Lecture 2 — Introduction to Signals and Systems

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- Signal Power and Energy
  - Definition
    * Consider signal $x(t)$ representing the voltage or current in a unit resistance. The signal power is defined as $p(t) = |x(t)|^2$
    * It is a common terminology to refer to $|x(t)|^2$ or $|x[n]|^2$ as the signal power even if the signal does not represent voltage or current
  - Total energy in a finite duration interval
    * The total energy in an interval $T = t_2 - t_1$ is given by:

$$\text{Continuous Time } \to E = \int_{t_1}^{t_2} \underbrace{|x(t)|^2}_{p(t)} \, dt$$

$$\text{Discrete Time } \to E = \Delta T \sum_{n=n_1}^{n_2} \underbrace{|x[n]|^2}_{p(t)} \text{ where } T = (n_2 - n_1 + 1)\Delta T$$

  - The average power in a finite duration interval

$$P_{avg} = \frac{E}{t_2 - t_1} = \frac{1}{t_1 - t_2} \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

or

$$P_{avg} = \frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

- Power and Energy over an infinite time interval

– Energy

$$\text{Continuous Time } \rightarrow E_\infty = \lim_{T\to\infty} \int_{-T}^{T} \underbrace{|x(t)|^2}_{p(t)}\, dt$$

$$\text{Discrete Time } \rightarrow E_\infty = \lim_{N\to\infty} \cancel{\Delta T} \sum_{n=-N}^{N} \underbrace{|x[n]|^2}_{p(t)}$$

– Power

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \underbrace{|x(t)|^2}_{p(t)}\, dt$$

or

$$P_\infty = \lim_{N\to\infty} \frac{E_\infty}{2N+1} = \lim_{T\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \underbrace{|x[n]|^2}_{p(t)}$$

- Energy Signals versus Power Signals
  - The energy or power of a signal quantifies the magnitude of the signal. For this measure to be meaningful, it must be finite. This requirement leads to the following classification of signals:
    - Energy
      - Signals with finite total energy ($E_\infty < \infty$)
      - They have zero average power

$$P_\infty = \lim_{T\to\infty} \frac{E_\infty}{2T} = 0$$
$$P_\infty = \lim_{N\to\infty} \frac{E_\infty}{2N+1} = 0$$

    - Power
      - Signals with finite average power ($P_\infty < \infty$)
      - They have infinite energy

$$E_\infty = \lim_{T\to\infty} 2T(P_\infty) \rightarrow \infty$$
$$E_\infty = \lim_{N\to\infty} (2N+1)(P_\infty) \rightarrow \infty$$

    - Any finite signal is automatically an energy signal (think: some value in range, 0 otherwise)

- Periodic Signals

  - Periodic signals are classified as power signals because they possess an infinite amount of energy
  - The average power of a periodic signal can be determined by averaging its power over one period:

$$P_\infty = P_{avg} = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x(t)|^2 \, dt$$

- Signals with neither finite power nor energy
  - Some signals have neither finite power nor energy
  - An example is a ramp signal, where $x(t) = t$, $t \geq 0$
  - Neither the energy nor the power can be defined for such signals
- Transformation of the independent variable
  - In this section, we will explore key elementary signal transformations that involve basic modifications of the independent variable for both discrete and continuous-time signals. These transformations include:
    * Time shifting
    * Time scaling
    * Time reversal
    * Combined operations
- Time Shifting
  - Given a signal $x(t)$, a shift could be written as $y(t) = x(t - t_o)$. This would mean that:

$$x(t - t_o) = x(t_{old}) \text{ at } t = t_{old} + t_o$$

  - For shift $x(t - t_o)$ the signal is shifted to the right, and for shift $x(t + t_o)$ the signal is shifted to the left
  - For discrete time, a shift has the same effect
- Time Reversal
  - Time reversal is performed using a 180° rotation around the vertical axis. If $x(t)$ represents an audio recording, $x(-t)$ is the audio recording played backward
  - For example $x(t) = t$, a linear line with slope one, would become $x(-t) = -t$ upon reversal, with slope negative one (reflected about vertical axis)
- Time Scaling

- Continuous Time Signals
  * $x(\alpha t)$ leads to linear compression if $\alpha > 1$ and linear stretching if $0 \leq \alpha \leq 1$
  * If $x(t)$ is an audio recording, the expression $x(2t)$ represents the recording played at twice the speed, and $x\left(\frac{1}{2}t\right)$ is the recording played at half the speed
  * In the time scaling operation, $t = 0$ serves as a fixed anchor point and remains unchanged, since $x(t) = x(\alpha t)$ at $t = 0$
  * The concepts of compression and expansions differ slightly for discrete time signals
- Discrete Time Signals
  * Zero remains as an anchor point
  * Take only equivalent values at integer n values, and assume $y[n] = 0$ for non-integer n values
  * For $x\left[\frac{n}{L}\right]$, which increases the sampling rate, (resample) the sequence by a factor of L, a process known as up-sampling
  * $L - 1$ zeros are inserted between each consecutive data points
  * Usually followed by a low pass filter to interpolate
  * Compression is much simpler, as a compression would lose information, but will only use integer n values
  * First, a low pass filter is used, and the values are down-sampled
  * For $x[Mn]$, the signal is decimated by a factor of M, which keeps only very M-th sample

- Combined Operations
  - For $y(t) = x(\alpha t + \beta) = x(t_{old}$ we can write:

$$t_{old} = \alpha t + \beta$$
$$t = \frac{1}{\alpha}\left(t_{old} - \beta\right)$$

  * This means the signal was scaled by $1/\alpha$ and shifted to the left β/α

- Euler Formula
  - $Ae^{j(\omega_o t+\phi)} = A\left\{\cos(\omega_o t + \phi) + j\sin(\omega_o t + \phi)\right\}$
  - $Ae^{-j(\omega_o t+\phi)} = A\left\{\cos(\omega_o t + \phi) - j\sin(\omega_o t + \phi)\right\}$
  - $A\cos(\omega_o t + \phi) = \dfrac{A}{2}\left\{e^{j\omega_o t+\phi)} + e^{-j(\omega_o t+\phi)}\right\} = \mathrm{Re}\left\{Ae^{j(\omega_o t+\phi)}\right\}$
  - $A\sin(\omega_o t + \phi) = \dfrac{A}{2j}\left\{e^{j\omega_o t+\phi)} - e^{-j(\omega_o t+\phi)}\right\} = \mathrm{Im}\left\{Ae^{j(\omega_o t+\phi)}\right\}$

- Sinusoidal Waveform, Discrete Time Case: The Discrete Time Frequency

- The continuous time signal $x(t) = A\sin(\omega_o t + \phi) = A\sin(2\pi f_o t + \phi)$
- The continuous time frequency (f_o) is in cycles/second and ω_o is in radians/second
- The discrete time signal can be expressed as:

$$x[n] = x(nT_s) = A\sin(2\pi f_o nT_s + \phi) = A\sin\left(2\pi\frac{f_o}{f_s}n + \phi\right) = A\sin(2\pi Fn + \phi)$$

- Where $F = \dfrac{f_o}{f_s} = \dfrac{T_s}{T_o}$ is the discrete time frequency, in cycles/sample

$$\frac{T_o}{T_s} = \text{ number of samples per one cycle of the signal}$$

$$x[n] = A\sin(\Omega n + \phi),\ \Omega \text{ is in radians/sample, } \Omega = 2\pi F$$

- $\Omega = 2\pi\dfrac{f_o}{f_s}$ is the discrete time frequency in radians/sample
- $\Omega = \pi$ is the largest discrete time frequency that corresponds to the lowest sampling rate (Nyquist rate)

- Is a Discrete Time Sinusoidal Waveform Always Periodic
  - A continuous time sinusoidal signal in the form $A\cos(\omega_o t + \phi)$ is always periodic with a fundamental period $T_o = \dfrac{2\pi}{\omega_o}$
  - For a discrete time sinusoidal signal, $x[n] = A\cos(\omega_o n)$ to be periodic, we must have an integer period N_o where $x[n + N_o] = x[n]$
  - $A\cos(\Omega_o(n + N_o)) = A\cos(\omega_o + 2m\pi),\ m = 0, 1, 2\cdots$
  - $N_o = \dfrac{2\pi}{\Omega_o}m$, which requires the ratio $\dfrac{2\pi}{\Omega_o}$ to be rational

- Exponential and Sinusoidal Waveforms
  - These signals occur frequently and serve as fundamental building blocks for constructing more complex signals
  - A continuous-time signal would be of form: $x(t) = Ce^{\alpha t}$
  - $\alpha > 0 \rightarrow$ rising exponential and $\alpha < 0 \rightarrow$ decaying exponential
  - A discrete-time signal would be of form: $x[n] = C\alpha^n$
  - $|\alpha| > 1 \rightarrow$ rising exponential and $|\alpha| < 1 \rightarrow$ decaying exponential
  - Imaginary Exponentials: $x(t) = Ce^{j\omega_o t}$, where ω_o is the fundamental frequency in radians/s
    - $x(t + T_o) = e^{j\omega_o(t+T_o)} = e^{j\omega_o t}e^{j\omega_o T_o} = e^{j\omega_o t}$ if $\omega_o T_o = 2\pi$
    - This complex exponential signal is periodic with fundamental period $T_o = \frac{2\pi}{\omega_o}$

- Imaginary Exponentials (Discrete Case): $x[n] = Ce^{j\Omega_o n}$
  - The complex exponential can be expressed using the Euler formula in terms of sinusoidal signals ${}^{j\Omega_o n} = C\{\cos(\Omega_o n) + j\sin(\Omega_o n)\}$
- Total energy in one period for such an exponential is $|C|^2 T_o$
- Total energy is infinite, since there is an infinite number of cycles
- Average power in a period is 1
- Average power in an infinite interval is $|C|^2$

- Harmonically Related Signals
  - Continuous Time Case
    - For a signal form of $\phi_k(t) = e^{jk\omega_o t}$, where $k = 0, \pm 1, \pm 2, \cdots$
      - The frequency $\omega = k\omega_o$ is an integral multiple of the fundamental frequency, ω_o
      - The period is $T_k = \frac{2\pi}{|k|\omega_o} = \frac{T_o}{|k|}$
  - Discrete Time Case
    - For a signal of form $\phi_k[n] = e^{jk\Omega_o n}$, where $k = 0, \pm 1, \pm 2, \cdots$
      - The frequency $\Omega = k\Omega_o$ is an integral multiple of the fundamental frequency, Ω_o
      - The period is $N_k = \frac{2\pi}{|k|\Omega_o} m = N_o \left(\frac{m}{|k|}\right)$

- Combining Complex Exponentials
  - Example:

$$x(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \rightarrow 2e^{.5j(\omega_1+\omega_2)t} \cos\left(\frac{(\omega_1 - \omega_2)t}{2}\right)$$

  - Multiply by a conjugate to more easily find the magnitude:

$$|x(t)|^2 = x(t)x^*(t)$$

- General Complex Exponential
  - Continuous Time
    - $x(t) = Ce^{\alpha t}$ (both C and α are complex)
    - $C = |C|e^{j\theta}$ and $\alpha = r + j\omega_o$
    - $x(t) = |C|e^{j\theta}e^{(r+j\omega_o)t} = |C|e^{rt}e^{j(\omega_o t+\theta)}$