Homework 1

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1. Express each of the following complex numbers in polar form and plot them

   (a) 8

$$r = \sqrt{8^2 + 0^2} = 8$$
$$\theta = 0$$
$$z(r,\theta) = r(\cos(\theta) + j\sin(\theta))$$
$$z(8,0) = 8(\cos(0) + j\sin(0))$$
$$\therefore \text{ In polar: } \boxed{z = 8}$$

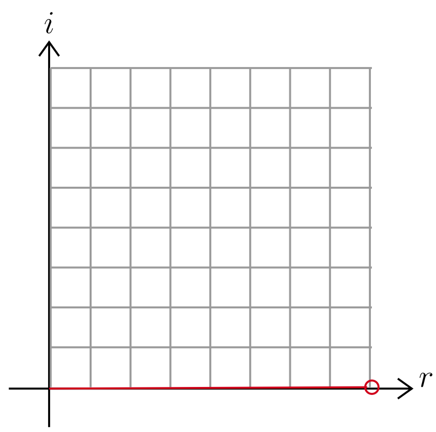


Figure 1: $z = 8$ Plotted on the Imaginary Plane

(b) -5

$$r = \sqrt{(-5)^2 + 0^2} = 5$$
$$\theta = \pi$$
$$z(r, \theta) = r(\cos(\theta) + j\sin(\theta))$$
$$z(5, \pi) = 5(\cos(\pi) + j\sin(\pi))$$
$$\therefore \text{ In polar: } \boxed{z = -5 = 5e^{\pi j}}$$

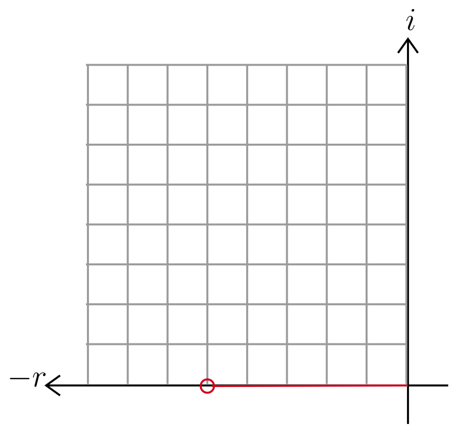


Figure 2: $z = -5$ Plotted on the Imaginary Plane

(c) $2j$

$$r = \sqrt{0^2 + (2)^2} = 2$$
$$\theta = \frac{\pi}{2}$$
$$z(r, \theta) = r(\cos(\theta) + j\sin(\theta))$$
$$z(2, .5\pi) = 2j$$
$$\therefore \text{ In polar: } \boxed{z = 2j = 2e^{.5\pi j}}$$

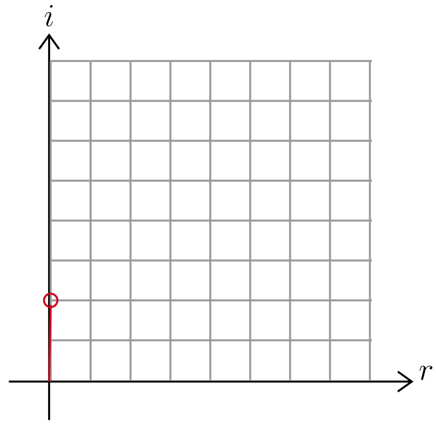


Figure 3: $z = 2j$ Plotted on the Imaginary Plane

(d) $\frac{1}{4}(1-j)^5$

$$.25(1-j)^2(1-j)^3$$
$$.25(-2j)(1-j)(1-j)^2$$
$$.25(-2-2j)(-2j)$$
$$z = j - 1$$
$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$
$$\theta = \frac{3\pi}{4}$$
$$z(r,\theta) = r(\cos(\theta) + j\sin(\theta))$$
$$z(\sqrt{2}, .75\pi) = j - 1$$
$$\therefore \text{ In polar: } \boxed{z = j - 1 = \sqrt{2}e^{.75\pi j}}$$

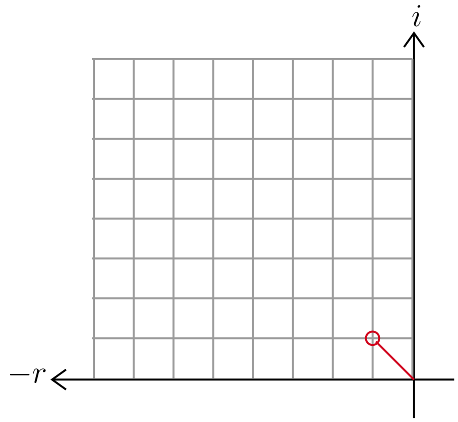


Figure 4: $z = \frac{1}{4}(1-j)^5$ Plotted on the Imaginary Axis

(e) $\frac{(1+j)}{j}e^{\frac{j\pi}{3}}$

$$\frac{(1+j)}{j} \cdot \frac{-j}{-j} = 1 - j$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{b}{a}$$

$$\frac{b}{a} = \sqrt{3}$$

$$b = a\sqrt{3}$$

$$\sqrt{(a\sqrt{3})^2 + a^2} = 1$$

$$4a^2 = \pm 1$$

$$a = \frac{1}{2}$$

$$b = \frac{\sqrt{3}}{2}$$

$$\frac{1}{2}(1-j)(1+\sqrt{3}j) \rightarrow \frac{1}{2}((\sqrt{3}+1) + (\sqrt{3}-1)j)$$

$$r = \frac{1}{2}\sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2} = \sqrt{(4+2\sqrt{3}) + (4-2\sqrt{3})}$$

$$r = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = .26179$$

$$z(r,\theta) = r(\cos(\theta) + j\sin(\theta))$$

$$z(\sqrt{2}, .26179) = \frac{1}{2}(\sqrt{3}+1) + \frac{1}{2}(\sqrt{3}-1)j$$

$$\therefore \text{ In polar: } \boxed{z = \frac{1}{2}\left[(\sqrt{3}+1)+(\sqrt{3}-1)j\right] = \sqrt{2}e^{.26179j}}$$

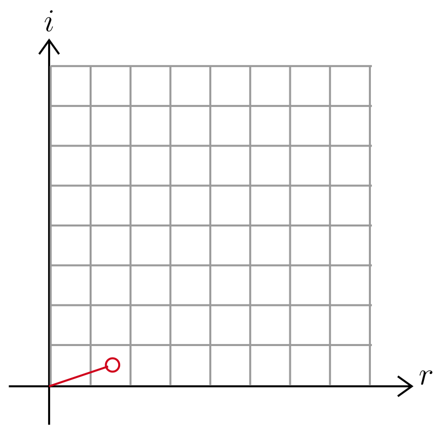


Figure 5: $z = \frac{(1+j)}{j}e^{\frac{j\pi}{3}}$ Plotted on the Imaginary Axis

(f) $(\sqrt{3}-j^5)(1+j)$

$$j^5 = j \rightarrow (\sqrt{3}-j)(1+j) = (\sqrt{3}+(\sqrt{3}-1)j+1)$$
$$(\sqrt{3}+1)+(\sqrt{3}-1)j$$
$$r = \sqrt{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2} = \sqrt{(4+2\sqrt{3})+(4-2\sqrt{3})}$$
$$r = \sqrt{8} = 2\sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = .26179$$
$$z(r,\theta) = r(\cos(\theta)+j\sin(\theta))$$
$$z(2\sqrt{2}, .26179) = (\sqrt{3}+1)+(\sqrt{3}-1)j$$
$$\therefore \text{ In polar: } \boxed{z = (\sqrt{3}+1)+(\sqrt{3}-1)j = 2\sqrt{2}e^{.26179j}}$$

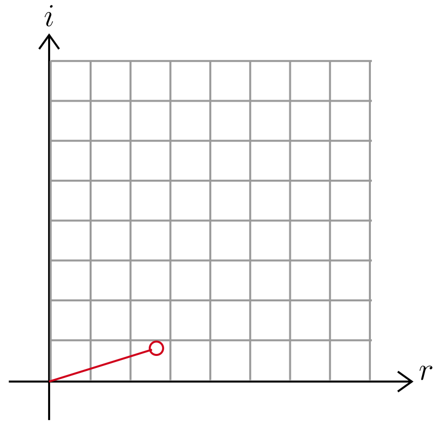


Figure 6: $z = (\sqrt{3} - j^5)(1 + j)$ Plotted on the Imaginary Axis

(g) $\frac{2(\sqrt{3}-j)}{1+j\sqrt{3}}$

$$\frac{2\sqrt{3} - 2j}{1 + j\sqrt{3}} \cdot \frac{1 - j\sqrt{3}}{1 - j\sqrt{3}} = -2j$$
$$r = \sqrt{0^2 + (-2)^2} = 2$$
$$\theta = \frac{3\pi}{2}$$
$$z(r, \theta) = r(\cos(\theta) + j\sin(\theta))$$
$$z(2, 1.5\pi) = -2j$$
$$\therefore \text{ In polar: } \boxed{z = -2j = 2e^{1.5\pi j}}$$

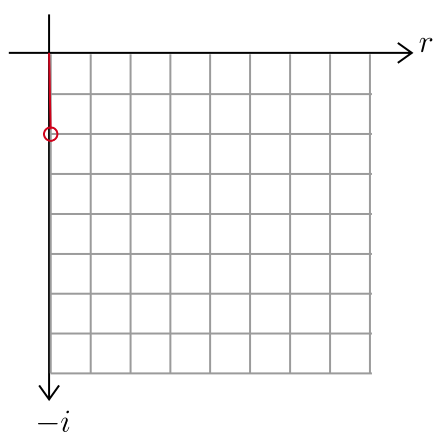


Figure 7: $z = \frac{2(\sqrt{3}-j)}{1+j\sqrt{3}}$ Plotted on the Imaginary Axis

2. Determine the value of E_∞ and P_∞ for each of the following signals and indicate whether the signal is a power or energy signal or neither.

   (a) $x_1(t) = \begin{cases} 5e^{j(4t+\pi/3)}, & t \geq 2 \\ 0, & \text{Otherwise} \end{cases}$

$$|x_1(t)| = \sqrt{\left(5\cos\left(4t+\frac{\pi}{3}\right)\right)^2 + \left(5\sin\left(4t+\frac{\pi}{3}\right)\right)^2}$$
$$|x_1(t)| = 5\sqrt{2}$$

$$E_\infty = \int_2^\infty 50\,dt$$
$$E_\infty = \infty$$
$$\therefore \text{ Energy is infinite}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} 50\,dt$$
$$P_\infty = 50$$
$$\therefore \text{ Power is finite}$$

$$\boxed{\text{Therefore, this is a power signal}}$$

   (b) $x_2(t) = \begin{cases} 2+2\cos(t), & 0 < t < 2\pi \\ 0, & \text{Otherwise} \end{cases}$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} (2+2\cos(t))^2\,dt$$
$$P_\infty = 6$$
$$\therefore \text{ Power is finite}$$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} (2+2\cos(t))^2\,dt$$
$$E_\infty = \infty$$
$$\therefore \text{ Energy is infinite}$$

$$\boxed{\text{Since power is finite and energy is infinite, this is a power signal}}$$

(c) $x_3[n] = \begin{cases} (.5)^n, & n \geq 0 \\ 0, & \text{Otherwise} \end{cases}$

$$E_\infty = \lim_{N\to\infty} \sum_{n=0}^{N} (.25)^n$$

A geometric series must be finite:

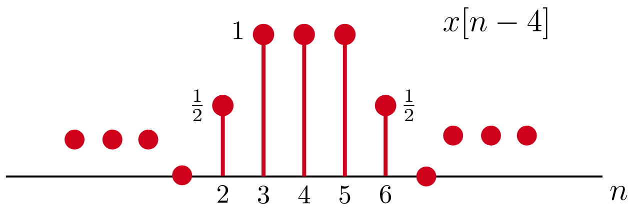
$$E_\infty \approx \left(\frac{1}{1-.25}\right) \approx \frac{4}{3}$$

$$P_\infty = \lim_{N\to\infty} \frac{4/3}{2N+1} \approx 0$$

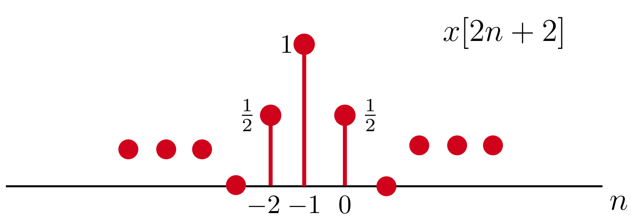
$\boxed{\text{As such, because energy is finite and average power is 0, this is an energy signal}}$

3. For the discrete time signal shown in Figure P1.3, sketch, and carefully label each of the following.

   (a) $x[n-4]$



   (b) $x[2n+2]$



4. For the continuous time signal shown in Figure P1.4, sketch, and carefully label each of the following.

   (a) $x(t+3)$

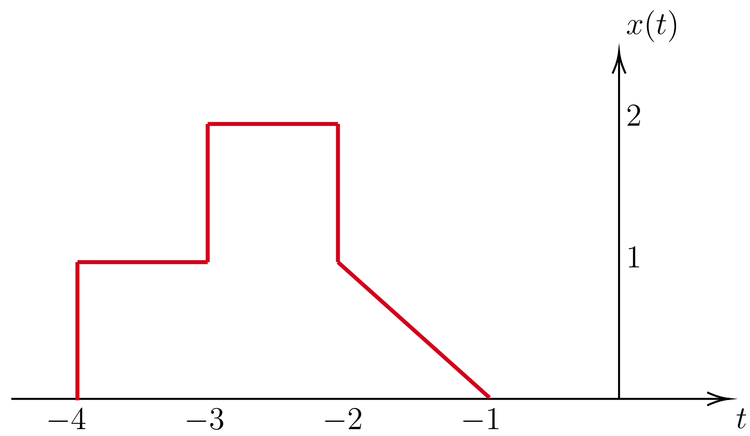


Figure 8: Figure Showing Transformation $x(t) \rightarrow x(t+3)$

(b) $x\left(3 - \frac{2}{3}t\right)$

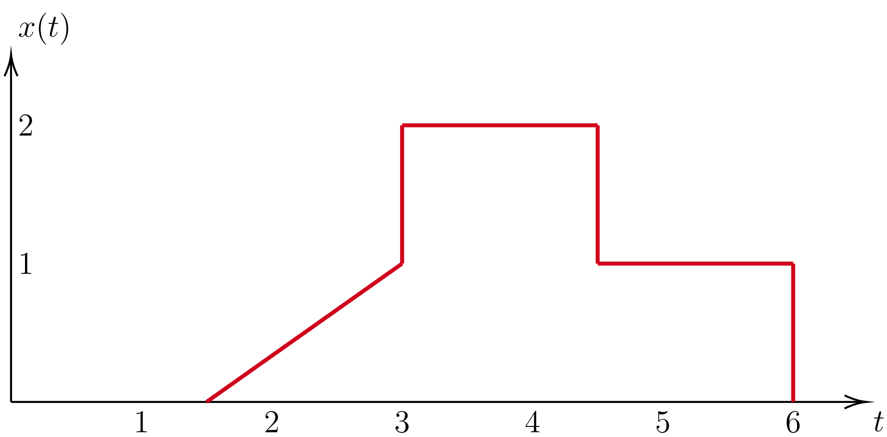


Figure 9: Figure Showing Transformation $x(t) \rightarrow x\left(3 - \frac{2}{3}t\right)$

5. Determine and sketch the even and odd parts of the signals depicted in Figure P1.5. Label your sketches carefully.

   (a) The original plot is as follows:

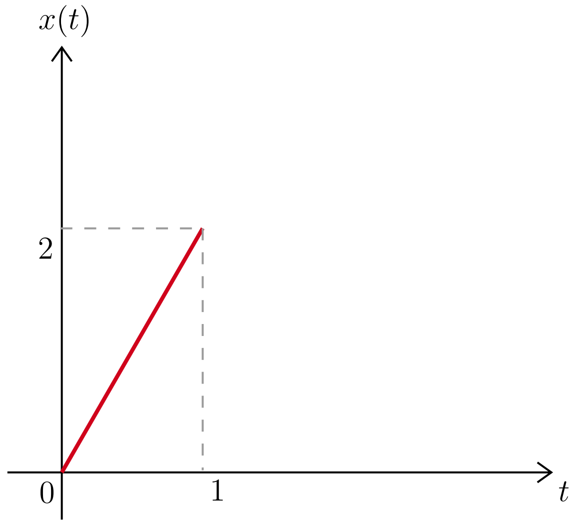


Figure 10: Original Plot

First, we begin by plotting $x(-t)$:

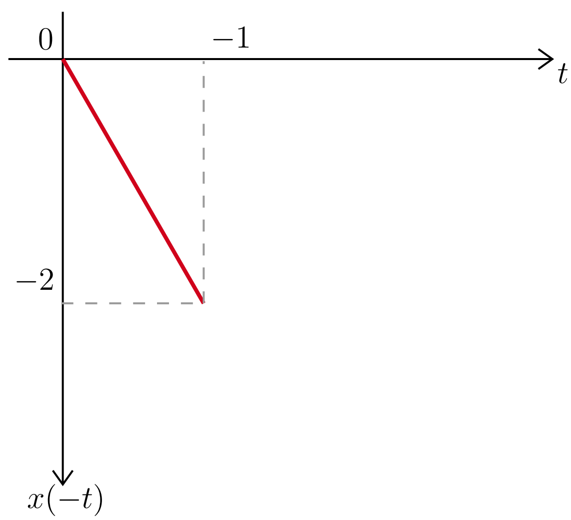


Figure 11: $x(t) \rightarrow x(-t)$

From here, we find the average of the two to find the even part:

$$\frac{x(t) + x(-t)}{2} \rightarrow \frac{-2t + 2t}{2} = 0$$

Thus, we see there is no even part (the plot would show $x(t) = 0$ for $0 < t < 1$).

From here, we subtract $x(-t)$ from $x(t)$ and then divide by two:

$$\frac{x(t) - x(-t)}{2} \rightarrow \frac{2t - (-2t)}{2} = 2t$$

We can plot this; however, as a result of being purely an even function, the plot would be the same as the original:

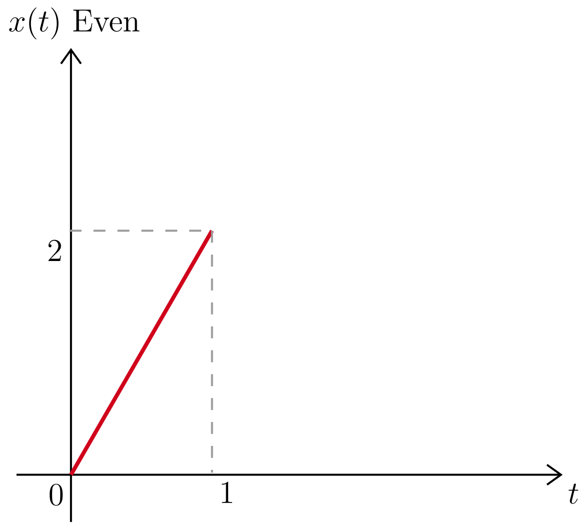


Figure 12: $x(t)$ Even (Original Plot)

(b) The original plot is as follows:

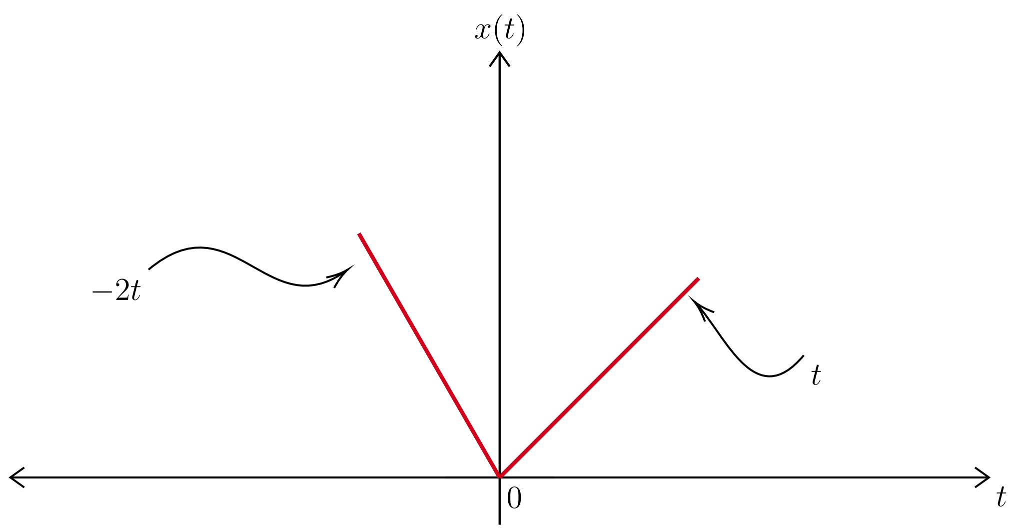


Figure 13: Original Plot

First, we begin by plotting $x(-t)$:

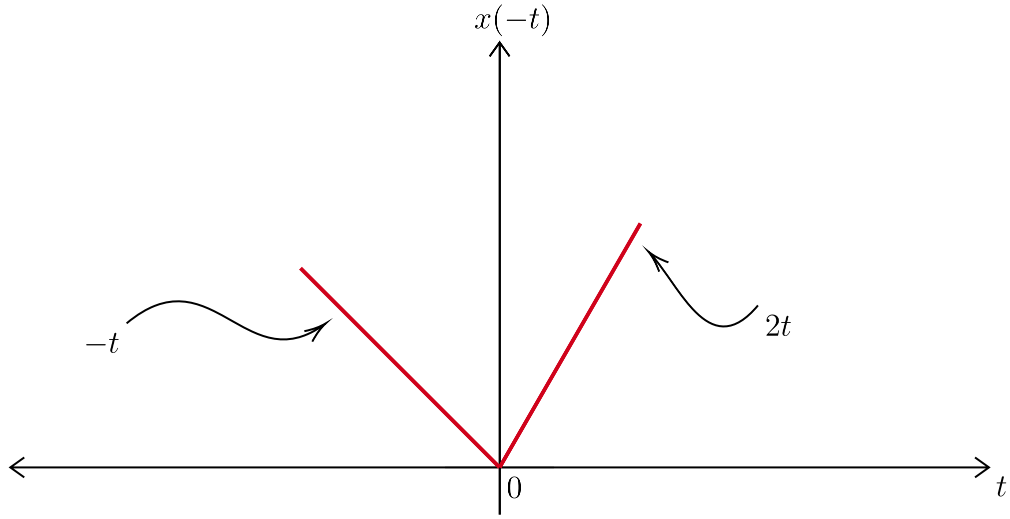


Figure 14: $x(t) \rightarrow x(-t)$

From here, we find the average of the two to find the even part:

$$\frac{x(t)+x(-t)}{2} \rightarrow \frac{-2t-t}{2} = -\frac{3t}{2} \text{ for } t<0$$
$$\frac{x(t)+x(-t)}{2} \rightarrow \frac{2t+t}{2} = \frac{3t}{2} \text{ for } t>0$$

Then, we plot this:

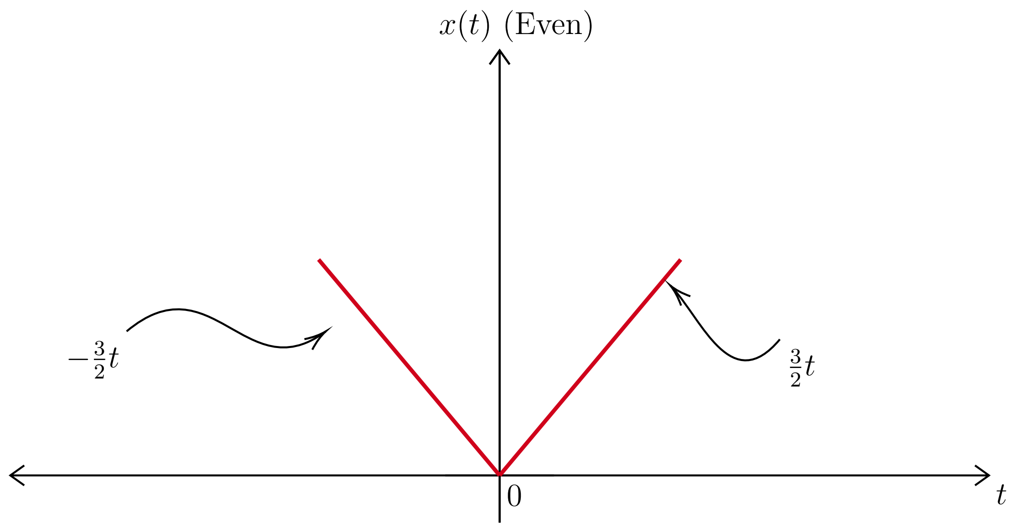


Figure 15: Even $x(t)$

From here, we subtract $x(-t)$ from $x(t)$ and then divide by two:

$$\frac{x(t)-x(-t)}{2} \to \frac{-2t+t}{2} = -\frac{t}{2} \text{ for } t<0$$
$$\frac{x(t)-x(-t)}{2} \to \frac{t-2t}{2} = -\frac{t}{2} \text{ for } t>0$$

Then, we plot this:

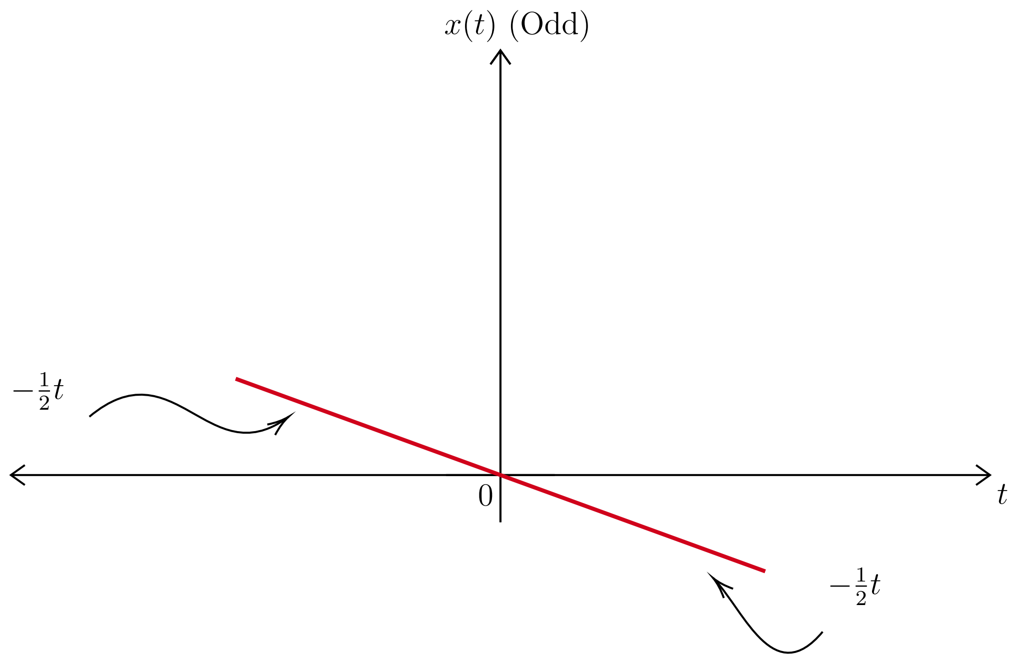


Figure 16: Odd $x(t)$

Hence, we have the even and odd figures here.

6. Determine and sketch the even and odd parts of the signal depicted in Figure P1.6. Label your sketches carefully.

   For this problem, we find even using:

$$\frac{x[n] + x[-n]}{2}$$

   and the odd part with:

$$\frac{x[n] - x[-n]}{2}$$

   The original signal is as follows:

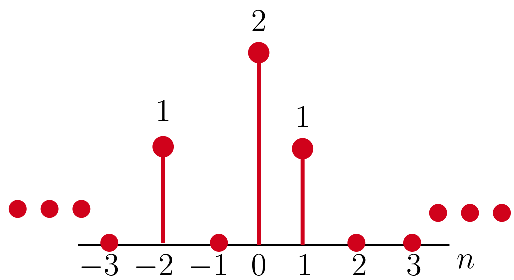


Figure 17: Original Plot

Using the above formulas, we can draw the even:

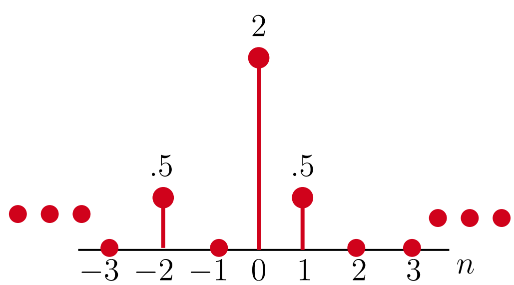


Figure 18: Even $x[n]$

and then the odd:

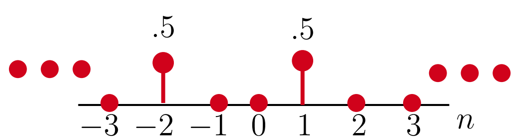


Figure 19: Odd $x[-n]$

7. Express the real part of each of the following signals in the form $Ae^{-at}\cos(\omega t + \phi)$ where A, a, ω and ϕ are real numbers with $A > 0$ and $-\pi < \phi \leq \pi$.

   (a) $x_1(t) = 4e^{-2t}\sin\left(10t + \frac{3\pi}{4}\right)\cos\left(10t + \frac{3\pi}{4}\right)$

   Per trig identities, we can rewrite this as:

$$2e^{-2t}\sin\left(20t + \frac{3\pi}{2}\right)$$

Per another identity, we can convert sin $\to$ cos:

$$2e^{-2t}\cos\left(\frac{\pi}{2} - 20t - \frac{3\pi}{2}\right)$$
$$x_1(t) = 2e^{-2t}\cos(-20t - \pi)$$

Since $\cos(x) = \cos(-x)$, we finally write:

$$\boxed{x_1(t) = 2e^{-2t}\cos(20t + \pi)}$$

(b) $x_2(t) = j(1-j)e^{(-5+j\pi)t}$

We can rewrite this in terms of exponentials:

$$x_2(t) = e^{\frac{\pi}{2}j}\left(\sqrt{2}e^{-\frac{\pi}{4}j}\right)\left(e^{(-5+j\pi)t}\right)$$
$$x_2(t) = \sqrt{2}e^{-5t}\left(e^{j\pi t+\frac{\pi}{2}j-\frac{\pi}{4}j}\right)$$
$$x_2(t) = \sqrt{2}e^{-5t}\left(e^{j(\pi t+\frac{\pi}{4})}\right)$$
$$\boxed{x_2(t) = \sqrt{2}e^{-5t}\cos\left(\pi t + \frac{\pi}{4}\right)}$$

8. Determine whether each of the following continuous time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x(t) = 5\cos\left(400\pi t + \frac{\pi}{4}\right)$

The function is periodic, with angular frequency $\omega = 400\pi\left[\frac{\text{rad}}{\text{s}}\right]$

This gives us the fundamental period:

$$\boxed{T = \frac{2\pi}{400\pi} = .005[\text{s}]}$$

(b) $x(t) = 20e^{j(\pi t - 2)}$

The function is periodic, with angular frequency $\omega = \pi\left[\frac{\text{rad}}{\text{s}}\right]$

This gives us the fundamental period:

$$\boxed{T = \frac{2\pi}{\pi} = 2[\text{s}]}$$

(c) $x(t) = 2\left[\sin\left(50\pi t - \frac{\pi}{3}\right)\right]^2$

The function is periodic, with angular frequency $\omega = 50\pi\left[\frac{\text{rad}}{\text{s}}\right]$

This gives us the fundamental period:

$$\boxed{T = \frac{2\pi}{50\pi} = .04[\text{s}]}$$

(d) $x(t) = \begin{cases} 2\sin(5\pi t), & t \geq 0 \\ -2\sin(-5\pi t), & t < 0 \end{cases}$

Per trigonometric identities, we know that:

$$\sin(t) = -\sin(-t)$$

Thus, the function presented is simply:

$$x(t) = 2\sin(5\pi t)$$

This function is periodic, with angular frequency $\omega = 5\pi \left[\frac{\text{rad}}{\text{s}}\right]$

This gives us the fundamental period:

$$\boxed{T = \frac{2\pi}{5\pi} = .4[\text{s}]}$$

9. Determine whether each of the following discrete time signals is periodic. If the signal is periodic, determine its fundamental period.

(a) $x[n] = 2\cos\left(\frac{7}{11}n + \frac{\pi}{2}\right)$

To be periodic, $(2\pi/\Omega_o)m$ must be rational. We see:

$$\Omega_o = \frac{7}{11} \rightarrow \frac{22\pi m}{7}$$

As a result of the π, this is never rational and therefore not periodic.

(b) $x[n] = \cos(\pi n) + 4\sin\left(\frac{\pi}{4}n^2\right)$

To be periodic, both sinusoids must have $2\pi/\Omega_o$ be rational. We see:

$$\Omega_1 = \pi \rightarrow \frac{2\pi}{\pi} = 2 \text{ and } \Omega_2 = \pi/4 \rightarrow \frac{2\pi}{\pi/4} = 8$$

Thus, the function is periodic. The period is the smallest number such that the two periods are a common divisor of the integer. Since 8 is divisible by 2, the fundamental period is 8.

(c) $x[n] = 3\sin\left(\frac{\pi}{3}n\right) + \cos\left(\frac{\pi}{4}n\right) - 3\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right)$

Once again, each of the sinusoids must be periodic:

$$\Omega_1 = \pi/3 \rightarrow \frac{2\pi}{\pi/3} = 6 \text{ and } \Omega_2 = \pi/4 \rightarrow \frac{2\pi}{\pi/4} = 8 \text{ and } \Omega_3 = \pi/6 \rightarrow \frac{2\pi}{\pi/6} = 12$$

Thus, we see these functions are all periodic. The smallest integer which is divisible by all of these values is 24, and, thus, the fundamental period is 24.