Inner Product Spaces

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- Let $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ be vectors in vector space V, and c is a scalar. The Inner Product on V is a function that associates a real number $\langle\overrightarrow{u},\overrightarrow{v}\rangle$ with each pair of vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ and satisfies the following axioms:
    1. $\langle\overrightarrow{u},\overrightarrow{v}\rangle = \langle\overrightarrow{v},\overrightarrow{u}\rangle$
    2. $\langle\overrightarrow{u},\overrightarrow{v}+\overrightarrow{w}\rangle = \langle\overrightarrow{u},\overrightarrow{v}\rangle + \langle\overrightarrow{u},\overrightarrow{w}\rangle$
    3. $c\langle\overrightarrow{u},\overrightarrow{v}\rangle = \langle c\overrightarrow{u},\overrightarrow{v}\rangle$
    4. $\langle\overrightarrow{v},\overrightarrow{v}\rangle \geq 0$ and $\langle\overrightarrow{v},\overrightarrow{v}\rangle = 0$ iff $\overrightarrow{v} = 0$
- A vector space V with an inner product is an inner product space
- The dot product is the standard inner product
- Standard Inner Products:
    1. $\mathbb{R}^n$: $\langle\overrightarrow{u},\overrightarrow{v}\rangle = \overrightarrow{u}_1\overrightarrow{v}_1 + \overrightarrow{u}_2\overrightarrow{v}_2 + \cdots + \overrightarrow{u}_n\overrightarrow{v}_n$
    2. P_n: $\langle\overrightarrow{u},\overrightarrow{v}\rangle = a_0b_0 + a_1b_1x + \cdots + a_nb_nx^n$
    3. $M_{2,2}$: $\langle\mathbf{A},\mathbf{B}\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$
- On $[a,b]$: $\langle\overrightarrow{f},\overrightarrow{g}\rangle = \int_a^b f(x)g(x)\,dx$
- Properties:
    1. $(1)\langle\overrightarrow{u},\overrightarrow{v}\rangle = \langle\overrightarrow{u},\overrightarrow{v}\rangle$
    2. $\langle\overrightarrow{u},\overrightarrow{v}+\overrightarrow{w}\rangle = \langle\overrightarrow{u},\overrightarrow{w}\rangle + \langle\overrightarrow{u},\overrightarrow{v}\rangle$
    3. $\langle\overrightarrow{u},0\rangle = 0$
- Norm or Distance
    1. $||\overrightarrow{u}|| = \sqrt{\langle\overrightarrow{u},\overrightarrow{u}\rangle}$
    2. $d\langle\overrightarrow{u},\overrightarrow{v}\rangle = ||\overrightarrow{u}-\overrightarrow{v}||$

- Angle Between Vectors:

    1. $\cos(\theta) = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{||\overrightarrow{u}||||\overrightarrow{v}||}, \quad 0 \leq \theta \leq \pi$

- $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal iff $\langle \overrightarrow{u}, \overrightarrow{v} \rangle = 0$
- The projection of $\overrightarrow{u}$ onto $\overrightarrow{v}$, or $\mathrm{proj}_{\overrightarrow{v}} \overrightarrow{u} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{\langle \overrightarrow{v}, \overrightarrow{v} \rangle} \overrightarrow{v} = \frac{\langle \overrightarrow{u}, \overrightarrow{v} \rangle}{||\overrightarrow{v}||} \overrightarrow{v}$