Homework 3

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1. Classifying systems as memory-less, time-invariant, linear, causal, and/or stable:

   (a) $y(t) = 5e^{4t}x(t-1)$

   - Memory: **not** memory-less; the $x(t-1)$ term means the system relies on values other than the present value; therefore, it is not memory-less
   - Time-Invariant: **not** time-invariant; we may see that, $y(t-t_o)$ changes the t value in the exponential and $x(t)$ statement, while $x(t-t_o)$ changes only the $x(t)$ statement; thus, it is not time-invariant, since $x(t-t_o) \neq y(t-t_o)$
   - Linear: the system **is** linear (see below) because $ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$

   $$ax_1(t) + bx_2(t) \to a5e^{4t}x_1(t-1) + b5e^{4t}x_2(t-1)$$
   $$ay_1(t) + by_2(t) \to a5e^{4t}x_1(t-1) + b5e^{4t}x_2(t-1)$$

   - Causal: the system **is** causal , because it only depends on past or present values (ex. $t = 0 \to y(t) = 5e^{4(0)}x(-1)$)
   - Stable: Given that the system depends on an exponential e^{4t}, its maximum value is unbounded and, therefore, it is **unstable**

   (b) $y(t) = \displaystyle\int_{-\infty}^{\frac{t}{2}} x(\tau)\,d\tau$

   - Memory: **not** memory-less; the system depends on a shift of the t parameter $(t/2)$, and, therefore, does not always depend on the current value of time
   - Time-Invariant: **not** time-invariant; $y(t-t_o) \neq x(t-t_o)$ (see below)

   $$x(t-t_o) \to \int_{-\infty}^{\frac{t}{2}} x(\tau - t_o)\,d\tau$$
   $$y(t-t_o) \to \int_{-\infty}^{\frac{(t-t_o)}{2}} x(\tau - t_o)\,d\tau$$
   $$\therefore x(t-t_o) \neq y(t-t_o)$$

- Linear: the system **is** linear; it follows both the superposition and homogeneity principles (see below)

$$ay_1(t) + by_2(t) \to a\int_{-\infty}^{\frac{t}{2}} x_1(\tau)\,d\tau + b\int_{-\infty}^{\frac{t}{2}} x_2(\tau)\,d\tau$$
$$ax_1(t) + bx_2(t) \to a\int_{-\infty}^{\frac{t}{2}} x_1(\tau)\,d\tau + b\int_{-\infty}^{\frac{t}{2}} x_2(\tau)\,d\tau$$
$$\therefore ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$$

- Causal: the system **is not** causal; integration depends on future values when $t < 0$
- Stable: the system **is not** stable (see below)

$$y(t) = \int_{-\infty}^{\frac{t}{2}} x(\tau)\,d\tau$$
$$h(t) = \int_{-\infty}^{\frac{t}{2}} \delta(\tau)\,d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$
$$h(t) \to u(t)$$
$$\int_{-\infty}^{\infty} h(t)\,dt = \infty$$

(c) $y(t) = 4 + 5\frac{d^2}{dt^2}x(t)$

- Memory: the system is **not** memory-less; the use of a differential implies that the system depends on past values
- Time-Invariant: the system **is** time-invariant (see below)

$$x(t - t_o) \to 4 + 5\frac{d^2}{dt^2}x(t - t_o)$$
$$y(t - t_o) \to 4 + 5\frac{d^2}{dt^2}x(t - t_o)$$
$$\therefore x(t - t_o) = y(t - t_o)$$

- Linear: the system is **not** linear (see below)

$$ax_1(t) + bx_2(t) = \left(4 + 5a\frac{d^2}{dt^2}x_1(t)\right) + \left(4 + 5b\frac{d^2}{dt^2}x_2(t)\right)$$
$$ay_1(t) + by_2(t) = a\left(4 + 5\frac{d^2}{dt^2}x_1(t)\right) + b\left(4 + 5\frac{d^2}{dt^2}x_2(t)\right)$$
$$\therefore ax_1(t) + bx_2(t) \neq ay_1(t) + by_2(t)$$

- Causal: the system **is** causal because it only depends on past or present values
- Stable: the system is **unstable** because it is unbounded

(d) $y(t) = \begin{cases} 0, & t < 0 \\ x(t-2) + 2x(t), & t \geq 0 \end{cases}$

- Memory: the system is **not** memory-less, since the $x(t-2)$ term depends on a past value
- Time-Invariant: the system is **not** time-invariant (see below)

$$x(t-t_o) \rightarrow \begin{cases} 0, & t < 0 \\ x(t-2-t_o) + 2x(t-t_o), & t \geq 0 \end{cases}$$

$$y(t-t_o) \rightarrow \begin{cases} 0, & t < 2 \\ x(t-2-t_o) + 2x(t-t_o), & t \geq 2 \end{cases}$$

$$\therefore x(t-t_o) \neq y(t-t_o)$$

- Linear: the system **is** linear (see below)

$$ax_1(t) + bx_2(t) \rightarrow \begin{cases} 0, & t < 0 \\ ax_1(t-2) + 2ax_1(t) + bx_2(t-2) + 2bx_2(t), & t \geq 0 \end{cases}$$

$$ay_1(t) + by_2(t) \rightarrow \begin{cases} 0, & t < 0 \\ ax_1(t-2) + 2ax_1(t) + bx_2(t-2) + 2bx_2(t), & t \geq 0 \end{cases}$$

$$\therefore ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$$

- Causal: the system **is** causal because it only depends on past or present values
- Stable: the system **is** stable, because it does not tend to diverge

Problem 1 can be tabulated as follows:

System	a	b	c	d
Memory-Less	no	no	no	no
Time-Invariant	no	no	yes	no
Linear	yes	yes	no	yes
Causal	yes	no	yes	yes
Stable	no	no	no	yes

2. Classifying systems as memory-less, time-invariant, linear, causal, and/or stable:

(a) $y[n] = x[n+1] - 2x[n-4]$

- Memory: system is **not** memory-less, as it depends on past and future values
- Time-Invariant: system **is** time-invariant (see below)

$$x[n-n_o] = x[n+1-n_o] - 2x[n-4-n_o]$$

$$y[n-n_o] = x[n+1-n_o] - 2x[n-4-n_o]$$

$$\therefore x[n-n_o] = y[n-n_o]$$

- Linear: system **is** linear (see below)

$$ax_1[n] + bx_2[n] = a(x_1[n+1] - 2x_1[n-4]) + b(x_2[n+1] - 2x_2[n-4])$$
$$ay_1[n] + by_2[n] = a(x_1[n+1] - 2x_1[n-4]) + b(x_2[n+1] - 2x_2[n-4])$$
$$\therefore ay_1[n] + by_2[n] = ax_1[n] + bx_2[n]$$

- Causal: system is **not** causal, as it depends on past and present values
- Stable: system **is** stable because $y[n]$ is finite

(b) $y[n] = \text{Even}\{x[n-1]\}$

To simplify analysis, we can express the even function as:

$$\frac{x[n] + x^*[-n]}{2} \rightarrow \frac{x[n-1] + x^*[-n+1]}{2}$$

- Memory: system is **not** memory-less, as it depends on future values
- Time-Invariant: system **is** time-invariant (see below)

$$x[n-n_o] = \text{Even}\{x[n-1-n_o]\}$$
$$y[n-n_o] = \text{Even}\{x[n-1-n_o]\}$$
$$\therefore x[n-n_o] = y[n-n_o]$$

- Linear: system is **not** linear (see below); note that this is because a or b may be complex. Given this, for a purely real signal, the system can be classified as linear; however, due to the need to use the 'conjugate' for the even function, in the case of a complex signal, this is non-linear

$$ax_1[n] + bx_2[n] = \frac{ax_1[n-1] + a^*x_1^*[-n+1]}{2} + \frac{bx_2[n-1] + b^*x_2^*[-n+1]}{2}$$
$$ay_1[n] + by_2[n] = \frac{ax_1[n-1] + ax_1^*[-n+1]}{2} + \frac{bx_2[n-1] + bx_2^*[-n+1]}{2}$$
$$\therefore ay_1[n] + by_2[n] \neq ax_1[n] + bx_2[n]$$

- Causal: system is **not** causal, as it depends on a future value
- Stable: system **is** stable because $y[n]$ is finite

(c) $y[n] = 5x[3n+1]$

- Memory: system is **not** memory-less, as it depends on non-present values
- Time-Invariant: system is **not** time-invariant (see below)

$$x[n-n_o] = 5x[3n-n_o+1]$$
$$y[n-n_o] = 5x[3n-3n_o+1]$$
$$\therefore x[n-n_o] \neq y[n-n_o]$$

- Linear: system **is** linear (see below)

$$ax_1[n] + bx_2[n] = 5ax_1[3n+1] + 5bx_2[3n+1]$$
$$ay_1[n] + by_2[n] = 5ax_1[3n+1] + 5bx_2[3n+1]$$
$$\therefore ay_1[n] + by_2[n] = ax_1[n] + bx_2[n]$$

- Causal: system is **not** causal, as it depends on a future value
- Stable: system **is** stable because $y[n]$ is finite

(d) $y[n] = \begin{cases} 0, & n = 2 \\ x[n], & \text{otherwise} \end{cases}$

- Memory: system **is** memory-less; it does not depend on past or present values
- Time-Invariant: system is **not** time-invariant (see below)

$$x[n-n_o] = \begin{cases} 0, & n = 2 \\ x[n-n_o], & \text{otherwise} \end{cases}$$
$$y[n-n_o] = \begin{cases} 0, & n = 2+n_o \\ x[n-n_o], & \text{otherwise} \end{cases}$$
$$\therefore x[n-n_o] \neq y[n-n_o]$$

- Linear: system **is** linear (see below)

$$ax_1[n] + bx_2[n] = \begin{cases} 0, & n = 2 \\ ax_1[n] + bx_2[n], & \text{otherwise} \end{cases}$$
$$ay_1[n] + by_2[n] = \begin{cases} 0, & n = 2 \\ ax_1[n] + bx_2[n], & \text{otherwise} \end{cases}$$
$$\therefore ay_1[n] + by_2[n] = ax_1[n] + bx_2[n]$$

- Causal: system **is** causal, as it depends on only past or present values
- Stable: system **is** stable because $y[n]$ is finite

3. (a) We may begin by constructing the expression for the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

This gets us:

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^{k-2} u[k-2]u[n-k+1]$$

We may observe the following:

$$x[n] \neq 0, \quad n \geq 2$$

$$h[n] \neq 0, \quad n \geq -1$$

Integrating this into the convolution sum, we may see that the expression is nonzero (unit functions both exist) for $k \geq 2$, and that $k \leq n+1$. Thus, we get:

$$y[n] = \sum_{k=2}^{n+1} \left(\frac{1}{4}\right)^{k-2}$$

Using the expansion for series, we obtain:

$$y[n] = \frac{1-(.25)^{n-1}}{1-.25}$$

$$\boxed{y[n] = \frac{4}{3}\left[1-\left(\frac{1}{4}\right)^{n-1}\right] \quad \text{for } n \geq 2}$$

This can be sketched as:

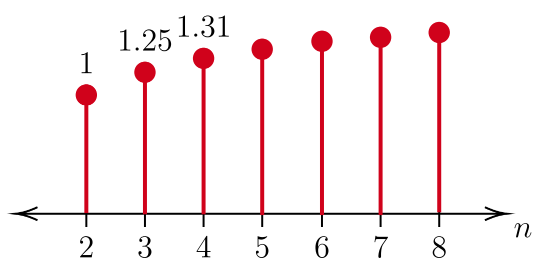

Figure 1: Sketch of $y[n]$

(b) Per the time-shifting property, we know that:

$$y[n] = x[n] * h[n] \rightarrow y_1[n] = y[n-3] = x[n-3] * h[n]$$

Which gives us:

$$\boxed{y_1[n] = \frac{4}{3}\left[1-\left(\frac{1}{4}\right)^{n-4}\right] \quad \text{for } n \geq 5}$$

(c) Per the time-shifting property, we know that:

$$y[n] = x[n] * h[n] \rightarrow y_2[n] = y[n-2] = x[n] * h[n-2]$$

$$\boxed{y_2[n] = \frac{4}{3}\left[1-\left(\frac{1}{4}\right)^{n-3}\right] \quad \text{for } n \geq 4}$$

(d) Per the time-shifting property, we know that:

$$y[n] = x[n] * h[n] \rightarrow y_3[n] = y[n+1] = x[n-2] * h[n+3]$$

$$\boxed{y_3[n] = \frac{4}{3}\left[1 - \left(\frac{1}{4}\right)^n\right] \quad \text{for } n \geq 1}$$

4. To simplify the analysis, we can defined the given functions as:

$$x[n] = u[n-1] - u[n-3]$$
$$h[n] = u[n-5] - u[n-10]$$

From here, we apply the sum:

$$y[n] = \sum_{k=-\infty}^{\infty} (u[k-1] - u[k-3])(u[n-k-5] - u[n-k-10])$$

$$y[n] = \sum_{k=-\infty}^{\infty} u[k-1]u[n-k-5] - u[k-3]u[n-k-5] - u[k-1]u[n-k-10] + u[k-3]u[n-k-10]$$

By observation, we find:

$$\begin{aligned}
\text{Term 1 } &\neq 0, \quad n-5 \geq k \geq 1 \\
\text{Term 2 } &\neq 0, \quad n-5 \geq k \geq 3 \\
\text{Term 3 } &\neq 0, \quad n-10 \geq k \geq 1 \\
\text{Term 4 } &\neq 0, \quad n-10 \geq k \geq 3
\end{aligned}$$

We see the smallest non-zero value for which a term exists is $k \geq 1$, and the largest value for which a term exists is $n-5$. We can apply these bounds:

$$y[n] = \sum_{k=1}^{n-5} u[k-1]u[n-k-5] - u[k-3]u[n-k-5] - u[k-1]u[n-k-10] + u[k-3]u[n-k-10]$$

We can break the sum down by looking at 'zones' for the terms. For $k = 1, 2$, two terms exist:

$$y_1[n] = \sum_{k=1}^{2} u[k-1]u[n-k-5] - u[k-1]u[n-k-10]$$
$$y_1[n] = u[n-6] - u[n-11] + u[n-7] - u[n-12]$$

5.

6. (a) We can express this as $x_1[n] = u[n-3]$

Since this is an LTI system, we know that the output will be shifted by the same amount (shifting property), which gives us:

$$\boxed{x_1[n] \rightarrow y_1[n] = \left(\frac{1}{4}\right)^{n-3} u[n-3] - \left(\frac{1}{4}\right)^{n-4} u[n-4]}$$

(b) This input can be expressed as: $x_2[n] = u[n] - u[n-4]$

Once again by the shifting property, as well as by homogeneity and superposition, we can write:

$$\boxed{x_2[n] \rightarrow y_2[n] = \left(\frac{1}{4}\right)^{n} u[n] - \left(\frac{1}{4}\right)^{n-1} u[n-1] - \left(\frac{1}{4}\right)^{n-4} u[n-4] + \left(\frac{1}{4}\right)^{n-5} u[n-5]}$$

(c) This input can be expressed as: $x_3[n] = 2u[n] - 2u[n-1]$

By the shifting, homogenous, and superposition principles of LTI systems, we get:

$$x_3[n] \rightarrow y_2[n] = 2\left(\frac{1}{4}\right)^{n} u[n] - 2\left(\frac{1}{4}\right)^{n-1} u[n-1] - 2\left(\frac{1}{4}\right)^{n-1} u[n-1] + 2\left(\frac{1}{4}\right)^{n-2} u[n-2]$$

Which can be simplifed:

$$\boxed{x_3[n] \rightarrow y_2[n] = 2\left(\frac{1}{4}\right)^{n} u[n] - 4\left(\frac{1}{4}\right)^{n-1} u[n-1] + 2\left(\frac{1}{4}\right)^{n-2} u[n-2]}$$

7. Per convolution properties, we know:

$$x(t) * \delta(t-t_o) \Rightarrow x(t-t_o)$$

Thus, looking at the input, in addition to the fact that this is an LTI system, we can write:

$$\boxed{y(t) = x(t) * h(t) \rightarrow y(t) = x(t+2) - 2x(t)}$$

Plotting this shift, we get:

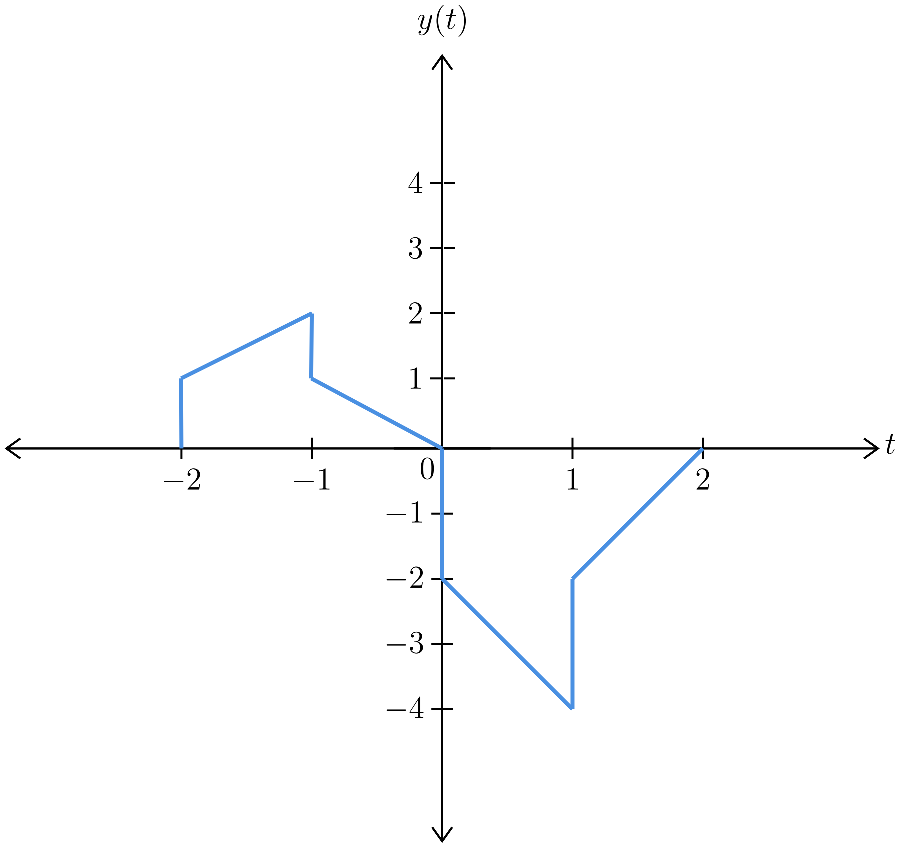

Figure 2: Plot for $y(t) = x(t+2) - 2x(t)$

8. (a) Using this, we can rewrite x as:

$$x(t) = u(t) - u(t-1)$$

Which would make h:

$$h(t) = u(t-2) - u(t-4)$$

This gives us:

$$y(t) = \int [u(t) - u(t-1)] * [u(t-2) - u(t-4)] \, dt$$

Which expands to:

$$y(t) = \int u(t)u(t-2) - u(t)u(t-4) - u(t-1)u(t-2) + u(t-1)u(t-4) \, dt$$

To analyze, we can use the following properties:

$$r(t) * r(t - T_1) = r(t - T_1)$$
$$r(t - T_1) * r(t - T_2) = r(t - T_1 - T_2)$$

This gives us:

$$y(t) = r(t-2) - r(t-4) - r(t-3) + r(t-5)$$
$$\boxed{y(t) = r(t-2) - r(t-3) - r(t-4) + r(t-5)}$$

And the plot becomes:

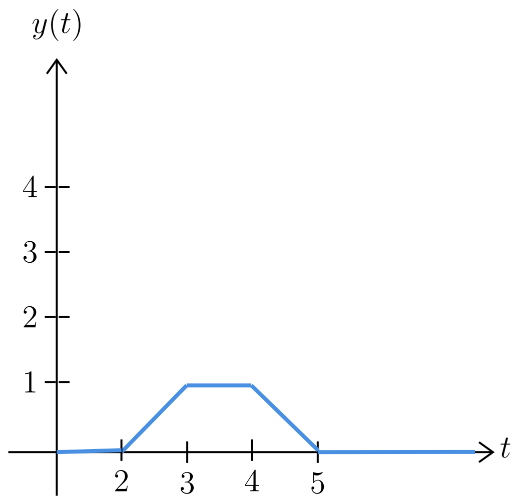

Figure 3: Plot for $y(t) = r(t-2) - r(t-3) - r(t-4) + r(t-5)$

(b) We can determine $\frac{d}{dt}[y(t)]$ to be:

$$\frac{d}{dt}[y(t)] = u(t-2) - u(t-3) - u(t-4) + u(t-5)$$

The discontinuities may be identified by differentiating again to get:

$$\frac{d^2}{dt^2}[y(t)] = \delta(t-2) - \delta(t-3) - \delta(t-4) + \delta(t-5)$$

We know that each impulse represents a discontinuity; Therefore, we can tell that there are 4 discontinuities, at $t = 2, 3, 4$, and 5, for $d/dt[y(t)]$

9. (a) We know that the impulse response can be determined if $x(t) = \delta(t)$:

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} x(\tau - 2)\, d\tau$$

$$y(t) = \int_{-\infty}^{t} e^{-(t-\tau)} \delta(\tau - 2)\, d\tau$$

We also know from the property of the impulse that:

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_o)\, dt = x(t_o)$$

Thus, we can see that:

$$\boxed{h(t) = e^{-(t-2)}}$$

(b)