Spanning Sets and Linear Independence

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- Linear Combinations $-$ If $\overrightarrow{\mathbf{w}} = a\overrightarrow{\mathbf{u}} + b\overrightarrow{\mathbf{v}}$, then $\overrightarrow{\mathbf{w}}$ is a linear combination of $\overrightarrow{\mathbf{u}}$ and $\overrightarrow{\mathbf{v}}$. $\overrightarrow{\mathbf{w}} = a\overrightarrow{\mathbf{u}} + b\overrightarrow{\mathbf{v}}$ is a plane (it spans a plane).

- Definition: Let $\overrightarrow{\mathbf{v}} \in \mathbf{V}$. Then $\overrightarrow{\mathbf{v}}$ is a linear combination of $\overrightarrow{\mathbf{u}}_1, \overrightarrow{\mathbf{u}}_2, \dots, \overrightarrow{\mathbf{u}}_n$ if $\exists$ scalars $c_1, c_2, \dots, c_n$ such that $\overrightarrow{\mathbf{v}} = c_1\overrightarrow{\mathbf{u}}_1 + c_2\overrightarrow{\mathbf{u}}_2 + \cdots + c_n\overrightarrow{\mathbf{u}}_n$

  1. ex. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a combination of $a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ (other solutions exist)

- The Span of a set of vectors: Let $S = \left\{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \dots, \overrightarrow{\mathbf{v}}_n\right\}$. Then, the span of S is a set of all linear combinations of vectors in S (i.e. span(S)$=\left\{c_1\overrightarrow{\mathbf{v}}_1 + c_2\overrightarrow{\mathbf{v}}_2 + \cdots + \overrightarrow{\mathbf{v}}_n\right\}$). Note: when span($S$)$=\mathbf{V}$, it means $\mathbf{V}$ is spanned by $\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \dots, \overrightarrow{\mathbf{v}}_n$

  1. ex. $\mathbb{R}^2 = \text{span}\left\{\hat{i}, \hat{j}\right\}, \mathbb{R}^3 = \text{span}\left\{\hat{i}, \hat{j}, \hat{k}\right\}$

- The span of S is always a subspace of $\mathbf{V}$, because closure is automatic.

- Linear Independence $-$ A set of vectors $S = \left\{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \dots, \overrightarrow{\mathbf{v}}_n\right\}$ in vector space $\mathbf{V}$ is linearly independent if vector equation $c_1\overrightarrow{\mathbf{v}}_1 + c_2\overrightarrow{\mathbf{v}}_2 + \cdots + c_n\overrightarrow{\mathbf{v}}_n = 0$ has trivial solution only ($c_1 = c_2 = c_n = 0$). Otherwise, it is linearly dependent.

  1. ex. in $\mathbb{R}^2$: $\begin{aligned} 2x + 3y = 1 \\ 4x + 6y = 2 \end{aligned}$
  2. ex. in $\mathbb{R}^3$: $\overrightarrow{\mathbf{u}}_1 = \langle 2, 1, 0\rangle$, $\overrightarrow{\mathbf{u}}_2 = \langle 3, 5, -2\rangle$, $\overrightarrow{\mathbf{u}}_3 = \langle 5, 6, -2\rangle$, then $\overrightarrow{\mathbf{u}}_3 = \overrightarrow{\mathbf{u}}_2 + \overrightarrow{\mathbf{u}}_1$

- $S = \left\{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2, \dots, \overrightarrow{\mathbf{v}}_k\right\}, k \geq 2$ is linearly dependent iff $\overrightarrow{\mathbf{v}}_i$ can be written as a linear combination of other vectors in S.

  1. $\left\{\overrightarrow{\mathbf{o}}\right\}$ is linearly dependent
  2. $\left\{\overrightarrow{\mathbf{v}}_1, \overrightarrow{\mathbf{v}}_2\right\}$ is linearly dependent iff $\overrightarrow{\mathbf{v}}_2 = c\overrightarrow{\mathbf{v}}_1$
  3. ex. Will 4 vectors in $\mathbb{R}^3$ be linearly independent? No, they will always be dependent.
  4. ex. Can 4 vectors span $\mathbb{R}^3$? Yes, they can span $\mathbb{R}^3$.