phi_n)$, meaning that the norm, or its generalized length, is $||\phi_n(x)|| = \sqrt{(\phi_n, \phi_n)}$

- The above means that (4)

$$\begin{gathered} ||\phi_n(x)||^2 = \int_a^b \phi_n^2(x)\, dx \\ ||\phi_n(x)|| = \sqrt{\int_a^b \phi_n^2(x)\, dx} \end{gathered} \tag{4}$$

- If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the additional property that $||\phi_n(x)|| = 1$ for $n = 0, 1, 2 \ldots$, then $\{\phi_n(x)\}$ is said to be an orthonormal set on the interval
- The norm of $\phi_0(x) = 1$ is $||\phi_0(x)|| = \sqrt{2\pi}$
- The process of normalizing a function set consists of dividing each function by its norm
- Given the components c_i where $i = 1, 2, 3$, $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$, each component may be found using (5)

$$\begin{aligned} c_1 &= \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{||v_1||^2} \\ c_2 &= \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{||v_2||^2} \\ c_3 &= \frac{\langle \mathbf{u}, \mathbf{v}_3 \rangle}{||v_3||^2} \end{aligned} \tag{5}$$

- In inner product notation, each component may be found using (6)

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n \phi_n(x) \\ c_n &= \frac{\int_a^b f(x)\phi_n(x)\, dx}{||\phi_n(x)||^2} \end{aligned} \tag{6}$$

- A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a, b]$ if (7), where $w(x)$ is usually greater than zero

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n \tag{7}$$