

# Homework 5

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1. (a) We may begin by rewriting  $S_2$  as:

$$w[n] = y[n] - \frac{1}{2}y[n-1]$$

$S_1$  may be rewritten in a similar format to get:

$$x[n] = w[n] - \frac{1}{4}w[n-1]$$

Substituting the first equation into the second, we get:

$$x[n] = y[n] - \frac{1}{2}y[n-1] - \frac{1}{4}\left[y[n-1] - \frac{1}{2}y[n-2]\right]$$

This can be simplified to:

$$x[n] = y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2]$$

- (b) We may find the impulse response by taking  $x[n] \rightarrow \delta[n]$ . This gives us:

$$w[n] = \frac{1}{4}w[n-1] + \delta[n]$$

Taking the  $z$  transform, we may write:

$$W(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)}, \quad |z| > .25$$

$$W(z) = \frac{z}{\left(z - \frac{1}{4}\right)}, \quad |z| > .25$$

$$w[n] = \left(\frac{1}{4}\right)^n u[n]$$

Finding the individual impulse response for  $S_2$ , we get:

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > .5$$

$$Y(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > .5$$

$$\boxed{y[n] = \left(\frac{1}{2}\right)^n u[n]}$$

- (c) We can once again use the  $z$  transform, this time using the equation obtained in (a), to get:

$$Y(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$Y(z) = \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$Y(z) = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}, \quad |z| > .5$$

We can use partial fraction decomposition, by rearranging:

$$\frac{Y(z)}{z} = \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}, \quad |z| > .5$$

$$\frac{Y(z)}{z} = \frac{A}{z - .25} + \frac{B}{z - .5}$$

We may find:  $A = -1$ ,  $B = 2$ , which gives us:

$$Y(z) = \frac{-z}{z - .25} + \frac{2z}{z - .5}, \quad |z| > .5$$

We take the inverse transform to get:

$$\boxed{y[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]}$$

2. (a) This gives us:

$$A\left(\frac{1}{4}\right)^n - \frac{A}{4}\left(\frac{1}{4}\right)^{n-1} = 0$$

$$A\left(\frac{1}{4}\right)^n - A\left(\frac{1}{4}\right)^n = 0$$

The two terms and subtract, and do equal zero ✓

(b) We may begin by substituting:

$$B \left(\frac{1}{2}\right)^n - \frac{B}{4} \left(\frac{1}{2}\right)^{n-1} = \left(\frac{1}{2}\right)^n$$

We subtract over:

$$\frac{B}{4} \left(\frac{1}{2}\right)^{n-1} = B \left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^n$$

From this, we can set up an expression for  $B$ :

$$\frac{B}{4} = \frac{1}{2} (B - 1)$$

And now we solve:

$$-\frac{B}{4} = -\frac{1}{2}$$

$B = 2$

(c) We first work to find the value of  $y[0]$ :

$$y[0] - \frac{1}{4}y[-1] = \left(\frac{1}{2}\right)^0$$

We know the term that occurs before  $n = 0$  is zero due to the rest condition, and, thus:

$$y[0] = 1$$

Now we plug this into the overall solution:

$$y[0] = A \left(\frac{1}{4}\right)^0 + 2 \left(\frac{1}{2}\right)^0$$

$$A + 2 = 1$$

$A = -1$

Thus, we see that the solution is:

$$y[n] = -\left(\frac{1}{4}\right)^n + 2 \left(\frac{1}{2}\right)^n$$

Which can be simplified:

$$y[n] = \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{4}\right)^n$$

3. (a) Setting up the Laplace Transform, we get:

$$X(s) = \int_2^{\infty} e^{-(4+s)t} dt$$

$$X(s) = -\frac{e^{-(4+s)t}}{4+s} \Big|_2^{\infty}$$

$$X(s) = \frac{e^{-2(4+s)}}{s+4}$$

$$\boxed{X(s) = \frac{e^{-2s-8}}{s+4}}$$

Since the equation is right-sided, the ROC is to the right of the right-most pole; there is one pole at  $s = -4$ , so the ROC is  $\text{Re}\{s\} > -4 \rightarrow \sigma > -4$  (since  $s = \sigma + j\omega$ )

- (b) We may find the Laplace Transform to be:

$$G(s) = \int_{-\infty}^{-2} A e^{-(4+s)t} dt$$

$$G(s) = -A \frac{e^{-(4+s)t}}{4+s} \Big|_{-\infty}^{-2}$$

$$G(s) = -\frac{A}{4+s} [e^{2(4+s)} - e^{\infty(4+s)}]$$

We may see that  $G(s)$  converge only when  $s$  reaches the ROC at  $\sigma < -4$

We can thus drop the term to get

$$G(s) = -\frac{A e^{2s+8}}{4+s}$$

We can check the value of  $A$ :

$$-A e^{2s+8} = e^{-2s-8}$$

We may see that, though the exponents will never be the same, we may take  $A = -1$  to create a similar algebraic form. Thus, we say:

$$\boxed{A = -1}$$

4. (a) We may calculate the Laplace transform for the time-shifted signal as:

$$X_1(s) = \int_{-\infty}^{\infty} x(t - t_o) e^{-st} dt$$

Taking  $t - t_o = n$ , we write:

$$X_1(s) = \int_{-\infty}^{\infty} x(n)e^{-s(n+t_o)} dn$$

$$X_1(s) = e^{-st_o} \underbrace{\int_{-\infty}^{\infty} x(n)e^{-sn} dn}_{X(s)}$$

Which gives us:

$$\boxed{X_1(s) = e^{-st_o} X(s)}$$

And we may see that the ROC remains to be  $R$ .

(b) We may express the signal as:

$$y(t) = u(t) - 2u(t-2) + u(t-4)$$

Thus, we use our table of known transforms to get:

$$\boxed{Y(s) = \frac{1}{s} - \frac{2e^{-2t}}{s} + \frac{e^{-4t}}{s}}$$

We may see that the ROC is:  $\boxed{\sigma > 0}$

5. We may rewrite  $x(t)$  as:

$$x(t) = e^t \sin(5t)u(-t)$$

Which gives us:

$$X(s) = \int_{-\infty}^0 e^{-(s-1)t} \sin(5t) dt$$

$$X(s) = -\frac{5}{(s-1)^2 + 25}$$

$$\boxed{X(s) = -\frac{5}{s^2 - 2s + 26}}$$

We may see by the second equation that the region of convergence is left-handed, and occurs at  $\boxed{\text{ROC: } \text{Re}\{s\} - 1 < 0 \longrightarrow \sigma < 1}$ . The poles will occur at the solutions to the quadratic in the denominator:

$$s^2 - 2s + 26 = 0$$

$$\frac{2 \pm \sqrt{4 - 4(1)(26)}}{2}$$

$$\frac{2 \pm 10j}{2}$$

$$\boxed{\text{Poles at: } s = 1 \pm 5j}$$

6. Using partial fraction decomposition, we may write:

$$\frac{s-1}{(s+1)(s+3)(s^2+4s+20)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4s+20}$$

And then:

$$(s+3)(s^2+4s+20)A + (s+1)(s^2+4s+20)B + (s+1)(s+3)(Cs+D) = s-1$$

$$As^3+7As^2+32As+60A+Bs^3+5Bs^2+24Bs+20B+Cs^3+4Cs^2+3Cs+Ds^2+4Ds+3D = s-1$$

From this, we may derive:

$$A + B + C = 0$$

$$7A + 5B + 4C + D = 0$$

$$32A + 24B + 3C + 4D = 1$$

$$60A + 20B + 3D = -1$$

Using a solver, we obtain:

$$\begin{cases} A &= -\frac{1}{17} \\ B &= \frac{2}{17} \\ C &= -\frac{1}{17} \\ D &= \frac{1}{17} \end{cases}$$

Now with our coefficients, we take the inverse Laplace transforms to get:

$$x(t) = -\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+4^2}\right\} + \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{4}{(s+2)^2+4^2}\right\}$$

And finally, we get:

$$\boxed{x(t) = -\frac{e^{-t}}{17} + \frac{2e^{-3t}}{17} - \frac{e^{-2t}\cos(4t)}{17} + \frac{e^{-2t}\sin(4t)}{17}}$$

For each term, in order, the ROCs may be identified as:  $\sigma = -1$ ,  $\sigma = -3$ , and  $\sigma = -4$ . Since all of the signals are causal, we know the ROCs are to the right. Thus, there

will be overlap when  $\sigma$  is greater than the greatest individual ROC, or  $\sigma = -1$ . This makes the combined ROC:  $\sigma > -1$

We may observe that four individual signals contribute to the Laplace Transform. Furthermore, we can find the zeroes and poles as:

$$\text{Zero: } s - 1 = 0 \rightarrow s = 1$$

$$\text{Poles: } \begin{cases} s + 1 = 0 \\ s + 3 = 0 \\ (s^2 + 4s + 20) = 0 \end{cases} \rightarrow \begin{cases} s = -1 \\ s = -3 \\ s = -2 \pm 4j \end{cases}$$

This can be plotted as:

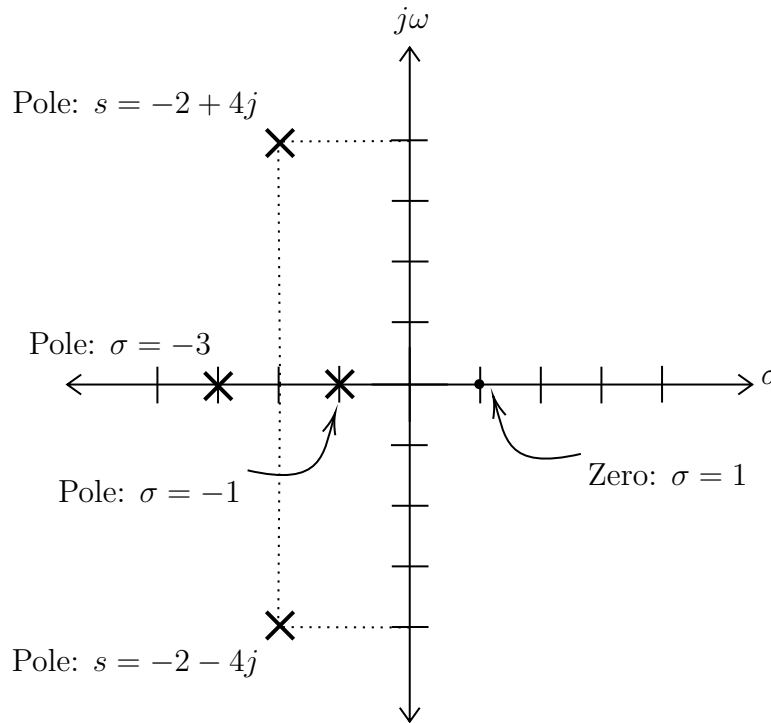


Figure 1: Pole-Zero Plot of  $X(s)$

7. (a) Per the basic Laplace transformation tables, we may write:

$$X(s) = \frac{1}{s+2} - \frac{1}{s-4}$$

We may observe that there are two ROCs,  $\sigma < 4$  and  $\sigma > -2$ , which gives us overlap in the region:

$$\boxed{-2 < \sigma < 4}$$

This gives us the following plot:

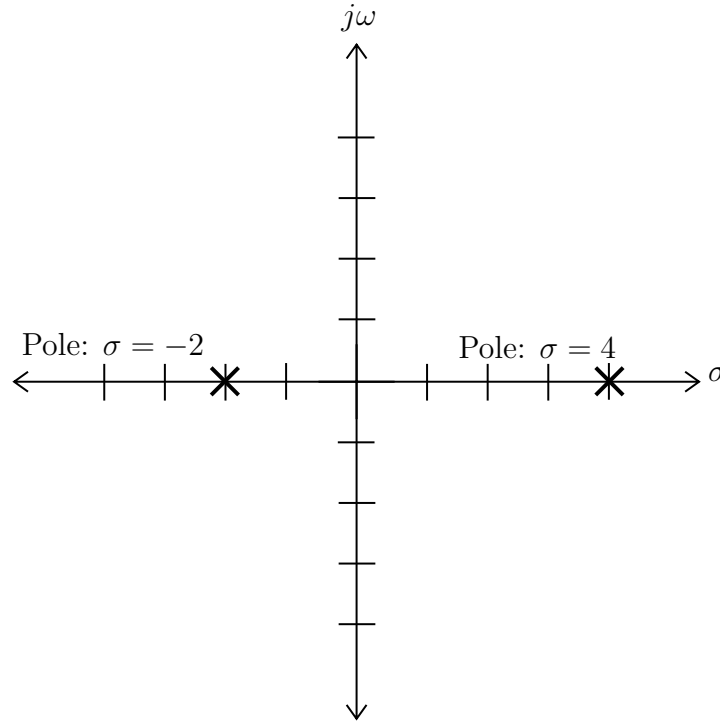


Figure 2: Pole-Zero Plot for 7(a)

(b) Once again employing the tables, we find:

$$X(s) = \frac{1}{s+3} + \frac{4}{(s+2)^2 + 16}$$

Rearranging to simplify ROC analysis, we get:

$$\boxed{X(s) = \frac{s^2 + 8s + 32}{(s+3)(s^2 + 4s + 20)}}$$

From this, we can determine that the zeros are at  $s = -4 \pm 4j$  and there are poles at  $-3$  and  $-2 \pm 4j$ . Since both are right-sided, we may notice that the ROC occurs to the right of the greatest pole, or  $\sigma > -3$



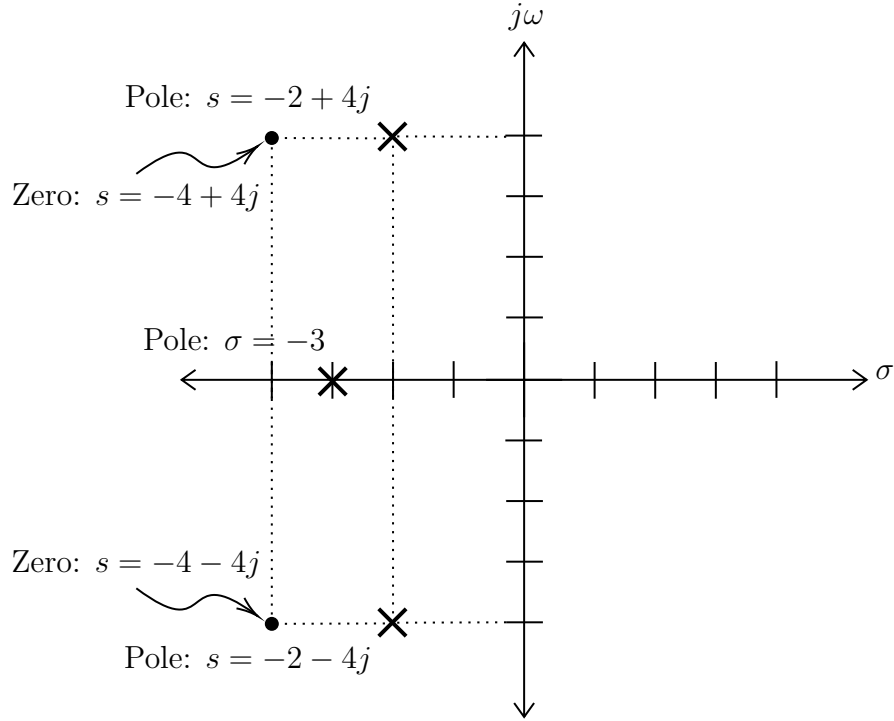


Figure 3: Pole-Zero Plot for 7(b)

(c) We may rewrite  $x(t)$  as:

$$x(t) = -te^{2t}u(-t) + te^{-2t}u(t)$$

Using our known transforms:

$$X(s) = \left[ \frac{d}{ds} \left( \frac{1}{s-2} \right) \right] + \frac{d}{ds} \left( -\frac{1}{s+2} \right)$$

$$X(s) = \frac{1}{(s+2)^2} - \frac{1}{(s-2)^2}$$

To simplify analysis, we rearrange to get:

$$X(s) = \frac{(s-2)^2 - (s+2)^2}{(s+2)^2(s-2)^2}$$

$$X(s) = \frac{-8s}{(s+2)^2(s-2)^2}$$

From this, we observe that there is a zero at  $s = 0$ , and there are poles (both of order 2)  $\sigma = \pm 2$ . Since both signals are right-handed, the ROC is in:  $\sigma > 2$ . This gives us the following plot:

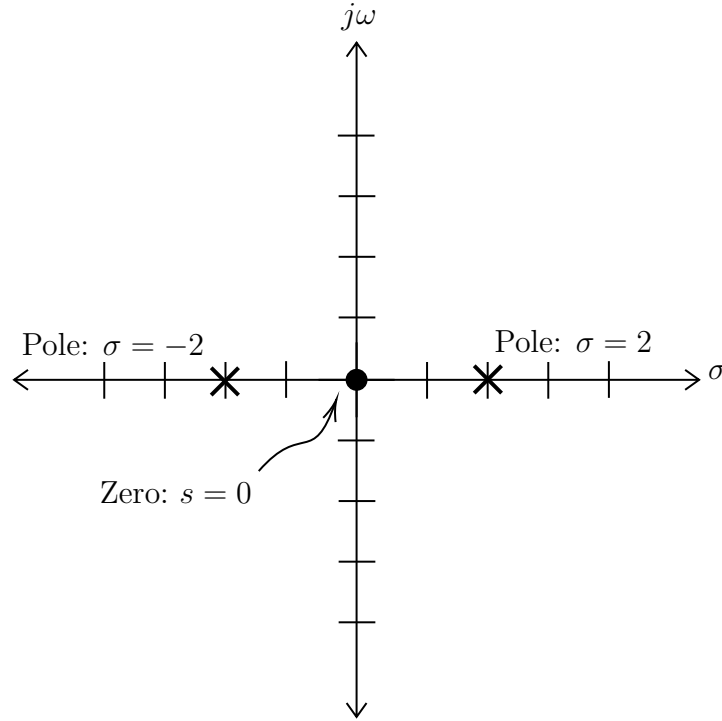


Figure 4: Pole-Zero Plot for 7(c)

(d) We may begin by writing:

$$x(t) = 3r(t) - 3r(t - 1)) - 3u(t - 2)$$

This gives us the transform as:

$$X(s) = \frac{3}{s^2} - \frac{3}{s^2}e^s - \frac{3}{s}e^{2s}$$

We may observe that there are no zeros, but there is a pole at  $s = 0$ , which gives an ROC of  $s > 0$  and the following plot:

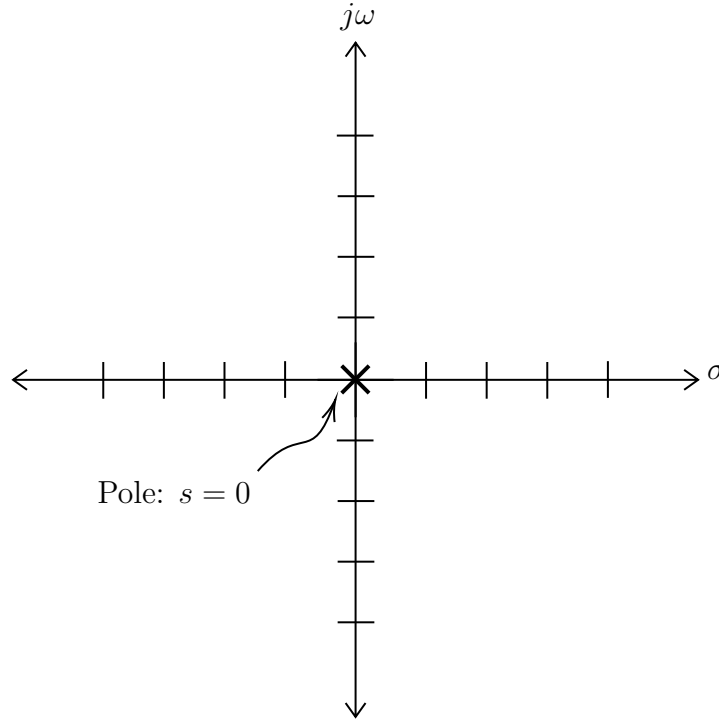


Figure 5: Pole-Zero Plot for 7(d)

8. Since the function is absolutely integrable, we know that it is bounded by BIBO stability. This means that the ROC contains the  $j\omega$  axis, and that the ROC does not have any poles. Knowing this, we can deduce:

(a) The function can not be finite

This is due to the fact that, since there is a pole at  $s = -4$ , we know that the signal contains  $x(t) = e^{-4t}$ . Since this function extends to  $t \rightarrow \infty$ , we know that the signal can not be finite. Also, note that a finite signal, by definition, does not contain poles.

(b) The function can not be left-sided

Since we know that, due to being absolutely integrable, the function must contain the  $j\omega$  axis, the right-most pole must be to the right of this axis. Since  $s = -4$  is to the left of the axis, the signal can not be left-sided.

(c) The function can be right-sided

An example of such a signal would be:

$$x(t) = e^{-4t}u(t)$$

- (d) The function can be two-sided (given that there may be multiple poles — the problem states that there *is* a pole at  $s = -4$ , but does not state that it is the only pole)

Take, for example, the signal (where  $n > 0$ ):

$$x(t) = e^{-4t}u(t) + e^{nt}u(-t)$$

This signal is two-sided, contains  $s = -4$  as a pole, is absolutely integrable (and thus contains the  $j\omega$  axis), which makes it a valid signal.

- (e)
- With the absolutely integrable rule change from  $x(t)$  to  $x(t)e^{-2t}$ , nothing would change. This is because it would mean that, instead of the  $j\omega$  axis, the ROC would now need to contain the  $\sigma = 2$  axis. Thus, the properties remain the same since this is to the right of the pole at  $s = -4$ .
  - Changing the condition such that  $x(t)e^{5t}$  is absolutely integrable would change things. This would mean that the ROC would need to contain the  $\sigma = -5$  axis, instead of the  $j\omega$  axis, which is left of the pole. Thus, the signal can now be left-sided, but not right-sided
- (f) We know that, due to the integrable rule, the ROC must be right of the right-most pole (to contain the  $j\omega$  axis). This means that the ROC is  $\sigma > -2$ . From here, we write  $X(s)$  as:

$$X(s) = \frac{N(s-1)}{(s+4)(s+2-j2)}$$

We can solve for  $N$  by using the initial condition:

$$\begin{aligned} X(0) &= \frac{-N}{4(2-j2)} \\ \frac{-N}{4(2-j2)} &= -1 \\ N &= 8-j8 \end{aligned}$$

This gives us:

$$X(s) = \frac{(8-j8)(s-1)}{(s+4)(s+2-j2)}$$

9. (a) Using the table, we get:

$$x(t) = \frac{1}{3} \sin(3t)u(t)$$

And we see that the ROC is  $\sigma > 0$

- (b) Using the table, we get:

$$x(t) = -\cos(2t)u(-t)$$

And we see that the ROC is  $\sigma < 0$

(c) Using the table, we get:

$$x(t) = -e^{-5t} \cos(4t)u(-t)$$

And we see that the ROC is  $\sigma < -5$

(d) To simplify analysis, we may rewrite this as:

$$X(s) = \frac{1}{(s+4)(s+2)}$$

We use partial fraction decomposition to get:

$$X(s) = \frac{A}{s+2} + \frac{B}{s+4}$$

Solving, we find:  $A = -.5$  and  $B = .5$ , which gives us:

$$X(s) = \frac{.5}{s+4} - \frac{.5}{s+2}$$

Finally, using the table, we get:

$$x(t) = -\frac{1}{2} [e^{-4t}u(t) + e^{-2t}u(-t)]$$

And we see the ROC is:  $-4 < \sigma < -2$

(e) This can be expanded to:

$$X(s) = \frac{(s+2)^2 - 5s - 10 + 8}{(s+2)^2}$$

We then break this into:

$$X(s) = 1 - \frac{5}{s+2} + \frac{8}{(s+2)^2}$$

We now use the table to find:

$$x(t) = \delta(t) - 5e^{-2t}u(t) + 8te^{-2t}u(t)$$

With ROC:  $\sigma > -2$