

Homework 3

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1. Classifying systems as memory-less, time-invariant, linear, causal, and/or stable:

(a) $y(t) = 5e^{4t}x(t-1)$

- Memory: **not** memory-less; the $x(t-1)$ term means the system relies on values other than the present value; therefore, it is not memory-less
- Time-Invariant: **not** time-invariant; we may see that, $y(t-t_o)$ changes the t value in the exponential and $x(t)$ statement, while $x(t-t_o)$ changes only the $x(t)$ statement; thus, it is not time-invariant, since $x(t-t_o) \neq y(t-t_o)$
- Linear: the system **is** linear (see below) because $ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$

$$ax_1(t) + bx_2(t) \rightarrow a5e^{4t}x_1(t-1) + b5e^{4t}x_2(t-1)$$

$$ay_1(t) + by_2(t) \rightarrow a5e^{4t}x_1(t-1) + b5e^{4t}x_2(t-1)$$

- Causal: the system **is** causal, because it only depends on past or present values (ex. $t=0 \rightarrow y(t) = 5e^{4(0)}x(-1)$)
- Stable: Given that the system depends on an exponential e^{4t} , its maximum value is unbounded and, therefore, it is **unstable**

(b) $y(t) = \int_{-\infty}^{\frac{t}{2}} x(\tau) d\tau$

- Memory: **not** memory-less; the system depends on a shift of the t parameter ($t/2$), and, therefore, does not always depend on the current value of time
- Time-Invariant: **not** time-invariant; $y(t-t_o) \neq x(t-t_o)$ (see below)

$$x(t-t_o) \rightarrow \int_{-\infty}^{\frac{t}{2}} x(\tau-t_o) d\tau$$

$$y(t-t_o) \rightarrow \int_{-\infty}^{\frac{(t-t_o)}{2}} x(\tau-t_o) d\tau$$

$$\therefore x(t-t_o) \neq y(t-t_o)$$

- Linear: the system **is** linear; it follows both the superposition and homogeneity principles (see below)

$$ay_1(t) + by_2(t) \rightarrow a \int_{-\infty}^{\frac{t}{2}} x_1(\tau) d\tau + b \int_{-\infty}^{\frac{t}{2}} x_2(\tau) d\tau$$

$$ax_1(t) + bx_2(t) \rightarrow a \int_{-\infty}^{\frac{t}{2}} x_1(\tau) d\tau + b \int_{-\infty}^{\frac{t}{2}} x_2(\tau) d\tau$$

$$\therefore ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$$

- Causal: the system **is not** causal; integration depends on future values when $t < 0$
- Stable: the system **is not** stable (see below)

$$y(t) = \int_{-\infty}^{\frac{t}{2}} x(\tau) d\tau$$

$$h(t) = \int_{-\infty}^{\frac{t}{2}} \delta(\tau) d\tau = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

$$h(t) \rightarrow u(t)$$

$$\int_{-\infty}^{\infty} h(t) dt = \infty$$

(c) $y(t) = 4 + 5 \frac{d^2}{dt^2} x(t)$

- Memory: the system is **not** memory-less; the use of a differential implies that the system depends on past values
- Time-Invariant: the system **is** time-invariant (see below)

$$x(t - t_o) \rightarrow 4 + 5 \frac{d^2}{dt^2} x(t - t_o)$$

$$y(t - t_o) \rightarrow 4 + 5 \frac{d^2}{dt^2} x(t - t_o)$$

$$\therefore x(t - t_o) = y(t - t_o)$$

- Linear: the system is **not** linear (see below)

$$ax_1(t) + bx_2(t) = \left(4 + 5a \frac{d^2}{dt^2} x_1(t)\right) + \left(4 + 5b \frac{d^2}{dt^2} x_2(t)\right)$$

$$ay_1(t) + by_2(t) = a \left(4 + 5 \frac{d^2}{dt^2} x_1(t)\right) + b \left(4 + 5 \frac{d^2}{dt^2} x_2(t)\right)$$

$$\therefore ax_1(t) + bx_2(t) \neq ay_1(t) + by_2(t)$$

- Causal: the system **is** causal because it only depends on past or present values
- Stable: the system is **unstable** because it is unbounded

$$(d) \ y(t) = \begin{cases} 0, & t < 0 \\ x(t-2) + 2x(t), & t \geq 0 \end{cases}$$

- Memory: the system is **not** memory-less, since the $x(t-2)$ term depends on a past value
- Time-Invariant: the system is **not** time-invariant (see below)

$$x(t-t_o) \rightarrow \begin{cases} 0, & t < 0 \\ x(t-2-t_o) + 2x(t-t_o), & t \geq 0 \end{cases}$$

$$y(t-t_o) \rightarrow \begin{cases} 0, & t < 2 \\ x(t-2-t_o) + 2x(t-t_o), & t \geq 2 \end{cases}$$

$$\therefore x(t-t_o) \neq y(t-t_o)$$

- Linear: the system **is** linear (see below)

$$ax_1(t) + bx_2(t) \rightarrow \begin{cases} 0, & t < 0 \\ ax_1(t-2) + 2ax_1(t) + bx_2(t-2) + 2bx_2(t), & t \geq 0 \end{cases}$$

$$ay_1(t) + by_2(t) \rightarrow \begin{cases} 0, & t < 0 \\ ax_1(t-2) + 2ax_1(t) + bx_2(t-2) + 2bx_2(t), & t \geq 0 \end{cases}$$

$$\therefore ax_1(t) + bx_2(t) = ay_1(t) + by_2(t)$$

- Causal: the system **is** causal because it only depends on past or present values
- Stable: the system **is** stable, because it does not tend to diverge

Problem 1 can be tabulated as follows:

System	a	b	c	d
Memory-Less	no	no	no	no
Time-Invariant	no	no	yes	no
Linear	yes	yes	no	yes
Causal	yes	no	yes	yes
Stable	no	no	no	yes

2. Classifying systems as memory-less, time-invariant, linear, causal, and/or stable:

$$(a) \ y[n] = x[n+1] - 2x[n-4]$$

- Memory: system is **not** memory-less, as it depends on past and future values
- Time-Invariant: system **is** time-invariant (see below)

$$x[n-n_o] = x[n+1-n_o] - 2x[n-4-n_o]$$

$$y[n-n_o] = x[n+1-n_o] - 2x[n-4-n_o]$$

$$\therefore x[n-n_o] = y[n-n_o]$$

- Linear: system **is** linear (see below)

$$ax_1[n] + bx_2[n] = a(x_1[n+1] - 2x_1[n-4]) + b(x_2[n+1] - 2x_2[n-4])$$

$$ay_1[n] + by_2[n] = a(x_1[n+1] - 2x_1[n-4]) + b(x_2[n+1] - 2x_2[n-4])$$

$$\therefore ay_1[n] + by_2[n] = ax_1[n] + bx_2[n]$$

- Causal: system is **not** causal, as it depends on past and present values
- Stable: system **is** stable because $y[n]$ is finite

(b) $y[n] = \text{Even}\{x[n-1]\}$

To simplify analysis, we can express the even function as:

$$\frac{x[n] + x^*[-n]}{2} \rightarrow \frac{x[n-1] + x^*[-n+1]}{2}$$

- Memory: system is **not** memory-less, as it depends on future values
- Time-Invariant: system **is** time-invariant (see below)

$$x[n - n_o] = \text{Even}\{x[n-1 - n_o]\}$$

$$y[n - n_o] = \text{Even}\{x[n-1 - n_o]\}$$

$$\therefore x[n - n_o] = y[n - n_o]$$

- Linear: system is **not** linear (see below); note that this is because a or b may be complex. Given this, for a purely real signal, the system can be classified as linear; however, due to the need to use the 'conjugate' for the even function, in the case of a complex signal, this is non-linear

$$ax_1[n] + bx_2[n] = \frac{ax_1[n-1] + a^*x_1^*[-n+1]}{2} + \frac{bx_2[n-1] + b^*x_2^*[-n+1]}{2}$$

$$ay_1[n] + by_2[n] = \frac{ax_1[n-1] + ax_1^*[-n+1]}{2} + \frac{bx_2[n-1] + bx_2^*[-n+1]}{2}$$

$$\therefore ay_1[n] + by_2[n] \neq ax_1[n] + bx_2[n]$$

- Causal: system is **not** causal, as it depends on a future value
- Stable: system **is** stable because $y[n]$ is finite

(c) $y[n] = 5x[3n+1]$

- Memory: system is **not** memory-less, as it depends on non-present values
- Time-Invariant: system is **not** time-invariant (see below)

$$x[n - n_o] = 5x[3n - n_o + 1]$$

$$y[n - n_o] = 5x[3n - 3n_o + 1]$$

$$\therefore x[n - n_o] \neq y[n - n_o]$$

- Linear: system **is** linear (see below)

$$ax_1[n] + bx_2[n] = 5ax_1[3n + 1] + 5bx_2[3n + 1]$$

$$ay_1[n] + by_2[n] = 5ax_1[3n + 1] + 5bx_2[3n + 1]$$

$$\therefore ay_1[n] + by_2[n] = ax_1[n] + bx_2[n]$$

- Causal: system is **not** causal, as it depends on a future value
- Stable: system **is** stable because $y[n]$ is finite

$$(d) \ y[n] = \begin{cases} 0, & n = 2 \\ x[n], & \text{otherwise} \end{cases}$$

- Memory: system **is** memory-less; it does not depend on past or present values
- Time-Invariant: system is **not** time-invariant (see below)

$$x[n - n_o] = \begin{cases} 0, & n = 2 \\ x[n - n_o], & \text{otherwise} \end{cases}$$

$$y[n - n_o] = \begin{cases} 0, & n = 2 + n_o \\ x[n - n_o], & \text{otherwise} \end{cases}$$

$$\therefore x[n - n_o] \neq y[n - n_o]$$

- Linear: system **is** linear (see below)

$$ax_1[n] + bx_2[n] = \begin{cases} 0, & n = 2 \\ ax_1[n] + bx_2[n], & \text{otherwise} \end{cases}$$

$$ay_1[n] + by_2[n] = \begin{cases} 0, & n = 2 \\ ax_1[n] + bx_2[n], & \text{otherwise} \end{cases}$$

$$\therefore ay_1[n] + by_2[n] = ax_1[n] + bx_2[n]$$

- Causal: system **is** causal, as it depends on only past or present values
- Stable: system **is** stable because $y[n]$ is finite

3. (a) We may begin by constructing the expression for the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

This gets us:

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^{k-2} u[k - 2]u[n - k + 1]$$

We may observe the following:

$$x[n] \neq 0, \quad n \geq 2$$

$$h[n] \neq 0, \quad n \geq -1$$

Integrating this into the convolution sum, we may see that the expression is nonzero (unit functions both exist) for $k \geq 2$, and that $k \leq n + 1$. Thus, we get:

$$y[n] = \sum_{k=2}^{n+1} \left(\frac{1}{4}\right)^{k-2}$$

Using the expansion for series, we obtain:

$$y[n] = \frac{1 - (.25)^{n-1}}{1 - .25}$$

$$y[n] = \frac{4}{3} \left[1 - \left(\frac{1}{4}\right)^{n-1} \right] \text{ for } n \geq 2$$

This can be sketched as:

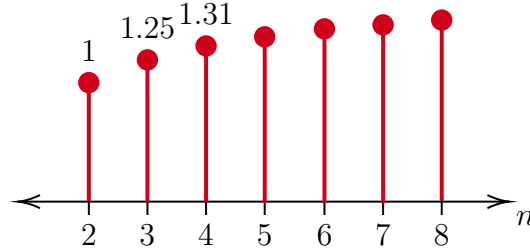


Figure 1: Sketch of $y[n]$

(b) Per the time-shifting property, we know that:

$$y[n] = x[n] * h[n] \rightarrow y_1[n] = y[n - 3] = x[n - 3] * h[n]$$

Which gives us:

$$y_1[n] = \frac{4}{3} \left[1 - \left(\frac{1}{4}\right)^{n-4} \right] \text{ for } n \geq 5$$

(c) Per the time-shifting property, we know that:

$$y[n] = x[n] * h[n] \rightarrow y_2[n] = y[n - 2] = x[n] * h[n - 2]$$

$$y_2[n] = \frac{4}{3} \left[1 - \left(\frac{1}{4}\right)^{n-3} \right] \text{ for } n \geq 4$$

(d) Per the time-shifting property, we know that:

$$y[n] = x[n] * h[n] \rightarrow y_3[n] = y[n+1] = x[n-2] * h[n+3]$$

$$y_3[n] = \frac{4}{3} \left[1 - \left(\frac{1}{4} \right)^n \right] \text{ for } n \geq 1$$

4. We may simplify analysis by using a tabulating method. Using this, we can see that:

$$x = \{0, 1, 1, 1\}$$

$$h = \{0, 0, 0, 0, 0, 1, 1, 1, 1, 1\}$$

This can be written as:

	0	1	1	1
0
0
0
0
0
1
1
1
1
1
1

Here, we multiply the corresponding terms to populate the table:

	0	1	1	1
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0
1	0	1	1	1
1	0	1	1	1
1	0	1	1	1
1	0	1	1	1
1	0	1	1	1
1	0	1	1	1

Now we add diagonals to find the product, which gives us:

$$y[n] = \{0, (0+0), (0+0+0), (0+0+0+0), (0+0+0+0), \\ (0+0+0+0), (0+0+1+0), (0+1+1+0), \\ (0+1+1+1), (0+1+1+1), (0+1+1+1), (1+1+1), (1+1), 1\}$$

This can be simplified to:

$$y[n] = \{0, 0, 0, 0, 0, 0, 1, 2, 3, 3, 3, 3, 2, 1, 0 \dots\}$$

Using the step function notation, we may write:

$$y[n] = u[n-6] + u[n-7] + u[n-8] - u[n-12] - u[n-13] - u[n-14]$$

5. We can use a method similar to question (4) to find N . Let us begin with a table:

	1	1	1	1	1	1
1						
1						
\vdots						

Using the tabulating method, we know that, for there to be a value of 4, $h[n]$ needs to exist for at least 4 different n . Therefore, let us begin with the minimum such value for $h[n]$, when $N = 3$, which gives us the following table:

	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1
1	1	1	1	1	1	1

This gives us $y[n] = \{1, 2, 3, 4, 4, 4, 3, 2, 1, 0, \dots\}$. With this result, we see that $y[3] = 4$, and $y[9] = 0$. Thus, we find that:

$$\boxed{N = 3}$$

6. (a) We can express this as $x_1[n] = u[n - 3]$

Since this is an LTI system, we know that the output will be shifted by the same amount (shifting property), which gives us:

$$\boxed{x_1[n] \rightarrow y_1[n] = \left(\frac{1}{4}\right)^{n-3} u[n - 3] - \left(\frac{1}{4}\right)^{n-4} u[n - 4]}$$

- (b) This input can be expressed as: $x_2[n] = u[n] - u[n - 4]$

Once again by the shifting property, as well as by homogeneity and superposition, we can write:

$$\boxed{x_2[n] \rightarrow y_2[n] = \left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{4}\right)^{n-1} u[n - 1] - \left(\frac{1}{4}\right)^{n-4} u[n - 4] + \left(\frac{1}{4}\right)^{n-5} u[n - 5]}$$

- (c) This input can be expressed as: $x_3[n] = 2u[n] - 2u[n - 1]$

By the shifting, homogenous, and superposition principles of LTI systems, we get:

$$x_3[n] \rightarrow y_2[n] = 2 \left(\frac{1}{4}\right)^n u[n] - 2 \left(\frac{1}{4}\right)^{n-1} u[n - 1] - 2 \left(\frac{1}{4}\right)^{n-1} u[n - 1] + 2 \left(\frac{1}{4}\right)^{n-2} u[n - 2]$$

Which can be simplified:

$$\boxed{x_3[n] \rightarrow y_2[n] = 2 \left(\frac{1}{4}\right)^n u[n] - 4 \left(\frac{1}{4}\right)^{n-1} u[n - 1] + 2 \left(\frac{1}{4}\right)^{n-2} u[n - 2]}$$

7. Per convolution properties, we know:

$$x(t) * \delta(t - t_o) \Rightarrow x(t - t_o)$$

Thus, looking at the input, in addition to the fact that this is an LTI system, we can write:

$$\boxed{y(t) = x(t) * h(t) \rightarrow y(t) = x(t + 2) - 2x(t)}$$

Plotting this shift, we get:

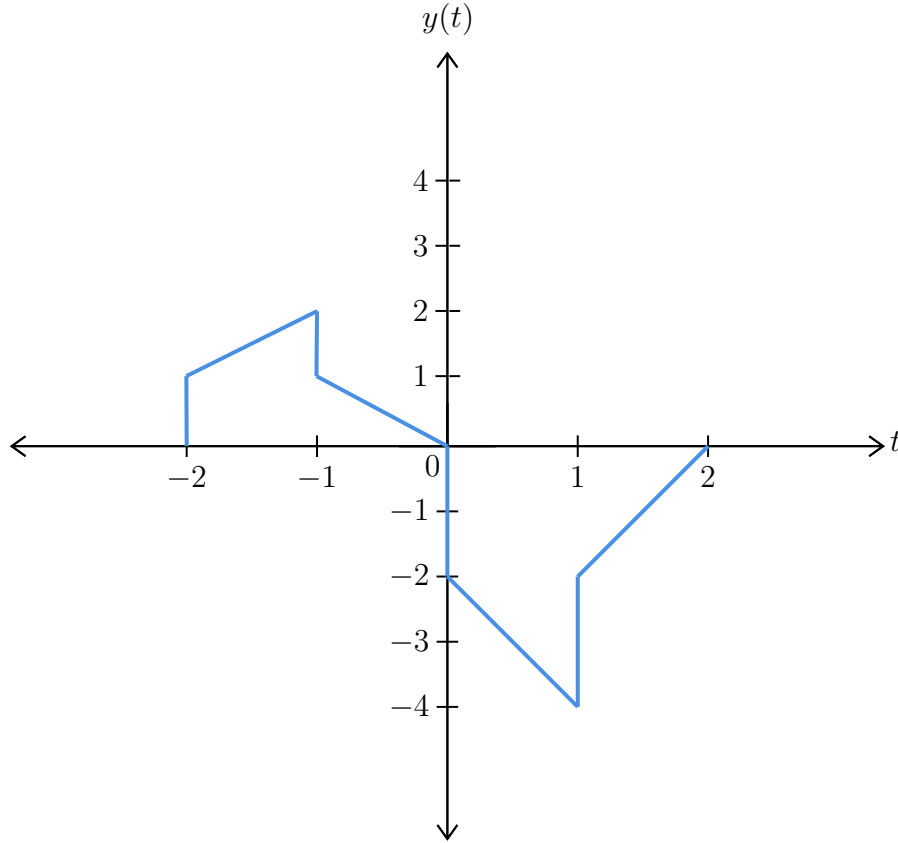


Figure 2: Plot for $y(t) = x(t+2) - 2x(t)$

8. (a) Using this, we can rewrite x as:

$$x(t) = u(t) - u(t-1)$$

Which would make h :

$$h(t) = u(t-2) - u(t-4)$$

This gives us:

$$y(t) = \int [u(t) - u(t-1)] * [u(t-2) - u(t-4)] dt$$

Which expands to:

$$y(t) = \int u(t)u(t-2) - u(t)u(t-4) - u(t-1)u(t-2) + u(t-1)u(t-4) dt$$

To analyze, we can use the following properties:

$$r(t) * r(t - T_1) = r(t - T_1)$$

$$r(t - T_1) * r(t - T_2) = r(t - T_1 - T_2)$$

This gives us:

$$y(t) = r(t - 2) - r(t - 4) - r(t - 3) + r(t - 5)$$

$$y(t) = r(t - 2) - r(t - 3) - r(t - 4) + r(t - 5)$$

And the plot becomes:

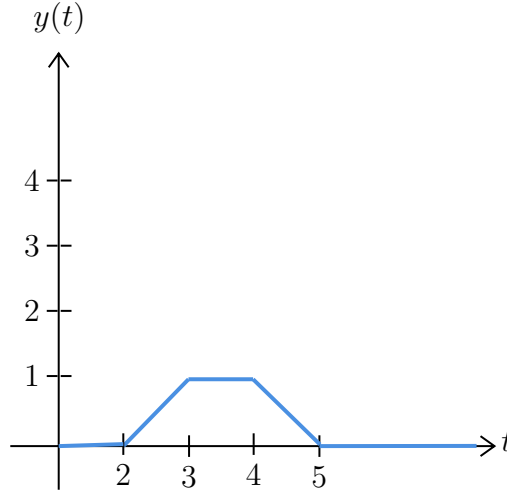


Figure 3: Plot for $y(t) = r(t - 2) - r(t - 3) - r(t - 4) + r(t - 5)$

(b) We can determine $\frac{d}{dt}[y(t)]$ to be:

$$\frac{d}{dt}[y(t)] = u(t - 2) - u(t - 3) - u(t - 4) + u(t - 5)$$

The discontinuities may be identified by differentiating again to get:

$$\frac{d^2}{dt^2}[y(t)] = \delta(t - 2) - \delta(t - 3) - \delta(t - 4) + \delta(t - 5)$$

We know that each impulse represents a discontinuity; Therefore, we can tell that there are 4 discontinuities, at $t = 2, 3, 4$, and 5 , for $d/dt[y(t)]$

9. (a) We know that the impulse response can be determined if $x(t) = \delta(t)$:

$$y(t) = \int_{-\infty}^t e^{-(t-\tau)} x(\tau - 2) d\tau$$

$$y(t) = \int_{-\infty}^t e^{-(t-\tau)} \delta(\tau - 2) d\tau$$

We also know from the property of the impulse that:

$$\int_{-\infty}^{\infty} x(t) \delta(t - t_o) dt = x(t_o)$$

Thus, we can see that:

$$\boxed{h(t) = e^{-(t-2)}}$$

(b) We may begin by writing $x(t)$:

$$x(t) = u(t + 1) - u(t - 2)$$

Alternatively, this may be written in laplace form as:

$$x(s) = \frac{e^s}{s} - \frac{e^{-2s}}{s + 1}$$

Which gives us:

$$h(s) = \frac{1}{s + 1}$$

Multiplying the two together, we get:

$$x(s)h(s) = e^s \left[\frac{1}{s} - \frac{1}{s + 1} \right] - \frac{e^{-2s}}{(s + 1)^2}$$

We then take the inverse laplace transform to finally find:

$$\boxed{y(t) = u(t + 1) - e^{-t-1}u(t + 1) - (t - 2)e^{2-t}u(t - 2)}$$