Homework 10

Michael Brodskiy
Professor: I. Salama

December 3, 2024

1. We first determine the sampling frequency to be:

$$f_s = \frac{1}{10^{-4}} = 10[\text{kHz}]$$
$$\omega_s = 2\pi \cdot 10^4 \left[\frac{\text{rad}}{\text{sec}}\right]$$

(a) Since the signal band is limited to 5000π[rad/s], the Nyquist frequency is:

$$\omega_N = 2\omega$$
$$\omega_N = 10000\pi$$

Since the sampling frequency is greater than the Nyquist frequency, the signal can be fully recovered

(b) Since the signal band is limited to 12000π[rad/s], the Nyquist frequency is:

$$\omega_N = 24000\pi$$

Since the sampling frequency is less than the Nyquist frequency, the signal can not be fully recovered

(c) Since we only know the behavior of the real part of the signal, we do not know if the signal is band-limited. Thus, the sampling theorem can not guarantee exact recovery

(d) Since the signal is real, $X(j\omega)$ must be symmetrical, and, therefore, the signal is band-limited to 5000π[rad/s]. This means that, for the same reasoning as (a), the signal can be fully recovered

(e) Similarly, since the signal is real, we know that $X(j\omega)$ must be symmetrical, and that, per the same reasoning as (b), it can not be exactly recovered

(f) We know that, if $X(j\omega) = 0$ for $|\omega| = \omega_1$, then $X(j\omega) * X(j\omega) = 0$ for $|\omega| > 2\omega$. Thus, we may write:

$$X(j\omega) = 0 \text{ for } |\omega| > 6000\pi$$

This means that the Nyquist frequency is:

$$\omega_N = 12000\pi$$

Since the Nyquist frequency is less than the sampling frequency, we know that the signal can be fully recovered

(g) From this, we may determine that, because:

$$|X(j\omega)| = 0 \text{ for } |\omega| > 6000\pi$$

The signal can be fully recovered, for the same reasoning as part (f).

2. (a) The maximum frequency of the signal, due to the multiplication in the frequency domain, becomes the minimum of the two bandwidths, such that:

$$\omega = 2000\pi$$

Thus, the Nyquist frequency becomes:

$$\omega_N = 4000\pi$$

This means that, to fully recover the signal, the period must be:

$$T \leq \frac{2\pi}{4000\pi}$$
$$T \leq \frac{1}{2000}$$
$$\boxed{T \leq .5[\text{ms}]}$$

(b) Multiplication leads to convolution in the frequency domain, which means that the bandwidth now limits the frequency to a maximum of:

$$\omega = 2000\pi + 4000\pi$$
$$\omega = 6000\pi$$

The Nyquist frequency becomes:

$$\omega_N = 12000\pi$$

This gives us a period range of:

$$T \leq \frac{2\pi}{12000\pi}$$

$$T \leq \frac{1}{6000}$$

$$\boxed{T \leq .166\bar{6}[\text{ms}]}$$

3. Per the Nyquist criterion for bandpass signals, we know that the sampling frequency must be twice the bandwidth of the signal. Thus, we may find:

$$\omega_N = 2(\omega_2 - \omega_1)$$

This gives us a maximum period of:

$$T \leq \frac{2\pi}{2(\omega_2 - \omega_1)}$$

$$\boxed{T \leq \frac{\pi}{\omega_2 - \omega_1}}$$

To accurately recover the signal, we should take the parameters as:

$$\boxed{\begin{cases} \omega_a & = \omega_1 \\ \omega_b & = \omega_2 \end{cases}}$$

This allows us to accurately obtain the initial signal. Furthermore, since we want the original signal, we want a gain of unity, or:

$$\boxed{A = 1}$$

4. (a) We may sketch $X_p(j\omega)$ as:

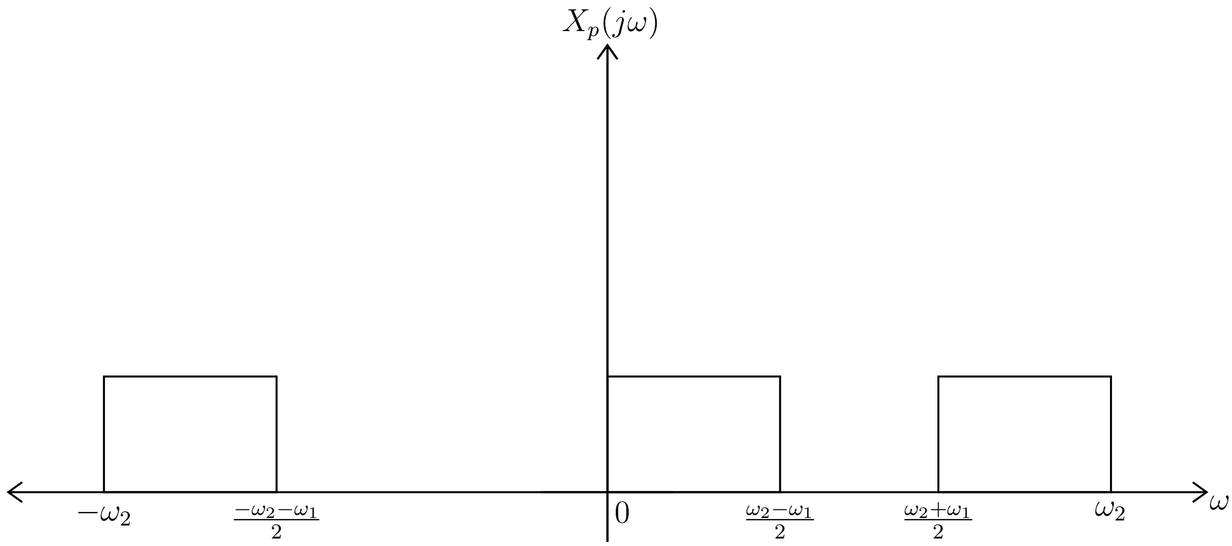

Figure 1: Sketch of $X_p(j\omega)$

(b) Given that the signal is limited to ω_2, we can take the minimum sampling frequency as:

$$\boxed{\omega_s \geq 2\omega_2}$$

(c) Given that $X(j\omega)$ is not present within $X_p(j\omega)$, the original signal can not be recovered using any kind of filter. Thus, this is a loss of information.

5. (a) We may sketch the given signals as:

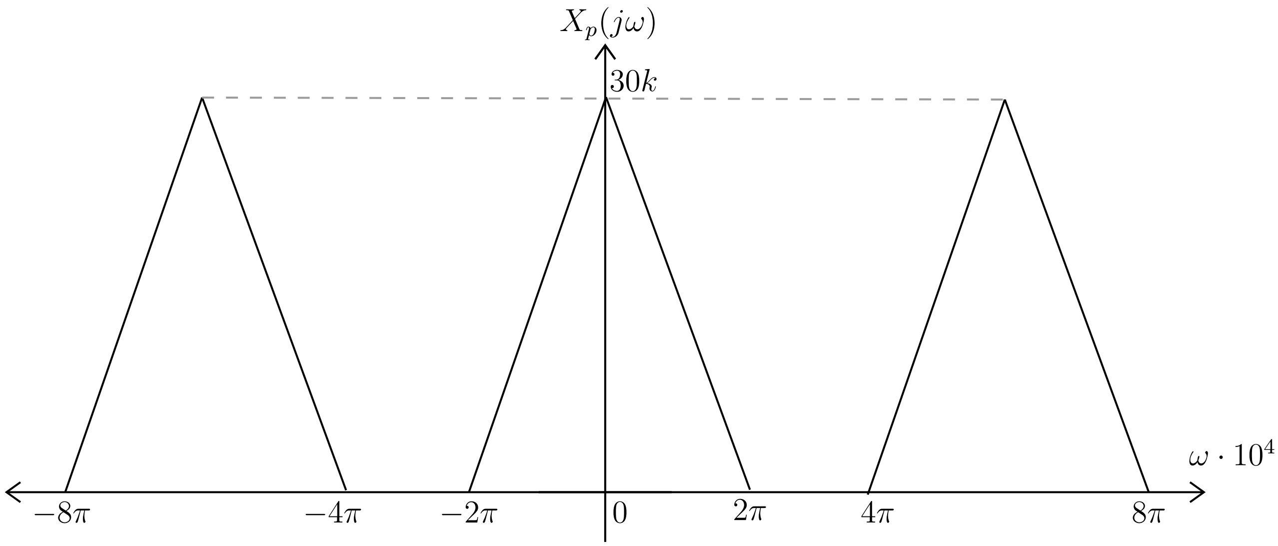

Figure 2: Sketch of $X_p(j\omega)$

We may see that the signal is constrained by:

$$x(n) = x_c(nT)$$

We can continue to sketch $X(e^{j\Omega})$, which looks quite similar; however, we first convert from analog to digital frequency:

$$\Omega = \frac{\omega}{f_s}$$

$$\Omega = \frac{\omega}{3 \cdot 10^4}$$

This gives us:

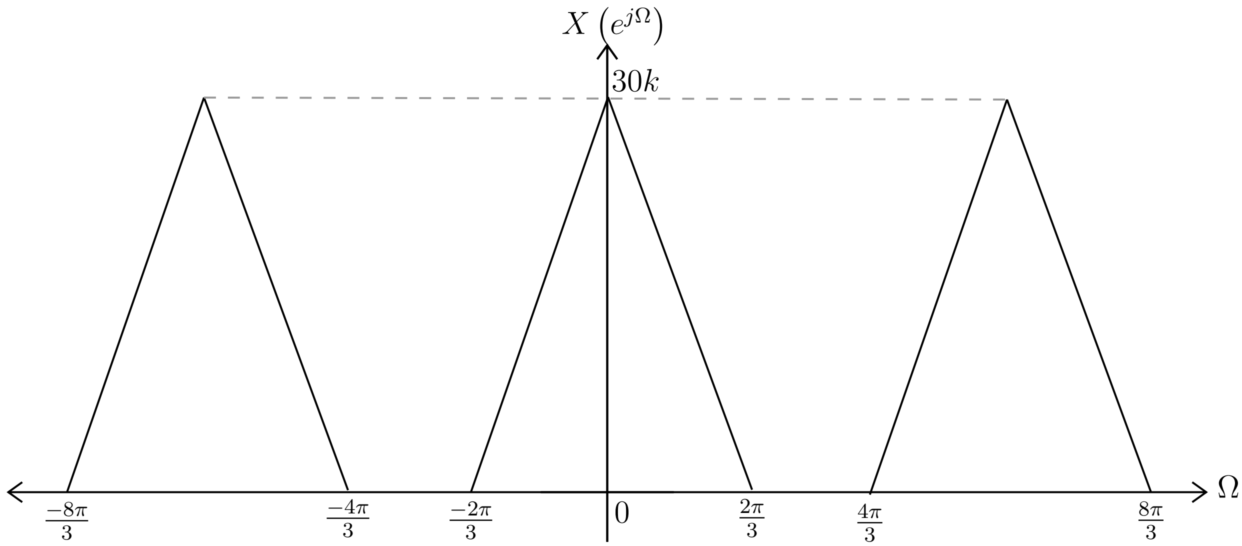

Figure 3: Sketch of $X(e^{j\Omega})$

From here, we may obtain $Y(e^{j\Omega})$ as:

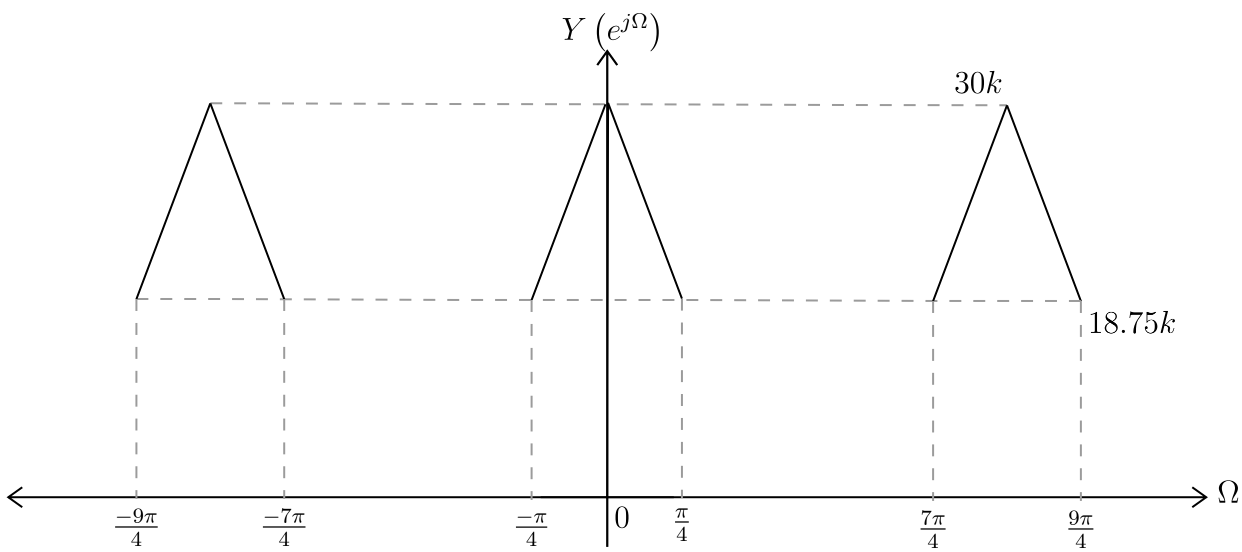

Figure 4: Sketch of $Y(e^{j\Omega})$

We convert back to digital frequency to get:

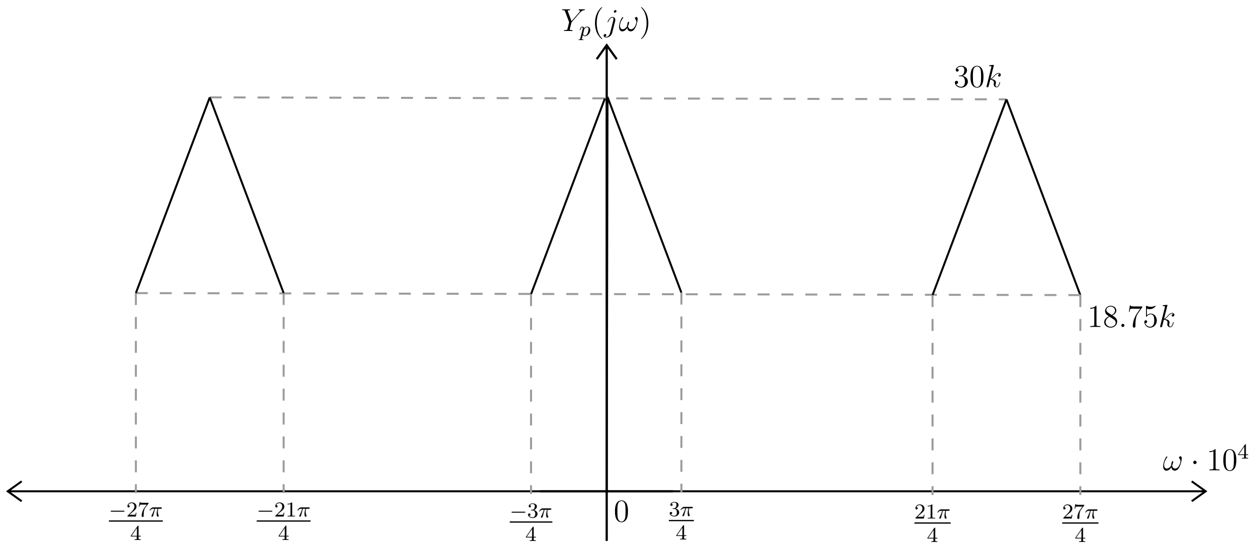

Figure 5: Sketch of $Y_p(j\omega)$

Passed through the filter, our response becomes:

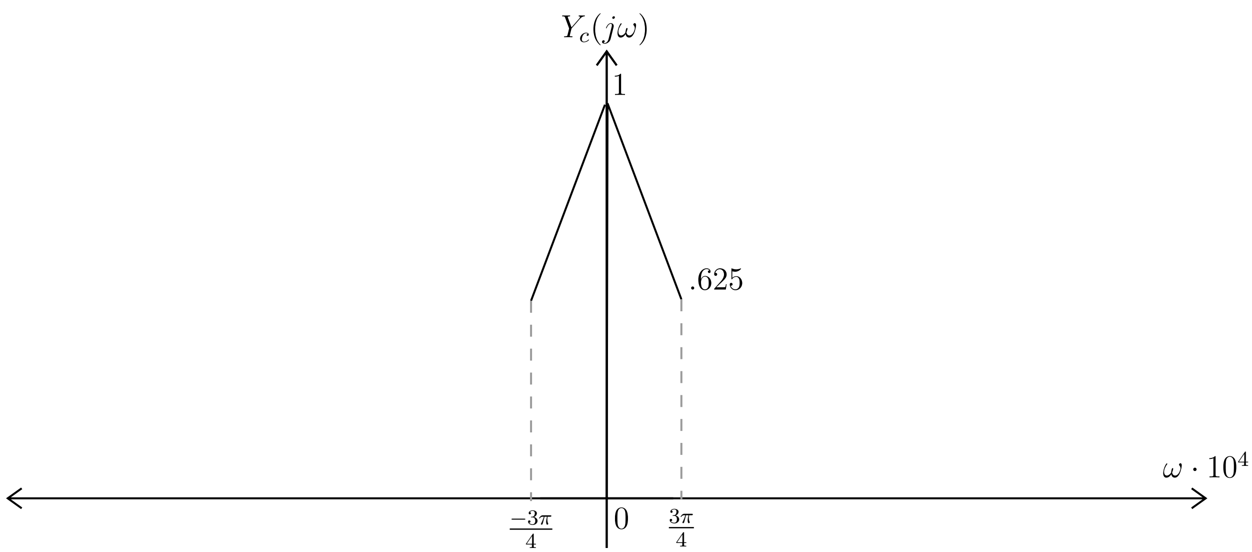

Figure 6: Sketch of $Y_c(j\omega)$

The frequency response of the continuous time equivalent would be:

$$\omega = \Omega f_s$$
$$\omega = \left(\frac{\pi}{4}\right)\left(3 \cdot 10^4\right)$$

$$\boxed{\omega = 7500\pi}$$

This gives us:

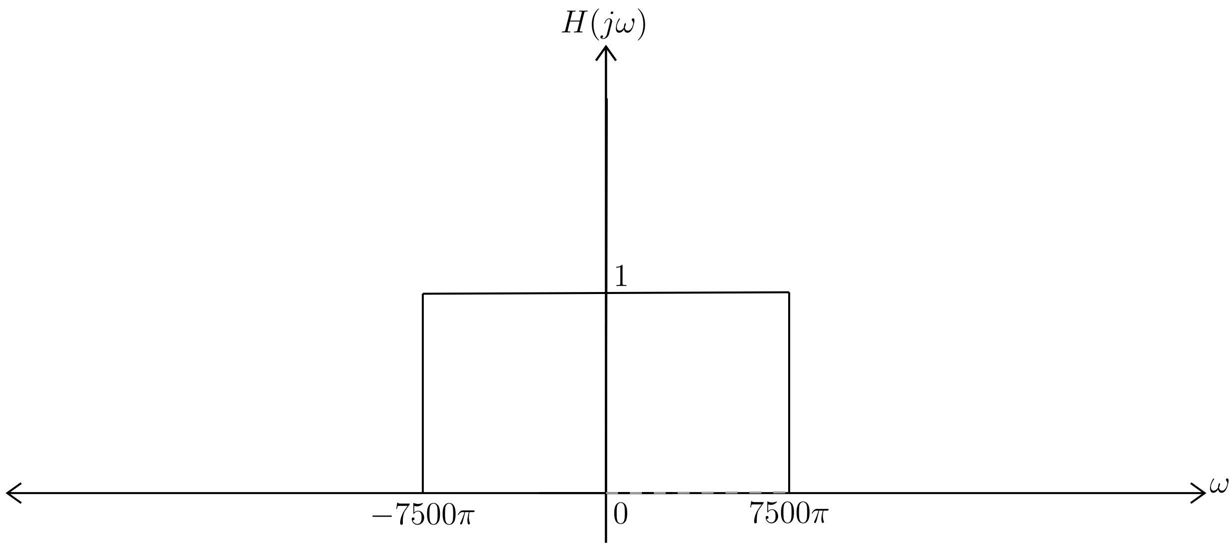

Figure 7: Continuous Time Response

(b) Only the first sinusoid is within the bandpass of the filter, which lets us determine:

$$\boxed{y(t) = 2\cos(4500\pi t + \pi/2)}$$

When converted to a discrete sequence, we may observe that the sine term becomes:

$$\Omega_N = 6\pi \cdot 10^4$$
$$6\pi \cdot 10^4 - 54000\pi = 6000\pi$$

This becomes aliased, which results in a discrete time result of:

$$\boxed{y(t) = 2\cos(4500\pi t + \pi/2) + \sin(6000\pi t)}$$

Thus, aliasing gives us a different result in discrete time.