Homework 4

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1. (a) We begin by taking the Laplace transform to get:

$$H(s) = -\frac{1}{s-1} \quad \text{and} \quad X(s) = \frac{2}{s}\left[e^{-s} - e^{-2s}\right]$$

We then multiply the two to get:

$$Y(s) = X(s)H(s)$$
$$Y(s) = -\frac{2}{s(s-1)}\left[e^{-s} - e^{-2s}\right]$$

Using partial fraction decomposition, we may write the equivalent such that:

$$-\frac{2}{s(s-1)} = \frac{A}{s-1} + \frac{B}{s} = As + B(s-1)$$

We use $s = 0, 1$ to get:

$$A = -2,\, B = 2 \to -\frac{2}{s-1} + \frac{2}{s}$$

We now distribute this in the above case to get:

$$Y(s) = -\frac{2}{s-1}\left[e^{-s} - e^{-2s}\right] + \frac{2}{s}\left[e^{-s} - e^{-2s}\right]$$
$$Y(s) = -\frac{2e^{-s}}{s-1} + \frac{2e^{-2s}}{s-1} + \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s}$$

Finally, we take the inverse transform to get:

$$\boxed{y(t) = -2u(1-t) + 2e^{t-1}u(1-t) + 2u(2-t) - 2e^{t-2}u(2-t)}$$

(b) Differentiating one of the inputs is the same as differentiating the output. Thus, we may say:

$$g(t) = \frac{d}{dt}[y(t)]$$

$$\boxed{g(t) = 2e^{t-1}u(1-t) - 2e^{t-2}u(2-t)}$$

(c) As stated in (b) $\underline{g(t) = (d/dt)[y(t)]}$

(d) $z(t)$ is the same as $g(t)$. Since taking the differential is a linear operation, it does not matter if this is done to the impulse response or to $x(t)$. Therefore, we get:

(e) By linearity of the transform, we may say:

$$y_1(t) = 2y(t-1)$$

Therefore, we may obtain:

$$\boxed{y_1(t) = -4u(2-t) + 4e^{t-2}u(2-t) + 4u(3-t) - 4e^{t-3}u(3-t)}$$

2. (a) We may write the convolution integral as:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Expressing this in terms of the provided functions, we may write:

$$y(t) = \int_{0}^{\infty} e^{-2\tau}e^{\tau-t}\,d\tau$$

$$y(t) = \int_{0}^{\infty} e^{-(\tau+t)}\,d\tau$$

Integrating, we obtain:

$$y(t) = -e^{-(\tau+t)}\Big|_0^{\infty}$$

$$y(t) = -\frac{1}{e^{\infty+t}} - \left(-e^{-t}\right)$$

Finally, we get (notice the step function returns to set boundaries):

$$\boxed{y(t) = e^{-t}u(t)}$$

(b) From the provided information for $x(t)$, we see that $x(t) \neq 0$ only for $0 \leq t \leq 4$. We can then set up the convolution integral as:

$$y(t) = \int_{-\infty}^{\infty} [u(\tau) - 2u(\tau-1) + u(\tau-4)](e^{t-\tau}u(1-t+\tau))\,d\tau$$

Here, we see that the term provided by $h(t)$ exists only for $\tau \leq t-1$. Thus, we may write:

$$y(t) = \int_0^{t-1} e^{-\tau+t}\, d\tau - 2\int_1^{t-1} e^{-\tau+t}\, d\tau + \int_4^{t-1} e^{-\tau+t}\, d\tau$$

We evaluate to get:

$$y(t) = e^t \left[-e^{-\tau}\Big|_0^{t-1} + 2e^{-\tau}\Big|_1^{t-1} - e^{-\tau}\Big|_4^{t-1} \right]$$

$$y(t) = e^t \left[(-e^{1-t}+1) + 2\left(e^{1-t} - \frac{1}{e}\right) - \left(e^{1-t} - \frac{1}{e^4}\right) \right]$$

$$y(t) = -e + e^t + 2e - e^{t-1} - e + e^{t-4}$$

And finally, this gets us:

$$\boxed{y(t) = e^t - e^{t-1} + e^{t-4}}$$

(c) We may break the overlap into cases to evaluate the impulse response. We know that $h(t-\tau)$ extends from $t-2$ to $t-1$, while $x(\tau)$ extends from 0 to 1. Thus, we may analyze:

- $t-1 < 0$ — There is no overlap
- $0 < t-1 < 1$ — There is overlap
- $0 < t-2 < 1$ — There is overlap
- $t-2 > 1$ — There is no overlap

Thus, we may solve for the impulse response using:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

$$y_1(t) = \int_0^{t-1} \sin(\pi\tau)\, d\tau$$

$$y_2(t) = \int_{t-2}^{1} \sin(\pi\tau)\, d\tau$$

Evaluating, we obtain:

$$y_1(t) = -\frac{1}{\pi}\left[\cos(\pi\tau)\Big|_0^{t-1} \right]$$

$$y_2(t) = -\frac{1}{\pi}\left[\cos(\pi\tau)\Big|_{t-2}^{1} \right]$$

$$y_1(t) = -\frac{1}{\pi}\left[(\cos(\pi t - \pi) - 1) \right]$$

$$y_2(t) = -\frac{1}{\pi}\left[(-1 - \cos(\pi t - 2\pi))\right]$$

Finally, we get:

$$y_1(t) = \frac{1}{\pi} - \frac{\cos(\pi t - \pi)}{\pi}$$

$$y_2(t) = \frac{1}{\pi} + \frac{\cos(\pi t - 2\pi)}{\pi}$$

Implementing boundaries from the two cases, we write:

$$y(t) = \begin{cases} 0, & t - 1 < 0 \\ \frac{1}{\pi} - \frac{\cos(\pi t - \pi)}{\pi}, & 0 < t - 1 < 1 \\ \frac{1}{\pi} + \frac{\cos(\pi t - 2\pi)}{\pi}, & 0 < t - 2 < 1 \\ 0, & t - 2 > 1 \end{cases}$$

This can be simplified as:

$$\boxed{y(t) = \begin{cases} 0, & t < 1 \\ \frac{1}{\pi} - \frac{\cos(\pi t - \pi)}{\pi}, & 1 < t < 2 \\ \frac{1}{\pi} + \frac{\cos(\pi t - 2\pi)}{\pi}, & 2 < t < 3 \\ 0, & t > 3 \end{cases}}$$

(d) We know that the step response can be defined as the integral with respect to time of the impulse response. Assuming zero-state initial conditions, we may write:

$$s(t) = \int y(t)\, dt$$

Evaluating the integral, we get:

$$\boxed{s(t) = \begin{cases} 0, & t < 1 \\ \frac{t}{\pi} - \frac{\sin(\pi t - \pi)}{\pi^2}, & 1 < t < 2 \\ \frac{t}{\pi} + \frac{\sin(\pi t - 2\pi)}{\pi^2}, & 2 < t < 3 \\ 0, & t > 3 \end{cases}}$$

3. (a) Given the set up, we may write:

$$\left(\frac{1}{4}\right)^n u[n] - A\left(\frac{1}{4}\right)^{n-1} u[n-1] = \delta[n]$$

We may redefine the delta as:

$$\left(\frac{1}{4}\right)^n u[n] - A\left(\frac{1}{4}\right)^{n-1} u[n-1] = u[n] - u[n-1]$$

Thus, we see that we need the exponential term to cancel. We can do this by simply taking:

$$\left(\frac{1}{4}\right)^n = A\left(\frac{1}{4}\right)^{n-1}$$

Dividing the exponential from one side to the other, we see:

$$A = \left(\frac{1}{4}\right)^1$$

$$\boxed{A = \frac{1}{4}}$$

(b) By definition, with $h[n]$ and $g[n] = h_{inv}[n]$, we know:

$$h[n] * g[n] = \delta[n]$$

Using the equation from part (a), we know:

$$h[n] - Ah[n-1] = \delta[n]$$

By the properties of convolution, we know that:

$$x[n] * \delta[n - n_o] = x[n - n_o]$$

Thus, we may expand to write:

$$h[n] * \delta[n] - Ah[n] * \delta[n-1] = \delta[n]$$
$$h[n] * (\delta[n] - A\delta[n-1]) = \delta[n]$$

Thus, combining this with the definition of inverse, we may write:

$$\boxed{g[n] = \delta[n] - \frac{1}{4}\delta[n-1]}$$

(c) To find the step response from the impulse response, we may simply sum with respect to n:

$$s[n] = \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^{n-k} u[n]$$

By our series simplification formulas, we may write:

$$s[n] = \frac{1 - \left(\frac{1}{4}\right)^{n+1}}{1 - .25} u[n]$$

$$\boxed{s[n] = \left[4 - \left(\frac{1}{4}\right)^n\right] u[n]}$$

4. (a) We may observe that the system is <u>not causal</u>, since it is non-zero for $n < 0$. Expressing the sum, we may see that:

$$\sum_{-\infty}^{3} 3^n u[3-n] \text{ is finite}$$

And, therefore, the system <u>is stable</u>

(b) We may see that, for $n < 0$, the system is zero, and, therefore, the system <u>is causal</u>. We can break the system apart to analyze stability:

$$\sum_{0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{3}^{\infty} (1.1)^n$$

We may see that, though the first term is finite, the second term is not. Therefore, the system is <u>not stable</u>.

(c) We may see that, due to the $u[2-n]$ term, there are values $n < 0$ for which the function is non-zero, meaning it is <u>not causal</u>. In terms of stability, we may write (taking the effect of cos as worst case magnitude, or 1):

$$\sum_{0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{-\infty}^{2} (1.1)^n$$

We may thus see that both terms are bounded, and, therefore, the system is <u>stable</u>.

(d) We may see that, for $n < 0$ the system is zero; therefore, the system <u>is causal</u>. Analyzing stability, we see:

$$\sum_{0}^{\infty} n \left(\frac{1}{3}\right)^n$$

$$\sum_{0}^{\infty} n(3)^{-n} \text{ is finite}$$

Thus, the system <u>is stable</u>

5. (a) For the given system, we may see that, for $t < 0$, the response may be non-zero (more precisely, it is non-zero for $-\infty < t < 5$); therefore, the system is <u>not causal</u>. We check for stability below:

$$\int_{-\infty}^{5} e^{-3t}\, dt$$

$$-\frac{e^{-3t}}{3}\Big|_{-\infty}^{5} = \infty$$

Therefore, the system is <u>not stable</u>

(b) We may see that, for the given system, for $t < 0$, the response may be non-zero (more precisely, it is non-zero for $t > -10$); therefore, the system is <u>not causal</u>. We check for stability below:

$$\int_{-10}^{\infty} e^{-4t}\, dt$$
$$-\frac{e^{-4t}}{4} - 10^{\infty} = \frac{e^{40}}{4} < \infty$$

Thus, we see that the system is <u>stable</u>

(c) We may rewrite the function as:

$$x(t) = \begin{cases} e^{-2t}, & t \geq 0 \\ e^{2t}, & t < 0 \end{cases}$$

Because the value of the function is non-zero when $t < 0$, we can see that it is <u>not causal</u>

We may check for stability below:

$$\int_{-\infty}^{0} e^{2t}\, dt + \int_{0}^{\infty} e^{-2t}\, dt$$
$$\left.\frac{e^{2t}}{2}\right|_{-\infty}^{0} - \left.\frac{e^{-2t}}{2}\right|_{0}^{\infty} = 1$$

Therefore, we may see that the system is <u>stable</u>

(d) We may see that, because the system is zero for $t \leq 0$, it <u>is causal</u>. We now check for stability:

$$\int_{2}^{\infty} 3e^{-2t} - e^{-.05t+5}\, dt$$
$$3e^{-2t} - e^{-.05t+5}\Big|_{2}^{\infty} = e^{4.9} - \frac{3}{e^{4}} < \infty$$

Therefore, we may see that the system is <u>stable</u>

6. We may begin by observing that $x(t)$ may be expressed as a summation of impulses, written as:

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Convolving this with $h(t)$, by the property that $x(t) * \delta(t - t_o) = x(t - t_o)$, we may write:

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - nT)$$

To expand this, we may write:

$$y(t) = \cdots + h(t+3T) + h(t+2T) + h(t+T) + h(t) + h(t-T) + h(t-2T) + h(t-3T) + \cdots$$

We begin by analyzing the $T = 1$ case. Summing these graphically, we obtain:

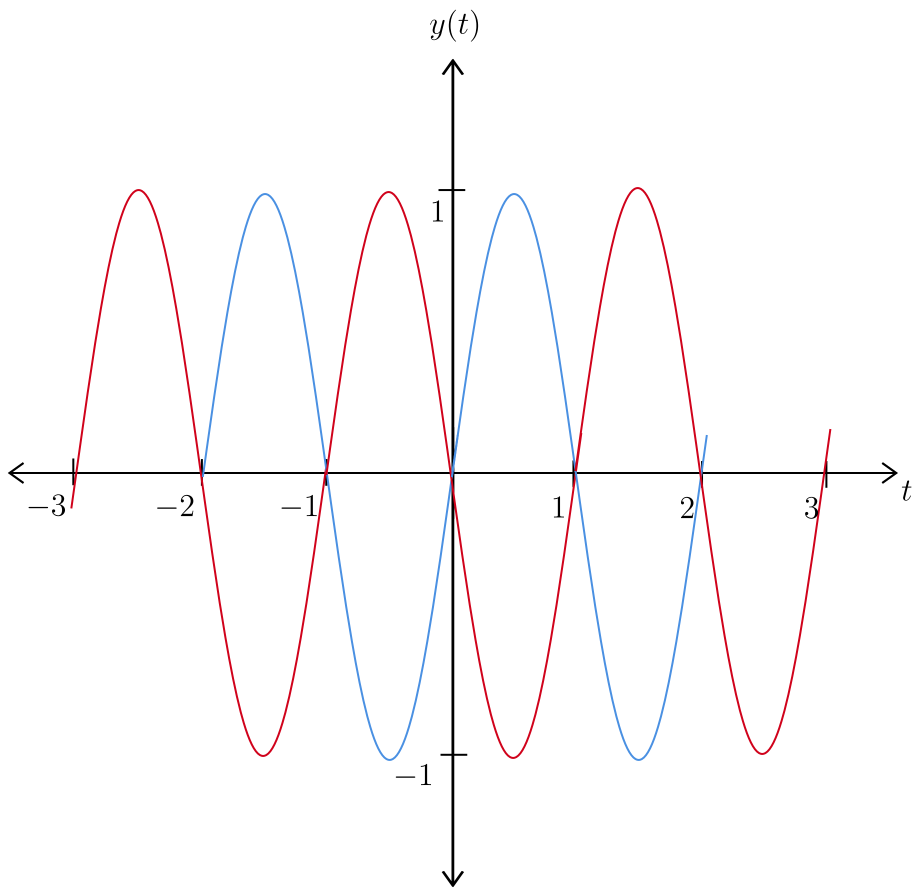

Figure 1: $T = 1$ Case Graphically Summed

For ease of interpretation, the graph above shows the sum expanded above, with n even cases in red and n odd in blue. We may thus see that, summing the two, we simply obtain $\boxed{y(t) = 0\Big|_{T=1}}$, or a flat line on the t axis. Applying similar logic to $T = 2$, we may draw the corresponding sinusoids to find:

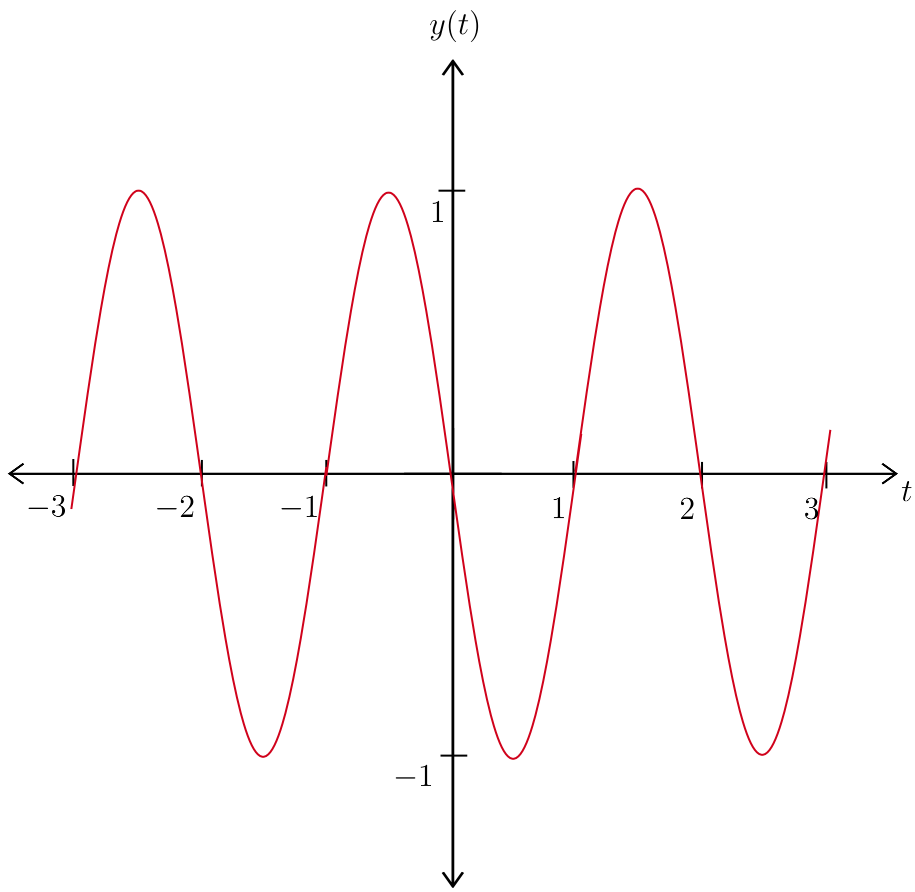

Figure 2: $T = 2$ Case Graphically Summed

We may observe that this sinusoid consists solely of the even sinusoids from Figure 1. This makes sense, as using $T = 2$ effectively doubles every n, making the shift even. Therefore, we obtain the solution shown in Figure 2. We may write this as:

$$y(t) = \cdots + h(t+6) + h(t+4) + h(t+2) + h(t) + h(t-2) + h(t-4) + h(t-6) + \cdots$$

$$\boxed{y(t) = \sum_{n=-\infty}^{\infty} h(t-2n)}$$

7. (a) Given the form of $x(t)$, we know $y(t)$ is of the form:

$$y(t) = Ae^{(-1+2j)t}$$

Plugging this into the given equation, we get:

$$A(-1+2j)e^{(-1+2j)t} + 3Ae^{(-1+2j)t} = e^{(-1+2j)t}$$

This simplifies to:

$$A(-1+2j) + 3A = 1$$
$$A = \frac{1}{2j+2}$$

And gives us the particular equation:

$$y_p(t) = \frac{e^{(-1+2j)t}}{2j+2}$$

We can now find the homogenous solution, with general form of $y(t)$:

$$y_h(t) = Ae^{\lambda t}$$

Using our equation, we may obtain:

$$A\lambda e^{\lambda t} + 3Ae^{\lambda t} = 0$$
$$\lambda = -3$$

Combing the two we get:

$$y(t) = Ae^{-3t} + \frac{e^{(-1+2j)t}}{2j+2}$$

Applying the initial rest condition, we get:

$$0 = A + \frac{1}{2j+2}$$
$$A = -\frac{1}{2j+2}$$

Thus, ensuring that there is a response only for $t > 0$, we finally get:

$$\boxed{y(t) = \frac{1}{2+2j}\left[e^{(-1+2j)t} - e^{-3t}\right]u(t)}$$

(b) We can expand our answer from (a):

$$y(t) = \frac{1}{2+2j}\left[e^{-t}\cos(2t) + je^{-t}\sin(2t) - e^{-3t}\right]u(t)$$

Multiplying by the conjugate, we get:

$$y(t) = (.25 - .25j)\left[e^{-t}\cos(2t) + je^{-t}\sin(2t) - e^{-3t}\right]u(t)$$
$$y(t) = \left[.25e^{-t}\cos(2t) + .25e^{-t}\sin(2t) - .25e^{-3t}\right]u(t)$$

Therefore, our output becomes:

$$\boxed{y(t) = \left[.25e^{-t}\sin(2t) + .25e^{-t}\cos(2t) - .25e^{-3t}\right]u(t)}$$

8. • We know that the equations governing time response of an inductor and capacitor are (respectively):

$$y(t) = L\frac{di(t)}{dt} \quad \text{and } i(t) = C\frac{dV_c(t)}{dt}$$

We know that the voltage across the capacitor will be the difference between the voltage supplied and the voltage across the inductor; thus, we may write:

$$i(t) = C\frac{d}{dt}\left[x(t) - y(t)\right]$$

Inserting this into the inductor equation, we get:

$$y(t) = L\frac{d}{dt}\left[C\frac{d}{dt}\left[x(t) - y(t)\right]\right]$$

$$y(t) = LC\frac{d^2}{dt^2}\left[x(t) - y(t)\right]$$

Putting similar terms to one side, we may write:

$$\frac{d^2y(t)}{dt^2} + \frac{1}{LC}y(t) = \frac{d^2x(t)}{dt^2}$$

Inserting known values:

$$\boxed{\frac{d^2y(t)}{dt^2} + 25y(t) = \frac{d^2x(t)}{dt^2}}$$

• Taking $x(t) \to 0$, we find the homogenous solution form as:

$$\frac{d^2y(t)}{dt^2} + 25y(t) = 0$$

Using the provided equation, we insert into the above:

$$\left[K_1\omega_1^2e^{j\omega_1 t} + K_2\omega_2^2e^{j\omega_2 t}\right] + 25\left[K_1e^{j\omega_1 t} + K_2e^{j\omega_2 t}\right] = 0$$

Dividing by the exponentials, we get:

$$K_1\omega_1^2 + K_2\omega_2^2 + 25K_1 + 25K_2 = 0$$

By observation, we may see that:

$$\boxed{\omega_1 = \omega_2 = \pm 5}$$

- From part (b), we see that the natural (homogenous) response may be modeled by:

  $$y(t) = Ae^{j5t} + Be^{-j5t}$$

  We may take $A = \frac{1}{2}(a+b)$ and $B = \frac{1}{2}(a-b)$. We then apply Euler's Law to get:

  $$\boxed{y(t) = a\cos(5t) + b\sin(5t)}$$

  We may thus observe that the natural response is sinusoidal in nature.