Homework 9

Michael Brodskiy

Professor: I. Salama

November 21, 2024

1. From the fundamental period, we may get:

$$\omega_o = \frac{2\pi}{N}$$
$$\omega_o = \frac{2\pi}{5}$$

We can express the signal as:

$$x[n] = \sum_{-\infty}^{\infty} a_k e^{jk\omega_o n}$$

We know:

$$a_o = 4,\ a_2 = 2je^{-\frac{j\pi}{4}} = a_{-2}^*,\ a_4 = e^{-\frac{j\pi}{8}} = c_{-4}^*$$

This allows us to write:

$$x[n] = 4 + a_2 e^{2j\omega_o n} + a_{-2} e^{-2j\omega_o n} + a_4 e^{4j\omega_o n} + a_{-4} e^{-4j\omega_o n}$$

Plugging in known values gets us:

$$x[n] = 4 + 2je^{-\frac{j\pi}{4}} e^{2j\omega_o n} - 2je^{\frac{j\pi}{4}} e^{-2j\omega_o n} + e^{-\frac{j\pi}{8}} e^{4j\omega_o n} - e^{\frac{j\pi}{8}} e^{-4j\omega_o n}$$

Per our exponential formulas, we may get:

$$x[n] = 4 - 4\sin\left(2\omega_o n - \frac{\pi}{4}\right) + 2\cos\left(4\omega_o n - \frac{\pi}{8}\right)$$

We then use:

$$\cos(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)$$

To get:

$$x[n] = 4 - 4\sin\left(2\omega_o n - \frac{\pi}{4}\right) + 2\sin\left(4\omega_o n + \frac{3\pi}{8}\right)$$

Inserting the fundamental frequency gets us:

$$\boxed{x[n] = 4 - 4\sin\left(\frac{4\pi}{5}n - \frac{\pi}{4}\right) + 2\sin\left(\frac{8\pi}{5}n + \frac{3\pi}{8}\right)}$$

2. (a) Using our trigonometric identities, we may expand to get:

$$\cos\left(\frac{2\pi}{3}n\right)\cos\left(\frac{2\pi}{4}n\right) = \frac{1}{2}\left[\cos\left(\frac{7\pi}{6}n\right) + \cos\left(\frac{\pi}{6}n\right)\right]$$

We then chance the sinusoids to exponentials to get:

$$\frac{1}{2}\left[\cos\left(\frac{7\pi}{6}n\right) + \cos\left(\frac{\pi}{6}n\right)\right] = \frac{1}{4}\left[e^{\frac{7j\pi}{6}n} + e^{-\frac{7j\pi}{6}n} + e^{\frac{j\pi}{6}n} + e^{\frac{-j\pi}{6}n}\right]$$

Thus, we write:

$$x[n] = \frac{1}{4}e^{\frac{7j\pi}{6}n} + \frac{1}{4}e^{-\frac{7j\pi}{6}n} + \frac{1}{4}e^{\frac{j\pi}{6}n} + \frac{1}{4}e^{\frac{-j\pi}{6}n}$$

From this, and the fact that the signal is real, we may write the coefficients as:

$$\boxed{c_n = a_n = \begin{cases} a_{-1} & = 1/4 \\ a_1 & = 1/4 \\ a_{-7} & = 1/4 \\ a_7 & = 1/4 \end{cases}}$$

(b) We can use the following formula:

$$c_k = \frac{1}{N_o}\sum_{n=0}^{N-1} x[n]e^{-\frac{2jk\pi}{N_o}n}$$

We may substitute in known values:

$$c_k = \frac{1}{3}\sum_{n=0}^{2}\left(1 - \sin\left(\frac{\pi}{3}n\right)\right)e^{-\frac{2jk\pi}{3}n}$$

We evaluate for all values to get:

$$c_k = \frac{1}{3}\left[1 + \left(1 - \frac{\sqrt{3}}{2}\right)e^{-\frac{2jk\pi}{3}} + \left(1 - \frac{\sqrt{3}}{2}\right)e^{-\frac{4jk\pi}{3}}\right]$$

We can simplify to get:

$$\boxed{c_k = \frac{1}{3} + \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right)\left[e^{-\frac{2jk\pi}{3}} + e^{-\frac{4jk\pi}{3}}\right]}$$

3. (a) To simplify the Fourier transform, we may express $x[n]$ in terms of delta functions, which gives us:

$$x[n] = \delta[n+2] + \delta[n+1] + \cdots + \delta[n-4]$$

Using our known transforms, we may write:

$$\boxed{X_1(e^{j\Omega}) = e^{2j\Omega} + e^{j\Omega} + 1 + e^{-j\Omega} + e^{-2j\Omega} + e^{-3j\Omega} + e^{-4j\Omega}}$$

(b) Per the time-shifting property, we may simply shift each signal by the frequency of the added exponential to get:

$$\boxed{X_2(e^{j\Omega}) = e^{2j\left(\Omega-\frac{\pi}{4}\right)} + e^{j\left(\Omega-\frac{\pi}{4}\right)} + 1 + e^{-j\left(\Omega-\frac{\pi}{4}\right)} + e^{-2j\left(\Omega-\frac{\pi}{4}\right)} + e^{-3j\left(\Omega-\frac{\pi}{4}\right)} + e^{-4j\left(\Omega-\frac{\pi}{4}\right)}}$$

(c) Combining several of our known Fourier transform properties, we may write:

$$\boxed{X_3(e^{j\Omega}) = \frac{e^{2j\Omega}}{1 - \frac{1}{8}e^{j\Omega}}}$$

(d) We can take:

$$\cos\left(\frac{\pi}{3}n\right) \to \pi\left[\delta\left(\Omega - \frac{\pi}{3}\right) + \delta\left(\Omega + \frac{\pi}{3}\right)\right]$$

And:

$$\left(\frac{1}{2}\right)^n u[n] \to \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

We combine the two to get:

$$\boxed{X_4(e^{j\Omega} = \frac{\pi\delta\left(\Omega - \frac{\pi}{3}\right)}{1 - \frac{1}{2}e^{-j\left(\Omega-\frac{\pi}{3}\right)}} + \frac{\pi\delta\left(\Omega + \frac{\pi}{3}\right)}{1 - \frac{1}{2}e^{-j\left(\Omega+\frac{\pi}{3}\right)}}}$$

(e) We take the known transforms for both sinusoids to write:

$$2\sin\left(\frac{\pi}{3}n\right) \to \frac{2\pi}{j}\left[\delta\left(\Omega - \frac{\pi}{3}\right) - \delta\left(\Omega + \frac{\pi}{3}\right)\right]$$
$$4\cos\left(\frac{2\pi}{5}n\right) \to 4\pi\left[\delta\left(\Omega - \frac{2\pi}{5}\right) + \delta\left(\Omega + \frac{2\pi}{5}\right)\right]$$

We sum the two to get:

$$\boxed{X_5(e^{j\Omega}) = \frac{2\pi}{j}\left[\delta\left(\Omega - \frac{\pi}{3}\right) - \delta\left(\Omega + \frac{\pi}{3}\right)\right] + 4\pi\left[\delta\left(\Omega - \frac{2\pi}{5}\right) + \delta\left(\Omega + \frac{2\pi}{5}\right)\right]}$$

4. (a) Based on our formulas, we may write:

$$x[n] = \frac{1}{2\pi}\left[\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} e^{jn\Omega}\, d\Omega\right]$$

From here, we simply evaluate:

$$x[n] = \frac{1}{2jn\pi}\left[e^{jn\Omega}\Big|_{-\frac{2\pi}{3}}^{-\frac{\pi}{3}} + e^{jn\Omega}\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right]$$
$$x[n] = \frac{1}{2jn\pi}\left[e^{-\frac{jn\pi}{3}} - e^{-\frac{2jn\pi}{3}} + e^{-\frac{2jn\pi}{3}} - e^{\frac{jn\pi}{3}}\right]$$

Using our formulas, we get:

$$\boxed{x[n] = \frac{1}{n\pi}\left[\sin\left(\frac{2\pi n}{3}\Omega\right) - \sin\left(\frac{\pi n}{3}\Omega\right)\right]}$$

(b) We may observe that this is simply a collection of delta signals:

$$\boxed{x_2[n] = 5\delta[n] + 4\delta[n-1] + 2\delta[n-3] - 5\delta[n-4] + \delta[n-7]}$$

(c) We may rewrite this as:

$$X_3(e^{j\Omega}) = 2\left[1 + \cos(6\Omega)\right] + 4\left[1 - \cos(4\Omega)\right]$$
$$X_3(e^{j\Omega}) = 6 + 2\cos(6\Omega) - 4\cos(4\Omega)$$

We may once again rewrite this as:

$$X_3(e^{j\Omega}) = 6 + e^{6j\Omega} + e^{-6j\Omega} - 2e^{4j\Omega} - 2e^{-4j\Omega}$$

This can easily be converted to delta signals:

$$\boxed{x_3[n] = 6\delta[n] + \delta[n-6] + \delta[n+6] - 2\delta[n-4] - 2\delta[n+4]}$$

(d) Skipped (4 Needed for Full Credit)

(e) We may rewrite this as:

$$\frac{1-\frac{1}{3}e^{-j\Omega}}{1-\frac{3}{4}e^{-j\Omega}+\frac{1}{8}e^{-2j\Omega}}=\frac{1-\frac{1}{3}e^{-j\Omega}}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)}$$

Using partial fraction decomposition, we may write this as:

$$X_5(e^{j\Omega})=\frac{2/3}{1-\frac{1}{2}e^{-j\Omega}}+\frac{1/3}{1-\frac{1}{4}e^{-j\Omega}}$$

From here, we can use our transform formulas to obtain:

$$\boxed{x_5[n]=\frac{2}{3}\left(\frac{1}{2}\right)^n u[n]+\frac{1}{3}\left(\frac{1}{4}\right)^n u[n]}$$

5. (a) Taking the transform, we get:

$$Y(e^{j\Omega})+\frac{1}{6}e^{-j\Omega}Y(e^{j\Omega})-\frac{1}{6}e^{-2j\Omega}Y(e^{j\Omega})=X(e^{j\Omega})-e^{-j\Omega}X(e^{j\Omega})$$

We know that the transfer function may be defined as the output divided by the input. This gives us:

$$\boxed{H(e^{j\Omega})=\frac{1-e^{-j\Omega}}{1+\frac{1}{6}e^{-j\Omega}-\frac{1}{6}e^{-2j\Omega}}}$$

(b) We may factor the denominator in the above to get:

$$H(e^{j\Omega})=\frac{1-e^{-j\Omega}}{-\frac{1}{6}(e^{-j\Omega}-3)(e^{-j\Omega}+2)}$$

Using partial fraction decomposition, we get:

$$H(e^{j\Omega})=\frac{18/5}{e^{-j\Omega}+2}-\frac{12/5}{e^{-j\Omega}-3}$$

We can rearrange this to a form that is easier to convert:

$$H(e^{j\Omega})=\frac{18/5}{1+\frac{1}{2}e^{-j\Omega}}-\frac{12/5}{1-\frac{1}{3}e^{-j\Omega}}$$

This allows to take the inverse transform to obtain:

$$\boxed{h[n]=\frac{18}{5}\left(-\frac{1}{2}\right)^n u[n]-\frac{12}{5}\left(\frac{1}{3}\right)^n u[n]}$$

(c) We can multiply the response in the frequency domain by the input in the frequency domain:

$$X_1(e^{j\Omega}) = \frac{1}{1 - \frac{1}{4}e^{-j\Omega}}$$

This gives us:

$$y_1(e^{j\Omega}) = \frac{18/5}{\left(1 + \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)} - \frac{12/5}{\left(1 - \frac{1}{3}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

We decompose to get:

$$y_1(e^{j\Omega}) = \frac{12/5}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{6/5}{1 - \frac{1}{4}e^{-j\Omega}} - \frac{48/5}{1 - \frac{1}{3}e^{-j\Omega}} + \frac{36/5}{1 - \frac{1}{4}e^{-j\Omega}}$$

$$y_1(e^{j\Omega}) = \frac{12/5}{1 + \frac{1}{2}e^{-j\Omega}} + \frac{42/5}{1 - \frac{1}{4}e^{-j\Omega}} - \frac{48/5}{1 - \frac{1}{3}e^{-j\Omega}}$$

And then we take the inverse:

$$\boxed{y_1[n] = \left[\frac{12}{5}\left(-\frac{1}{2}\right)^n + \frac{42}{5}\left(\frac{1}{4}\right)^n - \frac{48}{5}\left(\frac{1}{3}\right)^n\right] u[n]}$$

(d) We use the time-shifting property to write:

$$h_2(e^{j\Omega}) = e^{-\frac{j\pi}{2}\Omega}\left[\frac{18/5}{1 + \frac{1}{2}e^{-j\Omega}} - \frac{12/5}{1 - \frac{1}{3}e^{-j\Omega}}\right]$$

This gives us:

$$\boxed{y_2[n] = \frac{18}{5}\left(-\frac{1}{2}\right)^{n-\frac{\pi}{2}} u\left[n - \frac{\pi}{2}\right] - \frac{12}{5}\left(\frac{1}{3}\right)^{n-\frac{\pi}{2}} u\left[n - \frac{\pi}{2}\right]}$$

(e) We know that this sinusoid may be written as:

$$2\cos\left(\frac{\pi}{2}n\right) \to e^{-\frac{j\pi n}{2}} + e^{\frac{j\pi n}{2}}$$

Thus, we may write this input as:

$$x_3[n] = \frac{1}{2}x_2[n] + \frac{e^{-j\pi n}}{2}x_2[n]$$

This gives us:

$$\boxed{y_3[n] = \left[\frac{9}{5}\left(-\frac{1}{2}\right)^{n-\frac{\pi}{2}} - \frac{6}{5}\left(\frac{1}{3}\right)^{n-\frac{\pi}{2}}\right] u\left[n - \frac{\pi}{2}\right] +}$$

$$\boxed{\left[\frac{9}{5}\left(-\frac{1}{2}\right)^{n-\frac{3\pi}{2}} - \frac{6}{5}\left(\frac{1}{3}\right)^{n-\frac{3\pi}{2}}\right] u\left[n - \frac{3\pi}{2}\right]}$$

6. (a) First, we transform the response to get:

$$X(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}}$$

The response can then be written as:

$$Y(e^{j\Omega}) = j\frac{d}{d\Omega}\left[\frac{1}{1 - \frac{1}{3}e^{-j\Omega}}\right]$$

$$Y(e^{j\Omega}) = \frac{\frac{1}{3}e^{-j\Omega}}{\left[1 - \frac{1}{3}e^{-j\Omega}\right]^2}$$

We divide the response by the input to get:

$$\boxed{H(e^{j\Omega}) = \frac{\frac{1}{3}e^{-j\Omega} - \frac{1}{6}e^{-2j\Omega}}{\left[1 - \frac{1}{3}e^{-j\Omega}\right]^2}}$$

(b) We expand the response from above to get:

$$\frac{Y(e^{j\Omega})}{X(e^{j\Omega})} = \frac{\frac{1}{3}e^{-j\Omega} - \frac{1}{6}e^{-2j\Omega}}{1 - \frac{2}{3}e^{-j\Omega} + \frac{1}{9}e^{-2j\Omega}}$$

From this, we get:

$$\boxed{y[n] - \frac{2}{3}y[n-1] + \frac{1}{9}y[n-2] = \frac{1}{3}x[n-1] - \frac{1}{6}x[n-2]}$$

7. (a) The overall response can be found by multiplying the two individual responses:

$$H(e^{j\Omega}) = H_1(e^{j\Omega})H_2(e^{j\Omega})$$

$$H(e^{j\Omega}) = \left(\frac{1 - \frac{1}{2}e^{-j\Omega}}{1 + \frac{1}{3}e^{-j\Omega}}\right)\left(\frac{1}{\left[1 - \frac{1}{3}e^{-j\Omega}\right]^2}\right)$$

This gives us:

$$\boxed{H(e^{j\Omega}) = \frac{1 - \frac{1}{2}e^{-j\Omega}}{\left(1 + \frac{1}{3}e^{-j\Omega}\right)\left(1 - \frac{1}{3}e^{-j\Omega}\right)^2}}$$

(b) We can multiply out the denominator to get:

$$H(e^{j\Omega}) = \frac{1 - \frac{1}{2}e^{-j\Omega}}{\left(1 - \frac{1}{9}e^{-2j\Omega}\right)\left(1 - \frac{1}{3}e^{-j\Omega}\right)}$$

$$H(e^{j\Omega}) = \frac{1 - \frac{1}{2}e^{-j\Omega}}{1 - \frac{1}{3}e^{-j\Omega} - \frac{1}{9}e^{-2j\Omega} + \frac{1}{27}e^{-3j\Omega}}$$

This gives us the difference equation as:

$$\boxed{y[n] - \frac{1}{3}y[n-1] - \frac{1}{9}y[n-2] + \frac{1}{27}y[n-3] = x[n] - \frac{1}{2}x[n-1]}$$

(c) We can take the inverse of our result from (a) by first breaking it into partial fractions:

$$H(e^{j\Omega}) = \frac{15/8}{3 + e^{-j\Omega}} + \frac{15/8}{3 - e^{-j\Omega}} - \frac{9/4}{(-3 + e^{-j\Omega})^2}$$

$$H(e^{j\Omega}) = \frac{5/8}{1 + \frac{1}{3}e^{-j\Omega}} + \frac{5/8}{1 - \frac{1}{3}e^{-j\Omega}} - \frac{1/4}{(1 - \frac{1}{3}e^{-j\Omega})^2}$$

Taking the inverse, we get:

$$\boxed{h[n] = \left[\frac{5}{8}\left(-\frac{1}{3}\right)^n + \frac{5}{8}\left(\frac{1}{3}\right)^n - \frac{n}{4}\left(\frac{1}{3}\right)^n\right] u[n]}$$

(d) Taking the inverse of the first system, we get:

$$H_1^{inv}(e^{j\Omega}) = \frac{1 + \frac{1}{3}e^{-j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}$$

This can be separated into:

$$\boxed{H_1^{inv}(e^{j\Omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} - \frac{\frac{1}{3}e^{-j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}}$$

This gives us:

$$\boxed{h_1^{inv}[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{3}\left(\frac{1}{2}\right)^{n-1} u[n-1]}$$