

Homework 4

Michael Brodskiy

Professor: I. Salama

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1. (a) We begin by taking the Laplace transform to get:

$$H(s) = -\frac{1}{s-1} \quad \text{and} \quad X(s) = \frac{2}{s} [e^{-s} - e^{-2s}]$$

We then multiply the two to get:

$$Y(s) = X(s)H(s)$$
$$Y(s) = -\frac{2}{s(s-1)} [e^{-s} - e^{-2s}]$$

Using partial fraction decomposition, we may write the equivalent such that:

$$-\frac{2}{s(s-1)} = \frac{A}{s-1} + \frac{B}{s} = As + B(s-1)$$

We use $s = 0, 1$ to get:

$$A = -2, B = 2 \rightarrow -\frac{2}{s-1} + \frac{2}{s}$$

We now distribute this in the above case to get:

$$Y(s) = -\frac{2}{s-1} [e^{-s} - e^{-2s}] + \frac{2}{s} [e^{-s} - e^{-2s}]$$
$$Y(s) = -\frac{2e^{-s}}{s-1} + \frac{2e^{-2s}}{s-1} + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

Finally, we take the inverse transform to get:

$$\boxed{y(t) = -2u(1-t) + 2e^{t-1}u(1-t) + 2u(2-t) - 2e^{t-2}u(2-t)}$$

- (b) Differentiating one of the inputs is the same as differentiating the output. Thus, we may say:

$$g(t) = \frac{d}{dt}[y(t)]$$

$$\boxed{g(t) = 2e^{t-1}u(1-t) - 2e^{t-2}u(2-t)}$$

- (c) As stated in (b) $g(t) = (d/dt)[y(t)]$
 (d) $z(t)$ is the same as $g(t)$. Since taking the differential is a linear operation, it does not matter if this is done to the impulse response or to $x(t)$. Therefore, we get:
 (e) By linearity of the transform, we may say:

$$y_1(t) = 2y(t-1)$$

Therefore, we may obtain:

$$\boxed{y_1(t) = -4u(2-t) + 4e^{t-2}u(2-t) + 4u(3-t) - 4e^{t-3}u(3-t)}$$

2. (a)
 (b)
 (c)
 (d)
3. (a) Given the set up, we may write:

$$\left(\frac{1}{4}\right)^n u[n] - A \left(\frac{1}{4}\right)^{n-1} u[n-1] = \delta[n]$$

We may redefine the delta as:

$$\left(\frac{1}{4}\right)^n u[n] - A \left(\frac{1}{4}\right)^{n-1} u[n-1] = u[n] - u[n-1]$$

Thus, we see that we need the exponential term to cancel. We can do this by simply taking:

$$\left(\frac{1}{4}\right)^n = A \left(\frac{1}{4}\right)^{n-1}$$

Dividing the exponential from one side to the other, we see:

$$A = \left(\frac{1}{4}\right)^1$$

$$\boxed{A = \frac{1}{4}}$$

(b) By definition, with $h[n]$ and $g[n] = h_{inv}[n]$, we know:

$$h[n] * g[n] = \delta[n]$$

Using the equation from part (a), we know:

$$h[n] - Ah[n - 1] = \delta[n]$$

By the properties of convolution, we know that:

$$x[n] * \delta[n - n_o] = x[n - n_o]$$

Thus, we may expand to write:

$$h[n] * \delta[n] - Ah[n] * \delta[n - 1] = \delta[n]$$

$$h[n] * (\delta[n] - A\delta[n - 1]) = \delta[n]$$

Thus, combining this with the definition of inverse, we may write:

$$g[n] = \delta[n] - \frac{1}{4}\delta[n - 1]$$

(c)

4. (a)

(b)

(c)

(d)

5. (a) For the given system, we may see that, for $t < 0$, the response may be non-zero (more precisely, it is non-zero for $-\infty < t < 5$); therefore, the system is not causal. We check for stability below:

$$\int_{-\infty}^5 e^{-3t} dt$$

$$-\frac{e^{-3t}}{3} \Big|_{-\infty}^5 = \infty$$

Therefore, the system is not stable

- (b) We may see that, for the given system, for $t < 0$, the response may be non-zero (more precisely, it is non-zero for $t > -10$); therefore, the system is not causal. We check for stability below:

$$\int_{-10}^{\infty} e^{-4t} dt$$

$$-\frac{e^{-4t}}{4} - 10^\infty = \frac{e^{40}}{4} < \infty$$

Thus, we see that the system is stable

(c) We may rewrite the function as:

$$x(t) = \begin{cases} e^{-2t}, & t \geq 0 \\ e^{2t}, & t < 0 \end{cases}$$

Because the value of the function is non-zero when $t < 0$, we can see that it is not causal

We may check for stability below:

$$\begin{aligned} \int_{-\infty}^0 e^{2t} dt + \int_0^{\infty} e^{-2t} dt \\ \frac{e^{2t}}{2} \Big|_{-\infty}^0 - \frac{e^{-2t}}{2} \Big|_0^{\infty} = 1 \end{aligned}$$

Therefore, we may see that the system is stable

(d) We may see that, because the system is zero for $t \leq 0$, it is causal. We now check for stability:

$$\begin{aligned} \int_2^{\infty} 3e^{-2t} - e^{-.05t+5} dt \\ 3e^{-2t} - e^{-.05t+5} \Big|_2^{\infty} = e^{4.9} - \frac{3}{e^4} < \infty \end{aligned}$$

Therefore, we may see that the system is stable

6. (a)

(b)

7. (a) Given the form of $x(t)$, we know $y(t)$ is of the form:

$$y(t) = Ae^{(-1+2j)t}$$

Plugging this into the given equation, we get:

$$A(-1+2j)e^{(-1+2j)t} + 3Ae^{(-1+2j)t} = e^{(-1+2j)t}$$

This simplifies to:

$$A(-1+2j) + 3A = 1$$

$$A = \frac{1}{2j+2}$$

And gives us the particular equation:

$$y_p(t) = \frac{e^{(-1+2j)t}}{2j+2}$$

We can now find the homogenous solution, with general form of $y(t)$:

$$y_h(t) = Ae^{\lambda t}$$

Using our equation, we may obtain:

$$\begin{aligned} A\lambda e^{\lambda t} + 3Ae^{\lambda t} &= 0 \\ \lambda &= -3 \end{aligned}$$

Combing the two we get:

$$y(t) = Ae^{-3t} + \frac{e^{(-1+2j)t}}{2j+2}$$

Applying the initial rest condition, we get:

$$\begin{aligned} 0 &= A + \frac{1}{2j+2} \\ A &= -\frac{1}{2j+2} \end{aligned}$$

Thus, ensuring that there is a response only for $t > 0$, we finally get:

$$\boxed{y(t) = \frac{1}{2+2j} \left[e^{(-1+2j)t} - e^{-3t} \right] u(t)}$$

(b) We can expand our answer from (a):

$$y(t) = \frac{1}{2+2j} \left[e^{-t} \cos(2t) + je^{-t} \sin(2t) - e^{-3t} \right] u(t)$$

Multiplying by the conjugate, we get:

$$\begin{aligned} y(t) &= (.25 - .25j) \left[e^{-t} \cos(2t) + je^{-t} \sin(2t) - e^{-3t} \right] u(t) \\ y(t) &= [.25e^{-t} \cos(2t) + .25e^{-t} \sin(2t) - .25e^{-3t}] u(t) \end{aligned}$$

Therefore, our output becomes:

$$\boxed{y(t) = [.25e^{-t} \sin(2t) + .25e^{-t} \cos(2t) - .25e^{-3t}] u(t)}$$

8. • We know that the equations governing time response of an inductor and capacitor are (respectively):

$$y(t) = L \frac{di(t)}{dt} \quad \text{and} \quad i(t) = C \frac{dV_c(t)}{dt}$$

We know that the voltage across the capacitor will be the difference between the voltage supplied and the voltage across the inductor; thus, we may write:

$$i(t) = C \frac{d}{dt} [x(t) - y(t)]$$

Inserting this into the inductor equation, we get:

$$y(t) = L \frac{d}{dt} \left[C \frac{d}{dt} [x(t) - y(t)] \right]$$

$$y(t) = LC \frac{d^2}{dt^2} [x(t) - y(t)]$$

Putting similar terms to one side, we may write:

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{LC} y(t) = \frac{d^2 x(t)}{dt^2}$$

Inserting known values:

$$\boxed{\frac{d^2 y(t)}{dt^2} + 25y(t) = \frac{d^2 x(t)}{dt^2}}$$

- Taking $x(t) \rightarrow 0$, we find the homogenous solution form as:

$$\frac{d^2 y(t)}{dt^2} + 25y(t) = 0$$

Using the provided equation, we insert into the above:

$$[K_1 \omega_1^2 e^{j\omega_1 t} + K_2 \omega_2^2 e^{j\omega_2 t}] + 25 [K_1 e^{j\omega_1 t} + K_2 e^{j\omega_2 t}] = 0$$

Dividing by the exponentials, we get:

$$K_1 \omega_1^2 + K_2 \omega_2^2 + 25K_1 + 25K_2 = 0$$

By observation, we may see that:

$$\boxed{\omega_1 = \omega_2 = \pm 5}$$

- From part (b), we see that the natural (homogenous) response may be modeled by:

$$y(t) = Ae^{j5t} + Be^{-j5t}$$

We may take $A = \frac{1}{2}(a + b)$ and $B = \frac{1}{2}(a - b)$. We then apply Euler's Law to get:

$$\boxed{y(t) = a \cos(5t) + b \sin(5t)}$$

We may thus observe that the natural response is sinusoidal in nature.