Homework 6

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1. (a) Per the rules of Laplace Transforms, we can convolve two signals by the rule that:

$$y(t) = x_1(t) * x_2(t) \to Y(s) = X_1(s)X_2(s)$$

As such, we may obtain:

$$X_1(s) = \frac{1}{s+4} \quad \text{and} \quad X_2(s) = \frac{1}{s+2}$$

Now, we account for the shifts. We know that, for $x(t) \to x(t - t_o)$ the transform becomes $X(s) \to e^{st_o}X(s)$. Furthermore, we know that for $x(-t) \to X(-s)$. Thus, we find:

$$X_1(s) = \frac{e^{-3s}}{-s+4} \quad \text{and} \quad X_2(s) = \frac{e^{-2s}}{s+2}$$

Multiplying together, we find:

$$\boxed{Y(s) = \frac{e^{-5s}}{(4-s)(s+2)}, \quad \text{ROC: } -2 < \sigma < 4}$$

(b) We may write this integral as the convolution of two functions:

$$x_1(t) = \cos(2t)e^{-2t}$$
$$x_2(t) = u(-3t+1)$$

We now work to use the multiplicative property of the Laplace transform:

$$\mathcal{L}\{x_1(t)\} = \frac{s+2}{(s+2)^2+4}$$
$$\mathcal{L}\{x_2(t)\} = \frac{e^{-\frac{1}{3}s}}{s}$$

We multiply the two together to get:

$$\boxed{Y(s) = \frac{(s+2)e^{-\frac{1}{3}s}}{(s)[(s+2)^2+4]}}$$

Note that we may also use the last property of Laplace transforms on our equation sheet to write:

$$\int_t^{\infty} x(\tau)\, d\tau = \frac{1}{s}X(s)$$

This gives us the same result:

$$\boxed{Y(s) = \frac{(s+2)e^{-\frac{1}{3}s}}{(s)[(s+2)^2+4]}}$$

2. First, we know that the poles must be at plus or minus the imaginary value, so the two poles must be at $s = -1 \pm 3j$. Thus, we see that $X(s)$ can be expressed as:

$$X(s) = \frac{k}{(s+1-3j)(s+1+3j)}$$
$$X(s) = \frac{k}{(s+1)^2+3^2}$$

We then apply the condition given in statement (5) to get:

$$2 = \frac{k}{(1^2)+(3^2)}$$
$$k = 20$$

Then, because of statement (4), we know that $s = 4$ is NOT in the ROC of $X(s)$. This means that we obtain the transform as:

$$\boxed{X(s) = \frac{20}{(s+1)^2+3^2}, \quad \text{ROC: } \sigma < -1}$$

Taking the inverse transform, per our Laplace tables, we see:

$$\boxed{x(t) = -\frac{20}{3}e^{-t}\sin(3t)u(-t)}$$

3. (a) Using our tables, we may obtain (with $X(s)$ ROC: $\sigma < 3$ and $H(s)$ ROC: $\sigma > -2$):

$$\boxed{X(s) = -\frac{5}{s-3} \quad \text{and } H(s) = \frac{1}{s+2}}$$

(b) We may write the convolution transform as:

$$Y(s) = X(s)H(s)$$

Thus, we get:

$$Y(s) = \left(-\frac{5}{s-3}\right)\left(\frac{1}{s+2}\right)$$

$$\boxed{Y(s) = -\frac{5}{(s-3)(s+2)}}$$

(c) We begin by using partial fraction decomposition, which gives us:

$$Y(s) = \frac{A}{s-3} + \frac{B}{s+2}$$

From here, we get $A = -1$ and $B = 1$, which gives us:

$$Y(s) = \frac{-1}{s-3} + \frac{1}{s+2}$$

Using our inverse transforms, we obtain:

$$\boxed{y(t) = e^{3t}u(-t) + e^{-2t}u(t)}$$

(d) Explicit convolution gives us:

$$x(t) * h(t) = \int_0^t 5e^{3\tau}u(-\tau)e^{-2(t-\tau)}u(t-\tau)\,d\tau$$

$$x(t) * h(t) = \int_0^t 5e^{-2t+5\tau}u(-\tau)u(t-\tau)\,d\tau$$

We see that the function is bounded by:

$$\tau \leq 0 \quad \text{and} \quad \tau \leq t$$

From this, we may write:

$$y(t) = -5e^t \int_0^t e^{5\tau}\,d\tau$$

$$y(t) = -e^{-2t}\left[e^{5\tau}\right]\Big|_0^t$$

$$y(t) = -e^{-2t}\left[e^{5t} - 1\right]$$

This confirms:

$$\boxed{y(t) = e^{3t}u(-t) + e^{-2t}u(t)}$$

4. (a) Taking the Laplace transform, we get:

$$s^2Y(s) - sY(s) - 6Y(s) = sX(s)$$
$$Y(s)[s^2 - s - 6] = sX(s)$$
$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2 - s - 6}}$$

Thus, we see that there is a zero at $s = 0$ and poles at $s = -2, 3$. This allows us to plot:

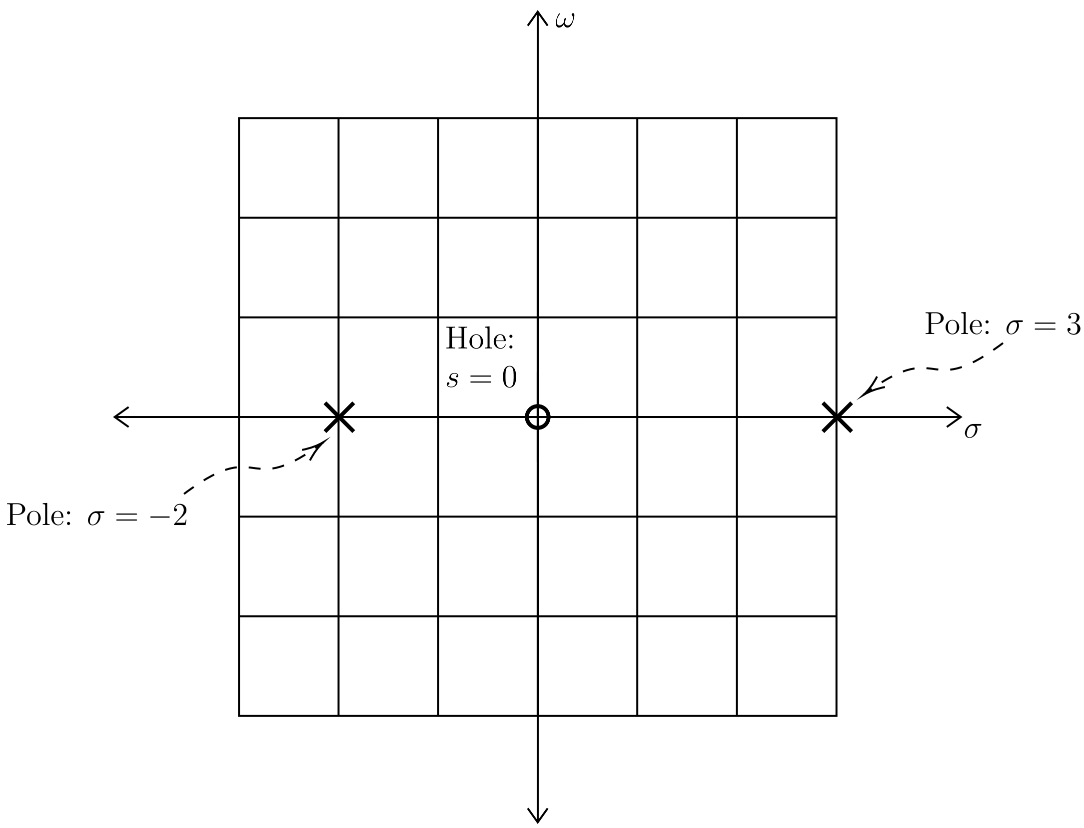

Figure 1: Pole-Zero Plot

(b) We may begin by using partial fraction decomposition:

$$\frac{s}{s^2 - s - 6} \Rightarrow \frac{A}{s-3} + \frac{B}{s+2}$$

Plugging in our values, we find $A = 3/5$ and $B = 2/5$, which gives us:

$$H(s) = \frac{3/5}{s-3} + \frac{2/5}{s+2}$$

i. When the system is stable, we know that the ROC must be bounded. Thus, we know that the ROC is $-2 < \sigma < 3$. Using our transform table, this gives:

$$\boxed{h(t) = -\frac{3}{5}e^{3t}u(-t) + \frac{2}{5}e^{-2t}u(t)}$$

ii. When the system is causal, we know that the ROC is right-sided, such that $\sigma > 3$. Thus, we see:

$$\boxed{h(t) = \frac{3}{5}e^{3t}u(t) + \frac{2}{5}e^{-2t}u(t)}$$

iii. When it is neither stable nor causal, the ROC must be left-sided and can not include the jω axis. This gives us:

$$\boxed{h(t) = \frac{3}{5}e^{3t}u(t) - \frac{2}{5}e^{-2t}u(-t)}$$

5. Given that this is the step response, and that it is multiplied by the step function, we know that:

$$X(s) = \frac{1}{s}$$

We take the transform of $y(t)$ to get:

$$Y(s) = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}$$

We know that:

$$H(s) = \frac{Y(s)}{H(s)}$$

Thus, we find the transfer function be:

$$H(s) = 1 - \frac{s}{s+2} - \frac{2s}{(s+2)^2}$$

$$H(s) = \frac{4}{(s+2)^2}$$

Then we can find:

$$Y_1(s) = \frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+4}$$

Knowing that this must be equivalent to the transfer function, we see:

$$\frac{Y_1(s)}{H(s)} = X_1(s)$$

This gives us:

$$X_1(s) = \frac{(s+2)^2}{4s} - .5(s+2) + \frac{(s+2)^2}{4(s+4)}$$
$$X_1(s) = \frac{(s+2)^2}{4s} - .5(s+2) + \frac{(s+2)^2}{4s+16}$$

We can simplify to get:

$$X_1(s) = \frac{2s+4}{s^2+4s}$$
$$X_1(s) = \frac{2}{s+4} + \frac{4}{s(s+4)}$$

We use partial fraction decomposition for the second term to get:

$$X_1(s) = \frac{2}{s+4} + \frac{A}{s+4} + \frac{B}{s}$$

We find $A = -1$ and $B = 1$ to get:

$$X_1(s) = \frac{1}{s+4} + \frac{1}{s}$$

Taking the inverse, we find:

$$\boxed{x_1(t) = [e^{-4t} + 1]u(t)}$$

6. We may express $x(t)$ as:

$$x(t) = e^t u(-t) + e^{-t} u(t)$$

This gives us:

$$X(s) = -\frac{1}{s-1} + \frac{1}{s+1}$$

We combine the two terms to get:

$$X(s) = \frac{-2}{s^2-1}$$

Multiplying by the system function, we get:

$$Y(s) = \left[\frac{s+1}{s^2+2s+9}\right]\left[-\frac{2}{s^2-1}\right]$$

We continue to simplify:

$$Y(s) = \left[\frac{1}{s^2+2s+9}\right]\left[-\frac{2}{s-1}\right]$$
$$Y(s) = \frac{-2}{(s^2+2s+9)(s-1)}$$

We then use partial fraction decomposition to write:

$$Y(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+9}$$

Thus, we get:

$$A(s^2+2s+9) + (Bs+C)(s-1) = -2$$
$$As^2 + 2As + 9A + Bs^2 - Bs + Cs - C = -2$$

From this, we can set up the following system:

$$A + B = 0$$
$$2A - B - C = 0$$
$$9A - C = -2$$

Solving the system, we see: $A = -\frac{1}{6}$, $B = \frac{1}{6}$, and $C = \frac{1}{2}$, which gives us:

$$Y(s) = \frac{-1/6}{s-1} + \frac{(1/6)s + (1/2)}{s^2+2s+9}$$

We break this up further to see:

$$Y(s) = \frac{-1/6}{s-1} + \frac{(1/6)s + (1/6)}{(s+1)^2+8} + \frac{(1/3)}{(s+1)^2+8}$$
$$Y(s) = \frac{-1/6}{s-1} + \frac{(1/6)s + (1/6)}{(s+1)^2+8} + \frac{1}{\sqrt{8}}\frac{3\sqrt{8}}{(s+1)^2+8}$$

We then use the Laplace tables, and the fact that the system is causal, to get:

$$\boxed{y(t) = -\frac{1}{6}e^t u(t) + \frac{1}{6}e^{-t}\cos\left(\sqrt{8}t\right)u(t) + \frac{1}{3\sqrt{8}}e^{-t}\sin\left(\sqrt{8}t\right)u(t)}$$

7. (a) We may observe that the left side of the diagram indicates poles, and the right side indicates zeros, while the s^{-1} blocks are delays. We know that the transfer function may be written as:

$$H(s) = \frac{\displaystyle\sum_{n=0}^{N} b_n s^{-n}}{1 + \displaystyle\sum_{n=1}^{N} a_n s^{-n}}$$

We may see that $N = 2$, which gives us:

$$H(s) = \frac{\displaystyle\sum_{n=0}^{2} b_n s^{-n}}{1 + \displaystyle\sum_{n=1}^{2} -a_n s^{-n}}$$

First, we work to get the numerator:

$$H(s) = \frac{1 - s^{-1} - 3s^{-2}}{1 + \sum_{n=1}^{2} a_n s^{-n}}$$

And then the denominator:

$$H(s) = \frac{1 - s^{-1} - 3s^{-2}}{1 + 2s^{-1} + 4s^{-2}}$$

Multiplying both the numerator and denominator by s^2, we get:

$$H(s) = \frac{s^2 - s - 3}{s^2 + 2s + 4}$$

We know that the transfer function is equivalent to:

$$H(s) = \frac{Y(s)}{X(s)}$$

This allows us to obtain the following:

$$s^2X(s) - sX(s) - 3X(s) = s^2Y(s) + 2sY(s) + 4Y(s)$$

Taking the invers transform, we see:

$$\boxed{\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} - 3x(t)}$$

(b) We know the system is stable if all poles are to the left of the s-plane. This gives us:

$$s^2 + 2s + 4 = 0$$

Using the quadratic equation, we get:

$$s = \frac{-2 \pm \sqrt{4 - 4(1)(4)}}{2}$$
$$s = -1 \pm 3j$$

Given the -1 real part, all poles are to the left of the s-plane, and, therefore, the system is stable.

8. (a) From the information, we know $Y(s) = (2s^2 + 3s - 4)Y_1(s)$, which allows us to write:

$$\boxed{y(t) = 2\frac{d^2 y_1(t)}{dt^2} + 3\frac{dy_1(t)}{dt} - 4y_1(t)}$$

(b) From the figure, we may observe the following relationship:

$$s^{-1} f(t) = y_1(t)$$

This allows us to obtain:

$$\boxed{f(t) = \frac{dy_1(t)}{dt}}$$

(c) Similar to part (b), we may observe the flow of the diagram to see:

$$s^{-2} e(t) = y_1(t)$$
$$e(t) = s^2 y_1(t)$$

This allows us to get:

$$\boxed{e(t) = \frac{d^2 y_1(t)}{dt^2}}$$

(d) Given parts (a), (b), and (c), we may write the original signal as:

$$\boxed{y(t) = 2e(t) + 3f(t) - 4y_1(t)}$$

(e) We may combine the diagram and result from (d) to form the following figure:

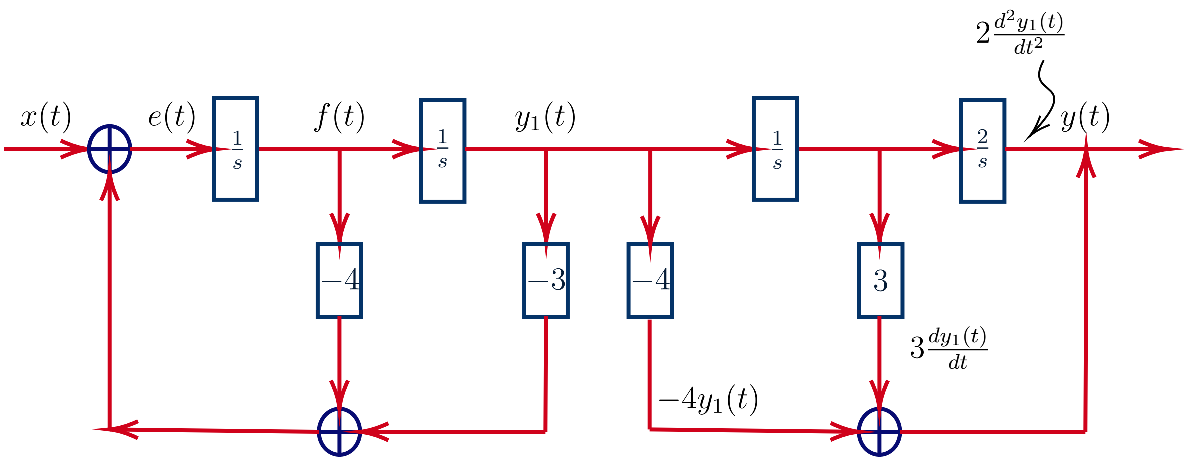

Figure 2: Full System (S) Diagram

(f) From the provided form, we may write:

$$H_1(s) = \frac{Y_1(s)}{X(s)} \quad \text{and} \quad H_2(s) = \frac{Y(s)}{Y_1(s)}$$

This gives us:

$$(s+3)Y_1(s) = (2s-4)X(s) \Rightarrow (s+2)Y_1(s) = (s+1)Y(s)$$

$$\frac{dy_1(t)}{dt} + 3y_1(t) = 2\frac{dx(t)}{dt} - 4x(t) \Rightarrow \frac{dy_1(t)}{dt} + 2y_1(t) = \frac{dy(t)}{dt} + y(t)$$

From $H_2(s)$, we may obtain:

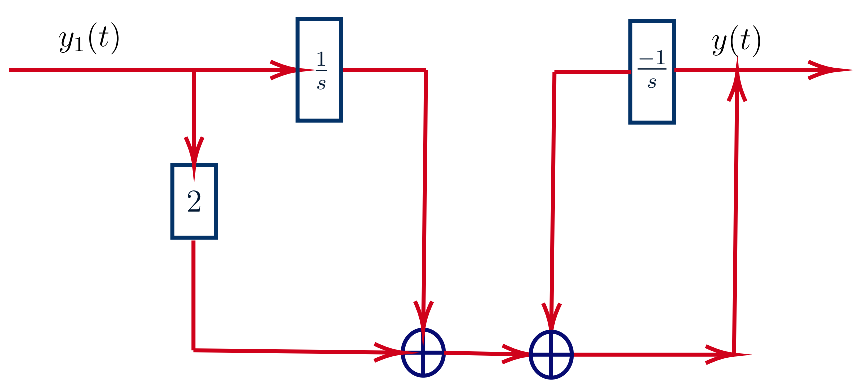

Figure 3: Representation of $H_2(s)$

We then combine this with $H_1(s)$ to get a full system flow:

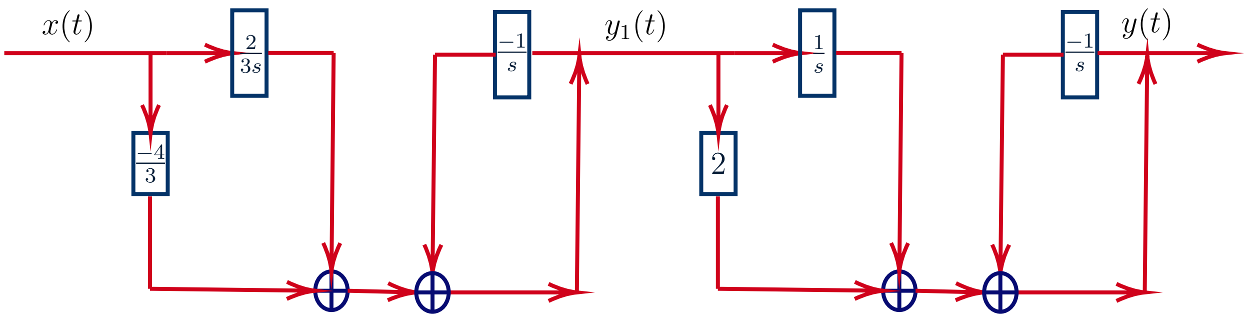

Figure 4: Representation of Full System, $H(s)$

(g) Based on the given system, we may obtain:

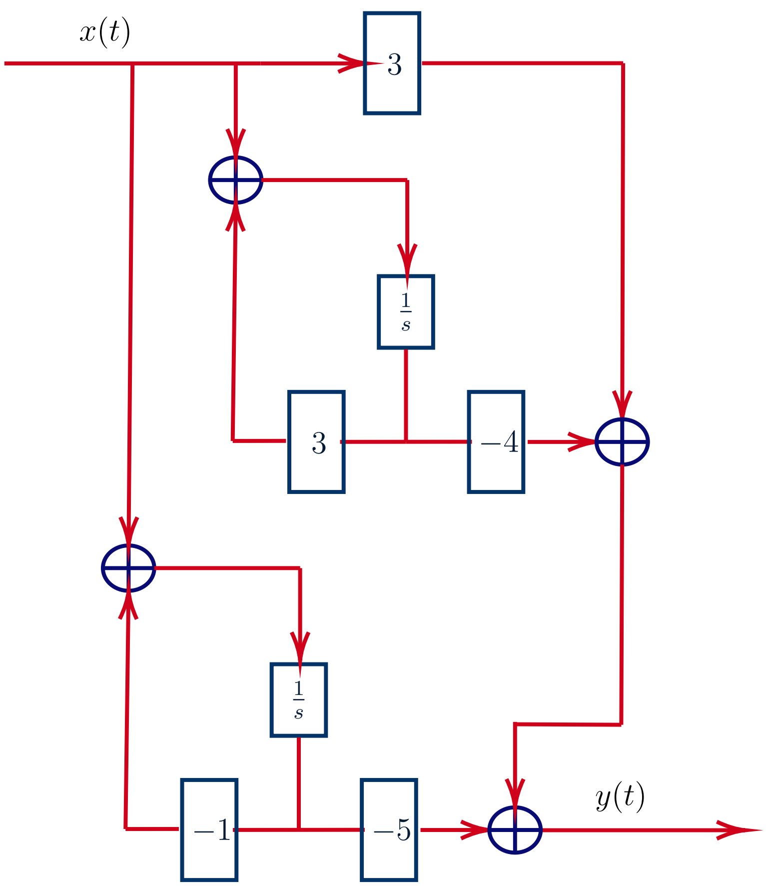

Figure 5: Representation of Full System (Part g), $H(s)$

9. (a) We may begin by converting all of the components to the frequency domain:

$$z_C = \frac{1}{sC} \quad \text{and} \quad z_L = sL$$

This gives us:

$$R = 1[\Omega],\ z_C = \frac{2}{s}[\Omega],\ z_l = .4s[\Omega]$$

Given that the components are in parallel, we can find the equivalent impedance:

$$z_{eq1} = \frac{z_C z_L}{z_C + z_L}$$
$$z_{eq1} = \frac{.8}{(2/s) + .4s}$$
$$z_{eq1} = \frac{.8s}{2 + .4s^2}$$

And then:

$$z_{eq} = \frac{R z_{eq1}}{R + z_{eq1}}$$
$$z_{eq} = \frac{z_{eq1}}{1 + z_{eq1}}$$
$$z_{eq} = \frac{.8s}{2 + .8s + .4s^2}$$

The voltage through each branch may be found by taking:

$$v(t) = z_{eq} i_g(t)$$
$$v(t) = \frac{.8s i_g(t)}{2 + .8s + .4s^2}$$

We then divide by the impedance of the capacitor to find the current:

$$i_o(t) = \frac{.4s^2 i_g(t)}{2 + .8s + .4s^2}$$

We then find the transfer function:

$$H(s) = \frac{i_o(s)}{i_g(t)}$$
$$H(s) = \frac{.4s^2}{2 + .8s + .4s^2}$$

To simplify, we multiply both the numerator and denominator by 5:

$$H(s) = \frac{2s^2}{10 + 4s + 2s^2}$$

$$\boxed{H(s) = \frac{s^2}{5 + 2s + s^2}}$$

(b) Given that we know:

$$H(s) = \frac{Y(s)}{X(s)}$$

We may obtain:

$$s^2 X(s) = [5 + 2s + s^2]Y(s)$$

This gives us:

$$\boxed{\frac{d^2 i_g(t)}{dt^2} = 5i_o(t) + 2\frac{di_o(t)}{dt} + \frac{d^2 i_o(t)}{dt^2}}$$

(c) We may obtain $I_o(s)$ using the transfer function:

$$I_o(s) = \left[\frac{s^2}{s^2 + 2s + 5}\right] I_g(s)$$

We take the Laplace transform for the current input to get:

$$I_g(s) = \frac{10s}{s^2 + 1}$$

Multiplying, we get:

$$\boxed{I_o(s) = \frac{10s^3}{(s^2 + 1)(s^2 + 2s + 5)}}$$

(d) We may begin by using partial fraction decomposition, such that:

$$I_o(s) = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 2s + 5}$$

Multiplying together, we find:

$$Cs^3 + Cs + Ds^2 + D + As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B = 10s^3$$

This allows us to generate the following system:

$$C + A = 10$$
$$D + 2A + B = 0$$

$$C + 5A + 2B = 0$$

$$D + 5B = 0$$

We solve the system and find: $A = -2, B = -1, C = 12$, and $D = 5$, which gives us:

$$I_o(s) = \frac{-2s - 1}{s^2 + 1} + \frac{12s + 5}{s^2 + 2s + 5}$$

Continuing to simplify to take the inverse, we get:

$$I_o(s) = -\frac{2s + 1}{s^2 + 1} + \frac{12s + 5}{(s + 1)^2 + 2^2}$$

$$I_o(s) = -\frac{2s}{s^2 + 1} - \frac{1}{s^2 + 1} + \frac{12(s + 1)}{(s + 1)^2 + 2^2} - \frac{7}{(s + 1)^2 + 2^2}$$

Per our tables, this gives us:

$$\boxed{i_o(t) = [\underbrace{-2\cos(t) - \sin(t)}_{\text{steady-state}} + \overbrace{12e^{-t}\cos(2t) - 7e^{-t}\sin(2t)}^{\text{transient}}]u(t)}$$

The transient terms fade with time and are expressed as those with decaying exponentials, while the steady state response is purely sinusoidal.