Homework 6

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1. (a) Per the rules of Laplace Transforms, we can convolve two signals by the rule that:

$$y(t) = x_1(t) * x_2(t) \to Y(s) = X_1(s)X_2(s)$$

As such, we may obtain:

$$X_1(s) = \frac{1}{s+4} \quad \text{and} \quad X_2(s) = \frac{1}{s+2}$$

Now, we account for the shifts. We know that, for $x(t) \to x(t - t_o)$ the transform becomes $X(s) \to e^{st_o}X(s)$. Furthermore, we know that for $x(-t) \to X(-s)$. Thus, we find:

$$X_1(s) = \frac{e^{-3s}}{-s+4} \quad \text{and} \quad X_2(s) = \frac{e^{-2s}}{s+2}$$

Multiplying together, we find:

$$\boxed{Y(s) = \frac{e^{-5s}}{(4-s)(s+2)}, \quad \text{ROC: } -2 < \sigma < 4}$$

(b)

2. First, we know that the poles must be at plus or minus the imaginary value, so the two poles must be at $s = -1 \pm 3j$. Thus, we see that $X(s)$ can be expressed as:

$$X(s) = \frac{k}{(s+1-3j)(s+1+3j)}$$
$$X(s) = \frac{k}{(s+1)^2 + 3^2}$$

We then apply the condition given in statement (5) to get:

$$2 = \frac{k}{(1^2) + (3^2)}$$
$$k = 20$$

Then, because of statement (4), we know that $s = 4$ is NOT in the ROC of $X(s)$. This means that we obtain the transform as:

$$\boxed{X(s) = \frac{20}{(s+1)^2 + 3^2}, \quad \text{ROC: } \sigma < -1}$$

Taking the inverse transform, per our Laplace tables, we see:

$$\boxed{x(t) = -\frac{20}{3} e^{-t} \sin(3t) u(-t)}$$

3. (a) Using our tables, we may obtain (with $X(s)$ ROC: $\sigma < 3$ and $H(s)$ ROC: $\sigma > -2$):

$$\boxed{X(s) = -\frac{5}{s-3} \quad \text{and } H(s) = \frac{1}{s+2}}$$

(b) We may write the convolution transform as:

$$Y(s) = X(s)H(s)$$

Thus, we get:

$$Y(s) = \left(-\frac{5}{s-3}\right)\left(\frac{1}{s+2}\right)$$
$$\boxed{Y(s) = -\frac{5}{(s-3)(s+2)}}$$

(c) We begin by using partial fraction decomposition, which gives us:

$$Y(s) = \frac{A}{s-3} + \frac{B}{s+2}$$

From here, we get $A = -1$ and $B = 1$, which gives us:

$$Y(s) = \frac{-1}{s-3} + \frac{1}{s+2}$$

Using our inverse transforms, we obtain:

$$\boxed{y(t) = e^{3t} u(-t) + e^{-2t} u(t)}$$

(d) Explicit convolution gives us:

$$x(t) * h(t) = \int_0^t 5e^{3\tau}u(-\tau)e^{-2(t-\tau)}u(t-\tau)\,d\tau$$

$$x(t) * h(t) = \int_0^t 5e^{-2t+5\tau}u(-\tau)u(t-\tau)\,d\tau$$

We see that the function is bounded by:

$$\tau \leq 0 \quad \text{and} \quad \tau \leq t$$

From this, we may write:

$$y(t) = -5e^t \int_0^t e^{5\tau}\,d\tau$$

$$y(t) = -e^{-2t}\left[e^{5\tau}\right]\Big|_0^t$$

$$y(t) = -e^{-2t}\left[e^{5t} - 1\right]$$

This confirms:

$$\boxed{y(t) = e^{3t}u(-t) + e^{-2t}u(t)}$$

4. (a) Taking the Laplace transform, we get:

$$s^2Y(s) - sY(s) - 6Y(s) = sX(s)$$

$$Y(s)[s^2 - s - 6] = sX(s)$$

$$\boxed{H(s) = \frac{Y(s)}{X(s)} = \frac{s}{s^2 - s - 6}}$$

Thus, we see that there is a zero at $s = 0$ and poles at $s = -2, 3$. This allows us to plot:

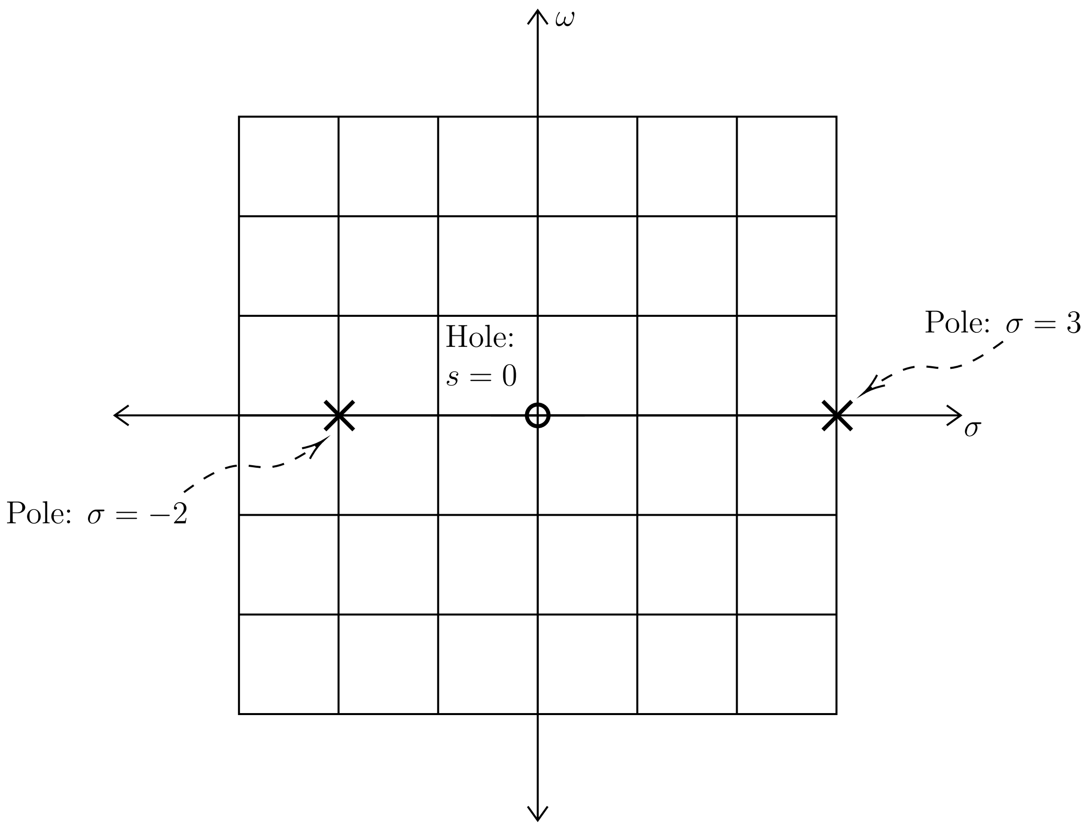

Figure 1: Pole-Zero Plot

(b) We may begin by using partial fraction decomposition:

$$\frac{s}{s^2 - s - 6} \Rightarrow \frac{A}{s-3} + \frac{B}{s+2}$$

Plugging in our values, we find $A = 3/5$ and $B = 2/5$, which gives us:

$$H(s) = \frac{3/5}{s-3} + \frac{2/5}{s+2}$$

i. When the system is stable, we know that the ROC must be bounded. Thus, we know that the ROC is $-2 < \sigma < 3$. Using our transform table, this gives:

$$\boxed{h(t) = -\frac{3}{5}e^{3t}u(-t) + \frac{2}{5}e^{-2t}u(t)}$$

ii. When the system is causal, we know that the ROC is right-sided, such that $\sigma > 3$. Thus, we see:

$$\boxed{h(t) = \frac{3}{5}e^{3t}u(t) + \frac{2}{5}e^{-2t}u(t)}$$

iii. When it is neither stable nor causal, the ROC must be left-sided and can not include the jω axis. This gives us:

$$\boxed{h(t) = \frac{3}{5}e^{3t}u(t) - \frac{2}{5}e^{-2t}u(-t)}$$

5. Given that this is the step response, and that it is multiplied by the step function, we know that:

$$X(s) = \frac{1}{s}$$

We take the transform of $y(t)$ to get:

$$Y(s) = \frac{1}{s} - \frac{1}{s+2} - \frac{2}{(s+2)^2}$$

We know that:

$$H(s) = \frac{Y(s)}{H(s)}$$

Thus, we find the transfer function be:

$$H(s) = 1 - \frac{s}{s+2} - \frac{2s}{(s+2)^2}$$
$$H(s) = \frac{4}{(s+2)^2}$$

Then we can find:

$$Y_1(s) = \frac{1}{s} - \frac{2}{s+2} + \frac{1}{s+4}$$

Knowing that this must be equivalent to the transfer function, we see:

$$\frac{Y_1(s)}{H(s)} = X_1(s)$$

This gives us:

$$X_1(s) = \frac{(s+2)^2}{4s} - .5(s+2) + \frac{(s+2)^2}{4(s+4)}$$
$$X_1(s) = \frac{(s+2)^2}{4s} - .5(s+2) + \frac{(s+2)^2}{4s+16}$$

We can simplify to get:

$$X_1(s) = \frac{2s+4}{s^2+4s}$$
$$X_1(s) = \frac{2}{s+4} + \frac{4}{s(s+4)}$$

We use partial fraction decomposition for the second term to get:

$$X_1(s) = \frac{2}{s+4} + \frac{A}{s+4} + \frac{B}{s}$$

We find $A = -1$ and $B = 1$ to get:

$$X_1(s) = \frac{1}{s+4} + \frac{1}{s}$$

Taking the inverse, we find:

$$\boxed{x_1(t) = [e^{-4t} + 1]u(t)}$$

6. We may express $x(t)$ as:

$$x(t) = e^t u(-t) + e^{-t}u(t)$$

This gives us:

$$X(s) = -\frac{1}{s-1} + \frac{1}{s+1}$$

We combine the two terms to get:

$$X(s) = \frac{-2}{s^2-1}$$

Multiplying by the system function, we get:

$$Y(s) = \left[\frac{s+1}{s^2+2s+9}\right]\left[-\frac{2}{s^2-1}\right]$$

We continue to simplify:

$$Y(s) = \left[\frac{1}{s^2+2s+9}\right]\left[-\frac{2}{s-1}\right]$$
$$Y(s) = \frac{-2}{(s^2+2s+9)(s-1)}$$

We then use partial fraction decomposition to write:

$$Y(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+9}$$

Thus, we get:

$$A(s^2+2s+9) + (Bs+C)(s-1) = -2$$
$$As^2 + 2As + 9A + Bs^2 - Bs + Cs - C = -2$$

From this, we can set up the following system:

$$A + B = 0$$
$$2A - B - C = 0$$
$$9A - C = -2$$

Solving the system, we see: $A = -\frac{1}{6}$, $B = \frac{1}{6}$, and $C = \frac{1}{2}$, which gives us:

$$Y(s) = \frac{-1/6}{s-1} + \frac{(1/6)s + (1/2)}{s^2+2s+9}$$

We break this up further to see:

$$Y(s) = \frac{-1/6}{s-1} + \frac{(1/6)s + (1/6)}{(s+1)^2+8} + \frac{(1/3)}{(s+1)^2+8}$$
$$Y(s) = \frac{-1/6}{s-1} + \frac{(1/6)s + (1/6)}{(s+1)^2+8} + \frac{1}{\sqrt{8}} \frac{3\sqrt{8}}{(s+1)^2+8}$$

We then use the Laplace tables, and the fact that the system is causal, to get:

$$\boxed{y(t) = -\frac{1}{6}e^{t}u(t) + \frac{1}{6}e^{-t}\cos\left(\sqrt{8}t\right)u(t) + \frac{1}{3\sqrt{8}}e^{-t}\sin\left(\sqrt{8}t\right)u(t)}$$

7. (a)
   (b)

8. (a)
   (b)
   (c)
   (d)
   (e)
   (f)
   (g)

9. (a)
   (b)
   (c)
   (d)