Homework 5

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1. (a) We may begin by rewriting S_2 as:

$$w[n] = y[n] - \frac{1}{2}y[n-1]$$

S_1 may be rewritten in a similar format to get:

$$x[n] = w[n] - \frac{1}{4}w[n-1]$$

Substituting the first equation into the second, we get:

$$x[n] = y[n] - \frac{1}{2}y[n-1] - \frac{1}{4}\left[y[n-1] - \frac{1}{2}y[n-2]\right]$$

This can be simplified to:

$$\boxed{x[n] = y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2]}$$

(b)

(c)

2. (a)

(b)

(c)

3. (a) Setting up the Laplace Transform, we get:

$$X(s) = \int_2^\infty e^{-(4+s)t}\,dt$$

$$X(s) = -\frac{e^{-(4+s)t}}{4+s}\bigg|_2^\infty$$

$$X(s) = \frac{e^{-2(4+s)}}{s+4}$$

$$\boxed{X(s) = \frac{e^{-2s-8}}{s+4}}$$

Since the equation is right-sided, the ROC is to the right of the right-most pole; there is one pole at $s = -4$, so the ROC is $\text{Re}\{s\} > -4 \longrightarrow \sigma > -4$ (since $s = \sigma + j\omega$)

(b) We may find the Laplace Transform to be:

$$G(s) = \int_{-\infty}^{-2} Ae^{-(4+s)t}\,dt$$

$$G(s) = -A\frac{e^{-(4+s)t}}{4+s}\Big|_{-\infty}^{-2}$$

$$G(s) = -\frac{A}{4+s}\left[e^{2(4+s)} - e^{\infty(4+s)}\right]$$

We may see that $G(s)$ converge only when s reaches the ROC at $\sigma < -4$

We can thus drop the term to get

$$G(s) = -\frac{Ae^{2s+8}}{4+s}$$

We can check the value of A:

$$-Ae^{2s+8} = e^{-2s-8}$$

We may see that, though the exponents will never be the same, we may take $A = -1$ to create a similar algebraic form. Thus, we say:

$$\boxed{A = -1}$$

4. (a)

   (b)

5. We may rewrite $x(t)$ as:

$$x(t) = e^t \sin(5t)u(-t)$$

Which gives us:

$$X(s) = \int_{-\infty}^{0} e^{-(s-1)t} \sin(5t)\,dt$$

$$X(s) = -\frac{5}{(s-1)^2 + 25}$$

$$\boxed{X(s) = -\frac{5}{s^2 - 2s + 26}}$$

We may see by the second equation that the region of convergence is left-handed, and occurs at $\boxed{\text{ROC: Re}\{s\} - 1 < 0 \longrightarrow \sigma < 1}$. The poles will occur at the solutions to the quadratic in the denominator:

$$s^2 - 2s + 26 = 0$$
$$\frac{2 \pm \sqrt{4 - 4(1)(26)}}{2}$$
$$\frac{2 \pm 10j}{2}$$
$$\boxed{\text{Poles at: } s = 1 \pm 5j}$$

6. Using partial fraction decomposition, we may write:

$$\frac{s-1}{(s+1)(s+3)(s^2+4s+20)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4s+20}$$

And then:

$$(s+3)(s^2+4s+20)A + (s+1)(s^2+4s+20)B + (s+1)(s+3)(Cs+D) = s-1$$

$$As^3+7As^2+32As+60A+Bs^3+5Bs^2+24Bs+20B+Cs^3+4Cs^2+3Cs+Ds^2+4Ds+3D = s-1$$

From this, we may derive:

$$A + B + C = 0$$
$$7A + 5B + 4C + D = 0$$
$$32A + 24B + 3C + 4D = 1$$
$$60A + 20B + 3D = -1$$

Using a solver, we obtain:

$$\begin{cases} A &= -\frac{1}{17} \\ B &= \frac{2}{17} \\ C &= -\frac{1}{17} \\ D &= \frac{1}{17} \end{cases}$$

Now with our coefficients, we take the inverse Laplace transforms to get:

$$x(t) = -\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}+\frac{2}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}-\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+4^2}\right\}+\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{4}{(s+2)^2+4^2}\right\}$$

And finally, we get:

$$\boxed{x(t) = -\frac{e^{-t}}{17}+\frac{2e^{-3t}}{17}-\frac{e^{-2t}\cos(4t)}{17}+\frac{e^{-2t}\sin(4t)}{17}}$$

For each term, in order, the ROCs may be identified as: $\sigma = -1$, $\sigma = -3$, and $\sigma = -4$. Since all of the signals are causal, we know the ROCs are to the right. Thus, there will be overlap when σ is greater than the greatest individual ROC, or $\sigma = -1$. This makes the combined ROC: $\boxed{\sigma > -1}$

We may observe that <u>four individual signals</u> contribute to the Laplace Transform. Furthermore, we can find the zeroes and poles as:

$$\boxed{\text{Zero: } s - 1 = 0 \rightarrow s = 1}$$

$$\boxed{\text{Poles: } \begin{cases} s+1 & = & 0 \\ s+3 & = & 0 \\ (s^2+4s+20) & = & 0 \end{cases} \rightarrow \begin{cases} s & = & -1 \\ s & = & -3 \\ s & = & -2 \pm 4j \end{cases}}$$

This can be plotted as:

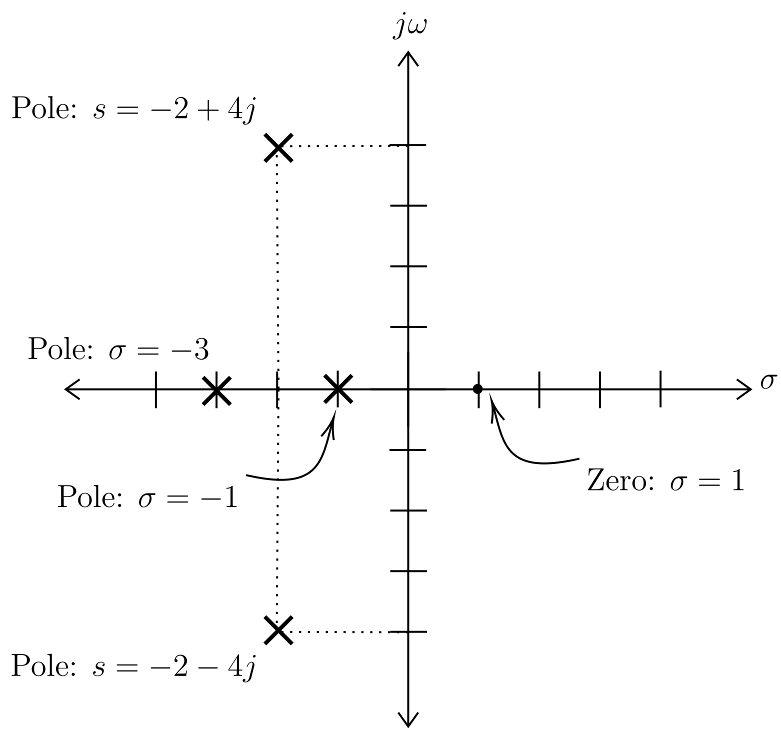

Figure 1: Pole-Zero Plot of $X(s)$

7. (a) Per the basic Laplace transformation tables, we may write:

$$\boxed{X(s) = \frac{1}{s+2} - \frac{1}{s-4}}$$

We may observe that there are two ROCs, $\sigma < 4$ and $\sigma > -2$, which gives us overlap in the region:

$$\boxed{-2 < \sigma < 4}$$

This gives us the following plot:

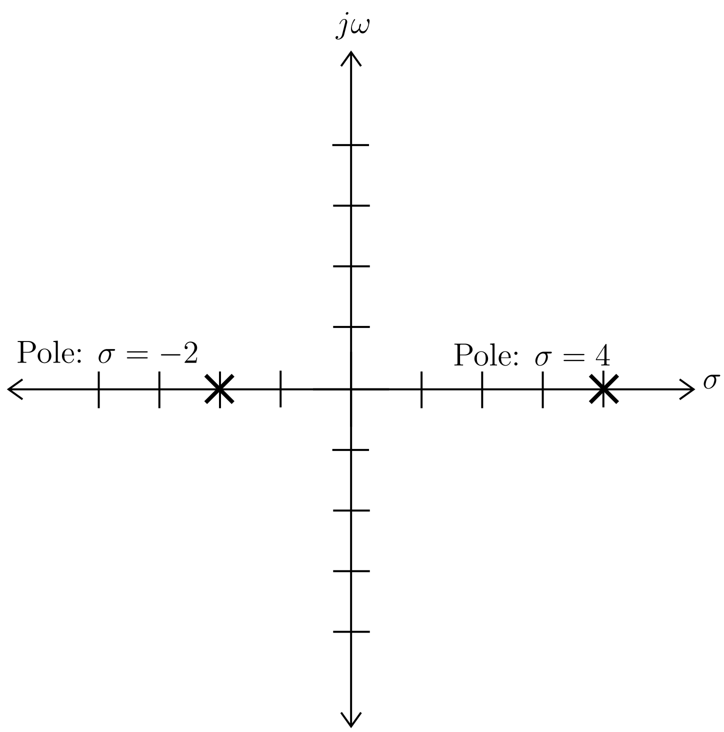

Figure 2: Pole-Zero Plot for 7(a)

(b) Once again employing the tables, we find:

$$X(s) = \frac{1}{s+3} + \frac{4}{(s+2)^2 + 16}$$

Rearranging to simplify ROC analysis, we get:

$$\boxed{X(s) = \frac{s^2 + 8s + 32}{(s+3)(s^2 + 4s + 20)}}$$

From this, we can determine that the zeros are at $s = -4 \pm 4j$ and there are poles at -3 and $-4 \pm 4j$. Since both are right-sided, we may notice that the ROC occurs to the right of the greatest pole, or $\sigma > -3$

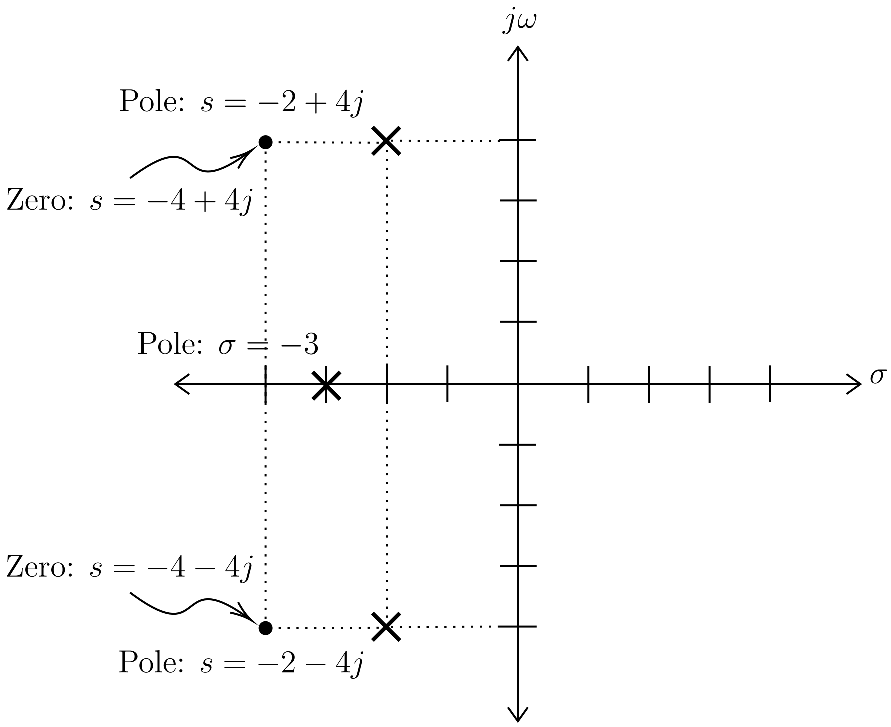

Figure 3: Pole-Zero Plot for 7(b)

(c) We may rewrite $x(t)$ as:

$$x(t) = -te^{2t}u(-t) + te^{-2t}u(t)$$

Using our known transforms:

$$X(s) = -\left[-\frac{d}{ds}\left(\frac{1}{s-2}\right)\right] - \frac{d}{ds}\left(\frac{1}{s+2}\right)$$

$$\boxed{X(s) = \frac{1}{(s+2)^2} - \frac{1}{(s-2)^2}}$$

To simplify analysis, we rearrange to get:

$$X(s) = \frac{(s-2)^2 - (s+2)^2}{(s+2)^2(s-2)^2}$$

$$X(s) = \frac{-8s}{(s+2)^2(s-2)^2}$$

From this, we observe that there is a zero at $s = 0$, and there are poles (both of order 2) $\sigma = \pm 2$. Since both signals are right-handed, the ROC is in: $\sigma > 2$. This gives us the following plot:

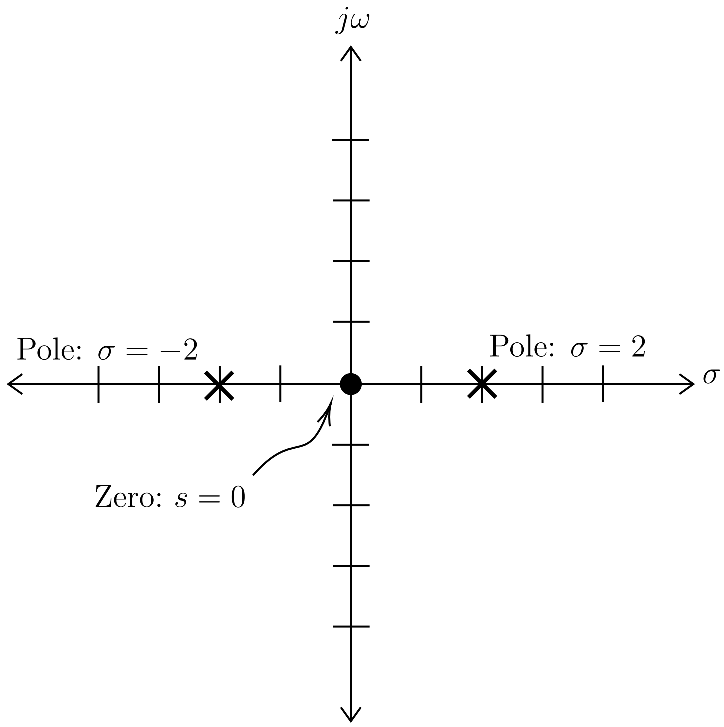

Figure 4: Pole-Zero Plot for 7(c)

(d) We may begin by writing:

$$x(t) = 3r(t) - 3r(t-1)) - 3u(t-2)$$

This gives us the transform as:

$$\boxed{X(s) = \frac{3}{s^2} - \frac{3}{s^2}e^{s} - \frac{3}{s}e^{2s}}$$

We may observe that there are no zeros, but there is a pole at $s = 0$, which gives an ROC of $s > 0$ and the following plot:

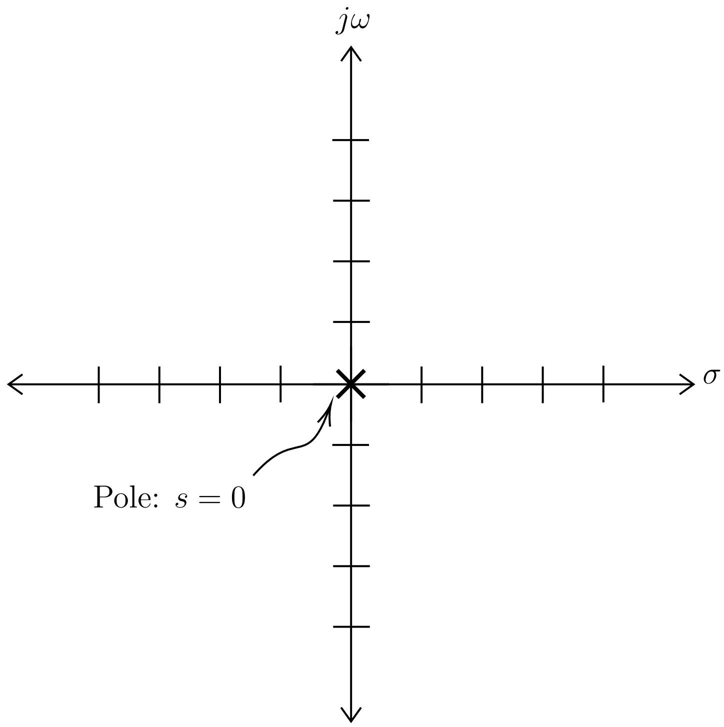

Figure 5: Pole-Zero Plot for 7(d)

8. (a)
   (b)
   (c)
   (d)
   (e)
   (f)
9. (a)
   (b)
   (c)
   (d)
   (e)