Linear Algebra 3.3 Homework

Michael Brodskiy
Instructor: Prof. Knight

1. (a) $|\mathbf{A}| = -2(-2) - 1(4) = 0$
   (b) $|\mathbf{B}| = -1(1) - 0(1) = -1$
   (c) $\mathbf{AB} = \begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix}$
   (d) $|\mathbf{AB}| = -2(6) - 4(3) = 0$

$$|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}| \checkmark$$

3. (a) $|\mathbf{A}| = -1(-1) - 2(0) + 1(1) = 2$
   (b) $|\mathbf{B}| = -1(2)(3) = -6$
   (c) $\mathbf{AB} = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 0 & 3 \\ 0 & 2 & 0 \end{bmatrix}$
   (d) $|\mathbf{AB}| = -12$

$$|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}| \checkmark$$

5. (a) $|\mathbf{A}| = 3$
   (b) $|\mathbf{B}| = 6$
   (c) $\mathbf{AB} = \begin{bmatrix} 6 & 3 & -2 & 2 \\ 2 & 1 & 0 & -1 \\ 9 & 4 & -3 & 8 \\ 8 & 5 & -4 & 5 \end{bmatrix}$
   (d) $|\mathbf{AB}| = 18$

$$|\mathbf{A}||\mathbf{B}| = |\mathbf{AB}| \checkmark$$

7. $5^2\left(1(-4) - 2(3)\right) = -250$

9. $3^3\left(-1(15-16) - 2(10-12) + 3(8-9)\right) = 54$

11. $2^3\left((15-16)+2(-10+12)+3(8-9)\right)=0$

13. $5^4\left(\begin{vmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{vmatrix}\right) = 5^4\left(\begin{vmatrix} -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 1 \end{vmatrix}\right) = -3125$

19. $5(8)-4(10)=0$. The matrix is singular.

23. $1(8)(0)(2)=0$. The matrix is singular.

25. (a) $\mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} \Rightarrow \frac{8}{25}-\frac{3}{25}=\frac{1}{5}$

(b) $\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 4(2)-3(1)=5 \Rightarrow \frac{1}{5}$

$$|\mathbf{A}^{-1}| = \tfrac{1}{|\mathbf{A}|}\ \checkmark$$

29. (a)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & -2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 1 & -3 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & -5 & -1 & 1 & 0 & 0 \\ 0 & 0 & 4 & -7 & -2 & 0 & 1 & 0 \\ 0 & -3 & 2 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{7}{4} & -\frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \frac{5}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{5}{6} & 0 & 0 & \frac{1}{6} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{3}{8} & \frac{7}{8} & -\frac{5}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{8} & -\frac{5}{8} & \frac{7}{8} & 0 \\ 0 & 1 & 0 & 0 & \frac{5}{12} & \frac{5}{12} & -\frac{1}{4} & -\frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{3}{8} & \frac{7}{8} & -\frac{5}{8} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\det\left(\mathbf{A}^{-1}\right) = -\frac{1}{3}\left[-\frac{1}{8}\left(-\frac{7}{16}+\frac{5}{16}\right)+\frac{5}{8}\left(-\frac{3}{16}+\frac{5}{16}\right)+\frac{7}{8}\left(\frac{3}{16}-\frac{7}{16}\right)\right] = \frac{1}{24}$$

(b) $|\mathbf{A}| = -3\left[2(2-9)-(1-6)+(-3+4)\right] = 24 \Rightarrow \frac{1}{24}$

$$|\mathbf{A}^{-1}| = \tfrac{1}{|\mathbf{A}|}\ \checkmark$$

31. $1+6=7$. The determinant does not equal zero, so there is a solution.

33.

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & -1 & 1 \\ 3 & -2 & 2 \end{vmatrix} = 1(-2+2)+1(4-3)+1(-4+3)$$
$$= 0$$
$$\det = 0, \text{ no solution}$$

37.

$$\begin{aligned}(k-1)(k-2)-6 &= k^2-3k-4 \\ (k-4)(k+1) &= 0 \\ k &= -1, 4\end{aligned}$$

41.

$$\begin{vmatrix} 0 & k & 1 \\ k & 1 & k \\ 1 & k & 0 \end{vmatrix} = -k(0-k)+1(k^2-1)$$
$$2k^2-1=0$$
$$k = \pm\sqrt{\frac{1}{2}}$$

47. (a) $|\mathbf{A}^\intercal| = |\mathbf{A}| \Rightarrow 2(-1-12)+5(8+3) = 29$

(b) $|\mathbf{A}^2| = |\mathbf{A}|^2 \Rightarrow 29^2 = 841$

(c) $|\mathbf{A}\mathbf{A}^\intercal| = |\mathbf{A}||\mathbf{A}^\intercal| = |\mathbf{A}|^2 \Rightarrow 29^2 = 841$

(d) $|2\mathbf{A}| = 2^{\mathrm{ord}(\mathbf{A})}|\mathbf{A}| \Rightarrow 2^3 \cdot 29 = 232$

(e) $|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} \Rightarrow \frac{1}{29}$

65. (a)

$$\begin{aligned}\det(\mathbf{A}) &= x\\ \det\left(\mathbf{A}^{-1}\right) &= y\\ \det\left(\mathbf{A}^{-1}\right) &= \frac{1}{\det(\mathbf{A})}\\ \det(\mathbf{A}) \cdot \frac{1}{\det(\mathbf{A})} &= xy = 1\end{aligned}$$

(b)

$$\begin{aligned}\det(\mathbf{A}) \text{ and } \det\left(\mathbf{A}^{-1}\right) &\text{ are integers}\\ \therefore\ x \text{ and } y &\text{ are integers too}\end{aligned}$$

(c)

$$\begin{aligned}x = \det(\mathbf{A}) &= -1 \text{ or } 1\\ \text{Because those are the only integer solutions to } xy &= 1\\ \therefore x = y &= \pm 1\end{aligned}$$

75.

$$\begin{aligned}\mathbf{A}^{\intercal} &= \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\\ &\frac{1}{-1-1}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\\ \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} &\neq \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\\ \therefore \mathbf{A} &\text{ is not orthogonal}\end{aligned} \tag{1}$$

80.

$$\begin{aligned}\text{If orthogonal: } \mathbf{A}^{\intercal} &= \mathbf{A}^{-1}\\ \det(\mathbf{A}) &= \det(\mathbf{A}^{\intercal})\\ \det(\mathbf{A}) &= \frac{1}{\det(\mathbf{A}^{\intercal})}\\ x = \det(\mathbf{A}) = \det(\mathbf{A}^{\intercal}) &= \det\left(\mathbf{A}^{-1}\right)\\ x &= \frac{1}{x}\\ x^2 &= 1\\ x &= \pm 1\end{aligned} \tag{2}$$