Rank of a Matrix

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- If $\mathbf{A}$ is an $m \times n$ matrix: $\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$, then row space = subspace of $\mathbb{R}^n$ spanned by rows $\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2, \dots, \vec{\mathbf{r}}_m$ and column space = subspace of $\mathbb{R}^m$ spanned by columns $\vec{\mathbf{c}}_1, \vec{\mathbf{c}}_2, \dots, \vec{\mathbf{c}}_n$

  1. Example: $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

     (a) The row space is the subspace of $\mathbb{R}^3$ spanned by $\{(1,0,1),(2,1,0)\}$

     (b) The column space is the subspace of $\mathbb{R}^2$ spanned by $\{(1,2),(0,1),(1,0)\}$

- Let dim($\mathbf{A}$)=dim($\mathbf{B}$)=$m \times n$ such that $\mathbf{A}$ is row equivalent to $\mathbf{B}$, then the row space of $\mathbf{A}$ = the row space of $\mathbf{B}$

- If $\mathbf{A}$ is row equivalent to $\mathbf{B}$, where $\mathbf{B}$ is in row-echelon form, then the non-zero row vectors of $\mathbf{B}$ form a basis for the row space of $\mathbf{A}$

- If $\mathbf{A}$ and $\mathbf{B}$ are row equivalent matrices, then a collection of columns of $\mathbf{A}$ are linearly independent or dependent iff corresponding columns of $\mathbf{B}$ are linearly independent or dependent, respectively

- To find a basis for a row space:

  – Reduce $\mathbf{A}$

  – Take non-zero rows in reduced form to create

- To find a basis for a column space:

  1. Reduce $\mathbf{A}$ to $\mathbf{B}$
  2. Take columns in $\mathbf{A}$ that correspond to identity columns in $\mathbf{B}$ to form a basis

- The dimension of the column space should equal the dimension of the row space, which is the rank of $\mathbf{A}$
- If $\mathbf{A}$ is an $n \times n$ matrix, then the following are equivalent:
    1. $\det(\mathbf{A}) \neq 0$
    2. $\mathbf{A}\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for any $n \times 1$ column matrix $\overrightarrow{b}$
    3. $\mathbf{A}\overrightarrow{x} = \overrightarrow{0}$ has a trivial solution only
    4. $\mathbf{A}$ is invertible
    5. $\mathbf{A}$ is a product of elementary matrices
    6. $\mathbf{A}$ is row equivalent to I_n
    7. The rank of $\mathbf{A}$ is n
    8. The n rows of $\mathbf{A}$ are linearly independent
    9. The n columns of $\mathbf{A}$ are linearly independent
- Null Space of a Matrix:
    1. If $\mathbf{A}$ is an $m \times n$ matrix, then set of all solutions of homogeneous system of linear equations $\mathbf{A}\overrightarrow{x} = \overrightarrow{0}$ is a subspace of $\mathbb{R}^n$ called the null space of A. That is, $N(\mathbf{A}) = \left\{\overrightarrow{x} \in \mathbb{R}^n | \mathbf{A}\overrightarrow{x} = \overrightarrow{0}\right\}$. The dimension of the null space is the nullity of $\mathbf{A}$
- In an $m \times n$ matrix, the nullity plus the rank equals n