Homework 8

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1. (a) By the duality property, we may write:

$$\cos(\omega_o t) \leftrightarrow \pi\left[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right]$$
$$\pi\left[\delta(t-t_o)+\delta(t+t_o)\right] \leftrightarrow 2\pi\cos(\omega t_o)$$

In tandem with the time shifting property, we may write:

$$\frac{1}{2}e^{-\alpha j\omega}\left[\delta(t-t_o)+\delta(t+t_o)\right] \leftrightarrow \cos(\omega t_o-\alpha)$$

This gives us:

$$\boxed{x(t)=\frac{1}{2}e^{-\frac{\pi j\omega}{3}}\left[\delta(t-4)+\delta(t+4)\right]}$$

(b) We may use the following known transform formulas:

$$\pi\left[\delta(\omega-\omega_o)+\delta(\omega+\omega_o)\right] \leftrightarrow \cos(\omega_o t)$$
$$\frac{\pi}{j}\left[\delta(\omega-\omega_o)-\delta(\omega+\omega_o)\right] \leftrightarrow \sin(\omega_o t)$$
$$\frac{1}{(\alpha+j\omega)^2} \leftrightarrow te^{-\alpha t}u(t)$$

In combination with the time shifting property, we may get:

$$\boxed{x(t)=\frac{2}{\pi}\cos(5t-3)-4j\pi\sin(2\pi t)+te^{-2t}u(t)}$$

2. (a) The signal $x(t)$ may be expressed in several parts: a step up by 1 at $t=-1$ to $t=1$, a slope of 1 between -1 and 1, and a step up of 2 for $t>1$. Thus, we get:

$$\boxed{x(t)=r(t+1)-r(t-1)}$$

This lets us find:

$$\boxed{\frac{dx}{dt} = u(t+1) - u(t-1)}$$

And finally:

$$\boxed{\frac{d^2x}{dt^2} = \delta(t+1) - \delta(t-1)}$$

(b) The DC component can be accounted for using:

$$x_{DC} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt$$

This gives:

$$x_{DC} = \lim_{T\to\infty} \frac{1}{T} \left[\int_{-1}^{1} (t+1)\,dt + \int_{1}^{\frac{T}{2}} 2\,dt \right]$$

$$x_{DC} = \lim_{T\to\infty} \frac{1}{T} \left[2 + 2\left(\frac{T}{2} - 1\right) \right]$$

$$x_{DC} = 1$$

Using the duality property, we may write:

$$1 \leftrightarrow 2\pi\delta(\omega)$$

We then transform the second order derivative to see:

$$\mathcal{F}\left\{\frac{d^2x(t)}{dt^2}\right\} = e^{j\omega} - e^{-j\omega}$$

$$(j\omega)^2 X(\omega) = e^{j\omega} - e^{-j\omega}$$

This gives us:

$$X(j\omega) = -\frac{2j\sin(\omega)}{\omega^2}$$

We then add in the DC component to get:

$$\boxed{X(j\omega) = 2\pi\delta(\omega) - \frac{2j\sin(\omega)}{\omega^2}}$$

(c) Given that the subtraction of the 1 removes the DC component, we simply get:

$$\boxed{G(j\omega) = -\frac{2j\sin(\omega)}{\omega^2}}$$

4. (a) Per the theorem, we may write:

$$\int_{-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$
$$2\pi \int_{-\infty}^{\infty} |x(t)|^2 = \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

This allows us to write:

$$A = 4\pi \int_{0}^{\infty} e^{-4t} \, dt$$
$$-\pi \left(e^{-4t}\right) \Big|_{0}^{\infty}$$
$$-\pi \left(e^{-4\infty} - 1\right)$$

Thus, we obtain:

$$\boxed{A = \pi}$$

(b) Per our Fourier transform properties, we may write:

$$tx(t) \to j \frac{d}{d\omega} X(\omega)$$

For $y(t) = te^{-2|t|}$ this gives us:

$$Y(j\omega) = j \frac{d}{d\omega} \left[\frac{4}{(4 + \omega^2)} \right]$$

Differentiating gives us the final answer as:

$$\boxed{Y(j\omega) = -\frac{8j\omega}{(4 + \omega^2)^2}}$$

(c) By the duality property, we know that if $x(t) \leftrightarrow X(j\omega)$, then:

$$x(t) \leftrightarrow 2\pi X(-j\omega)$$

As such, we may write:

$$-\frac{8jt}{(4 + t^2)^2} \leftrightarrow 2\pi(-\omega)e^{-2|\omega|}$$
$$\frac{t}{(4 + t^2)^2} \leftrightarrow -j\pi\omega e^{-2|\omega|}$$

Thus, we see that:

$$\boxed{\mathcal{F}\left\{ \frac{4t}{(4 + t^2)^2} \right\} = -j\pi\omega e^{-2|\omega|}}$$

5. We may compute the response of the system using the convolution; the convolution may be more easily computed using the Fourier transform such that:

$$y(t) = h(t) * x(t) \leftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

We can see that the given response may be written as:

$$H(j\omega) = \begin{cases} e^{-3j\omega}, & |\omega| \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) We may see that the transform becomes:

$$X_1(j\omega) = \pi e^{N\pi j\omega}\left[\delta(\omega - 10) + \delta(\omega + 10)\right]$$

We may observe that the response and signal do not have common values for which they are non-zero. This gives us:

$$Y_1(j\omega) = 0$$

This ultimately means:

$$\boxed{y_1(t) = 0}$$

(b) We may see that the transform becomes:

$$X_2(j\omega) = 5\pi\left[\delta(\omega - 2) + \delta(\omega + 2)\right]$$

We multiply the two together to get:

$$Y_2(j\omega) = 5\pi e^{-3j\omega}\left[\delta(\omega - 2) + \delta(\omega + 2)\right]$$

We see this introduces a delay of three units, which gives us:

$$\boxed{y_2(t) = 5\cos(2(t - 3))}$$

(c) We may see that the transform becomes:

$$X_3(j\omega) = \begin{cases} 1, & |\omega| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Multiplying the two together creates a delayed sink function, but with the boundaries of the input:

$$Y_3(j\omega) = \begin{cases} e^{-3j\omega}, & |\omega| \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Transforming back, we get:

$$\boxed{y_3(t) = \frac{\sin(t-3)}{\pi(t-3)}}$$

(d) We may observe that the input signals within the passband of the filter simply introduce a delay at the output, thus, we may conclude:

$$\boxed{y_4(t) = \left(\frac{\sin(t-3)}{\pi(t-3)}\right)^2}$$

6. (a) We may apply the Laplace transform to get:

$$s^2Y(s) + 5sY(s) + 4Y(s) = 3X(s)$$

Since we know that the response is the output over input, we may write:

$$H(s) = \frac{3}{s^2 + 5s + 4}$$
$$H(s) = \frac{3}{(s+4)(s+1)}$$

We use partial fraction decomposition in order to be able to apply the reverse transform. This gets us:

$$H(s) = -\frac{1}{s+1} + \frac{1}{s+4}$$

We take the inverse transform to conclude:

$$\boxed{h(t) = [-e^{-t} + e^{-4t}]u(t)}$$

(b) Given the input $x(t)$, we may write:

$$X(s) = \frac{1}{s+1}$$

We know the response may be written as:

$$Y(s) = H(s)X(s)$$

And so we get:

$$Y(s) = \frac{3}{(s+4)(s+1)^2}$$

We once again use partial fraction decomposition to write:

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+4}$$

$$Y(s) = \frac{-1}{3(s+1)} + \frac{1}{(s+1)^2} + \frac{1}{3(s+4)}$$

We take the inverse transform to get:

$$\boxed{y(t) = \left[-\frac{1}{3}e^{-t} + \frac{1}{3}e^{-4t} + te^{-t}\right]u(t)}$$

8. (a) We may take the Laplace transform of both to get:

$$X(s) = \frac{1}{s+1} + \frac{2}{s+4}$$
$$Y(s) = \frac{3}{s+1} - \frac{3}{s+3}$$

We may combine the fractions to get:

$$X(s) = \frac{3s+6}{(s+1)(s+4)}$$
$$Y(s) = \frac{6}{(s+1)(s+3)}$$

We know the response may be written as:

$$H(s) = \frac{Y(s)}{X(s)}$$

This gives us:

$$H(s) = \frac{6}{(s+1)(s+3)} \cdot \frac{(s+1)(s+4)}{3s+6}$$
$$H(s) = \frac{2s+8}{(s+2)(s+3)}$$

Given that $s = j\omega$, we may write:

$$H(j\omega) = \frac{2j\omega + 8}{(j\omega+2)(j\omega+3)}$$

We may simplify to get:

$$\boxed{H(j\omega) = \frac{2j\omega + 8}{6 + 5j\omega - \omega^2}}$$

(b) Using partial fraction decomposition, we may write:

$$H(s) = \frac{A}{s+2} + \frac{B}{s+3}$$

We may find that $A = 4$ and $B = -2$ to get:

$$H(s) = \frac{4}{s+2} - \frac{2}{s+3}$$

Taking the inverse transform, we find:

$$\boxed{h(t) = [4e^{-2t} - 2e^{-3t}]u(t)}$$

(c) From part (a), we know:

$$\frac{Y(s)}{X(s)} = \frac{2s+8}{(s+2)(s+3)}$$
$$Y(s)[s^2 + 5s + 6] = X(s)[2s + 8]$$

Taking the inverse transform, we get:

$$\boxed{\frac{d^2y(t)}{dt} + 5\frac{dy(t)}{dt} + 6y(t) = 2\frac{dx(t)}{dt} + 8x(t)}$$

(d) For an inverse system, we know:

$$H(s)H_i(s) = 1$$

As such, we get:

$$H_i(s) = \frac{(s+2)(s+3)}{2s+8}$$

Taking $s \to j\omega$, we get:

$$\boxed{H_i(j\omega) = \frac{(j\omega+2)(j\omega+3)}{2j\omega+8}}$$

9. To plot the signals, we may begin by converting them to their Fourier form. Moving in order, we may begin with $v_1(t)$ to observe:

$$v_1(t) = x(t) \cdot 2\cos(\omega_c t)$$
$$V_1(j\omega) = 2\pi\delta(\omega - \omega_c) + \delta(\omega + \omega - c)X(j\omega)$$

Since, at ω_c, $X(j\omega) = .5$, This gives us:

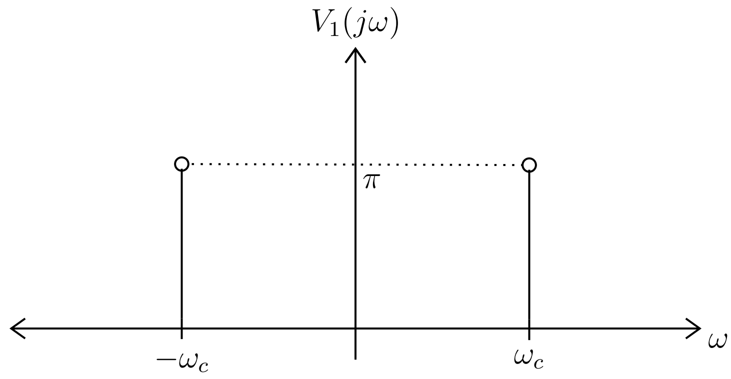

Figure 1: Spectrum of $V_1(j\omega)$

As a result of the filter, we may see that:

$$W_1(j\omega) = 2\pi\left[\delta(\omega - \omega_c) + \delta(\omega - \omega_c)\right]$$

Changing ω_c to ω_o per the given relationship, we get:

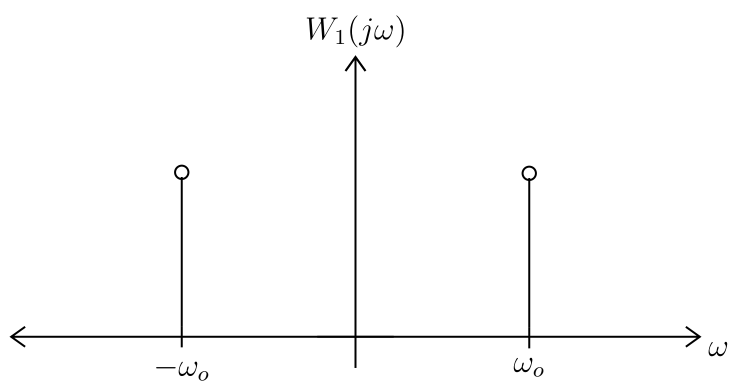

Figure 2: Spectrum of $W_1(j\omega)$

We then check $V_2(j\omega)$ to see that, because of the flipped sign in the transform of the sin term, the negative corner frequency has negative magnitude:

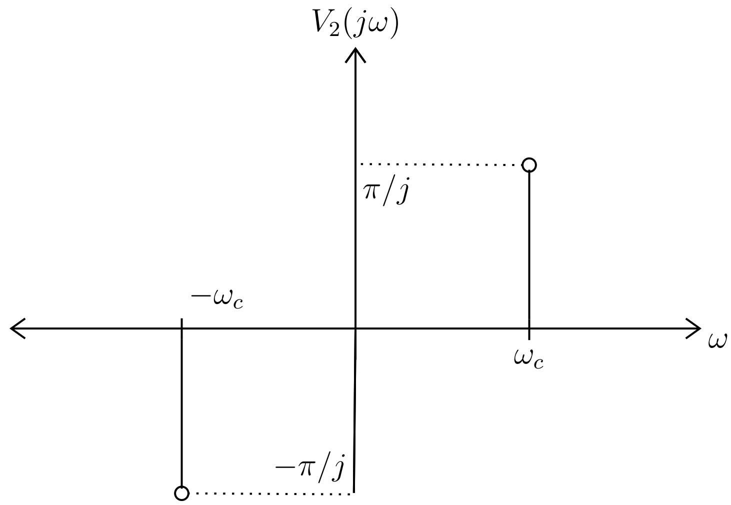

Figure 3: Spectrum of $V_2(j\omega)$

Because of the filter, however, we see that the spectrum of $W_2(j\omega)$ remains the same as that of $W_1(j\omega)$:

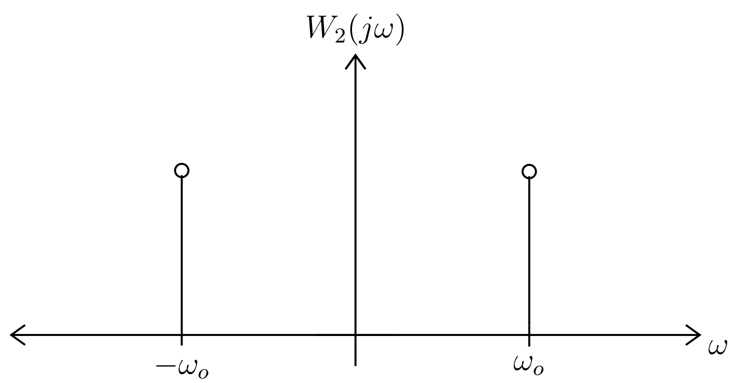

Figure 4: Spectrum of $W_2(j\omega)$

We see that combining the signals $W_{1,2}$ with the sinusoids results in Fourier responses consisting of rect functions, centered at $\pm\omega_c$, with a width of $2\omega_o$:

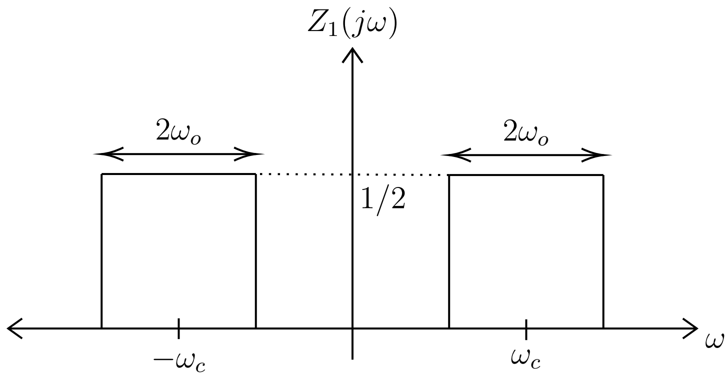

Figure 5: Spectrum of $Z_1(j\omega)$

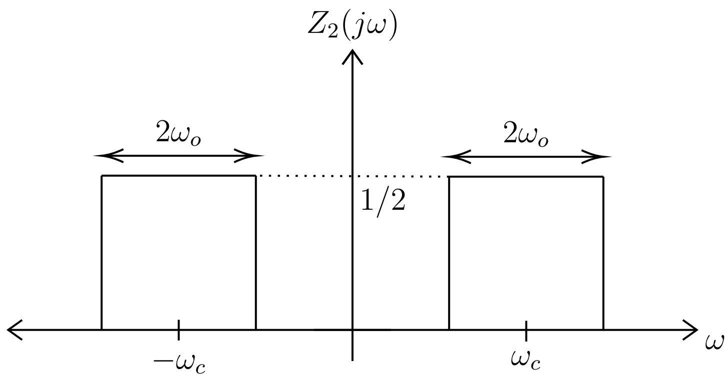

Figure 6: Spectrum of $Z_2(j\omega)$

Summing the signals gives us a height of 1, which indicates that this is a bandpass filter with center frequency ω_c and bandwidth $2\omega_o$:

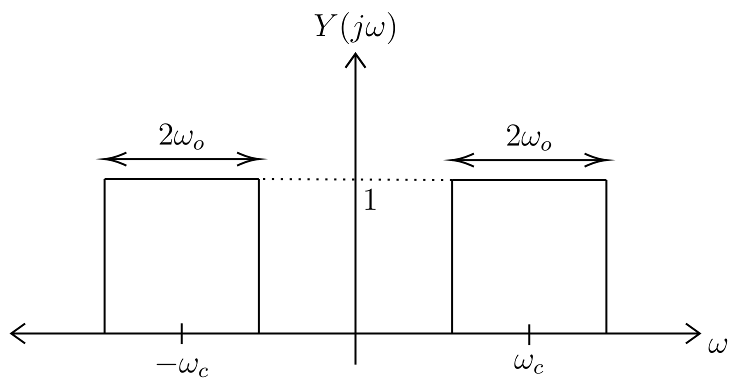

Figure 7: Spectrum of $Y(j\omega)$

10. We may write the equivalent impedances of all of the components as:

$$Z_L = .5s \quad \text{and } Z_C = \frac{1}{2s}$$

The equivalent impedance becomes:

$$Z_{eq} = .5s + 1 + \frac{1}{2s}$$

The current becomes:

$$I(s) = \frac{X(s)}{Z_{eq}}$$

Which means that the voltage across the capacitor, $y(t)$ is:

$$Y(s) = \frac{\frac{1}{2s}X(s)}{.5s + 1 + \frac{1}{2s}}$$

Since the response is output over input, we get:

$$H(s) = \frac{1}{s^2 + 2s + 1}$$
$$H(s) = \frac{1}{(s+1)^2}$$

This gives us the final response as:

$$\boxed{h(t) = te^{-t}u(t)}$$