Homework 5

Michael Brodskiy
Professor: I. Salama

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1. (a) We may begin by rewriting S_2 as:

$$w[n] = y[n] - \frac{1}{2}y[n-1]$$

S_1 may be rewritten in a similar format to get:

$$x[n] = w[n] - \frac{1}{4}w[n-1]$$

Substituting the first equation into the second, we get:

$$x[n] = y[n] - \frac{1}{2}y[n-1] - \frac{1}{4}\left[y[n-1] - \frac{1}{2}y[n-2]\right]$$

This can be simplified to:

$$\boxed{x[n] = y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2]}$$

(b) We may find the impulse response by taking $x[n] \to \delta[n]$. This gives us:

$$w[n] = \frac{1}{4}w[n-1] + \delta[n]$$

Taking the z transform, we may write:

$$W(z) = \frac{1}{\left(1 - \frac{1}{4}z^{-1}\right)}, \quad |z| > .25$$

$$W(z) = \frac{z}{\left(z - \frac{1}{4}\right)}, \quad |z| > .25$$

$$\boxed{w[n] = \left(\frac{1}{4}\right)^n u[n]}$$

Finding the individual impulse response for S_2, we get:

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > .5$$

$$Y(z) = \frac{z}{z - \frac{1}{2}}, \quad |z| > .5$$

$$\boxed{y[n] = \left(\frac{1}{2}\right)^n u[n]}$$

(c) We can once again use the z transform, this time using the equation obtained in (a), to get:

$$Y(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$Y(z) = \frac{z^2}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

$$Y(z) = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}, \quad |z| > .5$$

We can use partial fraction decomposition, by rearranging:

$$\frac{Y(z)}{z} = \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}, \quad |z| > .5$$

$$\frac{Y(z)}{z} = \frac{A}{z - .25} + \frac{B}{z - .5}$$

We may find: $A = -1$, $B = 2$, which gives us:

$$Y(z) = \frac{-z}{z - .25} + \frac{2z}{z - .5}, \quad |z| > .5$$

We take the inverse transform to get:

$$\boxed{y[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right) u[n]}$$

2. (a)

   (b)

   (c)

3. (a) Setting up the Laplace Transform, we get:

$$X(s) = \int_2^\infty e^{-(4+s)t}\, dt$$

$$X(s) = -\frac{e^{-(4+s)t}}{4+s}\Big|_{2}^{\infty}$$

$$X(s) = \frac{e^{-2(4+s)}}{s+4}$$

$$\boxed{X(s) = \frac{e^{-2s-8}}{s+4}}$$

Since the equation is right-sided, the ROC is to the right of the right-most pole; there is one pole at $s = -4$, so the ROC is $\text{Re}\{s\} > -4 \longrightarrow \sigma > -4$ (since $s = \sigma + j\omega$)

(b) We may find the Laplace Transform to be:

$$G(s) = \int_{-\infty}^{-2} Ae^{-(4+s)t}\,dt$$

$$G(s) = -A\frac{e^{-(4+s)t}}{4+s}\Big|_{-\infty}^{-2}$$

$$G(s) = -\frac{A}{4+s}\left[e^{2(4+s)} - e^{\infty(4+s)}\right]$$

We may see that $G(s)$ converge only when s reaches the ROC at $\sigma < -4$

We can thus drop the term to get

$$G(s) = -\frac{Ae^{2s+8}}{4+s}$$

We can check the value of A:

$$-Ae^{2s+8} = e^{-2s-8}$$

We may see that, though the exponents will never be the same, we may take $A = -1$ to create a similar algebraic form. Thus, we say:

$$\boxed{A = -1}$$

4. (a) We may calculate the Laplace transform for the time-shifted signal as:

$$X_1(s) = \int_{-\infty}^{\infty} x(t - t_o)e^{-st}\,dt$$

Taking $t - t_o = n$, we write:

$$X_1(s) = \int_{-\infty}^{\infty} x(n)e^{-s(n+t_o)}\,dn$$

$$X_1(s) = e^{-st_o} \underbrace{\int_{-\infty}^{\infty} x(n)e^{-sn}\, dn}_{X(s)}$$

Which gives us:

$$\boxed{X_1(s) = e^{-st_o} X(s)}$$

And we may see that the ROC remains to be R.

(b) We may express the signal as:

$$y(t) = u(t) - 2u(t-2) + u(t-4)$$

Thus, we use our table of known transforms to get:

$$\boxed{Y(s) = \frac{1}{s} - \frac{2e^{-2t}}{s} + \frac{e^{-4t}}{s}}$$

We may see that the ROC is: $\boxed{\sigma > 0}$

5. We may rewrite $x(t)$ as:

$$x(t) = e^{t} \sin(5t) u(-t)$$

Which gives us:

$$X(s) = \int_{-\infty}^{0} e^{-(s-1)t} \sin(5t)\, dt$$

$$X(s) = -\frac{5}{(s-1)^2 + 25}$$

$$\boxed{X(s) = -\frac{5}{s^2 - 2s + 26}}$$

We may see by the second equation that the region of convergence is left-handed, and occurs at $\boxed{\text{ROC: Re}\{s\} - 1 < 0 \longrightarrow \sigma < 1}$. The poles will occur at the solutions to the quadratic in the denominator:

$$s^2 - 2s + 26 = 0$$

$$\frac{2 \pm \sqrt{4 - 4(1)(26)}}{2}$$

$$\frac{2 \pm 10j}{2}$$

$$\boxed{\text{Poles at: } s = 1 \pm 5j}$$

6. Using partial fraction decomposition, we may write:

$$\frac{s-1}{(s+1)(s+3)(s^2+4s+20)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4s+20}$$

And then:

$$(s+3)(s^2+4s+20)A + (s+1)(s^2+4s+20)B + (s+1)(s+3)(Cs+D) = s-1$$

$$As^3{+}7As^2{+}32As{+}60A{+}Bs^3{+}5Bs^2{+}24Bs{+}20B{+}Cs^3{+}4Cs^2{+}3Cs{+}Ds^2{+}4Ds{+}3D = s{-}1$$

From this, we may derive:

$$A+B+C=0$$
$$7A+5B+4C+D=0$$
$$32A+24B+3C+4D=1$$
$$60A+20B+3D=-1$$

Using a solver, we obtain:

$$\begin{cases} A &= -\frac{1}{17} \\ B &= \frac{2}{17} \\ C &= -\frac{1}{17} \\ D &= \frac{1}{17} \end{cases}$$

Now with our coefficients, we take the inverse Laplace transforms to get:

$$x(t) = -\frac{1}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{17}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{(s+2)}{(s+2)^2+4^2}\right\} + \frac{1}{17}\mathcal{L}^{-1}\left\{\frac{4}{(s+2)^2+4^2}\right\}$$

And finally, we get:

$$\boxed{x(t) = -\frac{e^{-t}}{17} + \frac{2e^{-3t}}{17} - \frac{e^{-2t}\cos(4t)}{17} + \frac{e^{-2t}\sin(4t)}{17}}$$

For each term, in order, the ROCs may be identified as: $\sigma = -1$, $\sigma = -3$, and $\sigma = -4$. Since all of the signals are causal, we know the ROCs are to the right. Thus, there will be overlap when σ is greater than the greatest individual ROC, or $\sigma = -1$. This makes the combined ROC: $\boxed{\sigma > -1}$

We may observe that four individual signals contribute to the Laplace Transform. Furthermore, we can find the zeroes and poles as:

$$\boxed{\text{Zero: } s-1=0 \rightarrow s=1}$$

$$\boxed{\text{Poles: } \begin{cases} s+1 &= 0 \\ s+3 &= 0 \\ (s^2+4s+20) &= 0 \end{cases} \rightarrow \begin{cases} s &= -1 \\ s &= -3 \\ s &= -2 \pm 4j \end{cases}}$$

This can be plotted as:

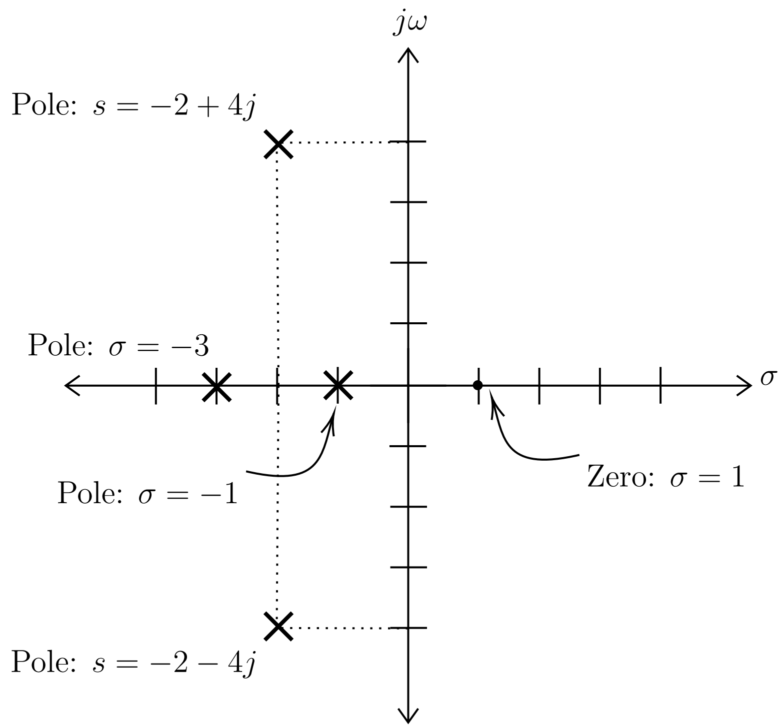

Figure 1: Pole-Zero Plot of $X(s)$

7. (a) Per the basic Laplace transformation tables, we may write:

$$\boxed{X(s) = \frac{1}{s+2} - \frac{1}{s-4}}$$

We may observe that there are two ROCs, $\sigma < 4$ and $\sigma > -2$, which gives us overlap in the region:

$$\boxed{-2 < \sigma < 4}$$

This gives us the following plot:

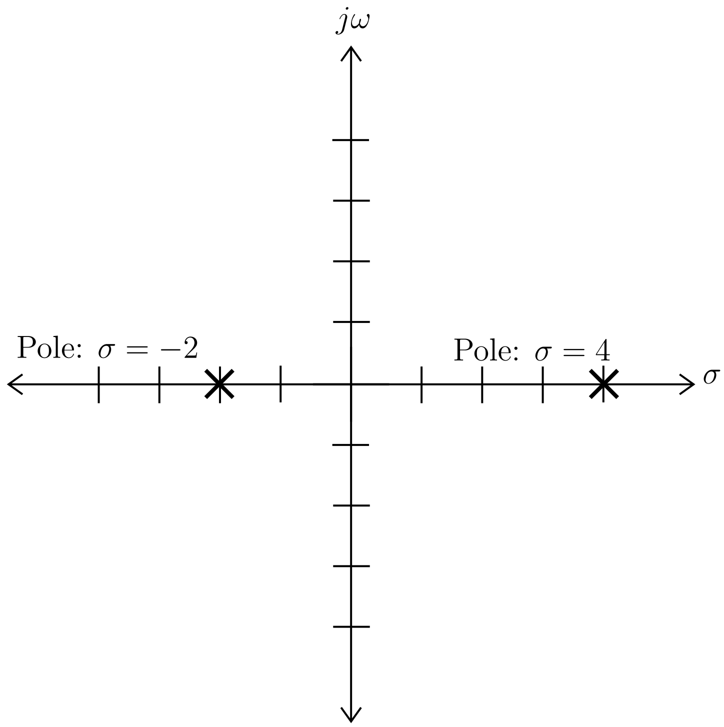

Figure 2: Pole-Zero Plot for 7(a)

(b) Once again employing the tables, we find:

$$X(s) = \frac{1}{s+3} + \frac{4}{(s+2)^2 + 16}$$

Rearranging to simplify ROC analysis, we get:

$$\boxed{X(s) = \frac{s^2 + 8s + 32}{(s+3)(s^2 + 4s + 20)}}$$

From this, we can determine that the zeros are at $s = -4 \pm 4j$ and there are poles at -3 and $-4 \pm 4j$. Since both are right-sided, we may notice that the ROC occurs to the right of the greatest pole, or $\sigma > -3$

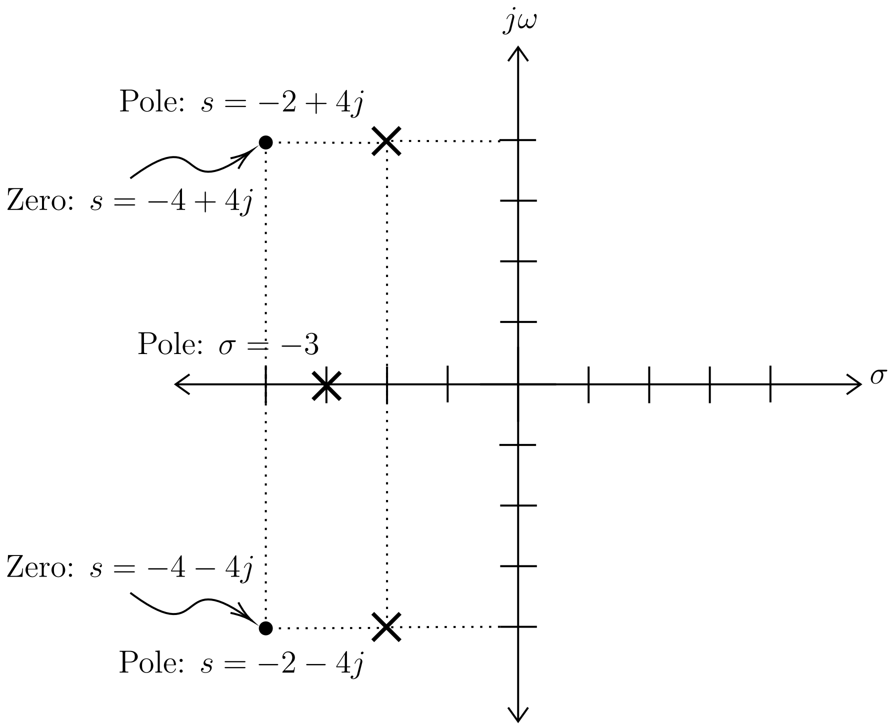

Figure 3: Pole-Zero Plot for 7(b)

(c) We may rewrite $x(t)$ as:

$$x(t) = -te^{2t}u(-t) + te^{-2t}u(t)$$

Using our known transforms:

$$X(s) = -\left[-\frac{d}{ds}\left(\frac{1}{s-2}\right)\right] - \frac{d}{ds}\left(\frac{1}{s+2}\right)$$

$$\boxed{X(s) = \frac{1}{(s+2)^2} - \frac{1}{(s-2)^2}}$$

To simplify analysis, we rearrange to get:

$$X(s) = \frac{(s-2)^2 - (s+2)^2}{(s+2)^2(s-2)^2}$$

$$X(s) = \frac{-8s}{(s+2)^2(s-2)^2}$$

From this, we observe that there is a zero at $s = 0$, and there are poles (both of order 2) $\sigma = \pm 2$. Since both signals are right-handed, the ROC is in: $\sigma > 2$. This gives us the following plot:

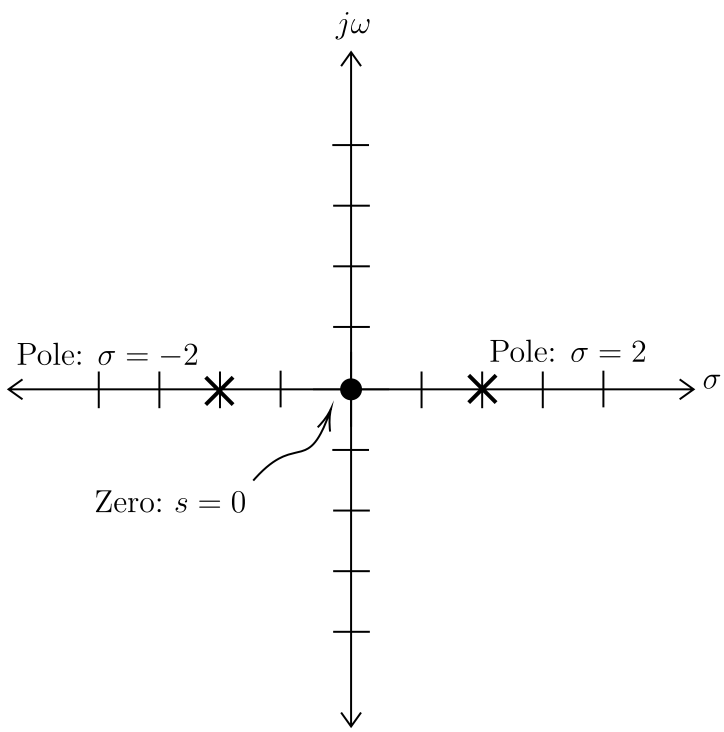

Figure 4: Pole-Zero Plot for 7(c)

(d) We may begin by writing:

$$x(t) = 3r(t) - 3r(t-1)) - 3u(t-2)$$

This gives us the transform as:

$$\boxed{X(s) = \frac{3}{s^2} - \frac{3}{s^2}e^{s} - \frac{3}{s}e^{2s}}$$

We may observe that there are no zeros, but there is a pole at $s = 0$, which gives an ROC of $s > 0$ and the following plot:

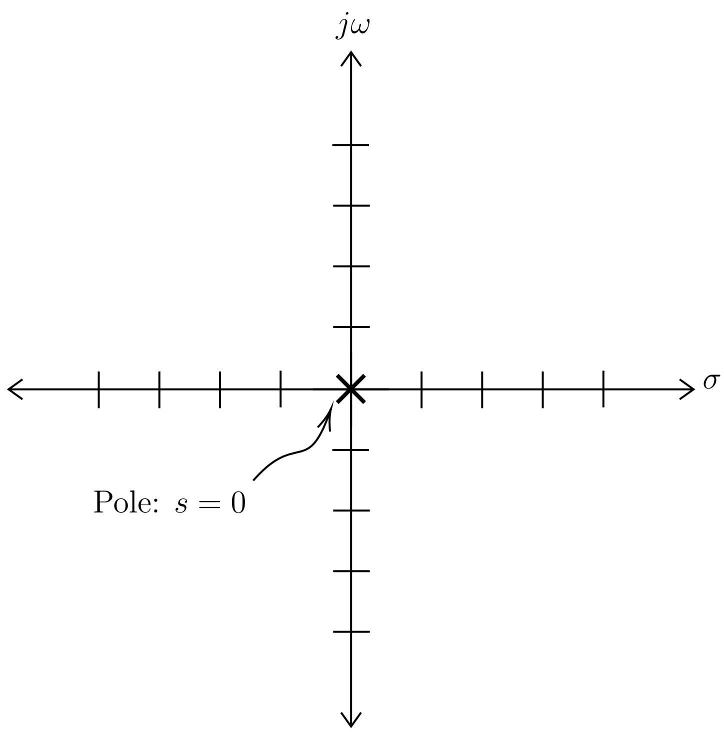

Figure 5: Pole-Zero Plot for 7(d)

8. Since the function is absolutely integrable, we know that it is bounded by BIBO stability. This means that the ROC contains the $j\omega$ axis, and that the ROC does not have any poles. Knowing this, we can deduce:

   (a) The function can not be finite

   This is due to the fact that, since there is a pole at $s = -4$, we know that the signal contains $x(t) = e^{-4t}$. Since this function extends to $t \to \infty$, we know that the signal can not be finite. Also, note that a finite signal, by definition, does not contain poles.

   (b) The function can not be left-sided

   Since we know that, due to being absolutely integrable, the function must contain the $j\omega$ axis, the right-most pole must be to the right of this axis. Since $s = -4$ is to the left of the axis, the signal can not be left-sided.

   (c) The function can be right-sided

   An example of such a signal would be:

   $$\boxed{x(t) = e^{-4t}u(t)}$$

   (d) The function can be two-sided (given that there may be multiple poles — the problem states that there *is* a pole at $s = -4$, but does not state that it is the only pole)

Take, for example, the signal (where $n > 0$):

$$x(t) = e^{-4t}u(t) + e^{nt}u(-t)$$

This signal is two-sided, contains $s = -4$ as a pole, is absolutely integrable (and thus contains the $j\omega$ axis), which makes it a valid signal.

(e)
- With the absolutely integrable rule change from $x(t)$ to $x(t)e^{-2t}$, nothing would change. This is because it would mean that, instead of the $j\omega$ axis, the ROC would now need to contain the $\sigma = 2$ axis. Thus, the properties remain the same since this is to the right of the pole at $s = -4$.
- Changing the condition such that $x(t)e^{5t}$ is absolutely integrable would change things. This would mean that the ROC would need to contain the $\sigma = -5$ axis, instead of the $j\omega$ axis, which is left of the pole. Thus, the signal <u>can now be left-sided, but not right-sided</u>

(f) We know that, due to the integrable rule, the ROC must be right of the right-most pole (to contain the $j\omega$ axis). This means that the ROC is $\sigma > -2$. From here, we write $X(s)$ as:

$$X(s) = \frac{N(s-1)}{(s+4)(s+2-j2)}$$

We can solve for N by using the initial condition:

$$X(0) = \frac{-N}{4(2-j2)}$$
$$\frac{-N}{4(2-j2)} = -1$$
$$N = 8 - j8$$

This gives us:

$$\boxed{X(s) = \frac{(8-j8)(s-1)}{(s+4)(s+2-j2)}}$$

9. (a) Using the table, we get:

$$\boxed{x(t) = \frac{1}{3}\sin(3t)u(t)}$$

And we see that the ROC is $\boxed{\sigma > 0}$

(b) Using the table, we get:

$$\boxed{x(t) = -\cos(2t)u(-t)}$$

And we see that the ROC is $\boxed{\sigma < 0}$

(c) Using the table, we get:

$$\boxed{x(t) = -e^{-5t}\cos(4t)u(-t)}$$

And we see that the ROC is $\boxed{\sigma < -5}$

(d) To simplify analysis, we may rewrite this as:

$$X(s) = \frac{1}{(s+4)(s+2)}$$

We use partial fraction decomposition to get:

$$X(s) = \frac{A}{s+2} + \frac{B}{s+4}$$

Solving, we find: $A = -.5$ and $B = .5$, which gives us:

$$X(s) = \frac{.5}{s+4} - \frac{.5}{s+2}$$

Finally, using the table, we get:

$$\boxed{x(t) = -\frac{1}{2}\left[e^{-4t}u(t) + e^{-2t}u(-t)\right]}$$

And we see the ROC is: $\boxed{-4 < \sigma < -2}$

(e) This can be expanded to:

$$X(s) = \frac{(s+2)^2 - 5s - 10 + 8}{(s+2)^2}$$

We then break this into:

$$X(s) = 1 - \frac{5}{s+2} + \frac{8}{(s+2)^2}$$

We now use the table to find:

$$\boxed{x(t) = \delta(t) - 5e^{-2t}u(t) + 8te^{-2t}u(t)}$$

With ROC: $\boxed{\sigma > -2}$