Augmented Matrices and Elementary Row Operations

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- Matrix $-$ An ordered array of numbers
- Examples of a Matrix
  1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (2x2 Matrix)
  2. $\begin{bmatrix} 9 & 0 & e \end{bmatrix}$ (1x3 Matrix)
  3. $\begin{bmatrix} 2 & 4 \\ \pi & 6 \\ 3 & 0 \end{bmatrix}$ (3x2 Matrix)
  4. Augmented Matrix: $\left[\begin{array}{cc|c} 1 & -2 & 6 \\ 2 & 3 & -2 \end{array}\right]$
- Elementary Row Operations:
  1. Interchange Rows ($R_i \leftrightarrow R_j$)
  2. Multiply a Row by a non-zero Constant ($kR_i \to R_i$)
  3. Add a Multiple of one Row to Another ($kR_i + R_j \to R_j$)
  4. Notation: $\widetilde{R}$ (Row Equivalent)
- Solving (Get $\widetilde{R}$ so that some entries are 0s)

- Example: $\begin{array}{r} x - 2y = 6 \\ 2x + 3y = -2 \end{array}$

$$\left[\begin{array}{cc|c} 1 & -2 & 6 \\ 2 & 3 & -2 \end{array}\right]$$

$$-2R_1 \to R_1$$

$$\left[\begin{array}{cc|c} -2 & 4 & -12 \\ 2 & 3 & -2 \end{array}\right]$$

$$R_1 + R_2 \to R_2 \text{ and } -\frac{1}{2}R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & -2 & 6 \\ 0 & 7 & -14 \end{array}\right]$$

$$\frac{1}{7}R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & -2 & 6 \\ 0 & 1 & -2 \end{array}\right]$$

$$y = -2$$

$$x - 2(-2) = 6 \longrightarrow 2$$

$$S = \{(2, -2)\}$$

- Organization

  1. $\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = c_1 & (E_1) \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = c_2 & (E_2) \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = c_3 & (E_3) \end{array}$

  2. $\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & c_1 \\ a_{21} & a_{22} & a_{23} & c_2 \\ a_{31} & a_{32} & a_{33} & c_3 \end{array}\right]$

- Example: $\begin{array}{r} x_1 - 2x_2 + x_3 = -1 \\ 2x_1 - 3x_2 + x_3 = -4 \\ 3x_1 - 4x_2 + 2x_3 = -3 \end{array}$

$$
\left[\begin{array}{ccc|c}
1 & -2 & 1 & -1 \\
2 & -3 & 1 & -4 \\
3 & -4 & 2 & -3
\end{array}\right]
$$

$$
\begin{aligned}
-2R_1 \widetilde{+ R_2} &\to R_2 \\
-3R_1 \widetilde{+ R_3} &\to R_3 \\
-2R_2 \widetilde{+ R_3} &\to R_3
\end{aligned}
$$

$$
\left[\begin{array}{ccc|c}
1 & -2 & 1 & -1 \\
0 & 1 & -1 & -2 \\
0 & 0 & 1 & 4
\end{array}\right]
$$

$$
\begin{aligned}
x_3 &= 4 \\
x_2 &= (4) - 2 \to 2 \\
x_1 &= 2(2) - 4 - 1 \to -1 \\
S &= \{(-1, 2, 4)\}
\end{aligned}
$$

- Gaussian Elimination and Gauss Jordan Elimination

  1. Reduced Row-Echelon Form
     (a) If a row does not consist entirely of 0s, and the first non-zero element in the row is a 1, then it is called a leading one.
     (b) For any two successive nonzero rows, the leading 1 in the lower row is farther to the right than the leading 1 in the higher row.
     (c) All the rows consisting entirely of 0s are at the bottom of the matrix. If the fourth property is also satisfied, a matrix is said to be in reduced row-echelon form:
     (d) Each column that contains a leading 1 has 0s everywhere else (above and below)
     (e) If properties $a-c$ are met, but d is not, then the matrix is in just Row Echelon Form
  2. A Matrix in Reduced Row-Echelon Form: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right]$

- For Gaussian Elimination

  1. Put Augmented Matrix in Row-Echelon Form
  2. Back Substitution

- For Gauss-Jordan Elimination

  1. Put Augmented Matrix in Reduced Row-Echelon Form

2. Back Substitution

- Homogeneous Systems:

  1. All constants are equal to zero
  2. Ex: $\begin{array}{l} 2x_1 - 3x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \end{array}$
  3. $x_i = 0$ is always a solution, and is called trivial (never inconsistent)
  4. If the number of variables in a homogeneous system is greater than the number of equations, then the system always has a nontrivial solution[1]

[1] Note: if the number of variables is less than or equal to the number of equations, then it may or may not have a nontrivial solution